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SOME ALGEBRAICALLY SPECIAL SOLUTIONS ON EINSTEIN'S EQUATIONS - II

P. C. Vaidya

Abstract : In the first part of this series of papers it was shown that algebraically special solutions of Einstein's equations expressed in the metric-form , $g_{ik} = \eta_{ik} + H \overset{*}{r}_i \overset{*}{r}_k$, $\overset{*}{r}_i \overset{*}{r}^i = o$, $\overset{*}{r}_{i;k} \overset{*}{r}^k = o$ and exhibitting symmetry about an axis split into two families - the Schwarzschild and the Kerr families. The most general solution of Schwarzschild family was obtained in I. The most general solution of the Kerr family is derived in the present paper. The general solution presented here describes the gravitational field of a rotating body which is radiating energy.

1. Introduction :

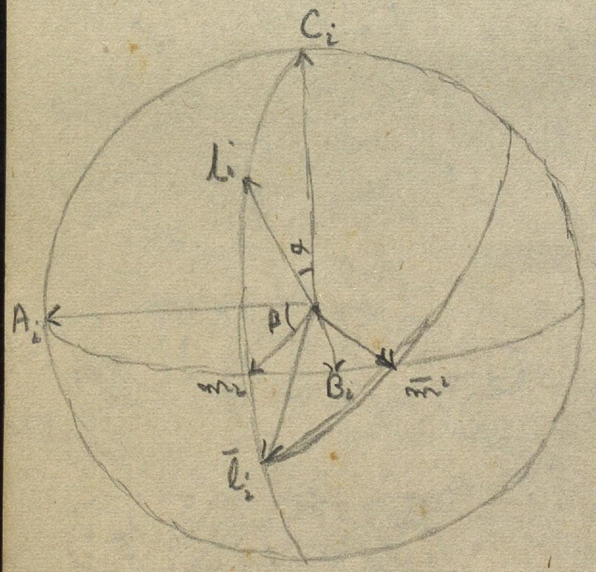
This paper is in continuation of the earlier paper [1] which will be referred to in the sequel as I. Section 2 of I forms a necessary introduction for the present investigation and so we begin with a brief outline of the geometrical notations introduced there.

In a Minkowskian space-time, we can find four uniform vector fields such that (i) any two of them are mutually orthogonal and (ii) one of them is time-like and the other three space-like. Let λ^i be the unit tangent to the time-like vector passing through a point P (coordinates x^i) and A^i, B^i, C^i be the unit tangents to the space-like vectors. We use the signature (- , - , - , +) and raise and lowever indices with the help of the metric tensor η_{ik} or η^{ik} of Minkowskian space-time. These four uniform vector fields give rise to a Euclidean reference frame with coordinates x , y, z, t for P where $x = x^1 A_1$, $y = x^1 B_1$, $z = x^1 C_1$ and $t = x^1 \lambda_i$ so that

$x_{,1} = A_1$, $y_{,1} = B_1$, $z_{,1} = C_1$ and $t_{,1} = \lambda_1$, a comma indicating an ordinary derivative.

If l^i is any space-like unit vector in the 3- flat II at right angles to λ^i at the point P then $\xi^i = \lambda^i + l^i$ is a null vector defining a null congruence in in the Minkowskian space-time.

In the 3- flat II at the point P, let α and β be the spherical angles of the direction l^i with reference to the triad A^1, B^1, C^1 at P. Then we can define 3 news space-like unit vectors $\bar{l}^i$, $\bar{m}^1$ and m^1 by the following relations (see Fig. 1)



$$m^1 = \cos\beta\, A^1 + \sin\beta\, B^1$$

$$l^i = \sin\alpha\, m^1 + \cos\alpha\, C^1$$

$$\bar{l}^i = \cos\alpha\, m^1 - \sin\alpha\, C^1$$

$$\bar{m}^i = -\sin\beta\, A^1 + \cos\beta\, B^1$$

We take α and β as functions of the coordinates x^i of the point P. It has been shown in I that the null congruence $\overset{*}{\xi}_k = \lambda_k + l_k$ is geodetic and shearfree if

(1.1) $\alpha_{,k} \overset{*}{\xi}{}^k = o \;, \quad \beta_{,k} \overset{*}{\xi}{}^k = o$

and

(1.2) $\bar{m}^i \alpha_{,i} + \bar{l}^i \sin\alpha \; \beta_{,i} = o$

(.3) $\bar{l}^i \alpha_{,i} - \bar{m}^i \sin\alpha \; \beta_{,i} = o.$

It can be verified that if

(1.4) $u = x \sin\alpha \cos\beta + y \sin\alpha \sin\beta + z \cos\alpha + t$

(1.5) $v = x \cos\alpha \cos\beta + y \cos\alpha \sin\beta - z \sin\alpha$

(1.6) $w = x \sin\beta - y \cos\beta$

then $u_{,i} \overset{*}{\xi}{}^i = o$, $v_{,i} \overset{*}{\xi}{}^i = o$, $w_{,i} \overset{*}{\xi}{}^i = o$. Therefore the conditions (1.1) for geodetic null congruence can be integrated and exhibited in the form

$$w = W(u, \alpha, \beta) \,, \quad v = V(u, \alpha, \beta)$$

W and V being undetermined functions of the arguments.

In I, the partial differential equations satisfied by the functions V and W [in view of (1.2) and (1.3)] are written down on the assumption that they do not depend on the angle β. (Symmetry about C^1) These equations are

(1.7) $W V_u + V W_u - W \cot\alpha + W_\alpha = o$

(1.8) $V V_u - W W_u - V \cot\alpha + V_\alpha = o$

a suffix denoting partial derivatives : e.g. $V_u = \partial V / \partial u$.
The general solution of these equations satisfying the restriction $W = o$ was discussed in I the corresponding gravitational fields being designated as fields of the Schwarzschild family.

2. Shearfree null congruence in Kerr family :

In order to solve equations (1.7) and (1.8) in the more general case $w \neq o$ we use the substitutions

$$v \operatorname{cosec} \alpha = p \,, \quad w \operatorname{cosec} \alpha = q \,, \quad \cos \alpha = s$$

and derive a single equation

$$(2.1) \quad (p - iq)_s - \frac{1}{2}\left[(p - iq)^2 \right]_u = o$$

from the two equations (1.7) and (1.8). Equation (2.1) can be treated as a linear partial differential equation in $p - iq$ as a function of u and s. The complete solution of (2.1) can be written as

$$(2.2) \quad u + is\,(p - iq) = f\,(p - iq)$$

where f is an arbitrary analytical function of the complex variable $p - iq$. From (2.2) we get the two solutions

$$u + ps + \varphi(p,q) = o$$
$$qs + \psi(p,q) = o \,,$$

where φ and ψ are ~~conjecta~~ conjugate harmonic functions of their arguments. Thus finally

$$(2.3) \quad u + v \cot \alpha + \varphi(p,q) = o \,,$$

$$(2.4) \quad w \cot \alpha + \psi(p,q) = o \,,$$

where

(2.5) $\quad p = v \operatorname{cosec} \alpha\ ,\ q = w \operatorname{cosec} \alpha\ ,\ \varphi_p = \psi_q\ ,\ \varphi_q = -\psi_p$

are the two equations which determine α and β as functions of the coordinates of the field point P in such a way

(2.6) $\quad \overset{*}{\xi}{}^i = \lambda^i + \sin\alpha \cos\beta\ A^i + \sin\alpha \sin\beta\ B^i + \cos\alpha\ C^i$

is a tangent vector at the point P (x,y,z,t) of a shearfree geodetic null congruence. From equations (2.3) and (2.4) and from the derivatives of the functions u, v, w recorded in Appendix (1), one can find that

(2.7) $\quad \alpha_{,i} = (A^2 + B^2)^{-1} \left[- B \sin\alpha\ \overset{*}{\xi}_i + (AC - BD) \bar{l}_i + (AD + BC)\ \bar{m}_i \right]$

(2.8) $\quad \sin\alpha\ \beta_{,i} = (A^2+B^2)^{-1} \left[- A \sin\alpha\ \xi_i - (AD + BC)\ \bar{l}_i + (AC - BD) \bar{m}_i \right]$

where

(2.9)
$$A = (\sec\alpha + \varphi_p) \psi - \varphi_q (t + \varphi)$$
$$B = \varphi_q \psi + (\cos\alpha + \varphi_p) (t + \varphi)$$
$$C = \varphi_q\ , \quad D = \cos\alpha + \varphi_p$$

with $\varphi = \varphi(p,q)$ and $\psi = \psi(p,q)$ satisfying (2.5).

The expansion $\theta = \overset{*}{\xi}{}^i{}_{,i}$ of the null congruence is given by

(2.10) $\quad \theta = - 2\ (AC - BD)\ (A^2 + B^2)^{-1}$.

It may be noted that $\varphi(p,q) = b\ (p^2 - q^2)$, $\psi(p,q) = 2bpq$, ~~kaxka~~ b = constant, will give $v = - \frac{1}{2}\ b \sin\alpha \cos\alpha$ which corresponds to the case of radiating Kerr metric reported eleswhere [2] . Again if $\varphi(p,q) = \frac{p}{a}$, $\psi(p,q) = \frac{q - k}{a}$, k and a constants, one obtains from (2.3) and (2.4) that

(2.11) $$v = au\sin\alpha\,(1 + a\cos\alpha)^{-1}, \quad w = k\sin\alpha\,(1 + a\cos\alpha)^{-1}$$

which correspond to solutions discussed by Patel [3]
If one puts $a = o$ in (2.11), one gets $v = o$, $w = k\sin\alpha$ which corresponds to Kerr solution.

3. <u>The Metric</u> :

Consider a Riemannian space-time described by the metric

(3.1) $$g_{ik} = \eta_{ik} + H\overset{*}{\xi}_i\overset{*}{\xi}_k ,$$

H being a function of the coordinates and $\overset{*}{\xi}_i$ the null congruence. It has been shown in I that if ξ_i is shearfree and geodetic in the Minkowskian space- time it will continue to be so in the Riemannian space-time described by (3.1). The christoffel symbols for the metric (3.1) have been recorded in I as appendix (ii).
In appendix (ii) of the present paper will be found the general form of R_{ik} for the metric (3.1) when $\overset{*}{\xi}_i$ is geodetic .

Taking $\overset{*}{\xi}_i$ given by equation (2.6) above and following Mas [4] we take

(3.2) $$H = M\overset{*}{\xi}^i{}_{,i} = M\theta = -2M(AC - BD)(A^2 + B^2)^{-1}$$

with

(3.3) $$M = M(x^1) \text{ such that } M_{,1}\overset{*}{\xi}^1 = o$$

We note, in passing, that any function of coordinates like $M = M(x^1)$ which satisfies the equation $M_{,i}\overset{*}{\xi}^i = o$ will be taken as independent of β and hence as functions of the arguments p and q only . This is consistent with our symmetry assumption .

With the above form of H, we find that

$$R_{ik}\overset{*}{\xi}{}^{i} = 0, \quad -R_{ik}\bar{l}^{k} = \mu\overset{*}{\xi}_{i}, \quad -R_{ik}\bar{m}^{k} = \nu\overset{*}{\xi}_{i}.$$

The forms of the scalars μ and ν are recorded in appendix (iii). In order to get $-R_{ik} = 8\pi\sigma\overset{*}{\xi}_{i}\overset{*}{\xi}_{k}$ we must take we must take $\mu = \nu = 0$ which will lead to the following two differential equations ~~6~~ for M.

$$(3.4) \qquad (C^2 - D^2)(\partial M/\partial p) - 2CD(\partial M/\partial q) = 0,$$

$$(3.5) \qquad 2CD(\partial M/\partial p) + (C^2 - D^2)(\partial M/\partial q) + 3M(C^2 + D^2)q^{-1} = 0.$$

As mentioned earlier in deriving (3.4) and (3.5) we have taken $M = M(p,q)$.

In order to find the condition which will ensure that (3.4) amd (3.5) are mutually consistent as equations for M, we use the fact that $C_{,i}\overset{*}{\xi}{}^{i} = 0$, $D_{,i}\overset{*}{\xi}{}^{i} = 0$ so that C and D can be taken as functions of p and q only. Then from the defining equations (2.9) for C and D we shall find that

$$C_p - D_q = D q^{-1} \; D/q; \qquad C_q + D_p = C q^{-1} \; C/q$$

Using these simplifying results we find that the two equations for M are mutually consistent if

$$(3.6) \qquad C C_p (C^2 - 3D^2) + D C_q (D^2 - 3C^2) = 0.$$

Thus in order that α and β defined by (2.3) and (2.4) determine ~~that~~ the shearfree geodetic null congruence leading to $R_{ik} = -8\pi\sigma \overset{*}{\xi}_i \overset{*}{\xi}_k$, it is necessary that the two conjugate functions φ and ψ should, in addition, satisfy equation (3.6).

Now it can be verified that

$$-R_{ik}\lambda^k = \overset{*}{\xi}_i \left[(H\theta + h)_{,k}\lambda^k - \frac{1}{2}\eta^{ab} H_{,ab} \right]$$

where $h = H_{,a} \overset{*}{\xi}{}^a$. Therefore if $R_{ik} = -8\pi\sigma \overset{*}{\xi}_i \overset{*}{\xi}_k$, we find that

$$8\pi\sigma = (H\theta + h)_{,k}\lambda^k - \frac{1}{2}\eta^{ab} H_{,ab} \tag{3.7}$$

Using several results mentioned in appendix (iv) we can rewrite (3.7) in the explicit form

$$\begin{aligned} 8\pi\sigma = {} & 6M(A^2+B^2)^{-2}\left[(AC-BD)(C^2+D^2+\sin^2\alpha+2Aq^{-1}) + (AC+BD)B_p \right. \\ & \left. + (BC-AD)B_q\right] - 6M(A^2+B^2)^{1}(C^2+D^2)^{-1}\left[2CD\,C_p + (C^2-D^2)C_q\right] \\ & -6Mq\sin^2\alpha\,(AC-BD)(A^2+B^2)^{-2}(C^2+D^2)^{-2}\left[(3C^2-D^2)DC_p + (C^2-3D^2)C_q\right]. \end{aligned} \tag{3.8}$$

Thus we have the general solution of Einstein's equations $R_{ik} = -8\pi \overset{*}{\xi}_i \overset{*}{\xi}_k$ where $\overset{*}{\xi}_i$ given by (2.6). The metric for the solution is given by $g_{ik} = \eta_{ik} + H \overset{*}{\xi}_i \overset{*}{\xi}_k$ H being given by the equation (3.2). The corresponding form of σ is given by equation (3.7) above, A, B, C, D being defined in (2.9) and M satisfying any one of the two equations (3.4), (3.5).

4. <u>Particular Cases</u>.

Case (1) $\varphi = ap$, $\psi = aq - b$, a, b constants.

$$C = o, \quad D = bq^{-1}$$

$$A = -q\,(1-a^2) - ab\ ; \quad B = b\,(ap + t\,)\,q^{-1}.$$

Since $C = o$, the condition of consistency (3.6) is easily seen to be satisfied, and (3.8) will now give $8\pi\sigma = o$ so that this case satisfies $R_{ik} = o$. This is the solution reported by Patel [3] and contains Lorentz transform of Kerr's solution as a particular case.

Case (2) $\varphi = b\,(p^2 - q^2)$, $\psi = 2bpq$, b = constant

$$C = -2bq\ , \quad D = o$$

$$A = -q \sin^2\alpha + 2bq\left[\,t + b\,(p^2 - q^2)\right], \quad B = -4b^2 pq^{-2}$$

Since $C_p = o$, $D = o$, the condition of consistency (3.6) is easily seen to be satisfied. (3.8) will now give $8\pi\sigma = -12M\,b\,(A^2 + B^2)^{-1}$. This is the case reported elsewhere as a radiating Kerr metric [2].

Case (3) A new case is

$$\varphi = (b-q)\,a^{-1}, \quad \psi = pa^{-1}, \quad a, b \text{ constants}, \ a \neq o$$

$$C = -a^{-1}, \quad D = -p\,(aq)^{-1}$$

$$A = a^{-2}\left[\,b + at - q\,(1 + a^2)\right], \quad B = -p(b+at)\,/\,a^2q\ \cancel{a^2q}\,.$$

The condition of consistency is obviously satisfied since C reduces to a constant.

$$8\pi\sigma = -\,(6M/aq)\,(A^2 + B^2)^{-1}.$$

In order to get a meaning for the excluded case a = o, we write down the functions v and w by solving the two equations (2.3) and (2.4) we shall then find that

$$w = (au + b)\sin\alpha\,(1 + a^2\cos^2\alpha)^{-1},$$

$$v = -a\sin\alpha\cos\alpha\,(au + b)(1 + a^2\cos^2\alpha)^{-1}.$$

It is easily seen that the excluded case a = o, gives v = o, w = b sin α , which gives Kerr's metric.

5. Conclusion. In section 3 we have an explicit solution of $R_{ik} = -8\pi\sigma\overset{*}{\xi}_i\overset{*}{\xi}_k$ which represents the most general solution of

what we have called here the Kerr family of Einstein's equations. One peculiar mathematical feature of the explicit solution is worth noting. If α and β are the spherical angles in the 3-flat II of the projection of $\overset{*}{\xi}_i$ ~~an~~ on this 3-flat then if α and β satisfy (2.3) and (2.4) the null congruence $\overset{*}{\xi}_i$ is geodetic and shearfree. But all geodetic and shearfree null congruences will not lead to solutions of Einstein's equations belonging to the kerr family. Only those members $\overset{*}{\xi}_i$ of the congruences for which α and β, in addition to satisfying the consistency condition (3.6) will lead to radiating Kerr type metrics satisfying Einstein's equations $R_{ik} = -8\pi\sigma\overset{*}{\xi}_i\overset{*}{\xi}_k$.

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Appendix (i)

The derivatives of the functions u, v, w are :

$$(A.1)\quad u_{,i} = \overset{*}{\xi}_i - w \sin\alpha\, \beta_{,i} + u\, \alpha_{,i}$$

$$(A.2)\quad v_{,i} = \bar{\ell}_i - w \cos\alpha\, \beta_{,i} - (u - t)\, \alpha_{,i}$$

$$(A.3)\quad w_{,i} = -\bar{m}_i + \left[u + v\cot\alpha - t\right] \sin\alpha\, \beta_{,i}\ .$$

Appendix (ii)

We give here the form of $\overset{*}{R}_{ik}$ for the metric (3.1)

$$-2\overset{*}{R}_{ik} = 2(H\theta+h)_{,(i}\xi_{k)} + 2(H\theta+h)\,\xi_{(i,k)} - 2\eta^{ab}H_{,b}\left(\overset{*}{\xi}_{i,a}\overset{*}{\xi}_{k} + \overset{*}{\xi}_{k,a}\overset{*}{\xi}_{i}\right)$$

$$(A.4)\qquad -2H\eta^{ab}\,\overset{*}{\xi}_{i,a}\,\overset{*}{\xi}_{k,b} - H\eta^{ab}\left(\overset{*}{\xi}_{i,ab}\,\overset{*}{\xi}_{k} + \overset{*}{\xi}_{k,ab}\,\overset{*}{\xi}_{i}\right)$$

$$-\left(\eta^{ab}H_{,ab} + H^2\,\overset{*}{\xi}{}^{a}_{,b}\,\overset{*}{\xi}{}^{b}_{,a} - H^2\eta^{ab}\,\overset{*}{\xi}_{\ell,a}\,\overset{*}{\xi}{}^{\ell}_{,b} - Hh_{,a}\overset{*}{\xi}{}^{a} - Hh\theta\right)\overset{**}{\xi_i\xi_k}$$

Here $h = H_{,i}\overset{*}{\xi}{}^{i}$ $\quad \theta = \overset{*}{\xi}{}^{i}_{,i}$.

Appendix (iii)

Forms of scalars μ and ν of section 3 are given by

$$(A.5)\ \mu(A^2+B^2) = (C^2+D^2)\left[\,M_{,k}\bar{\ell}^{k} + 3MB\sin\alpha\,(A^2+B^2)^{-1}\right]$$

$$-2(AC-BD)\,M_{,k}\alpha_{,\ell}\,\eta^{k\ell}\ .$$

$$(A.6) \quad \nu (A^2 + B^2) = (C^2 + D^2) \left[M_{,k} \bar{m}^k + 3MA \sin\alpha \, (A^2 + B^2)^{-1} \right]$$

$$-2(AC - BD) \eta^{kl} M_{,k} \sin\alpha \, \beta_{,l}$$

In deriving the above expressions, the following results are useful :

$$(A.7) \quad \eta^{ab} \alpha_{,ab} + \cot\alpha \, (C^2 + D^2)(A^2 + B^2)^{-1} = 0,$$

$$(A.8) \quad \eta^{ab} \sin\alpha \, \beta_{,ab} = 0 .$$

Appendix (iv)

In deriving the final expression (3.8) for σ, the following intermediary results are useful :

$$(A.9) \quad \eta^{ab} \overset{*}{r}_{i,ab} = 2 l_i (C^2 + D^2)(A^2 + B^2)^{-1}$$

$$\frac{1}{2} (H\theta + h)_{,i} + \frac{1}{2} (H\theta + h) \overset{*}{\xi}_{i,k} \lambda^k - H(C^2 + D^2)(A^2 + B^2)^{-1} l_i$$

$$(A.10) \quad -\eta^{ab} H_{,a} \overset{*}{r}_{i,b} = \frac{1}{2} (H\theta + h)_{,k} \lambda^k \overset{*}{\xi}_i .$$

$$(A.11) \quad \eta^{ab} \overset{*}{\xi}{}^i_{,a} \overset{*}{r}_{i,b} = 0 .$$

$$(A.12) \quad (A^2 + B^2) p_{,i} = \psi \tan\alpha \, (A \bar{l}_i + B \bar{m}_i) - \left[A\psi + B(\varphi + t) \right] \overset{*}{\xi}_i$$

$$(A.13) \quad (A^2 + B^2) q_{,i} = \psi \tan\alpha \, (B \bar{l}_i - A \bar{m}_i) + \left[A(\varphi + t) - B\psi \right] \overset{*}{\xi}_i$$

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