Effect of Concrete and Abstract aspects of Mathematics on its Teaching

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Abstract and Concrete

Amongst different subjects ~~the~~ which are taught in schools, ~~Mathema~~ mathematics is the one subject which is of great practical ~~use~~ utility and yet which is fundamentally of an abstract nature. In other words mathematics has both aspects – concrete as well as abstract. Every child begins to acquire an acquaintenance with concrete as well as abstract ideas from his mother. A toy which can be handled by a child is a concrete object. But when the mother gives the name of the ~~child~~ toy, the child comes to know a new "word". A process of one-one correspondence between the solid toy elephant in his hand (concrete) and the word "elephant" (abstract) heard from the mother begins to take root in the child's mind and in this manner he ~~starts to~~ begins to form a habit of building a bridge between the concrete and the abstract. Afterwards when the child sees a live elephant ~~w~~ walking down the road and he exclaims, "Mummy, ~~mummy~~, see that elephant", we get a proof of such a bridge being formed in the child's mind.

The formation of this habit ~~of~~ in the child of building a bridge between the concrete and the abstract initiated by the mother is extended to different branches of knowledge by the mathematics-teachers of primary and secondary schools because this teacher too, ~~by~~ organises ~~ing~~ the concrete illustrations of everyday life in terms of abstract numbers and operations on them. ~~helps to put on firm foundation this~~

He thus puts on a firm foundation this habit initiated by the mother.

But are both these aspects of mathematics properly emphasised in our present day teaching of mathematics? This is a point which requires careful consideration in any attempts to improve mathematics-teaching in our schools and colleges.

In the present day teaching of mathematics in schools a very large emphasis is given to the concrete aspect of the subject and its abstractness is side tracked, nay, almost forgotten. It is one thing to use a concrete ~~ill~~ example to illustrate an abstract mathematical concept, but if the abstractness of the concept is not referred to at all in any manner, not only that but if the teacher himself is quite innocent about this abstractness then what is being taught will be anything but mathematics. A little reflection will reveal that such a situation prevails in our teaching to a very large extent.

The development of the following characteristics in our teaching of mathematics is largely due to this over emphasis on the concrete form of an abstract subject:

(1) A distinct bias towards processes as against principles in teaching of mathematics.

(2) Over emphasis on Arithmetic in the curriculum of secondary schools.

(3) Absence of any relationship between the teaching of mathematics and the all round progress in science.

(4) A prevailing tendency amongst students to accept whatever the teacher says as correct rather than to attempt at an understanding of the fundamentals involved.

(5) Absence of axiomatic approach in any branch of mathematics (including geometry) throughout the curriculum right upto the higher secondary stage

Processes vs. Principles

The first ~~of~~ ~~and formos~~ of these characteristics is the very dominating role of mathematical processes in our teaching. Mathematics has come to be regarded as a collection of processes. A feeling has arisen that if one knows how to effectively use the processes of addition, subtraction, multiplication, division, root extraction, factorization, solutions of equations etc etc. one has mastered mathematics. The ~~to~~ Hindi word

~~Karmak~~ Karmakand (कर्मकांड) is very appropriate to describe this over emphasis on mathematical processes in our teaching. One can as well say that our mathematics-teaching has been as karmkandi as some of our religions. By over emphasizing the karmakand of a religion as against its principles intelligent people become averse to the karmkandi religion. On the same lines many intelligent children who has formed a strong habit of establishing correspondence between the concrete and the abstract from ~~in~~ ~~the~~ early childhood find later in their school days that in the karmakand-ridden mathematics which they are taught there is very little place of for any activity of corelating concrete with the abstract and so they start disliking the subject. If intelligent people who have been very successful in life remark that mathematics was one of their dislikes in the school-days, may be, it is largely due to this ~~over~~ emphasis on processes rather than on principles.

We shall now take one or two illustrations of Karmakandi teaching. I quote the following passage from the chapter on negative numbers of a widely used text-book of algebra for beginners.

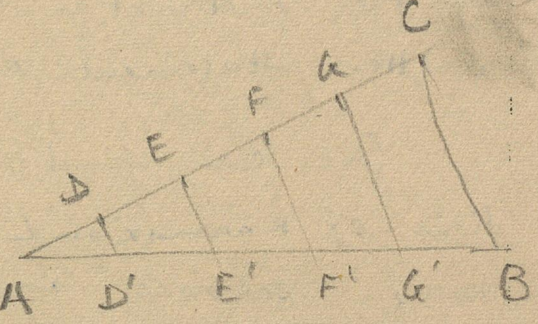
"It will be inconvinient to use ~~to~~ the number-line everytime that we have to work an example. Let us therefore evolve a simple rule for the addition of two directed numbers .... When we have to add two numbers with different signs, we take the difference between their numerical values and put the sign of the larger of the two numbers before this difference. E.g. $(-4) + (+2) = (-2)$."

When I read such passages, I am reminded of the family priest helping to perform Puja. He gives instructions on the same lines— "Hold water in your right ~~hand~~ palm. Add grains of rice and petals of flowers to it. Offer this water to the idol...." Niether the priest nor the familyman performing the Puja ever think for a while as to why these operations are to be performed. Both have the faith that ~~to~~ these operations are necessary to perform the Puja. By prescribing the above method of adding directed numbers both the teacher and the students are led to go by faith ~~in mathematics~~ rather than by conviction in mathematics. The number-line is forgotten and with it the abstract notion of negative numbers is also forgotten. What remains with the student is the memorised process of adding directed numbers.

The abstract notion of positive integers was first given to the child in the early childhood with the help of concrete illustrations. The notion of negative numbers is as abstract as the notion of positive integers. The use of number-line in giving a concrete form to this abstract notion is to be commended But after this stage, instead of strengthening the ~~effect~~ impression of the correspondence between the abstract notion of negative numbers with the concrete positions of numbers on the number line, we turn to the process of working with these numbers ~~The~~ with the result that the impact of the abstract notion slowly gets obliterated. Many of us might have been taught our negative numbers in the above Karmakandi way. In order to find out to what extent we have retained the abstract notion of negative numbers, let us read again the passage from algebra book quoted above. Do we find any thing wrong with the process of addition prescribed there? Let us now see how a child who was studying negative numbers for the first time with the help of this book, reacted to this prescription. The child said, "In (-4) + (+2) = (-2), the rule states that the sign of the larger of the two numbers is to be put before the difference. But between (-4) and (+2) the larger is +2 and its sign is +. ~~But yet~~ Yet the answer is ~~put~~ -2 and not +2!" It is obvious that the child's reaction is correct and that the statement in the book is incorrect. But let us face the situation squarely. How is it that we did not note this simple truth when we read that passage for the first time? The Karmakandi way of teaching ~~has~~ obliterated this abstract notion of negative ~~nu~~ numbers from our mind and its place an artificial notion of a mixed entity consisting of a "number" and a sign ~~is being~~ has been created. As a result, instead of looking upon -4 as a negative number we are trained to split it into the $-$ sign and the "number" 4 and since this "number" 4 is larger the prescription in the above passage mentions the choosing of ~~its~~ the negative sign of the "larger number" -4! For that child, the abstract notion of the negative number is still fresh in his mind and so he could immediately see that -4 is not the larger number. (Because he has not yet been trained to split up -4 into $-$ and 4!) But by the time this child completes his first course of algebra he will be trained in the processes with these numbers in such a way that this abstract notion of negative numbers will be completely wiped out from his mind and he will form a habit of splitting -4 into $-$ and 4 and then he will also make the same mistake in this concept as I or you or the author of that book made!

If one looks carefully through various mathematics text-books used in our schools one will find many such mathematically untrue statements there and in many cases ~~we~~ are ~~shall~~ will find that the child is advised (of course not directly but impliedly) to forget about an abstract notion and to remember the process of using that notion only.

We shall now take an illustration from teaching of geometry. The construction of dividing a given straight line into given number of (say 5) of equal parts is well-known. This construction was being taught in a class in the following manner.



AB is the given st. line. We want to divide it into 5 equal parts. Draw a ~~strai~~ st. line AC making any angle with AB. Divide AC into 5 equal parts at D, E, F, G. Join CB. Draw DD', EE'.. ∥ CB and so on.

The teacher explained this construction and wrote the above prescription on the board. We have to divide AB into 5 equal parts and for that purpose we talk of drawing another line AC and of dividing AC into 5 equal parts. Well, then the question arises — why not divide AB straightaway into five equal parts in the same manner in which you divided AC into five equal parts? Why should one undertake all these operations like "Join CB. Draw DD', EE' ... ∥ CB ..." If one could divide AC into 5 equal parts without any elaborate construction why not do the same thing with AB? ~~AB~~ AB is the given straight line. Therefore its length is fixed. We are drawing the straight line AC and so we can draw it of any convinient length. We can therefore take it of such a length that with any unit it can be ~~divided~~ easily divided into five equal parts. This abstract thinking which is at the back of this concrete construction must find a place in the teaching of the construction as well as in the written statement of this construction.

One can find several such illustrations of Karmakandi teaching of mathematics in schools and Colleges. The abstract notions behind mathematical concepts and operations are lost sight of and only the processes are being taught and remembered.

Mathematics and Progress in Science

Not the processes but the basic abstract notions of mathematics are useful in science. ~~Re~~ One result of our ~~fog~~ forgetting abstract notions and of emphasizing only processes in our teaching of mathematics, is that at present we do not find any relationship between our mathematics and the phenomenal growth of science in the last 50 years. "Of what use is this mathematics?"—

this is a very perplexing problem before the teacher or the student of mathematics in our schools. He knows only one use and that is the use of arithmetic in money-markets, share-bazaars, post-offices, banks or insurance houses. The unprecedented development of the abstract in mathematics is at the root of the present phenomenal growth of science. But the mathematics that we teach in schools has lost almost all ~~his~~ its abstractness and neither the student nor the teacher sees any link between the science of space-age and the mathematics that is being learnt or taught. In order to understand the concrete aspects of mathematics as revealed by modern scientific discoveries it has become necessary to know the abstract aspects more thoroughly. If we wish to bring the teaching of science in line with the requirements of the modern age, it is necessary that adequate emphasis is laid on the abstract aspect of mathematics.

~~Value of the ratio~~

Estimate of the ratio abstract : concrete

We have four principal stages of education—

primary, secondary, undergraduate and post-graduate. Mathematics flows as a continuous stream through all these 4 stages. One can formulate a broad principle of the following type: A teacher in the primary stage must have studied in at the secondary stage. A teacher at the secondary stage will be a graduate. A teacher in an undergraduate college will have a post graduate degree while those teaching at the post-graduate stage must be persons of standing and eminence in the subject. This general rule will enable us to make a rough estimate of the ratio ~~abstract:~~ abstract: concrete in mathematics at different stages of education.

We begin with the primary stage. At this stage we introduce positive integers and zero and the operations of addition, subtraction, multiplication and division with them and because of division positive fractions are also introduced at this stage. Thus at this stage, the child is exposed to a number of abstract notions. Of course we cannot introduce these abstract notions as abstract, but can only introduce the concrete form of these abstract notions. But again the purpose of introducing this concrete form is to see that the corresponding abstract notion is germinated and takes roots in the child's mind through his habit of building a bridge of one-one correspondence between the concrete and the abstract. This purpose could be achieved only if the primary teacher himself is aware of the abstract aspects of these ~~concrete illustrations~~ notions. That the teacher must be conversant with the abstract notions behind the various processes that he teaches is desirable not only from the point of view of mathematics but also from the point of view of pedagogy. If all the abstract ~~a~~ notions which are introduced in a concrete form at the primary stage are introduced in an abstract form at the next (secondary) stage, then the student at the secondary stage will be found ready to receive them and the primary teacher will get acquainted with these abstract notions during his studies at the secondary stage.

Let us ourselves be a little more concrete about this point. We shall take one illustration of an abstract notion behind a process taught at the primary stage. The process of multiplying 75 by 12 is taught at the primary stage as follows:

$$\begin{array}{r} 75 \\ \times\ 12 \\ \hline 750 \\ +150 \\ \hline =900. \end{array}$$

Let us write this process in the standard form:

$$75 \times 12 = 75\,(10+2) = 75\times10 + 75\times2$$
$$= 750 + 150 = 900.$$

Thus our process of multiplication uses the proposition $a(b+c) = ab+ac$. Is this proposition proved in algebra? Can it be proved at all? To these questions one could as well add another question – Is our primary teacher aware of any such proposition or that such a proposition is at the back of this Karmakand of multiplication?

$a(b+c) = ab+ac$ is one of the many

assumptions which our numbers satisfy. At the primary stage we introduce almost all the axioms on which our number system is based, but our teacher is blissfully ignorant of them. He must be made conscious of these assumed properties of our number system and then only will his teaching be purposeful. And again if these properties which were introduced through concrete illustrations in the primary stage are introduced in a little more abstract manner at the next stage then only the student will get acquainted with both aspects of mathematics and will not carry ~~wit~~ away the faulty impression that mathematics is a collection of processes. It is thus clear that all the properties of our number system (number field) should be introduced as properties at the secondary stage. But the concept of a set is basic in the enunciation of these properties. Of the

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the number field. Thus the fundamental concepts of sets, subsets, members of a set etc. should be introduced in mathematics at the secondary stage and ~~in their~~ the properties of the number field should be given in their context.

Now properties of our number ~~field~~ system flow from the concepts of groups, rings, integral domains and fields. So following our line of arguement, it will be clear that these abstract notions (whose concrete form was illustrated in the properties of the number system in the schools) should be introduced in their abstract form at the under-graduate college mathematics. The mathematics of these abstract concepts – Abstract Algebra and Topology – should find a place in post graduate teaching. If the continuous ~~too~~ flow of mathematics-teaching is to be properly channelized, we should introduce abstract concepts gradually (but with a positive definite gradient) in all the four stages of education. In the above we have indicated an ascending sequence for arithmetic and algebra. A similar sequence can be worked out for other branches as well.

It is clear that the task of freeing the present mathematical teaching and curriculum from the bonds of the concrete and of bringing it in a proper atmosphere of concrete as well as abstract (~~present~~ in proper proportions) is not the task of a single individual. The primary teacher, the secondary teacher, the college teacher and the mathematician familiar with modern trends, should all sit together and work out a continuous programme. A body like NCERT can ~~bring the above~~ take the initiative ~~4 experts to-gether~~ and collect the four types of experts ~~meet~~ to-gether to evolve a continuing curriculum with a purposeful teaching sequence.