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NON LINEAR BALLISTICS SEMINAR
(21-22 JUNE 74)

Dixon's Linear Algebra (3/10/74 at Institute).

(1) Suppose V is spanned by a set of $\underline{n}$ vectors. If $m > n$ then any set of $\underline{m}$ vectors in V is linearly dependent.

(2) If W is a subspace of a vector space V, then W is a vectorspace. If V is finite dimensional so is W.

(3) Any set of vectors that span a non-zero space V contains a basis of V.

(4) Any linearly independent set of vectors in V is contained in a basis of V.

(5) If $\{u_1, u_2, \ldots u_m\}$ and $(v_1, v_2, \ldots v_n)$ are two bases of a vector space V, then $m = n$.

(6) Set of linear transformations $L(V, W)$ is a vector space.

(6) If $(v_1 \ldots v_n)$ is a basis of V, and $\{w_1 \ldots w_n\}$ any set of $\underline{n}$ vectors in W, then there exists one & only one L.T. $A: V \to W$ for which $v_i A = w_i$ $(i = 1, 2, \ldots n)$

(7) $\dim L(V, W) = (\dim V)(\dim W)$

(8) Null space of a L.T $A: V \to W$ is defined to be the set $N(A)$ of all vectors $\underline{v}$ of V for which $\underline{v} A = 0$. $N(A)$ is a subspace of V.

(9) For $A : V \to W$ is a one to one mapping if & only if

$N(A)$ is the zero space

(10) Image space: of a L.T. $A : V \to W$ is the set $I(A)$ of all vectors $\underline{W}$ of W such that there exists a vector $\underline{\underline{v}}$ in V with $W = vA$. $I(A)$ is a subspace of W.

(11) $\dim N(A) + \dim I(A) = \dim V$.

(12) $L(V)$ stands for $L(V, V)$

(13) If A & B are L.T.'s of V, then AB is non-singular if & only if A & B both non-singular. In that case

$(AB)^{-1} = B^{-1}A^{-1}$

(14) A L.T of a vector space V has at most one inverse.

(15) The row space & the column space of a matrix A have the same dimension, and this common dimension is the rank of A.

(16) The set of solution vectors of a linear homogeneous system with $\underline{n}$ unknowns is a subspace K^n of dimension $n - \text{rank } A$, where A is the coefft. matrix of the system

(17) A linear system (general) has a solution if and only if its coefficient matrix and augmented matrix have the same rank.

(18) ~~Let~~ $(u_1 - u_n)$ & $(v_1 - v_n)$ ~~are~~ be bases of vector space V and $P = (p_{ij})$ ~~be~~ the matrix determined by the equations

$v_i = p_{i1}u_1 + p_{i2}u_2 + - + p_{in}u_n \quad (i = 1, - n)$

$v_i = p_{i1}u_1 + p_{i2}u_2 + \cdots + p_{in}u_n \quad (i = 1, \ldots n)$

If A represent a L.T. $T : V \to V$ with respect to the

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basis $(u_1, \ldots u_n)$ and B represent T w.r.t to the basis $(v_1 \ldots v_n)$ then $B = PAP^{-1}$ (similar A & B)

(19) A & B represent the same L.T. say $T : V \to V$ (w.r.t different bases) if & only if A & B are similar

(20) Adding a multiple of row $\underline{q}$ to row $\underline{p}$ $(p \neq q)$ does not change the value of the determinant of a matrix

(21) An $n \times n$ matrix is non-singular if & only if its determinant is not zero.

(22) If A & B are $n \times n$ matrices $\det AB = (\det A)(\det B)$

(23) If A be a non-singular matrix $\det A^{-1} = (\det A)^{-1}$

(24) If A & B are similar $n \times n$ matrices, then $\det A = \det B$.

(25) Transpose of the matrix (d_{ij}) of cofactors of A is called the adjoint of A & denoted by adj A.

(26) For each $n \times n$ matrix A, $(\operatorname{adj} A)\, A = A\, (\operatorname{adj} A) = (\det A)\, I$.

(27) If A is a non-singular matrix $A^{-1} = \frac{1}{\det A}$ (adj A)

(28) adj (adj A) $= (\det A)^{n-2} A$. (A is a $n \times n$ matrix)

Schaum's outline.

(1) If U & W, of dim^n $r \leq n$, & $s \leq n$ resp. are subspaces of a vectorspace V of dimension n and if $U \cap W$ and $U + W$ are of dimensions p & t respectively, then $t = r + s - p$.

(2) The set of all L.T's of a vector space into itself forms a ring w.r.t addition & multiplication (Proof long)

(3) The set of all non-singular linear transformations of a vectorspace into itself forms a group under multiplication. (Proof long)

(4) A vectorspace is an abelian additive group.

(5) Let $\xi = (1,1,1,1)$ & $\eta = (1,2,-3,0)$ be given vectors in $V_4(R)$.

(a) Show that they are orthogonal

(b) Find two linearly independent vectors λ & μ which are orthogonal to both ξ & η

[(a) $(\xi \cdot \eta) = 1.1 + 1.2 + 1.(-3) = 0$. Hence —

(b) Assume $(a, b, c, d) \in V_4(R)$ orthogonal to both ξ & η; then

$a + b + c + d = 0$ & $a + 2b - 3c = 0$

First take $c = 0$, then $a + 2b = 0$ is satisfied by $a = 2$, $b = -1$ & $a+b+c+d = 0$ now gives $d = -1$. We have $\lambda = (2, -1, 0, -1)$

$a+b+c+d=0$ & $a+2b-3c=0$

First take $c=0$, then $a+2b=0$ is satisfied by $a=2$, $b=-1$

& $a+b+c+d=0$ now gives $d=-1$. We have $\lambda=(2,-1,0,-1)$

Next take $b=0$, then $a-3c=0$ is satisfied by $a=3$, $c=1$ &

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$a+b+c+d=0$ now gives $d=-4$. We have $\mu=(3,0,1,-4)$.

Clearly λ & μ are linearly independent

(6) $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$, $B=\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}$

$A+B=\begin{pmatrix}6 & 8\\ 10 & 12\end{pmatrix}$, $10A=\begin{pmatrix}10 & 20\\ 30 & 40\end{pmatrix}$

$A\cdot B=\begin{pmatrix}10 & 22\\ 43 & 50\end{pmatrix}$, $B.A=\begin{pmatrix}23 & 34\\ 31 & 46\end{pmatrix}$

(7) The set of all n square matrices over a field F is itself a vector space

(8) [For $n=2$, $\underbrace{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}}_{E_{11}}$, $\underbrace{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}}_{E_{12}}$, $\underbrace{\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}}_{E_{21}}$, $\underbrace{\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}}_{E_{22}}$

is a basis of the vector space of all 2-square matrices over F & for any $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $A=aE_{11}+bE_{12}+cE_{21}+dE_{22}$

(9) The set of all matrices over F of order $m\times n$ is a vector space over F

(10) Solve

$$2x+3y+z=0$$
$$x-y+4z=0$$
$$4x+11y-5z=0$$

[corresp. matrix $\begin{pmatrix}2 & 3 & 1\\ 1 & -1 & 4\\ 4 & 11 & -5\end{pmatrix}$ can be reduced to the echelon form $\begin{pmatrix} 0 & 1 & -7/5 \\ 1 & 0 & 13/5 \\ 0 & 0 & 0 \end{pmatrix}$ — — (A)

gen. soln $z = r$, $x = -\frac{13r}{5}$, $y = \frac{7r}{5}$]

(11) Is the set $\rho_1 = (2, 3, 1)$, $\rho_2 = (1, -1, 4)$, $\rho_3 = (4, 11, -5)$ a basis of $V_3(R)$?

[using the above echelon form we get

$$(4x + 11y - 5z) + 2(x - y + 4z) - 3(2x + 3y + z) = 0$$

or $\rho_3 + 2\rho_2 - 3\rho_1 = 0$ ie set is not a basis.

(12) Show that the L.T

$$T : \begin{cases} \epsilon_1 \to (2, 3, 1) = \rho_1 \\ \epsilon_2 \to (1, -1, 4) = \rho_2 \\ \epsilon_3 \to (4, 11, -5) = \rho_3 \end{cases}$$

of $V = V_3(R)$ is singular & find a vector V whose image is 0

[using Ex. 10 above V_T has dimension 2 & T is singular this is implied by row 0, 0, 0 in A.

Since $3\rho_1 - 2\rho_2 - \rho_3 = 0$ image of $\eta = (3, -2, -1)$ is 0. ie

$$(3, -2, -1) \cdot \begin{pmatrix} 2 & 3 & 1 \\ 1 & -1 & 4 \\ 4 & 11 & 5 \end{pmatrix} = 0.$$

(13) Reduce $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 5 & 7 & 8 \end{pmatrix}$ to upper Δ^r, lower Δ^r

and diagonal form

and diagonal form

[using $H_{12}(-2/5)$, $H_{23}(-5/7)$, $H_{12}(-21/10)$, $H_{23}(-1/28)$

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$$A \approx \begin{pmatrix} -3/2 & 0 & 0 \\ 1/4 & -1/4 & 0 \\ 5 & 7 & 8 \end{pmatrix}$$ which is ~~upper Δ^r~~ lower Δ^r.

using $H_{21}(-4)$, $H_{31}(-5)$, $H_{32}(-1)$, we get

$$A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -1 \end{pmatrix}$$ which is upper Δ^r.

using $H_{21}(-4)$; $H_{31}(-5)$; $H_{32}(-1)$; $H_{12}(2/3)$, $H_{13}(-1)$, $H_{23}(-6)$

we get $$A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ which is diagonal]

(13) ~~The row rank & col rank of any matrix A are equal~~

Here interchange of i^{th} & j^{th} rows is denoted by H_{ij}

Multiplication of i^{th} row by a scalar k is " " $H_i(k)$

Addition ~~of~~ to the elements of the i^{th} row of k times the corresponding ~~terms~~ elements of the j^{th} row is denoted by $H_{ij}(k)$

For cols. K is used instead of H]

(14) The row rank & col. rank of any matrix are equal

(15) Any square matrix is non-singular if & only if it is row equivalent to the identity matrix I_n.

(16) The row rank of an $\underline{m} \times \underline{n}$ matrix is the number of non-zero elements in its row equivalent canonical matrix (echelon matrix).

(17) Normal form of a matrix A is that in which 1's occupy the diagonal positions in the first $\underline{r}$ rows and first $\underline{r}$ cols.

[eg: $A = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 2 & 5 & 3 & 1 \\ 3 & 8 & 4 & 2 \\ 2 & 7 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \approx \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$ is normal form]

(18) matrix resulting when an elementary row (col) trn is applied to the identity matrix I_n is called an elementary row (column) matrix

(19) Every elementary matrix is non-singular

(20) The product of 2 or more elem. matrices is non-singular

(21) For any A, there exist non-singular S & T such that
$S \cdot A \cdot T = N$ (normal form)

(22) Inverses of elementary row (col) matrix is an elementary row (col) matrix of same order & inverse is non-singular

(23) Find the minimum polynomial of

(23) Find the minimum polynomial of

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$ over Q (set of all rational nos)

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[N = set of all natural nos, I, integers, R real nos]

Ans ~~A^2~~ $A^2 = 4A + 5I$ & min. polynomial is

$\lambda^2 - 4\lambda - 5$

(24) If A is a $n \times n$ matrix, H (K) any n square row (col)

matrix, then

$$|H \cdot A| = |H| \cdot |A| \text{ and } |A \cdot K| = |A| \cdot |K|$$

[~~H~~ $|A| = \det A$]

(25) If $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $B = (4, 5, 6)$, find $A \cdot B$ & $B \cdot A$

$$A \cdot B = \begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{pmatrix}, \quad B \cdot A = 4\cdot1 + 5\cdot2 + 6\cdot3 = (32).$$

(26) If $A = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 3 & 2 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}$

find $A \cdot B$. Ans: $\begin{pmatrix} 4 & 5 & -2 & 1 \\ 6 & 4 & -1 & -4 \end{pmatrix}$

(27) Show that the L.T. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ is non-singular

[We find
$$A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & -4 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -4 & -8 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}]$$

(28) Reduce $A = \begin{pmatrix} 3 & 2 & 3 & 4 & 5 \\ 2 & -1 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2, & -3 \end{pmatrix}$ over R to normal form

$$\left[\text{Ans: } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \right.$$

(29) Find min. polynomial of $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ over R

[Ans $m(\lambda) = \lambda^3 - 4\lambda^2 + 4\lambda$

(30) Find dimensions of the vector space spanned by each of the set of vectors over Q.

(a) $\{(1,4,2,4), (1,3,1,2), (0,1,1,2), (3,8,2,4)\}$

(b) $\{(1,2,3,4,5), (5,4,3,2,1), (1,0,1,0,1), (3,2,-1,-2,-5)\}$

[Ans (a) 2, (b) 3]

(31) Show that if A is nonsingular $A \cdot B = A \cdot C$ implies $B = C$.

(32) Find inverse of $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$ over Q

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[Ans $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

(33) Find min. polynomial of

~~(a)~~ $\cancel{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}}$, (b) $\cancel{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 4 & 2 & 0 \end{pmatrix}}$

[~~Ans~~ (a) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 2 & -1 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

[Ans: (a) $\lambda^3 + \lambda^2 - 2\lambda - 1$
(b) $\lambda^2 - 3\lambda + 2$]

(34) Prove $(A + B)^T = A^T + B^T$

$(A \cdot B)^T = B^T \cdot A^T$,

where A & B are n-square matrices over F.

(35) Find char. polyn. of $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$

$\lambda^3 - 6\lambda^2 + 11\lambda - 6$

(36) The characteristic roots of a real symmetric

matrix are real.

(37) A real matrix H is orthogonal if

$$H \cdot H^T = H^T \cdot H = 1$$

[Proper or improper orthogonal accg as

$$|H| = 1 \text{ or } |H| = -1$$

(38) Find the char. roots & associated char. vectors of

$$A = \begin{pmatrix} 7 & -2 & -2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{pmatrix}$$

(39) Find a vector γ orthogonal to $\alpha = (2,1,3)$ &

$\beta = (1,1,-1)$

[$\gamma = \alpha \times \beta = (-4, 5, 1)$

use $\alpha_1, \beta_1, \gamma$ to form an orthogonal matrix S with $|S| = 1$

[Take $\rho_1 = \alpha/|\alpha|$, $\rho_2 = \beta/|\beta|$, $\rho_3 = \gamma/|\gamma|$

Then $\begin{vmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{vmatrix} = 1$ & $S = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} = \begin{pmatrix} 2/\sqrt{14}, & \cdots \\ 1/\sqrt{3}, & - \\ -4/\sqrt{42}, & — \end{pmatrix}$

(40) The characteristic polynomial of a square A is

(40) The characteristic polynomial of a square A is the product of invariant factors

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(41) If Q is orthonormal, $Q^T = Q^{-1}$!

(42) Every real 2-square matrix for which $|A|<0$ is similar to a diagonal matrix

(43) Prove by direct substitution that

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a zero of its characteristic polynomial

3584
2048
24064
2816
54
26934
13467

$n(47n^3 + 44n^2 + 7n - 2$

$8(47 \cdot 512 + 44 \cdot 64 + 56 - 2)$

$13A = 8(16 \cdot 14467$

$\Delta = 16 \cdot 4489.$

5 × 4937
4 × 3086
8 × 1543
5 × 1235
5 × 5 × 247
5 × 5 × 13 · 19
17 × 43
4 × 1030

256
256
2816

$3A = 47n^4 + 44n^3 + 7n^2 - 2n.$

$= ax^4 + bx^2 + cx^2 + dx + e$

4.86 $e = 0$
8.43 $= \square$

$3y^2 = \frac{a}{x^4} + \frac{b}{x^3} + \frac{c}{x^2} + \frac{d}{x}$

$y = Y/x^2$

$x^2 = Y/y$

$\frac{3Y^2}{x^4} = \frac{47}{x^4} + \frac{44}{x^3} + \frac{7}{x^2} - \frac{2}{x}$

$3y^2 = 47 + 44x + 7x^2 - 2x^3$

61	52	45	36	5	12	21	28
6	11	22	27	62	51	46	35
63	50	47	34	7	10	23	26
8	9	24	25	64	49	48	33
60	53	44	37	4	13	20	29
3	14	19	30	59	54	43	38
58	55	42	39	2	15	18	31
1	16	17	32	57	56	41	40

$S = 260$

$\frac{1}{2} \cdot 8 \cdot 65$

20, 30, 10, 40, 60, 80, 50, 70

58
70
71
4

7

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~~Dear Prof. Nick.~~

a (2, 1)
b -1)
c (-2, 1

	a		^					
				b				a'
	c	~~a~~				b'	d	
e					d'	f		
								g
	h							
8	11	23	35	38	50	62	65	77
61	64	76	a'	10	22	34	37	49

369

58	55
70	67
73	79
4	1
16	13
19	25
31	28
43	40
46	52
360	360

81	81		61	8
56	21		11	11
67	51		51	23
18	11		1	35
20	41		41	38
~~24~~	71		81	50
36	31		31	62
38	61		71	65
49	1		21	77
~~328~~	369		369	369
369				

35	37	51
16	24	5
81	56	67

36.0
2.9.8241
369

$\begin{pmatrix}2 & 2\\ 1 & 1\end{pmatrix}$ $\cancel{\begin{pmatrix}1, & 2\\ 1, & 1\end{pmatrix}\begin{pmatrix}2 & 2\\ & 1\end{pmatrix}}$ ×

$\begin{pmatrix}-1, & 3\\ -2, & 3\end{pmatrix}$ ✓

$\begin{pmatrix}2 & 0\\ 1 & 0\end{pmatrix}$ × $\begin{pmatrix}2 & 0\\ -1 & 2\end{pmatrix}$ × $\begin{pmatrix}-2 & 4\\ 1 & 0\end{pmatrix}$ × $\begin{pmatrix}-2, & 4\\ -1, & 2\end{pmatrix}$ × $\begin{pmatrix}1 & 1\\ 2, & -1\end{pmatrix}$ ✓, $\begin{pmatrix}1, & 1\\ -2, & 3\end{pmatrix}$ ×, $\begin{pmatrix}-1 & 3\\ -2, & 3\end{pmatrix}$ ×

a(2,1)
b(2,−1)
~~c(−2,1)~~
c(−2,2)
d(−1,−2)
e(2,1)

			a	
~~d~~ d	c			b
~~a~~	b'		e	
a'	d			c'

$i + 2x - 2y = 2$
$j + x - 2y = 2$

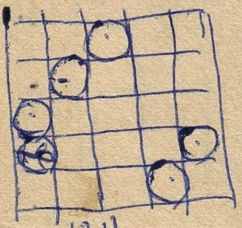
(2,0), (2,1), (2,2), (2,3), (2,4)
$(i,j) = (-2,0), (0,2), (2,4), (4,6)\ (6,8).$
$\equiv (3,0), (0,2), (2,4), (4,1), (1,3)$

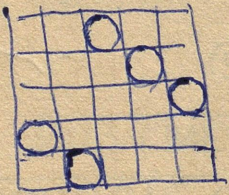
$i - 2x + 2y = 2$
$j + x - 2y = 2$

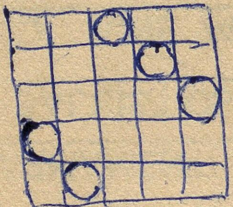
$(i,j) = (6,0), (4,2), (2,4), (0,6), (-2,8)$
$(1,0), (4,2), (2,4), (0,1), (3,3)$

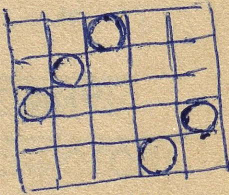
$i + x - y = 2$
$j - 2x + y = 2$

$(i,j) = (0,8), (1,5), (2,4), (3,3), (4,2)$


(2,1)


(−2,1)


~~(−1,2)~~ (1,−2)


(−1,−2)

$i - x + y = 2$
$j - 2x + y = 2$

(4,6), (3,5) (2,4), (1,3), (0,2)

(1,2), (−1,2)
(2,−1), (−3,−1)