What is required is the sum of the values of $\sin^2 \xi / \xi^2$, e... at equal intervals, α, and it can be shown that

$$\sum_{n=-\infty}^{+\infty} \frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2} = \frac{\pi}{\alpha}, \quad \text{-----}(11)$$

where n is an integer, $0 < \alpha \leq \pi$, and θ is a constant.

The proof is as follows :-

Consider the known series*

$$\sum_{n=-\infty}^{+\infty} \frac{\sin\{(n+\beta)y\}}{n+\beta} = \pi, \quad \text{---}(12)$$

in which β is a constant, n an integer, and $0 < y < 2\pi$. The series can be integrated term by term with respect to y in any closed interval (γ, δ), where $0 < \gamma < \delta < 2\pi$, since it is uniformly convergent in this ~~integral~~ interval. We then obtain

$$\sum_{n=-\infty}^{+\infty} \frac{\cos\{(n+\beta)\gamma\}}{(n+\beta)^2} - \sum_{n=-\infty}^{+\infty} \frac{\cos\{(n+\beta)\delta\}}{(n+\beta)^2} = \pi(\delta-\gamma). \quad \text{--}(13)$$

Keeping δ constant and making $\gamma \rightarrow 0$, it ~~can~~ is readily seen that (13) reduces to

$$\sum_{n=-\infty}^{+\infty} \frac{1-\cos\{(n+\beta)\delta\}}{(n+\beta)^2} = \pi\delta,$$

since the first series on the left hand side of (13) is uni~~ty~~formly convergent, and therefore represents a continuous function of γ. Putting now $\delta = 2\alpha$, and $\alpha\beta = \theta$, and dividing both sides by $2\alpha^2$ $(\alpha \neq 0)$, we obtain, for $0 < \alpha < \pi$

$$\sum_{n=-\infty}^{+\infty} \frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2} = \frac{\pi}{\alpha}. \quad \text{---}(11)$$

This can be ~~readily~~ seen to be true for $\alpha = \pi$ also, and hence (11) holds over the interval $0 < \alpha \leq \pi$.

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*See for example J. Bromwich, An introduction to the theory of infinite series, Macmillan 1931, p. 371, ex. 5.

We are thankful to Professor Norbert Wiener for the following elegant alternative proof of (11).

Let $$g(v) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(u)\, e^{iuv}\, du$$
be the Fourier transform of the function

$$f(u) = \frac{\sin^2(u+\theta)}{(u+\theta)^2} = \frac{\sin^2 w}{w^2}, \text{ say,}$$

then
$$g(v) = \frac{e^{-iv\theta}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\sin^2 w}{w^2}\, e^{iwv}\, dw$$
$$= \frac{e^{-iv\theta}}{\sqrt{2\pi}}\, \pi \left[1 - \frac{|v|}{2}\right], \neq 0 \text{ when } -2 < v < 2.$$

Now according to Poisson's formula,
$$\sum_{n=-\infty}^{+\infty} f(n\alpha) = \frac{\sqrt{2\pi}}{\alpha}\sum_{n=-\infty}^{+\infty} g\left(\frac{2\pi n}{\alpha}\right)$$
where n is an integer, and $\alpha > 0$. If further, as in our problem, α is not greater than π, there is only one value of n for which $g\left(\frac{2\pi n}{\alpha}\right)$ differs ~~form~~ from 0, namely $n = 0$; and $g(0) = \sqrt{\pi/2}$, and is independent of θ. Hence
$$\sum_{n=-\infty}^{+\infty} f(n\alpha) = \frac{\sqrt{2\pi}}{\alpha}\cdot\sqrt{\frac{\pi}{2}} = \frac{\pi}{\alpha}.$$

be). The proof is as follows.‡

Consider the known series*

$$\sum_{n=-\infty}^{+\infty} \frac{\sin\{(n+\beta)y\}}{n+\beta} = \pi, \qquad \text{----}(12)$$

in which β is a constant, n ~~and~~ integer, and $0 < y < 2\pi$. The series can be integrated term by term with respect to y in any closed interval (γ, δ), where $0 < \gamma < \delta < 2\pi$, since it is uniformly convergent in this interval. We then obtain

$$\sum_{n=-\infty}^{+\infty} \frac{\cos\{(n+\beta)\gamma\}}{(n+\beta)^2} - \sum_{n=-\infty}^{+\infty} \frac{\cos\{(n+\beta)\delta\}}{(n+\beta)^2} = \pi(\delta-\gamma). \qquad \text{---}(13)$$

Keeping δ constant and making $\gamma \to 0$, it is readily seen that (13) reduces to

$$\sum_{-\infty}^{+\infty} \frac{1-\cos\{(n+\beta)\delta\}}{(n+\beta)^2} = \pi\delta,$$

since the first series on the left hand side of (13) is uniformly convergent, and therefore represents a continuous function of γ. Putting now $\delta = 2\alpha$, and $\alpha\beta = \theta$, and dividing both sides by $2\alpha^2$ $(\alpha \neq 0)$, we obtain, for $0 < \alpha < \pi$,

$$\sum_{n=-\infty}^{+\infty} \frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2} = \frac{\pi}{\alpha}. \qquad \text{---}(11)$$

This can be seen to be true for $\alpha = \pi$ also, and hence (11) holds over the interval $0 < \alpha \leq \pi$.†

‡ Proof rewritten October 28, 1947.

* See for example J. Bromwich, "An Introduction to the Theory of Infinite Series", Macmillan, 1931, p.371, ex. 5.

† We are thankful to Professor Norbert Wiener for the following elegant alternative proof of (11). Let

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$$= \frac{e^{-iv\theta}}{\sqrt{2\pi}} \pi\left[1 - \frac{|v|}{2}\right], \neq 0 \text{ when } -2 < v < +2.$$

Now according to Poisson's formula,

be).. The proof is as follows :-

Consider the known series

in which is a constant, an integer, and . The series can be integrated term by term with respect to in any closed interval where , since it is uniformly convergent in this integral. We then obtain

Keeping constant and making it ~~xxx~~ is readily seen that (13) reduces to

since the first series on the left hand side of (13) is unity convergent, and therefore represents a continuous function of . Putting now = 2 , and = , and dividing both sides by 2 , we obtain, for

This can be readily seen to be true for = also, and hence (11) holds over the interval .