$A - a + D = 17$ $\quad A - a + \cancel{D} = B + \cancel{D}$

$B + D = 17$

(*) Such a long proof is not at all necessary, ~~the sqn~~ using the antipodal property of Rosser & Walker for a nasik square

$$A - a + D = 17$$

If also associated $B + D = 17$

ie $A - a + \cancel{D} = B + \cancel{D}$

ie $A - a = B$

which makes two elements in the magic square equal. Hence impossible

To get ~~cont~~ antipodal property from Bergholz's general form (1) see end. cover pl (180')

Continued from Book No. 11

Bergholtz's general form for any magic square of order 4 is

$A-a$	$C+a+c$	$B+b-c$	$D-b$
$D+a-d$	B	C	$A-a+d$
$C-b+d$	A	D	$B+b-d$
$B+b$	$D-a-c$	$A-b+c$	$C+a$

——— (1)

This is pandiagonal if $a = b = d - c = \frac{1}{2}(A - B - C + D)$, and is symmetrical if $a + c = d = b - c$, & $A + C = B + D$. Hence the square is never both pandiagonal & symmetric since otherwise we should have $A - a = B$. (*)

Bergholtz's prime magic square is obtained by taking

$A = 13,\ B = 11,\ C = 37,\ D = 41,\ a = 10,\ b = 18,\ c = 24,\ d = -2.$

Prime magic squares of orders 5, 6, ... 12 have been constructed by H. A. Sayles and J. N. Muncey [The Monist, 1913, Vol XXIII, 623-30]. Muncey's square of order 12 is remarkable in that the 144 primes involved are the first 144 odd primes 1, 3, 5, 7, 11, ... 827 (It has been proved that the first n^2 odd primes cannot be used when $n < 12$).

———

Can we find a scheme similar to the (1) above for constructing the general magic square of order three?

With the usual standard 3×3

8	1	6
3	5	7
4	9	2

is formed out of sets (1, 2, 3), (4, 5, 6), (7, 8, 9)

with $B = 5$, $A = 6$, $C = 4$ } in (2).

$b = -1$, $a = -2$

7, 9, 10, 11, 12, 13, 14, 15, 17

(7, 9, 11), (10, 12, 14), (13, 15, 17)

This appears possible if we take the scheme

$A-a$	$C+a+b$	$B-b$
$C+a-b$	B	$A-a+b$
$B+b$	$A-a-b$	$C+a$

— (2)

where all the rows & columns and the downward diagonal all sum up to $A+B+C$, but the upward diagonal sums up to $3B$. Hence if we choose $A+C=2B$, we get the required square. For eg $A=11$, $C=13$, $B=12$ $a=1$, $b=3$ give

10	17	9
11	12	13
15	7	14

with the magic number = 36.

In fact looking at p. 171 of Note book II, we find that this has actually being given by Kraitchik ie form (2) in a different ~~form~~. but equivalent form. The nine numbers here are (7, 9, 11), (10, 12, 14), (13, 15, 17) with C.D's 2 (internal) & 3 from set to set.

Chapter XIII – Calculating prodigies.

No need to make any detailed notes.

((40)) Math e Magic by Royal Vale Heath.

(magic : puzzles . games with numbers) – This appears to be quite an interesting book, specially in view of Heath being an expert on the subject of mathematical recreations.

(A) Mathemagic

(1) Central knows your age – Trivial

(2) The robber & the sheep – Too long winded

(3) Family Histrick:

Ask a brother or sister to:

(i) to write down the number of brothers he or she has living.

(ii) Multiply by 2

(iii) Add 3.

(iv) Multiply this number by 5.

(v) Add to this the number of living sisters

(vi) Multiply the total by 10.

(vii) Add the number of dead brothers & sisters.

(viii) Now look squarely at your victim, and demand the ~~answer~~ answer.

(ix) When he or she tells you the answer subtract 150 from it, and

(4)

then the first digit of the resulting number gives the number of living brothers, the middle one the number of living sisters, and the last digit the number of the dead brothers & sisters

[Also trivial but interesting. In fact instead of adding 3, one could add 2 & finally subtract 100 which would be easier for the final act by the stupid magician. See rough sheets p. (46). ~~And~~ "Multiply by 2 and add 2" would be more challenging! P.S. Instead of brothers & sisters living use the trick with two nos. which the victim does not like ($1 < n < 10$)]

(4) The lie detector – ~~Too~~ Rather longish.

(5) Arithmetical whoopee:

This trick illustrates an old East Indian theory – that the things which apparently appear most baffling are really the easiest to learn and do.

On a scrap of paper write down any number between 1 and 50. Fold the paper. Hand it to a friend and tell him to put it in his pocket without looking at it. Now give him ~~a piece of~~ some paper, and ask him to write down any number between 50 and 100, without letting you see it. Then tell him to add to the number

he wrote down, a number which you will give him. When he has done this, tell him to cross out the first left hand figure in the total, add it to the remaining number, and finally to subtract the ~~number~~ result from the number he originally wrote down

Now tell him to look at the folded paper you gave him, and he will see that the figure on it tallies with his result. Let's go behind the scenes: Takes 23 as number in folded sheet. Friend writes 86. Magician gives 99 − 23 = 76

86 + 76 = 162 → 62 + 1 = 63 ; 86 − 63 = 23.

[In choosing 23 states "any no. < 99; should be "any no < 50]

Health further gives ~~for~~ example of folded sheet number as between 100 and 200 & the friend's number between 200 & 1000 & works out a particular case. Further states the same holds for numbers between 300,000 and 1,000,000.

[All this is unnecessary heroics!

In fact for this to work with 2 digit numbers;

let x = folded sheet number & y = number written by friend.

Then the whole arithmetical whoopee boils down to

~~99 y+99-x~~ $y - \{(y + 99 - x) - 99\} = x$.

and in order that the whoopee should work it is

(b) necessary that $y + (99 - x) > 100$

ie $y > x + 1$. —— (1)

So all this bunkum about 1 & 50 and 50 and 100 though sufficient is not necessary. In fact for ~~2 digit no.~~ eg we can have ~~for example x between~~ $x < 10$, $y > 10$

then $99 - x \geqslant 90$ and $99 - x + y > 100$ so the whoopee works.

Further we may take $x < 5$, $y > 5$ for one digit numbers if we use 9 instead of 99 ie $9 - x + y > 10$ and the trick works. Even here, $(99 - x) + y > 100$.

~~Even with~~

and $(999 - x) + y > 1000$ & for higher cases.

The essential condition is (1) and the least bounds of x amenable to (1) is $x < 2$ & then we could have $y > 2$, and even here all the above tricks hold for eg:

$x = 1$, $y = 3$ $\quad y + (9 - x) = \cancel{1}1 \quad 3$

$\quad 1 \quad -2$

$\quad 2 \quad 1 = x.$

$y + (99 - x) = \cancel{1}01 \quad 3$

$\quad 1 \quad -2$

$\quad 2 \quad 1 = x$

998

3

1001

$y + (999 - x) = \cancel{1}001 \quad 3$

$\quad 1 \quad -2$

$\quad 2 \quad 1 = x.$

From the aesthetic point of view however, it looks more dignified to take $1 < x < 50$, $50 < y < 100$ for 2 digit cases

$100 < x < 200$, $200 < y < 1000$ for 3 digit cases

and so on].

(6) An eventric – trivial

(7) I'm tellin' you – trivial

(8) The dime & the penny – trivial

(9) Childs play.

Someday when conversation lags, and you find yourself with nothing to talk about, produce a pencil and paper and hand them to a friend and ask him to write down any number between 3 and 10. Now tell him to proceed as follows:–

1. Add 5 to this number, and write the result to the right of the original number.
2. Multiply the first number by itself.
3. Multiply the second number by itself.
4. Add these two totals together and multiply the result by 2

Now ask your friend for his answer and tell him the two numbers he thought of originally.

(8) In order to find out the two numbers he was thinking of, here's what you do:

1. Subtract 25 from his result.
2. Take the square root of the resulting number
3. Subtract 5.
4. Divide by 2 – and you'll have his first number; add 5 – and you will have his second ~~paper~~ number.

Then an illustration with the original ~~principle~~ number as taken as 6 – says " of course any difference will do [writing the added number to the right of the original number under 1 is rather confusing. The trick is based merely on $(a+b)^2 = a^2 + 2ab + b^2$. If the nos. be x and $x+5$, we have

$$2\{x^2 + (x+5)^2\} = 4x^2 + 20x + 50$$
$$= (2x+5)^2 + 25$$

So subtracting 25 → $(2x+5)^2$ & $\sqrt{}$ → $2x+5$

& subtracting 5 → $2x$ & ÷ 2 → x.

In general $2\{x^2 + (x+y)^2\} = 4x^2 + 4xy + 2y^2$

$$= (2x+y)^2 + y^2.$$

So take off y^2, take sq. root, subtract y & divide by 2.

(10) More Childs play

1. Take any number of 2 digits
2. Add 6 to it to get a second number
3. Multiply each of the two numbers by themselves, and subtract the lesser from the greater.
4. Tell the result, and I'll tell you both the original numbers

[Merely depends on $a^2 - b^2 = (a+b)(a-b)$.

$(x+6)^2 - x^2 = (2x+6)6 = 12x + 36$

Divide by 12 $\longrightarrow$ $x + 3$ & hence x and $x+6$ are known.

In fact, this works for any x & y: $(x+y)^2 - x^2 = y(2x+y)$

So the magician's rule would be:

1. Divide the result your friend gives by y.
2. Subtract y from the quotient.
3. Divide the result by 2.
4. This gives x & hence $x+4$.

In fact, this general rule holds for values of x and y, whether x be a one digit, 2 digit or n-digit number. But the 2 digit number is taken here to lessen the labour of the friend in squaring & a small value of y is chosen to lessen the labour of the stupid magician!]

(11) A - Numgram - Rather long winded - But the illustration of blowing out the tyres 1, 2, 3, 4, 5, 6, etc. is very attractive.

(B) Number Symphonies.

(10) In this section you will find some gorgeous number patterns - each pattern having its own particular motif and theme just as though it were a symphony in number. We ask you to consider in particular the two mystic numbers. They're quite amazing.

A close study of these numerical arrangements will begin to reveal to you some of the inexhaustible logical oddities which result from the simple fact that ours is a decimal system of counting.

(1) A symphony in fractions:

$11 \times 11 = 121$

$111 \times 11 = 1221$

$11 \times 11 = 121$

$111 \times 111 = 12321$

$1111 \times 1111 = 1234321$

$11111 \times 11111 = 123454321$

$111111 \times 111111 = 12345654321$

$1111111 \times 1111111 = 1234567654321$

$11111111 \times 11111111 = -1234567876543321$

$111111111 \times 111111111 = -123456789876543321$

[Put this in proper form]

$$121 = \frac{22 \times 22}{1+2+1}$$

$$12321 = \frac{333 \times 333}{1+2+3+2+1}$$

$$1234321 = \frac{4444 \times 4444}{1+2+3+4+3+2+1}$$

$$123454321 = \frac{55555 \times 55555}{1+2+3+4+5+4+3+2+1}$$

$$12345654321 = \frac{666666 \times 666666}{1+2+3+4+5+6+5+4+3+2+1}$$

$$1234567654321 = \frac{7777777 \times 7777777}{1+2+3+4+5+6+7+6+5+4+3+2+1}$$

$$123456787654321 = \frac{88888888 \times 88888888}{1+2+3+4+5+6+7+8+7+6+5+4+3+2+1}$$

$$12345678987654321 = \frac{999999999 \times 999999999}{1+2+3+4+5+6+7+8+9+8+7+6+5+4+3+2+1}$$

[Note that $1+2+\cdots+(n-1)+n+(n-1)+\cdots+1$
$= \frac{1}{2}n(n+1) + \frac{1}{2}(n-1)n = \frac{1}{2}n \cdot 2n = n^2$ [For diagramatic proof see p. 144 Note Book. II]

and using this, this and the original expn $11-1 \times 11--1$, we get the symphony shown here]

(2) The mystic number 76923

multiplications by 1, 10, 9, 12, 3, 4 give the same sequence of digits read up and down & across. Also every row & column adds up to 27. Note all the diagonals in the ↙ directions are the same numbers.

$76923 \times 1 = 076923$ ↓
$76923 \times 10 = 769230$ ↓
→ $76923 \times 9 = 692307$
$76923 \times 12 = 923076$
$76923 \times 3 = 230769$
$76923 \times 4 = 307692$ ↑

(12)

Again

$76923 \times 2 = 153846$

$76923 \times 7 = 538461$

$76923 \times 5 = 384615$

$76923 \times 11 = 846153$

$76923 \times 6 = 461538$

$76923 \times 8 = 615384$

getting a different scheme with 153846 and same properties as in previous case

(3) The magic number 142857. — This number = 999999×7

$142857 \times 1 = 142857$

$142857 \times 3 = 428571$

$142857 \times 2 = 285714$

$142857 \times 6 = 857142$

$142857 \times 4 = 571428$

$142857 \times 5 = 714285$

Properties just like 76923

(4) 1089 and all that.

A		B
$1089 \times 1 =$	1089	9801
$1089 \times 2 =$	2178	8712
$1089 \times 3 =$	3267	7623
$1089 \times 4 =$	4356	6534
" $\times 5 =$	5445	5445
" $\times 6 =$	6534	4356
" $\times 7 =$	7623	3267
" $\times 8 =$	8712	2178
" $\times 9 =$	9801	1089

Reversed

(B is A in reverse order)

(5) You'd never think it q 19 and 9109 — Omit.

(6) Some pretty Arrangements.

(i)

$$1 \times 8 + 1 = 9$$
$$12 \times 8 + 2 = 98$$
$$123 \times 8 + 3 = 987$$
$$1234 \times 8 + 4 = 9876$$
$$12345 \times 8 + 5 = 98765$$
$$123456 \times 8 + 6 = 987654$$
$$1234567 \times 8 + 7 = 9876543$$
$$12345678 \times 8 + 8 = 98765432$$
$$123456789 \times 8 + 9 = 987654321$$

(ii)

$$0 \times 9 + 1 = 1$$
$$1 \times 9 + 2 = 11$$
$$12 \times 9 + 3 = 111$$
$$123 \times 9 + 4 = 1111$$
$$1234 \times 9 + 5 = 11111$$
$$12345 \times 9 + 6 = 111111$$
$$123456 \times 9 + 7 = 1111111$$
$$1234567 \times 9 + 8 = 11111111$$
$$12345678 \times 9 + 9 = 111111111$$
$$123456789 \times 9 + 10 = 1111111111$$

(iii)

$$9 \times 9 + 7 = 88$$
$$9 \times 98 + 6 = 888$$
$$9 \times 987 + 5 = 8888$$
. . . . . .

$$9 \times 987654321 + 0 = 888888888$$
$$9 \times 9876543210 - 1 = 8888888888$$

(7) ~~All~~ THE SAME NUMBER

(i) $12345679 \times 9 = 111\ 111\ 111$

$12345679 \times 18 = 222\ 222\ 222$

$12345679 \times 27 = 333\ 333\ 333$

. . . . . .

$12345679 \times 81 = 999\ 999\ 999.$

(ii)

$1+2+1 = 2^2$

$1+2+3+2+1 = 3^2$

$1+2+3+4+3+2+1 = 4^2$

$1+2+3+4+5+4+3+2+1 = 5^2$

. . . . . .

$1+2+3+\ldots\ 9+8+\ldots\ +1 = 9^2.$

[Cf. Symphony in fractions, p. 10 – Also see. Note Book. No. 11, p. 144]

(C) Easy ways to multiply – omit this Section – too dull.

(D) Arithmetic klish.

(1) The lost digit – Did you ever lose a digit? It's a terrible feeling – especially when it is lost from a number of four figures you just don't know ~~what to~~ where to look for it. But if your friend loses ~~a digit~~ one and won't tell you what digit it is, it's easy easy for you to locate it. Here's how you do it.

Tell your friend to write down a 4 digit number. Don't let

him show you what he wrote.

Ask him next to find the sum of these digits on one & keep a record of it.

Now tell him to cross out one of the 4 original digits and write the 3 remaining ones above the sum of the digits he has recorded. Then ask him to subtract one from the other and tell you the result.

When you hear the answer, you add up the digits mentally and subtract the sum from the nearest multiple of 9 – this gives you the digit he crossed out.

A bird's-eye view with no. chosen = 7128, Sum of digits = 18

Crosses off 1: 7/28 & writes 728
18
710

He tells you 710 & the sum of its digits = 8. Nearest multiple of 9 is 9 nine itself and missing digit is 9 – 8 = 1.

If the final result happens to be add up to 9 or 18, number crossed out is a 9 or a zero. Then you can say it is "either a 9 or a 0" which he will still be willing to admit that you have his number.

[Vide Rouse Ball p. 47 – trivial but interesting]

(2) The Coo-Coo calculator – Appears good as a howler but rather a long-winded one. (Multiplying 39 by 42)

(3) Digitellus: Your patient writes down any number of four digits, and a moment later, you reveal what number he is thinking of.

(15) Steps: 1. Write down any number of 4 digits — 1895

2. Write down the first two digits — 18

3. Add to these first two digits — 18
The number next to it in sequence — 19
37.

4. Multiply this by 5 — 37 × 5 = 185

5. Place a 0 at end to make it a 4 digit no — 1850.

6. Now give any number between 10 and 99 & add it to the 4 digit no. — 1850 + 27 = 1877.

7. Add to this total the last two digits of the original number you wrote down — 1877 + 95 = 1972

When you hear this, subtract 50 plus the number he ~~chose~~ gave under instruction 6 viz 27 ie 1972 − (50+27) = 1895.

When you get to be an expert at this trick, you'll eventually be able to work with money. If you happen to have a friend who goes around with more than $10 in his pocket, you'll be able to tell him exactly how much he has with him

[For more than $10 would be dollars in 2 figures + cents in 2 figures ie 4 digit number]

[Too much of magic, see rough sheets p. 47 - Should find suitable variations of Heath's many tricks]

(4) Dice - Cyphering - Instead of dice, this trick will work

beautifully for brothers & sisters, uncles and aunts, or sons and daughters

1. Write down the number of brothers you have.
2. Double it and add 1.
3. Multiply by 5
4. Add the number of sisters
5. Tell the result.

You deduct 5 from this number & there are your brothers and sisters.

[Very trivial, See p(47')]

(5) Three in sequence:

Write down any three consecutive numbers (no number to be > 50), and add them together.

Now add any multiple of 3 you wish - but tell me the number you are adding

Multiply the result by 67.

Tell me the last two digits in your answer, and I'll tell you the original three numbers you chose, as well as your complete answer.

In working the trick in a parlor, it is best to add a touch of showmanship re. multiplication by 67 (rather tough). Instead of asking him to multiply by 67, pause and then say "multiply

(18) by 6 or 7" and then pause & say "Better yet, make it 67." If this is done neatly, the effect will be one of complete indifference to the importance of 67.

When you ask somebody to give a number under 40 divisible by 3. & he gives it and as soon as you hear the number, you mentally divide it by 3 and add one to the result. – this is your "key" number & you must remember it.

[Two examples worked with 8, 9, 10 and 16, 17, 18.]

Consider the latter 16, 17, 18

1. Adding these we get 51.
2. Calling for a multiple of 3, suppose 21 is suggested.
   Adding 21 to 51 we get 72
   (In the meantime divide 21 by 3, add 1 and get the key number 8)
3. Multiply 72 by 67 :

$$\begin{array}{r} 72 \\ 67 \\ \hline 504 \\ 432 \\ \hline 4824 \end{array}$$

When you hear 24 (the last two digits), deduct your key number 8, and you have 16, 17, 18 the three consecutive numbers. Double this 24 & you have 48 or the first

two numbers of the final answer.

[Trivial. Merely uses the fact that a two digit number multiplied by $3 \times 67 = 201$ gives a four digit number, with the original 2 digit no preceded by its double. The restrictions on x and y (see p. 47') appear meaningless so far as getting a 2-digit number).

(6) Birthday tricks – Rather longish

(7) How to make millions – Do –

(8) Multipractice – Some trick based on the interesting features of the number 12,345,679 (8 omitted) viz

$$\left\{\begin{array}{lll} 12,345,679 \times 3 & = & 037,037,037 \\ \quad " \quad \times 30 & = & 370,370,370 \\ \quad " \quad \times 57 & = & 703,703,703 \end{array}\right\} \text{Totals } 90$$

$$\left\{\begin{array}{lll} \quad " \quad \times 6 & = & 074,074,074 \\ \quad " \quad \times 33 & = & 407,407,407 \\ \quad " \quad \times 60 & = & 740,740,740 \end{array}\right\} \text{Totals } 33$$

Sing by 12, 39, 66
15, 42, 69
* 21, 48, 75
24, 51, 78

(9) Topsy Turvy – Rather long.

(10) Think of a card – No.

(E) Sleight of mind

(1) An amazing memory trick. – Too long

(2) Tele-phoney – " "

(3) At a glance – use of $1+2+\cdots+n = \frac{1}{2}n(n+1)$

$1+3+5+\cdots(2n-1) = n^2$.

(4) Psychic Bridge – Too long.

(5) Juggling like mine – Trivial

(6) A detective – Too involved

(7) Digitslips – For a change we have one that has no connection with arithmetic or geometry or mathemagic or the like. It merely goes to prove how silly our reactions at times are.

1. Corner your quarry.
2. Now tell him that you are going to give him ten numbers very quickly.
3. After each number he must say the next higher

number.

4. Tell him that you are going to go very fast, and offer to bet him anything he likes he won't get every number correct.

Here's the list:

66
12
125
7
149
315
212
1196
78
4099

Nine out of ten people will say 5000 after the last number.

[Childish, but funny if true!]

(8) Figure this out — Problem of ratios of ages of child & mother at different periods — trivial

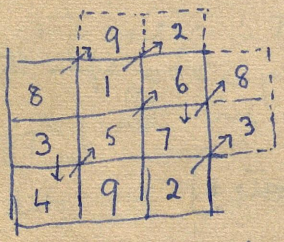
(F) Magic squares — [The figure alone this heading very attractive]

(1) Magic squares were known to the ancients, and were thought to possess mystic and magical powers because of their unusual nature. Astronomers of old tried vainly to find relations between

magic squares and the planets, and in India even at the present time magic squares are worn engraven on stone or metal as talismans. For centuries these wonderful arrangements have fascinated all who have the slightest interest in mathematics.

There are hundreds of different methods of forming magic squares but only the easiest and most intelligible ones have been given here

Three square – Describes Della Loubère's method for odd squares more clearly than Kraitchik or Rouse-Ball. —



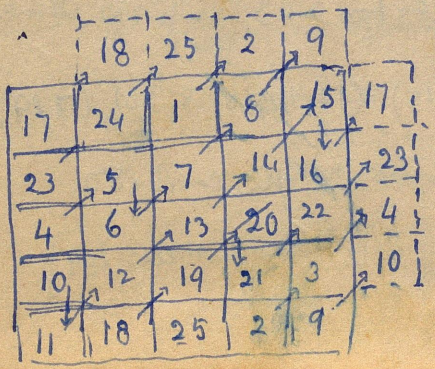
Always start by placing number 1 in the middle cell of the top row. Now go right oblique upward (11th to main diagonal) and you will note that the next number 2 comes <u>above</u> and outside the square in the dotted cell. In this case drop the 2 to the <u>lowest</u> cell in that column. Now go right oblique upwards, and you will find the next number 3 comes outside the square to the right in the dotted cell. In this case move 3 to the extreme <u>left</u> cell in that row. Now you will notice that the number 1 interferes with the next right oblique upward move; therefore drop down one cell and place the 4 directly below the 3. Now you have clear sailing for 5 and 6. It is impossible to go right oblique from this

corner (since there no cell corresponding to this move), hence drop down in the square as before, placing the 7 directly below 6, and continue as before, placing the ~~eight~~ 8 and 9 in the proper locations as shown in the figure.

Practice this 9-cell magic square by yourself – without the dotted lines as guides. When you have mastered this, start with any number instead of the number 1. Remember whatever number you start with must always be placed in the middle cell of the top row.

[Mention here the three sequences of 3 numbers in A.P. with same C.D as the first nos in the sequences also in A.P. with any C.D.]

[In the above case also Loubère's rule holds provided you take in the numbers as in the sequence order & not in increasing order of numbers – Vide Rouse p. 118]



Gives Loubère's Construction for 5×5 squares & mentions same is true for other odd number magic squares.

(24) To find magic constant for $n \times n$ square starting from 1 to n^2 gives the paraphrase of the rule $\frac{1}{2}n(n^2+1)$ viz:

1. Cube the number of cells in a row,
2. Add the number of cells in a row to the result.
3. Divide the result by 2.

[Loubère's Const$^{\underline{n}}$ holds for any n^2 consecutive numbers ($\underline{n}$ odd) – Vide for 5×5 case, rough sheet p. 48']

[Does Loubère's rule work 5×5 squares with 25 numbers in 5 sequences in A.P with ~~same~~ same C.D and the first nos. of the 5 sequences also in A.P. with some other C.D. This is not mentioned in Kraitchik.

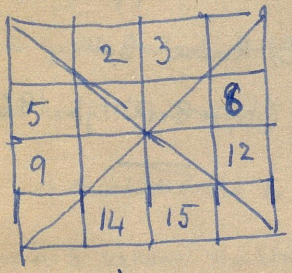
Obviously the rule works for the set of 25 nos. in 5 sequences provided the numbers are taken as in the sequence & not in ascending order (as for 3×3, see p. 23). For eg.

(2, 4, 6, 8, 10), (7, 9, 11, 13, 15), (12, 14, 16, 18, 20), (17, 19, 21, 23, 25), & (22, 24, 26, 28, 30) with magic const = 80

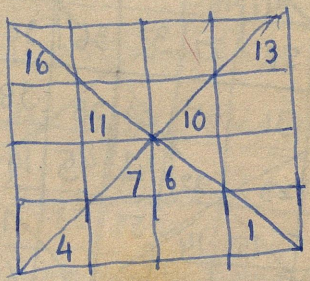
ie total of 400/5. [See rough sheet, 49].

The even-cell squares are formed entirely differently from the odd cell magic squares. Take a 4-cell & examine the way to construct
The first thing is to draw the 2 diagonals. In the upper left hand corner start with number 1, but do not put it in because of the diagonal. Now running directly across along the first row put place 2 and 3 in the next 2 cells, but omit 4 from the 4th cell since it also on the diagonal. Continue this right along in the 2nd, 3rd & 4th rows, putting in numbers where there is no diagonal & omitting numbers where there is a diagon

giving Fig (i)



(i)



(ii)

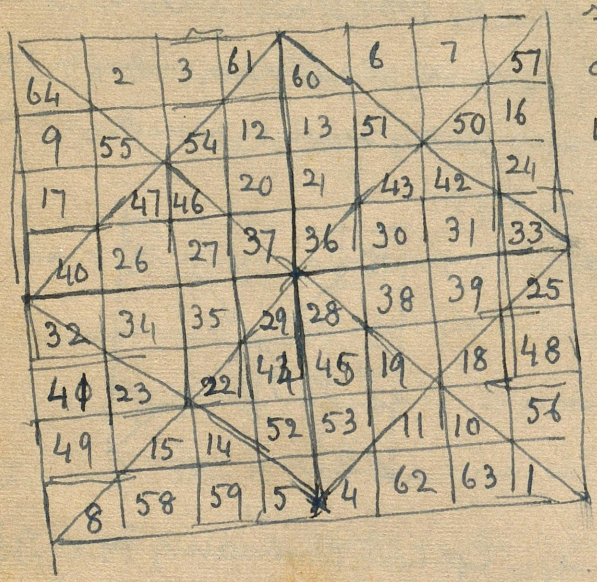
Now start with 16 in the upper left hand corner cell of Fig (ii) & put numbers in the diagonal crossed cells counting backwards, getting nos. as in Fig (ii). Now complete the square with by putting numbers in (i) & (ii) in the proper places getting (iii)

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

with magic number 34

This is equivalent to the method given in Rouse-Ball (Vide Note Book 11, p. 185 & it is difficult to say which is simpler

For the 8-cell square, the same method applies with the additional diagonals of the four 4-cell squares inserted, and starting from 1 to 64 and backwards from 64 to 1



Here Heath's method is certainly preferable to Rouse-Ball's (Note Book 11, p. 186)

For the 6 × 6 square gives the same method as in Rouse Ball (Bk. 11, p. 184) without however showing the squares A, B, C, D as there are

A	C
D	B

Heath has not given the method of construction of the 2(2m+1) square with m = 2 → 10 × 10 square which is given in Ball (p. 190)

For A → D, there are two bookkeeping lines and from C → B there are changes in additional columns [Rough sheet, p. 29].

(2) Fun with magic squares.

Ask someone to give you any number of two digits. Suppose he gives you 24. You can tell him that you will start with his number and build up a magic square of 4 cells on a side with the magic constant 216. (You arrive at 126 by mentally multiplying the number he gave you by 4 and adding 30$; if he had given you 16 instead of 24, the magic constant would be 94 instead of 216). The next thing to do is to construct the square & show it to your friend as the one starting with the number he gave you and with the magic constant predicted by you. (viz. 216).

A simple rule to remember in this connection is as follows:-

For a 3 square	multiply the number given you by	3	and add 12
" 4	"	4	" 30
" 5	"	5	" 60
" 7	"	7	" 168

It is not necessary to go any higher than this unless your friend is a perfect fool for magic squares.

[Magic constant for a $n \times n$ square starting with a is given by

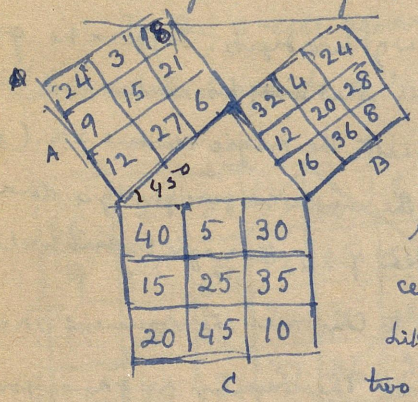
$$\frac{1}{n}\left\{a + (a+1) + \cdots + (a+n^2-1)\right\}$$

$$= \frac{1}{n}\left\{n^2 a + \frac{1}{2}n^2(n^2-1)\right\} = na + \frac{1}{2}n(n^2-1)$$

which is Heath's rule]

(3) Transposing magic squares — Too involved and not interesting.

(4) The magic triangle: —



A, B are squares on the two equal sides of the rt $\angle^d$ Δ^{le} – A, B, C are each magic squares. The square of any cell of C is equal to the sum of the squares of the corresponding cells in A & B (for eg: $40^2 = 24^2 + 32^2$) likewise the square of the sum of any two or more cells or any diagonal or horizontal row or any $\perp^r$ column in C is equal to the square of the sum of the corresponding two or more cells, rows, columns or diagonals of A plus B.

Also note that the square of the total of all the cells in C equals the square of the total of all the cells in A plus the square of the total of all the cells in B.

[$(40+5)^2 = (24+3)^2 + (32+4)^2$ ie $45^2 = 27^2 + 36^2$ ($5^2 = 4^2 + 3^2$)

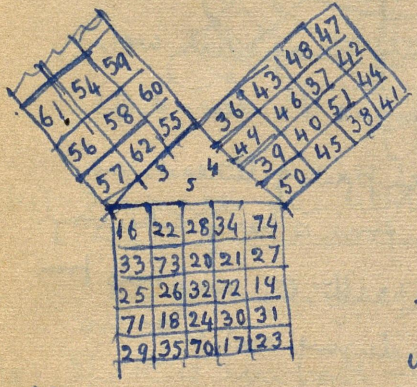
$(40+30)^2 = (24+18)^2 + (32+24)^2$ ie $70^2 = 42^2 + 56^2$ ($5^2 = 3^2 + 4^2$)

$(40+5+30)^2 = (24+3+18)^2 + (32+4+24)^2$ ie $75^2 = 45^2 + 60^2$ (")

When two or more cells are mentioned, there appears to be no need to mention rows, columns & diagonals separately]

(5) The magic square on the hypotenuse of a triangle

Old Pythagoras forgot that some day his theorem would get into the hands of a magician. If he had mentioned that the magic square on the hypotenuse is equal to the sum of the magic squares of the other two sides, he would have saved a lot of argument, and this book would be two pages shorter



(Bad figure)

Looking at these magic squares you will notice that the ~~so~~ magic constants of all the three is 174, and no two numbers are duplicated.

The sum of all the numbers in the 3-square squared, plus the sum of all the numbers in the 4-square squared, equals the sum of all the numbers in the 5-square squared

$$\left[(3\times174)^2+(4\times174)^2=(5\times174)^2.\right]$$

(6) The magic lovers – Too far fetched & rather vulgar.

(7) The magic cauldron – Does not very impressive.

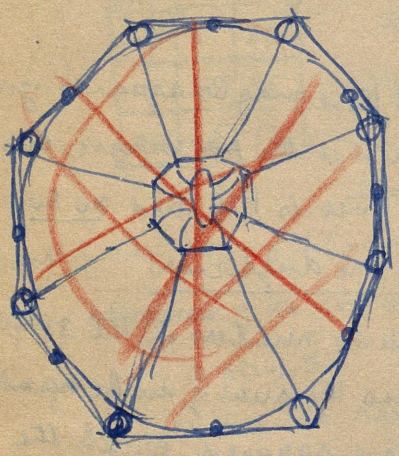
(8) The magic explosion – Picture taken in a number factory which blew up – Not very impressive.

(30) (9) The magic square of squares — Rather complicated

(10) The Complementary magic square of squares — Do —

(11) The magic honeycomb —— Do ——

(12) The magic web (Figure to be taken out from Health when required)



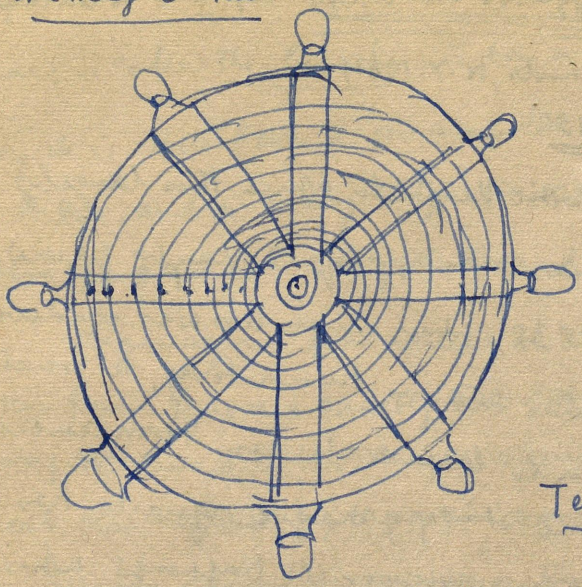
(Head figure)

What is that is the secret woven into the spiders web? He has pushed the dew-drops along each pair of threads so that they form a very interesting pattern, with the numbers from 1 to 64 inclusive and with no duplication.

This intelligent spider has arranged these dew-drops so that they will add up to 65, 130, 195 and 260 in ways known only to him.

Dust the cob-webs off your brain by trying to find how many combinations can be formed from each one of the four above totals (appears rather inscrutable)

(13) The magic wheel



8 numbers along each spoke handle

– Rather complicated

– sum of nos. along spoke handles & along concentric circles

Tentatively chosen

(14) The magic quintuplets – Interesting but unfortunately too very complicated.

(15) The three acrobats — Do —

(16) The magic block – Magic cubes – omit

(G) Calendar Trick — omit

(H) Short-cut interest method – omit

(I) Raffles – omit

Finis (funny figure (quite a beautiful book)

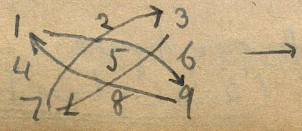
((41)) Mathematical puzzles & diversions by Martin Gardner.

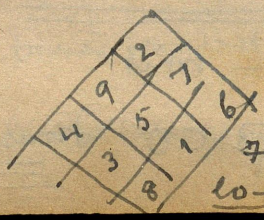
(Simon & Schuster, N.Y. 1961)

Magic squares, p. 130.

~~Studying~~ Fantastic lengths to which this largely frivolous topic has been analysed maybe gained from the fact that in 1838, when much less was understood about magic squares than is known today, a French work on the subject ran to three volumes. From ancient times until now the study of magic squares has flourished as a kind of cult, often with occult trappings, whose initiates range from such eminent mathematicians as Arthur Cayley & Oswald Veblen to laymen such as Benjamin Franklin

The traditional magic square is a set of integers in serial order, beginning with 1, arranged in square formation so that the total ~~sum~~ of each row, column and main diagonal are the same. — Gives this "easy" way for the 3 x 3 square as





74. 53. How the lo-shu can be formed.

[magic const = ½ sum of n^3 and n]. In China, where this square is called the Lo-shu, it has a long history as a charm. Today it is still found on amulets worn in the Far East and India, and on many large passenger ships it is the pattern for games of shuffle-board

Magic squares grow quickly in complexity when we turn to order four. There are exactly 880 different types, again ignoring rotations & mirror images, many of which are much more magical than required by the definition of a magic square — Eg Dürer's square & remarks on the same —

Fourth-order magic squares were linked to Jupiter by Renaissance astrologers and were believed to combat melancholy (which was Saturnian in origin) — Dürer's square is called symmetric because each number added to the number symmetrically opposite the square's centre yields 17. The four corner cells, the four central cells, the two-by-two squares at each corner all total 34. A square of this type can be constructed by an absurdly simple method. Merely write in square array and in serial order the numbers 1 to 16, then invert the diagonals. The result is a symmetrical magic square (Dürer then interchanges 2nd & 3rd cols to give the

the year in which he made the square)

The earliest recorded fourth-order square, found in an 11th or 12th-Century inscription at Khajuraho, India is

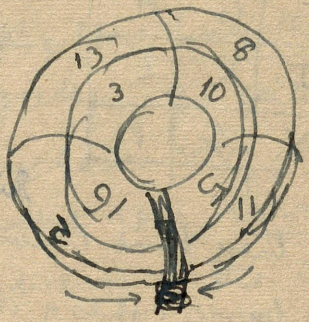
7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

(Khajuraho acc'g to Gardner)

It belongs to a species known as diabolic squares (also called "pandiagonal" and "Nasik") which are even more astonishing than the symmetric ones. In addition to the usual properties, diabolic squares are also magic along all "broken diagonals" [rows, cells (2, 12; 15, 5), and (2, 3, 15; 4)] are broken diagonals that can be restored by putting two duplicate squares alongside each other (?). A diabolic square remains diabolic if a row is shifted from top to bottom or from bottom to top, and if a column is moved from one side to the other. If we form a mosaic by fitting together a large number of duplicate diabolic squares, we have a field on which any four-by-four group of cells ~~on the field up and~~ will be diabolic. Any four adjacent ~~squares~~ cells on the field, up & down, left & right or diagonally, will yield the constant

Perhaps the most dramatic way of exhibiting the diabolic properties of such a square is described by mathematicians Rosser & Walker in 1938, (perhaps Bull. Amer. Soc. paper). We simply bring together the top and bottom of the square to make a cylinder, then stretch and bend the cylinder into a torus

2	13	8	11
16	3	10	5



Diabolic cylindrical doughnut

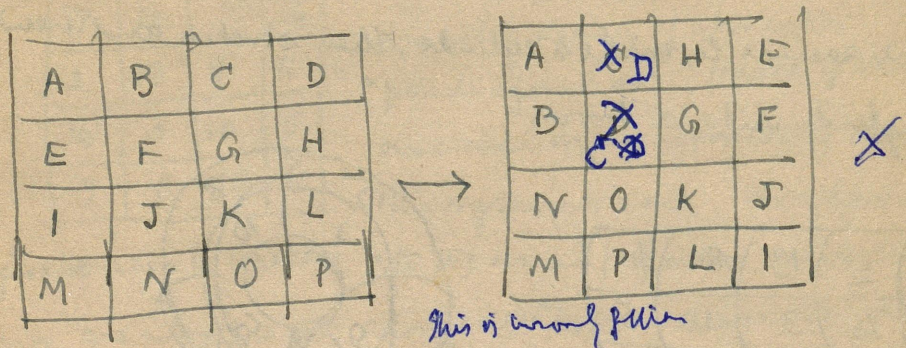
All rows & cols. & diagonals now become closed loops. If we start at any cell & move two squares away in any direction along a diagonal, we always arrive at the same cell. This cell is called the "antipode" of the cell where we ~~started~~ began. Every pair of antipodes on this diabolic doughnut will total 17. Every loop of four cells, diagonally orthogonally adds up to 34, as does any square group of four cells.

A diabolic square remains diabolic under five different transformations: (1) a rotation (2) a reflection,

(3) a transfer of row from top to bottom or vice versa

(4) - a transfer of a column from one side to the other

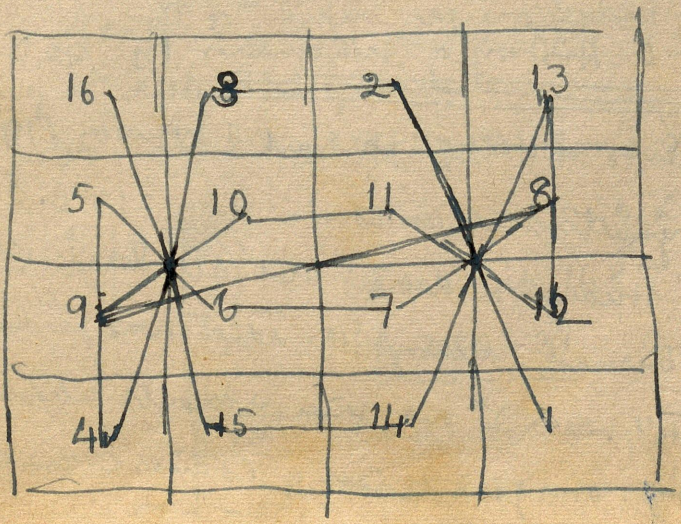
(5) a rearrangement of cells according to the plan below



By continuing these five transformations one can obtain 48 basic types of diabolic squares (384 if rotations & reflections are included). Rosser & Walker show that these five transformations constitute a group that is identical with the group of transformations of the hypercube (4 dimensional cube) into itself (rather far fetched).

Claude Fayette Bragdon, a prominent U.S. architect and occultist (who died in 1946) was fascinated by his discovery that on most magic squares

a line traced from cell to cell in serial order will produce an artistically pleasing pattern. Other patterns can be found by tracing only the odd or only the even cells. Bragdon used "magic lines" obtained in this manner as a basis for textile patterns, book covers, architectural ornaments, and the decorative chapter headings of his autobiography More lives than one. His design for the ventillating grill in the ceiling of the Chamber of Commerce in Rochester N.Y., where he lives, is derived from the magic line of the 16-star. A typical example of a magic line drawn on the 4×4 square is as follows:



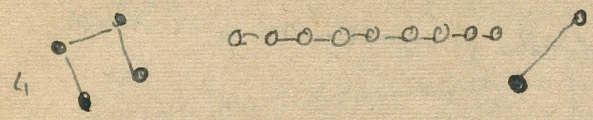
(Do this for the Khajuraho as given by Gardner)

(38) one of the great unsolved problems of recreational mathematics is that of finding a method for calculating the number of different squares of a given order. At present not even the number of fifth-order squares is known, though it is estimated at more than 13 million. The number of 5th-order diabolic squares has, however, been established by Rosser & Walker as 28,800 (including rotations & reflections) diabolic squares are possible in all orders above four except those divisible by 2 but not by 4. There is none, for example, of order six. — Diabolic cubes — (quite an interesting article).

((42)) Intro to History of Mathematics (3rd Edn) by Howard Eves (Holt, Rinehart & Winston (1969)

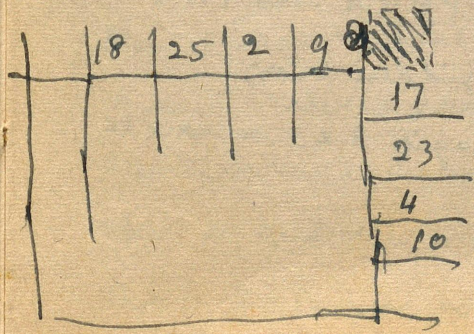
pp. 45-47 Lacking of original sources in China — In 213 B.C. the Emperor Shi Huang-ti decrees that all books in the country be burned. Although the edict was perhaps not completely carried out, there is doubt about earlier dates. one of the oldest of Chinese mathematical

Classics is the l-King or Book on permutations. In it appears the configuration (below) known as the lo-shu.



This is the oldest known example of a magic square, and myth claims that it was first seen by the Emperor Yu, in about 2200 B.C., decorated upon the back of a divine tortoise along a bank of the Yellow River. It is a square array of numerals indicated by knots in strings, black knots for even numbers & white knots for odd numbers.

An nth order magic square is a square array of n^2 distinct integers — — — — Gives rule for 5×5 square due to De la Loubère when he was envoy of Louis XIV to Siam in 1687 – 1688 — pp. 230 – 31 — Gives Heath's construction for 4 × 4 square & says that the resulting magic



(40) square is slightly different from Dürer's – The same rule applies to any magic square of order 4n if we visualise, as drawn, the diagonals of the n^2 principal four-by-four sub-blocks (same construction as in Heath given for the 8 × 8 square.

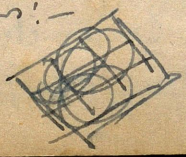
((43) Encyclopaedia Britannica (1973) – Article on magic squares. pp. 573–575.

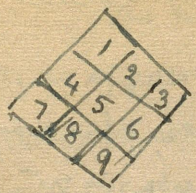
Lettered squares (other the Sator square) were used by Jews & Arabs who often used repeated sacred words or holy names, but on many of these squares the letters actually stood for numbers. – Mysterious mathematical properties associated with such squares

R	A	T	O	R
A	R	E	P	O
T	E	N	E	T
O	P	E	R	A
R	O	T	A	S

[meaningless phrase of SATOR Rotas AREPO TENET OPERA ROTAS holding a hidden cross]

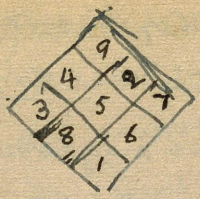
– mention of magic 3 × 3 square & gives figure as follows:–





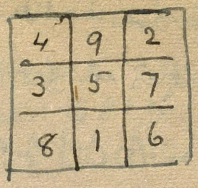
A

(natural square of three slanted to make a diamond)



B

(reversing centre row & column (rather diagonal)



C

Magic square of three

– vague memory that the squares had once been holy symbols caused people to retain them as charms.

Eg. 5 to 13 called augmented magic square

Eg. 2, 4, 6; 5, 7, 9; 8, 10, 12 & separate sequences called a serial magic square.

China: 3 × 3 can be traced back to 4th Century B.C. in China & it does not appear elsewhere until centuries later. Chinese made it from diamond A to diamond B & then collapsing the diamond to form square C. – Called Lo-shu claimed to have been discovered by Emperor Yu in prehistoric times. To them it was a symbolic representation of China itself – or the greater Universe & ~~whatever~~ – still further symbolisation like the balanced the Yin & Yang, the successive the five elements; and other workings of the Tao & so on — This square used by the Chinese as a pattern for altars, cosmic plans

 and divination boards & they discussed its occult properties in books called the wei-shu.

A set of old Chinese magic squares, already ancient when Yang Hui published them in A.D. 1275, shows that the Chinese quite early learned to make magic squares of orders 5 and 7 by taking the nine middle numbers of the respective sequences ($1, 2, 3 \ldots n^2$ with $n = 5$ or $n = 7$) and arranging these in a magic square of the third order to form a core, then placing round this the remaining numbers in opposing complementary* pairs, to complete the required sums (as in A). Squares of this type are called bordered magic squares. They also learned to construct magic squares of five & seven by taking the first four numbers, the middle one, and the last four of the standard sequence and arranging them in the order of a Lo-shu as a skeleton or frame work around which to place the rest (Vide B). [not clear] (Vide Bk. 13, p. 124)

[* $[n^2+1-x]$ complements]

23	1	2	20	19
22	12	17	10	4
5	11	13	15	21
8	16	9	14	18
7	25	24	6	3

(A)

4	19	25	15	2
20	10	5	18	12
3	17	13	9	23
14	8	21	16	6
24	11	1	7	22

(B)

The most spectacular development of the Lo-shu pattern appears in the Old Chinese magic square of nine. Apparently they wrote the sequence 1 to 81 in nine parallel columns - - - used the Lo-shu to make the huge 9 x 9 square (fig. given).

In the 9 x 9 square each opposing pair of nos. equidistant from the centre on a line drawn through the middle number is complementary ie the sum of each pair is twice the middle number - called associated (symmetric?) magic square - So are the Lo-shu & the 5x5 & 7x7 squares - The middle number M gives the magic constant = nM = C and the total sum T of all nos. in the square = MH where H is the highest number. These special properties of the middle number help to explain why the Han dynasty Chinese had so much respect for it, associating it with one of their chief gods T'ai-i, and also with their emperor.

Magic squares of even nos depending on doubly even or singly even cases. The latter (for eg 6, 10, 14, etc) present special problems. By contrast ~~odd squares~~ squares of doubly even order are comparitively easy.

Description of how Yang Hui showed the Chinese method for 4 x 4 squares. Taking a natural square of four (sequence 1 to 16) arranged in the four columns) and reversing

(44) the diagonals to obtain a balanced distribution. To make a second one, they split this first square vertically and exchanged its halves. Using a variation of the same system, they went on to make a magic square of eight. Taking a natural sequence of eight, (1-8), they reversed its two main diagonals, then they split it in half vertically and exchanged the halves; finally they reversed the main diagonals of this new square & it was "magic".

In their devotion to the Lo-shu, the Chinese again employed it to make two magic squares of order 6. For each of these they took from the sequence 1 to 36 nine sets of four numbers each and arranged them (in two separate ways) in nine 2×2 subsquares which they then set out in Lo-shu form; finally, they were forced to break the too-rigid symmetry of each by exchanging pairs of numbers in three of the subsquares (Because this last step destroyed the harmony & broke the Lo-shu pattern, these squares were considered inauspicious). The Chinese did not succeed in creating a proper magic square of ten until the 17th century, and they never were interested in squares for higher numbers.

[Here explanations of Chinese methods for 4, 8 ~~Towards the end of the three~~

and 6 squares are too ~~[illegible]~~ sketchy, specially the case for n = 6. & ~~shows that the author of the article in the Encyclopaedia has~~ really not ~~understood them properly~~)

Toward the end of the 9th century, the Chinese lost their ~~interest~~ earlier reverence for the Lo-shu and hence for the squares related to it. Until then these had apparently been kept hidden as occult lore. When the survey relaxed, the idea of making magic squares seems to have ~~been~~ leaked out to Persian and Arab merchants and to Indian traders in the port towns. It is not known which people got it first (Both people Arabs & Indians make claims to having had squares from ancient times, but neither can prove this tradition by actual examples) [This must be looked into] At any ~~rate~~ interest in magic squares now shifted to western & southern Asia.

~~The Islamic world~~.

(*) 8/4/74 Inst.

1	5	9	13
2	6	10	14
3	7	11	15
4	8	12	16

(i) 1 to 16 in 4 columns

16	5	9	4
2	11	7	14
3	10	6	15
13	8	12	1

(ii) Reversing diagonals (magic)

9	13	1	5
10	14	2	6
11	15	3	7
12	16	4	8

(iii) ~~Splitting~~ (i) vertically & Splitting exchanging the halves

(46) (iii) is not magic. Reversing diagonals in (iii) gives (iv) which is magic

8	13	1	12
10	3	15	6
11	2	14	7
5	16	4	9

(iv)
(magic)

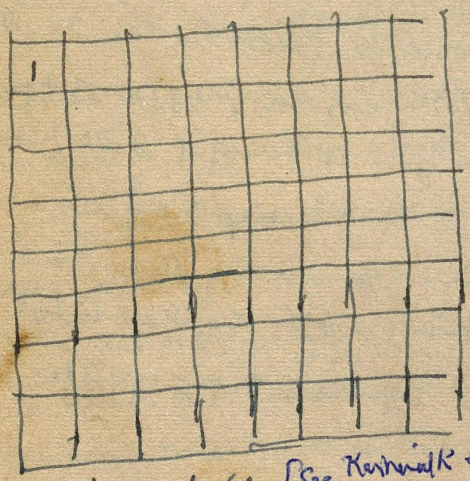
Again splitting (ii) vertically & exchanging halves gives (v) which is magic

9	4	16	5
7	14	2	11
6	15	3	10
12	1	13	8

(v)
(magic)

(iv) & (v) by suitable exchange of rows & cols.
[Valid for algebraic squares]

Square of eight



1							

(i) 1 to 64 in cols.

(ii) Reverse diagonals of (i)

(iii) Split (ii) vertically and exchange the halves

(iv) Reverse main diagonals of (iii) & get a magic square

(v) ~~which~~ which is associated

(to be tried at leisure)

[Vide p. (82') of rough sheets]

(i) 1 to 64 [See Kashmir K & Co. notebook - Forward Planning 1978 - July - Sept page]

The Islamic world - use of decimal system & availability of paper - learning from captured Chinese artisans to make paper and learning positional system from the Hindus - 900 A.D - 989 A.D - In 15th century A.D. the Arabs found the fastest method for making magic squares of odd numbers (order ?) by consecutive numbering. Here the 1 is placed immediately below the centre cell, and from there consecutive numbers proceed diagonally downward to the right. On reaching

(6) Slice Knot types form a subgroup $\mathcal{G}$ of the abelian semi-group of K of all oriented Knot types and the factor semi-group $\mathcal{C}$ is actually a group, called the Knot cobordism group. [Cobordism of Knots]

(7) Structure of the commutator subgroup of a Knot group.

[K. Murasugi { Amer. J. Math 85 (1963), 544–550]

(8) Darstellungen von Knotengruppen.

[Ref. to Fox, same ref. as (2) above pp. 120–167]

the bottom they resume at the top of the next column; and from the right edge they move over to continue from the left side; it is as if the top and the bottom of the square could meet to form a cylinder so that the numbers would pass right arround it; but also as though the left and right sides were joined to make a vertical cylinder around which the numbers would spiral downward (these two divergent requirements would be satisfied by a circular tube, a torus or a doughnut – vide Gardner). After every multiple of the base number, the next ~~cells~~ cell is found to be occupied, so that one must drop two cells before setting down the next number to continue as before. This slight interruption is called a break move. A final break move, after the highest number of ~~the highest number~~ of the sequence leads back to 1 – through the middle number – to begin the cycle again, for a potentially endless round. — Familiarity ~~who~~ with chess led Arab & Persian scholars to use knights moves to make magic squares – old Sufi concept that Allah is both source & destination. Standard knights move squares are also pandiagonal – making Islamic unit-centred squares very & mysterious".

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

Persian continuous method for a 5 x 5.

After 10 imagine top & bottom joined & make a 2-break ~~Also~~ after 15 (ie after 5, 10, 15, 20 & 25)

(48) Experimenting with the variations of the 4 x 4, some Persian or Arab happened upon a pandiagonal one (not explaining how they constructed it). This became the most popular magic square in the Islamic world.

8	11	14	1
13	2	7	12
3	16	9	6
10	5	4	15

A. (pandiagonal)

By using the pattern of this pandiagonal magic square in which to arrange the numbers from 1 to 16 in every possible combination, one can build 432 different magic squares of four, including 47 other pandiagonals. Then by taking certain of the squares from this initial group and shifting or exchanging various numbers within them, one can eventually get construct all other possible magic squares of four. It has been shown that there are precisely 880 different magic squares of four with the regular sequence not counting rotations or reflections of these magic squares. Other 4 x 4 pandiagonals given by Arabs & Persians are

8	11	14	1
10	5	4	15
3	16	9	6
13	2	7	12

(B)

14	7	12	1
11	2	13	8
5	16	3	10
4	9	6	15

(C)

[It is a pity it is not stated how A, B, C were constructed]

— Magic squares engraved on wood blades to increase their cutting power or engraved in metal bowls to purify their contents etc.

Many ways of making squares of eight and twelve (using 4 × 4) — Taking magic squares of 3rd & 5th order, combining them in patterns of the 4th order to form giant composite numbers — Thus to construct a square of twenty, they took 1 to 400 and made 16 magic squares of 5th order, which they then arranged in the pattern of their pandaigonal square of four

Another distinctive Islamic contribution was a further development of Old Chinese method of bordered magic squares — description of Muhyi'l-Din Abu'l Abbas al-Buni's (d. 1225) method, creating another symbol of the soul's journey outward from the Source of Life and its eventual return — figures given of Arabic bordered squares of 7 and 8 from Al-Buni.

Some remarks about Arabic work relating to "singly even order" squares by changing the pattern for numbering in the alternating frames, still remaining valid in the 20th century.

India — Modern writers often say that the Hindus had some of the earliest magic squares, but the oldest known examples from India date only from the 12th or 13th century A.D., after the early Muslim invasions. There are two isolated magic

(50) squares of four. As one of them was the favourite Islamic square of four (A on p. 48), and the other was a derivation of it, they were probably introduced from the Islamic world.

The Hindus experimented widely with magic squares of four which they considered to be especially efficacious, and often inscribed them on charms and amulets. A 19th-century English missionary in India, A. H. Frost, was introduced to pandiagonal magic squares of four in the old city of Nasik. Assuming incorrectly that they had been invented there, he named them "Nasik squares" or "Nasiks" — misleading terms that have been repeated by others

Although claims for the great antiquity of magic squares in India have no tangible basis, the Hindu development of this art was already well advanced by A.D. 1356, when Narayana Pandita presented a large group of them in his Ganita Kaumudi, ascribing them to a legendary King who had been taught by the god Siva. Beginning with a wide selection of magic squares of four and eight — some of them probably based upon earlier Islamic experiments — he went on to demonstrate some predominantly Indian methods

One Hindu technique described by the pandit Narayana combined two auxiliary squares

to make the magic square proper. For instance, to make a magic square of five, he first took the first 5 x 5 square and wrote in its central column the numbers 1 to 5 (in any order) and he repeated each digit diagonally to lower left five times (vide A

10
15
20
0
5

Then in the second or "root" square he placed in the centre column (0, 5, 10, 15, 20) and extended them out diagonally exactly as he had.

first square. Lastly columns to the right), onding cells of these sum in the equivalent

[To look this up later)

[Vide p. 103 of this notebook & rough sheet (79')]

[Also sheet 89 onwards]

$\cancel{(4^2+1)}\,5\,(5^2+1)$

1 2 3 4

13

22, 23, 24, 25

1, 2, 3 1

4, 13, 22 2

23, 24, 25 3

4 25 2

3 13 23

24 1 22

65 39 26 65 31 34

65 39 26

65 47 18

4 . 25 . 2

. 11 . 12 .

3 8 13 18 23

. 14 . 15 .

24 . 1 . 22

5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21

26

21+5 = 6+20
= 7+19 = 8+18
= 9+17 = 10+16
= 11+15
= 12+14
=

21	17	13	9	5
2	23	19	15	6
8	4	25	16	12

(C)

(50) squares of four. As one of them was the favourite Islamic square of four (A on p. 48), and the other was a derivative of it, they were probably introduced from the Islamic world.

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Beginning with a wide selection of ...ic squares of four and eight — some of them probably based upon earlier Islamic experiments — he went on to demonstrate some predominantly Indian methods

One Hindu technique described by the pandit Narayana combined two auxiliary squares

INDIAN MATHEMA...

PROF. B. S. MADHAVA RAO
President

(1) If the fundamental group Π of a center, then $\underline{K}$ is a torus knot.

(2) Topology of 3-manifolds & rela...

[Stallings — Proc. Univ. Geo...

(3) Theorem of Seifert [Acta. Math. ...

(4) Seifert & Threlfall — Lehrbuch...

[Teubner, Stuttgar...

(5) Policy that intelligibility shou...

to make the magic square proper. For instance, to make a magic square of five, he first took the first 5 × 5 square and wrote in its central column the numbers 1 to 5 (in any (?) order) and he repeated each digit diagonally to lower left and upper right until each appeared five times (vide A then in the second or "root" square he placed in the centre column (0, 5, 10, 15, 20) and extended them out diagonally exactly as he had done with the primary number in the first square. Lastly he reversed the second square (left columns to the right), then added the numbers in the corresponding cells of these two auxiliary squares, entering each sum in the equivalent cell of the third sequence (C)

4	5	1	2	3
5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2

(A)

15	20	0	5	10
20	0	5	10	15
0	5	10	15	20
5	10	15	20	0
10	15	20	0	5

(B)

14	10	1	22	18
20	11	7	3	24
21	17	13	9	5
2	23	19	15	6
8	4	25	16	12

(C)

[To look this up later]

[Vide p. 103 of this notebook & rough sheet (79')]

[Also sheet 89 onwards]

(52). This Hindu method of combining squares can yield magic squares of any odd order by simply filling the primary square with the numbers 1 to n repeated n times, and filling the "root" square with the set n(0, 1, 2, .. n-1) again repeating each number n times; then combining the sums of both as described above. The process can be done more swiftly and easily ~~of the~~ if the numbers in the second square are slanted on the other diagonal; then one can simply add the numbers in the corresponding cells, without having to reverse either square.

Nārāyaṇa also constructed magic squares of four and eight by a similar method of adding two auxiliary squares, though he set out the numbers in another order: usually in 2 × 2 subsquares. According to him, the Indians considered sacred this method of combining squares. Perhaps that was because the two initial squares with this of numbers running in opposite directions, intersecting to make a third pattern, reminded them of the way in which they believed the cosmic warp & weft were woven together to form the fabric of the Universe.

[It is a pity that this 4 × 4, 8 × 8 method of Nārāyaṇa is not explained more fully. I wonder where one can get this. — Also remarks made above re: cosmic warp & weft" look ridiculous & no Hindu would attach any superstitious meaning to such magic squares]

– Three centuries later, the French scientist Phillipe de la Hire (1640 – 1718) worked with this same method which he admitted having learned from someone else; but western writers usually credit him with having invented this old Indian process and speak of "the method of De la Hire" or of "La Hirean squares" strictly

As this method is ~~slightly~~ asymmetrical, it cannot be used to make the asymmetric singly-even magic squares. Another distinctive contribution of the Hindus was a special method to handle these, called the system of broken reversion. For a magic square of six they would take a natural square of six and reverse the main diagonals (as they did for a simple magic square of four); then they would break the too-regular symmetry by interchanging six pairs of the remaining numbers. For magic squares of ten and sixteen, they would first create a balance in the natural square by exchanging certain numbers across the main diagonals – carefully leaving the latter intact. Then they broke the symmetry by interchanging ten or sixteen pairs of the other numbers – again respecting the main diagonals

In the 19th century, this method was hailed in Europe as a new discovery. However, an example of the magic square

(54) of ten, made by means of it, had been brought there in the late 17th century by Simon de la Loubère, French ambassador to Siam, and he reported that he had learned it from a M. Vincent, a physician who had been taught the art at Surat

[Again the description of the Hindu method for ten and sixteen is very sketchy & how the broken reversion method is not properly explained]

Sometime before 1356, the Hindu mathematicians had also devised a continuous system for building odd-number ~~squares~~ magic squares. Although this method was based on the same general scheme as that of the Persians – which was also used by the Muslims in India – they may have discovered it independently by studying the magic square of three in its inverted form with the 1 at the top (they preferred this version of it). For in the standard Hindu ~~method~~ squares of this type the numbering begins in the centre cell of the top ~~half~~ row and proceeds upwards to the right with the break move one step down (see Fig A below), which is essentially what happens in the inverted magic square of three

In all Hindu magic squares made by the continuous method, the last number falls at the outer

edge of the square, and from there a final break move leads back to reenter the square on the opposite edge at the 1. to begin the cycle again for another succession of rounds. This figuratively unending, but recurrently broken, procession of numbers made these squares a graphic representation of the Hindu (and Buddhist) conception of "Endless Becoming" during which the individual dies, leaving the world for a moment to be reborn back into into it; or, in a much longer process – the world itself is destroyed and then recreated.

The Hindu continuous method was introduced to Europe in the 17th Century by La Loubère, who had learned this, as well as the broken reversion method, from Vincent. He called it "the method of the Indians", but European writers usually speak of it as the "method of La Loubère".

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

(A)

(Encyc. Brit. p. 575 Vol. 14, 1973)

(Magic square of five Constructed by the Hindu continuous method) – (Sauveur Loubère)

The Western World - In early 16th century H. C. Agrippa von Nettesheim in Germany and Girolamo Cardan in Italy, each published a set of seven magic squares associated with the "Seven Planets" (Sun, Moon, Mars, Mercury, Jupiter, Venus & Saturn") All these must have had Islamic sources, but the actual links ~~are not~~ have not been traced - traces of older religious interpretation in Agrippa's work, cell containing 1 representing the serenity of God.

Dürer's square of 1514 in his famous engraving "Melencolia I" - This particular square was already centuries old in the Middle East and in India, so it did not originate with him

A few years later Stifel (1544) (German) explained the old Islamic method of making bordered squares by an inverse process - work of Riese (about 1550) giving Persian continuous method of constructing odd number magic squares.

Meanwhile German interest in magic squares spread in France. Pierre de Fermat, C. G. Bachet de Méziriac, and Phillip de la Hire studied magic squares as problems in placement of numbers. Bernard Frénicle de Bessy produced a noted monograph on the construction of squares in which he presented for the first

twice all 880 of the magic squares of four.

In the 18th century, the study of magic squares interested such men as the Swiss mathematician Leonhard Euler and the American genius Benjamin Franklin who invented a type called "Franklin squares"

During the late 19th & early 20th centuries, the study of magic squares attracted considerable ~~attention~~ interest in the U.S as an intellectual pastime. A more practical development was the adaption of Latin squares (containing letters), made by the old Indian method of combining auxiliary squares, for designing statistically controlled experiments in Agronomy, medicine, psychology, and other disciplines. Some investigators also concentrated on the problem of determining the number of magic squares that could be constructed for $n > 4$, a question of enumeration that had never been solved. There also remained the even more difficult task of discovering a single "elegant" method of constructing all magic squares. In short, after more than 2,000 years of experiments with magic squares, much still remained to be learned.

Bibliography - Kraitchik & Rouse-Ball's books - W.S. Andrew's book (1960) on Magic squares & cubes - Meltzer's Magische Quadrate (1952), long bibliography

(S.V. R.C) (V)

((44)) Ency. Britt, (1961 Edn), Vol. 14 on Magic squares by H. E. Du.; X (obviously Dudeney & X).

These squares are of very great antiquity and appear to have been known from very ancient times in China and India where, as at the present time, magic squares were worn engraved on metal or stone as amulets or talismans

[This article of Dudeney appears more interesting than the 1973 Edition account & should be read again being also shorter]

10/4/74

The three main lines of enquiry are construction, enumeration and classification —

Construction: — Method of Loubère for odd produces only one type

3	1	4	2	5
5	3	1	4	2
2	5	3	1	4
4	2	5	3	1
1	4	2	5	3

Fig 4

15	0	20	~~20~~ 5	10
0	20	5	10	15
20	5	10	15	0
5	10	15	0	20
10	15	0	20	5

Fig. 5

18	1	24	~~22~~ 7	15
5	23	6	14	17
22	10	13	16	4
9	12	20	3	21
11	19	2	25	8

Fig 6

of square. Starting with 1 in any cell & using the L-method, we get a square magic in rows & cols, but not necessarily in diagonals

– Method of de la Hire using subsidiary squares. In Fig 4, the nos. 1, 2, 3, 4, 5 are arranged so that every number appears once, once only in every row & col, and in one diagonal, the other diagonal having repetitions of 3. In Fig. 5, the numbers 0, 5, 10, 15, 20 are similarly treated only the repeated 10's are in the other diagonal. These squares superimposed and added produce Fig. 6, which is a perfect (or strictly) magic square. — Most of the writers develop their own favourite schemes. An ideal complete general method of solution is probably not discoverable. Magic squares of strictly even order ($n = 6, 10, 14, 18$, etc) are generally the most difficult to construct, and their treatment is generally empirical

Classification. — This is a matter of individual taste, and may be called the elegant branch of the subject. Taking the case of the 4th order, the solutions can all be classified under the four types shown in Figs 7 to 10. —

1	14	7	12
15	4	9	6
10	5	16	3
8	11	2	13

Nasik

1	14	12	7
8	11	13	2
15	4	6	9
10	5	3	16

Associated

16	3	10	5
1	12	7	14
8	13	2	11
9	6	15	4

Simple

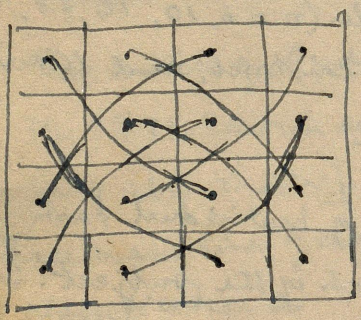
1	14	12	7
4	15	9	6
13	2	8	11
16	3	5	10

Semi-Nasik.

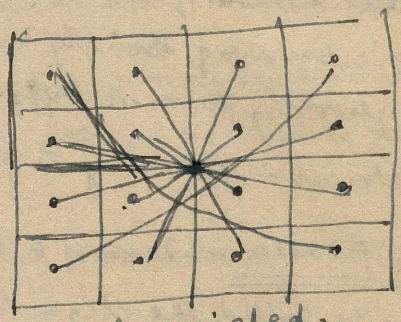
Nasik square (named after Rev. A. H. Frost who lived in that town in India) is the most perfect magic square – Broken diagonals

(60) — Property relating to field of such squares duplicated in a plane given by Gardner — Nasik squares can be constructed for any doubly-even order, and ~~many~~ for any odd
#3 order. But a Nasik square of singly even order is impossible. The number of fundamentally different Nasik squares of the 5th order is 15.

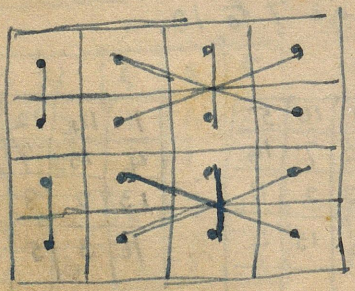
Graphic illustration of each of the types Figs 7–10 is as follows. - The graphic forms for Nasik & associated



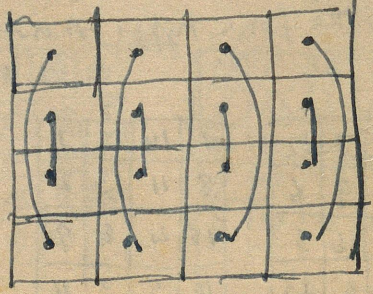
Nasik.



Associated.



Simple



Semi-Nasik

[Dudeney gives all four ((((← Here I have changed the last two from ((to)) — makes it look better]

squares are unique, but varied for the other two. There are (61)
12 such graphic forms [ie connected points sum up to 17 = ½
magic const] in all of the 4th order (Ref. to Dudeney's
Amusements in Mathematics). [Note that associated square
is same as Kraitchiks symmetric square]. A simple square
is one which satisfies the simple conditions & no more
[ie sums of rows, cols & diags. same]. The semi-Nasik
square has the property that the opposite short diagonals
of two cells together sum to 34. [not of 3 cells & the single
cell in short diagonals] .ie 14+4+11+5 = 34, 12+6+13+3 = 34

In this order there are 48 Nasiks, 384 semi-Nasiks,
(which include 48 associated) [Note that in associated
also the two-cell short diagonals sum property also holds
ie associated is semi-Nasik], and 448 simple,
making in all a total of 880. [This classification
of Dudeney is preferable to Kraitchiks algebraic etc]
[but Kraitchiks is deeper]
and is more elementary]. The Nasik squares are also sometimes
called diabolique and pandiagonal [or panmagic] and the
semi-Nasik semi-diabolique [Note that Diner's square
is semi-Nasik]. [Although these graphic illustrations

(62) and classification of 4×4 squares by Dudeney is elegant, ~~it does~~ this does not throw light on how

× | to construct such a Nasik square, say for eg. ~~based on~~ similar to Loubère's for odd squares. In fact this is just what I require.] Read Kraitchik again]

See Gamble Kaunas BJM 10/3/82

— Bordered magic squares — Composite squares —

Enumeration — Little progress during last 200 years — only 4th order with 880 fundamentals [Refs as to how to construct all these 880 to Queen, Jan. 15, 1910 & Frolow's Carré's magiques] — No complete enumeration for higher orders —

There are 28,800 Nasik and 174,240 bordered squares of order 5, and 507,705,600 bordered squares of order 6 — Actual formulas for counting the socalled uniform step-square of any order of order n (due to J. Willis) have been given by D. N. Lehmer ~~&~~

Extensions & variations — 3×3 with

$$\begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 13 & 14 & 15 \end{pmatrix}$$

[trivial] — A Nasik square from

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{pmatrix} \rightarrow \text{magic constant} = 40$$

$$\frac{34+38+42+46}{4} = \frac{160}{4} = 40$$

is

1	17	8	14
18	4	11	7
12	6	19	3
9	13	2	16

1	9	16	14
17	13	2	8
4	6	19	11
18	12	3	7

Alternately using G.K. method of Knight moves

BM 24/6/63

x [Here again how the above is constructed is not indicated. – Generalising the 16 nos. chosen we can we take a, b, c, d

$$\left(\begin{array}{l} A+a,\ A+b,\ A+c,\ A+d \\ 2A+a,\ 2A+b,\ - \\ 3A+a,\ - \end{array}\right)$$

with a, b, c, d in A.P

use G.K. method of Knight moves

? and get a Nasik square out of them] – Constn of squares with primes only & with smallest possible constants – from consecutive composite numbers – Subtracting, multiplying & dividing magic squares (Ref. to Amusements] – other extensions of the problem such as magic polygons, magic cubes, magic circles and magic stars have also attracted attention – Chess Board & Knights tour problems – Constructions with dominoes & playing cards – with coins and postage stamps – These interesting questions are of a

(64) puzzle character and often call for considerable mathematical analysis, though as a man said of Paradise Lost, they "prove nothing" and lead nowhere. They are therefore strictly speaking of no direct mathematical importance

[Good remarks supporting Hardy's contention about number curios].

References. Rouse-Ball, W. S. Andrews, Dudeneys Amusements, D.N. Lehmer etc etc - Rosser and Walker, Duke Math. Journal (1939).

((45)) Ramanujan's Note Books.

Vol I. — Note Book 1. — Chapter I — Magic squares.

(ii) Let a be the array, s a row or column, m middle of rows or cols., d a diagonal and w the whole sum

(i) when square contains 3 rows & 3 cols.

(i) If s & d are equal, write a in the middle and supply the other figures.

Soln $d_1 + d_2 + m_1 + m_2 = w + 3x$, where x is the required figure in the middle

$\therefore 4s = 3s + 3x$ or $s = 3x$ or $x = a$.

Cor. The figures in d are in A.P

Soln:- The sum of numbers in d = s or $3a$ and $2 \times d = a$ $\therefore$ 1st + 3rd = $2a$ = twice the second (?) - are in A.P. Sum is in m also.

Ex 1. Fill up the square when $s = 15$

6	1	8
7	5	3
2	9	4

2. when $s = 27$ and all numbers are odd

15	1	11
5	9	13
7	17	3

(ii) when s and d are unequal, write $\frac{d_1 + d_2 - s}{3}$ in the middle.

Ex: Show that the numbers in m are in A.P. here also

Soln - Proceed as in I 1.1.

2. Fill up the square when $s = 20$, $d_1 = 16$ & $d_2 = 19$

10	2	8
4	5	11
6	13	1

(iii) when the diagonals, cols. & rows are all different, write

$\frac{1}{3}(d_1 + d_2 + m_1 + m_2 - w)$ in the middle

Soln - As in I.1.1

Ex. Fill up the square when $d_1 = 15$, $d_2 = 19$ and the cols. and rows are 16, 17, 12, 6, 21 and 18.

1	2	3
8	9	4
7	6	5

2. When an oblong contains 3 rows & 4 cols.

$A+C = 2B+3D$			
A	C+D	A+2D	C+3D
B+6D	B+4D	B+2D	B
C	A+D	C+2D	A+3D

Ex: Fill up the oblong when $a = 8$

1	13	3	15
11	9	7	5
12	2	14	4

Figure struck off.

3. When a square contains 4 rows & 4 cols.

(i) When the diagonals, cols. & rows are all different, arrange the central four so that the sum may be

$$\frac{1}{2}(d_1 + d_2 + m_1 + m_2 - w)$$

(ii) When $s = d$

Fig I

A+P	C+S	D+Q	B+R
D+R	B+Q	A+S	C+P
B+S	D+P	C+R	A+Q
C+Q	A+R	B+P	D+S

Fig II - A + D = B + C, P + S = Q + R

A+P	D+Q	D+R	A+S	
B+S	C+R	C+Q	B+P	
C+S	B+R	B+Q	C+P	
D+P	A+Q	A+R	~~B~~+S	D

[Cf. Kraitchik's (A, B, C, D) & (a, b, c, d)

Ex:

1	10	15	8
16	7	2	9
6	13	12	3
11	4	5	14

[Semi. Nasik]

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

[Like Dürer's]

[Semi-Nasik & associate

(68) 4. when a square contains 5 rows & 5 cols

A+P	E+R	D+T	C+Q	B+S
C+T	B+Q	A+S	E+P	D+R
E+S	D+P	C+R	B+T	A+Q
B+R	A+T	E+Q	D+S	C+P
D+Q	C+S	B+P	A+R	E+T

[Remarkable indeed !]

Similarly we can form squares containing more rows & cols

[unfortunately Ramanujan has not explained how this "similarly" works]

1	58	59	4	5	62	63	8
16	55	54	13	12	57	50	9
24	47	46	21	20	43	42	17
25	34	35	28	29	38	39	32
33	26	27	36	37	30	31	40
48	23	22	45	44	19	18	41
56	15	14	53	52	11	10	49
57	2	3	60	61	6	7	64

[1 to 64 - magic
const $= \frac{1}{2}(8^3+8)$
$= \frac{1}{2}(520)$
$= 260$]

[How is this obtained ?]

Note Book II. – Chapter I – pp. 5 – 12.

Magic squares can be constructed by combining two sets letters so that the same letter may not appear in a row, a column or a corner, for example if we want to construct squares containing n rows & n columns, we should take two sets of letters A, B, C, D, E etc, and P, Q, R, S, T etc, combine them as A + P, A + Q, A + R, etc, B + P, B + Q, B + R, etc, C + P, C + Q, C + R, etc, and ~~so on~~ etc. & etc, and arrange them in such a way that any letter, say (?) may not appear in the same row, column or corner. Now we have algebraically constructed a square and the sum of the figures in the rows and ~~columns~~ corners & cors. will be equal if we give any value to the letters. Let us Letters & yet the same figure may likely appear again owing to arithmetical values; this difficulty receives from the following truths:

A + M differens from ~~A + N~~ A + N, as B + M from B + N, as C + M from C + N – – – unclear – –

Cor 1. If A + P, A + Q, A + R etc are in A . P, then B + P, B + Q, B + R etc are also in A . P, ∴

Cor 2. If the values of A + P, A + Q, A + R, A + S, etc are known, then from the value of B + P (?), those of B + Q, B + R, B + S, etc, are also known.

N.B: we should not give separate values to A, B, C, etc and to P, Q, R, etc, but we should give values to A+P, A+Q, etc

Ex. 1. The values of A+P, B+P, C+P, D+P are 8, 10, 11 and 14 and C+R = 25, find A+R, B+R, D+R.

2. A+P, A+Q, A+S are 5, 7, 17 respy and B+Q = 23 and B+R = 26, find A+R, B+P and B+S?

2. To construct a square containing three rows . . . .

(i) $m_1 + m_2 + c_1 + c_2 = S + 3x$

— — — —

(ii) If the cols & rows are equal & corners are diff. Gives Ex. 1

6	1	8
7	5	3
2	9	4

15	1	11
5	9	13
7	17	3

all odd

14	4	18
16	12	8
6	20	10

all even

24	9	30
27	21	15
12	33	18

All multiples of 3

N.B – Soln fails when given sum is not a multiple of 3

4. To construct a square of A+B+C+P+Q+R

C+Q	A+P	B+R
A+R	B+Q	C+P
B+P	C+R	A+Q

	∧	✓	
∧	✓	×	∧
✓	×	∧	✓
×	∧	✓	

13

A, B, C & P, Q, R must both be sets in A.P.

Neglecting only one corner construct a square for

(i) for 19, (ii) for 31 when all are odd

10	2	7
4	6	9
5	11	3

16	5	12
7	11	13
10	15	6

2 (i) corners are 16 & 19 and rest 20

(ii) " 15 & 19, cols 16, 17, 12 & rows 6, 21, 18

10	2	8
4	5	11
6	13	1

1	2	3
8	9	4
7	6	5

To construct an oblong containing 3 rows and 4 cols.

— — — — —

6. To construct a square containing 4 rows & 4 cols.

(i) When the corners, rows & cols. are all ~~equal~~ different so that the sum may be equal to half the difference between the whole sum and the sum of the corners, the middle rows & the middle cols

(ii) When the rows, cols & corners are all equal

(72)

A	B	C	D
D	C	B	A

P	Q	R	S
R	S	P	Q

Add these two as A + P, B + Q, etc, and fill up the other two rows. Or we may construct the oblong as in I, (?)

6

A + P	D + S	C + Q	B + R
C + R	B + Q	A + S	D + P
B + S	C + P	D + R	A + Q
D + Q	A + R	B + P	C + S

A + P	D + Q	D + R	A + S
B + S	C + R	C + Q	B + P
C + S	B + R	B + Q	C + P
D + P	A + Q	A + R	D + S

when A + D = B + C
P + S = Q + R

N.B: If A + D = B + C & P + S = Q + R, the extreme middle four in the 1st square also satisfy conditions

Ex-1. Construct for 34 & 35

(Nasik)

1	14	11	8
12	7	2	13
6	9	16	3
15	4	5	10

1	14	15	4
8	11	10	5
12	7	6	9
13	2	3	16

1	15	11	8
12	7	2	14
6	9	17	3
16	4	5	10

Ex. 2 - Construct two diff. squares for 66

1	30	27	8
28	7	2	29
6	25	32	3
31	4	5	26

9	22	19	16
20	15	10	21
14	17	24	11
23	12	13	18

Ex. 3 - Construct two diff. 3rd order squares for 60

23	6	31
28	20	12
9	34	17

25	3	32
27	20	13
8	37	15

This is connected with

3 8 13
15 20 25
27 32 37.

→ Connected with 6, 9, 12
17, 20, 23
28, 31, 34

- Gives two 8 x 8 squares.

8. To construct a square of odd rows & cols.

A, B, C, D, E, F, G, H, etc
A B C D E F, etc
A B C D etc
A B etc

P, Q, R, S, T, U, V, W, etc.
R, S, T, U, V, W, etc
T, U, V, W, etc
V, W, etc

Thus arranging the letters and adding the two sets a square of any number odd rows can be formed, and we can find many ways of constructing a square and the peculiarities are common to all the odd squares.

P.T.O

E/

A+P	B+R	D+T	C+Q	B+S
C+T	B+Q	A+S	E+P	D+R
E+S	D+P	C+R	B+T	A+Q
B+R	A+T	E+Q	D+S	C+P
D+Q	C+S	B+P	A+R	E+T

Same as on p. 68

D+Q	E+S	A+P	B+R	C+T
E+R	A+T	B+Q	C+S	D+P
A+S	B+P	C+R	D+T	E+Q
B+T	C+Q	D+S	E+P	A+R
C+P	D+R	E+T	A+Q	B+S

?
?
?
?

In this square A + B + D + E must be equal (to 4C)?

Exc.: To construct a 5-rowed square for 65 & 66

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

1	24	20	12	9
15	7	4	22	18
25	16	13	10	2
8	5	23	19	11
17	14	6	3	26

Ex. 2. Construct a 7-rowed square for 170 & 175

— Gives the two squares

See p. 138 for revised opinion

[Rather disappointed with Ramanujan's work on magic squares without proper explanations.]

((4 b)) Mathematical diversions by Hunter & Madachy

Chap. 3 — Mystic Arrays.

p. 215 p. 27 Constr for 4th order magic square — Write 1–4 as in (a) then use the same numbers so that rows & cols = 10 as in (b)

1			4
	2	3	
	2	3	
1			4

(a)

1	3	2	4
4	2	3	1
4	2	3	1
1	3	2	4

(b)

Now reverse the row-col positions of (b) to form (c)

The nos. appearing (b) & (c) we shall refer to as primary numbers.

1	4	4	1
3	2	2	3
2	3	3	2
4	1	1	4

(c)

Form ~~for~~ a root square as in (d)

0	12	12	0
8	4	4	8
4	8	8	4
12	0	0	12

(d)

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

(e)

Primary Nos	Root Nos
1	0
2	4
3	8
4	12

Then add the corresponding numbers (Semi-nasik).

(76) of (b) to those of (d) to give the 4th order magic square in (e) [Related to Narayana Pandithas method?]

Gen: Primary nos ($p-1, 2, 3 \ldots n$)

root nos $= n(p-1)$ when n = order (even) & the p the primary number corresponding

p. 32 – Magic word squares – Crosswords a primitive form of these

– Talks of Latin squares & agricultural research

((47)) Mathematical Magic by William Simon (1964)

Chap. 5 – Magic squares. Nothing very interesting or new – only with dots etc & moving a number to form a magic square

((38)) Further notes on Magic squares from Kraitchik.

on magic squares of order 4

p. 183 – Magic series – There are 86 magic series (Any set of n distinct numbers from 1 to n^2 whose sum is the nth magic constant is called a magic series). There are of 3 kinds, which we shall call algebraic, symmetric, and

1	2	3	4
5	6	7	8
13	14	15	16
9	10	11	12

16	8	1	[illegible]
10	4	15	5
7	13	2	12

1	2	4	3
5	6	8	7
13	14	16	15
9	10	12	11

1	2	4	3
5	6	8	7
9	10	12	11
13	14	16	15

1	16	7	10
11	6	13	4
14	3	12	5
8	9	2	15

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

(I) diagonal

8	21	14	2	20
25	13	1	19	7
12	5	18	6	24
4	17	10	23	11
16	9	22	15	3

23	6	19	2	15
10	18	1	14	22
17	5	13	21	9
4	12	25	8	16
11	24	7	20	3

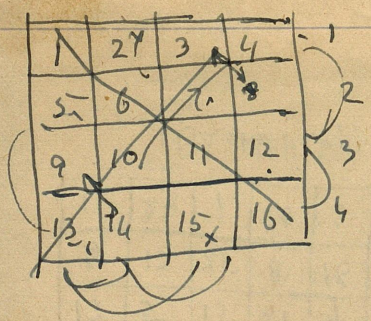
(Bachet diag)

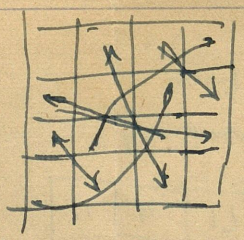
3	16	9	22	15
20	8	21	14	2
7	25	13	1	19
24	12	5	18	6
11	4	17	10	23

(Reichmann [illegible] using (OTH diag)

12

12





from 3 to 18

1 to 16.

9.

4! × 4!

24 × 24

576 × 2

1	15	8	10
12	6	13	3
14	4	11	5
7	9	2	16

3	17	10	12
14	8	15	5
16	6	13	7
9	11	4	18

5	6	7	8
1	2	3	4
9	10	11	12
13	14	15	16

1	12	14	7
15	6	4	9
8	13	11	2
10	3	5	16

5	15	4	10
12	2	13	7
14	8	11	1
3	9	6	16

(1234) (1243) (1324) (1342) (1423) (1432)

(2134) (2143) (2341) (2314) (2413) (2431)

(3124) (3142) (3241) (3234) (3412) (3421)

(4123) (4132) (4231) (4213) (4312) (4321)

arithmetical. Algebraic squares are those formed by taking an element from each ~~set~~ orthogonal of an addition table. This can be done in $3 \cdot 4!$ ways, but each series appears twice they are

[from (α) as in rough sheets take ~~1, 6, 7, 16~~ for eg. 1, 4, 14, 15
ie are from each column. This appears in (β) when taking one from each row. ie each series appears twice]
or col

1 4 13 16* [from (β)
14 15 [from β and α]
6 11 16*
(6) 12 15
7 10 16*
12 14
8 10 15
11 14
12 13
(9)

2 3 13 16
14 15*
5 11 16
12 15*
7 9 16
11 14
12 13
8 9 15*
11 13
(9)

3 5 10 16
12 14*
6 9 16
10 15
12 13
8 9 14*
10 13
(7)

4 5 9 16
10 15
(8) 11 14
6 9 15*
11 13
7 9 14*
10 13
(7)

5 8 9 12*
10 11
(2)

6 7 9 12
10 11*
(2)

$9+9+7+7+2+2$
$= 36$

Symmetric magic squares are those that contain only pairs of complementary numbers (two nos. are complementary if their sum is n^2+1 i.e 17 in this case). All possible sym. magic series of order 4 will be formed by taking all possible pairs of numbers not greater than eight, together with their complements. Hence there are $\frac{8 \cdot 7}{2} = 28$ distinct symmetric magic series, of which 12 are also algebraic and so have already been listed (on p. 77) and are marked in that table with an asterisk. The other sixteen are

1	2	15	16
	3	14	
	5	12	
	8	9	

2	4	13	15
	6	11	
	7	10	

3	4	13	14
	6	11	
	7	10	

4	5	12	13
	8	9	

3	5	6	12
	7	10	

6	8	9	11
7	8	9	10

The remaining magic series which are neither algebraic nor symmetric, we shall call arithmetic. They are due merely to the chance of arithmetic combinations and contain many anomalies. They are the following 34:

[36 + 16 + 34 = 86]
(24+12)

1	5	13	15 (o)
	6	13	14
	7	11	15 (o)
	9	10	14 (l)
		11	13 (lo)
	10	11	12 (l)

⑥

2	4	12	16 (e)
	5	13	14
	6	10	16 (e)
	6	12	14 (e)
	8	10	14 (e)
	9	10	13 (l)
		11	12 (l)

⑦

3	4	11	16
		12	15
	5	11	15 (o)
	7	8	16 (s)
		9	15 (o)
		11	13 (o)
	8	11	12
	9	10	12 (l)

⑧

4	6	8	16 (se)
		10	14 (e)
	7	8	15 (s)
		11	12
	8	10	12 (e)
	9	10	11 (l)

⑥

5	6	7	16 (s)
		8	15 (s)
		9	14
		10	13
	7	8	14 (s)
		9	13 (o)

⑥

6	7	8	13 (s)

①

$$[1 + 6 + 6 + 8 + 7 + 6 = 34]$$

Every magic series contains two large numbers (> 8) and two small numbers (< 9), except the arithmetic series marked (l) or (s) respectively. Likewise each magic series contains two even nos. & two odd numbers, except the ~~marked~~ arithmetic series marked with (e) & (o) respectively [These are the anomalies].

Except for the symmetric series, magic series occur

(80) in complementary pairs, the numbers of one series of such a pair being the complements of the numbers of the other series

[for eg. in table on p. 77, (1, 4, 14, 15) and (16, 13, 3, 2) which come under (2 3 13 16) are complementary while the symmetric pairs in that table for eg. (1 6 11 16) has complement (16, 11, 6, 1) ie complement to itself. Similar symmetric non-algebraic squares for eg. (1 2 15 16) has complement (16, 5, 12, 1) ie itself]

p. 186. **Literal magic squares.** [Literal means with which letters as Ramanujan revelled in.]. Writing letters as

a	b	c	d
e	f	g	h
i	j	k	l
m	n	p	q

(A)

Values of 10 sums are fixed (4 cols, 4 rows & 2 diagonals). But sum of sums of rows = Sum of sums of cols. Hence only 9 out of the 10 relations are independent. There are thus 16 − 9 = 7 independent elements (and the magic constant S). We find that the square may be written as follows:

a	b	c	$S-(a+b+c)$
e	f	g	$S-(e+f+g)$
i	$2a+b+c+e-g+i-S$	$2S-2a-b-c-e-f-i$	$f+g-i$
$S-(a+e-i)$	$2S-2a-2b-c-e-f+g-i$	$2a+b+e+f-g+i-S$	$a+b+c+e+i-S$

(B)

From this one can readily verify (?) that every magic square contains 4 magic series in addition to 10 prescribed. In the notation of fig (A) these are

$$a+d+m+q = f+g+j+k = b+c+n+p = c+i+h+l = S$$

There are many other relationships connecting the elements of a magic square. The following relations are typical (?):

$$a+d = n+p,\quad c+h = j+k,\quad a+q = g+j$$

$$a+b+e+f = c+d+g+h$$
$$i+j+m+n = k+l+p+q$$

} X diagonally consecutive squares have same sum

If we want the square to be panmagic

(82) apparently six more corollaries [corresp. to b. diagonals] are added. Actually, however, only three of these independent, so that there, so that there are [7-3] four independent constants left, and the square may be written as in (C) below

a	b	c	$S-(a+b+c)$
e	$S-(a+b+e)$	$a-c+e$	$b+c-e$
$\frac{S}{2}-e$	$(a+b+c)-\frac{S}{2}$	$\frac{S}{2}-a$	$\frac{S}{2}-b$
$\frac{S}{2}-(a-c+e)$	$\frac{S}{2}-(b+c-e)$	$\frac{S}{2}-e$	$(a+b+e)-\frac{S}{2}$

(C)

[Perhaps e maybe ⟶ d]

Algebraic squares. one of the above sets of relations on p. 81 [which one ?] holding for every magic square of order 4 states that the sum of the numbers in every quadrant of the square is the same. If this sum is also the magic constant, we shall call the square algebraic.

[Does not follow from the last two relations on p. 81, that

this sum is always S? ie if $x+y=2S$ & $x=y$, $x=y=S$]. This adds one further condn & we can write an algebraic square with six constants (a, b, c, e, g, i, S).

An algebraic magic square contains 6 magic series in addition to the 18 already noted [where ?]:

$a+c+i+k = b+d+j+l = e+g+m+p = f+h+n+q$
$= c+h+i+n = b+c+l+p = S$. All 24 series are algebraic.

It can be written in form (D) below:

a	b	c	$S-(a+b+c)$
e	$S-(a+b+e)$	g	$a+b-g$
i	$2a+b+c+e-g+i-S$	$S-(a+c+i)$	$S-a-b-e+g-i$
$S-(a+e+i)$	$S-a-b-c+g-i$	$a-g+i$	$a+b+c+e+i-S$

It is easily shown (?) that an algebraic square may be formed from an addition table by the methods given on p. 183 [Note-Bk. 11, pp. 178-79, ie using (α), (β), (γ)]

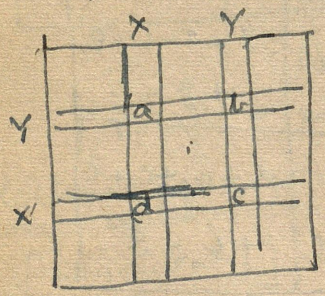
If $A+D=B+C$ and $a+d=b+c$ only the first transformation

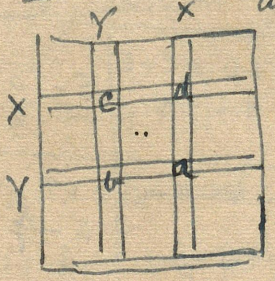
(84) [ie transfer of symmetrical] is necessary. If not the second form is certainly magic. Both forms are clearly algebraic.

In addition to the transformations (pp. 144-45 of Kraitchik) to which every magic square may be subjected

These are (i) Rotations & reflections

(ii) ~~Change of any~~ Interchange of any two rows equidistant from the centre plus the interchange of two columns at the same distance from the centre as in Fig on left showing manner in which such a transformation affects the

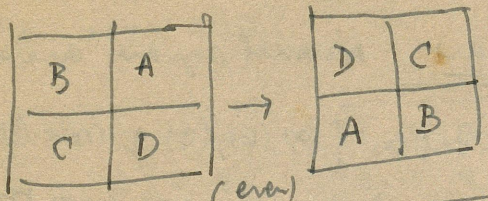




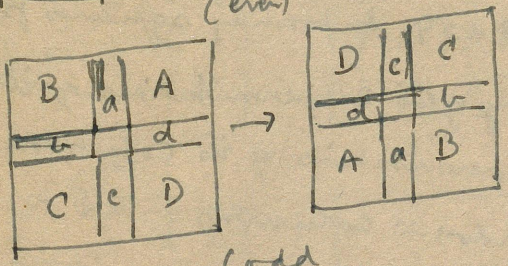
elements at the intersections of the transported rows and columns. Repetition of this transformation for all such sets of orthogonals results in a rotation of the original square through 180°

(iii) Interchange of diagonally opposite quarters

of the square. This description is exact for squares of even order (as below) ; but this modified slightly for squares of odd order in the manner shown below.



(even)



(odd)

(iv) In a normal magic square of order n, ~~outchange~~ if every element x is substituted by its complement ie (n^2+1-x), the square remains magic.

— In general when nos. are subtracted from any fixed number.

(v) In particular, transpan magic square, Cyclic interchange of rows or cols or both keeps the square still magic (as in Gardner)] an algebraic magic square remains both algebraic and magic if we

(86) (vi) interchange the first two orthogonals with the last two

For eg. cols 1, 2, 3, 4 may be put in the order 3, 4, 1, 2.

Number of algebraic squares. Normal squares are all derivable from the addition tables [(a), (b), (c) of Note BK-11, pp. 178–79] of Figs. 58 & 59. In forming squares from these tables we may use any of the 24 permutations of its rows with any of the 24 permutations of its columns and we may interchange rows & columns [This is mere repetition of what has already been said without any explanation] [~~[illegible]~~ & Set 432.

Classification of normal algebraic squares.

The distribution of the pairs of complementary numbers forms the basis of a complete classification of algebraic squares. Six types ~~[illegible]~~ may be discerned, in the first three of which the pairs of complements lie on diagonals, while in the others they lie on the orthogonals

We represent these six cases in figures by

designating a number and its complement by the same digit.

[for eg. 1 stands 1+16 or 16+1 and 6 stands for 6+11 or 11+6, --
8 for 8+9 or 9+8].

1	2	3	4
5	6	7	8
8	7	6	5
4	3	2	1

(1)

1	2	3	4
2	1	4	3
5	6	7	8
6	5	8	7

(2)

1	2	3	4
5	6	7	8
3	4	1	2
7	8	5	6

(3)

1	1	2	2
3	3	4	4
5	5	6	6
7	7	8	8

(4)

1	2	1	2
3	4	3	4
5	6	5	6
7	8	7	8

(5)

1	2	2	1
3	4	4	3
5	6	6	5
7	8	8	7

(6)

(1) Central -- The complementary numbers are symmetrically placed with respect to the centre.

(2) Diagonal — The complementary numbers form the diagonals of the quadrants.

(3) Panmagic — The complementary numbers lie on the diagonals in pairs whose elements separate each other.

[In fact, this can be taken true for the broken diagonals also]

(4) and (5) orthogonal (adjacent & alternating) - The complementary numbers lie in two columns, adjacent or alternating.

(6) Symmetric – The complementary numbers are symmetric with respect to the vertical median

It is interesting that an algebraic magic square can have no other distribution of the complementary numbers.

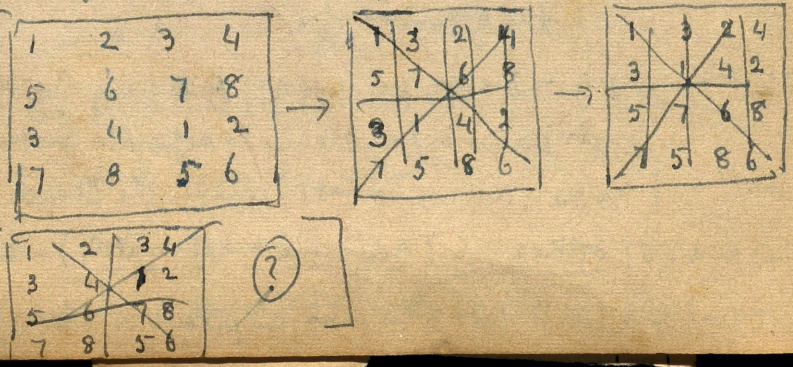
The distribution of the squares in the various classes is :–

Central	48
Diagonal	48
Panmagic	48
Adjacent orthogonal	96
Alternating orthogonal	96
Symmetric	96
Total no. of algebraic squares =	432.

[The nos. to be looked into using Tejs of p. 87]

A list of basic squares of the various types is given below. Since a panmagic square is converted into a diagonal square by interchanging the two central orthogonals

[not clear]



These diagonal squares have not been listed separately. Neither have the alternating orthogonal squares, since they may be similarly derived from the adjacent orthogonal squares.

Central:

1	8	12	13
14	11	7	2
15	10	6	3
4	5	9	16

1	8	14	11
12	13	7	2
15	10	4	5
6	3	9	16

1	8	15	16
12	13	6	3
14	11	4	5
7	2	9	16

Panmagic:

1	8	10	15
12	13	3	6
7	2	16	9
14	11	5	4

1	8	11	14
12	13	2	7
6	3	16	9
15	10	5	4

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

Adjacent orthogonal:
[or alternating?]

1	4	16	13
14	15	3	2
7	6	10	11
12	9	5	8

1	6	16	11
12	15	5	2
7	4	10	13
14	9	3	8

1	7	16	10
12	14	5	3
6	4	11	13
15	9	2	8

Symmetric:

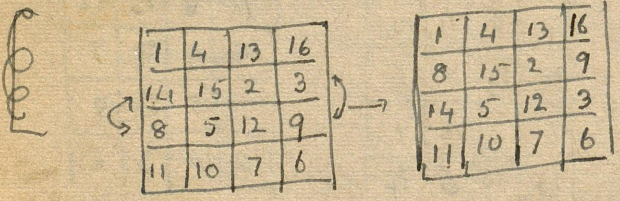
1	4	13	16
14	15	2	3
8	5	12	9
11	10	7	6

1	6	11	16
12	15	2	5
8	3	14	9
13	10	7	4

1	7	10	16
12	14	3	5
8	2	15	9
13	11	6	4

(90) [It would be interesting to find out how these are related to the squares obtained from (α), (β), (γ), specially the pan magic ones].

The 432 are the full set of normal algebraic squares. However if one interchanges the middle numbers of the end columns of a normal symmetric algebraic square it remains symmetric magic but not algebraic [for eg.



1	4	13	16
14	15	2	3
8	5	12	9
11	10	7	6

→

1	4	13	16
8	15	2	9
14	5	12	3
11	10	7	6

is symmetric magic but not algebraic since sum of elements in every quadrant is not $\neq 34$.]

Such a square will be called derivative symmetric. There are 96 of them [corresp to 96 symmetric] one for each symmetric square. Its literal ~~In addition to these there are 112 a~~ form it becomes

a	b	17 − b	17 − a
d	34 − a − b − c	a + b + c − 17	17 − d
c	a − b + d	17 − a + b − d	17 − c
34 − a − c − d	b + c − d	17 − b − c + d	a + c + d − 17

In addition to these there are 112 arithmetic symmetric squares, 224 semi-orthogonal, and 16 irregular squares. [There is no definition of arithmetic symmetric, semi-orthogonal and irregular squares, nor is there at least one simple example given of each]

Thus the total number of normal magic squares of order 4 is 880: viz.

Algebraic	432
Derivative symmetric	96
Arithmetic symmetric	112
Semiorthogonal	224
Irregular	16
	880

[or 880 × 8 = 7040 if reflections & rotations are not included]

Note: Total symmetric = 96 + 96 + 112 = 304
(alg.) + (Der.) (arith)

p. 174 – Panmagic squares: (of order 5 in particular)

It is impossible to form panmagic squares of order 3 or regular panmagic squares of order divisible by 3. Raynor has proved that it is not possible to form a panmagic square of order 4k+2. There are 3 essentially distinct

(92) pan magic squares of order 4 [See p. 89], 16 regular panmagic squares of order 5, and 54 of order 7.

[Some notions about constellations, not at all clear - not even a single example]

[From Rouse-Ball, p. 204 - there are ~~4800~~ exactly 3600 pandiagonals of 5th order, more than 38 million of the seventh order, and more than 6 ½ billion (~~6½ billion~~ (million + million) of the eighth order as shown by Rosser & Walker] - this is also mentioned in Note Bk. 11, p. 188. An example of a pandiagonal of order 5 is also given on this page. Here is another of order 7 from Rouse-Ball (p. 206)

26	21	9	4	48	36	31
44	39	34	22	17	12	7
20	8	3	47	42	30	25
38	33	28	16	11	6	43
14	2	46	41	29	24	19
32	27	15	10	5	49	37
1	45	40	35	23	18	13

$a + a' = 2$
$b + b' = -1$

By continuing step method regular move is $(1, 2)$ & break move is $(2, -1)$
$(a, b) = (1, 2)$
$(a', b') = (1, -3)$
$a \pm b$, $a' \pm b'$
$(3, -1)$, $(-2, 4)$ are all prime to 7
ie no sets
a, b, a', b' prime to 7
$ab' - a'b = -5$ also prime to 7

Kraitchik, p. 192 – Literal form used (for 4×4 sqs) (93)
to make an arithmetic magic square with four given numbers
that form a famous date – eg. the date of the first Independence
Day, 7/4/17-76. These four numbers determine the magic
~~sq~~ constant 104. We may now take four more rather arbitrarily
selected numbers to fill out the square, but avoiding repetitions
and negative numbers

7	4	17	76
51	50	1	2
33	14	39	18
13	36	47	8

((48)). Modern Mathematics in the Junior School by
C.E. Drew (Book brought by Chilton from British
Council Library). (1970).

Preface: the most elementary ideas of simple arithmetic
are a necessary prerequisite – none of the notation which
makes mathematics so absurdly difficult to the average
person –

p. 20 - magic ~~numbers~~ squares.

Magic squares were, we think, invented by the early Chinese mathematicians, and again their interest lies in ~~peculiarity~~ peculiarity and patterns. This is a magic square using the numbers 1 to 9 in a special order

8	1	6
3	5	7
4	9	2

Its magic lies in the fact that whether we add across, up and down or on the diagonal the answer is always 15. This magic square is called of order 3 because it has three rows & 3 cols. Its middle number 5 is very important because when we multiply it by 3 we get 15. the answer to the square is 15.

It is very easy to ~~use~~ manufacture, for class use, hundreds of them of order 3 from the original one above

Ex 1. ~~Add~~ first 5 to every number and we get

13	6	11
8	10	12
9	14	6

The answer to the addition in this case is 30

$= (3 \times 10)$

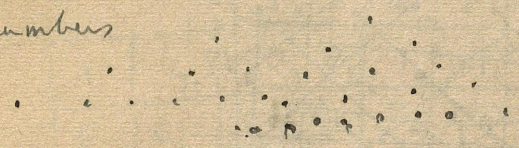
[why this insistence on central row?]

Ex. 2. Multiply every number by 4 & add 2. [6, 10, 14, 18, .. 38]

34	6	26
14	22	30
18	38	10

The answer to addition is 66 again (3 × 22)

[The author has only considered nos. in A.P., not the more general case of 3 sets in 2 different A.P's – Perhaps it is good to use what the other has given on p. 29 on figurate numbers

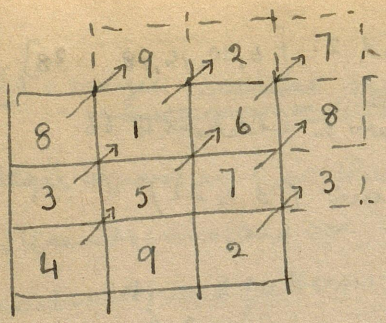


leading to simple formula $\frac{1}{2}n(n+1)$.]

Ex. 3. It is also possible to interchange rows or cols, as long as those interchanged are equidistant from the centre of the square. Thus interchange of two rows in Ex. 2 we get a magic square of constant 66

18	38	10
14	22	30
34	6	26

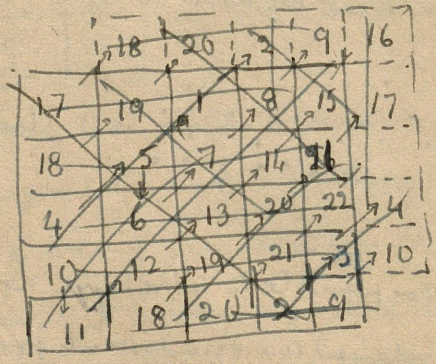
How to make up an odd magic square



Explain the ladder method or diagonal method —

At first sight this appears very complicated, but the average person finds it can be conquered in a few minutes and then it becomes simple to make up any odd magic square

How to make up an order 5 magic square:



Sum = 65 = 13 × 5. This original order 5 can now, of course, be used to manufacture hundreds more as we demonstrated with order 3 [Rather too trivial].

An even order magic square — These are worked out on an entirely different basis, and we demonstrate here an order 4.

Draw out a 4 by 4 square, and enter numbers 1 to 16 in natural order, or any other 16 consecutive numbers such as 8 – 23, 5 – 20, etc

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

To convert this into a magic square:

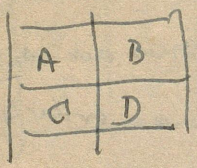
1. Change over the corner numbers on the diagonals
2. Then change over the four central numbers on the diagonal [Must be diagonals]
3. Fill in the remaining numbers in the same positions as the original square.

[1 & 2 separately are unnecessary. As remarked by Gardner, simply invert both the diagonals gives the magic square]

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

[only that adding four middle numbers gives 34 is mentioned. In fact, all these having

(98) algebraic magic squares, the more important to notice is that sums in the quadrants are equal. A mention might



have been made here of interchanges
$A \leftrightarrow D$ & $B \leftrightarrow C$]

The fact that we can use any 16 consecutive numbers allows us to make up any number of ~~ordinary~~ order 4 magic squares, or we can take the original order 4 & alter every number in it according to some rule [most ridiculous]. we could multiply by 2 and add 1 for instance.

[All trivial. The emphasis should be on normal magic squares only & the numbers 880 and 880 × 8 should have been mentioned]

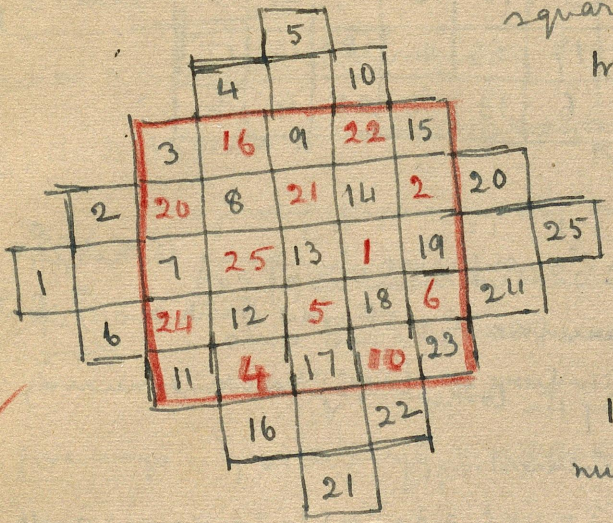
[Anyway a poor book even for the level aimed at]

((49)) Mathematical games and pastimes by A. P. Domoryad

§11. p. 97 – Magic squares.

Semi-magic if only sums of rows & cols. are equal.

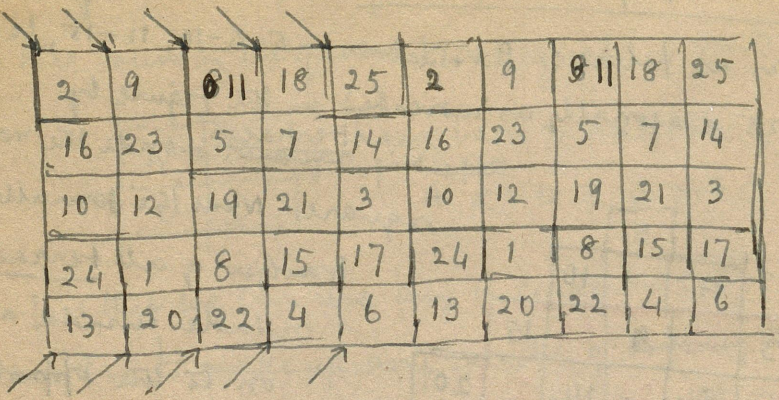
Method of terraces proposed by Bachet – For the 5-square (99) the method is as follows:– Put numbers 1–5, 6–10, 11–15, 16–20 and 21–25 diagonally as indicated in the figure by appending ~~squares~~ terraces outside the main square. Next the parallel transfer of all terraces outside, inside and on to the opposite side of the square leads to a magic square. The transferred terrace numbers inside are



indicated in red. [This appears to be a variation of the Indian diagonal method? Anyway gives a different arrangement of the square. Looks more symmetrical & ornamental

Ball's method for even squares – Looks rather complicated

Supermagic square – when an identical magic square is added to such a square on the right all sums obtained by addition in diagonal directions are the same. An example of a 5×5 square of this sort is given here.



2	9	11	18	25	2	9	11	18	25
16	23	5	7	14	16	23	5	7	14
10	12	19	21	3	10	12	19	21	3
24	1	8	15	17	24	1	8	15	17
13	20	22	4	6	13	20	22	4	6

[Obviously Domoryad's supermagic square is nothing but a Nasik square (since a 5th order Nasik exists). But the method of showing the broken diagonals continuously by this duplication is interesting]

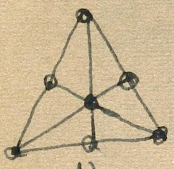
[The upward broken diagonals are (20, 8, 21, 14; 2), (22, 15, 3; 16, 9) (4, 17; 10, 23, 8) and (6; 24, 12, 5, 18). and the downward broken diagonals are (9, 5, 21, 17; 13), (8, 7, 3; 24, 20), (18, 14; 10, 1, 20) and (25; 16, 12, 8, 4) and all these are shown continuously and all add up to 65 ie Nasik.]

[But whether this square is Nasik or not, the method may be employed to show broken diagonals continuously]

magic 9^2-square – Possible to construct such a one by disintegrating into 9 magic squares of nine little squares each [Ref. Schubert – Math. Games & Pastimes, 1923, p. 106).

Magic hexagon and stars – 3 Examples given – Also the central projection of a regular dodecahedron drawn inside the circle.

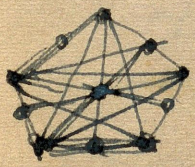
Ex x: (1) Is it possible to construct a ~~triangle~~ magic triangle in the adjoining scheme by using numbers 1 to 7?



(1)

Ans. [If the numbers used be all different, there is no soln since there are 6 directions and $\frac{1}{2} \cdot 7 \cdot 8 = 28$ is not divisible by 6]. The number 12 can be ~~constructed~~ represented in five ways only as the sum of three different nos. viz $12 = 1+4+7 = 1+5+6 = 2+3+7 = 2+4+6 = 3+4+5$. There are 3 ways only for similar sums for 11 and 13, 4 ways for 10 & 14, 3 ways for 9 and 5, etc. [I fail to see the relevance of all this]

(2) Is it possible to construct a reg. pentagon similar with nos 1 to 11?



Ans: Not possible since the number at its centre must be a part of five equal sums, each of which consists of three different numbers not exceeding 11. It is easy to verify directly that it is impossible to find such five sums.

(102) Corner Sums: – Arrange 1 to 9 in a 3-square such that the sum S of the corner squares are equal to eg. in adjoining figure

4	1	8
4	2	5
7	3	6

$3+6+5 = 5+8+1 = 1+9+4 = 4+7+3 = 14 = S.$

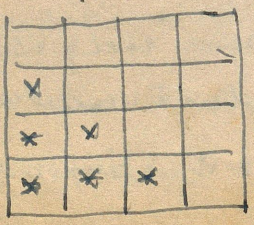
Ref: Ahrens – Altes und neues aus der Unterhaltungen mathematik.

where it is shown that the number n of substantially different solutions of the problem depends on S in the following manner

S	12	13	14	15	16	17	18
n	3	6	10	9	10	6	3

and when $S < 12$ & > 18, the problem has no solution

Ex: Try and ~~solve this~~ investigate the analogous problem for the 4^2-square where the subdivisions containing corner-squares are marked as in the adjoining figure.



[No solution given. Postpone trying to find a solution]

((43)) The 1973 Encyclopaedia Britt article – for a more detailed Examination. (p. 40 of this Note Book).

(1) Narayana Pandithas method for the odd squares – other than the Indian diagonal method. (p. 51 of this note book). This method of adding squares gives another magic square except for order 3.

$n = 3$

3	1	2
1	2	3
2	3	1

(1)

3	0	6
6	3	0
0	6	3

(2)

(1) + (2) =

6	1	8
7	5	3
2	9	4

Ladder method gives

8	1	6
3	5	7
4	9	2

which is merely reflection of (1) + (2) along vertical line ie 1st col $\longleftrightarrow$ 3rd col. (See rough p (89''))

$n = 5$ – The result of the addition method is the square (C) on p. 51 and the result of the Ladder method is on p. 23 (Heath) and they are quite different.

The same remarks obviously hold good for $n = 5$, $n = 7$.

p. 47 – Persian method for odd squares or Islamic method or Arabic method – Explained on p. 80, Rough sheet for a 5×5 square & also a 7×7 square – appears rather

(103) complicated ~~[illegible]~~ Also it is remarkable that the Muslim method of putting 1 below centre (as opposed to Hindu method of putting 1 at the middle of the top above centre) and going downward diagonally (instead of the Hindu method of going downward upward diagonally) are just opposite to the Hindu method. Also note that the 5×5 Arabic square obtained this way (Islamic) & the one on p. 51

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

Islamic (I)

14	10	1	22	18
20	11	7	3	24
21	17	13	9	5
2	13	19	15	6
8	4	25	16	12

Hindu (H)
(Narayana Pandithas)

viz. by Narayana Pandithas (not the upward diagonal one) can be obtained from each other by interchange of rows & columns in a suitable manner i.e. I & H are almost identical & does not give a new type of magic square of order 5, unlike H & the one obtained by the upward diagonal method (at least two out of the > 13,000,000 = 13×10^6 possible ones).

In fact the Ency. article calls the Islamic method the

fastest. Also at the end of the last broken move i.e. after reaching n^2 we come back to original 1 i.e. the Sufi concept that Allah is the source and destination.

But actual working out with 7 × 7 we find (rough sheet 80) that both methods are equally fast. Also the Islamic square has the same elements in the 1st. - - 7th cols as the elements in the 1st — 7th respectively of the Hindu square, so that mere interchange of rows gives makes the squares identical. Nothing new as for eg. what we get in diag method 4 additional methods of the Hindus.

Also in p. 80. rough, a mistake in taking 1 in the middle of the last but one row (instead of being immediately below the centre) and applying the Islamic method gave a semi-magic square (i.e. wrong in diagonal sums, vide Damoryan) instead of a magic square.

Trying with 1 in all possible places in middle column we find the (I) method works & gives semi-magic squares in all the four wrong cases with Sufi-Concept satisfied in two of these cases.

The (H) method fails even to work in the four wrong cases meeting with deadlock after reaching 5, 7, 13 & 19.

[Rough, p (80')]

(105) In the first case where we start with 1 as in the (I) method, use the (II) method except ~~when~~ after reaching 5, 10, 15, 20, when the (I) method* is employed (Rough p. 81), we get a correct magic square (not semi-magic), showing the efficacy of the Simla Agreement (!) [* this is called a "break move"- appears like regular war tactics!]

Chinese methods for order 5, say.

(1) p. 42 - The actual method of bordering is not explained. In fact in Rouse Ball p. 201 the remark is made "It only remains to make the sum of the numbers in each of the border lines also have the value $\frac{1}{2}n(n^2+1)$. Such an arrangement is easily made by trial and error, with a little patience a magic square of any order can be thus built up, border upon border - - This method of construction is much favoured by self-taught mathematicians".

"definite rule for arranging the numbers in the border cells have been indicated (Ref Japanese Maths. D. E. Smith & Y. Mikami, Chicago, 1914, pp. 116-120) though usually not in a precise form"

"J. Travers has proposed a simple rule

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20,
22, 23, 24, 25, 0

Question: _____

Title of the paper under discussion: _____

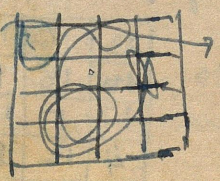
Note: Please hand over to Chairman after the Technical Session.

Signature

1, 2, 3, ... 24 (11. 29)
25
26, 27, ... 49.

	16	9	14	4
5	11	13	15	6
8	12	17	10	
7		6	6	3

indicator. The inner 3 x 3 is formed by taking



8	1	6
3	5	7
4	9	2

and adding $2n-2=8$ to each element. This is the same as taking

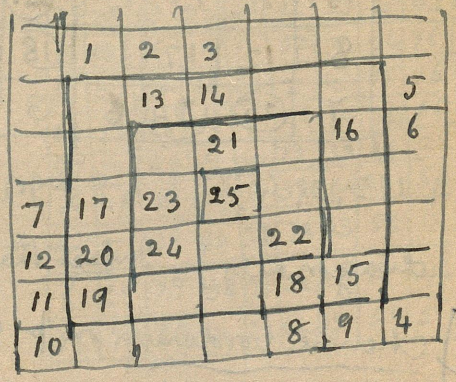
For constructing an odd square, when $n = 2m+1$. Rather than describe it in words, I illustrate its application to the case of $n = 7$ ($m = 3$). The figure on right below indicates

46	1	2	3	42	41	40
43	35	13	14	32	31	5
44	34	28	21	26	16	6
7	17	23	25	27	33	43
12	20	24	29	22	30	38
11	19	37	36	18	15	39
10	49	48	47	8	9	41

This is one bordering number

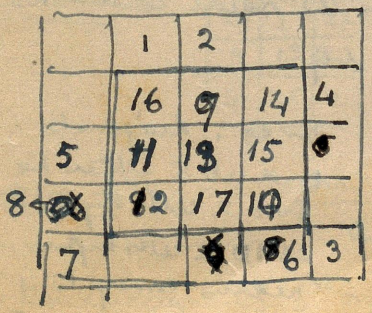
The inner 5×5 is formed

$2n-2 = 14-2 = 10$ is added to each element. Then complementary pairs are added by trial & error

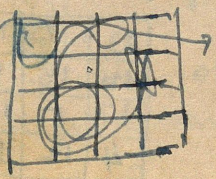


the proper positions for the numbers $1, 2, \ldots m;\ m+1;\ m+2, \ldots 2m;\ 2m+1;\ 2m+2, \ldots 3m;\ 3m+1;\ 3m+2, \ldots 4m$. The complementary numbers are easily inserted afterwards

[The same rule applied to the case of $n = 5$ ($m = 2$) would give this scheme in the borders viz



1, 2; 3; 4; 5; 6; 7; 8 as indicated. The inner 3×3 is found by taking



8	1	6
3	5	7
4	9	2

and adding $2n-2 = 8$ to each element. This is the same as taking

(107)

23 ~~19~~	1	2	20	19
22	16	9	14	4
5	11	13	15	21
8	12	17	10	18
7	25	24	6	3

the middle nine elements of the set

1, 2, 8, 9, 10 - - 17, 18, - - 25

as mentioned in the Ency. article (p. 42, fig). Now completing the putting down the complement of 1, 2, 3, 4, 5, 6, 7, 8 resp (putting complement to 7 and 3 diagonally). In fact the above figure is exactly the Chinese bordering on p. 42. The Chinese must have got it by trial & error — Also Trawer's rule itself appears arbitrary & it is not explained how it has been obtained; rule appears merely empirical] This is a case of bordered square

[Bachet's method as in Kraitchik, p. 150 for odd squares as given in Kraitchik's book p. 149. viz starting point is the cell directly ~~above the~~ over the centre. When a diagonal ends the new diagonal begins in the same columns two cells higher. Figs for 5, 7 & 9

23	6	19	2	15
10	18	1	14	22
17	5	13	21	9
4	12	25	8	16
11	24	7	20	3

Bachet n=5

For eg: diagonals end at 5 & then he goes 2 cells higher to 6. At 10, 2 cells higher means coming to 11 at bottom end at 15

to go to 16, at 20 there is no difficulty. This looks like a combination of Indian & Islamic method using only 2 break moves proper.

46	15	40	9	34	3	28
21	39	8	33	2	27	45
38	14	32	1	26	44	20
13	31	7	25	43	19	37
30	6	24	49	18	36	12
5	23	48	17	42	11	29
22	47	16	41	10	35	4

Bachet: $n = 7$

77	28	69	20	61	12	53	4	45
36	68	19	60	11	52	3	44	76
67	27	59	10	51	2	43	75	35
26	58	18	50	1	42	74	34	66
57	17	49	9	41	73	33	65	25
16	48	8	40	81	32	64	24	56
47	7	39	80	31	72	23	55	15
6	38	79	30	71	22	63	14	46
37	78	29	70	21	62	13	54	5

Bachet: $n = 9$

[written without looking into Kraitchik]

29/4/74 29/4/74

((38)) ~~Further notes from Kraitchik~~.

(2) Chinese method No. 2 (p. 42) (for $n = 5$).

(1, 2, 3, 4) (13) (22, 23 24, 25) taken & arranged as a lo-shu ie

24	1	22
3	13	23
4	25	2

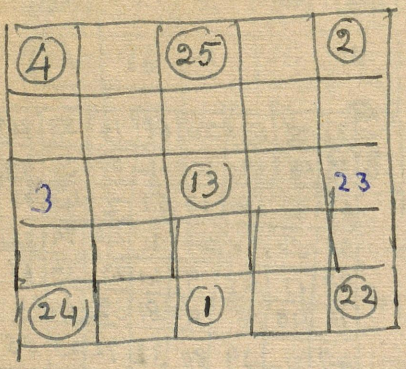
by Indian method. By r_n this →

4	25	2
3	13	23
24	1	22

[Note – This is a loshu only in the method of obtaining it. It is not a real lo-shu ie it is not a magic square, since for a magic square of 3rd order, we should have 3 sequences of 3 nos

(109) each in usual double A.P. form. Then the loshu form is obtained with figures encircled (with double lines below in Ency. article) & this gives the skeleton.

4		25		2
3		13		23
24		1		22



[It is not explained how the skeleton is to be filled by the remaining numbers since it is not described how to choose them – perhaps must be by trial & error (this appears to be characteristic of Chinese work); anyway not explained in the Ency. article.

((44)) From Dudeney's Ency. article – On bordered & Composite squares (not only mentioned on p. 62)

Ex. of 5×5 as a bordered square given by Dudeney. The Central 3×3 is a lo-shu & the bordering here appears different from the Chinese bordering of on p. 42 – What about Travers? Nor has Dudeney explained how the bordering is to be done (perhaps trial & error)

2	23	25	7	8
4	16	9	14	22
21	11	13	15	5
20	12	17	10	6
18	3	1	19	24

Frenicle's example of a 9-square, 7-, 5- & 3-squares

obtained by gradually stripping off of the borders starting with 9

16	81	79	78	77	13	12	11	2
76	28	65	62	61	26	27	18	6
75	23	36	53	51	35	30	59	7
74	24	50	40	45	38	32	58	8
9	25	33	39	41	43	49	57	73
10	60	34	44	37	42	48	22	72
14	63	52	29	31	47	46	19	68
15	64	17	20	21	56	55	54	67
80	1	3	4	5	69	70	71	66

Frenicle (1602–75)

Composite – from squares of m & n orders to construct one of mn-order.

Ex: m = n = 3 – Each ~~successive~~ subsidiary square is ~~constructed~~ successively constructed in the same order as the smallest one and each successive square is placed ~~in placed~~ in the larger square in the same order i.e. for the nos. in the

71	64	69	8	1	6	53	46	51
66	68	70	3	5	7	48	50	52
67	72	65	4	9	2	49	54	47
26	19	24	44	37	42	62	55	60
21	23	25	39	41	43	57	59	61
22	27	20	40	45	38	58	63	56
35	28	33	80	73	78	17	10	15
30	32	34	75	77	79	12	14	16
31	36	29	76	81	74	13	18	11

8	1	6
3	5	7
4	9	2

bor-loo-shu here read 1st, 2nd – 9th subsidiary squares.

(111) A more difficult but very elegant type of square is that of two degrees as below. Here we have a magic square of the 8th order with the constant 260 & every square of four cells summing up to 130. If for every number we substitute in it's allotted place in this square then it will also be entirely magic with the constant 11180. A Vasili square of this kind and order has been constructed. No square of two degrees is possible for any order smaller than 8. [No explanation given]

7	53	41	27	2	52	48	30
12	58	38	24	13	63	35	17
51	1	29	47	54	8	28	42
64	14	18	36	57	11	23	37
25	43	55	5	32	45	50	4
22	40	60	10	19	33	61	15
45	31	3	49	44	26	6	56
34	20	16	62	39	21	9	59

A square of two degrees (ie bimagic?)

[For 3rd order square general type, Dudeney's method of working for eg.

1 2 3
7 8 9
13 14 15

saying horizontal differences are all alike (equal to 1) and the vertical differences are all alike (equal to 6) but not necessarily the same as the horizontal is quite simple for an elementary book]

((43)) Continued — The method used of having white dividing lines separating individual cells & black lines when one wants to emphasise (as for eg. in composite squares etc) is quite elegant ✓

(3) Chiuso composite magic square of ninth order (p. 43)

31	76	13	36	81	18	29	74	11
22	40	58	27	45	63	20	38	56
67	4	49	72	9	54	65	2	47
30	75	12	32	77	14	34	79	16
21	39	57	23	41	59	25	43	61
66	3	48	68	5	50	70	7	52
35	80	17	28	73	10	33	78	15
26	44	62	19	37	55	24	42	60
71	8	53	64	1	46	69	6	51

They wrote the sequence 1 to 81 in nine parallel columns, to form a natural square; then they took each horizontal row of nine numbers to form a secondary magic square of the third order; finally taking as a guide the lowest number of each small square, they set out the latter in the pattern of the Lo-shu to make a single huge one.

[For the 9 × 9, Dudeney's method for eg gives the last lo-shu formed from 73, 74 . . . 81 as

80	73	78
75	77	79
76	81	74

& this is the 9th & is placed in the position 9 as given by the fundamental lo-shu viz

8	1	6
3	5	7
4	9	2

& similarly for the other lo-shus — In the Chinese method the lo-shu formed by nos. in the 1st horizontal row is

1	10	19	28	37	46	55	64	73
2	11	20	29	38	47	56	65	74
3	12	21	30	39	48	57	66	75
4	13	22	31	40	49	58	67	76
5	14	23	32	41	50	59	68	77
6	15	24	33	42	51	60	69	78
7	16	25	34	43	52	61	70	79
8	17	26	35	44	53	62	71	80
9	18	27	36	45	54	63	72	81

64	1	46
19	37	55
28	73	10

or after reflection in r_h

28	73	10
19	37	55
64	1	46

and this is placed in the posn 1 of the fundamental lo-shu in the Chinese form (i.e. after r_h of the Indian form) — Anyway both methods are equally elegant and equivalent.

4	9	2
3	5	7
8	1	6

In Dudeney's method (p. 110) the magic constants for the squares formed by the successive 9 sets of 9 numbers are

15, Using the formula in general case viz

$\frac{1}{2} n \{ 2a + d(n^2-1) \}$ ie $\frac{1}{2} \cdot 3 \{ 2a + 8 \}$ $(d=1)$. $n = 3$.

$= 3(a+4)$, with $a = 1, 10, \cancel{20, 30, 40, 50, 60, 70, 80}$ 19, 28, 37, 46, 55, 64 resp.

ie 15, ~~42, 72, 102~~, 132, 162, 192, ~~222~~, 252, ~~282~~

15, 42, 69, 96, 123, 150, 177, 204, 231. (diff = 27)

Sum of magic constants of 8th, 1st & 6th are $204 + 15 + 150 = 369$

of 3rd, 5th & 7th are $69 + 123 + 177 = 369$

4th, 9th & 2nd are $96 + 231 + 42 = 369$

For cols sum of 8th, 3rd & 4th = 369. — Best thing would be to ~~take~~ put these in the square form

204	15	150
69	123	177
96	231	42

with 15, ~~42~~ (15, 42, 69, 96, 123, 150, 177, 204, 231) (magic square of magic constants) as elements ~~having~~ C.D = 27

which itself is magic with constant 369 ie same as for a 9th order square viz $\frac{1}{2}(9^3 + 9) = \frac{1}{2}(729 + 9) = 369$.

For the Chinese form (p. 112) the subsquare in which 1 appears has magic const (see table, p. 113) = ~~$\frac{1}{2}\{ 6 \cdot 3 \{ a + 4 \}$~~

$= \frac{1}{2} \cdot 3 \{ 2 + 9 \cdot 8 \} = 111 \; (37 \times 3)$

For 2 appearing square $= \frac{1}{2} \cdot 3 \{ 4 + 9 \cdot 8 \} = 3(38) = 114$ & so on

ie in form of loshu (111, 114, 117, . . . with C.D = 3).

4	9	2
3	5	7
8	1	6

we get ~~magic~~ the square of magic consts as

120	135	114
117	123	129
132	111	126

which is magic with $k = 369$

(115)
1/5/74

In addition to the (H), (I) & (C) in Chinese trial & error bordering methods (2) based on the Lo-shu, for odd orders, we also have Bachet method — thus we have altogether six methods, two Hindu (one continuous & one adding 2 squares) two Chinese (really one by bordering the Lo-shu by trial & error & the elegant one for 9 x 9 also based on the Lo-shu), one Islamic (Continuous) & one Western one to Bachet (also continuous) — This last method called (B) appears to be a sort of a combination of the (I) and (H) - methods. Unlike both (I) & (H), B starts with 1 in the cell above the centre, then applies the (H)-method, except that when a diagonal ends inside the square, the new diagonal begins in the same column two cells higher (instead of coming down one cell lower in the same column as in H, and instead of going down two cells lower as in I) in the I-sense. In all the three cases the diagonals end at the numbers $n, 2n, \ldots n(n-1)$, but the (H) method appears most unambiguous as regards the next cell to start with after the diagonal ends.

To make a comparison of these four methods it is best to take a 9 x 9 square. [Vide Rough, pp. 81 - 82]

For the 4 x 4 square by Chinese method, the method of splitting vertically & exchanging halves holds good only for algebraic squares. Taking for example a
(Sums of A, B, C, D all = 34)

derivative symmetric square (i) [Eu·p·90]; this gives

1	4	13	16
8	15	2	9
14	5	12	3
11	10	7	6

(i)

→

13	16	1	4
2	9	8	15
12	3	14	5
7	6	11	10

(ii)

(ii) by splitting (i) vertically & exchanging halves, and (ii) is not a magic square (diagonal sums are 46 and 22) [Rough(82')]

Chinese method for $n = 8$: (p·44) – This method carried out on p(82') rough sheet gives a square different from that of Heath (Mathematics) (p·26 of this note book) – Chinese method is quite elegant.

Islamic methods for even squares.

((43)) 2|5|74 · al– Buni's bordered squares for even numbers.

The core is a magic square of the fourth order, using the first eight and last eight numbers of the sequence. Placing the first eight digits in the core, he set out the remaining numbers from the first half of the sequence in an elaborate zigzagging pattern to fill half of each bordering frame to the outer rim. Then taking the numbers in the second half of the sequence, he worked back, repeating the pattern in reverse, to the core of the square where

(117) he placed the final eight numbers, thus located as in fig below.

16	33	18	31	20	29	28
39	7	42	9	40	27	11
12	46	2	47	26	4	38
37	5	49	25	1	45	13
14	44	24	3	48	6	36
35	23	8	41	10	43	15
22	17	32	19	30	21	34

26	45	36	27	40	23	44	19
34	14	55	50	13	54	9	31
33	17	7	62	1	60	48	32
30	47	2	59	8	61	18	35
28	49	64	5	58	3	16	37
41	12	57	4	63	6	53	21
22	56	10	15	52	11	51	43
46	20	29	38	25	42	21	39

n = 8 (Suppl. p. 116)

Bordered magic square for $n=7$

[core a magic square of 3rd order, made with the first three nos. of the sequence, the three middle ones, and the last three. Then he took the other nos. and worked outward from the core to the outermost rim, following a regular pattern that filled half of each frame on the way. Finally he returned by the reverse of the same pattern, to complete the frames back to the inner core (sense of continuity as distinguished from the Chinese static method) [what is this wonderful pattern?]

For singly even numbers the irregularities presented were overcome by changing the pattern for numbering in the alternate frames. The effectiveness of this solution is especially apparent in the squares for higher numbers (Late Middle Eastern scholars

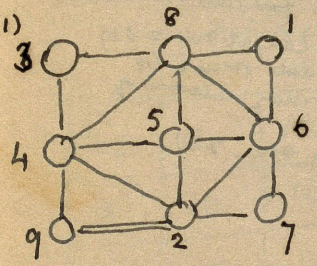
published examples up to 30 × 30). This remains the most convenient simple system [But what is this ?] by which magic squares of any even order can be constructed. Since the method for making bordered squares for odd nos. is closely related, this is the closest approach yet known to a general system for constructing all magic squares. — — . . . — still remaining valid in the 20th century.

((50)) Play Mathematics by Harry Langman (1962)

Chap VI – Magic number arrangements

p. 76 – other magic arrays

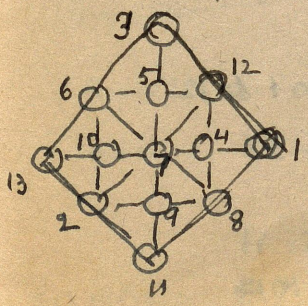
(1)



Distribute numbers from 1 to 9 one in each ○ so that every four forming a square shall have the same sum.

Solⁿ : Nos as marked in the figure
[After a bit of literal work]

(2) Arrange 1 to 13 in array below such that every four forming a square shall have the same sum.



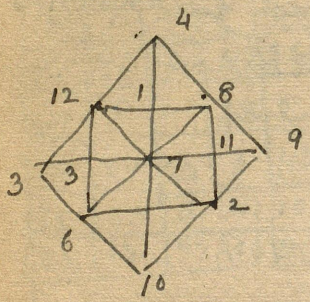
There are ten squares here

As shown with sum of each square = 28

[Again after bit of literal work]

(119) (2') Same array with conds that (a) sum of every three nos in alignment to be same (b) each set of five in alignment is same

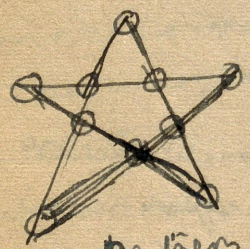
Soln – As below taking out ♂ – gains after lot of literal work.



(3) cross pentagons (Smarahara)

Every four in a st. line should have the same sum.

Soln (a) For numbers 1 to 10, soln is impossible [Long proof]

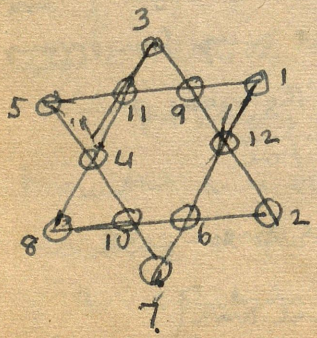


cross pentagon

(b) There exist solns for non-consecutive integers (min. largest no = 12) (line sum = 69)

(c) Soln with minimum sum of all primes (The line sum 72)

(4) cross hexagons.



(a) Sum of every four in alignment to be the same.

Soln – as shown with nos 1 to 12

Magic const = 26

Another soln is

7

9 1 5 11

6 10

12 8 2 4 1 → 3.

Many solns – Also replace each no by its complement
ie $x \longrightarrow (13-x)$. ~~$\frac{1}{2}n(n+1) = \frac{1}{2}$~~ (

(b) Six triads of numbers forming the small outer Δs to have the same sum

Many solns – Also prime no. soln given

Examples 1 to 107 with different arrays. preceded by comments on exercises.

[To choose some of the easier ones if necessary later on]

4|5|74

((51)) General Mathematics - A problem solving approach (From CC.B)
by Kinney, Ruble & Brown

10	2	12
6	10	14
8	18	4

Gives this as ex: to define a magic square
(this is merely Co-Sec x 2)

Gives the diagonal method ~~to~~ turns this ↑ ↑ ↑

(11 (121) Ex. 2. Make a magic square with even nos. beginning with 10 and ending with 26.

Ex. 3 = Make one for $n = 5$.

Ex. 4 – Check if magic const $= n \times c$

(Stupid book)

((52)). Introduction to modern elementary mathematics by Ladis D. Kovach (1966)

p. 199 – Magic squares.

3 × 3 and 4 × 4 given – A famous magic square is one called "Nasik" which is carved in Sanskrit characters on the gate of a fort in India

15	10	3	6
4	5	16	9
14	11	2	7
1	8	13	12

This has all the properties of the usual 4 × 4, and also all adjacent squares add to 34, for eg: 15 + 10 + 5 + 4 – 5 + 16 + 2 + 11 = 3 + 6 + 9 + 16 all add to 34. Can you find the 25 different ways in which this square gives a sum of 34?

[rows, cols & diginels = 10 ways, adjacent squars = 9 ways
broken diginals = 6 ways & total = 25].

Another unusual magic square is the following one:

1	23	137	223	263
167	229	191	7	53
197	37	83	173	157
89	101	163	227	67
193	257	73	17	107

[illegible] not only are all the entries, except 1, prime numbers, but the sum 647 is also a prime

[in increasing order nos. are 1, 7, 17, 23, 37, 53, 67, 73, 83, 89, 101, 107, 137, 157, 163, 167, 173, 191, 193, 197, 223, 227, 229, 257, 263 - How about the formation?]

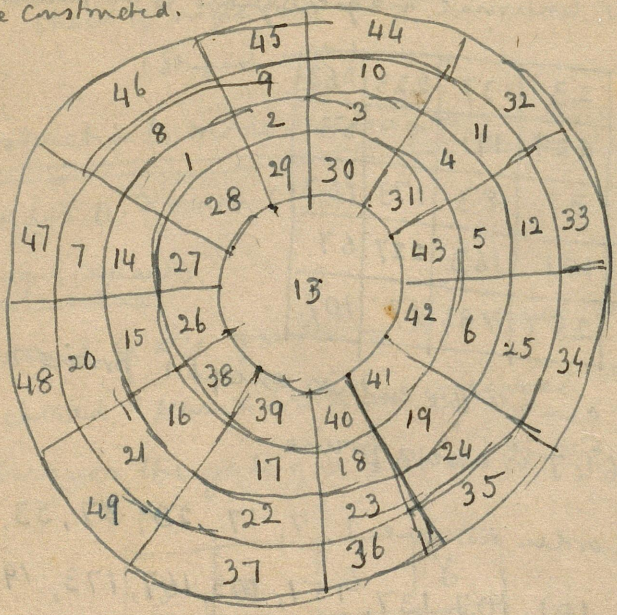
A variation of the magic square is the following magic circle: [next P]

Here the sum of the numbers along any one of the 12 diameters is 215. But even more remarkable is the fact that this sum will remain unchanged if any one (or more) of the rings is revolved to any position

Ref. to magic squares & W. S. Andrews - Magic Sqs & Cubes
Dover Pub. Inc. N. Y. 1960

(123) 2. A. A. Merrill – Math. Excursions – Dover Pub., Inc. N. Y. 1967. pp 67-76

Latter gives a simple explanation of how magic squares can be constructed.



[Nothing very extraordinary ?]

((53)) Alte und neue mathematische Spiele by G. Kowalewski (1930)

– Nothing about magic squares

((38)) Kraitchik – further notes

Lattices & magic squares, pp. 152–167:

Regular squares of odd order p ie $1, \ldots p^2$ (p a prime).

21	22	23	24	25
16	17	18	19	20
11	12	13	14	15
6	7	8	9	10
1	2	3	4	5

$p = 5$

In lattice notation $(1,1)$ is 1 & for eg $(5,1)$ is 5

$(4,3)$ is 14, $(5,5)$ is 25, $(2,5)$ is 22 & so on

For lattice of two dimensions with

$$(x,y) = (a,b) + r(t,u) + s(v,w)$$

Condns (i) $tw - uv$ not divisible by p

(ii) p does not divide t, u, v or w.

(iii) $t+u$, $t-u$, $v+w$, $v-w$ are not divisible by p.

If these condns be satisfied, the ~~magic~~ square given by the lattice points is ~~primarily~~ at least semi-magic, and if ~~but~~ also magic pan magic.

Eg: $(x,y) = (1,1) + r(2,1) + s(1,2) + \cancel{(a,b)}$

Putting $r = 0, 1, 2, 3, 4$; $s = 0, 1, 2, 3, 4$ } the 25 values of (x,y) give the 25 magic nos

Case $s = 0$, $r = 0, 1, 2, 3, 4$. $(x,y) = (1,1), (3,2), (5,3), (7,4), (9,5)$

ie $(1,1), (3,2), (5,3), (2,4), (4,5)$

ie $(1, 8, 15, 17, 24)$ (last row)

20	22	4	6	13
9	11	18	25	2
23	5	7	14	26
12	19	21	3	10
1	8	15	17	24

Since $s = 1$, $r = 0, 1, 2, 3, 4$ give

$(x,y) = (2,3), (4,4), (6,5), (3,6), (5,7)$

ie $(2,3), (4,4), (1,5), (3,1), (5,2)$

ie $(12, 19, 21, 3, 10)$ ie 2nd row from bottom

& so on, we get the square which is also pannapi since (iii) is also satisfied. The elements of the diagonals are given :-

For the upper diagonals : $(t+v, u+w)$ is $(3,3)$ since here (t,u) & (v,w) are $(2,1)$ & $(1,2)$ resply. ie taking a

[one-dimensional lattice for the main upper diagonal,

$$(x,y) = (1,1) + r(3,3)$$

with $r = 0, 1, 2, 3, 4$, $(x,y) = (1,1), (4,4), (7,7)$ or $(2,2)$ $(10,10)$, $(13,13)$ or $(5,5)$ or $(3,3)$

ie $(1, 19, 7, 25, 13)$

[For the lower main diagonal $(t-v, u-w)$ is $(1,-1)$

~~$(x,y) = (1,1) + r(1,-1)$~~ – not clear]

~~ie $(x,y) =$~~]

Taking $(x,y) = (1,1) + r(3,3) + s(1,-1)$

with $(r=0, s=0,-4)$, $(r=1, s=0,-4)$ etc we get the four lower diagonals. with $(s=0, r=0,-4)$, $(s=1, r=0,-4)$ etc we get the 5 upper diagonals (Rough, p. 83).

---

p. 159 – Similar investigation for $n=7$ with lattice empty $(x,y) = (1,1) + r(1,1) + s(2,-2)$

[$tw - uv = 4$, t, u, v, w not divisible by 7

$t+u = 2$, $t-u = 0$, $v+w = 0$, $v-w = 4$

[obvious 0 is not considered divisible by 7!]

$175 = K$]

Taking values for $r = 0, \ldots 6$, $s = 0, 1 \ldots 6$ we get the

7×7 panmagic square (Fig. 35, p. 159, Kraitchik) with magic constant 175.

20	28	29	37	45	4	12
32	40	48	7	8	16	24
44	3	11	19	27	35	36
14	15	23	31	39	47	6
26	34	42	43	2	10	18
38	46	5	13	21	22	30
1	9	17	25	33	41	49

$(t+v, u+w)$ is $(3,-1)$

$(t-v, u-w)$ is $(-1,3)$

So the diagonals are generated by

$(x,y) = (1,1) + r(3,-1) + s(-1,3)$

for eg: $s=0$, $r = 0, 1, \ldots 6$ give

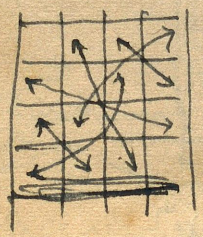
$(x,y) = (1,1), (4,0), (7,-1), (10,-2), (13,-3, (16,-4), (19,-5)$

ie $(1,1), (4,7), (7,6), (3,5), (6,4), (2,3), (5,2)$

ie $(1, 46, 42, 31, 27, 16, 12)$ main diagonal

and so on. [Another Nasik of order 7 on p. 92]

11/5/74 Regarding the literal form of the 4×4 square in Kraitchik the following simple transformation of the 3 addition tables to get magic squares is quite simple (Indicated by the arrows.



This can be worked for all the three addition tables & found to give correct results

[Vide rough, p. 83']

 13|5|74

((54)) The fascinating numbers by W.J. Reichmann (1963, Methuen)

Chapters on perfect numbers & oddities, Fallacies etc

Chapter 15 – Magic squares.

A magic square is composed of a number of integers so arranged within the square formation that the sum of the integers in each row, each col. & each diagonal is identical – To deal comprehensively a whole volume required – only simple principles involved considered

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

(A)

15	10	3	6
4	5	16	9
14	11	2	7
1	8	13	12

(B)

(i) magic const = 34 in (A) & (B)

(ii) total each smaller square = 34

(iii) In (B) among any 4 numbers in square formation total 34. This is not found in (A)

[(B) is Nasik & (A) semi-Nasik]

– formula $\frac{1}{2}n(n^2+1)$ for magic const – Heath's Constn for 4 × 4 square – Indian diagonal method for odd squares – For a square of order 5 gives an alternative method as follows.

3	5	4	1	2
1	2	3	5	4
5	4	1	2	3
2	3	5	4	1
4	1	2	3	5

(A)

0	10	20	5	15
20	5	15	0	10
15	0	10	20	5
10	20	5	15	0
5	15	0	10	20

(B)

Form (A) with any order of (1, 2, 3, 4, 5) to start with; then in 2nd row 4 & 5th elements of 1st row & then 1st, 2nd & 3rd elements & so on for all rows – Square (B) formed by 0, 5, 10, 15 & 20 placed in any order in 1st row & then constructed as in (A) but with 3rd, 4th & 5th elements [instead of 4th & 5th elements as in (A)]. (A) + (B) → (C)

3	15	24	6	17
21	7	18	5	14
20	4	11	22	8
12	23	10	19	1
9	16	2	13	25

(C)

Note that (A), (B) & (C) are all magic with constants 15 [mis. repeated] 50 & 65 resp.

~~(15 + 50~~ [See Rough p. 89' – 90]

Ramanujan's method

[This ~~resembles~~ is Narayana's method – ~~to be tested~~ into slightly modified] Vide p. 51

(129) – Example of bordered square vis (D)

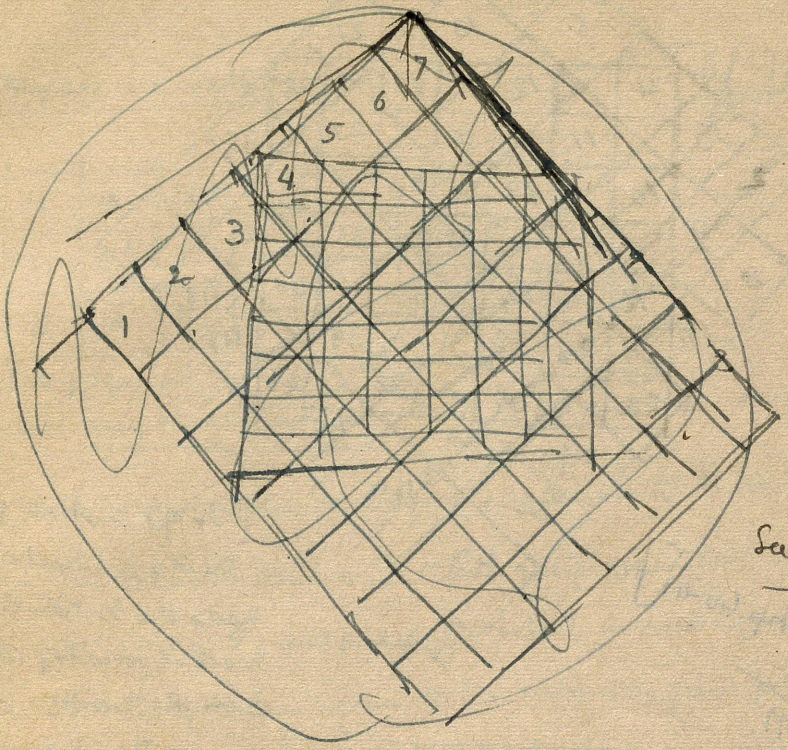
2	23	25	7	8
4	15	9	14	22
21	11	13	15	5
20	12	17	10	6
18	3	1	19	24

(D)

The inner square is formed by the nine central digits [No cuph. Same as Chinese method of bordering]

– Another method of using a preliminary square to obtain a final square which shall be magic, is to construct a larger square balanced on one of its edges, with the framework of the required square drawn inside the first square so that the sides of the first square are bisected by the corners of the smaller square. Numbers are then written consecutively in the first square. Some of the integers automatically appear in the smaller square (Fig (E)), because the subsquare of the two main squares partly coincide in places. These numbers are left as they are, but the numbers outside are transferred to positions within the square. Each such number is, in fact, moved to the furthest vacant subsquare

along the same row or column of the larger square,* numbers nearest to the central square being moved first



Bad figure

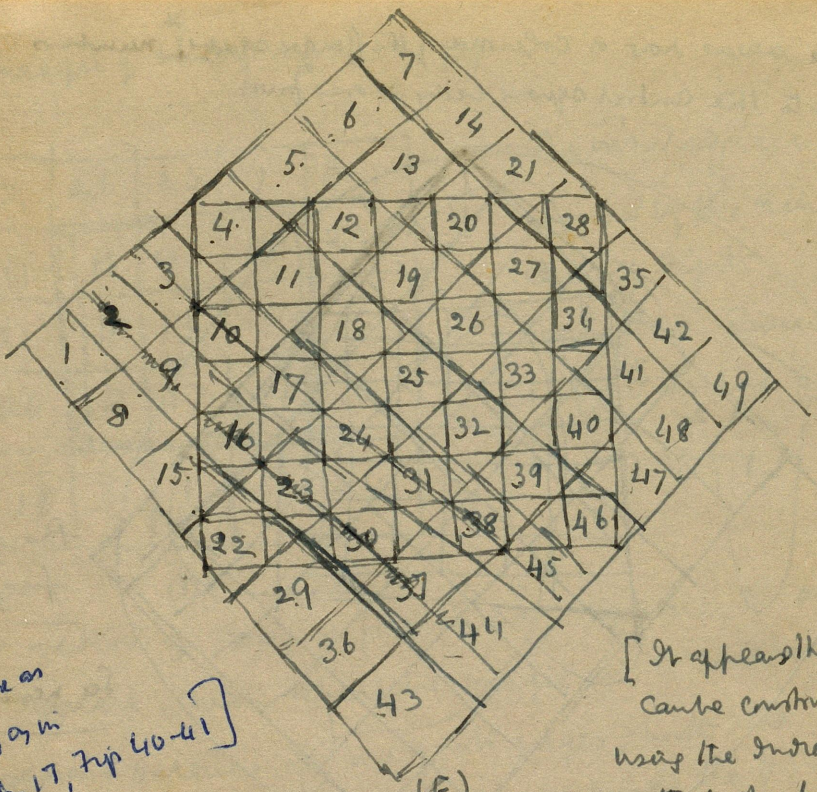
See next page

[This appears to be new apparently. Perhaps it is related to the Chinese, Islamic or Hurion methods given in the Encyclopaedia article]

The finial square as in Fig (F)

[P. T-O

[* not quite clear]



(E)

[This is done as the method as in Andrews p. 17, Fig 40-41]

Bim
23/11/77

4	29	12	37	20	45	28
35	11	36	19	44	27	3
10	42	18	43	26	2	34
41	17	49	25	1	33	9
16	48	24	7	32	8	40
47	23	6	31	14	39	15
22	5	30	13	38	21	46

(F)

[It appears that F can be constructed by using the Indian diagonal method starting with 1 in the position given, but the "break moves" at 7, 14, 21 & 28, 35, & 42 being "2 places to the right in same row" with vertical edges cylindrically closed] [To be looked up more carefully.]

– this example to be shown with

1	2	3
7	8	9
13	14	15

– Key square method. ($n = 3$) (Literal form)

A	B	C
D	E	F
G	H	K

(Key square) (α)

66	47	70
65	61	57
52	75	56

(β)

Select any three numbers, say 5, 9 and 47. neglect, say 47 is placed in the subsquare relative to B in the key square. Add this same number to either of the other two selected numbers, (say 5) and place the total in subsquare G. Add the second number again to the number first placed in G and place the new total in F. This gives us (β) with three numbers. We now treat each of these numbers in turn. Add the third number (9) to the first number (47) and place the total in the subsquare K. Add 9 again & write the result in D. Again add 9 ~~starting~~ in C to the number in G & place result in E. Again add 9 & place in C. Next add 9 to number in F (57) & place result in A. Add 9 again & place result in H. All subsquares are filled & we have (β)

(133) The reason why this must be a magic square is best shown by indicating how each individual number is obtained

* [Same as the form in Trautschill (Bk. II, p. 171) with A+a+b = m, x = a, y = b]

47+5 +5+9	47	47+5 +9+9
47+ 9+9	47+ 5+9	47+ 5+5
47 +5	47+5 +5+9	47+9 +9

*

A +2a+b	A	A+a+2b
A+2b	A+a+b	A+2a
A+a	A +2a +2b	A+b

(Literal form)

– more reference to doubly magic squares. Total 3 (A+a+b)

Subtracting & multiplying magic squares.

In forming a constant 'total' result results for each row, col or diagonal by deducting its first digit from the second, and the result from the third digit & so on. An example of such a square is

2	1	4
3	5	7
6	9	8

& in this square the constant is 5

4 – (1 – 2) = 5
7 – (5 – 3) = 5
8 – (9 – 6) = 5
} rows

6 – (3 – 2) = 5
9 – (5 – 1) = 5
8 – (7 – 4) = 5
} cols

8 – (5 – 2) = 5
6 – (5 – 4) = 5
} diags

12	1	18
9	6	4
2	36	3

Ex of multiplying square
12 × 1 × 18 = 9 × 6 × 4
= 2 × 36 × 3 & etc.
= 216.

– Ref. to music no. magic squares.

((38)) Kraitchik continued. (from p. 126 of this note book)

It is clear that distinct lattices may generate equivalent squares. For eg: the squares generated by

$$(x,y) = (a,b) + r(b,u) + s(v,w).$$
$$(x,y) = (b,a) + r(u,t) + s(w,v)$$

will be equivalent - - - - By such means one can determine how many non-equivalent magic squares of a particular prime order p there are. Below are given the four non-equivalent panmagic squares of order 5 from which all others can be derived by certain permutations of rows & columns. (?)

18	24	5	6	12
10	11	17	23	4
22	3	9	15	16
14	20	21	2	8
1	7	13	19	25

(A)

(-2,1) R.M
(1,0) D.M

15	18	21	4	7
24	2	10	13	16
8	11	19	22	5
17	25	3	6	14
1	9	12	20	23

(B)

(1,-2) R.M
(-1,-1) D.M

20	22	4	6	13
9	11	18	25	2
23	5	7	14	16
12	19	21	3	10
1	8	15	17	24

(C)

(-1,-2) R.M
(2,2) D.M

13	17	21	5	9
25	4	8	12	16
7	11	20	24	3
19	23	2	6	15
1	10	14	18	22

(D)

(2,1) R.M
(0,2) D.M

(135) A panmagic square of order 35 can be generated by $(x, y) = (1,1) + r(2,1) + s(1,2)$. Square is panmagic because its diagonals are generated by $(3,3)$, $(1,-1)$.

There are no regular panmagic squares of order $3k$.

Border squares: To form a 6×6 from a 4×4. ~~Add Form a 4 x 4 form~~ (Given in Bk. 11, p. 175) {This appears rather arbitrary. Kraitchik's method ~~says~~ (Bk. 11, p. 174 bottom): It is shown in Rough pp. 86–88 that the choices are not unique by considering three cases. The number of possible bordered squares of 6th order is given (Rough, p. 88 as 142,318,080 while Dudeney in Ency. article (Bk. 12, p. 62) gives number as 507,705,600. It is not clear how this is obtained.

The ~~squ~~ border squares of orders 6, 8, 10 & 12 in Kraitchik (Bk. 11, p. 176) are more complicated ~~& can be~~ but can be easily investigated. ie in 1 – – 144 choose the middle 16 ie [$\overbrace{1 \text{ to } 22}^{22}$, $\overbrace{23 \text{ to } 40}^{18}$, $\overbrace{41 \text{ to } 54}^{14}$, $\overbrace{55 \text{ to } 64}^{10}$, $\underbrace{65 \text{ to } 80}_{16}$, $\underbrace{81 \text{ to } 90}_{10}$, $\underbrace{91 \text{ to } 104}_{14}$, $\underbrace{105 \text{ to } 122}_{18}$, $\underbrace{123 \text{ to } 144}_{22}$]

Put 65 to 80 in a 4×4; then border with 55 to 64 & 81 to 90 for a 6×6; then border with 41 to 54 & 91 to 104 for a 8×8; then with

2 3 to 40 & 105 to 122 for a 10×10 & finally with border with

1 to 22 & 123 to 144 for a 12×12. [See Rough. pp. (88)–(88')].

– The work is tedious but quite simple. Of course incorrects exists,

but one can be found [Rough pp. 88–88']

[K. p. 168] Squares formed so far consist of consecutive integers

but that is not necessary. We give below 2 border squares whose

interior squares do not have consecutive numbers

7	32	6	18	35	13
1	29	10	9	26	36
34	20	15	16	23	3
33	14	21	22	17	4
12	11	28	27	8	25
24	5	31	19	2	30

7	36	8	11	31	18
9	2	33	34	5	28
27	17	22	21	14	10
25	23	16	15	20	12
24	32	3	4	35	13
19	1	29	26	6	30

Inner square with

$$\begin{pmatrix} 8, 9, 10, 11, \\ 14, 15, 16, 17, \\ 20, 21, 22, 23 \\ 26, 27, 28, 29 \end{pmatrix}$$

& $K = \frac{1}{4}(38 + 62 + 86 + 110) = \frac{1}{4}(296) = 74$

and the bordered 6×6 has nos from 1 to 36 i.e. $K = 111$. A method of bordering is to add complementary pairs. The interior 4×4 can be found using the literal form

Inner square with

$$\begin{pmatrix} 2, 3, 4, 5 \\ 14, 15, 16, 17 \\ 20, 21, 22, 23 \\ 32, 33, 34, 35 \end{pmatrix}$$

using magic square using literal form

& leads to bordered 6×6 with

which is normal & has $K = 111$

The above inner has $K = \frac{1}{4}(14 + 62 + 86 + 134) = \frac{1}{4}(296) = 74$

& hence bordering using complements.

one can also border squares of odd order. Below is an an example of a 9×9 obtained by successively bordering a 3×3.

(Vide Bk. II, p. 177).

Necessary to follow all possible ways of bordering the normal magic square of order 3 are obtained from the 10 borders given in the figures below

3, 22, 21, 18, 1		
7		19
24		2
6		20
25, 4, 5, 8, 23		

3, 22, 20, 19, 1		
8		18
24		2
5		21
25, 4, 6, 7, 23		

6, 24, 23, 8, 4		
25		1
7		19
5		21
22, 2, 3, 18, 20		

6, 25, 23, 7, 4		
24		2
5		21
8		18
22, 1, 3, 19, 20		

7, 25, 24, 6, 3		
22		4
8		18
5		21
23, 1, 2, 20, 19		

7, 22, 25, 8, 3		
6		20
5		21
24		2
23, 4, 1, 18, 19		

7, 6, 23, 24, 5		
25		1
8		18
4		22
21, 20, 3, 2, 19		

7, 25, 24, 4, 5		
23		3
8		18
6		20
21, 1, 2, 22, 19,		

8, 25, 23, 7, 2		
22		4
5		21
6		20
24, 1, 3, 19, 18		

8, 24, 23, 4, 6		
25		1
5		21
7		19
20, 2, 3, 22, 18		

by (a) permuting the middle three rows, (b) permuting the middle three columns, and (c) subjecting each border to the

1 3 5 7

4	9	2
3	5	7
8	1	6

8	1	6
3	5	7
4	9	2

and Δ Bachet [illegible]

1 15 8 10 10

12 6 13 3

14 4 11 5

7 9 2 16

12	14	1	7
15	6	4	9
8	13	11	2
10	3	5	16

3	16	9	22	15
20	8	21	14	2
7	25	13	1	19
24	12	5	18	6
11	4	17	10	23

15	2	19	6	23
22	14	1	18	10
9	21	13	5	17
16	8	25	12	4
3	20	7	24	11

3 . 576 . 2

3456 magic square

3 30,800

4

7 9 2 16

14 4 11 5

12 6 13 3

1 15 8 10

8 4

4 3

3456 × 8

27648

330

11	24	7	20	3
4	12	25	8	16
17	5	13	21	9
10	18	1	14	22
23	6	19	2	15

$$\frac{10^{10}}{10^{2}}$$

10^{8} days.

4×10^{2}

$$\frac{\cancel{10^{8}}\ 10^{6} \times \cancel{10^{2}}}{\cancel{4} \times \cancel{10^{2}}}$$

25×10000

250,000

250,000 years

2500 generations

25	36	2	13	24	100
34	10	11	22	23	100
8	9	20	31	32	100
17	18	29	30	6	100
16	27	38	4	15	100
100	100	100	100	100	100

18	7	21	15	4
5	19	8	22	11
12	1	20	9	23
24	13	2	16	10
6	25	14	3	17

	20	8		4
5		16	9	12
13	1	21	17	10
6	14	2	22	18
19	7	15	3	11

8	7	3	11	
6		2		10
	1		9	
5		8	9	
	7	8		4

8 rotations & reflections. Since there is just one square of order 3, we get $10 \cdot (3!)^2 \cdot 8 = 2880$ non-equivalent border squares of order 5.

These are not all, however, since these include only those border squares of order 5 formed from a magic square of order 3 composed of the numbers 9 to 17, the middle nine of the first 25 numbers. Violle has completed the study of the question by showing that there are 26 different magic squares that can be formed with 9 of the first 25 numbers, and by finding for each case the total number of possible borders. He finds $594 \cdot 288 = 171,072$ squares in all.

((45)) Ramanujan's note books (continued) - We have remarked on p. 75 that Ramanujan's work is disappointing. It is really not so as the following considerations show :-

Starting with method of construction given by Ramanujan for squares of odd order (pp. 73-74), we find on using the form for the 5×5 on p. 74) and on taking as second two squares in (A, B ... F) and (a, b, ... e) (using Kraitchik's a, ... e instead of R's P, Q, ... T), we find the meaning of the statement on p. 73, specially looking at the method of construction of 5×5 squares given in Reichmann's Fascination of numbers ((54)) [p. 128, Bk. 12]. Also taking into consideration the four equivalent non-equivalent

nasik squares of Kraitchik [p. 134, Bk. 12] we find that Ramanujan's general literal form for 5×5 corresponds to case (B) of Kraitchik taking A = 0, B = 5, C = 10, D = 15, E = 20, and a = 1, b = 2, c = 3, d = 4, e = 5, except for interchanges of rows & columns. [Vide rough sheets pp. 89 – .. for further work on Ramanujan's methods] Ramanujan's scheme

A B C D E F G H . . P Q R S T U V W – .
A B C D E – R S T U V W —
A B – T U – –

is equivalent to taking in the 5×5 case the two squares (hint given by Reichmann's book)

A	B	C	D	E
D	E	A	B	C
B	C	D	E	A
E	A	B	C	D
C	D	E	A	B

&

a	b	c	d	e
c	d	e	a	b
e	a	b	c	d
b	c	d	e	a
d	e	a	b	c

Both these are magic in the narrow sense since there is repetition

& adding

A+a	B+b	C+c	D+d	E+e
D+c	E+d	A+e	B+a	C+b
B+e	C+a	D+b	E+c	A+d
E+b	A+c	B+d	C+e	D+a
C+d	D+e	E+a	A+b	B+c

Which is magic, but different from that given by Ramanujan p. 74 (this book) where A+a, B+b, — are in the diagonal. This as well as that form are both nasik. Rather surprising in view of Kraitchik giving

only 4 non-equivalent nasiks. Kraitchik's statement holds only for the normal 5×5 squares ie (A, .. E) & (a, .. e) are to be chosen that the 16 numbers are only from 1 to 25.

Putting a, .. e = 1, 5 and A = 0, B = 5, C = 10, D = 15, E = 20 in (X) we get , and this is exactly the nasik form (A) of Kraitchik (p. 134)

1	7	13	19	25
18	24	5	6	12
10	11	17	23	4
22	3	9	15	16
14	20	21	2	8

with the first row brought last.

As shown in rough p. (89') the other form in Ramanujan (p. 74) with the same values of for a, .. A, .. gives Kraitchik's form (B). So no surprise.

For comparison with Reichmann's statement (p. 128) about A, B, C, D, E being any order of 0, 5, 10, 15, 25 and a, b, c, d, e being 1, 2, 3, 4, 5 in any order; vide p (90) Rough, onwards.

~~Also Ramanujan~~

Ramanujan gives another form of the 5×5 square on p. 74 where A + B + D + E must = 4 C since the upper diagonal should have the magic constant $\frac{1}{5}\Sigma A + \Sigma a$. This condn is satisfied by A = 0, B = 5, C = 10, D = 15, E = 20. The square obtained with these A, .. E and a .. e = 1, .. 5 is the first one p. 74 bottom. This form can be obtained

(141) by adding the two squares

2	4	1	3	5
3	5	2	4	1
4	1	3	5	2
5	2	4	1	3
1	3	5	2	4

(x_1)

15	20	0	5	10
20	0	5	10	15
0	5	10	15	20
5	10	15	20	0
10	15	20	0	5

(x_2)

Note that Ramanujan's corresp. square is identical with Pandit Narayana's on p. 51 – Also the two auxiliary squares are the same except for changes in rows & columns. ~~In~~ (x_2) is in fact identical with Narayana Pandith's "root-square". In (x_2) putting the fourth row as the first, we have Reichmann's conditions also satisfied. In (x_1) ~~putt~~ making 2nd col → 1st row, 3rd col → 3rd row, 4th col → 5th row, 5th col → 2nd row and 1st col → 4th row, we get Narayana's first auxiliary square on p. 51. As remarked in rough p. 90, this square is obtainable from the diagonal method also. It can be shown that Ramanujan's, Narayana Pandith's, Reichmann's ~~also~~ methods all lead to the 4 types of Kraitchik's Nasik [Rough, p. (90')]

Perhaps this "root-square" method is applicable to all odd order squares. In fact Ramanujan has remarked this (p. 73). Before going to this let us come back to the 4×4 "root-square" method [also mentioned by Ramanujan (pp. 71-72)]. Also this N.P's method is true, in general, for any odd n from 1 to n [p. 52 of this Bk from Ency. Article] — Before doing this we can investigate the 3rd order squares construction by using Narayana Pandithas method and get all the 8 equivalent squares (done on p. 89" rough sheets) — The method given by Hunter & Madachy (p. 75 of this Book) uses auxiliary squares viz primary no. square & root no square, and is perhaps the one used by Narayana, but not fully explained in the Ency. article (p. 52) [Rough sheets (98) & (98')] — In the sheets it is also shown that the two squares of Ramanujan (pp. 67 & 72) also give Correct squares when A, B, C, D are primary nos 1, 2, 3, 4 and P, Q, R, S (i.e a, b, c, d in K's notation) are the root nos 0, 4, 8, 12. Perhaps Ramanujan's method is same as Narayanas. ~~perhaps~~. The further question of comparing these results of R with the numerous 4×4 squares given by Kraitchik is investigated in the further rough sheets — this only the additn table (α) of K is used here

(143) with a, b, c, d = 0, 4, 8, 12 and A, B, C, D = 1, 2, 3, 4 and then using R's first formula. This formula is same as Kraitchik's final magic square listed form with only the rows & cols are interchanged. Hence changing rows & cols. in R's formula we get K's and we can interchange A and a. Thus all this Narayana's auxiliary square additive method leads to same results as before giving 144 magic squares [$= 432/3$ algebraic squares] with $\frac{48}{3} = 16$ results corresponding to (α) of Kraitchik.]. Anyway this is a simple confirmation of earlier work viz 4! permutations of primary rows with 4! permutations of root numbers giving $(4!)^2$ multiplied by 2 for row - col interchange & divided by 8 for reflections & rotations giving 144. Same for results we have 4 choices for a, b, c, d & 4 for A, B, C, D (using a + c = b + d) & thus leading to 16 results. [Rouse pp. 92-92']

2/6/74 ((38)) Kraitchik (continued).

Multiple magic squares — m x n magic square is one which can be decomposed into m^2 magic squares each of order n

8/8/74 – References from Ogilvy & Anderson

(1) Conjecture on reversals – Amer. Math. Monthly 64 (1957), p. 434, E-1243.

(2) Amer. Math. Monthly 58 (1951), 418, E-742

(3) " 70 (1963), 1101, E-1565

(4) Mathematics of Computation, 18 (1964), 646.

(5) Scripta Mathematica, 21 (1955), 201.

(6) Amer. Math. Monthly 64, 434, E-1243.

(1) (a) Problem proposed by M. A. Rashid & M. A. Uffal, West Pakistan

Prove that the product of a number consisting of two digits and its reverse is never a square except when the two ~~digits~~ integers are equal.

(i) Solution by D. C. B. Marsh, Colorado School of Mines. – Let no. be $10t+u$ (t, u non-negative < 10). Then product $(10t+u)(10u+t)$

$= 10t^2 + 101tu + 10u^2 = 10(t-u)^2 + 121tu \equiv 10(t-u)^2 \pmod{11}$

Since 10 is a quadratic non-residue, modulo 11, it follows that $t-u \equiv 0 \pmod{11}$ and $t = u$

(ii) Soln by Alan Wayne, Baldwin N.Y – Consider $(10t+u)(10u+t) = k^2$ to be solved in positive integers. Obviously when $t = u$, then $k = 11t$ defines solutions. Suppose $t > u$ & let $x = t+u$, $y = t-u$; then $(9y)^2 + (2k)^2 = (11x)^2$ an eqn whose primitive solutions are ~~$9y = p^2 - q^2$~~ $9y = p^2 - q^2$, $2k = 2pq$, $11x = p^2 + q^2$, where p & q are integers of opposite parity & relatively prime. It follows that

(145) $10t + u = p^2$, $10u + t = q^2$. However there are no two digit squares ($t \neq u$) whose reversals are also squares. This completes the proof.

The property does not extend to three or four digit numbers, as is shown by the examples $(961)(169) = (403)^2$ and $(9801)(1089) = (3267)^2$. There arises the following conjecture for proof or disproof. When an integer & its reversal are unequal, their product is never a square except when both are squares

[In above $169 = 13^2$ & $961 = 31^2$; $9801 = 99^2$, $1089 = 33^2$]

Dudley solves the problem of two digits by actually enumerating & testing the possible 45 products.

(3) Ibid, 70, 1101, E-1565 — Proposed by ~~Leon Bankoff~~ R.L. Graham ~~Los Angeles~~ Univ of California and F.D. Parker, Univ. of Alaska.

~~Show that the~~

Show that for any +ve integer n there exists a power of 2 with a string of more than n successive zero

(i) Soln by E.J. Burr (Australia) — Let $\epsilon = \log_{10}(1 + 10^{-n-1})$. Since $\log_{10} 2$ is irrational, there exist integers $a > 0$ and $b > n$ for which

$$0 < s \log_{10} 2 - b < a$$

whence $10^b < 2^a < 10^b(1 + 10^{-n-1})$

$\therefore$ 2^a, in the scale of ten, begins with a one and $n+1$ or more successive zeros.

(1) Soln by Schatz, N.M. — For any +ve integer t, there are +ve integers K and r such that $2^K = 1 + r \cdot 5^t$, since 2 is a non-divisor of 0 in the ring of integers modulo 5^t. Taking $t = 2n$ and multiplying by 2^{2n} we obtain $2^{K+2n} = r \cdot 10^{2n} + 2^{2n}$ which has a string of at least n consecutive zeros since $2^{2n} < 10^n$.

[Ref to a paper by Moser & Macon in Scripta Mathematica 16 (1950), 290-2.]

(2) Not available in the Library.

(4) On maximal gaps between successive primes by Daniel Shanks — [only a summary taken — Proof uses no number theory results to-day: Prime number that $\pi(p) \sim p/\log p$ and Bertrand's "postulate" that there is a prime between any N and $2N$]

Carlson's question to Shanks as to whether one could find a run of a million or more consecutive (prime) integers.

Let $p(g)$ be the first prime that follows a gap g or more consecutive composites. Thus $p(1) = 5$, $p(2) = p(3) = 11$, $p(4) = p(5) = 29$, $p(6) = p(7) = 97$ etc. We seek to estimate $p(10^6)$ [Carlson's question]. Conversely, by $g(n)$ we mean the largest gap that occurs below any prime $p \leq n$. We may call these values of g maximal gaps.

(147) That $p(g)$ is finite for every g is well known. The famous proof by Lucas merely notes that the g consecutive integers

$$(g+1)!+2,\ (g+1)!+3,\ \ldots\ (g+1)!+g+1$$

are divisible respectively by 2, 3, 4, etc. Therefore $p(g) \leq$ the first prime greater than $(g+1)(g!+1)$. Using Bertrand's postulate we have a rigorous what weak answer to Carlson's question viz

$$p(10^6) < 2000002(1000000!+1)$$
$$= 1.65 \cdot 10^{5565715}.$$

– Alternatively $p(10^6) < 3.70 \cdot 10^{434295}$ (another proof using p-n th.) which is again not satisfactory, the R.H.S. being a gross over-estimate etc

– Using Prime number theorem & probability arguement that

$$\log p(g) \sim \sqrt{g}$$

& using also other results of Lehmer on dist of primes etc finally obtains the result

$$10^{300} < p(10^6) < 10^{600}.$$

Almost surely $p(10^6) >$ a googol. On the other hand eight known Mersenne primes

namely $M_p = 2^p - 1$ for $p = 2203, 2281, 3217, 4253,$ 4423, 9689, and 11213 exceed 10^{600}, and almost surely $p(10^6)$ is less than any of them. In fact, it is probable that there is a gap somewhere below M_{11213} there is a gap of fifty million or more consecutive composites.

Ibid p. 508 relating to (3) ie problem of string of zeros in powers of 2.

The first power of 2 with 8 consecutive zeros by E. Karst and D. Kanet.

We have written and run a fast programme for the standard IBM 1620 discovering on 1/1/64 the first string of 8 zeros at $n = 14007$ after 1 hr. 18 min. There were no string of 9 zeros of 2^n up to $n = 50,000$ which limit was reached after 13 hrs 37 min. The first occurrence of 4, 5, 6 and 7 consecutive zeros at $n = 377, 1491, 1492$ and 6801 respty. as noted by Gruenberger [Ref. at bottom] were checked and found correct. The string of eight zeros in 2^{14007} starts at the 7291st decimal digit position, reading from right to left.

on 1/5/64, $n = 60,000$ was reached. It takes now about one hour machine time to raise the upper bound by 2000.

(149) Ibid p. 324 : New factors of Fermat numbers by Claude P. Wrathall.

Gives a table for m & character of F_m – This is reproduced in Ogilvy & Anderson's book.

9/8/72 At Institute

Mathematics of computation 26, No. 120, Oct. 1972

(1) p. 1011 – Note on representing a prime as a sum of two squares. by John Brillhart.

Ex (i) If $p = 10006721$, $p = 3136^2 + 415^2$

(ii) The number $N = (2^{691} - 2^{346} + 1)/5$ has been recently shown by the author & Selfridge to be prime. Hence we can write

$$N = \left[(3 \cdot 2^{345} - 1)/5\right]^2 + \left[(2^{345} - 2)/5\right]^2$$

(iii) The identity $V_{2k+1} = V_k^2 + V_{k+1}^2$ where V_k is the nth Fibonacci number provides such a representation for Fibonacci primes in terms of Fibonacci numbers themselves.

(2) p. 998 – Hugh Williams & Zarnke – Some prime numbers of the form $2A3^n + 1$.

Remarks: The largest twin primes obtained till date are

$10 \cdot 3^{102} \pm 1$, $68 \cdot 3^{30} \pm 1$, $70 \cdot 3^{25} \pm 1$, $76 \cdot 3^{139} \pm 1$,

$82 \cdot 3^{26} \pm 1$, $94 \cdot 3^{55} \pm 1$

The largest pair here $76 \cdot 3^{139} \pm 1$ seems to be the largest pair of twin primes currently known.

Of the numbers analogous to Cullen numbers ie integers of the form $n \cdot 3^n + 1$, only $2 \cdot 3^2 + 1$, $8 \cdot 3^8 + 1$, $32 \cdot 3^{32} + 1$ are primes for $n \leq 100$. Unfortunately $128 \cdot 3^{128} + 1$ is composite.

Ibid - July 72 (Vol. 26, No. 119)

Ibid - April 72 (" No. 118)

p. 567 - Factorisation of $K! \pm 1$, and $2 \cdot 3 \cdot 5 \cdot \ldots \cdot p \pm 1$ by Alan Borning, $K = 2(1)30$

Remarks. In Sierpinski 3! + 1 is omitted from table of primes
Both Sierpinski & Kraitchik erroneously list
20! − 1 as a prime
Kraitchik fails to give the factor 5171 of 21! − 1.
In the factorisations, author gets only 27! + 1 = 1088. 88694. 50418, 35216, 07680, 00011
as a prime & all others from 2! + 1 to 30! + 1 composite & gives factors.
For $K! - 1$: ~~$3 \nmid 1$~~, $K = 3, 4, 6, 7, 12, 14$, and next $K = 30$ give primes

p. 571 - On Thabit ibn Kurrah's formula for amicable numbers by Walter Borho.

For [Amicable nos m & n viz sum of factors of $m = n$ & sum of factors $n = m$ eg. 220, 284 - According to Frenkel known to the Hindus]

51) Euler's ~~generalisation~~ result: $2^n pq$ and $2^n r$ are amicable, if the three integers $p = 2^{n-m} f - 1$, $q = 2^n f - 1$ and $r = 2^{2n-m} f^2 - 1$ are primes, where $n > m$, and $f = 2^m + 1$

Thabits ~~rule~~ (Arabian mathematician 9th century) rule If $p = 3 \cdot 2^{n-1} - 1$, $q = 3 \cdot 2^n - 1$ and $r = 9 \cdot 2^{2n-1} - 1$ are primes and $n > 2$, then $2^n pq$ and $2^n r$ are amicable [Ref. Scripta Mathematica V. 12, 1946, 61-72 by Escott also ibid 23, 1957, pp. 167-71 by Garcia — Also Cohen — on amicable and Sociable numbers in Math. Comp. 24, 1970, 423-28]. Thabits rule yields amicable numbers for $n = 2, 4, 7$ but for no other value $n < 200$. With Euler's generalisation, there is still one more solution $m = 7$, $n = 8$. Whether there are any further solutions is not known [In footnote mention's result of E.g. Lee that from Riesel's tables in Math. Comp. 23, 1969, 869-75 that there is no further solution for $200 \leq n \leq 1000$ (Friends are rare!!)

Various rules are given of Thabits type - only two new pairs are actually given (pp. 573 & 578).

(14|8|74) At Institute - Mathematics of Computation

Vol 27, No. 121 - January 1973.

p. 183 - Note on catalan - Dickson conjecture by Shanks & Serafin

Vol 27, No. 122, p. 397. April 73, p. 397

Three Thirteens by L. J. Lander. — A recent computer search yielded (152) solns to the Tarry-Escott problem $\sum_{i=1}^{n} a_i^j = \sum_{i=1}^{n} b_i^j$, $j = 1, 2, \cdots k$ with $k = 14$, $n = 26$. The terms $a_1, a_2, \cdots a_{26}$ are 1, 8, 9, 22, 23, 34, 36, 48, 50, 62, 75, 83, 87, 89, 95, 97, 109, 130, 132, 134, 136, 156, 157, 158, 171, 173, and $b_i = 175 - a_i$, $i = 1, 2, \cdots 26$. Previously the solution with fewest terms for $k = 14$ had $n = 30$ [Ref. to Golden]. From the new solution it is possible to derive

$$\sum_{i=1}^{13} |b_i - a_i|^x = \sum_{i=14}^{26} |a_i - b_i|^x \text{ or } 1^x + 9^x + 25^x + 51^x + 75^x + 79^x + 103^x + 107^x + 129^x + 131^x + 157^x + 159^x + 173^x = 3^x + 15^x + 19^x + 43^x + 85^x + 89^x + 93^x + 97^x + 137^x + 139^x + 141^x + 167^x + 171^x$$ for $x = 1, 3, 5, 7, 9, 11, 13$ in which there are thus 13 terms on each side of the equation.

NO. 123. p. 491 — More about four biquadrates equal one biquadrate (July 73) by Rose & Brudno.

Solns of the Diophantine eqn $A^4 + B^4 + C^4 + D^4 = E^4$ in least integers. The following table includes the first 82 consecutive primitive solutions of the above equation. The very first entry in the table is $30^4 + 120^4 + 315^4 + 272^4 = 353^4$ & the remaining are colossal numbers. Authors make a conjecture re

(153) Every Diophantine equation

$$\sum_{i=}^{p-1} x_i^{p-1} = y^{p-1}, \quad p \text{ a prime}$$

has at least one solution in integers such that $x+y = p^{p-1}$, where x is the sum of the x_i.

The known examples are

$p = 2$, $1^1 = 1^1$, $1+1 = 2$

$p = 3$, $3^2+4^2 = 5^2$, $4+5 = 3^2$

$p = 5$, $30^4 + 120^4 + 272^4 + 315^4 = 353^4$, $272+353 = 5^4$!

for $p = 7$, no solution is known, the problem of six integers to the sixth power equal to one to the sixth was attacked and a computer search up to $y = 38314$ was unfortunately negative.

However, as indicated by this conjecture, a possible solution may be found for $y > 588825 \; (= 7^6/2 + 1)$.

No. 124. (October 73). p. 95) (ii) A lower bound for the set of odd perfect numbers by Peter Hagis.

(ii) p. 943 – A search procedure and lower bound for odd perfect numbers by B. Tuckerman

(iii) p. 955 – on the largest prime divisor of an odd perfect number.

(b)

(i) Summary - Odd perfect number denoted by Opn. Any Opn has either some component $p^a > 10^{18}$ with a even, or no divisor < 7; hence any Opn must be $> 10^{36}$.

(ii) Summary - It is proved here that if n is odd and perfect, then $n > 10^{50}$.

(iii) Summary - It is shown here that if n is odd and perfect, then n has a prime divisor which exceeds 11,200

Def: $\sigma(n) = 2n$.

24 even perfect numbers are known at present, but whether or not an odd perfect number exists is an open question

Mathematical Wrinkles by S.A. Jones (1930 Edn) - from B.U. Library

Ch. Section on Mathematical Recreations, pp. 58-112

Ex. 5, p. 58 (i) Take any number, (ii) reverse its digits, (iii) find the difference of numbers in (ii) & (i). (iv) multiply this difference by any number you please, (v) cross out any digit except a 0, (vi) give me the sum of the remaining digits, and I will give you the figure struck out

[Ans. p. 206: To find out the digit crossed out subtract the sum of remaining digits from the next higher multiple of 9

(155) [Quite trivial. Difference between number & its reversal is always divisible by 9. Multiple of this by any number is again divisible by 9. For ~~a numb~~ divisibility by 9, the test is sum of digits should be divisible by 9. If a digit is struck off & the sum of remaining digits given, the struck off digit is obviously given by the answer in the book.

The injunction of not striking off a zero is really not necessary. In case of 0 being struck off, the next highest multiple of 9 is equal to the sum of the remaining digits itself]

Note: – Heath's Mathemagic contains some ~~exam~~ tricks based on this same principle.

Ex. 6, p. 58. Same as above but with (ii) replaced by subtracting from the number the sum of its digits.

[This difference is also divisible by 9

$$(a_1 + 10a_2 + \cdot + 10^{n-1}a_n) - (a_1 + \cdot + a_n)$$
$$= 9a_2 + (10^2-1)a_3 + \cdot + a_n(10^{n-1}-1).]$$

Ex. 31, p. 63. Select any two numbers less than 10, (i) choose either of them and multiply it by 5, (ii) add 7 to the result, (iii) double this result, (iv) to this add the other number, (v) give me the result and I will give the numbers originally selected, and also tell you which one you multiplied by 5.

Ans. subtract 14 from the given final result and the resulting number of two digits has its first digit the number multiplied by 5 & the ~~other~~ second digit is the other number

[Trivial, x & y, $\rightarrow 5x+7 \rightarrow 10x+14 \rightarrow 10x+14+y$
$\rightarrow (10x+y)+14$. & $10x+y$ can be written number xy]

[This might be extended to three nos x, y, z. Multiply x by 50 add α, Multiply be 2, multiply y by 5, add β, multiply by 2 Add z to the sum of the two earlier results.

$$100x + 2\alpha + 10y + 2\beta + z - 2(\alpha+\beta) \rightarrow xyz.$$

Looks rather artificial.

Might again be extended to two 2 digit numbers < 100 each. say ab, cd = Multiply ab by 5 add α, multiply 2 & add cd $\rightarrow 10(10a+b)+10c+d+2\alpha \ldots = x$ say (known

Further if $b+d = y$ is given. then.

$$100a + 10b + 10c = x - 2\alpha - d$$
$$10b = \cancel{(y-10d)}.10(y-d). =$$
$$100a + 10c = \text{given ie } ac \text{ is known.}$$
$$= x - 2\alpha - 10b - d.$$

So pose question as follows: 'Take two nos of 2 digits. (i) Multiply one of them by 5, (ii) add α, (iii) double the result, (iv) to these add the other number. (v) Give me this result & also the sum of the last digits of the numbers taken and I will give you the numbers &

(157) and the one multiplied by 5.

Ans to be as follows :- Subtract 2α from the first given. From this ~~subtract~~ remove the last digit. ~~Multiply~~ second given by 10 and subtract this last digit

3/4/74 While looking at magic squares of order 4 and 8 as given by Heath (Mathemagic), it was easily seen that for order 4, Heath's method & Gardiner's method of reversing the diagonals were identical. For the 8×8 square there was no simple analogue of Gardiner's method, but it was found that Heath's method is identical with replacing the diagonal elements (primary & subsidiary) in (diagonals of all the four subsquares of 4×4) by their complements ie whose sum is n^2+1 ie in this case $8^2+1 = 65$. In fact, even the 4×4 square is obtained by replacing the diagonal elements $\frac{x}{2}$ by their complements ie by n^2+1-x ie $17-x$ in this case. Observing this method ~~must be~~ may be true for 16×16 squares etc. yes. It holds as found later ✓ — Again it is found that this rule holds for all squares whose orders are multiples of 4 for eg. orders of 12, 16, 20, etc.

29/5/75 at Institute (158)

(1) D. D. Kosambi Comm. Vol – D.D.K (31-7-07 to 29-6-1966)

Poor fellow! Very versatile & remarkable Chap.

Article by Kosambi on combined methods in Indology – Excerpts

Scriptures being committed to memory without alterations of a single syllable or accent – The profusion of written contracts and registers in Greece afforded a striking contrast with India, where the natives' honesty and truthfulness in the absence of written agreements astounded Greek observers (Remark by Megasthenes, forceful version of Arrian viz "No Indian is ever known to lie". – Very complicated article.

Problems of Marxist historical analysis by Irfan Habib. p 34.

Marx's remark: "Indian Society has no history at all, at least no known history. What we call its history is but the history of the successive intruders who founded their empires on that unresisting and unchanging society."

The astronomical method and Indian Chronology by D. G. Sharale, p. 186

B. G. Tilak work on the Research into the antiquity of the Vedas (1955 Eon). Tilak has endeavoured to show that the traditions recorded in the Rigveda (Ṛgveda) unmistakably point to a period not later than 4000 B.C, when the vernal equinox was in Orion" (?) – Tilak's work fallen into disfavour – Bentley, an orientalist of the last century came to the conclusion that Varahamira lived in the 16th Century

(159) A.D. – ignoring the fact that Alberuni who visited India in the 11th century A.D dwells at considerable length in describing the works and opinions of Varāhamira

J. D. Bernal on Kosambi, p. 332.

"Indians were not themselves historians: they left few documents and never gave dates"!

Kosambi's approach to human problems by A. R. Vasudeva Murthy. p. 338

Some personal reminiscences of Prof. Kosambi by M. R. Siddiqi. p. 340.

One passion of Kosambi was "weightlifting". He was a strong man physically and was in the habit of boasting to his intimate friends that he was the "strongest" mathematician of his time! Perhaps overdoing this exercise must have led to damage to his heart

D. D. Kosambi by Nandan Khan, p. 344 (Governor Reserve Bank)

D. D. K. could speak in any one of French, German, Italian & Russian as if it were his mother tongue. He had enough knowledge of Greek & Hebrew also – In spite of his being a Brahmin he never hesitated in taking meat openly and even in the presence of his co-religionists

(2) Van der Waerden – Science Awakening – nil

(3) A Concise history of Mathematics by D. J. Struik

Stressing continuity & and affinity of the Oriental Civilisations rather than the mechanical division between Egyptian, Babylonian, Chinese, Indian and Arabic cultures – Mahavira comes from Mysore (C. 850), Brahmagupta (Ujjain, C. 625), Aryabhatta (c. 500), Bhaskara (C. 1150)

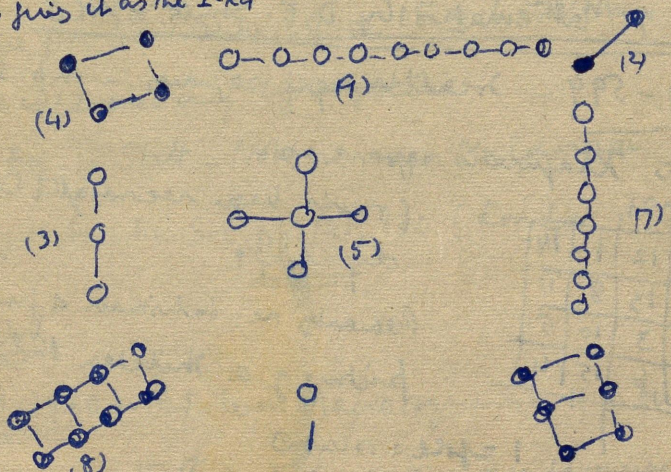
(14) History of Mathematics by D. E. Smith

pp. 153 – 164 – Nice description of the works of Aryabhattas, Brahmagupta, Mahavira (Rashtrakuta monarch's court – Nripatunga (Amoghavarsha) [Ref. to M. Rangacharya – the Ganitasara Sangraha of Mahaviracharya] – Large Number of references also. – on p. 165 gives the Bakshali Manuscript from Kaye's Indian Mathematics

p. 28 – Refers to the Lo-shu

4	9	2
3	5	7
8	1	6

and also gives it as the I-King



World's oldest specimen of Magic square. The black ○ are used in representing feminine (even) numbers, the white ones representing masculine (odd) numbers.

found on the back of a tortoise which appeared to Emperor Yu (c. 2200 B.C.) – Remarks on pp. 33 – 35 about paucity of ancient

(161) information re. ancient Indian Mathematics; similar claims about dates given by many Indians

pp. 274 – 282 – Gives detailed account of the work of Sridharacha (Ganita Sara) and Bhaskara; gives picture of palm-leaf MSS of Lalavati & the story of Lilavati's marriage

p. 435. – India has always been content to take her time. Not since Bhaskara (12th century) has she produced a single native genius in the field. Mathais Ranganatha (1621) contributing nothing original

(5) History of Mathematics by D. E. Smith – Vol II

pp. 591 – 599 – treats magic squares. – on p. 594 gives the Khajuraho square as (runs near records of the Chandel dynasty)

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Remarks re. talisman & gives a picture of a Thibetan talisman

1 = gold = sun ☉
2 = silver = moon ☽
3 = tin = Jupiter ♃ (Hand grasping the thunderbolt)

Again 4 = gold again = sun
5 = mercury = ☿ = Mercury
6 = copper = Venus ♀

7 = Silver agari = moon ☽

8 = lead = Saturn ♄

9 = iron = Mars ♂

p. 596 — gives account of a talisman found in the Maldivian Islands in the Indian ocean, where the charm to protect a virgin sums to 18, whose digits sum to 9, the number of Mars the protector. In these numbers there also enters the idea of congruence, particularly to the modulus 9. Thus a talisman to keep out Satan has its rows $80 \equiv 8$, $1600 \equiv 7$, $180 \equiv 9 \ldots$ which have for their sum $69 \equiv 15 \equiv 6$ = Venus — a vagary that can be explained only by conjecture

[Ref. to Stitt — Notes on some Maldivian talismans.

new J. Royal. Asiatic. Soc, pp. 121; 130 (London, 1906)]

30/5/75 B.U.L

Ency. Britt. 1960 — Article on Magic squares by H.E. Dudeney X

Most of the writers on the subject develop their own favourite schemes. It is not difficult to construct squares of particular types but the ideal solution is, of course, a method completely general for squares of every order, that will include every possible arrangement of the order dealt with. Such a solution is probably not discoverable. Magic squares of singly even orders (such as

(163) those where $n = 6, 10, 14, 18$ etc) are generally the most difficult to construct & their treatment is largely empirical

Classification (Constn, classification & enumeration) is the ~~most~~ elegant branch of the subject & is a matter of individual taste. – 4th order (Nasik, associated, Simple & Semi-Nasik) Nasik possible for $4m$ & $2n+1$ but not for $2(2n+1)$. Number of fundamentally different Nasiks of order 5 is ~~10~~. 16:

Associated as in Andrews, Simple only simple condns Semi-Nasik ras opp. short diagonals sum to 34. – There are 48 Nasiks, 384 semi-Nasiks (2,2) (including 48 associated) and 448 simple making a total of 880. – Bordered & composite

Enumeration – difficult question, little progress made in the last 200 years – 3rd order only one – 4th order 880 (Ref. to Frenicle), (confirmed later – Frolow's Carré's magiques) – 880 are only different fundamentals. All told there are 549,504 4th order magic squares in rows & columns. – Enumeration for larger squares not complete – Particular types counted. Thus there are 28,800 Nasik & 174,240 bordered squares of order 5 and 567,705,600 bordered squares of order 6. – Ref. to D. N. Lehmer (Trans. Amer. Math. Soc., 1929 & Bull. Amer. Math. Soc, 1933, 34)

De la Hire method for squares of any order – for eg. for $n = 5$,

(1644)

The numbers 1, 2, 3, 4, 5 are arranged in 1st square so that every number appears once & once only in every row and column, and in one diagonal, the other diagonal having repetitions of 3. In the 2nd square the numbers 0, 5, 10, 15, 20 are treated similarly, only the 10's are in the other diagonal. The two squares superimposed & added produce a perfect magic square.

Formation of squares consisting of consecutive composite nos. - These can be formed for any order without using a table of primes. For eg. for $n = 3$, write 2, 3, 4, 5, 6, 7, 8, 9, 10. The only prime factors of these numbers are 2, 3, 5, 7. Add the product of these four numbers, 210 to each of the nine numbers getting nos from 212 to 220 inclusive leading to But a magic square of constant smaller than this is found by 114, to 122.

It would be difficult for 4th order to do this empirically (Vide Andrews Book & also my sequence of 19 composites 888 – 906). The above rule gives 510 512 to 510527.

Knight's move squares (1)

23/6/77

A

Summary of Andrew's Book

P. T. O

(A) Chap I – Devoted to associated or regular (also called symmetrical by Rouse Ball) magic squares

odd (1) Diagonal method for odd squares.
(2) Knight's move "
(3) de la Hire's method "
(4) Bachet de Mezeriac's method " (moving from outside).

B even (a) (1) Substitution of complements for non-diagonal & diagonal elements in the case of 4 × 4 – Geometrical diagram

(2) Similar but more complicated method for the 6 × 6 square. Geometrical diagrams too complicate – the square obtained is not associated

(3) Similar method for 8 × 8 square – Geometrical diagram method of (1) [Verify if easier method of taking complements as in (1) works]

(4) (10 × 10) square – construction by four diagrams as in (2) too complicate

(5) (12 × 12) square – " as in (1) & (3) quite simple [By ordinary geometrical method]

(6) (14 × 14) – analogous to (2) & (4) – merely mentioned [(6 × 6), (10 × 10), & (14 × 14) – it is thus not associative]

(b) Application of de la Hire's method to even magic squares

(1) Case of (4 × 4) – two cases – both lead to associated squares

(2) " (6 × 6) – Starting with the (6 × 6) square of (a)(2) constructs the corresp. primary & next squares; gets two primary squares – By analogy constructs by variations of primary squares of the 6 × 6 square – quite complicated but better

than the method in (a)(2) – Also the squares obtained are not associated

(3) 8 × 8 squares – Similar to (b)(2) – gives associated squares.

[For illustration just pick out one primary & one ~~secondary~~ root square]

(c) Composite magic squares – [All lead to associated ones if subsquares be associated?]

(1) 9 × 9 – constructs as in the Chinese method (i.e. heredity business) – Compares this with the (9 × 9) of the diagonal method & remarks that since for each of the 9 subsquares there are 8 possible configurations, one gets an immense number of combinations

[Could this be really 8^9? yes it appears so.] ~~since~~

(2) Next is 15 × 15 square – constructed out of 25 (3 × 3) or 9 (5 × 5)

[~~Note that 12 × 12 cannot be~~ what about 12 × 12?

$$12 \times 12 = 144 = 4^2 \times 3^2$$

(1, 9), (10, 18), (19, 27), (28, 36), (37, 45), (46, 54), (55, 63), (64, 72), (73, 81), – – – – – – – – – , (136, 144)

Consider ~~may~~ the (3 × 3) squares constructed out of these 16 sets & treating them as 1, ... 16 in a 4 × 4 square, why should not one get a composite square? Try one case in ~~detail~~ detail] – Also may be built of 9 subsquares of 4 × 4, (*) try this also

(3) Case of (16 × 16) – built with 16 subsquares of 4 × 4

(4) 18 × 18 – 36 subsquares of 3 × 3 or ~~nine~~ 9 subsquares of 6 × 6

~~Concentric~~ [In fact Fig 98, p. 45 of Andrews corresponds to this each subsquare of 4 × 4 (*) being of the type in Fig 42, p. 18 – Andrews has forgotten to mention 12 × 12]

(D) Concentric magic squares [Bordered magic squares]

Quite complicated methods

Better omit bordering from the Book.

(E) General notes.

(1) formulae for magic constant

$S = \frac{1}{2}n(n^2+1)$

$S = An + \frac{1}{2}\beta n(n^2-1) = An + \beta K$

A = initial number

β = increment

Trivial applications of these formulae for several values of A & β & n

Works a lot of useless examples for cases n = 3, 4 – ridiculously trivial

(2) Remarks about magic series other than having constant β.

Considers case n = 4, & the variations of the arrangement

a	1	2	3	4	5
b	6	7	8	9	10
c	11	12	13	14	15
d	16	17	18	19	20
e	21	22	23	24	25

(i) Taking arrangement b e c a d & using the folding method (Bachet)

gives the square without constructor with 1 in the second cell from the left in the upper row, with right-hand upward diagonal sequence and a knight move given by ⌐ (knight's move)

(ii) gives another arrangement

6	7	8	9	10	b
21	22	23	24	25	e
11	12	13	14	15	c
1	2	3	4	5	a
16	17	18	19	20	d

This is only a rearrangement of 1 to 25 for applⁿ to the following method.
Here we have five series in A.P. with the first terms in each A.P being

6, 21, 11, 1, 16
(i.e. (+15 −10 −10 +15.)

Suppose we start with the first A.P. series as

3, 10, 6, 2, 9 (i.e. (+7, −4, −4, +7)

and take the series as

					(C.D = 5)
3	8	13	18	23	
10	15	20	25	30	"
6	11	16	21	26	"
2	7	12	17	22	"
9	14	19	24	29	"

so that there is no overlapping & then construct a 5×5 by diagonal method as follows starting from 2

20	29	2	11	18	80
26	8	10	17	19	80
7	9	16	23	25	80
13	15	22	24	6	80
14	21	30	3	12	80
80	82	80	78	80	↘80

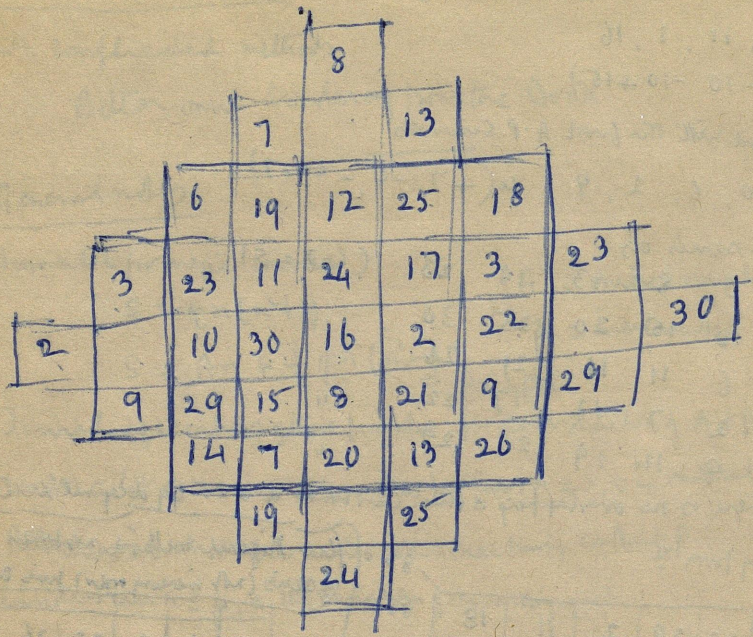
But diagonal method not taken in order of series (not increasing order) hence correct result

7	24	3	20	26
19	23	15	21	2
18	10	16	22	14
30	11	17	9	13
6	12	29	8	25

✓

Appears to fail narrowly in just 2 columns only. In the series of 25 numbers we have the breaks from continuity viz 3–6, 26–29. i.e. two breaks. Perhaps we should have consecutive nos. only. Does this succeed if we take the fold up method?

[P. T. O



This is not magic at all.

So it looks as if the nos. must be consecutive. But consider the series

→ A.P(2)

2	4	6	8	10
9	11	13	15	17
16	18	20	22	24
23	25	27	29	31
30	32	34	36	38

↓ A.P(7)

Scheme (A)

Suppose we construct a square where the numbers are not consecutive at all. Let us construct a square by the Indian diagonal method taking nos in increasing order starting from 2

25	36	2	13	23	→ 99 ×
34	9	11	22	24	100
8	10	20	30	32	100
16	18	29	31	6	100
17	27	38	4	15	→ 101 ×
100	100	100	100	100	→ 100

[Note Later addn If Ind. diag. method is applied not in increasing order but in the order as in Scheme (A) ie 9 after 10, 16 after 17, 23 after 24, 30 after 31 etc. we get correct square]

Here again there is failure in two rows only. But for $n = 3$ take for eg

→ AP(2)

2	4	6
9	11	13
16	18	20

↓ AP(7)

we have by the Ind. diagonal method

18	2	13	33
6	11	16	33
9	20	4	33
33	33	33	33

we get correct result.

Perhaps for $n = 5$, etc a method other than the Ind. diag. method will work. Anyway quite a complicated business]

(iii) consider

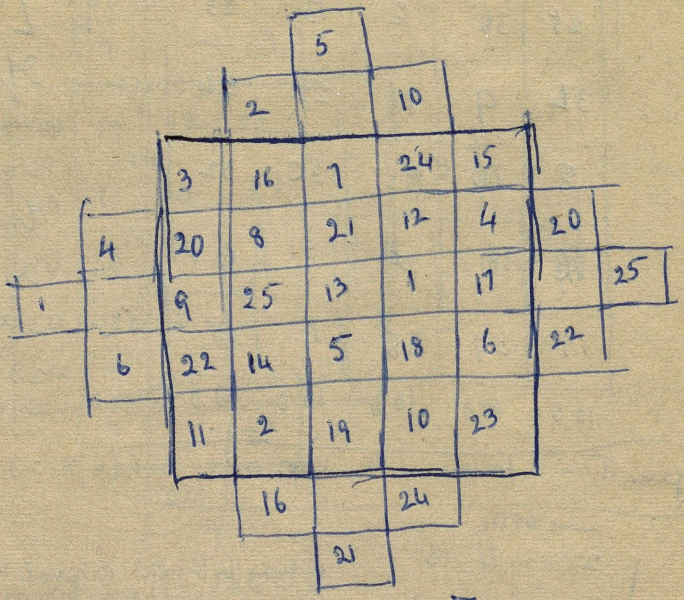
$a = 1, 4, 3, 2, 5$
$b = 6, 9, 8, 7, 10$
$c = 11, 14, 13, 12, 15$
$d = 16, 19, 18, 17, 20$
$e = 21, 24, 23, 22, 25$

~~all consecutive~~ ie no discontinuity

of course Ind. diag method works.

(iii) is given for folding method to work.

(171) [Let us verify this

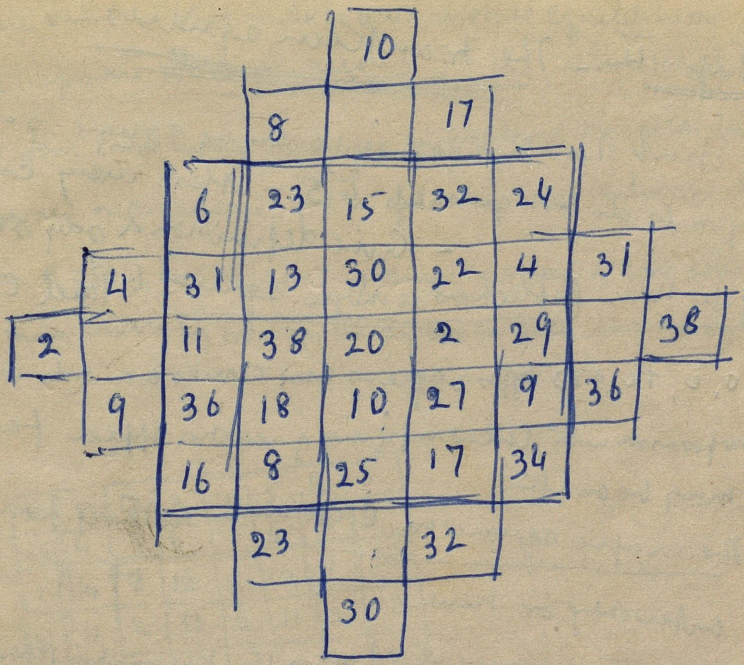


This gives magic square all right with $S = 65$.]

[Let us ~~verify~~ examine if like AP → 2 gives by the following method

↓
7

and see if it gives a magic square



This gives a magic square all right with $S = 100$ So that the const for nos. being consecutive is not necessary. The following method works but not the 2nd. diagonal method.]

Chap II - Magic cubes.

omit this

(171)

Let us verify this

This gives map

Let us ...

and sel...

INDIAN INSTITUTE OF SCIENCE
BANGALORE 560012

I.I.Sc (16/10/74)

Center for Theoretical Studies

(1) Impact of basic research on technology: Edited by Behram Kursunoglu & Arnold ~~Perlmutter~~ Perlmutter – Studies in Natural Sciences Vol I – Plenum Press New York, London, 1973

Centre for Theoretical Studies, Univ. of Miami, Coral Gables, Florida.

1st Article by P.A.M. Dirac, pp. 1–13. – ~~Early~~ Some of the early developments of quantum theory (Dirac's address is the above C.T.S, at Miami) – A sort of reminiscent article discussing his scientific work in a general and non-technical way. (with a fine photograph of Dirac)

– Read the whole article through – quite a beautiful one

Many other beautiful articles – one on superconductivity another by Teller on the atom bomb etc. etc

(2) Read a book on thermodynamics – Ultimate temp. of Universe 10^{12} K – a limit something like c in relativity?

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10

38

(172)

(1711)

Let us verify this

(3) Fundamental interactions in Physics (1973). by the same authors as in (1) (Proc. of a Conference at C.T.S, Miami)

1st article by Dirac pp. 1–19 on Long range forces and broken symmetries: Dirac's address is given as Dept of Physics, Florida State University, Tallahassee, Florida and C.T.S. Univ. of Miami. – A photograph of Dirac with a microphone attached to his mouth (?) delivering the first address at the Conference.

10

38

(1721)

Chap. IH – The Franklin squares.

omit This. It is not really a magic square since the diagonals do not sum up to 5. – Also very complicate – Give definition & say some words]

Chapter IV. – Reflections on magic squares by Paul Carus.

(1) 0, 1, π and 2 types of squares used to make even order magic squares – Comparison with Chladni figures of acoustics appear far fetched & may be omitted

(2) The magic square in China – nothing particularly interesting or new.

(3) The Jaina square – This is the Khajuraho NASIK square called Jaina square for short – Suggestions re magic squares must have gone out over the whole world in the most primitive types This perhaps from Mesopotamia, the cradle of Babylonian civilisation or later from India, the centre of a most brilliant development of scientific & religious thought.

Jaina square not associated, but has other remarkable features. It represents a distribution

of the figure values in what might be called absolute equilibrium.

The Jaina square is continuous in the sense that wherever we may start four consecutive numbers in whatever direction, backwards or forward, upwards or downward, and in slanting lines always yields the same sum $S = 34$, and so does any small square of 2×2 cells. These results can be made visible by extending the square in each direction by half its own size.

10	5	16	3	10	5	16	3
15	4	9	6	15	4	9	6
1	14	7	12	1	14	7	12
8	11	2	13	8	11	2	13
10	5	16	3	10	5	16	3
15	4	9	6	15	4	9	6
1	14	7	12	1	14	7	12
8	11	2	13	8	11	2	13

(Jaina equilibrium square (12th or 13th century A D)

After a lot of manipulation by using letters A, B, C, D and Nos 1, 2, 3, 4 the square

1	8	10	15
14	11	5	4
7	2	16	9
12	13	3	6

(175) [There is nothing wonderful about this, however. In fact it can be obtained from the above one by interchanging rows and columns and then by a cyclic interchange of rows viz (1 2 3 4) to (3 4 1 2) (Vide Kraitchik, p. 145]

Chapter V – A mathematical study of magic squares by L. S. Frierson.

Says "magic squares are not simple puzzles to be solved by the old rule of 'try & try again' but are visible results of 'order' as applied to numbers their construction is therefore governed by laws that are as fixed and immutable as the laws of geometry"! But his further considerations using trivial mathematics do not at all justify these remarks.

(1) 3×3 square

a	b	c
d	e	g
h	m	n

$h+e+c = b+e+m \longrightarrow h+c = b+m$

$h+m+n = g+c+n \longrightarrow h+m = g+c$

$\therefore 2h = b+g$

Simly $2n = b+d$

$2c = d+m$

$2a = g+m$

Hence least quantity cannot be in a corners cell i.e. because of invariance of rotation about the centre. Least quantity may be put in the middle of top row.

Adding $a+e+n$, $d+e+g$ and $h+e+c \longrightarrow 3e+(a+d+h)+(n+g+c)$

i.e. $3S = 3e+2S$ or $3e = S$ or $e = S/3$.

Thus e is A.M of the two quantities with which it forms a st: l.s.

$x+y+2v$	x	$x+2y+v$
$x+2y$	$x+y+v$	$x+2v$
$x+v$	$x+2y+2v$	$x+y$

Putting the least number x in middle of top row & writing $x+y$ & $x+v$ as shown in fig. we get $x+2y$ in middle of 1st col. since $x+y$ should be mean of x & this middle number. Similarly $x+2v$ should be middle of 3rd column. Mean of $x+2y$ & $x+2v = x+y+v$ will be in the centre. This leads to $x+2y+2v$ in middle of bottom row & the numbers $x+y+2v$ & $x+2y+v$ in the corners of 1st row are easily obtained.

[Note – This is exactly Ex. 28, p. 148 of Kraitchik in different notation on p. 147, Kraitchik says that the necessary & sufficient condition that any nine distinct numbers be capable of forming a magic square is that they form three, three-termed A.P's whose first terms are in A.P. In fact this is exactly what Frenicle has shown].

Some remarks about the 9×9 square quite trivial

(ii) (4×4)-square

Euler's six laws governing such squares these being

a	b	c	d
g	h	n	m
k	o	p	s
t	v	x	y

(i) $h+n+o+p = S$

(ii) $a+d+t+y = S$

(iii) ~~$b+c+s+t+x$~~ $b+c+v+x = g+m+k+s = S$

(iv) $a+d = v+x$

(v) $g+m = o+p$

(vi) $t+d = h+p$.

If in addition the square is associated some more laws

[He does not explain how these laws enable construction of 4×4 squares. Also

(177) Further special types like vasith are not considered.

Finally remarks "Squares of larger dimensions do not seem to be reducible to laws on account of their complexity" – [Wonderful 'mathematical study' indee

Notes on number series used in the construction of magic squares –

(i) For 4 × 4 squares – may be formed of a series of numbers whose interrelations are such as to permit them to be placed as in Fig (i). This figure is deduced f the six laws mentioned above. Takes the special cases of the Fig (i) [from nos 61 to 16] as

a	v	s	d
|	|	|	|
l	o	n	g
||	||	||	||
p	k	h	m
|	|	|	|
t	b	c	y

Fig (i)

(a)

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

(b)

1	2	11	12
3	4	9	10
6	5	16	15
8	7	14	13

and constructs the corresponding squares

a	b	c	d
g	h	k	l
m	n	o	p
t	v	s	y

gives an other example

2	9	7	14
10	15	21	26
12	11	19	18
20	17	33	30

for Fig (i)

& constructs the corresp.

2	17	33	14
26	19	11	10
18	21	15	12
20	9	7	30

with S = 66

Next considers associated squares & finds lots of rules for their construction on type of Eq (i) – rather obscure – one example ~~of~~ of the type of Eq (i) for associated squares is given viz

$$\left|\begin{array}{cccc} 3 & 7 & 13 = & 18 & 28 \\ \| & | & & | & | \\ 4 & - & 14 = & 19 & 29 \\ \| & \| & & \| & \| \\ 21 & - & 31 = & 36 & 46 \\ | & & & | & | \\ 22 & - & 32 = & 37 & 47 \end{array}\right|$$ leading to

3	32	37	28
29	36	31	4
46	19	14	21
22	13	18	47

(associated) ($S = 100$)

Written in order of magnitude is

3 – 4 – 13 – 14 – 18 – 19 – 21 – 22 – 28 – 29 – 31 – 32 – 36 – 37 – 46 – 47

& the differences 1 9 1 4 1 2 1 (6) 1 2 1 4 1 9 1

ie this rule with (6) in centre & the other differences symmetrical about it. Further says "It is the belief of the writer that this rule applies to all associated squares whether odd or even" – calls this a mathematical study!

(ii) for 5×5 squares – finds the only example

1 · 4 · 7 · 8 · 10 · 11 · 13 · 14 · 15 · 17 · 18 · 20 · 21 · 22 · 24 · 25 · 27 · 28 · 29 · 31 · 32 · 34 · 35 · 38 · 41

& difference scheme being

3 3 1 2 12 1 12 1 21 | 1 2 12 1 12 1 21 2 1 33 showing symmetry

[After all this amounts to choosing the series

→ A.P. 3

1	4	7	10	13
8	11	14	17	20
15	18	21	24	27
22	25	28	31	34
29	30	35	38	41

↓ A.P. 7

& the writer could have easily avoided unnecessary labour [The square is easily obtained by the 2nd diagonal method]

(179) Next considers unnecessarily 9 x 9 squares.

Chapter VI - (a) Magics & Pythagorean numbers by ~~Paul Carus~~ C. A. Browne

A series of non-sensical speculations.

(b) Mr Browne's square and lusus numerorum by Paul Carus

Refers to Browne's 27 x 27 square & its method of construction using one of the 8 possible arrangements for a 3 x 3 square - Suggests taking other forms - Next consider arithmetical curiosities having no connection with magic squares - Wonderful flourishing language with no meaning.

Chapter VII - Some curious magic squares and combinations by W. S. Andrews

(a) Ref. to book "Games, Ancient & Oriental" by Edward Falkener.

(a) Case of a 6 x 6 not impressive

(b) 8 x 8 squares - Ref. to Gavalier gate & extracts from Falkener's book

Lots of things about 8 x 8 squares including Jaina or Indian patterns

(pp. 164 - 170)

(c) Peculiar bordered squares of C. Planck.

(d) A 9 x 9 magic square of Frierson.

(e) magic squares composed of purely prime numbers - mere mention saying that it is proposed another book on this?

(f) magic squares with composites - Schuldham's 4 x 4 & 5 x 5 squares with such lowest composites - Planck's table for composites for squares of orders 6 to 12 [Hyp nos and even failed to get one for 6th order ie 36 consecutive

composites going up to 10,000. Planck gives for this 15,684 to 15,719 – for 12th order 2010734 to 2010877 !]

(g) magic square formed by continuous Knight's tour – not solved completely – ~~Frierson~~ Falkener's example failing in its two diagonals (semi-magic?) – Knight's move possible if consecutive numbers be disregarded – Diagram of the move for the Gwalior gate square (?) (Fig. 261, p. 165 & diagrams in Fig. 283, p. 175

(h) 18 x 18 square of H. A. Sayles produced by values of n/19 ie the lines of recurring decimals for 1/19, 2/19, ... 18/19 are arranged one below the other so as to form a magic square

Chapter VIII (a) Various constructive plans by which magic squares may be classified by W. S. Andrews.

(i) 12 plans (11 by Frierson & one by Dudeney) for classifying 4 x 4 squares with diagrams for the plans – Plan 5 for Jaina square is interesting – 6 of Frierson's plans are distinct & the remaining 5 are combinations of these – These classify the 4 x 4 squares [Note Dudeney's remarks about classification problem] – Plan 4 also shown for 8 x 8 squares.

(ii) Plans for 6 x 6 squares – 7 plans by Frierson (?) and 4 plans by E. Black (Plan in 327 appears simplest) – these plans more complicated.

(180) (h) mathematical value of magic squares

Quotation from Mac Mahon - first paragraph this may be used in the Preface.

Chapter IX - Magic cubes of sixth order by four people

~~omit~~ omit this

Chapter X - Various kinds of magic squares

(a) overlapping magic squares by W. F. Savage

(b) Complicated & not too interesting - omit ~~except~~

(b) oddly-even magic squares by W.F.S.

for the special case of the 4×4 square formed from (i) & (ii) to

1	5	13	9
13	9	1	5
1	5	13	9
13	9	1	5

(i)

0	3	0	3
1	2	1	2
2	0	3	0
2	1	2	1

(ii)

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

(iii)

give (iii) by addition - Something like adding primary and root squares of Narayana Panditha

~~(b) oddly-even magic squares~~

Similarly for the 6×6 square Savage uses the two auxiliary squares

[Contd. in Bk. 13, p. 94]

(180')

"Lakshmi"

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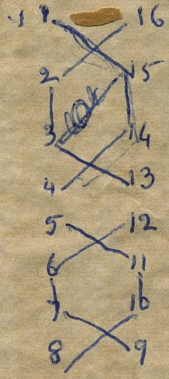
1	16
2	15
3	14
4	13
5	12
6	11
7	10
8	9



$D + a - d + C + a + c$
$+ A - b + c + B + b - d = 34$
$2a - 2d + 2c = 0$ $\quad d - c = a$

$\cancel{C - b + d + D - a - c}$
$\cancel{+ B + b - c + A - a + d = 34}$
$\cancel{2d - 2a - 2c = 0}$

$d - c = a$
$- c + d =$

$A - a + D - a - c + D + A - a + d = 34$
$2A + 2D - 3a - c + d = 34$
$2A + 2D - 2a = 34$
$A - a + D = 17$

9 | 9801 (99
81
189 | 1701
1701

1 | 1089 (5
1 | 089

$\frac{49 \times 50}{2 \cdot 7} = 175$

33 x

121 3.11