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# The distribution of temperature along a thin rod electrically heated *in vacuo*. V. Time lag

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On the basis of the expressions obtained in parts I and II of this series for the distribution of temperature in the *steady* state along a filament electrically heated *in vacuo*, the growth of temperature accompanying a small increase in the heating current is investigated in the present part. Over a considerable region about the centre of the filament, which is the region of practical interest, it is found that to a close approximation the growth of temperature can be completely represented by a simple exponential law involving a *single* relaxation time, whose magnitude is readily calculated.

This method of investigating the time lag, which is general and applicable to any filament, is compared with the well-known method of Fourier expansion developed by Straneo for the special case where the temperature everywhere in the filament is only slightly higher than the room temperature, and hence the loss by radiation conforms to Newton's law of cooling. Each of the Fourier terms is assigned in his method a separate relaxation time that will make the term separately satisfy the differential equation and the boundary conditions. In principle the Fourier method also should be applicable to any filament. But the actual temperature distribution is in general too complicated for an analytical Fourier expansion.

In the special case treated by Straneo the temperature distribution over practically the whole length of the filament is parabolic. The actual distribution near the centre of any filament is also known to be parabolic. Hence a comparison of the results obtained by the two methods in the above special case suggests a convenient adaptation of the Fourier method also to the calculation of the time lag near the centre of any filament. The adaptation lies essentially in the use of a certain effective length to determine the period of the Fourier expansion, instead of the actual length generally used. The magnitude of this length is obtained from the results of the present investigation. The distinction between the two lengths is not significant in the special case treated by Straneo, but it is in other cases.

Though the occurrence of a single effective relaxation time is not directly obvious from the Fourier expansion, it is shown to follow from it as a close approximation. This result is convenient for practical application.

For a given central temperature the relaxation time is found to vary inversely as the ratio of the surface to the volume, and is therefore smaller for a ribbon filament than for one of circular cross-section, as observed by Prescott & Morrison. For a given central temperature and length, the ribbon filament is found to approximate closer than one of circular section, to an infinitely long one.

The variation of the relaxation time near the centre with the length of the filament is investigated in some detail.

## 1. Introduction

Many years ago Prescott & Morrison (1939) made the following important observation on a filament electrically heated *in vacuo*. When the heating current is changed, there is naturally a time lag between the change of current and the attainment of the corresponding steady state. Their observation was that this time lag is considerably shorter in a ribbon filament than in a filament of circular cross-section. Denoting by $\omega$ the area, and by $p$ the periphery, of the cross-section of the filament, they found that the time lag is the shorter the larger the value of $p/\omega$. Obviously $p/\omega$ will be a minimum when the cross-section is circular.

There has been some difference of opinion between these authors on one side and Wensel on the other regarding the mechanism by which an increase in $p/\omega$ effects a reduction in the time lag. The two views are stated in detail in the discussion accompanying a paper by Prescott (1941). Whereas according to Prescott & Morrison the effect of the large ratio of the surface of the filament to the volume, which obtains in a ribbon filament, on the time lag, is through its 'minimizing the heat capacity', according to Wensel it is through its 'minimizing the heat conduction loss (from the central part) to the leads'. The general impression left on the reader after a perusal of this discussion is that, independently of the controversy, there is need for a detailed theoretical investigation of the growth of temperature in a filament with time when a current is switched on it, or when the heating current is varied. The investigation will also be useful otherwise. In the optical pyrometer, for example, which is used extensively for the measurement of high temperatures, the heating current through the filament is adjusted so as to make the brightness of the filament match that of the background whose temperature is being measured. The efficiency of this matching depends naturally on how small is the time lag referred to above. Indeed, the original use of the ribbon filament by Prescott & Morrison was in an optical pyrometer, and was intended specifically to reduce the time lag.

Some special cases, where either the loss of heat due to conduction through the ends of the filament, or the loss due to radiation from the surface, is wholly eliminated, have been investigated by Fischer (1938*a*, *b*, *c*), and in connexion with the study of the bearing loads of fuse wires by Wintergerst (1950). But the most important earlier work is that of Straneo (1898) on a filament whose temperature at every point is only slightly above the room temperature, such that the rate of cooling due to radiation may be taken to conform to Newton's law. Though the temperatures that we shall be concerned with in the optical pyrometer are naturally of a different order of magnitude, and the loss due to radiation is by no means Newtonian, the method developed by him is a general one. The method is this. Since the temperature distribution is known for his filament both in the final and in the initial steady states, the difference between them is known as a function of the distance. When the temperature is growing the expression for the difference between the final temperature at any point and the actual temperature there can be expressed as a Fourier sine series, *each term in the series having its own characteristic relaxation time*, the relaxation time chosen being such that each term separately satisfies the differential equation and the boundary conditions. Hence the growth of the current may be followed at every point of the filament in detail.

In Straneo's problem the temperature distribution along the filament is practically parabolic, and can be expressed conveniently as a Fourier sine series. But the filaments that are of practical interest are the long ones, worked at high temperatures, in which the temperature distribution is much more complicated, and hence the Fourier method in its usual form becomes a laborious one.

In parts I and II (Jain & Krishnan 1954*a*, *b*) was given a theoretical investigation of the distribution of temperature in an electrically heated filament in the steady state. On the basis of the results obtained therein the time lag in electrical heating in the central portion (which is the region of practical interest) of any filament, long or short, is investigated in the present part, taking into account the loss of heat due both to the conduction at the ends, and to the radiation from the surface, which is not Newtonian. The direct method, besides being applicable generally to a filament of any length, has also some advantages over the Fourier method. Even where, as in Straneo's problem, the temperature distribution along the filament is parabolic, and the Fourier expansion convenient, there are some features which may be missed altogether by the Fourier method. For example, it is found by the present method that the whole course of the growth of temperature near the centre of the filament, in which we are particularly interested, corresponds practically to a single relaxation time. Though this result should be implicit in the Fourier expansion, it is by no means obvious from the expansion.

## 2. Formulation of the differential equation

Now in a heated filament the heat generated per second per unit length of the filament is $I^2\rho/\omega$, and the rate of loss of heat per unit length is $p\epsilon\sigma(T^4-T_0^4)$, where $T$ is the temperature of the element, and $T_0$ that of the walls of the chamber, $I$ is the heating current, $\rho$ is the specific resistance, $\epsilon$ is the total (as distinguished from the spectral) emissivity of the surface, and $\sigma$ is Stefan's constant of radiation. The heat conducted away from this element, per unit length per second, is $-\kappa\omega \mathrm{d}^2T/\mathrm{d}x^2$, where $x$ is the distance along the filament, and $\kappa$ is the thermal conductivity.

When the conditions are *steady*, the temperature distribution along the filament will be given by the well-known differential equation (see Carslaw & Jaeger 1947)

$$-\kappa\omega \mathrm{d}^2T/\mathrm{d}x^2 = I^2\rho/\omega - p\epsilon\sigma(T^4-T_0^4). \tag{1}$$

Had there been no loss due to conduction, the temperature of the filament at every point would have been equal to $T_m$, where

$$p\epsilon\sigma(T_m^4-T_0^4) = I^2\rho/\omega. \tag{2}$$

Obviously $T_m$ will be the value to which the temperature $T_l$ at the centre of the filament tends as the length $2l$ tends to infinity, the heating current being kept constant.

In view of (2) the equation for the steady state may be written in the convenient form

$$-\kappa\omega\, \mathrm{d}^2T/\mathrm{d}x^2 = p\epsilon\sigma(T_m^4-T^4). \tag{3}$$

It will be seen from (3) that the temperature $T_0$ of the walls of the chamber does not appear explicitly in the expression, and the whole effect of this temperature is exercised through its influence on $T_m$.

Now if a heating current is switched on or off, or the current is changed, the temperature distribution at any later instant $t$ can be obtained from the following considerations. Since the temperature $T$ in the element is changing, a quantity equal to $c\delta\omega\, \mathrm{d}T/\mathrm{d}t$, where $\delta$ is the density and $c$ is the specific heat of the material of the filament, will be taken up in this process per second per unit length of the element, and this quantity has to be included on the left-hand side of (3). Hence the differential equation defining the growth of temperature at any point of the filament is

$$-\kappa\omega\, \partial^2T/\partial x^2 + c\delta\omega\, \partial T/\partial t = p\epsilon\sigma(T_m^4-T^4), \tag{4}$$

which may be written in the form

$$\partial T/\partial t = \alpha\, \partial^2 T/\partial x^2 + \beta(T_m^4 - T^4), \tag{5}$$

where

$$\alpha = \kappa/(c\delta) \tag{6}$$

and

$$\beta = p\epsilon\sigma/(c\delta\omega). \tag{7}$$

To $T_m$ appearing in (4) is to be assigned the value appropriate to the final steady state. $\alpha$ is the thermal *diffusivity* as usually defined.

It follows directly from (4) that the time lag should be the greater the smaller the value of $p/\omega$, as observed by Prescott & Morrison. Further, the first term on the left-hand side of (4) is the one associated directly with the loss due to the conduction towards the ends. What part this conduction plays in determining the time lag depends on the magnitude of its contribution to $\partial T/\partial t$ as compared with that from the term on the right-hand side.

Let $T_1$ and $T_2$ be the initial and the final steady temperatures at the point $x$, i.e. when $t = 0$ and $t \to \infty$ respectively. Following the usual procedure adopted in solving problems of this type, we may put $T = T_2 - \tau$, where $\tau$ is a function of the time (and also of the distance $x$). Substituting this value of $T$ in (5) one obtains from it *two* differential equations, one defining the temperature distribution in the final steady state, i.e. defining $T_2$ as a function of $x$, and the other defining the variation of $\tau$ with time. These equations are

$$\alpha \mathrm{d}^2 T_2/\mathrm{d}x^2 + \beta(T_m^4 - T_2^4) = 0 \tag{8}$$

and

$$-\partial\tau/\partial t = \beta[T_2^4 - (T_2 - \tau)^4] - \alpha\, \partial^2\tau/\partial x^2 \tag{9}$$

respectively.

## 3. The relaxation time at the centre of a long filament

Our main objective is to solve (9), and equation (8) for the steady state is invoked for the purpose of evaluating $\partial^2\tau/\partial x^2$, which is required for the solution of (9). In the special case when $\partial^2\tau/\partial x^2$ is zero, the solution of (9) may be obtained directly without the aid of (8), though even here one has to invoke (8) in order to determine the conditions under which $\partial^2\tau/\partial x^2$ vanishes. It will be seen from (8) that *in the steady state*, the value of $\mathrm{d}^2T/\mathrm{d}x^2$ at any point $x$ is proportional to $T_m^4 - T^4$, where $T$ is the temperature at the point. At the centre of a long filament, where $T = T_l$, and $T_l$ is practically the same as $T_m$, $\mathrm{d}^2T/\mathrm{d}x^2$ should be negligible. This is due essentially to the central portions of such a long filament being too far removed from the ends, through which the heat is conducted away, to be affected by the conduction. Hence when the heating current is growing, the value of $\partial^2\tau/\partial x^2$ in this region should remain negligible. The differential equation (9) defining the growth of temperature now reduces to

$$-\partial\tau/\partial t = \partial T/\partial t = \beta(T_m^4 - T^4). \tag{10}$$

This equation is identical with the equation for the rate of cooling of a mass of water kept at a high temperature, the temperature being maintained the same throughout the mass, which has been discussed by Jaeger (1951). This is also the expression obtained by Wintergerst for a filament in which the loss in heat is due wholly to radiation. Since the solution is well-known, we shall merely give the final result here namely,

$$\frac{T_m - T}{T_m + T} e^{-2 \arctan (T/T_m)} = C e^{-t/t_0}, \tag{11}$$

where

$$t_0 = 1/(4\beta T_m^3) = c\delta\omega/(4pe\sigma T_m^3) \tag{12}$$

is a constant having the dimension of time, which may be called the relaxation time, and $C$ is the constant of integration which can be determined from the boundary conditions $T = T_0$ when $t = 0$ and $T \to T_m$ when $t \to \infty$. One thus obtains

$$C = \frac{T_m - T_0}{T_m + T_0} e^{-2 \arctan (T_0/T_m)}. \tag{13}$$

It may be mentioned immediately that the relaxation time $t_0$ obtained here is not directly that of $\tau$, but of a certain function of $\tau$.

In deriving expression (11) for the growth of the temperature near the centre of a long filament, the whole current is taken to be switched on directly, i.e. $T$ is made to grow from the room temperature $T_0$ to a high temperature $T_m$. But in actual practice one is not generally concerned with such a large change in the heating current, but with a small adjustment of the current so as to vary $T$ in the neighbourhood of the final temperature. In other words, the initial value of the temperature at the centre, which ultimately grows to $T_m$, is not $T_0$, but a certain temperature $T_1$ close to $T_m$. In this case the growth of temperature is given by the simpler expression

$$\partial T/\partial t = 4\beta T_m^3 (T_m - T) \tag{14}$$

with the boundary conditions

$$T = T_1 \quad \text{when} \quad t = 0$$
$$T \to T_m \quad \text{when} \quad t \to \infty.$$

The solution is

$$T_m - T = \tau = (T_m - T_1) e^{-\nu_0 t}, \tag{15}$$

where the relaxation time $t_0 = 1/\nu_0$ has the same value as in (12). For convenience we shall refer to the reciprocal of the relaxation time as the attenuation coefficient.* The relaxation time is obviously a measure of the time lag. We shall note in passing that for a given material for the filament, and a given central temperature $T_m$, $t_0$ is *inversely proportional to* $p/\omega$, *as observed by Prescott & Morrison. It is also inversely proportional to the cube of the temperature at the centre.*

## 4. The magnitude of the time lag in a long filament

Wensel gives some numerical data regarding the performance of an optical pyrometer, in which the filament can be regarded as a long one. Using a tungsten three-mil filament (of diameter 0·0075 cm) he makes the heating current drop from the value corresponding to $T_m = 1200°$ K to the value corresponding to $T_m = 1000°$ K. He finds that the time taken for the temperature to approach to within 1° of the final temperature is 10 to 20 s.

* The same expressions (11) and (15) will obviously hold when the heating current is switched off or reduced.

Using the following data for tungsten at about 1100° K, taken from Worthing & Halliday (1948), namely, $c = 0{\cdot}14\,\mathrm{J\,g^{-1}\,deg.^{-1}}$, $\delta = 19\,\mathrm{g\,cm^{-3}}$ and $\epsilon = 0{\cdot}12$, we obtain the value $t_0 = 1{\cdot}8\,\mathrm{s}$, from which the above-mentioned time comes out as 10s. This compares well with the estimate of 10 to 20s made by Wensel. Since the accuracy of measurement of temperature with an optical pyrometer is of the order of a degree, Wensel's estimate, which refers to the time needed for approach to the final temperature to within a degree, must be regarded as very rough.

## 5. Criterion for regarding the filament as long

The relaxation time calculated in the previous section for the growth of temperature at the centre of a long filament naturally raises the question how long the filament should be in order that the temperature $T_l$ at the centre might approximate to $T_m$ to any given degree of approximation. In parts I and II we have discussed in detail the distribution of temperature along a filament electrically heated *in vacuo*. For a long filament in which $T_m - T_l$ is small in comparison with $T_m$, the temperature at the centre is given by

$$T_l = T_m - 2\Delta_0\, \mathrm{e}^{f(\Delta_0)}\, \mathrm{e}^{-l\sqrt{A}}, \tag{16}$$

where $l$ is the semi-length of the filament,

$$A = 4p\epsilon\sigma T_m^3/(\kappa\omega), \tag{17}$$

$$\Delta_0 = T_m - \Theta, \tag{18}$$

and $\Theta$ is the temperature of the ends. Till now we had taken the temperature of the ends of the filament to be that of the room. For generality we shall henceforth take the temperature of the ends, which is kept constant, to be $\Theta$, as distinguished from the temperature $T_0$ of the room which occurs in the expression for the radiation. $f(\Delta_0)$ in equation (16) is defined by

$$f(\Delta_0) = \tfrac{1}{2}\Delta_0/T_m + \tfrac{1}{16}\Delta_0^2/T_m^2 - \tfrac{1}{240}\Delta_0^3/T_m^3 + \ldots. \tag{19}$$

It is significant that it is not directly the length $2l$, but $2l\sqrt{A}$, which may be called the reduced length, whose magnitude determines how close $T_l$ is to $T_m$. For a given length the approximation will be the closer the larger the value of $A$, i.e. the larger the value of $p/\omega$ or of $T_m^3$. Here again the ribbon filament has the advantage over a filament of circular cross-section.

Indeed, the two effects, namely, the reduction of the relaxation time, and the reduction of the actual length of filament needed for securing a given approximation to the infinitely long filament, both accompanying an increase in $p/\omega$ or in $T_m^3$, are closely related. It can be seen from the expressions for $t_0$ and $A$ that

$$t_0 = c\delta/(A\kappa), \tag{20}$$

and hence any factor like $p/\omega$ or $T_m^3$ that tends to increase $A$ will automatically reduce the relaxation time.

## 6. The criterion for a definite relaxation time for $\tau$

Coming back to the differential equation (9) defining the growth of temperature, when once we decide to confine attention to small values of $\tau$ only, it reduces to the simple form

$$-\partial\tau/\partial t = 4\beta T_2^3\tau - \alpha\partial^2\tau/\partial x^2. \tag{21}$$

Now if $\tau$ is to have a definite relaxation time, $-\partial\tau/\partial t$ should be proportional to $\tau$. In other words, the two terms on the right-hand side of (21) should be separately proportional to $\tau$. The first term can be directly seen to be so. We now proceed to investigate whether the proportionality of the second term also, namely, $\partial^2\tau/\partial x^2$, to $\tau$ can be secured at least in the central region, which is of practical interest.

## 7. The relaxation time near the centre of any filament

We have shown in parts I and II that the steady temperature distribution near the centre of any filament, long or short, is given by*

$$T_2 = T_{l2} - \tfrac{1}{2}Q_2 q^2, \tag{22}$$

where $q = l - x$ denotes the distance measured from the centre,

$$Q = (\beta/\alpha)(T_m^4 - T_l^4) = (p\epsilon\sigma/\kappa\omega)(T_m^4 - T_l^4), \tag{23}$$

and $Q_2$ and $T_{l2}$ are the values of $Q$ and $T_l$ respectively appropriate to the final steady state. From (22) and a similar expression for the distribution of the initial temperature $T_1$, one obtains

$$T_2 - T_1 = (T_{l2} - T_{l1}) - \tfrac{1}{2}(Q_2 - Q_1)q^2, \tag{24}$$

which may be written in the convenient form

$$T_2 - T_1 = \Delta T_l\left(1 - \frac{1}{2}\frac{\Delta Q}{\Delta T_l}q^2\right). \tag{25}$$

It can be seen directly from the form of (25) that $\dfrac{\partial^2}{\partial q^2}(T_2 - T_1)$ cannot be rigorously proportional to $T_2 - T_1$, e.g., $\partial^2\tau/\partial q^2$ cannot be *rigorously* proportional to $\tau$, even for small values of $q$. But at the same time one can also see that for small values of $q$, $1 - \frac{1}{2}(\Delta Q/\Delta T_l)q^2$ can be put equal to $\cos(\Delta Q/\Delta T_l)^{\frac{1}{2}} q$, and hence *to this approximation* $\partial^2\tau/\partial q^2$ may be taken to be proportional to $\tau$. Therefore one may write

$$\tau = (T_2 - T_1)\,e^{-\nu t} = \Delta T_l[1 - \tfrac{1}{2}(\Delta Q/\Delta T_l)q^2]\,e^{-\nu t}, \tag{26}$$

in which, however, $\nu$ will not now be quite independent of $t$, and for any given $q$ may be expressed in the form

$$\nu = \nu_1 + (\partial\nu/\partial t)\,t. \tag{27}$$

Hence from (21), (26) and (27) one obtains

$$(1/\tau)\,\partial^2\tau/\partial q^2 = -(\Delta Q/\Delta T_l)[1 + 2q(\partial\nu/\partial q)\,t]/[1 - \tfrac{1}{2}(\Delta Q/\Delta T_l)q^2] + (\partial\nu/\partial q)^2 t^2 + (\partial^2\nu/\partial q^2)\,t, \tag{28}$$

from which it will be seen that $\nu_1$ which will be the value of $\nu$ for small values of $t$, is given by

$$\begin{aligned} \nu_1 &= 4\beta T_2^3 - \alpha(1/\tau)\,\partial^2\tau/\partial q^2 \\ &= 4\beta T_2^3 + \alpha(\Delta Q/\Delta T_l)/[1 - \tfrac{1}{2}(\Delta Q/\Delta T_l)q^2] \\ &= \nu_a + \nu_{b1}, \quad \text{say.} \end{aligned} \tag{29}$$

* (22) could also have been obtained directly as a solution of (8) in the special case when $T$ is close to $T_l$.

The expression will obviously hold over the whole range of $q$ over which the parabolic law (22) or (24) holds. As $t$ increases (29) continues to hold, as we shall show in a later section, but its validity is restricted to a progressively decreasing range of $q$, and though it is not now a precise result, it is a close approximation. At the centre of the filament, in particular, which is of practical interest, (29) holds as a close approximation for all values of $t$, the approximation being the closer the smaller the value of $t$.

When the filament is a long one, such that $T_l$ is practically the same as $T_m$, both $Q_2$ and $Q_1$ vanish, and hence $\nu_b$ also vanishes. This confirms our earlier finding based on certain very plausible direct considerations that $\partial^2 T/\partial x^2$ should be equal to zero at the centre of a long filament, not only for the steady temperature distribution but also for a growing one.

## 8. The case when the temperature distribution is parabolic throughout

We referred in the previous section to the temperature distribution near the centre being always parabolic, irrespective of the length of the filament or the heating current. Under suitable conditions the whole length of the filament can be included in this region. In other words (22) and (24) will then be applicable over the whole of the range $0 \leqslant q \leqslant l$. Before considering the time lag in such a filament it will be useful to know the criterion for the distribution to be parabolic over the whole length of the filament.

In part II it was shown that the temperature distribution over a wider region than the parabolic region is given by

$$T_{l2} - T_2 = (Q_2/P_2)\,[\cosh q\sqrt{P_2} - 1], \tag{30}$$

where

$$P = 4p\epsilon T_l^3/(\kappa\omega). \tag{31}$$

It can be seen immediately that when $q\sqrt{P_2}$ is sufficiently small that the higher powers of $q\sqrt{P_2}$ than the second are negligible in comparison with unity, (30) reduces to the parabolic expression (22) obtained previously. The range of validity of the parabolic law is therefore determined by this criterion.

Just as in a long filament it is not the absolute length $2l$, but $2l\sqrt{A}$ that determines how close $T_l$ is to $T_m$, so also in the shorter filaments it is not directly $l$ but $l\sqrt{P}$ that determines the parabolic range. $P$ differs from $A$ in having the factor $T_l^3$ in place of the $T_m^3$ that appears in the expression for $A$. If the central temperature $T_l$ is small, the actual length $l$ can be quite large, and yet the whole of it can be well within the parabolic range. This is the case, for example, when the temperature over the whole filament is close to the room temperature, as in Straneo's problem.

The hyperbolic expression (30) is invoked here, not merely for determining the extent of the parabolic region, which can be done otherwise, but also because we need this expression in a later section.

Considering now such a filament, in which the parabolic relation (22) holds over the whole length of the filament $0 \leqslant q \leqslant l$, $T_2$ should now become equal to the end-temperature $\Theta$ when in (22) $q$ is put equal to $l$. We thus obtain

$$Q_2/(T_{l2} - \Theta) = 2/l^2. \tag{32}$$

Obviously this should also be the value of $Q_1/(T_{l1}-\Theta)$ corresponding to the initial steady state, from which it follows that

$$(Q_2-Q_1)/(T_{l2}-T_{l1}) = \Delta Q/\Delta T_l = 2/l^2. \tag{33}$$

Hence for the parabolic filament, the expression for $\nu_{b1}$ (see (29)) reduces to the simple form

$$\nu_{b1} = 2\alpha/(l^2-q^2), \tag{34}$$

and the expression for $\tau$ reduces to

$$\tau = \tfrac{1}{2}\Delta Q l^2(1-q^2/l^2)\exp-[4\beta T_2^3+2\alpha/(l^2-q^2)]\,t, \tag{35}$$

as compared with the corresponding general expression for any filament

$$\tau = \Delta T_l[1-\tfrac{1}{2}(\Delta Q/\Delta T_l)\,q^2]\exp-\left[4\beta T_2^3+\alpha\left(\frac{\Delta Q}{\Delta T_l}\right)\Big/\left(1-\frac{1}{2}\frac{\Delta Q}{\Delta T_l}q^2\right)\right]t. \tag{36}$$

Before taking up the general case we shall consider expression (35) obtained here for the wholly parabolic distribution, in relation to the corresponding expression obtained by Straneo's method.

## 9. Time lag in parabolic distribution by Straneo's method

As was mentioned in the Introduction the growth of temperature in a filament in which the temperature at any point is only slightly higher than the room temperature such that the loss by radiation may be taken to conform to Newton's law has been discussed by Straneo. A good account of Straneo's method is given in Carslaw & Jaeger (1947). The expression given by Carslaw & Jaeger for the temperature distribution, after making it conform to the present notation and to a finite end-temperature $\Theta$, is

$$T_2-\Theta = \frac{Q_2}{P_2}\cosh l\sqrt{P_2}\left[1-\frac{\sinh(l+q)\sqrt{P_2}+\sinh(l-q)\sqrt{P_2}}{\sinh 2l\sqrt{P_2}}\right], \tag{37}$$

which will be seen to reduce to

$$T_2-\Theta = \frac{Q_2}{P_2}\cosh l\sqrt{P_2}\left[1-\frac{\cosh q\sqrt{P_2}}{\cosh l\sqrt{P_2}}\right], \tag{38}$$

which in its turn can be seen to follow from (30), since $T_l$ is now sufficiently small to make the whole of the filament conform to (30).

The expressions for the initial and the final temperatures as functions of the distance may be expanded into the appropriate sine series, and they yield*

$$\tau = \frac{16l^2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}(A_{n2}-A_{n1})\cos\frac{(2n-1)\pi q}{2l}\exp-\left[4\beta T_2^3+\frac{\alpha(2n-1)^2\pi^2}{4l^2}\right]t, \tag{39}$$

where

$$A_n = \frac{Q}{(2n-1)^2\pi^2+l^2P}. \tag{40}$$

* In Straneo's filament the loss by radiation, which conforms to Newton's law, is proportional to $T_2-T_0$, and the constant of proportionality is $4\beta T_2^3$. Though it is not quite independent of $q$, it is practically so, since the maximum variation of temperature over his filament is small.

By making $l^2P$ much smaller than $\pi^2$, equation (37), on the basis of which (39) has been expanded, reduces to the parabolic law over the whole length of the filament.* Hence by making the corresponding approximation in (32), by which $A_n$ becomes equal to $Q/[(2n-1)^2\pi^2]$, equation (39) may be made to represent the growth of temperature in a filament whose length is wholly in the parabolic region.

Incidentally it may also be mentioned that in the experimental arrangement intended to be covered by (37), where the temperature everywhere is close to the room temperature, $T_l$ is actually small enough to make the whole of the filament conform to the parabolic law. The parabolic distribution obtains also in the special case when the radiation from the surface is completely eliminated. This is secured for example in the well-known method of Kohlrausch for the determination of thermal conductivities. The problem has been investigated theoretically by Wintergerst using the method of Laplace transforms. He obtains the same infinite series as (39) without $l^2P$, and naturally without $\nu_a$ too.

Comparing now equation (35) obtained by the present method with the corresponding expression (39) one obtains the following results. For small values of $t$, for which $\exp\{-\alpha(2n-1)^2\pi^2t/4l^2\}$ can be put equal to $1-\alpha\{(2n-1)^2\pi^2/4l^2\}t$, it can be shown that (39) is identical with (35) for all values of $q$. Making this substitution we obtain

$$\tau = \frac{16l^2}{\pi^3}\Delta Q\exp(-4\beta T_l^3 t)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\cos\frac{(2n-1)\pi q}{2l}$$
$$-\frac{4}{\pi}\Delta Q\alpha t\exp(-4\beta T_l^3 t)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\frac{(2n-1)\pi q}{2l}. \quad (41)$$

Using the well-known relations (see Sommerfeld 1949)

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^3}\cos\frac{(2n-1)\pi q}{2l} = \sin\frac{\pi x}{2l}+\frac{1}{3^3}\sin\frac{3\pi x}{2l}+\frac{1}{5^3}\sin\frac{5\pi x}{2l}+\dots$$
$$=\frac{\pi}{8}\left[\frac{\pi^2 x}{2l}-\left(\frac{\pi x}{2l}\right)^2\right]=\frac{\pi^3}{32}(1-q^2/l^2) \quad (42)$$

and†

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\frac{(2n-1)\pi q}{2l} = \sin\frac{\pi x}{2l}+\tfrac{1}{3}\sin\frac{3\pi x}{2l}+\tfrac{1}{5}\sin\frac{5\pi x}{2l}+\dots=\frac{\pi}{4}, \quad (43)$$

(41) can be seen to reduce to

$$\tau = \tfrac{1}{2}\Delta Q l^2(1-q^2/l^2)\exp(-4\beta T_l^3 t)\left(1-\frac{2\alpha t}{l^2-q^2}\right)$$
$$= \tfrac{1}{2}\Delta Q l^2(1-q^2/l^2)\exp-\left[4\beta T_l^3+\frac{2\alpha}{l^2-q^2}\right]t, \quad (44)$$

which is identical with (35).

This is indeed very gratifying especially since we are now dealing with a single term having a single relaxation time, instead of an infinite series each term of which has its own characteristic relaxation time. Equation (44) also enables us to follow in detail the variation of the relaxation time with $q$ for small values of $t$.

* This can be seen to be consistent with the criterion obtained earlier for the whole of the filament to be in the parabolic range, namely that higher powers of $l\sqrt{P}$ than the second are negligible in comparison with unity.

† Equation (43) may be shown to follow from (42) by differentiating both sides of (42) twice with respect to $q$.

As $t$ increases it can be seen from (39) that the terms involving high values of $n$ get progressively weakened out, because of the exponential factor involving $n$ and $t$, and the expression for $\tau$ ultimately settles down to

$$\tau = \frac{16l^2}{\pi^3}\Delta Ql^2 \cos\frac{\pi q}{2l} e^{-\nu_2 t}, \tag{45}$$

where $$\nu_2 = 4\beta T_2^3 + \alpha\pi^2/(4l^2). \tag{46}$$

From the following considerations it will be seen that in the *central regions where q is small*, (45) is not very different from (44). (1) The limiting value $\nu_2$ to which $\nu$ tends as $t \to \infty$ is not very different from the value

$$\nu_1 = 4\beta T_2^3 + 2\alpha/l^2, \tag{47}$$

which holds for small values of $t$. The difference between them is in the second term only, and corresponds to dropping a factor $\frac{1}{8}\pi^2$ from it, which differs only slightly from unity. (2) As $t \to \infty$ the coefficient of the exponential term in the Fourier expansion tends to $(16/\pi^3)\Delta Ql^2 \cos \pi q/(2l)$. The corresponding coefficient from the present calculation, which remains independent of $t$, is $\frac{1}{2}\Delta Ql^2(1-q^2/l^2)$. Remembering that $32/\pi^3 \approx 1$, the two coefficients again can be seen to agree to the same approximation to which $\nu_{b1}$ and $\nu_{b2}$ agree, provided that $q$ is small enough to make

$$1 - q^2/l^2 \approx \cos(q\sqrt{2}/l).$$

Confining attention to regions close to the centre, it is thus seen that expression (35) for the growth of temperature will hold for all values of $t$; for small values of it as a precise result, and for larger values as a close approximation, the approximation remaining as close as $\frac{1}{8}\pi^2$ to 1, or $\frac{1}{32}\pi^3$ to 1, even when $t \to \infty$.

We have discussed at some length the special case where the temperature distribution over the whole filament is parabolic, for two reasons. It is the only case where the Fourier method can be conveniently applied, and where the results obtained in the present paper can be compared with those obtained by the Fourier method. But the more important reason is that the actual distribution in any filament near the centre is parabolic, and the comparison made just now between the two methods as applied to the special case, enables us to adapt the Fourier method for the investigation of the time lag near the centre of any filament, though the temperature distribution taken over the rest of the filament would be too complicated for a direct Fourier expansion, besides involving a relaxation time that varies markedly with $q$.

## 10. General case of any filament

Returning to the general case for which the growth of temperature is given by (36), it can be readily seen that in the region where the temperature distribution is parabolic, it has precisely the same validity as the corresponding expression (35) for a shorter filament which was discussed in the previous section. Considering first the expression for $T_2 - T_1$, namely,

$$T_2 - T_1 = \Delta T_l[1 - \tfrac{1}{2}(\Delta Q/\Delta T_l)q^2],$$

one may, without invoking any special mechanism, identify it with the infinite series

$$T_2 - T_1 = \frac{32}{\pi^3}\Delta T_l \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} \cos\frac{(2n-1)\pi q}{2\lambda}, \tag{48}$$

where $\lambda$ is now a certain *effective length* defined by

$$\lambda = \left(\frac{2}{\Delta Q/\Delta T_l}\right)^{\frac{1}{2}}. \tag{49}$$

Assigning to each of the terms in (48) the appropriate time factor that will satisfy the differential equation,* we obtain

$$\tau = \frac{32}{\pi^3}\Delta T_l \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^3} \cos\frac{(2n-1)\pi q}{2\lambda} \exp\left[-\{4\beta T_2^3 + \alpha(2n-1)^2\pi^2/(4\lambda^2)\}t\right]. \tag{50}$$

Equation (50) is a complete and precise description of the growth of the temperature at every point in the parabolic range of any filament, just as (39) is for the special case when the whole filament is within the parabolic range. Indeed, (50) can be seen to reduce to (39) when $\Delta Q/\Delta T_l$ is put equal to $2/l^2$, which is its appropriate value for the latter filament. In the general case considered just now, the usual Fourier representation, in which the actual length of the filament determines the period, would have involved coefficients $A_n$ which would have been difficult to evaluate. On the other hand by using the effective length $\lambda$ the representation over the whole of the parabolic range takes the elegant form (50). In the choice of the appropriate $\lambda$ we are guided by the results of the present investigation.

Comparing now (36) with (50) one can see that the relation between them is exactly the same as the relation between (35) and (39). For small values of $t$, (36) and (50) are identical over the whole range of $q$ over which the temperature distribution in the steady state is parabolic. Further (36) can be seen to be a close approximation to (50) at any value of $t$, if $q$ is made sufficiently small. The degree of approximation is exactly the same as that of (35) to (39).

This is a very satisfactory result, since it implies that the growth of temperature near the centre of any filament, long or short, can always be represented to a close approximation by a single relaxation time $t_r = 1/\nu$, where

$$\nu = 4\beta T^3 + \alpha \Delta Q/\Delta T_l.$$

The first term on the right-hand side is a constant contribution which is independent of time. The correct value of the second term, however, should vary slightly with time, and since we are trying to represent it also by a constant term the approximation, which is perfect for small values of $t$, slightly deteriorates with time, the approximation, however, remaining as close as $\frac{1}{8}\pi^2$ is to 1 even when $t$ tends to infinity.

The possibility of representing the growth of temperature near the centre of any filament by a simple exponential law involving a single relaxation time effects naturally a considerable simplification of the problem, and further makes the practical application convenient.

* The value of $\nu_a$, equal to $4\beta T^3$, appearing in (50) is not quite independent of $q$. It will be, however, practically so, since over the range where $4\beta T_2^3$ is significant it can be shown to be close to $4\beta T_l^3$.

The simplification is due ultimately to the following circumstances:

(1) The distribution of temperature both in the initial and in the final states, near the centre of any filament, is parabolic.

(2) The parabola can be represented by a Fourier series which is very simple when the length involved is suitably chosen.

(3) With the proper choice of the *effective* length the first term in the Fourier expansion is by itself a close approximation to the parabola.

(4) Hence the growth of temperature near the centre can be represented by a simple exponential law involving a single effective relaxation time.

## 11. Variation of the time lag at the centre of a filament with its length

The general expression for $\nu$ at the centre of the filament may also be written in the following convenient form. Using relation (23) for $Q$, namely, $\alpha Q = \beta(T_m^4 - T_l^4)$, one obtains

$$\alpha \Delta Q/\Delta T_l = 4\beta[T_m^3 \Delta T_m/\Delta T_l - T_l^3], \tag{51}$$

whence

$$\nu = \nu_0 \Delta T_m/\Delta T_l, \tag{52}$$

where $\nu_0 = 4\beta T_m^3$, as we have seen, is the attenuation coefficient at the centre of an infinitely long filament having the same $T_m$. Equation (52) is a convenient relation since it combines both $\nu_a$ and $\nu_b$ into a single term.

It can be further shown that for a filament in which the temperature distribution is wholly parabolic

$$(\beta/\alpha)(T_m^4 - T_l^4) = (2/l^2)(T_l - \Theta),$$

and if $\Theta$ is small* in comparison with $T_l$,

$$\Delta T_m/\Delta T_l = \frac{T_m^4 + 3T_l^4}{4T_m^3 T_l} = \frac{1+3\theta^4}{4\theta}, \tag{53}$$

where $\theta = T_l/T_m$ depends on the length of the filament.

For long filaments, as $T_l \to T_m$, $\Delta T_m/\Delta T_l \to 1$, and this is also the value to which (53) tends. Presumably the variation of $\Delta T_m/\Delta T_l$ in the intermediate range of lengths also is given by the same expression, namely (53).

The filaments used in practice are long ones in which $\theta$ is close to unity. In such filaments obviously even a major change in length will not much affect the relaxation time. Thus the major factors that affect it then are $p/\omega$ and $T_m^3$.

## 12. Dependence of the time lag on $p/\omega$ and on $T_m^3$

The expression for the relaxation time for any filament, namely,

$$t_r = \frac{c\delta\omega}{p\epsilon\sigma T_m^3} \frac{\Delta T_l}{\Delta T_m}, \tag{54}$$

shows that for a given value of $T_m$, the relaxation time is inversely proportional to $p/\omega$. If, however, the heating current through the filament is kept constant when $p/\omega$ is varied, an increase in $p/\omega$ will reduce $T_m$, since $(T_m^4 - T_0^4)p/\omega$ has now to

* Equation (52) will obviously hold for any value of $\Theta$.

remain constant. If $T_0^4$ can be neglected in comparison with $T_m^4$, then $t_0$ will be proportional to $(p/\omega)^{-\frac{1}{2}}$, and the effect of an increase in $p/\omega$ is then not quite so striking. But in practice it is no inconvenience to maintain the same $T_m$ by suitably increasing the heating current in the ribbon, and thus ensure a great reduction on the time lag. We should also emphasize here the inverse proportionality of $t_0$ to $T_m^3$, and the consequent advantage of using high central temperatures in reducing the time lag.

It will be seen from (54) that the expression for the relaxation time involves in addition to $p/\omega$ and $T_m^3$ two other factors, namely, $c\delta$ and $\Delta T_l/\Delta T_m$. An increase in $p/\omega$ will in effect be equivalent to a decrease in the heat capacity $c\delta$, and hence the effect of an increase of $p/\omega$ on the time lag may be taken to be through minimizing the heat capacity. This effect will be independent of whether the filament is long or short. On the other hand the magnitude of the factor $\Delta T_l/\Delta T_m$ depends on the conduction losses through the ends, and it tends to unity at the centre of a long filament. Hence in a short filament where $\Delta T_l$ will be much larger than $\Delta T_m$, the effect of an increase in $p/\omega$ is equivalent to making $\Delta T_l$ approach $\Delta T_m$, and so would be equivalent to minimizing also the heat loss due to conduction. When the filament is long and we are considering a region near the centre, the question of conduction loss does not arise. But at other points, particularly near the ends it would.

In any case, the advantage of a ribbon filament over one of circular cross-section for use particularly in an optical pyrometer is very definite. Apart from the reduction of the time lag secured by the use of a ribbon filament, the actual length of the filament needed for ensuring a given approximation to the ideally long one is also smaller, as we have shown, in a ribbon filament than in a filament of circular cross-section. The larger area which the ribbon can present is also a help in matching it against the background whose temperature is to be measured, if the background is a broad one.

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# The distribution of temperature along a thin rod electrically heated *in vacuo*

## VI. End-losses

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On the basis of the known distribution of temperature along a filament electrically heated *in vacuo*, the distributions of some of the other physical characteristics that are known functions of the temperature were investigated in part IV. By means of the expression thus obtained for the distribution of any given characteristic, the over-all performance of the filament for this particular purpose can be obtained by integration, and can be expressed in terms of a certain equivalent length of an ideal filament in which the loss by conduction at the ends is completely eliminated. The difference between the actual and the equivalent lengths, which is generally referred to as the correction for the end-losses, is calculated in the present part for some typical physical properties that are of practical interest. Among them are

(1) properties like the heat capacity, the heat generated, and the total output of radiation, which are proportional to known powers of the temperature $T$;

(2) properties like brightness and the rate of evaporation, which are proportional to known powers of exp $(1/T)$;

(3) properties like thermionic emission, which are proportional to known powers of $T$ and of exp $(1/T)$.

The corrections for the end-losses thus calculated for a long filament of tungsten for the different properties are found to agree closely with those computed graphically by Worthing from the observed distributions of these properties along the filament.

It is further shown that the correction for a filament of *medium* length, such as is used in high-frequency power-tubes, is practically the same as for a similar *long* filament heated by the same current.

### 1. Introduction

In parts I and II of this paper (Jain & Krishnan 1954 *a*, *b*) the distribution of temperature along a thin filament electrically heated *in vacuo* was investigated in detail, and simple expressions were obtained for the temperature as a function of the distance. Parts III and IV (Jain & Krishnan 1954 *c*, *d*) were concerned with the experimental verification of the various results obtained in I and II. On the basis of the known distribution of temperature along the filament, the distributions of other physical characteristics which depend on the temperature, and which can be expressed as explicit functions of the temperature, were also investigated in IV.

The most convenient way of appraising the over-all performance of a heated filament used for any specific purpose is to compare the performance with that of an ideal filament in which the loss of heat by conduction through the ends is eliminated, and in which, therefore, the temperature remains constant at $T_m$ throughout its length. Obviously $T_m$ would be the temperature at the centre of a filament similar to the filament actually used, but very long, and heated by the same current.

Let $F_i$ be the particular physical property under consideration, which is a known function of the temperature $T$. Now denoting by $\psi_i$ the ratio $F_i/F_{i,m}$, where $F_{i,m}$ is the value of $F_i$ at $T = T_m$, the effective length is given by

$$L_e = \int_0^L \psi_i \,\mathrm{d}X, \tag{1}$$

where $X = x\sqrt{A}$ is the reduced distance as defined in part IV, and $L = l\sqrt{A}$ is the reduced semi-length of the filament. Now $L - L_e$, which we shall denote by $\eta$, is generally called the correction for the end-loss at each end. It is a useful physical quantity which enables us to appraise the efficiency of performance of the filament for the particular purpose for which it is to be used. The present part is concerned with the calculation of the corrections for the end-losses for some of the typical temperature-dependent properties that are of practical interest.

## 2. General expression for the correction in a long filament

We shall first consider the case of a long filament in which the temperature $T_l$ at the centre is sensibly the same as $T_m$. As was shown in II and IV, the temperature distribution along such a filament is given by

$$v \exp f(v) = v_0 \exp\{f(v_0) - X\}, \tag{2}$$

where
$$v = (T_m - T)/T_m \tag{3}$$

and
$$f(v) = A_1 v + A_2 v^2 + A_3 v^3 + \ldots, \tag{4}$$

and $v_0$ denotes the value of $v$ at the ends. The coefficients $A_n$ in (4) have the same significance as in I, and are readily evaluated. Thus one obtains

$$\mathrm{d}X/\mathrm{d}v = -v^{-1}(1 + A_1 v + 2A_2 v^2 + 3A_3 v^3 + \ldots). \tag{5}$$

Under the usual conditions of working, $v$ is considerably smaller than unity, even at the ends of the filament, and since the coefficients $A_n$ also decrease with increasing $n$, the terms in (5) involving $A_3, A_4, \ldots$ are found to be negligible.

In view of (1) and (5), one obtains the following expression for the correction for the end-loss,

$$\eta = \int_0^{v_0} (1 - \psi_i)\, v^{-1}(1 + A_1 v + 2A_2 v^2)\,\mathrm{d}v. \tag{6}$$

Several physical properties have been studied in detail for their temperature-dependence, and many of them, as Langmuir (1930) has shown, may be expressed, at any rate empirically, by the relation

$$F_i = A_i T^{\gamma_i} \exp(-\Theta_i/T), \tag{7}$$

where $A_i$, $\gamma_i$ and $\Theta_i$ are constants characteristic of the particular property $F_i$ under consideration. Some of these properties were tabulated in IV. For such properties the value of $1 - \psi_i$ appearing in expression (6) can be expressed as a power series in $v$, namely,

$$1 - \psi_i = a_1 v + a_2 v^2 + a_3 v^3 + \ldots, \tag{8}$$

and the integration in (6) can be done term by term. One thus obtains

$$\eta = a_1 v_0 + \tfrac{1}{2}(a_2 + a_1 A_1)\, v_0^2 + \tfrac{1}{3}(a_3 + a_2 A_1 + 2a_1 A_2)\, v_0^3 + \tfrac{1}{4}(a_4 + a_3 A_1 + 2a_2 A_2)\, v_0^4 + \dots \qquad (9)$$

Hence one needs to know only the magnitudes of $a_1, a_2, a_3, \dots$ for the different properties, in order to be able to calculate the corresponding corrections for the end-losses.

### 3. Properties for which $\Theta_i = 0$

We shall first consider the end-losses for the special case of those properties for which $\Theta_i = 0$, as, for example, the electrical resistance, or the energy generated (which is proportional to the resistance), or the total output of radiation, all of which have been studied in detail for tungsten. These quantities are also of practical importance. For example, since the energy input is proportional to the electrical resistance, the end-loss correction for the resistance will enable us to determine the voltage to be maintained between the terminals of a filament in order that the temperature $T_m$ at the centre may have any specified value. Similarly, the end-loss correction for total radiation will enable us to estimate the efficiency of performance of a sealed-in filament lamp to be used for radiation heating.

Equation (1) now reduces to the simple form

$$F_i = A_i T^{\gamma_i}, \qquad (10)$$

so that

$$\psi_i = (1-v)^{\gamma_i}. \qquad (11)$$

(From now on we shall drop the subscripts $i$ for convenience.) For these properties the expression for $1-\psi$ will obviously conform to (8), and that for $\eta$ to (9), in which

$$a_1 = \gamma, \quad a_2 = -\tfrac{1}{2}\gamma(\gamma-1), \quad a_3 = \tfrac{1}{6}\gamma(\gamma-1)(\gamma-2). \qquad (12)$$

Worthing (1922) has studied experimentally the distribution of several physical properties along a long filament of tungsten electrically heated *in vacuo*. By plotting the experimental values of $F/F_m$ against $X$, and by finding graphically the area subtended between the curve and the $X$-axis, he has computed the effective lengths $L_e$ for these properties, and thence the end-loss corrections. In all his measurements the temperature $T_m$ at the middle point of the filament was 2400° K, and the temperature of the point from which the distances $X$ were measured was $0{\cdot}25T_m$, so that $v_0 = 0{\cdot}75$. The values of $\eta$ obtained graphically by Worthing for all the properties of tungsten studied by him for which $\Theta = 0$ are entered in table 1.

The values of $\eta$ calculated from (9) and (12) are also entered in the table for comparison. Now the values of $A_1$ and $A_2$ appearing in (9) are readily calculated, as was shown in I, from the known values of $T_m$ and of the temperature coefficients of the physical quantities involved. Among the latter the coefficient of the thermal conductivity $\kappa$ exercises the predominant influence. The temperature coefficient of $\kappa$, which we shall denote by $\alpha$, has been determined by Worthing himself for the specimen of tungsten used by him, and he obtains a small *positive* value. Later measurements by Osborne (1941), on the other hand, yield a small *negative* value for $\alpha$, which appears more probable from general theoretical considerations. Osborne attributes the positive $\alpha$ obtained by Worthing to the inadequate ageing of the specimen used by him.

Whatever may be the reason for the discrepancy, since we are calculating the end-losses for Worthing's specimen (for comparison with the values computed by him graphically for the same specimen) we use here his $\alpha$ in preference to Osborne's, remarking, however, that the choice of the one or the other value of $\alpha$ makes very little difference to $\eta$. The agreement between the values obtained by Worthing graphically and those calculated is satisfactory.

TABLE 1. VALUES OF $\eta$ FOR A LONG FILAMENT OF TUNGSTEN

| property | heat content | energy input | total radiation |
|---|---|---|---|
| $\Theta = 0;\ \gamma =$ | 1 | 1·2 | 5·1 |
| $\eta$ calculated | 0·9 | 1·1 | 2·4 |
| $\eta$ obtained graphically by Worthing | 0·9 | 1·0 | 2·3 |

## 4. PROPERTIES FOR WHICH $\gamma = 0$

We shall consider next the other special case in which $\gamma = 0$ but $\Theta$ remains finite. The brightness and the rate of evaporation, for example, belong to this category. In this case

$$\psi = \exp\{-\Theta(1/T - 1/T_m)\} = \exp\{-\theta v/(1-v)\}, \tag{13}$$

where

$$\theta = \Theta/T_m. \tag{14}$$

Expression (13) can then be seen to conform to (8), and the corresponding expression for $\eta$ to (9), where

$$a_1 = \theta, \quad a_2 = \theta - \tfrac{1}{2}\theta^2, \quad a_3 = \theta - \tfrac{1}{2}\theta^2 + \tfrac{1}{6}\theta^3. \tag{15}$$

Since we shall have occasion to use separately the values of $a_n$ given by (12) and by (15), we shall denote them by $b_n$ and $c_n$ respectively.

## 5. THE CONVERGENCE OF THE SERIES FOR $\eta$

There is an essential difference between the two special cases $\Theta = 0$ and $\gamma = 0$ considered in the last two sections, which has an important bearing on the convergence of the power series representing $\eta$. In Worthing's measurements $v_0 = 0{\cdot}75$. Hence in the power series of $v_0$ the decrease of $v_0^n$ with the increase of $n$ is not by itself rapid enough for convenient numerical computation. In the case when $\Theta = 0$, however, this is not a serious inconvenience, since the coefficients of $v_0^n$, namely, $b_n$, decrease rapidly with the increase of $n$. Even for the highest value of $\gamma$ that we have to deal with, namely, 5·1, which occurs in the expression for the total radiation, it is found that the fourth term in the expression for $\eta$ has dropped much below 2 % of the first term, and the fifth term is practically negligible.

On the other hand, when $\gamma = 0$ and $\Theta$ is finite, the corresponding coefficients $c_n$ that occur in the expression for $\eta$ actually increase in the early stages of the increase of $n$ before decreasing. This is due to $\theta$ being greater than unity. Indeed, it is much greater, being about 10·5 for brightness, and about 39 for the rate of evaporation. Hence the coefficients $c_n$ continue to grow till we reach very high values of $n$.

For the same reason, namely, that $\theta \gg 1$, for which the coefficients $c_n$ of the terms in the expansion of $\eta$ as a power series in $v_0$ remain large till we reach high values of $n$, the value of $F$, which is proportional to $\exp(-\Theta/T)$ will drop down rapidly with the decrease of $T$, i.e. with the increase of $v$. Hence the contribution to $L_e$ may become negligible long before $v$ has approached $v_0$. In other words, the significant portion of the contribution to $L_e$ may be confined to the region $0 \leqslant v \leqslant v_s$, where $v_s \ll v_0$. Or from the point of view of $\eta$ the contribution to it from the range $0 \leqslant X \leqslant X_s$, where $X_s$ is the distance of the point $v_s$, will be just the length $X_s$ of the range, and the contribution from the rest of the filament can be expressed as a power series in $v_s$, which will be rapidly convergent.

Aiming at an accuracy of say 1 % in the evaluation of $\eta$, one obtains

$$\exp\{-\theta v_s/(1-v_s)\} \approx 0{\cdot}01, \tag{16}$$

which for optical brightness, for which $\theta = 10{\cdot}5$, yields $v_s = 0{\cdot}3$. With these values it is found that the number of terms to be retained in the evaluation of $\eta$ gets reduced to seven, as against a number of the order of 100 to be retained if the expansion had been in powers of $v_0$. When $\theta$ is larger, as is the case with the rate of evaporation, the saving in the number of terms is correspondingly larger.

The values of $\eta$ for brightness and for the rate of evaporation calculated in this manner are entered in table 2. The value obtained by Worthing by graphical computation for the former case can be seen to be close to the calculated value. No experimental data are available for evaporation.

TABLE 2. END-LOSS CORRECTIONS FOR TUNGSTEN

| property | brightness | rate of evaporation |
|---|---|---|
| $\gamma = 0;\ \Theta =$ | 25200 | 94100 |
| $\eta$ calculated | 3·2 | 4·6 |
| $\eta$ obtained graphically by Worthing | 3·0 | — |

## 6. PROPERTIES FOR WHICH $\Theta$ AND $\gamma$ ARE BOTH FINITE

We shall next consider the more general case where both $\Theta$ and $\gamma$ are finite. Thermionic emission is a typical example of such a property. In this case

$$\begin{aligned}\psi &= (1-v)^\gamma \exp\{-\theta v/(1-v)\}\\ &= (1-b_1v-b_2v^2-b_3v^3-\dots)(1-c_1v-c_2v^2-c_3v^3-\dots),\end{aligned} \tag{17}$$

which can be seen to reduce to the same form as (8), where now

$$a_1 = b_1+c_1, \quad a_2 = b_2+c_2-b_1c_1, \quad a_3 = b_3+c_3-b_1c_2-b_2c_1. \tag{18}$$

Here again the filament can be divided conveniently into two regions $0 \leqslant v \leqslant v_s$ and $v_s \leqslant v \leqslant v_0$ respectively, the contribution to $\eta$ from the latter region being practically the same as the length $X_s$ of the region, and the contribution from the former region being expressible as a rapidly convergent power series in $v_s$. Taking

$\Theta = 52\,600$ and $\gamma = 2$, the correction for thermionic emission from an electrically heated long tungsten filament calculated in this manner, comes out as 4·0, as compared with the value 3·9 computed by Worthing graphically.

### 7. Same expressions applicable to filaments of medium length

Even when the filament is not long, but long enough to include the logarithmic region, it was shown in IV that the temperature distribution over the whole length of the filament will conform to an additive formula of the type proposed by Stead. Now the temperature-dependent characteristics that we are considering vary more rapidly than in proportion to the temperature, and hence they too should conform to a similar additive formula, namely

$$1-\psi_L(X) = [1-\psi_\infty(X)]+[1-\psi_\infty(2L-X)], \tag{19}$$

where $\psi_L(X)$ denotes the value of $\psi$ at distance $X$ in a filament of reduced semi-length $L$.

Multiplying both sides of (19) by $\mathrm{d}X$ and integrating between the limits $X = 0$ and $X = L$, one obtains for the end-loss correction in the finite filament

$$\eta_L = \int_0^L [1-\psi_L(X)]\,\mathrm{d}X = \int_0^L [1-\psi_\infty(X)]\,\mathrm{d}X + \int_0^L [1-\psi_\infty(2L-X)]\,\mathrm{d}X. \tag{20}$$

The second integral on the right-hand side of (20) can be shown to reduce to $\int_L^{2L} [1-\psi_\infty(X)]\,\mathrm{d}X$, and the whole of the right-hand side to $\int_0^{2L} [1-\psi_\infty(X)]\,\mathrm{d}X$.

Consistently with the basic assumption that the additive law holds, the value of $\psi_\infty(X)$ will remain at unity for all values of $X$ beyond $2L$. Hence the upper limit of the integral may be extended from $2L$ to $\infty$, and $\eta_L$ will have practically the same value as $\eta_\infty$.

The filaments used for example in high-frequency power-tubes belong to this category.

### 8. Short filaments

In the case of filaments which are too short to include the logarithmic region, the distribution of temperature is given by

$$t = T_l - T = \frac{Q}{P}[\cosh(q\sqrt{P})-1], \tag{21}$$

in which the constants $Q$ and $P$ are known (see II), and $q = l-x$ is the actual distance measured from the centre. The correction due to the end-loss can be calculated in the same manner as for the longer filaments, but it will now be a large fraction of the total length. Since, however, such short filaments are not of practical interest, we do not give here the detailed calculations.

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