On the Equivalence of Certain Infinite Series and the corresponding Integrals.

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§1. In calculating the intensity of light scattered from a homogeneous medium, one comes across the infinite series

$$\alpha \sum_{n=-\infty}^{+\infty} \frac{\sin^2 (n\alpha+\theta)}{(n\alpha+\theta)^2},$$

where θ is a constant, and n an integer. α is a positive number which under the conditions under which light-scattering is generally studied, can be made arbitrarily small, and hence the sum is usually replaced by the corresponding integral*

$$\alpha \sum_{n=-\infty}^{+\infty} \frac{\sin^2 (n\alpha+\theta)}{(n\alpha+\theta)^2} = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \pi. \qquad (1)$$

It is easy to show†, however, ~~(Krishnan~~ that (1) holds not only in the limit when α tends to zero, but for any value of α in the range $0 < \alpha \leq \pi$. The proof is as follows.

Consider the series‡

$$\sum_{n=-\infty}^{\infty} \frac{\sin (n+\beta) z}{n+\beta} = \pi, \qquad (2)$$

where β is a constant, n an integer, and $0 < z < 2\pi$. The series can be integrated term by term with respect to z in any closed interval (γ, δ), where $0 < \gamma < \delta < 2\pi$, since it is uniformly convergent in this interval. We thus obtain

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* Einstein. Ann. der Physik.

† Krishnan and Bhatia, (1947).

‡ Bromwich. Infinite Series (1931), 371, Ex. 5.

$$\sum_{n=-\infty}^{\infty} \frac{\cos(n+\beta)\gamma}{(n+\beta)^2} - \sum_{n=-\infty}^{\infty} \frac{\cos(n+\beta)\delta}{(n+\beta)^2} = \pi(\delta-\gamma). \quad (3)$$

Keeping δ constant and making $\gamma \to 0$, ~~it is readily seen that~~ (3) reduces to

$$\sum_{n=-\infty}^{\infty} \frac{1-\cos(n+\beta)\delta}{(n+\beta)^2} = \pi\delta,$$

Since the first series on the left side of (3) is uniformly convergent at $\gamma = 0$ and therefore represents a continuous function of γ. Putting now $\delta = 2\alpha$, $\alpha\beta = \theta$, and dividing both sides by 2α ($\neq 0$), we obtain for $0 < \alpha < \pi$,

$$\alpha \sum_{n=-\infty}^{\infty} \frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2} = \pi. \quad (4)$$

This can be seen to be true for $\alpha = \pi$ also, and hence (4) holds over the interval $0 < \alpha \leq \pi$.

The above result implies that the area subtended between the curve $y = \sin^2 x/x^2$ and the x-axis may be obtained just as well by adding up the ordinates at equal intervals α, and multiplying by α (i.e. by ~~single~~ simple rectangulation), as by integration, provided ~~that~~ $0 < \alpha \leq \pi$. It may be noted here that the sum in (4) is independent of θ, which shows that we may start the division of the x-axis into equal steps α from any value of x. In particular, $x = 0$ need not be one of the points of division.

§2. As we shall show presently, the same property, viz.

$$\alpha \sum_{n=-\infty}^{\infty} f(n\alpha+\theta) = \alpha \sum_{n=-\infty}^{\infty} f(n\alpha) = \int_{-\infty}^{\infty} f(x)\,dx \quad (5)$$

for a suitable range of values of α, $0 < \alpha \leq l$, say, holds for several other functions too. Before considering such functions, we shall refer here to an alternative proof of (4), ~~which was~~ given ~~as~~ by Prof. ~~Professor~~ Norbert Wiener‡, ~~and is quoted by us in the paper referred to (Krishnan and Bhatia, 1947)~~. The proof is very suggestive, and enables us to determine the conditions for the validity of the equations (5)*.

Consider an even function $f(x) = f(-x)$, and its Fourier transform defined by

$$g(v) = \cancel{\frac{1}{\sqrt{2\pi}}} \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(x)\, e^{ivx}\, dx. \qquad (6)$$

Let $f(x)$ be such that $g(v)$ has non-zero values in the range $-a < v < a$, where a is a positive number. Obviously $f(x) = \sin^2 x / x^2$ satisfies this condition, since

$$g(v) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (2 - |v|), \text{ if } 0 \le |v| \le 2, \; \cancel{\text{and}}$$
$$= 0 \qquad , \text{ if } |v| \ge 2.$$

According to Poisson's summation formula†,

$$\sum_{n=-\infty}^{\infty} f(n\alpha) = \frac{(2\pi)^{\frac{1}{2}}}{\alpha} \sum_{N=-\infty}^{\infty} g\left(\frac{2\pi N}{\alpha}\right), \qquad (7)$$

where N is an integer. If now $0 < \alpha \le \frac{2\pi}{a}$, there is only one value of N, viz. $N = 0$, for which $g\left(\frac{2\pi N}{\alpha}\right)$ differs from 0. Hence

$$\sum_{n=-\infty}^{\infty} f(n\alpha) = \frac{(2\pi)^{\frac{1}{2}}}{\alpha} g(0), \qquad (8)$$

whence substituting for $g(0)$ from (6), we obtain

$$\alpha \sum_{n=-\infty}^{\infty} f(n\alpha) = \int_{-\infty}^{\infty} f(x)\, dx. \qquad (9)$$

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‡ Quoted by us in the paper by Krishna and Bhatia.

* $f(x)$ = constant is a trivial example that satisfies (5).

† See Titchmarsh, Introduction to the Theory of Fourier Integrals, p. 60, where the Poisson summation formula is proved with reference to the Fourier cosine transform $g_c(v) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^{\infty} f(x) \cos vx\, dx$; but, for the even functions that we are considering, the exponential and the cosine transforms become identical.

Further, for a fixed θ, it can be seen that

$$\int_{-\infty}^{\infty} f(x+\theta)\, e^{ivx}\, dx = e^{-iv\theta}\int_{-\infty}^{\infty} f(x)\, e^{ivx}\, dx, \qquad (10)$$

which shows that for $v = 0$, the Fourier transform of $f(x+\theta)$ is the same as that of $f(x)$. Hence it follows that if $f(x)$ be such that its Fourier transform $g(v)$ has non-zero values if $|v| < a$, and zero value otherwise, then

$$\alpha \sum_{n=-\infty}^{\infty} f(n\alpha+\theta) = \alpha \sum_{n=-\infty}^{\infty} f(n\alpha) = \int_{-\infty}^{\infty} f(x)\,dx, \qquad (11)$$

provided that $0 < \alpha \leq 2\pi/a$.

We have seen that $\sin^2 x/x^2$ is such a function, the range of α over which (11) holds being $0 < \alpha \leq \pi$. More generally, the functions $f_{m,n}(x) = \sin^m x/x^n$, where m and n are positive integers, both odd or both even, and $n \leq m$, are examples of such functions; their Fourier transforms $g_{m,n}(v)$ can be seen to have zero value if $|v| \geq n$ ($|v| > n$ when $n = 1$), and non-zero values otherwise, and hence for these functions, relations (11) will be valid if $0 < \alpha \leq 2\pi/n$ ($0 < \alpha < 2\pi/n$ when $n = 1$).

The Fourier transforms of ~~the~~ $(\sin x/x)^m$, $m = 1, 2, 3$ are plotted in Fig. 1, a, b, c.

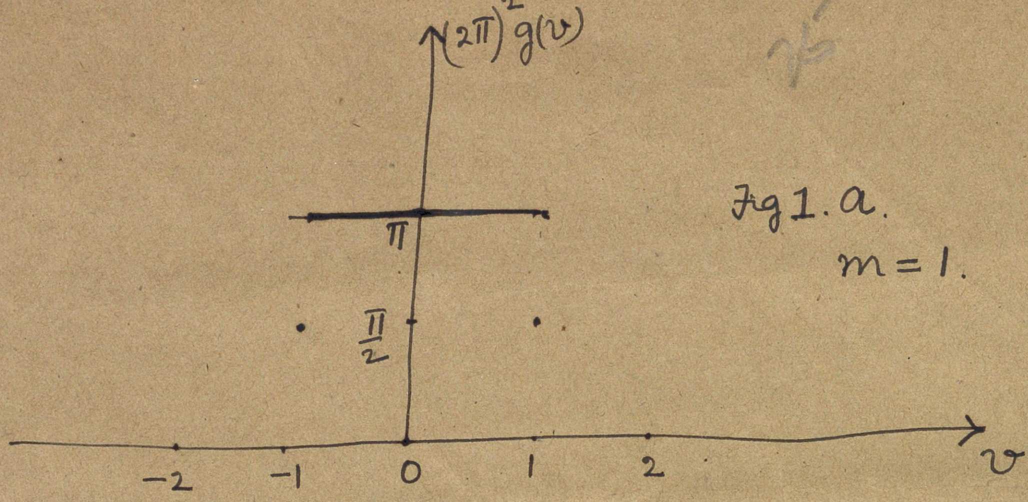

Fig 1. a.
$m = 1$.

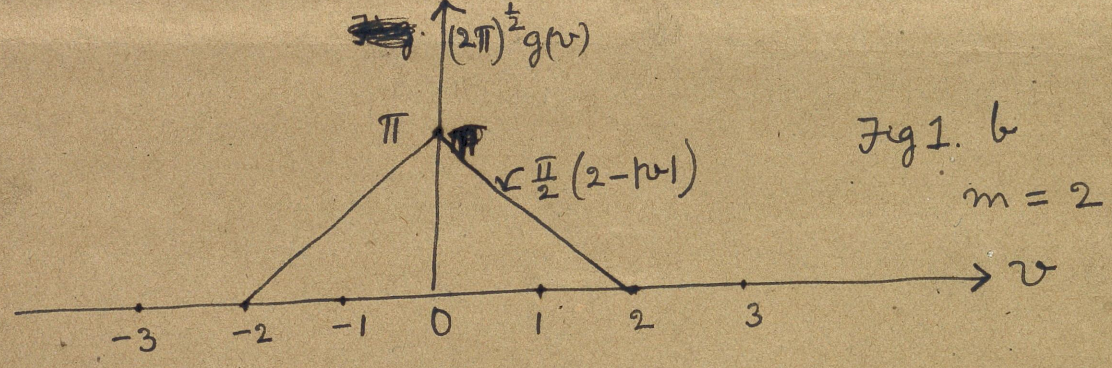

Fig 1. b
$m = 2$

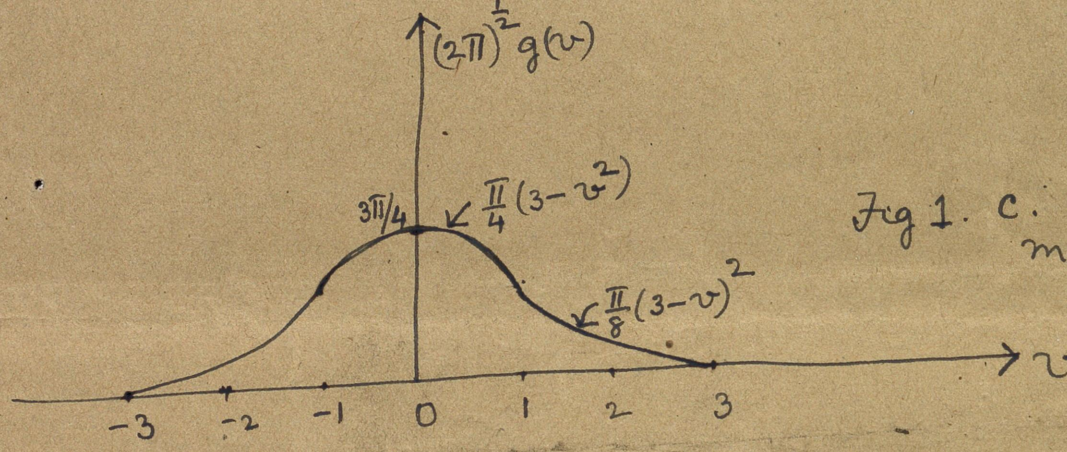

Fig 1. c.
$m = 3$

(11)?

§3. Since the Fourier transforms are known for a large number of functions, and many of them have been conveniently tabulated*, it is easy to select other examples of functions that satisfy the criterion stated above, and for which therefore relations (10) are valid. We give in the following tables a few. Those given in Table I are taken from a paper by Ramanujan†, ~~entitled~~ A class of Definite integrals, in which among others are given the values of several Fourier integrals, some of which are found to satisfy the above criterion. Those given in Table II are taken from Campbell

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* see for ex. G. A. Campbell and R. M. Foster, Fourier Integrals for Practical Applications, Bell Telephone Publications, Monograph B-584 (1931).

† Quarterly Jour. Math, 48 (1920), 294; Collected Papers, Cambridge ~~Univ. Press~~, 1927, 216.

and Foster's Tables referred to.

Table I

$f(x)$	$(2\pi)^{\frac{1}{2}} g(v)$, when $\lvert v\rvert < \pi$ $[g(v) = 0$ otherwise.$]$
$\dfrac{1}{\Gamma(\alpha+x)\Gamma(\beta-x)}$	$\dfrac{\left(2\cos\frac{v}{2}\right)^{\alpha+\beta-2} e^{\frac{1}{2}iv(\beta-\alpha)}}{\Gamma(\alpha+\beta-1)}$, convergence condition $R(\alpha+\beta)>1$.
$\dfrac{J_{\alpha+x}(\lambda)\, J_{\beta-x}(\mu)}{\lambda^{\alpha+x}\,\mu^{\beta-x}}$	$\left(\dfrac{2\cos\frac{v}{2}}{\Omega}\right)^{\frac{1}{2}(\alpha+\beta)} e^{\frac{1}{2}v(\beta-\alpha)i}\, J_{\alpha+\beta}\left\{\sqrt{\left(2\,\Omega\cos\frac{v}{2}\right)}\right\}$ where $\Omega = \lambda^2 e^{\frac{1}{2}vi} + \mu^2 e^{-\frac{1}{2}vi}$, Convergence condition $R(\alpha+\beta)>-1$.

Table II

$f(x$	$(2\pi)^{\frac{1}{2}} g(v)$, when $\lvert v\rvert \leq a$ $[g(v) = 0$ otherwise.$]$
$\dfrac{\sin a(x^2+\lambda^2)^{\frac{1}{2}}}{(x^2+\lambda^2)^{\frac{1}{2}}}$	$\pi J_0\left[\lambda(a^2-v^2)^{\frac{1}{2}}\right]$
$\dfrac{\sin a(x^2-\lambda^2)^{\frac{1}{2}}}{(x^2-\lambda^2)^{\frac{1}{2}}}$	$\pi I_0\left[\lambda(a^2-v^2)^{\frac{1}{2}}\right]$
$\cos a(x^2+\lambda^2)^{\frac{1}{2}} - \cos ax$	$-\dfrac{\pi a\lambda J_1\left[\lambda(a^2-v^2)^{\frac{1}{2}}\right]}{(a^2-v^2)^{\frac{1}{2}}}$
~~$\sin\dfrac{a(1-x)}{1-x} + \sin\dfrac{a(1+x)}{1+x}$~~	
$\dfrac{\sin a(1-x)}{1-x} + \dfrac{\sin a(1+x)}{1+x}$	$2\pi\cos v$
$\dfrac{\sin^2\frac{a}{2}(1-x)}{1-x} + \dfrac{\sin^2\frac{a}{2}(1+x)}{1+x}$	$\pi\sin v$
$\dfrac{\sin ax}{x^2} - \dfrac{a\cos ax}{x}$	$i\pi v.$

In the above, a is a positive real finite number, and λ is a complex number, not infinite. Relations (11) will be valid for all the functions in Tables I and II. Taking for example the first function entered in Table II, (11) will read as follows:

If $0 < \alpha \leq \frac{2\pi}{a}$,

$$\alpha \sum_{n=-\infty}^{\infty} \frac{\sin a\{(n\alpha+\theta)^2+\lambda^2\}^{\frac{1}{2}}}{\{(n\alpha+\theta)^2+\lambda^2\}^{\frac{1}{2}}} = \alpha \sum_{n=-\infty}^{\infty} \frac{\sin a(n^2\alpha^2+\lambda^2)^{\frac{1}{2}}}{(n^2\alpha^2+\lambda^2)^{\frac{1}{2}}}$$

$$= \int_{-\infty}^{\infty} \frac{\sin a(x^2+\lambda^2)^{\frac{1}{2}}}{(x^2+\lambda^2)^{\frac{1}{2}}}\,dx = \pi J_0(\lambda a). \qquad (12)$$

When $\lambda = 0$ and $a = 1$, this reduces to the case $f(x) = \sin x / x$.

The last two functions given in Table II however differ from the rest in that $g(v)$ besides being zero when $|v| > a$ is zero at $v = 0$ also. Considering the last ~~two~~ function, one obtains, if $0 < \alpha < \pi$,

$$\alpha \sum_{n=-\infty}^{\infty} \left[\frac{\sin(n\alpha+\theta)}{(n\alpha+\theta)^2} - \frac{\cos(n\alpha+\theta)}{n\alpha+\theta}\right] = \alpha \sum_{n=-\infty}^{\infty} \left[\frac{\sin n\alpha}{n^2\alpha^2} - \frac{\cos n\alpha}{n\alpha}\right]$$

$$= \int_{-\infty}^{\infty} \left(\frac{\sin x}{x^2} - \frac{\cos x}{x}\right) dx \qquad (13)$$

$$= g(0) = 0.$$

That the value of the integral in (13) is zero is otherwise obvious. We thus obtain when $0 < \alpha < \pi$ and $n\alpha + \theta$ is not equal to θ for any value of n,

$$\sum_{n=-\infty}^{\infty} \frac{\sin(n\alpha+\theta)}{(n\alpha+\theta)^2} = \sum_{n=-\infty}^{\infty} \frac{\cos(n\alpha+\theta)}{n\alpha+\theta}. \qquad (14)$$

The corresponding integrals $\int_{-\infty}^{\infty} \frac{\sin x}{x^2}\,dx$ and $\int_{-\infty}^{\infty} \frac{\cos x}{x}\,dx$ are however infinite.

The last but one function entered in Table II, for which also $g(v) = 0$ at $v = 0$ as also when $|v| > a$, yields similarly, if $0 < \alpha < \frac{\pi}{a}$,

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2 a(n\alpha+\theta)}{n\alpha+\theta} = \sum_{n=-\infty}^{\infty} \frac{\sin^2 a(n\alpha+\theta+2)}{n\alpha+\theta+2}, \qquad (15)$$

though ~~the integral~~ $\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x}\,dx = 0.$

§4. The class of functions that we have been considering here, which is characterized by the Fourier transforms being zero when $|v|$ is greater than a certain positive number a, has other interesting properties. Analogous to Poisson's summation formula which we have used, and which we may write in the more familiar form

$$\sqrt{\alpha}\left\{\tfrac{1}{2}f(0) + f(\alpha) + f(2\alpha) + \cdots\right\} = \sqrt{\beta}\left\{\tfrac{1}{2}g_c(0) + g_c(\beta) + g_c(2\beta) + \cdots\right\}, \qquad (16)$$

where $\alpha\beta = 2\pi$, there are others of the same type due to Ramanujan*. ~~viz.~~ Two of the typical formula are given below.

$$\sqrt{\alpha}\left\{f(\alpha) - f(3\alpha) - f(5\alpha) + f(7\alpha) + f(9\alpha) - \cdots\right\} = \sqrt{\beta}\left\{g_c(\beta) - g_c(3\beta) - g_c(5\beta) + g_c(7\beta) + g_c(9\beta) - \cdots\right\}, \qquad (17)$$

where $\alpha\beta = \frac{\pi}{4}$;

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* Collected papers, p.163.

$$\sqrt{\alpha}\left\{f(\alpha)-f(5\alpha)-f(7\alpha)+f(11\alpha)+f(13\alpha)-\cdots\right\}$$
$$=\sqrt{\beta}\left\{g_c(\beta)-g_c(5\beta)-g_c(7\beta)+g_c(11\beta)+g_c(13\beta)-\cdots\right\}, \quad (18)$$

where ~~α+~~ $\alpha\beta=\frac{\pi}{6}$, and 1, 5, 7, 11, 13, ⋯ are the numbers prime to 6. ;

~~and other forms~~ Unlike in Poisson's formula (16), in which the first term on the right side is $\frac{1}{2}g_c(0)$, the first term in (17), (18) and similar formulae is $g_c(\beta)$. If now $\beta > a$, i.e. if α is chosen small enough to make $\beta > a$, then all the terms on the right side of (17) and (18) vanish, and we get the following interesting results. From (17), for example, we obtain, if $0<\alpha<\frac{\pi}{4a}$,

$$f(\alpha)+f(7\alpha)+f(9\alpha)+f(15\alpha)+f(17\alpha)+\cdots \quad (19)$$
$$=f(3\alpha)+f(5\alpha)+f(11\alpha)+f(13\alpha)+\cdots$$

Similarly, from 18, if $0<\alpha<\frac{\pi}{6a}$,

$$f(\alpha)+f(11\alpha)+f(13\alpha)+f(23\alpha)+f(25\alpha)+\cdots \quad (20)$$
$$=f(5\alpha)+f(7\alpha)+f(17\alpha)+f(19\alpha)+\cdots$$

Taking $\sin x/x$ as an example of such a function, we obtain from (19), if $|\alpha|\le\frac{\pi}{4}$,

$$\frac{\sin\alpha}{\alpha}+\frac{\sin 7\alpha}{7\alpha}+\frac{\sin 9\alpha}{9\alpha}+\cdots$$
$$=\frac{\sin 3\alpha}{3\alpha}+\frac{\sin 5\alpha}{5\alpha}+\frac{\sin 11\alpha}{11\alpha}+\cdots \quad (21)$$
$$=\frac{1}{8\alpha}\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx=\frac{\pi}{8\alpha}.$$

More generally,

$$\frac{\sin^m \alpha}{\alpha^n} + \frac{\sin^m 7\alpha}{(7\alpha)^n} + \frac{\sin^m 9\alpha}{(9\alpha)^n} + \cdots$$

$$= \frac{\sin^m 3\alpha}{(3\alpha)^n} + \frac{\sin^m 5\alpha}{(5\alpha)^n} + \frac{\sin^m 11\alpha}{(11\alpha)^n} + \cdots$$

$$= \frac{1}{8\alpha}\int_{-\infty}^{\infty} \frac{\sin^m x}{x^n}\,dx, \qquad (22)$$

where ~~θ~~ $|\alpha| \leq \frac{\pi}{4n}$, m and n are positive integers, both odd or both even, and $0 < n \leq m$.

Now the Fourier transform of $f(x+\theta)$ differs from that of $f(x)$ by a multiplying factor $e^{-iv\theta}$, or $\cos v\theta$ in the case of cosine transforms, and hence the g's on the right side of (17) and (18) will continue to be zero even when $f(x)$ is changed to $f(x+\theta)$. Hence, we obtain from (22), even more generally

$$\frac{\sin^m(\alpha+\theta)}{(\alpha+\theta)^n} + \frac{\sin^m(\alpha-\theta)}{(\alpha-\theta)^n} + \frac{\sin^m(7\alpha+\theta)}{(7\alpha+\theta)^n} + \frac{\sin^m(7\alpha-\theta)}{(7\alpha-\theta)^n}$$

$$+ \frac{\sin^m(9\alpha+\theta)}{(9\alpha+\theta)^n} + \frac{\sin^m(9\alpha-\theta)}{(9\alpha-\theta)^n} + \cdots$$

$$= \frac{\sin^m(3\alpha+\theta)}{(3\alpha+\theta)^n} + \frac{\sin^m(3\alpha-\theta)}{(3\alpha-\theta)^n} + \frac{\sin^m(5\alpha+\theta)}{(5\alpha+\theta)^n} + \frac{\sin^m(5\alpha-\theta)}{(5\alpha-\theta)^n} + \cdots \qquad (23)$$

under the same conditions as before.

Similar series can be constructed from (20) and the other formulae analogous to Poisson's, and for all the functions ~~tabulated~~ in Tables I and II.

Indeed, equation (19) ~~and the subsequent ones~~ (21), (22), and (23) ~~which follow therefrom~~, can be seen to be special cases of

$$\sum_{n=-\infty}^{\infty} f(nA+\Theta) = \frac{1}{A}\int_{-\infty}^{\infty} f(x)\,dx, \qquad (24)$$

and therefore independent of Θ, when $0 < A < \frac{2\pi}{a}$, and $g(v) = 0$ when $|v| \geq a$. By putting $A = 8\alpha$, it can be seen that the left sides of (19), (21) and (22) correspond to $\Theta = \alpha$, and the right sides to $\Theta = 3\alpha$; the left side of (23) corresponds to $\Theta = 3\alpha + \theta$, $\Theta = \alpha + \theta$ while the right side corresponds to if we remember that f is an even function, so that $f(-nA+\Theta) = f(nA-\Theta)$.

Equation (24) moreover enables us to evaluate the series in all these equations.

Similarly, (20) corresponds to $A = 12\alpha$ and $\Theta = \alpha$ and 5α respectively.

§5. Till now, we have confined ourselves to the Fourier cosine or exponential transforms. There are formulae analogous to Poisson's, applicable to Fourier sine transforms, also due to Ramanujan*, of which we shall quote here just one.

If $g_s(v) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty} f(x)\sin vx\,dx$, then

$$\alpha\{f(\alpha) - f(3\alpha) + f(5\alpha) - \cdots\} = g_s(\beta) - g_s(3\beta) + g_s(5\beta) - \cdots \qquad (25)$$

where $\alpha\beta = \frac{\pi}{2}$.

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* Collected Papers, p. 64

If $f(x)$ be such that its Fourier sine transform $g_s(v)$ ~~b~~ is zero for $|v|>a$, and if $\beta>a$, i.e. if $0<\alpha<\frac{\pi}{2a}$, all the terms on the right side of (25) vanish, and we obtain

$$\sum_{n=0}^{\infty} f\left[(4n+1)\alpha\right] = \sum_{n=0}^{\infty} f\left[(4n+3)\alpha\right] \qquad (26)$$

As examples of such functions, we may mention*

$$(1)\quad \begin{cases} f(x) = 2^{\nu-\frac{3}{2}}\,\Gamma\!\left(\nu-\frac{1}{2}\right) x^{1-\nu} J_\nu(x) \\ g_s(v) = x(1-x^2)^{\nu-\frac{3}{2}} & \text{if } 0<v<1 \\ \quad = 0 & \text{if } v>1\,; \end{cases} \qquad (27)$$

$$(2)\quad \begin{cases} f(x) = 2^{\nu-\frac{1}{2}}\,\Gamma\!\left(\nu+\frac{1}{2}\right) x^{-\nu} H_\nu(x) \\ g_s(v) = (1-x^2)^{\nu-\frac{1}{2}} & \text{if } 0<v<1 \\ \quad = 0 & \text{if } v>1\,, \end{cases} \qquad (28)$$

where $H_\nu(x)$ is Struve's function of order ν.

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* Titchmarsh : *loc. cit.* p. 179.

On the equivalence of ~~some~~ certain infinite series and the~~ir~~ corresponding integrals

~~A PROBLEM IN QUADRATURE~~

By K. S. Krishnan

[Received 15 March 1948]

Proof sheets to be sent to Dr T. Vijayaraghavan

In ~~the~~ calculati~~on~~ng (the intensity) of light scatter~~ing~~ed from a homogeneous medium one comes across the infinite series

$$\alpha \sum_{n=-\infty}^{+\infty} \frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2},$$

where θ is a constant, and n an integer. α is a positive real number ~~constant~~, which under the conditions under which light-scattering is generally studied can be made ~~infinitesimally~~ arbitrarily small, and hence the sum is usually replaced by the corresponding integral,† ~~which gives~~

$$\alpha \sum_{n=-\infty}^{+\infty} \frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2} = \int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\,dx, \quad \text{---(1)}$$

which can be seen to be equal to π.

It is easy to show, however, (Krishnan and Bhatia, 1947) that (1) holds not only in the limit when $\alpha \to 0$, but for any value of α in the range $0 < \alpha \leq \pi$. The proof is as follows.

Consider the known series*

$$\sum_{n=-\infty}^{+\infty} \frac{\sin[(n+\beta)Z]}{n+\beta} = \pi, \quad \text{---(2)}$$

in which β is a constant, n an integer, and $0 < Z < 2\pi$. The series can be integrated term by term with respect to Z ~~α~~ in any closed interval (γ, δ) where $0 < \gamma < \delta < 2\pi$, since it is uniformly convergent in this interval. We then obtain

$$\sum_{n=-\infty}^{+\infty} \frac{\cos[(n+\beta)\gamma]}{(n+\beta)^2} - \sum_{n=-\infty}^{+\infty} \frac{\cos[(n+\beta)\delta]}{(n+\beta)^2} = \pi(\delta-\gamma)\ldots \quad (3)$$

Full reference

† See Einstein, Ann. der Physik,

* J. Bromwich, An introduction to the theory of infinite series, Macmillan, 1931, p. 371, ex. 5.

keeping δ constant and making $\gamma \to 0$, it is readily seen that (3) reduces to

$$\sum_{n=-\infty}^{+\infty} \frac{1-\cos[(n+\beta)\delta]}{(n+\beta)^2} = \pi\delta,$$

since the first series on the left-hand side of (3) is uniformly convergent, at $\gamma = 0$ also and therefore represents a continuous function of γ. Putting now $\delta = 2\alpha$, $\alpha\beta = \theta$, and dividing both sides by 2α ($\alpha \neq 0$), we obtain for $0 < \alpha < \pi$,

$$\alpha \sum_{n=-\infty}^{+\infty} \frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2} = \pi. \qquad \ldots \qquad (4)$$

This can be seen to be true for $\alpha = \pi$ also, and hence (4) holds over the interval $0 < \alpha \leq \pi$.

The above result ~~un~~implies that the area subtended between the curve $y = \sin^2 x / x^2$ and the x-axis may be obtained just as well by adding up the ordinates at equal intervals α, and multiplying by α (i.e. by simple rectangulation), as by integration, provided that $0 < \alpha \leq \pi$. It may be noted here that the sum in (4) is independent of θ, which shows that we may start the division~~s~~ of the x-axis into equal steps α from any value of x. In particular, $x = 0$ need not be one of the points of ~~divisio~~ division.

As we shall show presently, the same property, namely

$$\alpha \sum_{n=-\infty}^{+\infty} f(n\alpha+\theta) = \alpha \sum_{n=-\infty}^{+\infty} f(n\alpha) = \int_{-\infty}^{+\infty} f(x)\,dx \quad \ldots (5)$$

for a suitable range of values of α, $0 < \alpha \leq l$ say, holds for several other functions too. Before considering such functions, we shall refer here to an alternative proof of (4), which was given us by Professor Norbert Wiener, and is quoted (by us in the paper referred) to (Krishnan and Bhatia, 1947). The proof is very suggestive, and enables us to ~~find~~ determine the conditions ~~under which (5) holds.*~~ for the validity of equations (5).*

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* $f(x)$ = constant is a trivial example ~~which~~ that fies (5).

And g equal to zero for all other values q.v.

Consider ~~a symmetric~~ an even function $f(x) = f(-x)$, and its Fourier transform defined by

$$g(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{ivx}\, dx. \quad \ldots \qquad (6)$$

Let $f(x)$ be such that $g(v)$ ~~differs from zero~~ has non-zero values in the range $-a < v < a$, ~~only~~ ~~and zero value elsewhere~~, where a is a positive real number, ~~(Obviously f(x) =~~ (Obviously $f(x) = \sin^2 x/x^2$ satisfies ~~such a~~ this condition, since ~~its~~ then g~~(v)~~ $=$

$$g(v) = \sqrt{\tfrac{\pi}{2}}\,(2 - |v|) \text{ if } 0 \leq |v| \leq 2, \text{ and } = 0 \text{ if } |v| \geq 2.)$$

~~Then by~~ According to Poisson's summation formula,*

$$\sum_{n=-\infty}^{+\infty} f(n\alpha) = \frac{\sqrt{2\pi}}{\alpha} \sum_{N=-\infty}^{+\infty} g\left(\frac{2\pi N}{\alpha}\right), \quad \ldots (7)$$

where N is an integer. ~~and~~ If now $0 < \alpha \leq 2\pi/a$, there is only one value of N, namely $N = 0$, for which $g(2\pi N/\alpha)$ differs from 0. Hence

$$\sum_{n=-\infty}^{+\infty} f(n\alpha) = \frac{\sqrt{2\pi}}{\alpha}\, g(0), \quad \ldots \qquad (8)$$

from which, ~~in view of~~ (substituting for $g(0)$ its value from) (6), we obtain

$$\alpha \sum_{n=-\infty}^{+\infty} f(n\alpha) = \int_{-\infty}^{+\infty} f(x)\, dx. \quad \ldots \qquad (9)$$

New para

Further, for a fixed θ, it can be seen that

$$\int_{-\infty}^{+\infty} f(x+\theta)\, e^{ivx}\, dx = e^{-iv\theta} \int_{-\infty}^{+\infty} f(x)\, e^{ivx}\, dx, \ldots \qquad \cancel{(9)}\ (10)$$

which shows that for $v = 0$, the Fourier transform of $f(x+\theta)$ is the same as that of $f(x)$. Hence it follows that ~~for~~ if ~~such functions~~ $f(x)$ be such that its ~~whose~~ Fourier transform~~s~~ $g(v)$ ~~differs from zero over the range $-a \leq v < +a$ only;~~ has non-zero values if $|v| < a$ and zero value otherwise, then

$$\alpha \sum_{n=-\infty}^{+\infty} f(n\alpha + \theta) = \alpha \sum_{n=-\infty}^{+\infty} f(n\alpha) = \int_{-\infty}^{+\infty} f(x)\, dx; \ldots \qquad \cancel{(10)}\ (11)$$

~~if 0 <~~ provided that $0 < \alpha \leq 2\pi/a$.

---

* See E.C.Titchmarsh, Introduction to the theory of Fourier ~~I~~ integrals, Oxford, 1937, p.60, where the Poisson summation formula is ~~stated and~~ proved ~~specifically~~ with reference to the Fourier cosine transforms; $g_c(v) = \sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} f(x) \cos vx\, dx$; but for the even functions that we are considering the and the cosine transforms become identical.

We have seen that ~~f(x) =~~ $\sin^2 x/x^2$ is ~~an example~~ ~~of~~ such a function, the range of α over which (10) holds for this function being $0 < \alpha \leq \pi$. More generally the functions $f_{m,n}(x)$ = $\sin^m x / x^n$, where m and n are positive integers, ~~either~~ both ~~of them~~ odd, or both ~~of them~~ even, and $n \leq m$, are examples of such functions; ~~since~~ their Fourier transforms ~~g( )~~ $g_{m,n}(\nu)$ can be seen to have zero value if $|\nu| \geq n$ ($|\nu| > n$ when $n=1$), and non-zero value otherwise, and hence for these functions relations ~~(10)~~ (11) will be valid if $0 < \alpha \leq 2\pi/n$ ($0 < \alpha < 2\pi/n$ when $n = 1$)

The Fourier transforms of the first three powers of $\sin x/x$, that is of $f(x) = (\sin x/x)^m$, m = 1, 2, 3, are plotted in Fig. 1, a b c. (m = 0 corresponds to the trivial case f(x) = constant, noticed already.)

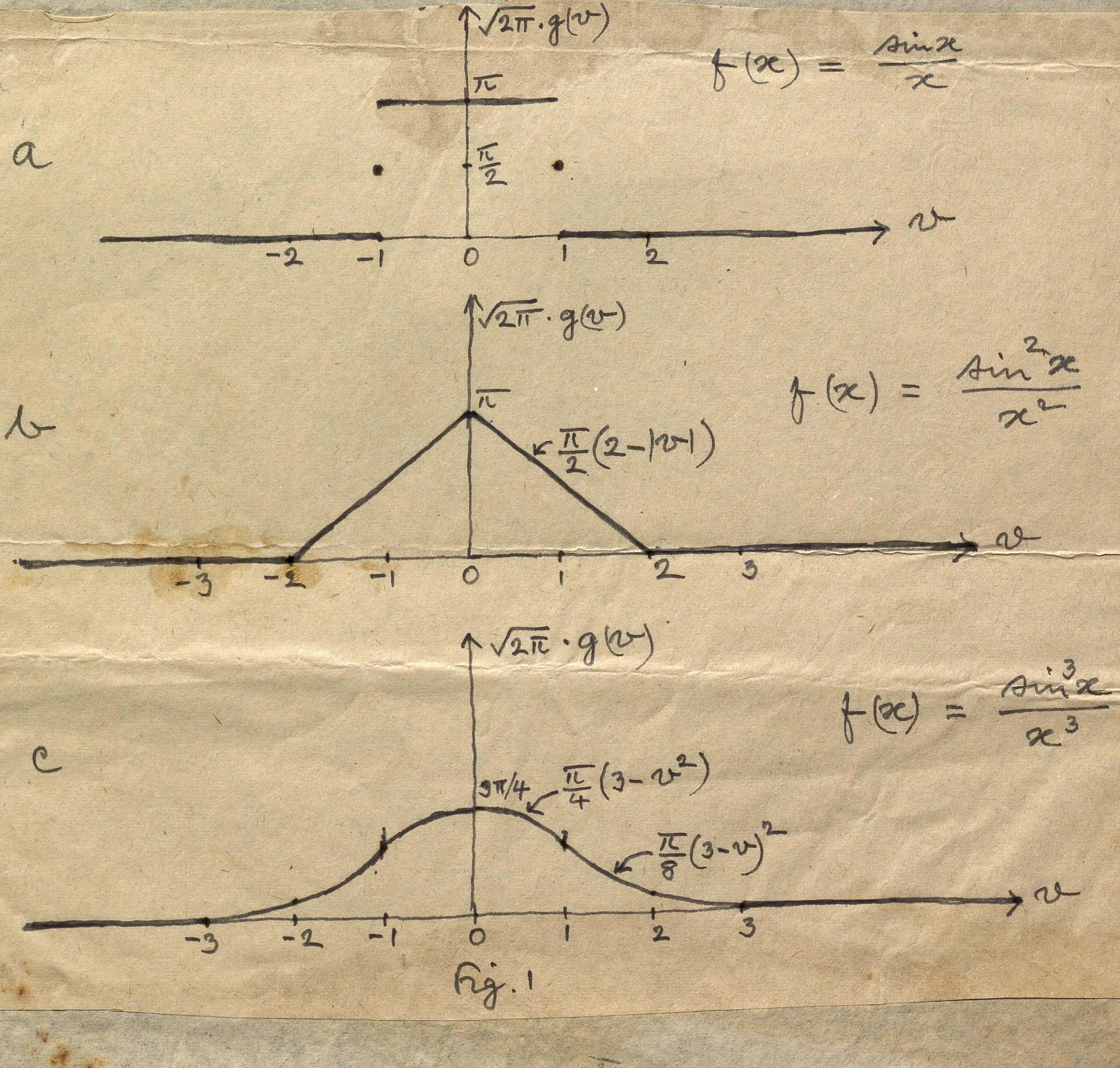

Fig. 1

Any others?

$f = \frac{\sin^3 x}{x}$

$\frac{\sin^4 x}{x^2}$?

Since the Fourier transforms are known for a large number of functions, and many of them have been conveniently tabulated*, it is easy to select other examples of functions that ~~which~~ satisfy the criterion stated above, and for which therefore relations (10) are valid. We give in the following tables a few. Those given in Table I are taken from a paper by Ramanujan†, entitled A class of definite integrals, in which among others are given the values of several Fourier ~~in~~ integrals, some of which are found to satisfy the above criterion. Those given in Table II are taken from Campbell and Foster's Tables of Fourier integrals.

Table I.

$f(x)$	$\sqrt{2\pi}\cdot g(v)$ when $\lvert v\rvert < \pi$ [$g(v)=0$ otherwise]
$\dfrac{1}{\Gamma(\alpha+x)\,\Gamma(\beta-x)}$	$\dfrac{\left(2\cos\frac{1}{2}v\right)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\,e^{\frac{1}{2}iv(\beta-\alpha)}$ convergence condition $R(\alpha+\beta)>1$
$\dfrac{J_{\alpha+x}(\lambda)}{\lambda^{\alpha+x}}\cdot\dfrac{J_{\beta-x}(\mu)}{\mu^{\beta-x}}$	(R. p. 200.) $\left(\dfrac{2\cos\frac{v}{2}}{\sqrt{2}}\right)^{\frac{1}{2}(\alpha+\beta)} e^{\frac{1}{2}v(\beta-\alpha)i}\,J_{\alpha+\beta}\left\{\sqrt{2\Omega\cos\frac{v}{2}}\right\}$ where $\Omega=\lambda^2 e^{\frac{1}{2}vi}+\mu^2 e^{-\frac{1}{2}vi}$ Convergence condition $R(\alpha+\beta)>-1$. (p. 216)

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*(See for example) G.A.Campbell and R.M.Foster, Fourier integrals for practical applications, Bell Telephone Publications, Monograph B-584 (1931)

† Quarterly Journal of Mathematics, 48 294 (1920); Collected papers of Srinivasa Ramanujan, Cambridge University Press, 1927, p. 216.

TABLE II

$f(x)$	$\sqrt{2\pi}\, f.\, g(v)$ when $\lvert v\rvert \leq a$ ($g(v) = 0$ otherwise.) ~~{$\neq 0$ when $\lvert v\rvert > a$~~ ~~$\sqrt{2\pi}\,.\,g(v)$ {$= 0$ when $\lvert v\rvert \leq a$~~ otherwise has the following value
$\dfrac{\sin\left[a(x^2+\lambda^2)^{1/2}\right]}{(x^2+\lambda^2)^{1/2}}$	$\pi\, J_0\left[\lambda(a^2-v^2)^{1/2}\right]$
$\dfrac{\sin\left[a(x^2-\lambda^2)^{1/2}\right]}{(x^2-\lambda^2)^{1/2}}$	$\pi\, I_0\left[\lambda(a^2-v^2)^{1/2}\right]$
$\cos\left[a(x^2+\lambda^2)^{1/2}\right]-\cos ax$	$-\dfrac{\pi a \lambda J_1\left[\lambda(a^2-v^2)^{1/2}\right]}{(a^2-v^2)^{1/2}}$
$\dfrac{\sin\left[a(1-x)\right]}{1-x}+\dfrac{\sin\left[a(1+x)\right]}{1+x}$ ξ	$2\pi\cos v$
$\dfrac{\sin^2\left[\frac{a}{2}(1-x)\right]}{1-x}+\dfrac{\sin^2\left[\frac{a}{2}(1+x)\right]}{1+x}$	$\pi\sin v$
$\dfrac{\sin ax}{x^2}-\dfrac{a\cos ax}{x}$	$i\pi v$

* a, ~~as before~~, is a positive real number not infinite.
λ is a complex quantity, not infinite.
~~is a complex quantity~~, not infinite, ~~the real part being greater than zero~~.
$J_\nu(z)$ is the Bessel function of the first kind.
$I_\nu(z)$ is the Bessel function of the first kind for imaginary argument.

Relations ~~(12)~~ (11) will be valid for all the functions entered in Tables I and II.

Taking for example the first function entered in ~~the~~ Table II, (11) will read as follows:— ~~π, (11) may be translated as follows:—~~ ~~table, we may deduce the following results from the point of view adopted here~~. New para [If ~~where~~ $0 < \alpha \leq 2\pi/a$,† ξ

$$\alpha\sum_{n=-\infty}^{+\infty}\frac{\sin\left[a\left\{(n\alpha+\theta)^2+\lambda^2\right\}^{1/2}\right]}{\left\{(n\alpha+\theta)^2+\lambda^2\right\}^{1/2}} = \alpha\sum_{n=-\infty}^{+\infty}\frac{\sin\left[a(n^2\alpha^2+\lambda^2)^{1/2}\right]}{(n^2\alpha^2+\lambda^2)^{1/2}}$$

$$= \int_{-\infty}^{+\infty}\frac{\sin\left[a(x^2+\lambda^2)^{1/2}\right]}{(x^2+\lambda^2)^{1/2}}\,dx$$

$$= \pi J_0(\lambda a). \quad \ldots(12)^{*} \quad \cancel{(11)}$$

~~Similar results hold for the other functions given in Tables I and II.~~

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† When $\lambda = 0$ and $a = 1$ this example reduces to the case $f(x) = \sin x/x$

~~When $\lambda = 0$ and $a = 1$, the above function obviously reduces to $f(x) = \sin x/x$.~~

Similar results ~~follow~~ hold for all the other functions too, entered in Tables I and II.

The last two functions given in Table II, however, differ from the rest in that g(v), besides being zero ~~is~~ when ~~the range~~ $|v| > a$, is ~~also~~ zero at v = 0 also. Considering the last function one obtains, ~~then~~ if $0 < \alpha < \pi$,

$$\alpha \sum_{n=-\infty}^{+\infty} \left[\frac{\sin(n\alpha+\theta)}{(n\alpha+\theta)^2} - \frac{\cos(n\alpha+\theta)}{n\alpha+\theta} \right] = \alpha \sum_{n=-\infty}^{+\infty} \left[\frac{\sin n\alpha}{n^2\alpha^2} - \frac{\cos n\alpha}{n\alpha} \right]$$

$$= \int_{-\infty}^{+\infty} \left[\frac{\sin x}{x^2} - \frac{\cos x}{x} \right] dx \quad (13)$$

$$= g(0)$$

$$= 0$$

(The zero value of the integral in (13) is also otherwise obvious.)

Hence we obtain if $0 < \alpha < \pi$, and $n\alpha + \theta$ is not equal to 0 for any value of n,

$$\sum_{n=-\infty}^{+\infty} \frac{\sin(n\alpha+\theta)}{(n\alpha+\theta)^2} = \sum_{n=-\infty}^{+\infty} \frac{\cos(n\alpha+\theta)}{n\alpha+\theta} \quad \ldots (14)$$

The corresponding integrals $\int_{-\infty}^{+\infty} \frac{\sin x\,dx}{x^2}$ and $\int_{-\infty}^{+\infty} \frac{\cos x\,dx}{x}$ are, however, infinite.

The ~~previous~~ last but-one function entered in Table II, for which also g(v) = 0 at v = 0, and again for $|v| > a$, yields similarly, ~~for~~ if $0 < \alpha < \pi/a$,

$$\sum_{n=-\infty}^{+\infty} \frac{\sin^2 a(n\alpha+\theta)}{n\alpha+\theta} = \sum_{n=-\infty}^{+\infty} \frac{\sin^2 a[n\alpha+\theta+2]}{n\alpha+\theta+2} \quad (15)$$

though the integral $\int_{-\infty}^{+\infty} \frac{\sin^2 ax}{x} dx$ is equal to zero.

The class of functions that we have been considering here, which ~~are~~ is characterised by the~~ir~~ Fourier transforms being zero when $|v|$ is greater than a certain positive real number a, has other interesting properties. Analogous to ~~the~~ Poisson's summation formula which we have used, and which we may write in the more familiar form

$$\sqrt{\alpha}\left\{\tfrac{1}{2}f(0)+f(\alpha)+f(2\alpha)+\cdots\right\}$$

$$=\sqrt{\beta}\left\{\tfrac{1}{2}g_c(0)+g_c(\beta)+g_c(2\beta)+\cdots\right\}\quad\cdots(16)$$

where $\alpha\beta=2\pi$, there are others of the same type due to Ramanujan,* namely,

$$\sqrt{\alpha}\left\{f(\alpha)-f(3\alpha)-f(5\alpha)+f(7\alpha)+f(9\alpha)-\cdots\right\}$$

$$=\sqrt{\beta}\left\{g_c(\beta)-g_c(3\beta)-g_c(5\beta)+g_c(7\beta)+g_c(9\beta)-\cdots\right\}\quad(17)$$

where $\alpha\beta=\pi/4$;

$$\sqrt{\alpha}\left\{f(\alpha)-f(5\alpha)-f(7\alpha)+f(11\alpha)+f(13\alpha)-\cdots\right\}$$

$$=\sqrt{\beta}\left\{g_c(\beta)-g_c(5\beta)-g_c(7\beta)+g_c(11\beta)+g_c(13\beta)-\cdots\right\}\quad(18)$$

where $\alpha\beta=\pi/6$, and 1, 5, 7, ~~9~~, 11, 13, .. are the numbers prime to 6; ~~other~~ and other formula similar to (17) and (18).

[Unlike in Poisson's formula (16), ~~which~~ in which the first term on the right-hand ~~hand~~ side is $g_c(0)$, in (17), ~~a~~ (18) etc, the first term is $g_c(\beta)$. If now β is greater than a, i.e. if α is chosen small enough to make $\beta > a$, then all the terms on

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* Collected Papers, p. 63.

the right hand side of (17) and (18) become zero, and we get the following interesting results. From (17) for example we obtain, if $0 < \alpha < \pi/(4a)$,

$$f(\alpha) + f(7\alpha) + f(9\alpha) + f(15\alpha) + f(17\alpha)$$
$$= f(3\alpha) + f(5\alpha) + f(11\alpha) + f(13\alpha) + \cdots \quad (19)$$

~~and~~ Similarly from (18), if $0 < \alpha < \pi/(6a)$,

$$f(\alpha) + f(11\alpha) + f(13\alpha) + f(23\alpha) + f(25\alpha) + \cdots$$
$$= f(5\alpha) + f(7\alpha) + f(17\alpha) + f(19\alpha) + \cdots \quad (20)$$

Taking $\sin x/x$ as an example of such a function, $a = 1$ in this case, we obtain from (19), if $|\alpha| \le \pi/4$

$$\frac{\sin\alpha}{\alpha} + \frac{\sin 7\alpha}{7\alpha} + \frac{\sin 9\alpha}{9\alpha} + \cdots$$
$$= \frac{\sin 3\alpha}{3\alpha} + \frac{\sin 5\alpha}{5\alpha} + \frac{\sin 11\alpha}{11\alpha} + \cdots \quad (21)$$
$$= \frac{1}{8\alpha}\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{8\alpha}.$$

or more generally

$$\frac{\sin^m\alpha}{\alpha^n} + \frac{\sin^m 7\alpha}{(7\alpha)^n} + \frac{\sin^m 9\alpha}{(9\alpha)^n} + \cdots$$
$$= \frac{\sin^m 3\alpha}{(3\alpha)^n} + \frac{\sin^m 5\alpha}{(5\alpha)^n} + \frac{\sin^m 11\alpha}{(11\alpha)^n} + \cdots \quad (22)$$

$$= \frac{1}{8\alpha}\int_{-\infty}^{+\infty} \frac{\sin^m x}{x^n}\,dx.$$

where $|\alpha| \le \pi/(4m)$ and m and n are positive integers both odd, or both even, and $0 < n \le m$.

Now ~~since~~ the Fourier ~~cosine~~ transform of $f(x+\theta)$ differs from that of $f(x)$ by a multiplying factor $e^{-iv\theta}$, or $\cos v\theta$ in the case of cosine transforms, and hence the g's on the right-hand side of (17) and (18) will continue to be zero even when $f(x)$ is changed to $f(x+\theta)$. Hence we obtain from (22), even more ~~*~~ generally

$$\frac{\sin^m(\alpha+\theta)}{(\alpha+\theta)^n}+\frac{\sin^m(\alpha-\theta)}{\cancel{\theta}(\alpha-\theta)^n}+\frac{\sin^m(7\alpha+\theta)}{\cancel{\alpha}(7\alpha+\theta)^n}$$
$$+\frac{\sin^m(7\alpha-\theta)}{(7\alpha-\theta)^n}+\frac{\sin^m(9\alpha+\theta)}{(9\alpha+\theta)^n}+\frac{\sin^m(9\alpha-\theta)}{(9\alpha-\theta)^n}$$
$$=\frac{\sin^m(3\alpha+\theta)}{(3\alpha+\theta)^n}+\frac{\sin^m(3\alpha-\theta)}{(3\alpha-\theta)^n}+\frac{\sin^n(5\alpha+\theta)}{(5\alpha+\theta)^n}$$
$$+\frac{\sin^m(5\alpha-\theta)}{(5\alpha-\theta)^n}+\cdots \qquad (23)$$

where α, m and n satisfy the conditions stated under (22).

New piece.

Insert the Appendix A

[Similar series can be constructed from (20) and the other formula analogous to Poisson's, and for all the functions tabulated in I and II.

~~Appendix A~~

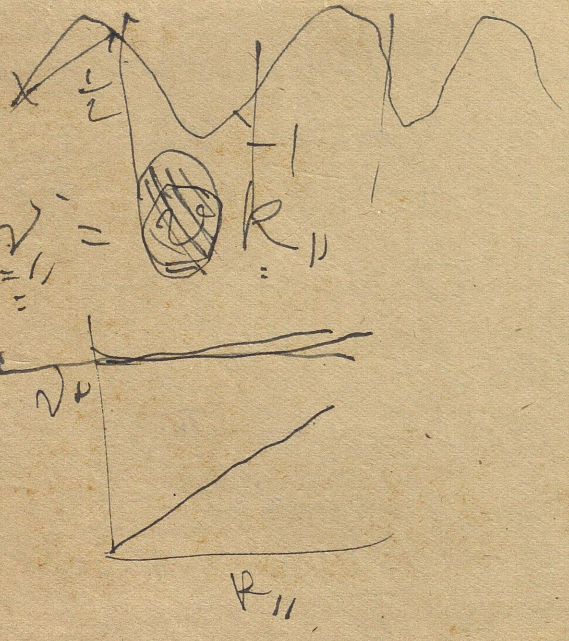
Indeed equation (19) and the subsequent one (21), 22 and (23) ~~that~~ which follow therefrom, can be seen to be special cases of

$$\sum_{n=-\infty}^{+\infty} f(nA + \Theta) = \frac{1}{A}\int_{-\infty}^{+\infty} f(x)\,dx \quad \ldots (24)$$

~~if~~ and ~~hence being~~ therefore independent of Θ ~~if~~ when $0 < A < 2\pi/a$, and $g(v) = 0$ when $|v| > a$. By putting $A = 8\alpha$, it can be seen that the left-hand sides of (19), (21) and (22) correspond to $\Theta = \alpha$ and the right-hand sides to $\Theta = 3\alpha$; the left-hand side of (23) corresponds to $\Theta = \alpha + \theta$, and the right-hand side to $3\alpha + \theta$, if we remember that f is an even function, and that $f(-nA + \Theta) = f(nA - \Theta)$.

Equation (24) ~~also~~ further enables us to evaluate the series in all these ~~same~~ equations.

Similarly equation (20) corresponds to $A = 12\alpha$, and $\Theta =$ ~~4~~ α and 5α respectively.

$\frac{1}{2}$
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$\nu_{\parallel} =$ [illegible] $k_{\parallel}$
$\nu_{\parallel}$
$k_{\parallel}$

Till now we have confined ourselves to the Fourier cosine or exponential transforms. There are formulae analogous to Poisson's, applicable to Fourier sine transforms, also due to Ramanujan*, of which we shall quote here just one.

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* Collected papers, p. 64.

again due to Ramanujan ~~There are analogous formulae for sine transforms.~~*

If $g_s(v) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin vx\, dx$, then

$$\alpha\left\{f(\alpha) - f(3\alpha) + f(5\alpha) - \cdots\right\} = g_s(\beta) - g_s(3\beta) + g_s(5\beta) - \cdots \quad (24)$$

where $\alpha\beta = \pi/2$.

~~If The Hence for those functions f(x)~~ (If $f(x)$ be such that its Fourier) ~~whose~~ sine transform~~s~~ $g_s(v)$ ~~become~~ is zero for $|v| > a$, ~~we have, for~~ and if $\beta > a$, i.e. if $0 < \alpha < \pi/(2a)$, all the terms on the right-hand side of (~~18~~ 24) vanish, and we obtain

$$\sum_{n=0}^{\infty} f\left[(4n+1)\alpha\right] = \sum_{n=0}^{\infty} f\left[(4n+3)\alpha\right] \quad (25)$$

~~* Collected papers of Ramanujan, pp. 63 and 64.~~

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ON THE EQUIVALENCE OF CERTAIN INFINITE SERIES AND THE CORRESPONDING INTEGRALS

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Laboratory of India/

[Received 15 March, 1948.]

1. In calculating the intensity of light scattered ~~from~~ a homogeneous medium, one comes across the infinite series

in/

$$\alpha \sum_{n=-\infty}^{+\infty} \frac{\sin^2 (n\alpha+\theta)}{(n\alpha+\theta)^2},$$

(nα)

where θ is a constant, and n an integer. α is a positive number which under the conditions under which light-scattering is generally studied, can be made arbitrarily small, and hence the sum is usually replaced by the corresponding integral*

$$\alpha \sum_{n=-\infty}^{+\infty} \frac{\sin^2 (n\alpha+\theta)}{(n\alpha+\theta)^2} = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi. \qquad (1)$$

It is easy to show†, however, that (1) holds not only in the limit when α tends to zero, but for any value of α in the range $0 < \alpha \leqslant \pi$. The proof is as follows.

Consider the series‡

$$\sum_{n=-\infty}^{\infty} \frac{\sin (n+\beta)\, z}{n+\beta} = \pi, \qquad (2)$$

where β is a constant, n an integer, and $0 < z < 2\pi$. The series can be integrated term by term with respect to z in any closed interval (γ, δ), where $0 < \gamma < \delta < 2\pi$, since it is uniformly convergent in this interval. We thus obtain

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* Einstein. *Ann. der Physik*, 33(1910) 1294.

† Krishnan and Bhatia, (1947).

‡ Bromwich, *Infinite Series*, (1931), 371, Ex. 5.

Give name of journal + title of paper + page

$$\sum_{n=-\infty}^{\infty} \frac{\cos (n+\beta)\gamma}{(n+\beta)^2} - \sum_{n=-\infty}^{\infty} \frac{\cos (n+\beta)\delta}{(n+\beta)^2} = \pi(\delta-\gamma). \qquad (3)$$

Keeping δ constant and making $\gamma \to 0$, (3) reduces to

$$\sum_{n=-\infty}^{\infty} \frac{1-\cos (n+\beta)\delta}{(n+\beta)^2} = \pi\delta,$$

live

since the first series on the left side of (3) is uniformly convergent at $\gamma = 0$ and therefore represents a continuous function of γ. Putting now $\delta = 2\alpha$, $\alpha\beta = \theta$, and dividing both sides by 2α $(\neq 0)$, we obtain for $0 < \alpha < \pi$,

$$\alpha \sum_{n=-\infty}^{\infty} \frac{\sin^2 (n\alpha+\theta)}{(n\alpha+\theta)^2} = \pi. \qquad (4)$$

This can be seen to be true for $\alpha = \pi$ also, and hence (4) holds over the interval $0 < \alpha \leqslant \pi$.

The above result implies that the area subtended between the curve $y = \sin^2 x/x^2$ and the x-axis may be obtained just as well by adding up the ordinates at equal intervals α, and multiplying by α (i.e. by simple rectangulation), as by integration, provided $0 < \alpha \leqslant \pi$. It may be noted here that the sum in (4) is independent of θ, which shows that we may start the division of the x-axis into equal steps α from any value of x. In particular, $x = 0$ need not be one of the points of division.

short -/

2. As we shall show presently, the same property, viz.

$$\alpha \sum_{n=-\infty}^{\infty} f(n\alpha+\theta) = \alpha \sum_{n=-\infty}^{\infty} f(n\alpha) = \int_{-\infty}^{\infty} f(x)\,dx \qquad (5)$$

for a suitable range of values of α, $0 < \alpha \leqslant l$, say, holds for several other functions too. Before considering such functions, we shall refer here to an alternative proof of (4), given by Prof. Norbert Wiener.* The proof is very

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* Quoted by us in the paper by Krishna and Bhatia, *loc. cit.*

E

suggestive, and enables us to determine the conditions for the validity of the equations (5).*

Consider an even function $f(x) = f(-x)$, and its Fourier transform defined by

$$g(v) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(x) e^{ivx} dx. \qquad (6)$$

Let $f(x)$ be such that $g(v)$ has non-zero values in the range $-a < v < a$, where a is a positive number. Obviously $f(x) = \sin^2 x/x^2$ satisfies this condition, since l.c/

$$g(v) = (\pi/2)^{\frac{1}{2}}(2 - |v|), \text{ if } 0 \leqslant |v| \leqslant 2, \text{ and}$$
$$= 0 \text{ if } |v| \geqslant 2.$$

According to Poisson's summation formula,†

$$\sum_{n=-\infty}^{\infty} f(n\alpha) = \frac{(2\pi)^{\frac{1}{2}}}{\alpha} \sum_{N=-\infty}^{\infty} g\left(\frac{2\pi N}{\alpha}\right), \qquad (7)$$

where N is an integer. If now $0 < \alpha \leqslant 2\pi/a$, there is only one value of N, *viz.* $N = 0$, for which $g(2\pi N/\alpha)$ differs from 0.

Hence

$$\sum_{n=-\infty}^{\infty} f(n\alpha) = \frac{(2\pi)^{\frac{1}{2}}}{\alpha} g(0), \qquad (8)$$

whence substituting for $g(0)$ from 6, we obtain

$$\alpha \sum_{n=-\infty}^{\infty} f(n\alpha) = \int_{-\infty}^{\infty} f(n) dx. \qquad (9)$$

Further, for a fixed θ, it can be seen that

$$\int_{-\infty}^{\infty} f(x+\theta) e^{ivx} dx = e^{-iv\theta} \int_{-\infty}^{\infty} f(x) e^{ivx} dx, \qquad (10)$$

which shows that for $v = 0$, the Fourier transform of $f(x+\theta)$ is the same as that of $f(x)$. Hence it follows that

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* $f(x)$ = constant is trial example that satisfies (5). a/ vi/

† See Titchmarsh, *Introduction to the Theory of Fourier Integrals*, p. 60, where the Poisson Summination formula is proved with reference to the Fourier cosine transform, mm/

$$g_c(v) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int^{\infty} f(x) \cos vx\, dx; \qquad 0]$$

but, for the *even* functions that we are considering, the exponential and the cosine transforms become identical.

if $f(x)$ be such that its Fourier transform $g(v)$ has non-zero values if $|v| < a$, and zero value otherwise, then

$$\alpha \sum_{n=-\infty}^{\infty} f(n\alpha+\theta) = \alpha \sum_{n=-\infty}^{\infty} f(n\alpha) = \int_{-\infty}^{\infty} f(x) dx, \qquad (11)$$

provided that $0 < \alpha \leqslant 2\pi/a$.

We have seen that $\sin^2 x/x^2$ is such a function, the range of α over which (11) holds being $0 < \alpha \leqslant \pi$. *More generally,* the functions $f_{m,n}(x) = \sin^m x/x^n$, when m and n are positive integers, both odd or both even, and $n \leqslant m$, are examples of such functions; their Fourier transforms $g_{m,n}(v)$ can be seen to have zero value if $|v| \geqslant n$ ($|v| > n$ when $n = 1$), and non-zero values otherwise, and hence for these functions, relations (11) will be valid if, re/

$$0 < \alpha \leqslant 2\pi/n \; (0 < \alpha < 2\pi/n \text{ when } n = 1).$$

The Fourier transforms of $(\sin x/x)^m$, $m = 1, 2, 3$ are plotted in Fig. 1, *a*, *b*, *c*.

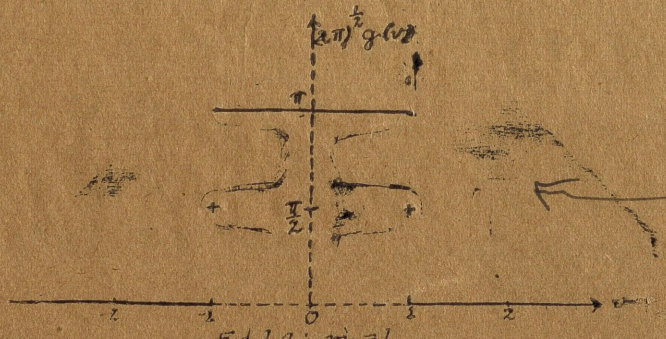

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Fig 1 a; m = 1 g/

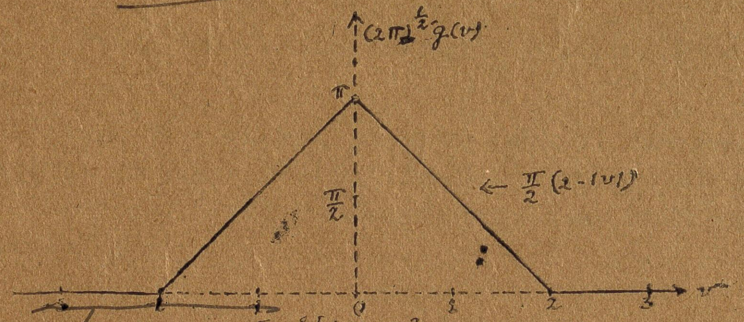

Fig 1 b; m = 2

-3 -2 -1

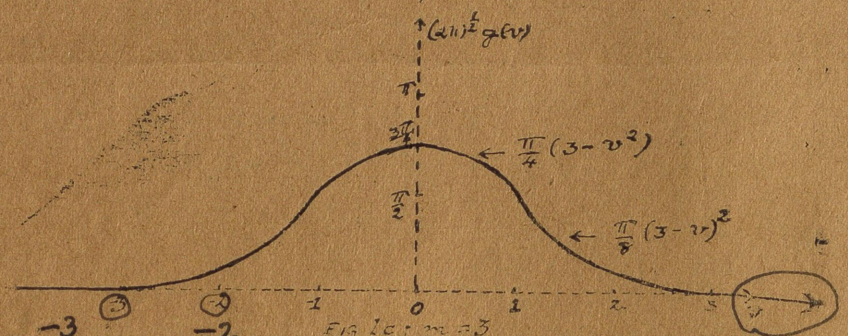

Fig. 1c; m = 3

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3. Since the Fourier transforms are known for a large number of functions, and many of them have been conveniently tabulated,* it is easy to select other examples of functions that satisfy the criterion stated above, and for which therefore relations (10) are valid. We give in the following tables a few. Those given in Table I are taken from a paper by Ramanujan,† *A class of Definite Integrals*, in which among others are given the values of several Fourier integrals, some of which are found to satisfy the above criterion. Those given in Table II are taken from Campbell and Foster's Tables referred to.

ought to be (11)

TABLE I.

$f(x)$	$2\pi)^{\frac12} g(v)$, when $\|v\| < \pi$ $[g(v)=0$ otherwise.$]$
$\frac{1}{\Gamma(\alpha+x)\,\Gamma(\beta-x)}$	$\frac{(2\cos v/2)^{\alpha+\beta-2}}{\Gamma(\alpha+\beta-1)}\, e^{\frac12 iv(\beta-\alpha)}$, convergence condition $R(\alpha+\beta)>1$
$\frac{J_{\alpha+x}(\lambda)\, J_{\beta-x}(\mu)}{\lambda^{\alpha+x}\,\mu^{\beta-x}}$	$\left(\frac{2\cos v/2}{\Omega}\right)^{\frac12(\alpha+\beta)} \boxed{e^{\frac12 v(\beta-\alpha)i}} \xrightarrow{\quad}$ raise $\boxed{e^{\frac12 v(\beta-\alpha)i}}$ $\cancel{\times}\ \cancel{J_{\alpha+\beta}}\,[\sqrt{(2\,\Omega\cos v/2)}]$, × $J_{\alpha+\beta}$ where $\Omega=\lambda^2 \cancel{e^{\frac12 vi}} + \mu^2 e^{-\frac12 vi}$ $e^{\frac12 vi}$ converge condition $R(\alpha+\beta) > -1$.

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[*] See for ex. G. A. Campbell and R. M. Foster, *Fourier Integrals for Practical Applications*, Bell Telephone Publications, Monograph B—(1931) 584.

[†] *Quarterly Jour. Math.* 48 (1920), 294; *Collected Papers*, Cambridge (1 27), 216.

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TABLE II.

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$f(x)$	$(2\pi)^{\frac12} g(v)$, when $\|v\| \leqslant a$ $[g(\cancel{v})=0$ otherwise.$]$
l.c $\frac{\cancel{S}\text{in}\ a(x^2+\mu^2)^{\frac12}}{(x^2+\lambda^2)^{\frac12}}$	$\pi J_0[\lambda(a^2-v^2)^{\frac12}]$
l.c $\frac{\cancel{S}\text{in}\ a(x^2-\lambda^2)^{\frac12}}{(x^2-\lambda^2)^{\frac12}}$	$\pi I_0 \cancel{/}\lambda(a^2-v^2)^{\frac12}]$ [
l.c $\cancel{C}\text{os}\ a(x^2+\lambda^2)^{\frac12}-\cos ax$	$-\frac{\pi a\lambda J_1[\lambda(a^2-v^2)^{\frac12}]}{(a^2-v^2)^{\frac12}}$
l.c/ l.c/ $\frac{\cancel{S}\text{in}\ a(1-x)}{1-x}+\frac{\cancel{S}\text{in}\ a(1+x)}{1+x}$	$2\pi\cos v$
l.c/ l.c $\frac{\cancel{S}\text{in}^2\ a(1-x)/2}{1-x}+\frac{\cancel{S}\text{in}^2\ a(1+x)/2}{1+x}$	$\pi \cancel{S}\text{in}\ v$ l.c/
l.c/ $\frac{\cancel{S}\text{in}\ ax}{x^2}-\frac{a\cos ax}{x}$	$i\pi v$.

In the above, a is a positive real finite number, and λ is a complex number, not infinite. Relations (11) will be valid for all the functions in Tables I and II. Taking for example the first function entered in Table II, (11) will read as follows:

If $0<\alpha\leqslant 2\pi/a$,

l.c/

$$\alpha\sum_{n=-\infty}^{\infty}\frac{\cancel{S}\text{in}\ a\{(n\alpha+\theta)^2+\lambda^2\}^{\frac12}}{\{(n\alpha+\theta)^2+\lambda^2\}^{\frac12}}=\alpha\sum_{n=-\infty}^{\infty}\frac{\cancel{S}\text{in}\ a(n^2\alpha^2+\lambda^2)^{\frac12}}{(\cancel{\lambda}^2\alpha^2+\lambda^2)^{\frac12}}$$
$$=\int_{-\infty}^{\infty}\frac{\sin a(x^2+\lambda^2)^{\frac12}}{(x^2+\lambda^2)^{\frac12}}dx=\pi J_0(\lambda a).\quad(12)$$

l.c n/

When $\lambda=0$ and $a=1$, this reduces to the case $f(x)=\sin x/x$.

The last two function~~s~~ given in Table II however differ from the rest in that $g(v)$ besides being zero when $|v|>a$ is zero at $v=0$ also. Considering the last function, we obtain, if $0<\alpha<\pi$,

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$$\alpha\sum_{n=-\infty}^{\infty}\left[\frac{\sin(n\alpha+\theta)}{(n\alpha+\theta)^2}-\frac{\cos(n\alpha+\theta)}{n\alpha+\theta}\right]$$

$$=\alpha\sum_{n=-\infty}^{\infty}\left[\frac{\sin n\alpha}{n^2\alpha^2}-\frac{\cos n\alpha}{n\alpha}\right]$$

$$=\int_{-\infty}^{\infty}\left(\frac{\sin x}{x^2}-\frac{\cos x}{x}\right)dx \qquad (13)$$

$$=g(0)=0.$$

That the value of the integral in (13) is zero is otherwise obvious. We thus obtain when $0<\alpha<\pi$ and $n\alpha+\theta$ is not equal to θ for any value of n,

$$\sum_{n=-\infty}^{\infty}\frac{\sin(n\alpha+\theta)}{(n\alpha+\theta)^2}=\sum_{n=-\infty}^{\infty}\frac{\cos(n\alpha+\theta)}{n\alpha+\theta}. \qquad (14)$$

The corresponding integrals $\int_{-\infty}^{\infty}\frac{\sin x}{x^2}dx$ and $\int_{-\infty}^{\infty}\frac{\cos x}{x}dx$ are however infinite. X

The last but one function entered in Table II, for which also $g(v)=0$ at $v=0$ as also when $|v|>a$, yields similarly, if $0<\alpha<\pi/a$, v| X

$$\sum_{n=-\infty}^{\infty}\frac{\sin^2 a(n\alpha+\theta)}{n\alpha+\theta}=\sum_{n=-\infty}^{\infty}\frac{\sin^2 a(n\alpha+\theta+2)}{n\alpha+\theta+2}, \qquad (15)$$

though

$$\int_{-\infty}^{\infty}\frac{\sin^2 ax}{x}dx=0.$$

4. The class of functions that we have been considering here, which is characterized by the Fourier transforms being zero when $|v|$ is greater then a certain positive number a, has other interesting properties. Analogous to Poisson's summation formula which we have used, and which we may write in the more familiar form a

$$\surd\alpha\left\{\tfrac{1}{2}f(0)+f(\alpha)+f(2\alpha)+\ldots\right\}$$
$$=\surd\beta\left\{\tfrac{1}{2}g_c(0)+g_c(\beta)+g_c(2\beta)+\ldots\right\}, \qquad (16)$$

where $\alpha\beta=2\pi$, there are others of the same type due to Ramanujan*. Two of the typical formula are given below:

$$\surd\alpha\{f(\alpha)-f(3\alpha)-f(5\alpha)+f(7\alpha)+f(9\alpha)-\ldots\}$$
$$=\surd\beta\{g_c(\beta)-g_c(3\beta)-g_c(5\beta)+g_c\,7(\beta)+g_c(9\beta)-\ldots\}, \qquad (17)$$

where $\alpha\beta=\pi/4$;

$$\surd\alpha\{f(\alpha)\,f(5\alpha)-f(7\alpha)+f(11\alpha)+f(13\alpha)-\ldots\} \quad -|$$
$$=\surd\beta\{g_c(\beta)-g_c(5\beta)-g_c(7\beta)+g_c(11\beta)+g_c(13\beta)-\ldots\}, \qquad (18)$$

where $\alpha\beta=\pi/6$, and 1, 5, 7, 11, 13, ..., are the numbers prime to 6.

Unlike in Poisson's formula (16), in which the first term on the right side is $\frac{1}{2}g_c(0)$, the first term in (17), (18) and similar formulae is $g_c(\beta)$. If now $\beta>a$, i.e. if α is chosen small enough to make $\beta>a$, then all the terms on the right side of (17) and (18) vanish, and we get the following interesting results. From (17), for example, we obtain, if $0<\alpha<\pi/4a$,

$$f(\alpha)+f(7\alpha)+f(9\alpha)+f(15\alpha)+f(17\alpha)+\ldots$$
$$=f(3\alpha)+f(5\alpha)+f(11\alpha)+f(13\alpha)+\ldots \qquad (19)$$

Similarly, from (18) if $0<\alpha<\pi/6a$, ( )h

$$f(\alpha)+f(11\alpha)+f(13\alpha)+f(23\alpha)+f(25\alpha)+\ldots$$
$$=f(5\alpha)+f(7\alpha)+f(17\alpha)+f(19\alpha)+\ldots \qquad (20)$$

Taking $\sin x/x$ as an example of such a function, we obtain from (19), if $|\alpha|\leqslant\pi/4$,

$$\frac{\sin\alpha}{\alpha}+\frac{\sin 7\alpha}{7\alpha}+\frac{\sin 9\alpha}{9\alpha}+\ldots$$
$$=\frac{\sin 3\alpha}{3\alpha}+\frac{\sin 5\alpha}{5\alpha}+\frac{\sin 11\alpha}{11\alpha}\ldots+$$
$$=\frac{1}{8\alpha}\int_{-\infty}^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{8\alpha}. \qquad (21)$$

More generally,

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*Collected papers, p, 63.

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$$\frac{\sin^m\alpha}{\alpha^n}+\frac{\sin^m 7\alpha}{(\cancel{4}\alpha)^n}+\frac{\sin^m 9\alpha}{(9\alpha)^n}+\dots$$
$$=\frac{\sin^m 3\alpha}{(\cancel{7}\alpha)^n}+\frac{\sin^m 5\alpha}{(5\alpha)^n}+\frac{\sin^m 11\alpha}{(11\alpha)^n}+\dots$$
$$=\frac{1}{8\alpha}\int_{-\infty}^{\infty}\frac{\sin^m x}{x^n}\,dx, \qquad (22)$$

where $|\alpha|\leqslant\frac{\pi}{4n}$, m and n are positive integers, both odd or both even, and $0<n\leqslant m$.

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Now the Fourier transform of $f(x+\theta)$ differs from that of $f(x)$ by a multiplying factor $e^{-iv\theta}$, or $\cos v\theta$ in the case of cosine transforms, and hence the g's on the right side of (17) and (18) will ~~continue to be~~ zero even when $f(x)$ is changed to $f(x+\theta)$. Hence, we obtain from (22), even more generally,

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$$\frac{\sin^m(\alpha+\theta)}{(\alpha+\theta)^n}\boxed{\cancel{+\frac{\sin^m(\alpha-\theta)}{(\alpha-\theta)^n}+\frac{\sin^m(7\alpha+\theta)}{(7\alpha+\theta)^n}}}+\frac{\sin^m(7\alpha-\theta)}{(7\alpha-\theta)^n}$$
$$+\frac{\sin^m(9\alpha+\theta)}{(9\alpha+\theta)^n}\boxed{\cancel{+\frac{\sin^m(9\alpha-\theta)}{(9\alpha-\theta)^n}}}+\dots$$
$$=\frac{\sin^m(3\alpha+\theta)}{(3\alpha+\theta)^n}+\frac{\sin^m(3\alpha-\theta)}{(3\alpha-\theta)^n}+\frac{\sin^m(5\alpha+\theta)}{(5\alpha+\theta)^n}+\frac{\sin^m(5\alpha-\theta)}{(5\alpha-\theta)}+\dots \qquad (23)$$

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under the same condirions as before.

Similar series can be constructed from (20) and the other formulae analogous to Poisson's and for all the functions in Tables I and II.

Indeed, equations (19), (21), (22) and (23) can be seen to be special cases of

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$$\sum_{n=-\infty}^{\infty} f(nA+\phi)=\frac{1}{A}\int_{-\infty}^{\infty} f(x)\,dx, \qquad (24)$$

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and therefore independent of ϕ when $0<A<2\pi/a$, and $g(v)=0$ when $|v|\ngtr a$. By putting $A=8\alpha$, it can be seen that the left sides of (19), (21) and (22) correspond to $\phi=\alpha$, and the right sides to $\phi=3\alpha$; the left side of (23)

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corresponds to $\phi=\alpha+\theta$ while the right side corresponds to $\phi=3\alpha+\theta$, if we remember that f is an even function, so that $f(-nA+\phi)=f(nA-\phi)$.

Equation (24) moreover enables us to evaluate the series in all these equations.

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Similarly, (20) corresponds to $A=12\alpha$ and $\phi=\alpha$ and 5α respectively.

5. Till now, we have confined ourselves to the Fourier cosine or exponential transforms. There are formulae analogous to Poisson's, applicable to Fourier sine transforms, also due to Ramanujan*, of which we shall quote here just one.

If $g_s(v)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^{\infty} f(x)\sin vx\,dx$, then

$$\alpha\{f(\alpha)-f(3\alpha)+f(5\alpha)-\dots\}=g_s(\beta)-g_s(3\beta)+g_s(5\beta)-\dots \qquad (25)$$

where $\alpha\beta=\pi/2$.

If $f(x)$ be such that its Fourier sine transform $g_s(v)$ is zero for $|v|>a$, and if $\beta>a$, i.e. if $0<\alpha<\pi/2a$, all the terms on the right side of (25) vanish, and we obtain

$$\sum_{n=0}^{\infty} f\left[(4n+1)\alpha\right]=\sum_{n=0}^{\infty} f\left[(4n+3)\alpha\right]. \qquad (26)$$

As examples of such functions, we may mention†

$$(1)\begin{cases} f(x)=2^{\nu-\frac{3}{2}}\,\Gamma(\nu-\tfrac{1}{2})\,x^{1-\nu}J_\nu(x) \\ g_s(v)=x(1-x^2)^{\nu-\frac{3}{2}} & \text{if } 0<v<1 \\ \quad=0 & \text{if } v>1\ ; \end{cases} \qquad (27)$$

$$(2)\begin{cases} f(x)=2^{\nu-\frac{1}{2}}\,\Gamma(\nu+\tfrac{1}{2})\,x^{-\nu}H_\nu(x) \\ g_s(v)=(1-x^2)^{\nu-\frac{1}{2}} & \text{if } 0<v<1 \\ \quad=0 & \text{if } v>1\ , \end{cases} \qquad (28)$$

where $H_\nu(x)$ is Struve's function of order ν.

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* *Collected Papers*, p. 64.

† Titchmarsh : *loc. cit.* p. 179.