Book No. 22

19/6/79 C.C.B.

(A) Math. Magazine - Vol. 49, No. 5

p. 253 Q. 642 - Prove that the numbers 49, 4489, 444889 -- obtained by inserting 48 into the middle of the preceding numbers are all perfect squares

p. 257 Q. 654 - Show that any even perfect number > 28 can be represented as the sum of at least two perfect numbers.

Soln. Solution quite trivial

ibid - Vol. 50. No. 3.

ibid - Vol 50. No. 5.

p. 258 - Defining a tree - A tree is a graph on n nodes in which each of the $\binom{n}{2}$ pairs of nodes is connected by a unique path.

p. 268 - Reference to a "Problem Solving Group", Berne, Switzerland

(B) Scripta Mathematica - Bound Vols 3 - 8. (1935-41)

Vol. 6. No. 1.

p. 56 Curiosa

No. 42 - Perfect nos as sums of cubes by R.V. Heath

Beginning with $P_2 = 28 = 1^3 + 3^3$

$P_3 = 496 = 1^3 + 3^3 + 5^3 + 7^3$

$P_4 = 8180 = 1^3 + 3^3 + \cdots + 15^3$.

No. 44 - There are only two rational Δ^s integer right Δ^s having their area equal to perimeter. They are (10, 8, 6), (13, 12, 5). Heronian

triangles having the same property are (17, 10, 9), (20, 15, 7) & (29, 25, 6)

from N. Allistone's Mathematical Snack-Ver.

p. 59 – Berry's remarkable relationship (S. M, V. 5, 1939, p. 176)

$$\frac{a^{10}+a^5+1}{a^2+a+1} = \frac{a^{12}+a^9+a^6+a^3+1}{a^4+a^3+a^2+a+1}$$

is a special case ($m=5$) of the following:

$$\frac{a^{2m}+a^m+1}{a^2+a+1} = \frac{a^{3m}-1}{a^m-1}\cdot\frac{a-1}{a^3-1} = \frac{a^{3m}-1}{a^3-1}\cdot\frac{a-1}{a^m-1} =$$

$$\frac{a^{3m-2}+a^{3m-6}+\cdots+a^3+1}{a^{m-1}+a^{m-2}+\cdots+a+1}$$ (Lt. Cd. Harry Jones)

p. 60. Soln to diophantine eqn $x^2-kxy+y^2=1$:

obvious soln is $x=0, y=1$ (or $x=1, y=0$). Putting

$x_1=y,\ y_1=ky-x$

we get a new soln. A continuous appln of this substitution will yield an infinite number of other solns. For eg if $k=3$, i.e. for $x^2-3xy+y^2=1$

we get $x=0, y=1$

$x_1=1, y_1=3\cdot1-0=3$ | $x_2=3,\ y_2=3\cdot3-1=8$

$x_3=8, y_3=3\cdot8-3=21$ | $x_4=21, y_4=3\cdot21-8=55$ etc.

(Soln due to Kiveliovitch. L'Ens. Math, 1938, p. 126)

Editor's note – It is interesting to note that the numbers

1, 3, 8, 21, 55, ... etc

are satisfying recurrence relation $U_n+1=3U_n-U_n-1$

6) are the coefficients of

$$\frac{1}{1-3x+x^2} = 1+3x+8x^2+21x^3+\cdots$$

Vol. 6. No. 2 – p. 95 – The life of Evariste Galois by Gustav Davidson

The most tragic story in the annals of mathematics

[Was not Ramanujan's another tragic story?]

p. 110 – Review of work on Surya – Siddhantha by W. E. Van Wijk

by G. Waldo Dunnington – Quite interesting

p. 114 – The anatomy of magic squares

(a) Squares of prime order with irregular knight's move formations

p. 120 Pythagorean numbers pleasantries:

If $x^2+y^2=z^2$, $(y+x)^2+(y-x)^2=2z^2$. If we now put

$$A=\frac{y+x+1}{2},\ B=\frac{y-x+1}{2},\ C=D=\frac{z-1}{2}$$

$$E=\frac{y+x-1}{2},\ F=\frac{y-x-1}{2},\ G=H=\frac{z+1}{2}$$

we have the identities

$$\left.\begin{aligned} A+B+C+D &= E+F+G+H \\ A^2+B^2+C^2+D^2 &= E^2+F^2+G^2+H^2 \end{aligned}\right\} \quad \text{Ex: } 8^2+15^2=17^2$$

In particular if $z-y=1$ each of the above sums will be a square

Ex: $5^2+12^2=13^2$ [Alfred Moessner

Vol. 6. No. 3 – The anatomy of magic squares by J. C. McCoy

Deals with squares of orders 6, 10, 14 ie whose orders are an even multiple of a prime number – To be read later.

p. 179. curiosa

(1) Lefkin's mental multiplication:

```
   77
 × 77
 ----
   49
 4949
   49
 ----
 5929

  666
 ×666
 ----
   36
 3636
363636
 3636
   36
 ----
443556
```

A^r variations

```
   77
 × 77
 ----
    7
  777
 ----
  847
 ×  7
 ----
 5929

  666
 ×666
 ----
    6
  866
66666
 ----
73926
 ×   6
 ----
443556
```

Same further

```
777777777777
777777777777

   49
  4949
 494949
49494949
 ----

   49
 ----
60493827160372839506 1729
```

Also its triangular variation

```
    7
  777
77777
 ----
 ----
 ----
864 —— 7
       7
 ----
6049 —— 29
```

(2) Curious identities

$$1^2+28^2+31^2+32^2+55^2+61^2+68^2 = 17^2+20^2+23^2+44^2+49^2+64^2+67^2$$

$$1^4+28^4+ \quad - \quad +68^4 = 17^4+20^4+ \quad - \quad +67^4$$

$$1^6+28^6+ \quad +68^6 = 17^6+20^6+ \quad - \quad +67^6$$

$$1^8+28^8+ \quad +68^8 = 17^8+20^8+ \quad - \quad +67^8$$

$$1^{10}+28^{10}+ \quad . \quad +68^{10} = 17^{10}+20^{10}+ \quad . \quad +67^{10}$$

(Alfred Moesner)

p. 184 – Beauty of geometrical forms by R. E. Moritz gives beautiful pictures of the curve $r = a\cos 4\theta$ for different value of a.

Vol. 6. No. 4.

p. 218 Curiosa

(i) Nos. with reversed order of digits.

Problem posed by A. A. K. – Existence of numbers which may be represented in more than one way as a product of nos. with reversed digits

Such a number is 101556 because $651 \times 156 = 372 \times 273 = 101556$.

Solution Sitavamanyalu Math. Student, Sept. 1939

Let (x', y') be reversals of (x, y). The numbers must be so selected that

(i) The product of any digit in x by a digit in y should not be > 9

(ii) When x is multiplied by y or y' in the ordinary way, the sum of the nos. in any col. should not exceed 9. For eg if $x = 12$, $y = 13$

$x' = 21$, $y' = 31$. These nos. satisfy (i) & (ii) & hence

$$(12.13)(21.31) = (12.31)(21.13)$$

ie $156 \times 651 = 273 \times 372$.

Mention obtaining palindromes by addition of nos. to reversals & continuing the process – Kaprekar's business – Doubtful! (G)

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p. 238 – The Lewis Carrol Problem

. . . . .

. . . . .

The problem is to move four dots only & make five rows with four in a row. Problem set in verse – Dudeney attributes the problem to Newton – Dudeney's four solutions

Ref (i) Hoffman's Puzzles old & new (1893?)

(ii) Sam Lloyd – Cyclopaedia of puzzles (N.Y. 1914).

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p. 246 – Curiosa

No. 57 – A family of giant Pythagorean Δs

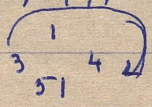
$$95,098,509,561,418,487^2 + 27,682,998,614,693,184^2$$
$$= 99,045,822,390,973,705^2$$

This is one 1024 primitive right Δs having a common hypotenuse. In addition to these 1024 primitive Δs, there are 87,549 more non-primitive Δs totalling 88,573 sharing the same hypotenuse

The fertility of this hypotenuse is that it is ~~the~~ a product of the prime numbers 5, 13, 17, 29, 37, 41, 53, 61, 73, 89, 97 all of the form $4k+1$ and it therefore yields $1+3+3^2+\dots+3^{10} = 88,573$ Pythagorean Δ^s of which $2^{10} = 1024$ are primitive (ie ~~having~~ sides having no factor in common)

No. 58 . π as a mystic number

The numerical values ~~of the~~ (Gematria) of the letters of the Hebrew word for God (Elohim) are 1, 30, 5, 10, 40 or dropping the zeros 1, 3, 5, 1, 4 which when put in the form of a ☉



1
3 4
5 1

Can be made to read 3.1415 ≃ π. [Pollak—Farfetcher]

Vol. 7. Nos 1–4.

p. 33 – Waring's Problem – Squares by R. G. Archibald

p. 68 – Curiosa

(i) From Problems in Biblical Hebrew:

$37(3+7) = 3^3+7^3$

$48(4+8) = 4^3+8^3$

$111(11+1) = 11^3+1^3$

$147(14+7) = 14^3+7^3$

$148(14+8) = 14^3+8^3$

× $123(1+2+3) = 1^3+2^3+3^3$

(ii)

$4 \times 1738 = 6952$

$4 \times 1963 = 7852$

$47 \times 198 = 5346$

$42 \times 138 = 5796$

~~12 × 297 =~~

$12 \times 483 = 5796$

$18 \times 297 = 5346$

$28 \times 157 = 4396$

$39 \times 186 = 7254$

Scripta Math (Contd) Vols 1-4

Another illegal operation — due to a student

The progression $s = 242$, $a = 2$, $n = 5$. Find r

$$s = \frac{a(r^n - 1)}{r - 1}, \quad 242 = 2\,\frac{r^5 - 1}{r - 1}; \quad \frac{r^5 - 1}{r - 1} = \frac{242}{2},$$

$$\frac{r^5 - 1}{r - 1} + \frac{1}{1} = \frac{242}{2} + \frac{1}{1} \quad \text{Then } \frac{r^5}{r} = \frac{243}{3} = 81, \; r^4 = 81, \; r = 3$$

[W. J. Hagard

A mnemonical rule. Write in succession the square of seven, seven, twice seven, twice the square of seven, seven; before this group of figures place a decimal point & before that zero as a characteristic. The result

0.49714987, is $\log_{10} \pi$ to eight decimal places

[Bernard H. Dawson]

An interesting property of prime numbers:

If k^{th} prime be P_k.

(i) P_{2n} may be represented as an algebraic sum of all the preceding primes each taken once & only once

(ii) P_{2n+1} may be represented in the same way, except that the last addend is to be taken twice. Thus

$P_1 = 2 = 2 \cdot 1$, $P_2 = 3 = 1 + 2$

$P_3 = 5 = 1 - 2 + 2 \cdot 3$, $P_4 = 7 = 1 - 2 + 3 + 5$

$P_5 = 11 = 1 - 2 + 3 - 5 + 2 \cdot 7$

$P_6 = 13 = 1 + 2 - 3 - 5 + 7 + 11$

$P_7 = 17 = -1 - 2 + 3 - 5 + 7 - 11 + 2 \cdot 13$

$P_8 = 19 = 1 - 2 - 3 - 5 + 9 - 11 + 13 + 17$

wrong 9 is not prime

Renato Dela Torre

Mathematical quintuplets:

In Sphinx V, 5, 1935, p. 141, Gloden & Zanetti give the following interesting relationships which are correct for $n = 1, 2, 3, 4, 5$

a) $1^n + 5^n + 10^n + 18^n + 23^n + 27^n = 2^n + 3^n + 13^n + 15^n + 25^n + 26^n$

b) $1^n + 6^n + 7^n + 17^n + 18^n + 23^n = 2^n + 3^n + 11^n + 13^n + 21^n + 22^n$

Not content with these, M. Marenholz, in the same vol (p. 176) gives the following formula from which any number of quintuplets may be obtained

$$a^n + (a+4b+c)^n + (a+b+2c)^n + (a+9b+4c)^n + (a+6b+5c)^n + (a+10b+6c)^n$$
$$= (a+b)^n + (a+c)^n + (a+6b+2c)^n + (a+4b+4c)^n + (a+10b+5c)^n$$
$$+ (a+9b+6c)^n$$

correct for $n = 1, 2, 3, 4, 5$

p. 14 Curiosa:

(1) A simple proof of a seemingly complicated proposition. Some of the most difficult propositions in number theory are stated in very simple form for eg. Fermat's & Goldbach's theorems. It is easy to reverse the process, namely to build up complicated looking propositions the proof of which is surprisingly simple

(i) It is impossible to find integral solns of

$$x^6 + y^6 + z^6 + x^2y + y^2z + z^2x + xyz = 0$$

Proof:– Unless x, y, z are even the value of the above expn is always odd

(ii) The same applies to

$$x^4 + y^4 + z^4 - x^4y^2 - y^4z^2 - z^4x^2 \pm x^2y^2z^2 = 0$$

Proof: – Unless x, y, z are divisible by 3, the above expn $\equiv \pm 1 \pmod 3$.

(Emanuel Lasker)

Triplets again

Euler's error

$$\left.\begin{aligned} 15+42+48 &= 16+35+54 = 21+24+60 \\ 15\cdot 42\cdot 48 &= 16\cdot 35\cdot 54 = 21\cdot 24\cdot 80 \end{aligned}\right\}$$

Another integers of

$$a_1+a_2+a_3 = b_1+b_2+b_3 = c_1+c_2+c_3$$

$$a_1a_2a_3 = b_1b_2b_3 = c_1c_2c_3$$

is $a_1 = Aax$, $a_2 = Bby$, $a_3 = Ccz$

$b_1 = Bcx$, $b_2 = Cay$, $b_3 = Abz$

$c_1 = Cbx$, $c_2 = Acy$, $c_3 = Baz$

where A, B, C, a, b, c are arbitrary, and

$$x = Bca^2 + ABb^2 + ACc^2 - B^2ab - C^2ac - A^2bc$$

$$y = ABa^2 + ACb^2 + BCc^2 - A^2ab - B^2ac - C^2bc$$

$$z = ACa^2 + BCb^2 + ABc^2 - C^2ab - A^2ac - B^2bc$$

Approximations to π. [A. Moessner]

(A) $\dfrac{104348}{33215} = 3.1415\,92653\,|\,92142$

(B) $\dfrac{833719}{265381} = 3.1415926535\,8\,|\,108$

B is 38 times as good as A, i.e. its error is $1/38$ of A's

[W. Hope-Jones in Math. Gaz. v. 23 p. 284]

p. 15. Article by Moschopoulos – magic squares by J. A. C. McCoy

pp. 47–48 – The old problem of multiplying 142857 by each of 1, 2, 3, 4, 5, 6, 7 and of getting the 1st six products as cyclic permutations & the last as 999999, has been generalised here – [Mannis Charosh]

pp. ~~49 – The~~ 49–55 – The anatomy of magic square by J. C. McCoy

pp. 56–57 – Interlocked mathematical progressions by A. A. K – gets a new formula & applies it to a result of Mahavirachaya (Ganitasara-Sangraha tr. by M. O. Rangacharya)

Numbers – Fun & facts by J. Newton Friend – Book in C. C. B (Scribner's, N. Y. 1954)

p. 36 Gives 220, 284 as amicable numbers – mentions discovery of a second pair 17,296 & 18,416 by Fermat in 1636 – many others on the high side found since then

p. 40 Digital root (refers to Dudeney's Amusements in Mathematics)
Eg: 43769 – Sum of digits = 29, 2+9 = 11, 1+1 = 2, the digital root

p. 41 perfect nos – mentions 6, 28, 496, 8128, 33,550,336

" Prime nos

pp. 42–43 – Sieve of Eratosthenes – Mentions that between 1 and 1000 there are 169 primes and of these 16 are palindromes – Between 1 & 10 million there are 664,580 primes

p. 44 Gives Euler's expn x^2+x+41 giving primes for $x = 0, 1, 2, \ldots 39$, but when $x = 40$, expn $= 1681 = 41^2$ – Says –ve x also allowed, & shows values

only for $x = -1, -2, -5$ - It can be verified that from $x = -1$ to $x = -40$ we get primes (last one being 1601), and $x = 1$ to $x = 39$ we get primes (last one being also 1601) - The author does not mention this - In fact taking $f(x) = x^2 + x + 41$, we have

$f(-n) = f(n-1)$ for all n from 1 to 40 - Also the formula does not give all the primes between 1 and 1601 but only 80 of them - Also we get only 5 prime palindromes between 1 and 1601 instead of 16 palindromes between 1 and 1,000 - Also there are only 15 palindromic primes between 1 and 1000 (excluding 1).

Mentions $2^{127} - 1$ as highest prime known & states that Lucas took 19 years(?) in checking (this number 170,141,183,460,469,231,731,687,303,715,884,105,727 is 170×10^{36} approx).

Classification of primes - All primes can be divided into two classes $4n+1$ or $4n-1$ but not every $4n+1$ or $4n-1$ is a prime -

p. 45 Prime curios: The following four

(i) 19,109, 1009, 10009

are primes & no other digit can replace the 9 and yield four primes.

(ii) Primes are the only numbers which when cubed yield palindromes.

(X) No composite number can do so - Gives example

$11^3 = 1331$.

This statement is wrong since $2201^3 = 10662526601$ is a palindrome, but $2201 = 31 \times 71$ is composite

Re(i) one may raise the question whether this applies to only four numbers of this type and also whether one can find more than four such numbers (or less than four) with same end digits & same properties

The answer to the first question is no since, if we consider 100009 (irrespective of whether it is prime or not) replacing 9 by 1 gives 100001 which is divisible by 11 ~~so, second question appears difficult~~ ~~even~~ If we replace "more than four" by "three". The answer is "no" since out of 11, 101, 1001 ~~the~~ 13, 103, 1003; 17, 107, 1007 the last third numbers are not prime.

For two numbers note that 11, 101; 13, 103; 17; 107 are all primes & hence so replacing 9 in 19, 109 by any other digit also gives primes.

So (i) is a real curio although trivial.

---

p. 52 Mnemonics on π

(i) How I wish I could recollect pi easily today

3 1 4 1 5 9 2 6 5

figures below corresponding to numbers of the letters of the word above

(ii) Que j'aime à faire connaitre ce nombre

3 1 4 1 5 9 2 6

utile aux vrais amateurs Archimède

5 3 5 8 9

π = 3.14159265358979323846264338327 9 . . .

The following is a Kannada version of this mnemonic:

ಇಲ್ಲಿಯೋ ~~ಈ~~ ~~ಅನೇಧಾರಣಿ~~ ತರಹದ ~~ಈ~~ ಅನೇಧಾರಣಿ

3 1 4 1 5

ಆರೇಶವದಾದಿಹಿಡಿದ ಸಂಖ್ಯೆ ಜ್ಞಾಪಿಸಿಕೊಳ್ಳುವ ದಾರಿ ಕೋರಿದೆ

9 2 6 5

(3.14159265)

---

p.55 - Chap V - The old curiosity shop.

(1) Nine eqns each containing nine digits 1 to 9 once

4 × 1738 = 6952 ; 12 × 483 = 5796

4 × 1963 = 7852 ; 42 × 138 = 5796

18 × 297 = 5346 ; 28 × 157 = 4396 ; 48 × 159 = 7632

27 × 198 = 5346 ; 39 × 186 = 7254

(2) Including 0:

2 × 3485 = 1 × 6970 ; 3 × 5694 = 17082 ; 78 × 345 = 26910

64 × 915 = 80 × 732 27 × 594 = 16038

(3) 3 × 51,249,876 = 153,749,628

6 × 32,547,891 = 195,287,346.

9 × 16,583,742 = 149,253,678.

(4) 987654321 − 123456789 = 864197532

[This is an example of nine-digit R-S. invariant. There are heaps more]

~~123456789 x~~

(5) $2 \times 78 = 156 = 4 \times 39$ – Double eqn involving all nine digits

(6) $123,456,789 \times 3 = 246,913,578$

" $\times 4 = 493,827,156$

" $\times 5 = 617,283,945$

" $\times 7 = 864,197,523$

" $\times 8 = 987,654,312$

(7) $3839 \times 2 = 7678$; $2789 \times 3 = 8367$ – what is curious about this?

(8) $8 \times 473 = 3784$; $21 \times 87 = 1827$

$9 \times 351 = 3159$; $27 \times 81 = 2187$

$15 \times 93 = 1395$; $35 \times 41 = 1435$

$14 \times 926 = 12964$; $78 \times 624 = 48672$

$24 \times 651 = 15624$; $87 \times 435 = 37845$

$42 \times 678 = 28476$; $72 \times 936 = 67392$

$51 \times 246 = 12546$; $65 \times 281 = 18265$

$57 \times 834 = 47538$; $65 \times 983 = 63895$

$75 \times 231 = 17325$; $86 \times 251 = 21586$.

(9) It occasionally happens that . .

$2178 \times 4 = 8712$

$1089 \times 9 = 9801$ [Rouse – Ball's problem – under R-S numbers]

(10) $362 - 236 = 14 \times 9$; $632 - 263 = 9 \times 41$

(11) Following eqns are correct whether the digits are read as usual from left to right or

From right to left:

112 × 113 = 12656 & 65621 = 311 × 211

112 × 124 = 13888 & 88831 = 421 × 211

112 × 133 = 14896 & 69841 = 331 × 211

112 × 223 = 24976 & 67942 = 322 × 211

122 × 213 = 25986 & 68952 = 312 × 211

[This appears to be a special case of my result of product of reverses being reverse of the products]

Says it is not difficult to find many others & if the cipher be included the number may be considerably increased.

Miscellaneous oddities

(12) (1+2−3−4)(5−6−7−8−9) = 100

(13) The following 3 digit nos are each divisible by the product of their digits

111, 112, 115, 128, 132, 135, 144

175, 212, 216, 224, 312, 315, 384

432, 612, 624, 672, 735, 816.

[There is something approaching the notion of persistence of nos. ~~some [illegible] are divisible by their persistence number, [illegible] except those with [illegible]~~ ~~[illegible]~~ Persistence no of 135 = 2 (1.3.5 = 15, 1.5 = 5)

35
15
5

but it is not divisible by 2. p(175) = 3 ✗ 175, p(315) = 2 ✗ 315 – nothing interesting about this division by persistence number]

96
54
20
0

(14) Digital triangle

$$1 \times 8 + 1 = 9$$
$$12 \times 8 + 2 = 98$$
$$123 \times 8 + 3 = 987.$$
$$1234 \times 8 + 4 = 9876$$
$$\text{— — — — — —}$$
$$123456789 \times 8 + 9 = 987654321.$$

(15) £98,765, 4s, $3\frac{1}{2}$d (involving 1 — 9) is the largest sum in English Currency

£2,567, 18s, $9\frac{3}{4}$d — — " smallest such sum.

(Do people count $\frac{1}{2}$d & $\frac{3}{4}$d?)

(16) $79 + 5\frac{1}{3} = 84 + \frac{2}{6} = 84\frac{1}{3}$.
(odd digits) (even digits)

(17) Reciprocal of Perelman's fractions (see p. 9 of this note book) using all 1 to 9; but in addition we have $7 = \frac{16758}{2394}$

(18) For 5 gives in addition to $5 = \frac{13485}{2697}$ (same as Perelman's $\frac{1}{5}$) gives

$$5 = \frac{13845}{2769} = \frac{14865}{2973} = \frac{18645}{3729}.$$

(19) Similarly two other different ways for 9 (in addition to Perelman's $\frac{1}{9}$)

$$9 = \frac{95249}{8361} = \frac{58239}{6471}.$$

(20) Three more ways for 9 if 0 be included

$$\frac{97524}{10836} = \frac{95823}{10647} = \frac{95742}{10638} = 9.$$

(21) Nine digits arranged in 11 different ways to represent 100

$3\frac{69258}{74} = 3 + 97$, $91 + \frac{5742}{638} = 91 + 9$, $81\frac{5643}{297} = 81 + 19$

$96\frac{1428}{357} = 96 + 4$ $\quad 91 + \frac{5823}{647} = 91 + 9$, $81\frac{7524}{396} = 81 + 19$

$96\frac{1752}{438} = 96 + 4$ $\quad 94 + \frac{1578}{263} = 94 + 6$, $82\frac{3546}{197} = 82 + 8$.

$96\frac{2148}{537} = 96 + 4$. $\quad 91 + \frac{7524}{836} = 91 + 9$

(22) trivial

(23) $\frac{2 \times 3485}{1 \times 6970} = 1$, $123456789^0 = 1$

Curious numbers – depending upon divisibility by 9

~~States~~ Every R. S. no divisible by 9.

Re. 37; 37 multiplied by 3, 6, 9, 12, 15, 18, 21, 24, 27, -- gives

111, 222, 333, .. 999

[Does not go beyond 27 – 30 gives 1110, ~~33 gives 1221~~ 37×33 = 1221

a palindrome, 37×36 = 1332, etc. no special relations]

Re. 153. 153 = 3 × 51 a palindromic eqn.

[I find that pp. 65 – 68 have been torn off by some miscreant]

pp. 69 – 70 – Fallacies all trivial

pp. 71 – 76 – other problems like Apples sale, Achilles & the tortoise, ancestors etc all trivial

p. 77. Chapter VI – The powers that be. (Good title).

Stay about Professor's ~~CLASSES~~ CLASSES

Prelate, Relate, elate, late, ate

$275,625 = 525^2$

$75,625 = 275^2$

$5,625 = 75^2$

$625 = 25^2$

$25 = 5^2$

Rearrangements: (Now, own, won), (Amen, Mean, Mane), (Caters, Crates, Reacts, Recast).

$169 = 13^2$, $196 = 14^2$, $691 = 31^2$

$1296 = 36^2$, $2916 = 54^2$, $9216 = 96^2$

$11664 = 108^2$, $16641 = 129^2$, $41616 = 204^2$

$23716 = 154^2$, $32761 = 181^2$, $72361 = 269^2$

$36481 = 191^2$, $38416 = 196^2$, $43681 = 209^2$

$42849 = 207^2$, $49284 = 222^2$, $82944 = 288^2$.

A useful observation:

25 & 76 are the only two pairs of digits in the first & second places of a number that reappear when that number is squared, cubed or raised to a higher power:

$25^2 = 625$	$76^2 = 5,776$
$25^3 = 15,625$	$76^3 = 438,776$
$25^4 = 390,625$	$76^4 = 33,362,176$

$325^2 = 105,625$, $376^2 = 141,376$

Hence (i) the only two numbers that are the squares of the first & second digits are

$625 = 25^2$

$5776 = 76^2$

(ii) the only two numbers which are squares of their last three digits are

$141,376 = 376^2$

$390,625 = 625^2$

Reversals:

$12^2 = 144$, $21^2 = 441$; $13^2 = 169$, $31^2 = 961$

Higher numbers are

$112^2 = 12544$, $211^2 = 44521$

$113^2 = 12769$; $311^2 = 96721$

$122^2 = 14884$, $221^2 = 48841$

$1112^2 = 1236544$; $2111^2 = 4456321$

$1113^2 = 1238769$; $3111^2 = 9678321$

$1121^2 = 1256641$; $1211^2 = 1466521$

$1212^2 = 1468944$; $2121^2 = 4498641$.

Freaks & curiosities

(1) $18 \times 9^2 = 2592$

(2) 48 & 1680 are unique in the sense

$$48+1 = 49 = 7^2, \quad 1680+1 = 1681 = 41^2$$

$$\frac{48}{2}+1 = 25 = 5^2; \quad \frac{1680}{2}+1 = 841 = 29^2$$

(3) The following nos. are peculiar in that each is exactly divisible not merely by the sum of its digits, but also by the square of the sum of its digits

162, 243, 324, 392, 512, 648, 972

1296, 2156, 2916, 3564, 4536, 5184, 5632, 5832, 7128, 7452

Prog: Take 392, $3+9+2 = 14$, $\frac{392}{14} = 28$, $\frac{392}{14^2} = 2$.

(4) It is worth noting that

$10 = 1^2+3^2$, $10,000 = 28^2+96^2$

$100 = 8^2+6^2$ $100,000 = 12^2+316^2$

$1000 = 18^2+26^2$ $1,000,000 = 352^2+936^2$

Also while $10 = 1+2+3+4$

$100 = (1+2+3+4)^2$

$1000 = (1+2+3+4)^3$

— trivial

p.82 (5) Pythagorean Δ^s: $\rightarrow 33^2+56^2=65^2$

$33^2 = 65^2 - 56^2$ ✓ (56 & 65 reverse)

$41^2 - 14^2 = 87 - 78^2$ } what have these to do with
$62^2 - 26^2 = 97^2 - 79^2$ } Pythagorean theorem?
$72^2 - 27^2 = 96^2 - 69^2$ }

Hoppono's rule: Corresponds

$3^2+4^2=5^2$

one has $10^2+11^2+12^2 = 13^2+14^2$ (This is the case $n=$ in my Vigdis's investigation)

Square roots.

(1) $\sqrt{81} = 8+1=9$ $(8+1)^2 = 81$ only one given by Maclaurin

$\sqrt{2025} = 20+25=45$ $\sqrt{(88,209)} = 88+209$

$\sqrt{(3025)} = 30+25=55$; $\sqrt{(494,209)} = 494+209=703$

$\sqrt{(9801)} = 98+01=99$, $\sqrt{(998,001)} = 998+1=999$

(2) Square roots of 6561 and 8281 can be obtained by summing up its digits & reversing:

6561 .. $6+5+6+1=18$, $81 = \sqrt{(6561)}$

8281 — $8+2+8+1=19$, $91 = \sqrt{(8281)}$

(3) 784 is unique in that its square root is the product of its first & last digits: $\sqrt{(784)} = 7\times4 = 28$.

Cubes — mentions Fermat's theorem

Square cubes — should be (cube cubes)

(1) Solve $x^3+y^3+z^3 = w^3$:

$3^3+4^3+5^3 = 6^3$
$1^3+6^3+8^3 = 9^3$ } 3^{3s} is trivial following from the 1st.
$6^3+8^3+10^3 = 12^3$

(2) other more complex expressions are

$1^3+5^3+7^3+12^3 = 13^3$

$2^3+3^3+8^3+13^3 = 14^3$

$1^3+2^3+7^3+10^3+14^3 = 16^3$

(3) $153 = 1^3+5^3+3^3$
$370 = 3^3+7^3+0^3$
$371 = 3^3+7^3+1^3$
$407 = 4^3+0^3+7^3$
} 3rd order P.D. 9's

(4) 1729 is unique — (Ramanujan taxi-cab no. problem)

$1729 = 12^3+1^3 = 10^3+9^3$

(5) $10 = 1+2+3+4$

$100 = 1^3+2^3+3^3+4^3$.

(6) The only digit to yield a square when its cube is added to its divisors is 7. Thus

$7^3+(1+7+7^2) = 400 = 20^2$, $(1, 7, 7^2$ divisors of $7^3)$

(7) 1, 2, 5 are the only three digits that can be arranged to yield two cubes

$125 = 5^3$, $512 = 8^3$.

Cube roots:

(1) $\sqrt[3]{(512)} = 5+1+2 = 8$ (only case found by Madathy)

$\sqrt[3]{(4913)} = 4+9+1+3 = 17$

$\sqrt[3]{(5832)} = 5+8+3+2 = 18$

$\sqrt[3]{(17576)} = 1+7+5+7+6 = 26$

$\sqrt[3]{(19683)} = 1+9+6+8+3 = 27$

(2) Cube roots of nine nos (each $< 10^6$) given by the reversal of the sum of their digits

148,877	sum of digits	35	cube root	53
238,328	"	26	"	62
373,248	"	27	"	72
531,441	"	18	"	81
551,368	"	28	"	82

(3) Cube roots obtained by multiplying first & last digits (sums $< 10^6$)

$\sqrt[3]{(125)} = 1 \times 5 = 5$

$\sqrt[3]{(3375)} = 3 \times 5 = 15$

$\sqrt[3]{(91,125)} = 9 \times 5 = 45$

Higher Powers. ~~with i of obtained roots to digits~~ Highest nos. obtained

For 1 — 1^{11} — 4 ones

2 — 2^{22} — 3 twos

3 — 3^{33} — 3 threes

4 — 4^{4^4} — 3 fours

9 — 9^{9^9} — 3 nines

Googol & Googolplex — Spelt it Googal. names by 10^{100} names by a nephew of Kasner it seems

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p. 94 — Squares — magic & otherwise

Chinese do — the seven — 4th order Durer's magic square (consecutive)

Prime squares

67	1	43
13	37	61
31	73	71

(Dudeney's square

3	71	5	23	
53	11	37	1	
17	13	41	31	
29	7	19	47	

Bergholt & Shuldham's square

Reversible squares

15	96	93	38
94	37	16	95
36	91	98	17
97	18	35	92

(i)

83	39	69	51
59	61	73	49
71	89	19	63
29	53	81	79

(ii)

Rows of (ii) are reverses of rows of (i) both magic with S = 242

Also further the 1st place digits of all rows, cols & diagonals all add to 22 in both squares

& so also the second place digits.

(Quite interesting squares)

Chapter VIII — Palindromes

Greek palin backwards, dromos running.

(1) Reversal - ad dition process for palindromes - doubtful

(2) 16 out 99 palindromes between 1 & 1000 are prime - trivial

(3) Power palindromes.

(1) mentions $11^2, 22^2, 101^2, 111^2, 121^2, 202^2, 212^2$ as palindromes trivial

$26^2, 264^2, 307^2, 836^2$ are palindromes — trivial

(2) $2772 = 86^2 - 68^2$, $5445 = 83^2 - 38^2$, $6336 = 80^2 - 08^2$

(Interesting, to be looked into for more such)

(3) mentions $7^3, 11^3$ as only two $< 10^6$

& says next highest is $101^3 = 1030301$ — trivial

(Does not know 2201^3 is a palindrome).

(4) only 4th power one $< 10^6$ is $11^4 = 14641$
There is no 5th power palindrome $< 10^7$. } trivial

(5) $11, 11^2, 11^3, 11^4$ are palindromes but not 11^5, etc — trivial

Products Expns palindrome as a whole

(1)

$3 \times 51 = 153$, $6 \times 21 = 126$, $8 \times 86 = 688$

$12 \times 42 = 24 \times 21$, $21 \times 36 = 63 \times 12$

$13 \times 62 = 26 \times 31$, $23 \times 64 = 46 \times 32$

$14 \times 82 = 28 \times 41$, $24 \times 84 = 48 \times 42$

(Can this be generalised?)

(2) $111 \times 111 = 12321$, $121 \times 121 = 14641$

$111 \times 121 = 13431$, $121 \times 212 = 25652$

$111 \times 131 = 14541$, $121 \times 222 = 26862$, $212 \times 212 = 44944$

& so on to $111 \times 171 = 18981$

(anything special in this?)

101. $101^2, 101^3, 101^4$, & so on all palindromes ×

(wrong gives $101^4 = 104040401$ should be 104060401

& 101^5 not a palindrome

1001 = 7 × 11 × 13 — probably ~~most~~ most famous numerical palindrome

Arabian nights' entertainments — or Thousand & one nights — I have a "Thousand and one things to do".

X Says 100_1^2, 100_1^3 & so on are palindromes like 101 — wrong again

69696 = 6336 × 11 & 6336 = 8 × (63 + 36) × 8

(Are there more like this?

Palindromes of like digits:

$\sqrt{121} = 11$

$\sqrt{(123321)} = 111 = 3 \times 37$

$\sqrt{(1234321)} = 1,111 = 11 \times 101$

$\sqrt{(12345432 1)} = 11,111 = 41 \times 271$

$\sqrt{(1234565432 1)} = 111,111 = 3 \times 7 \times 11 \times 13 \times 37$

$\sqrt{(12345676543 21)} = 1,111,111 = 239 \times 4649$

$\sqrt{(1234567876543 21)} = 11,111,111 = 11 \times 73 \times 101 \times 137$

$\sqrt{(123456789876543 21)} = 111,111,111 = 9 \times 37 \times 333,667$

and so on (what does this 'so on' mean?)

[These above relations connects repunits & polyndromic square roots

11, 111, 111, 111, 111, 111 = 2,071,723 × 5,363,222,357

(17 ones)
111, 111, 111, 111, 111, 111 = 7 × 9 × 11 × 13 × 19 × 37 × 52,579 × 333,667

(18 ones)
No one has however as yet resolved the number of 19 repunits

(To look into Yates work on repunits).

$1 \times 9 + 2 = 11$

$12 \times 9 + 3 = 111$

$123 \times 9 + 4 = 1111$

$1234 \times 9 + 5 = 11111$

— — — — —

$12345678 \times 9 + 9 = 111,111,111$ (*)

& this last is equivalent to $12345679 = 111,111,111$ (L.H.S not containing 8)

Take ~~on~~ the sequence 1, 2, 3, 4, 5, 6, 7, 9 (without 8)

$4 \times 9 = 36$] $12345679 \times 36 = 444,444,444$

$7 \times 9 = 63$ " $\times 63 = 777,777,777$

$8 \times 9 = 72$ " $\times 72 = 888,888,888$

(*)

trivial since $12345679 \times 9 = 111,111,111$ as author himself remarks

---

Using $a^2 - b^2 = (a-b)(a+b)$:, $a = 6$, $b = 5$ gives

$a^2 - b^2 = 11$, $a = 56$, $b = 45 \rightarrow a^2 - b^2 = 101 \times 11 = 1,111$

ie $6^2 - 5^2 = 11$

$56^2 - 45^2 = 1111$

$556^2 - 445^2 = 111,111$

$5556^2 - 4445^2 = 11,111,111$

$55556^2 - 44445^2 = 1,111,111,111$

& so on

Simly with $a = 7$, $b = 4$, $7 - 4 = 3$

$7^2 - 4^2 = 33$

$57^2 - 54^2 = 333$

$557^2 - 554^2 = 3333$

$5557^2 - 5554^2 = 33333$

— — & so on

with $a = 8$, $b = 3$, $8 - 3 = 5$

$8^2 - 3^2 = 55$

$58^2 - 53^2 = 555$

$558^2 - 553^2 = 5555$

& so on

with $a = 9$, $b = 2$, $9 - 2 = 7$

$9^2 - 2^2 = 77$

$59^2 - 52^2 = 777$

$559^2 - 552^2 = 7777$

etc.

$a = 10$, $b = 1$, $10 - 1 = 9$

$10^2 - 1^2 = 99$

$60^2 - 51^2 = 999$

$560^2 - 551^2 = 9999$

$5560^2 - 5551^2 = 9999$

etc.

for polynomials with 8, $a^2 - b^2$ doesn't appear to apply

use $9 \times 9 + 7 = 88$

$98 \times 9 + 6 = 888$

$987 \times 9 + 5 = 8888$

$9876 \times 9 + 4 = 88888$

— — — —

$98765432 \times 9 + 0 = 888,888,888$

(All these must be connected with repunits)

$11^2 - 0^2 = 121$

$61^2 - 50^2 = 1221$

$561^2 - 550^2 = 12221$

$5561^2 - 5550^2 = 122221$

etc.

(obviously this man is like Kaprekar. Anyway no striking results re polyndromes are obtained)

Chap 9

pp. 111 – 157 – Numbers – Tradition & Superstition – mostly from western culture – would be interesting if we thought of alternative oriental, specially Indian culture

Chap. 10 – Numerical problems – Claims some new – good many cannot be solved by arithmetical or algebraic methods – Taxing ones ingenuity – All told 40 problems are given

No. 21 is ~~about~~ from an Indian manuscript of 850 A.D about 100 birds.

No 37. is about a Hindu merchant

No. 34 – Diffce between the squares of two whole numbers is 12345. What are the numbers? Ans given 1237 & 1232 – Some reader of this book has ~~mentioned~~ added 419 & 404 in ink.

Chap. 11 – Problems with digits & letters (~~[illegible]~~)

(1) Given $(ab)^3 = cadb$ (a, b, c, d ~~are different~~ distinct)

and $(cd)^2 = cabd$, find a, b, c, d

22^3 has 5 digits so $ab < 21$ also last digit of $(ab)^3$ is b & b = 1, 4, 5, 6, 9.

∴ Possible ab = 11, 14, 15, 16, 19, 21. – of these only 21 satisfies $(cd)^2 = cabd$

~~$11^3 = 1331$. $(31)^2 = 961$ ×, $14^3 = 2744$, $94^2 = 1936$.~~

~~$15^3 = 3375$~~

2. Given $(bc)^2 = abc$ find the appropriate digits. only 25, & 76 when squared end with 25 & 76. Trial shows that 25 is the required no since 76^2 has 4 digits

otherwise

$$\begin{array}{r} bc \\ bc \\ \hline xxc \\ xx \\ \hline abc \end{array}$$

(no we carry)

c must be 1, 5, 6, bc not > 31 since $32^2 = 1024$ has four digits. Hence bc = 15, 16, 21, 25, 26, 31

of these 25 is the only one that satisfies above condn.

(3) $(abc)^2 = bcdbc$, abc not > 316 since 317^2 has 6 digits

square of abc ends in bc, hence bc = 25 or 76.

∴ abc must be 125, 176 or 276 since a, b, c are different

of these $(276)^2 = 76176$ satisfies the condns

(4) $(ab)^c = cdb$, 10^3 has four digits hence $c < 3$, $c = 0, 1, 2$, $c \neq 0$ or 1.

∴ $c = 2$ ie $(ab)^2 = 2db$. ~~since Hence b~~ $(ab)^2$ ends with 1, 4, 5, 6, 9

~~ie (ab)² ends with~~ ie b = 5 or 6, Trial gives $16^2 = 256$.

(5) $(ed)^b = ebad$
$(cd)^b = abed$
$(ba)^c = ebda$

Since 10^4 has 5 digits neither b nor c is > 3.

neither can be 0 or 1, hence b = 2, c = 3

or b = 3, c = 2

Since $(ba)^c$ yields a number with final digit a

For c = 2, b = 3, a is 1, 5 or 6, ie we have to consider $(31)^2, (35)^2, (36)^2$

c = 3, b = 2, a is 1, 5, 6, or 9 & $(21)^3, (25)^3, (26)^3, (29)^3$.

$(31)^2$ has 3 digits, $(25)^3, (26)^3, (29)^3$ have more than 4 digits

∴ only $(35)^2, (36)^2, (21)^3$ are left out & trial gives only 21.

(8) (i) $(cc)^2 = braa$, (ii) $(aaa)^2 = abcba$

(iii) $a+d=9$, $cbca = (da)^2$

$c+b+c+a = ad$

(iv) $(acd)^2 = aebbd$, $(bed)^2 = adbeab$, (v) $(db)^2 = beab$

(vi) $(ed)^2 = ceed$, $(ed)^3 = bafged$ $(bab)^2 = adbeab$.

[not worked out]

(i) Looking at table of squares just $88^2 = 7744$

(ii) $111^2 = 12321$ is a palindromic square. 222, 333, 444, 555, 666, 777, 888, 999 are not palindromes as only unique.

(iii) Now $(da)^2 = cbca$ we have $da > 31$, & $a = 1, 5$ or 6

& from $a+d=9$, we have $(a,d) = (8,1), (4,5), (3,6)$

The last cond^n $a+b+2c$ is satisfied by $(81)^2 = 6561$ since in this $= ad$

$c=6$, $b=5$ & $a=1$, thus $2c+b+a = 18 = ad$.

(iv) acd & bcd should both be < 320 so as to have three squares in 5 digits. Also $d = 1, 5$ or 6.

from $(acd)^2 = aebbd$, we should have $a=1$.

& $(135)^2 = 18225$ satisfies the first eqn with $a=1, e=8, b=2, d=5$ & $c=3$

& the second eqn $(285)^2 = 81225$ is also satisfied with these values

(v) $db > 32$ & $b = 1, 5$ or 6

$(41)^2 = 1681$ satisfies first eqn but not the second

$(75)^2 = 5625$ satisfies first eqn with $b=5, d=7$,

& second eqn $(525)^2 = 275625$ is also satisfied $a=2, c=6$

(vi) Both the eqns give ed = 25 or 76. in view of this repeating ~~phen~~ as last two digits when squared or cubed. It is easily verified that ~~9 2~~ ed = 76 satisfies both the eqns.

[worked (i) to (vi) independently & found answers tallying with those given on p. 193 of the book.

(7) A simple addition sum involving all 9 digits (0 excluded)

Given
$$\begin{array}{r} a\,b\,9 \\ c\,1\,d \\ \hline 5\,e\,f \end{array}$$
to find a, b, c, d, e, f

a = 2, c = 3 or a = 3, c = 2,

ie
$$\begin{array}{r} 2\,b\,9 \\ 3\,1\,d \\ \hline 5\,e\,f \end{array} \quad \text{or} \quad \begin{array}{r} 3\,b\,9 \\ 2\,1\,d \\ \hline 5\,e\,f \end{array}$$
(i) (ii)

, b, d, e, f must take values 4, 6, 7, 8

~~[illegible]~~ ie d = 4 or 6 is not possible

using (i) & hence d = 7 or 8, d = 7 gives f = 6 & 2 + b = e & trying with b = 4 or 8 gives e = 6 or 0 which is impossible. Hence taking d = 8, f = 7

~~and~~ we ~~must have~~ we must have 2 + b = e, b = 4 gives e = 6.

which is correct.

Interchange of 3 & 2 in (ii) also gives correct soln ie

$$\begin{array}{r} 2\,4\,9 \\ 3\,1\,8 \\ \hline 5\,6\,7 \end{array} \quad \text{or} \quad \begin{array}{r} 3\,4\,9 \\ 2\,1\,8 \\ \hline 5\,6\,7 \end{array}$$

(involving all 9 digits)

(8) Division sum. Given f is an even digit, find soln for

$$\begin{array}{l} ab\,)\,cdeeb\,(\,bfb \\ \quad\;\; ceb \\ \hline \quad\;\; gge \\ \quad\;\; gch \\ \hline \quad\;\; \cancel{ceb} \\ \quad\;\; \cancel{ceb} \\ \quad\;\; ceb \\ \quad\;\; ceb \end{array}$$

In the third line e − h gives e, hence h = 0

f × b in 4th line gives a number ending in 0

b = 5 since f is even

e in line 2 must therefore be 2 or 7 according as a is even or odd

If e = 2, g = 7 which makes d = 0 which is impossible since h = 0

Taking e = 7, g = 2 and d = 9. Hence we reach the scheme

```
a5) c9775 (5f5
    c75
    ---
     227
     2c0
     ---
      c75
      c75
```

This gives easily a = 3, c = 1 & f = 6.

(9) a multiplication sum given by, where x is any digit

```
  345      | 345 x ab
   xx      |  cdef
 ----      |  jihg
 xxxx      |  ----
xxxx       | Kl76m
 ----      |
xx76x      |
```

Since cdef is of 4 figures b ≥ 3.

Trying with b = 3, ~~345 x a3~~, 5, 6, ..

and using e + g = 6, d + h = 7 we find that for b = 7 we have the scheme

```
345 x a7
 2415
jihg
 ----
Kl76m
```

we have m = 5,
g = 5
h = 3

which requires a = 3 leading to i = 0, j = 1, K = 2, l = 2

leading correctly to 345 x 37 = 12765.

Chap XII – Problems with matches & coins – Trivial

Quite a useful book from which some things can be ~~incorporated in~~ taken up for consideration

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Number recreations from Kraitchik: Some of these are collected on pp. 156–168 of NMBook. No. 11. Here are some more

p. 77 (Kraitchik) Automorphic numbers

Number N is automorphic if all of its powers end in the same digits. In scale 10, other than trivial cases, the true automorphic numbers are given by $N \equiv 0 \pmod{2^n}$ and $1 \pmod{5^n}$ or $N \equiv 1 \pmod{2^n}$ and $0 \pmod{5^n}$. Thus numbers ending in 6, 76, 376, ... or in 5, 25, 625 ... are automorphic

The last sixteen digits may be either

$$\left.\begin{array}{l} 3,740,081,787,109,376 \\ 6,259,918,212,890,625 \end{array}\right\} \quad (?)$$

p. 78 – Prime nos – Gives history of several large prime nos

p. 79 – Multigrade relation

p. 83 – Arithmetical games – Nothing useful for my purpose

p. 95 – ~~Arithmetics~~ – Chap IV – Arithmetics – geometrical problems

Primitive right-triangle is one with no common factors of the sides.

Rt Δ^{le} primitive if & only if $x = 2ab$, $y = a^2 - b^2$, $z = a^2 + b^2$, where a & b are any two relatively prime integers such that $a > b$, & one is even & the other odd.

Problems (1) Given hypotenuse z, find the legs

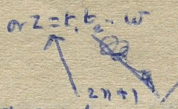
From $z = a^2 + b^2$, find a, b & then x & y

Case (a) z prime (i) $z = 2$, $a = b = 1$. No primitive Δ^{le}

(ii) z prime of the form $4k+1$, unique a & b can be found

(iii) z " $4k-1$, no solns.

or $z = t_1 t_2 \cdots w$ (not of the form $4K-1$)



(b) z composite (i) If $z = t^{2n+1} w$ (t of the form $4K-1$), there are no solns

If $z = t^{2n} w$ (,) solns exist

Then we can work $z = t^2 w$ & solns are (ta', tb') where (a', b') are solns of $w = x^2 + y^2$.

(c) If z has no prime factors of the form $4K-1$, there are several solns

Let $z = uv$, & let $u = a^2+b^2$, $v = c^2+d^2$, then

$z = uv = (a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2 = (ac-bd)^2 + (ad+bc)^2$

Thus from two prime factors we can find solns for their product, from these & from the solns of another prime we can find solns for product of 3 primes & so on.

Examples (i) $z = 29 = 4\cdot 7+1 = 5^2+2^2$ only, $x = 2\cdot 5\cdot 2$, $y = 5^2-2^2$

$(x, y, z) = (20, 21, 29)$

(ii) $z = 145 = 5\cdot 29 = (2^2+1^2)(5^2+2^2) = 12^2+1^2 = 8^2+9^2$. Hence $(x, y, z) =$ $(24, 143, 145)$ or $(144, 17, 145)$

(iii) $z = 58 = 7^2+3^2$ only (not primitive since z is even

(iv) $z = 1105 = 5\cdot 13\cdot 17 = (2^2+1^2)(3^2+2^2)(4^2+1)^2$

$= (8^2+1^2)(4^2+1^2) = (7^2+4^2)(4^2+1)^2 = 33^2+4^2 = 31^2+12^2$

$= 32^2+9^2 = 24^2+23^2$. Hence $(x, y, z) = (264, 1073, 1105)$,

$(744, 817, 1105)$, $(576, 943, 1105)$, $(1104, 47, 1105)$

(v) $z = 385 = 5\cdot 7\cdot 11$ — Problem has no solution

[$7 = 4\cdot 2-1$, $11 = 4\cdot 3-1$ ie of forms $4K_1-1$ & $4K_2-1$ ie z is of the form $t_1 t_2 w$ & not $t^2 w$].

(2) Given the even ~~leg to~~ leg to find the other two sides.

Solvable only when $x = 2ab$ is divisible by 4. $\therefore 2 = 4m$, factor $2m$ in all possible ways into two unequal factors (without common factor), take the larger as a & the smaller as b, determine y & z.

$4m$	a	b	(x,y,z)
4	2	1	(4,3,5)
8	4	1	(8,15,17)
12	6	1	(12,35,37)
12	3	2	(12,5,13)
16	8	1	(16,63,65)
20	10	1	(20,99,101)
20	5	2	(20,21,29)

(3) Given the odd leg, find the other two sides.

Factor $a^2 - b^2 = (a-b)(a+b)$ where $(a-b, a+b) = 1$ & take larger factor $a+b = u$, smaller $a-b = v$, $a = \frac{1}{2}(u+v)$, $b = \frac{1}{2}(u-v)$

& other both are integers since both u & v are odd, ($a^2 - b^2$ is odd).

y	$a+b$	$a-b$	a	b	(x,y,z)
3	3	1	2	1	(4,3,5)
5	5	1	3	2	(12,5,13)
15	15	1	8	7	(112,15,113)
15	5	3	4	1	(8,15,17)

[For primes only one set of (a,b), for eg: $11 = 11 \cdot 1$, $a = 6, b = 5$, and $(x,y,z) = (60, 11, 61)$

(4) Find right $\triangle$ when legs are consecutive integers.

$y - x = \pm 1$, $(a^2 - b^2) - 2ab = (a-b)^2 - 2b^2 = \pm 1$

Putting $a - b = u$, $b = v$, we have to solve the Pellian eqn

$$u^2 - 2v^2 = \pm 1.$$

Soln by using contd. fractions is obtained. Convergents are $1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$

even or odd convergents correspond to ± 1. Solns are

$a-b$	b	a	(x, y, z)
1	1	2	(4, 3, 5)
3	2	5	(20, 21, 29)
7	5	12	(120, 119, 169)
17	12	29	(696, 697, 985)

[In fact this is a special case of the problem I want to solve

$$k^2 + (k+1)^2 + \cdots + (k+m-1)^2 = l^2$$

in addition to the two other types. This has been solved in general by somebody & I have to give only a few special solns].

(5) find a primitive rt. $\triangle$ whose hypotenuse exceeds the even leg by 1.

From $z - x = 1$, $a^2 + b^2 - 2ab = 1$ ie $(a-b)^2 = 1$ as $a > b$, $a - b = 1$ ie b & a must be consecutive integers

(6) one leg is divisible by 3.

If a or b is divisible by 3, $2ab$ is divisible by 3

If neither a nor b is divisible by 3, let $a = 3k+1$, $b = 3l+1$, then $a^2 - b^2$ is divisible by 3

(7) one of the sides is divisible by 5.

The product of the three sides $= 2ab(a^4 - b^4)$. If $5 \mid a$ or b, $5 \mid ab$

If $5 \nmid a$ or b, $a = 5k \pm 1$, $b = 5l \pm 1$, then $5 \mid a^4 - b^4$.

Since ~~product of~~ $5 \mid xyz$, $5 \mid x$ or y or z.

(8) The area of a primitive rt $\angle$le is divisible by 6 & product of sides by 60

$\underline{x} = 2ab$ (one of $\underline{a}$ or $\underline{b}$ is even). Hence $4 \mid x$, $3 \mid x$ or y & $5 \mid x$ or y or z

$\therefore$ area $\Delta = \frac{xy}{2}$ is divisible by $\frac{4 \cdot 3}{2}$ & xyz is divisible by $4 \cdot 3 \cdot 5 = 60$.

(9) Primitive rt $\angle$le is called elementary or composite acc'g as the hypotenuse is prime or composite. An angle is called arithmetical if all its trig ratios are rational

(9) An angle is arithmetical if and only if it is an angle of a primitive rt $\angle$le. or differs from such an angle by a multiple of $90°$

(Proof not elegant)

Heronian triangles (rational Δs) are those whose sides & area are rational

(10) The inradius, the circumradius and radii of escribed circles of a rational triangle are all rational.

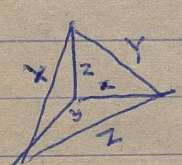
$r = \frac{\Delta}{s}$, $\tan\frac{A}{2} = \frac{r}{s-a}$, etc. & so all these rational & angles arithmetical

$R = \frac{abc}{4\Delta}$, $r_1, r_2, r_3 = \frac{\Delta}{s-a}, \frac{\Delta}{s-b}, \frac{\Delta}{s-c}$ are all rational

[Carmichael's diophantine relations are better suited to prove these]

(11) Any ~~[illegible]~~ desired Heronian triangle can be formed by combining two rational right Δs. — Fig. 15 on p. 104 is quite beautiful

(12) To find a cuboid (rect. $\square^d$) in which edges & diagonals are rational:



Take $x = a(4b^2 - c^2)$, $y = b(4a^2 - c^2)$, $z = 4abc$

where a, b, c are sides of a rational rt $\angle$le

Then $X = b(4a^2 + c^2)$, $Y = a(4b^2 + c^2)$, $Z = c^3$.

$(a, b, c) = (m^2 - n^2, 2mn, a^2 + b^2)$

These give only a particular soln — Nine examples of (x, y, z) given

(p. 108) not of this particular type. Two of the simple solns are

(x, y, z) = (85, 132, 720), (187, 1020, 1584).

Diophantine problem!

[What is the general solution — Certainly a problem worth investigating.] ~~At least one integral soln of x, y, z, x, y, z~~ [Particular integral solns can however be found from the above formulae for x, y, z].

Number recreations from Rouse-Ball.

Chapter 1 — p. 13. 8712 = 4 × 2178, 9801 = 9 × 1089

312 × 221 = 68952; 213 × 122 = 25986.

153 = $1^3 + 5^3 + 5^3$, sums 370, 371, 407.

Properties of recurring decimal for 1/7 are notorious. Troitsky

p. ~~Prof.~~ has shown that 142857 and 285714 are the only numbers < 10^6 which become multiplied when we remove the first digit & put it after the last

p. 14. (1) abc − cba = 99x, for any x, if the last digit is known the no. is known

(2) for an R.S. number, multiply by any number, cross out any digit except 0, read the remainder. Then the sum of the digits in the remainder subtracted from the next highest multiple of nine will

will give the figure struck out.

[If $f(x,y,z,\ldots)$ be the R.S. number, $\delta(x,y,z,\ldots)$ is also an R.S. no of the sum of its digits = Multiple of 9 say $9n$. If a digit m of this be struck off the sum of remaining digits $= 9n - 6n$ & the next highest multiple of 9 none is $9n$ itself ie $9n - (9n - m) = m$ — Quite trivial. In fact this is true for any number divisible by 9 & not necessarily an R-S. number]

Empirical problems — with the ten digits 0, 1, — 9 express numbers whose sum is unity, each digit being used only once & the weights usual fractional notation being allowed — with the same ten digits express numbers whose sum is 100 : In the making of such questions there is no limit, but their solution involves little or no mathematical skill

[This remarks throws doubt about my idea of choosing several properties involving family of 9 & 10 digits as for eg. the cubes 69^3 & 69^3. But Ball's remarks are justified ; perhaps for trivial cases]

Four digits problem — using 1, 2, 3, 4, once & only once, decimal notation, algebraic signs prescribed differences, integral powers, we can express all numbers up to 88. If the square root symbol & factorials (repeated if desired a finite number of times) we can go to 264. If negative integral indices are also permitted to 276, and if fractional indices are permitted to 312. Many similar questions such as using four out of the five digits 1, 2, 3, 4, 5 with the five digits 1, 2, 3, 4, 5 each being used once & only once, I have got to 3832 and 4282 accordingly as

negative & fractional indices are included or allowed.

[Ball considers this as more interesting than the above empirical problems. My view is just the opposite]

Four fours form – Similar to above using '4' only four times – or four 1's, four 2's, four 3's, ... four 9's, how far one can go?

Problems with a series of numbered things

(1) Watch-dial problem

(2) Another watch dial problem

(3) Problem of pack of cards

(4) – Do –

(5) – Do –

(6) – Do –

(I consider these quite trivial. Ball must be a card player!)

Arithmetical ~~recreations~~ restorations

~~Class A~~: Reconstruction of arithmetical sums from which various digits have been erased. (Newton Friends type?)

Class A: (i) 417 × .1... is 9...57. Find missing digits

$$\begin{array}{r} \overset{2}{a}\,\overset{1}{1}\,\overset{9}{b}\,\overset{2}{c}\,\overset{1}{d} \times 417 \\ \hline 153447 \\ 21921 \\ 87684 \\ \hline 9{,}141{,}057 \end{array}$$

$d = 1$, $c = 2$, $b = 9$ } $a = 2$ since result has 7 digits

(ii) 70..34. is exactly divisible by 792. Find missing digits

$792 = 8 \times 9 \times 11$, $70ab34c$

$7 - 0 + a - b - 3 + 4 - c = M(11)$

~~$7+a+3+c = 6+b+4$~~ ~~divisible by~~ 11

$7 + 0 + a + b + 3 + 4 + c = M(9)$

$c = 4$ because $34c$ must be divisible by 8

~~14+a=b+4~~ ⇒ $10+a-b=M(11)$ —(1)

~~18+a+b=M(9)~~ $18+a+b=M(9)$ or $a+b=M(9)$ —(2)

~~∴ a+b=M(9)~~ $a=5, b=4$ satisfies both (1) & (2)

~~b=a~~ = Number is 7,054,344.

(iii) Five digit number 4·18· is divisible by 101. Find the digits

Let no be $4x18y$

~~To~~ A number $a+10b+10^2c+10^3d+\cdots$ can be written as

$M(9)+a+b+c+d+\cdots$ or

$M(11)+a-b+c-d+\cdots$

$M(33)+(a+10b)+(c+10d)+(e+10f)+\cdots$

$M(101)+(a+10b)-(c+10d)+(e+10f)-\cdots$

$M(27)$ or $M(37)$ or $M(111)+(a+10b+10^2c)+(d+10e+10^2f)+\cdots$

$M(7)$ or $M(13)$ or $M(77)$ or $M(91)$ or $M(143)+(a+10b+10^2c)-(d+10e+10^2f)+\cdots$

Here $M(101)+(y+80)-(1+10x)+4$ is the given number

$y-10x+83=0$ & $y=7, x=9$.

No required is 49,187

(iv) The four digit number ·8· is divisible by 1287. Find the digits

Let no be $x8yz$, $1287=9\times11\times13$

$$\left.\begin{aligned} z+y+8+x &= M(9) \\ z-y+8-x &= M(11) \\ (z+10y+800)-x &= M(13) \end{aligned}\right\}$$

Taking $x=3, y=6, z=1$ all three equations are satisfied

i.e required no is 3861

(v) Ahardrexample; example.

Given that 6 . 80 . 8 . . 51 is exactly divisible by 73 & 137

73 × 137 = 10001 & the other factor has 6 digits, say $x_1 x_2 x_3 x_4 x_5 x_6$

Multiplying $x_1 x_2 x_3 x_4 x_5 x_6 \times 1001$

$$\begin{array}{r} x_1\ x_2\ x_3\ x_4\ x_5\ x_6 \\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0 \\ x_1\ x_2\ x_3\ x_4\ x_5\ x_6 \\ \hline 6\ .\ 8\ 0\ .\ 8\ .\ .\ 51 \end{array}$$

$x_6 = 1,\ x_5 = 5$

$x_2 + x_6 = 8 \therefore x_2 = 7.$

$x_1 = 6,\ x_5 = 5$ & hence $x_4 = 9.$

$1 + x_3 = 8$ or $x_3 = 7.$

Hence no reqr is 6,780,187,951.

[Ref to (iii), (iv) & (v) – P. Delens: Problèmes d'Arithmétique Amusante. Paris 1914, p. 55, p. 57 & p. 60 resp.]

Class B: Division problems with restorations

(i) Hindoo origin (Note Hindoo!) [Ref. Amer. Math. Monthly, vol. 28, p. 37, 1921]

$$\begin{array}{l} \ldots)\ldots\ldots(\ldots \\ \quad .0\ .\ . \\ \hline \quad\ \ .\ .\ .\ . \\ \quad\ \ .5\ 0\ . \\ \hline \quad\ \ \ .\ .\ . \\ \quad\ \ \ .4\ . \\ \hline \end{array}$$

Says "The soln involves no difficulty. Answer is that the divisor is 215 and the quotient is 573. & soln is unique

[Could not make out anything. Should look up reference viz Amer. Math. Monthly, 1921, vol. 28, p. 37]

(ii) Another division problem with no digits given one involving a repeating decimal of last nine digits. [Ref. Amer. Math. Monthly 1921, ~~Vol 23~~ Vol 28, p. 278 – Should look this up]

(iii) – (v) – Three additional examples due to W. E. H. Berwick. Solns are lengthy and involve much empirical work.

[Ref (iii) School World – July & Oct. 1906, Vol. 8, pp. 280, 320

(iv) Math. Gazette – 1920, Vol. 10, pp. 43, 359–60

(v) – do – , Vol 10, p. 361 & Vol. 11, p. 8]

Class C: Problems involving symbols (like Newton friends problem)

(i) $bc \times bc = abc$

Ans $bc = 25$, $a = 6$, [~~What~~ $bc = 76$ does not hold: $76^2 = 5776$ (4 digits)]

(ii)
$$\begin{array}{r|l} ab & cdeeb \;(\; bfb \\ & eeb \\ \hline & fqe \\ & fch \\ \hline & ceb \\ & ceb \end{array}$$

[Ref. Strand Magazine – Oct. 1921]

Soln: $b = 1, 5$ or 6. Since $ab \times b$ has 3 digits $b \neq 1$, $e - h = e$ & hence $h = 0$ and therefore if $b = 5$, we have f even, & if $b = 6$ we have $f = 5$. Also $q - c = c$ & hence $q = 2c$ & hence $c \neq 4$, from which it follows that $b \neq 6$. A few trials now show that $ab = 35$, $cdeeb = 19,775$.

It is possible to frame digit restoration examples of mixed type involving the dots & the symbols & increase the difficulty by expressing them in a non-denary scale. But such elaborations do not add

to the interest of the questions

[Thank God! I think I shall have to think carefully about inclusion or otherwise of such restoration problems after looking at the references given]

Calendar pr

Calendar problems – Medieval problems (4 or 5 given)

Exploration problem – The Josephus problem – The game of Nim – Wythoff's game – Game of Kayles.

Chap II – Arithmetical recreations (contd)

(1) Arithmetical fallacies – 8 fallacies all trivial – 2 paradoxes – Other questions on probability (2 problems) – Miscellaneous problems.

Permutation problems – Knights of the round table – The Ménage problem – Bachet's weight problem

Expressions for $1/n$ as recurring decimals :– Simplified method for writing down the periods of recurring decimals like $1/7$, $1/47$, $1/87$ etc etc – Ref. to Schiffner – Sphinx, 1937, p. 115, ibid 1938, p. 206

Decimals & continued fractions.

Change of decimals with scale of notation used. Thus in scale 7, $\frac{1}{7} = 0.1$ writes of 0.142857 in scale 10.

Rule for expressing an integer in scale 10 to in scale p. Divide by p again and again noting the successive remainders. These remainders when read backwards are the digits of the required new expression

[Thus, take 87, expect 1110101 as remainders. Hence

$\frac{87}{2}$, 1 $\quad 87 = 1010111$ in binary scale

$\frac{43}{2}$, 1 $\quad = 2^6 + 2^4 + 2^2 + 2^1 + 2^0 = 64 + 16 + 4 + 2 + 1 = 87$]

The rule for a fraction is multiply by p again and again, operating only on the fractional part at each stage. The successive integral parts are the digits of reqd new expn (in their proper order).

For eg: $\sqrt{2} = 1.41421356$

Operating only on the fractional part) 0.82842712
1.65685424
1.31370848
0.62741696
1.25483392
0.50966784
etc

$\therefore$ In binary notation $\sqrt{2} = \sqrt{10} = 1.0110101000001001111 00\ldots$

(continuing the etc).

Sing $\pi = 3.14159265$
0.28318530
etc.

Writing 3 as 11 we get in binary notation

$\pi = 11.0010010000111111011 01\ldots$

In the expn for $\sqrt{2}$, the existence of five 0's shows that

1.0110101 is a good approxn for $\sqrt{2}$.

In fact squaring $(1.0110101)^2$ in the binary notation ie multiplying

$1.0110101 \times 1.0110101$ (~~ie~~ carrying over ~~[illegible]~~ 1 ~~by taking~~ if

$1+1$ ~~[illegible]~~ appears & putting $1+1=0$) we get

1.1111111111101

Similarly we note that for π a good approxn is $11.\dot{0}0\dot{1} = 3\frac{1}{7}$

[To see that $.\dot{0}0\dot{1} = 1/7$ in binary notation. write $\frac{1}{7} = 0.\dot{1}4285\dot{7}$

& apply the rule of multiplying & multiplying by 2, then we get $.\dot{0}0\dot{1}$.

In continued fractions

$$\sqrt{2} = 1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+} \qquad \text{———} (1)$$

[Special case of contd. fm expn of $(\sqrt{N} + b_1)/r$, which I have used in solving Pell's eqn].

$$e = 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{8+}\,\frac{1}{1+} \quad \text{———} (2)$$

$$\pi = 3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{292+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+} \cdots \qquad (3)$$

[Expn for e follows a simple law as evident ~~in (2)~~ from (2). No law has, however, yet been found for π. The fourth quotient $a_4 = 292$ is the largest known, the runner up being $a_{73} = 161$.

In fact even the method of expansion of e as a continued fraction is not to be found in Barnard & Child.

This method of getting expansions for transcendental numbers like e & π in continued fractions should be found & if necessary included in a chapter on transcendental numbers]

[According to Lehmer (Sphinx, 1938, pp. 185–187) the A.M. of the first 100 coefficients of π is 10.2 & 41 of them are equal to unity]

The first half dozen convergents to $\sqrt{2}$, e & π are

$\sqrt{2}$: $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}$

e: $\frac{2}{1}, \frac{3}{1}, \frac{8}{3}, \frac{11}{4}, \frac{19}{7}, \frac{87}{32}$

π: $\frac{3}{1}, \frac{22}{7}, \frac{333}{100}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}$

There must be connection with Ramanujan's work.

The simplest regular (simple) C.F is

$$\tau = 1 + \frac{1}{1+} \frac{1}{1+} - \qquad (4)$$

Easy rise to Fibonacci sequence

Convergents of (4) are $\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}$. Both the numerators & denominators form the Fibonacci nos

$$1, 1, 2, 3, 5, 8, 13, 21, -$$

$\tau = 1 + \frac{1}{\tau}$ where $\tau = \frac{1}{2}(\sqrt{5}+1)$

Ratios of alternate Fibonacci numbers are said to measure the fraction a turn between successive leaves on the stalk of a plant, $\frac{1}{2}$ for grasses, $\frac{1}{3}$ sedges, $\frac{2}{5}$ for the apple, Cherry etc, $\frac{3}{8}$ for the common plantain, $\frac{5}{13}$ for the leek & soon

[Ref. H. E. Licks, Recreations in Maths, N.Y. 1936, p. 107]

Here are the convergents to

$$\frac{1}{\tau} = \frac{1}{2+} \frac{1}{1+} \frac{1}{1+} \frac{1}{1+}$$

τ is intimately connected with the metrical properties of the pentagon, decagon, dodecahedron and icosahedron since $\tau = 2\cos \pi/5$. A line-segment is said to be divided in golden section if one part is τ times the other, or $1/\tau$ times the whole. The nth Fibonacci number is $\{\tau^n - (-\tau)^{-n}\}/\sqrt{5}$.

[I am afraid I have to omit Fibonacci numbers since it involves colossal no. of properties & I found a

whole Journal viz The Fibonacci Quarterly is devoted to this topic & this Journal does not appear to be available in Bangalore]

Rational rt Ld Δ^s. [$x = b^2 - c^2$, $y = 2bc$, $z = b^2 + c^2$]

Lehmer's proof that the number of primitive triangles with hypotenuse less than X is approximately $X/2\pi$ and the number with perimeter less than X is approx $(X \log 2)/\pi^2$. [Amer. J. Math, 1900, Vol 22, p. 38]

If c and b are consecutive terms of the sequence 1, 2, 5, 12, 29, 79, ... ie if b/c is a convergent to the continued fraction for $\sqrt{2}+1$, then $|x-y| = 1$. [Bracket

Hoppenot's result (ie special case of Vigdeen's investigation with $m = n$) given

observation re $3^3+4^3+5^3 = 6^3$ analogous to $3^2+4^2 = 5^2$

Euler's conjecture a corresponding result for $x^4+y^4+z^4 = u^4$ that if a soln exists then $v > 1040$.

$x^4+y^4 = z^4+v^4$ has infinitely many solns [Sphinx, 1937, p. 98]

Triangular & Pyramidal numbers (Ref. to Dudeney's Amusements in Mathematics, London, 1917)

Δ^r nos – $\frac{1}{2}n(n+1)$, ~~& the nos~~

The nos 1, 36, 1225, 41616, 1413721, 48024900, ... are simultaneously Δ^r & square. The general formula for such a number is b^2c^2 where b/c is any convergent to the C.F of $\sqrt{2}$

Tetrahedral nos – $\frac{1}{6}n(n+1)(n+2)$

Pyramidal nos – $\frac{1}{6}n(n+1)(2n+1)$. [I thought these were called tetrahedral nos. Note the my error]

The only number simultaneously square and pyramidal is 4900.
This result was conjectured by Lucas in 1875 (Nouv. Ann. de Math (2), Vol 14, p. 336), and proved by ~~Watson~~ G.N. Watson in 1918 (Mess. of Math (new series), Vol. 48, pp. 1 – 22). The proof is by no means elementary.

Automorphic numbers (Ref. to Kraitchik, p. 66 (Sap. 46 of this note book). Gives the ~~last~~ two last 16 digits of an automorphic numbers as given by Kraitchik, but adds that for the one 625. 890625 the last five digits are the same as those of $5^8 = 390625$

Finite arithmetics

Arithmetics modulo p. (p a prime)

Lehmer's machine

Became obsolete in view of development of computers

Distribution of Primes

Euler's quadratic form $x^2 + x + 41$ for primes ($x = 0, \ldots 39$)

Lehmer's proof that if there be ~~a~~ such a form

$x^2 + x + A$ representing primes for $A-1$ consecutive values of x

then x must exceed 1250,000000 (Sphinx, 1936, pp. 212-14)

Use criteria to test whether N is prime or not.

(i) Wilson's theorem (Sir J. Wilson)

N is prime if & only if $(N-1)! + 1$ is divisible by N

only a theoretical test.

(ii) Fermat (1640) & Euler's proof (1736) of Fermat's theorem that if p is prime & a not divisible by p, then $a^{p-1} - 1$ is divisible by p. known to the Chinese (500 B.C.) when $a = 2$. They stated the converse ie if N divides $2^{N-1} - 1$, then N is prime. This was rediscovered & "proved" by Leibniz (1680). However it is false; it fails for $N = 341 = 11 \times 31$ & for an infinity of other N's. Modern tests on primality are the following converse of Fermat's theorem due to Lucas (Theorie de nombres, 1891) viz if $a^{x} - 1$ is divisible by N when $x = N-1$, but not when x is a proper divisor of $N-1$, then N is a prime (Bull. Amer. Math. Soc., 1927, Vol. 33, pp. 327-40). The numbers up to 10^7 have all been factorised (Lehmer's factor tables for the first ten millions, Washington, 1909).

Hoppenot & Kraitchik (Sphinx, 1936, p. 164, 1938, p. 84) have tabulated the smallest prime factors of nos. from $10^{12} - 10^4$ to $10^{12} + 10^4$

By an ingenious process due to Meissel (Math. Ann. 1870, Vol. 2, p. 636; 1871, Vol. 3, p. 523) it is possible to calculate the number of primes less than X without finding the primes themselves & counting them

the no. of primes between X and $X + x$ (X large & x comparatively small) or between $X - x$ & X, is approximately $x / \log X$ (de la Vallée Poussin, 1839) For numbers between 10,000,000 & 10,005,000, this formula gives $5000/7 \log 10 = 310.210$, while the exact number is known to be 305. Actually Hoppenot-Kraitchik tables show there are exactly 336 primes between $10^{12} - 10^4$ & 10^{12} whereas the approx. formula gives $10^4 / 12 \log 10 = 361.912$

Riemann's approx formula (Rep. to fascinating introduction to D. N. Lehmer's list of Prime Nos from 1 to 10,006,721, Washington, 1914) which is quite complicated gives no. of primes less than x is so close that its error is only 24 when $x = 10^9$ (According to Meissel the no. of primes less than 10^9 is exactly 50,847,479)

Polignac's Conjecture (Now. Ann. Math, 1849, Vol. 8, p. 428) that every even no = diffce of two consecutive primes in infinitely many ways. For example if $n = 2$, this means that there infinitely many twin primes. This conjecture has not been proved or disproved. As evidence in favour, the Hoffert-Kraitchik table shows that the interval from $10^{12} - 10^4$ to 10^{12} contains 15 such pairs (20 pairs in interval from 10^{12} to $10^{12} + 10^4$), of which the largest is 999999, 999960 ± 1. Since $10^4/(12 \log 10)^2 = (3.61912\ldots)^2 = 13.097\ldots$, we may guess that the number of prime pairs between $x - x$ & x is approximately $x/(\log x)^2$

It can be proved quite easily (?) that the sum of the reciprocals of all the primes less than x increases without limit as x increases. If, however, we consider only the reciprocals of twin primes less than x, then this remains bounded as x increases. This fact, due to Brun, shows that there are not "too many" primes which differ by 2.

Mersenne's numbers, Perfect numbers, & Fermat nos

These have been summarised on pp. 4-7 of big notebook No. 2 devoted to number recreations.

Fermat's last theorem

Legendre's proof for $n = 5$ (1823) – Dirichlet's proof for $n = 14$ (1832) – Lamé & Lebesgue's proof for $n = 7$ (1840)

In 1849, Kummer proved the theorem for all "regular" primes (A prime p is said to be regular if it does not divide any of the numerators of the Bernoulli numbers $B_1, B_2, \ldots B_{\frac{1}{2}(p-3)}$.

(Bernoulli numbers appear in the series

$$\frac{x}{2}\cot\frac{x}{2} = 1 - \frac{B_1}{2!}x^2 - \frac{B_2}{4!}x^4 - \frac{B_3}{6!}x^6 - \ldots$$

the values of the first eight B's are $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \frac{7}{6}, \frac{3617}{510}$.) (Kummer's work – Adj. H. S. Vandiver – Trans. Amer. Math. Soc., 1929, Vol. 31, pp. 613 – 642).

Kummer found that 37, 59, 67, 101, 103, 131, 149, 157 are the only "irregular" primes less than 164 & H. S. Vandiver (Duke Math. J, 1937, Vol 3, pp. 569 – 84) found 27 more irregular primes between 164 and 616. The theorem can be proved for these 35 cases other tests: Wieferich (Crelle's Journal, 1909, Vol. 136, pp. 293 – 302) has shown that if the eqn is soluble in integers prime to n (n odd prime) then $2^{n-1} - 1$ is divisible by n^2. This restricted problem has been pushed as far as $n = 8,332,600$ by Barkley & Rosser (Bull. Amer. Math. Soc., Vol 45, 1939, p. 90) using a theorem of Morishima.

A prize of 100,000 marks has been offered for a general proof to be given before 2007 (L'Intr. des. math., Vol. 15, pp. 217–18)

Remarks re. Fermat's conjecture – Ball writes he used continued fractions (like Ramanujan?) – mentions a problem proposed by Fermat to English mathematicians viz. to prove that

$x^2 + 2 = y^3$

has one & only one integral soln viz. $x = 5, y = 3$.

[His proof of ~~the~~ of a prime of the form $4n+1$ being expressible in one & only one way as sum of two squares may be established with comparative ease by using properties of continued fractions]

[Why not include a chapter on continued fractions in Part II?]

Galois arithmetics

A short note not too important

3/7/79 at B.V. Library

(1) Science Today – June 1979

Harshad numbers by Rajekar.

If sum of digits of $N = d$ & $d \mid N$, then N is called a Harshad number. H-numbers can be found for any value of d, but how to find the least such number is of great interest – for $d = 13$, least H-no = 247, for $d = 21$, it is 588, & for $d = 29$ it is 7588.

Ans prev (1) find least H for $d = 81$

(2) " $d = 89$, & $d = 99$. What are the next two numbers for $d = 89$ & next three for $d = 99$

(3) find all H for $d = 47$. Which of them has 6 digits? Find the smallest & highest no. of 6 digits

(4) Find two H-nos, one with 3 digits & the other with 5 digits such that the product of those two is also a H-number.

(5) Find 7 H-nos for the same d in A.P.

New Scientist (11/1/79) - Vol. 81, No. 1137 Weekly

p. 115. puzzle by Eric Emmet.

In the following addition sum all the digits are wrong. But the same wrong digit stands for the same correct digit wherever it appears, and the same correct digit is always represented by the same wrong digit.

$$\begin{array}{r} 1939 \\ 1079 \\ \hline 6856 \end{array}$$

Find the correct addition sum

Soln in No. 1138, p. 195

$$\begin{array}{r} 2767 \\ 2137 \\ \hline 4904 \end{array}$$

Mathematical Reviews.

Vol. 55 No. 6 - June 1978

(1) p. 1702 Review of Zahlentheorie by Koch Helmut; Pieper, Herbert
Rev by W. Schwarz (Frankfurt)
Last para - Ohne Zweifel liegt hier ein lesenswertes Buch vor, das als Grundlage für Vorlesungen sehr geeignet sein dürfte

(2) Some new prime of the form $K \cdot 2^n + 1$ by G. Mathews & H. C. Williams
Math. Comp. 31 (1977) no. 139, 797-98
The authors give tables of all primes of the form $K 2^n + 1$, K odd with $9 \le K \le 99$, $512 \le n \le 1000$ & also with $101 \le K \le 129$, $1 \le n \le 1000$.
These tables extend earlier tables of Hallyburton & Brillhart.
The new primes were tested as divisors of Fermat numbers, and just two were found:

$$17 \cdot 2^{744} + 1 \mid F_{744} \quad \& \quad \left(127 \cdot 2^{558} + 1\right) \mid F_{556}$$

(3) Multiply perfect numbers, Mersenne primes & effective computability
by Carl Pomerance (Math. Ann. 226 (1977) no. 3, 195-206.
Sei $\sigma(n) = \sum_{d \mid n} d$. Die Verfasser diskutiert die Gleichung (1)
$\sigma(n) \cdot n^{-1} = \alpha$, dabei α bedeutet eine feste rational Zahl.
(1) Can't for having finitely many solns.
(2) If $n = em$, e = gerade vollkommen Zahl $(e, m) = 1$ sind

(4) The depth of a rational number by M. France Mendes

Topics in Number Theory (North-Holland, 1976)

— Something to do with associated continued fraction $[a_0; a_1, a_2 \ldots a_2]$ normed by $a_2 \geq 2$.

(5) Palindromic pentagonal numerals by C. W. Trigg

J. Rec. Math. 6 (1973), no. 3, 186–189

The nth pentagonal number N is of the form $\frac{1}{2}n(3n-1)$. In base 10 there are 15 numbers less than 10^{10} that are both palindromic & pentagonal. — The author lists such nos that in base 2 have $n <$ . . .

— Anyway write to Trigg for reprints.

(6) Proof that perfect nos. are ∞ in number & that no odd no. can be perfect by D. M. Patel — Math. Edn 11 (1977), no. 2, B. 29–B. 33.

Reviewer states that the various proofs are not correct.

(7) Pythagorean twins by Erik Kristensen — Results re. Chains of Pythagorean twins (?) — ~~D. Math~~ Nordisk Mat. Tidskr 24 (1976) no. 2. 61–62, 76. (Ref. M. R. 54. # 10124

Found this.

Pythagorean twins by A. Bronman

If a, b, c, k are +ve integers with $a^2+b^2=c^2$ and $b=a+k$, then the pair (a, b) is called a pair of Pythagorean ~~twins~~ k-twins. In this paper the author gives a complete construction of all such twins by means of a recurring sequence. The rate of growth of k-twins & the relative

& their ~~other~~ ref. frequency for different values of $\underline{k}$ are also studied

Reviewer: S.P. Mohanty (Kanpur)

(8) Lecture notes in maths - Springer Verlag, 1977.

Important work of Cudnovskii in the theory of transcendental nos.

(a) A ~~result~~ remarkable result on elliptic functions which includes the transcendence of $\Gamma(1/4)$ & $\Gamma(1/3)$. Each of these nos. is algebraically independent of π.

Math. Reviews Vol. 51, No. 5, Nov. 1977 — #9899 - 12411

(1) A. H. Beiler — Consecutive hypotenuses of Pythagorean Δs

J. Recr. Math 7 (1974), 120 - 33

It is a well known result that for any +ve integer $\underline{r}$ there exist $\underline{r}$ consecutive +ve integers, each of which is the hypotenuse of a right Δ^6. In fact if p_i denotes the ith prime of the form $4t+1$, and if $\underline{z}$ is chosen via the Chinese remainder theorem, to be the least +ve integer satisfying $z+i \equiv 1 \pmod{p_i}$; $i = 1, 2, \ldots r$, then the numbers $z, z+1, \ldots z+r-1$ have the desired property. However the values obtained this way are far from being minimal. Define $z = z(r)$ to be the least integer such that $z, z+1, \ldots, z+r-1$ are reqd hypotenuses, but $z-1$ & $z+r$ are not. The author calculates $z(r)$ for $2 \leq r \leq 22$, and also $r(27)$. The fn exhibits a great deal irregularity which is to be expected since the value of $z(r)$ is intimately connected with the prime factorisation of the nos $z, z+1, \ldots z+r-1$

(2) On palindromic squares of non-palindromic nos. by Simmons — Already noted — The major result of the author (J. Rec. Math. 5 (1972), No. 1, pp 11–19) is to show that $Q(3,2)$ & $Q(4,2)$ are infinite & he conjectures that $Q(m,2)$ is infinite for all $m > 1$.

(3) Sylvester primes by S. Yates. (Ibid 8 (1973), no 3, 159–68

Elementary remarks on primes of the form $(b^n-1)/(b-1)$ & lists all known primes of this form for $2 \leq b \leq 10$ (H. L. Abbott reviewed Edmonton, Alta)

(4) I. I. Mihailov — on the eqn $x(x+1) = y(y+1)z^2$

Gaz. Mat. Ser A-78 (1973), 28–30

Author shows that the eqn has an ∞ set of solns (the Δ^2 nos associated to y).

(5) P. Erdös — Problems & results on number-theoretic properties of consecutive integers & related questions # 10138

Proc. 5th Manitoba Conf. on Numerical Maths

An old conjecture of Catalan states that 8 and 9 are the only consecutive powers. R. Tijdeman (to appear) just proved that for $n > 10^{10^{500}}$ it is impossible to have $n = x^l$, $n+1 = y^s$ if $l > 1$, $s > 1$.

Another old conjecture states that the product of consecutive integers is never a ~~power~~ power. This conjecture was proved by J. L. Selfridge & Erdös (Illinois J. Math [J. M. Gandhi, Macomb Ill]

(6) Rameswara Rao's Ph. D. Thesis on Diophantine eqns
(Published by S. Chand & Co) reviewed by E. L. Cohen
(Ottawa, Ont)

This being a Thesis there is a random arrangement of Diophantine eqns - there are many misprints. The most annoying is the use of the same symbol for 1 and l on several occasions

(7) Ref to the "famous theorem of Thue - Siegel - Roth" - To find out what this is?

(8) Review of Baker's book on "Transcendental number theory" by Tijdeman (Leiden)

Contains proofs of Liouville's theorem (?), the transcendence of e and π and the theorem of Lindemann - Weierstrass

Math. Reviews - Vol. 56, No. 1 July 1978

(1) 17 terms in A.P. by Weintraub, S.H.
Math. Comp. 31 (1977), no. 140, 1030
The author announces a sequence of 17 terms in A.P.; the first term is 3430751869 and the common difference is $87297210 = 2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$. He also claims to have found 7 new sequences of 16 terms in A.P., and a sequence of six consecutive terms in A.P. in the vicinity of 10^{15}.

(2) A remarkable class of continued fractions by Adams & Davison
Proc. Amer. Math. Soc. -
Gives ~~a series which~~ the series which on the basis of the

Thue-Siegel-Roth theorem, represent transcendental numbers.

Let α be irrational, a an integer & $\alpha > 0$ and $|a| > 1$. The authors compute the continued fraction expansion of the series

$$(a-1)\sum_{r=1}^{\infty} 1/a^{[r\alpha]}$$ in terms of the C.F. expn of α

(3) Prime numbers & the Brownian motion

Wiadom Mat (2) 20 (1976), no. 1. 35-53

English version [Amer. Math. Monthly 80 (1973), 1099-1115,

M.R. 49 #9883.

(4/7/79) at B.U. Library [The fool should have given the method]

Kaprellar's article on H-numbers — Tables given by him

a	7	11	12	13	15	16	17	19	21	22	23	24
	70	209	48	247	195		476	874	588	2398	1679	1896
	133	308	84	364	285	592	629		777	2596		1968
	322	407	156	481								
	511		192	715								
			228									

d	25	28	29	31	37	41	43
	~~139~~	7588	4988	17887	37999	177899	99889
	4975	8596	55883	25699			397879
				25978			

Math. Reviews. 41, No. 5, May 1971.

(1) Siegel's T.I.F.R. Lecture on Analytic No. Theory – No. 23 of the series (1965)
~~Edited by~~ Notes by S. Raghavan
Became unavailable in 1961 & republished ~~[illegible]~~ in 1965.

(2) On odd multiply perfect numbers by McDaniel Wayne #6764.
n is $(s-1)$ fold multiply perfect number if $\sigma(n) = sn$. McCarthy has shown that the number of distinct prime factors that an odd m.p. no of $(s-1)$ multiplicity ~~can~~ can have $\geq s^2-1$. The author improves this to $\geq \frac{1}{3}(s^3-5)$. If s is odd, this is improved to $\geq \frac{1}{3}s^3$. He also shows the non-existence of odd multiply perfect nos.

(3) Samuel S. H. Young – An arithmetical problem involving the sum of integers in A.P. (Math. Mag. 43 (1970) 89–92 #6767
Considers finite a.p's of natural numbers of first term L, last term M such that the sum is equal to "LM", this being the number obtained by juxtaposition of the decimals (?) for L & M. For eg

$$17 + 19 + 21 + \cdots + 85 = 1785$$

It is shown that for such an A.P., one has $10^m d < L + M + (10^m - 1)d < 10^m(1 + \sqrt{(d^2+1)})$, where d is difference & m is the number of digits in m (?).

(4) Eine Eigenschaft aufeinanderfolgen Zahlen by H. Harbott #6771
For $2 \leq n \leq 16$, it is known that in any sequence of consecutive integers, there must be at least one integer relatively prime to each of the others. The falsity of this statement for $n > 16$ was proved in 1941

Brauer & Pillai & again in 1969 by R. J. Evans. The author gives a new proof of the same result

(5) ~~Prob~~ Pisot numbers & Salem nos. in #6786

(6)

Math. Reviews. Vol. 55, No. 5.

(1) Factors of repunits by Yates #10369 [J. R. Math. & nos. 2. 99-107]
Tables dealing with the factorization of $R_n = (10^n - 1)/9$. Table 2 gives the decimal value of the product of the primitive factors of R_n for $n = 1(1)100$ together with the no. of these prime factors, when known, and the inequalities for this number otherwise. More precisely the table gives $P_n = \Phi_n(10)$ where $\Phi_n(x)$ is the cyclotomic polynomial. Table 1 gives the "algebraic factors" of R_n, that is, R_n/P_n as products of the P_n thus giving unfactored the proper divisors of n for $n = 1(1)100$.

D. H. Lehmer - Berkeley, Calif.

(2) A note on an eqn related to the Pell eqn by Morris Newman #10376.
The cases under which $x^2 - dy^2 = -1$ has solns are not still fully known. (Review by H. Gupta)

(3) $27y^2 + 4x^3 = -\Delta^2$ (Diophantine) has no solns

(4) on $x^3 - Dy^2 = 1$ by R. J. Stroeker #10378

Solves for $D = 5, -5, 7, -7$

$x^2 + 7y^2 = 1$ has solns $(-3, 2)$

$x^2 - 7y^2 = 1$ has solns $(2, 1), (4, 3), (22, 39)$.

(5) on the eqn ~~$4x^3 = 27y^2$~~ $4x^3 = 27y^2 + N$ by R. J. Stroeker. #10379

Bombieri ~~showed~~ mentioned that $4x^3 = 27y^2 + 49$ has the soln $x = 7$, $y = \pm 7$ but was notable to decide whether there are any other integral solns. The paper confirms that there are in fact no other solns

(6) Ref: 5 or 6 other references to which the author has not referred

(6) Ref. to book by Schneider-Lang on Intn to transcendental numbers, 1966

(7) Sur le nombre $[\alpha, \alpha+\beta, \alpha+2\beta, \ldots \alpha+n\beta \ldots]$ by Jean Eliann #10399

Central result proved is that the C.F. expn in the title denoted by $A(\alpha, \beta)$ is a transcendental number — A C.F. is transcendental if its quotients are natural numbers that form an A.P. from some point on. The proof uses some deep results of Siegel [Transcendental no. Ann. of Math. Studies No. 16, Princeton Univ. Press 1949

(8) The numbers $\sum_{n=0}^{\infty} F_n^{-k}$ $(k = 1, 2, \ldots)$ $(F_n = 2^{2^n} + 1)$ are algebraically independent

(9) R. C. Vaughan (J. R. A. Math. 290, 93–108. (1977) #10408

Theorem, Each +ve even integer can be represented as the sum of at most 26 primes, and consequently every natural number can be expressed in more than one way as a sum of at most 27 primes

Math. Reviews — Vol. 55. No. 3 — March 1978

(1) Ein Analogon zu den ganzzahligen pythagorischen Dreiecken by Helmut Hasse

Elem. Math. 32 (1977), no. 1, 1–6 #5521

The author considers the Diophantine eqn $a^2 + ab + b^2 = c^2$ and deduces

all of its primitive integral solns using the sixth root of unity $(1+\sqrt{-3})/2$ surveying all the solns which may be interpreted as integer sides of Δs with angles 120° $(ab>0)$ & 60° $(ab<0)$

(2) ~~a^x+b^y~~ $a^x = b^y + c^2$ * #5526

Solver for all a, b, c being distinct primes ≤ 17 & where $b<c$.

except $2^x = 3^y + 13^z$ which is solver by Uchiyama (#5527) by showing that the only solns are $(1,0,0)$, $(2,1,0)$, $(4,1,1)$ and $(8,5,1)$

[* Math. J. Okayama Univ. 19 (1976/77) no. 1, 25–29]

(3) Waring's problem by Charles Small

Math. Mag. 50 (1977) no. 1, 12–16.

Survey of the history of Waring's problem. The main results are described (of course without proof). & also the most important of the newest results are quoted. Though short, this paper is very interesting to read

Math. Reviews Vol. 56, No. 2, August 1978

(1) Review of book "Explorations in number theory" by ~~Jean~~ Jeanne Agnew, 1972

Alt ref. to Le Veques's book "Topics in number theory" Vol I & II

(2) On the elementary proof of the prime no. theorem by B. Novak #2937

The development of the famous formula $\pi(x) \sim x/\log x$ & its development from Gauss, Legendre, Cebyser to Lavrik and Sobirov ending with $\pi(x) = \mathrm{li}\, x + O\left(x \exp\{-\log^{1/6} x \log^{-3} \log x\}\right)$ gives the Diamond–Steinig result

$$\pi(x) = \mathrm{li}\, x + O\left(x \exp\{-\log^{1/7} x \log^{-2} \log x\}\right)$$

(3) No. of triples of +ve integers with $a^2+b^2=c^2$ $(0 \le c \le x)$ & Pythagorean triples. If $P(x)$ = no. of such triples, the author proves (F. Fricke) proves that $P(x) = (4/\pi)\, x \log x + \alpha x + O(x^{3/4} \log x)$ etc. — Very complicated expn for α.

Math. Reviews — Vol. 56, No. 4, October, 1978

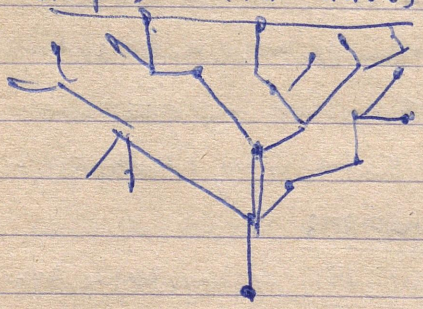
(1) Some identities for Δ^r nos by Trotter

J. R. Math 6 (1973), no. 2, 127–35

Identities which fascinate him — no reviewer!

(2) 5/7/79 At I. I. Sc

Graphs in Math. models



Now from the cover of a book on Graph Theory on E. V. K's table

(1) Problems in the theory of numbers by Dynkin & Uspenski

Nothing in particular now

(2) The fascinating numbers by W. J. Reichman —

Must have seen this earlier

p 39 — Perfect number are triangular — Converse not true

p. 90 $232 = 4 \times 58$

$23432 = 4 \times 5858$

$2345432 = 4 \times 586358$

$234565432 = 4 \times 58641358$

$= 4 \times 5864191358$

$= 4 \times 580419691358$

$23456789876543 2 = 4 \times 58041974691358$

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p. 91 $12345679 \times 99999999 = 1234567887654321.$

" $3025 = 55^2$ & $30 + 25 = 55$

$9801 = 99^2$ & $98 + 01 = 99$

$2025 = 45^2$ & $20 + 25 = 45$

p. 92 In addition to $2178 \times 4 = 8712$

$1089 \times 9 = 9801$

gives $4356 \times \frac{3}{2} = 6534.$ (ie my extn with λ & μ)

p. 123 Stirling's theorem

$\sqrt{(2\pi n)} \cdot e^{-n} n^n = n!$ approx

J. R. Math - Vol. 10, No. 4

(1) Consecutive-digit primes (Finale) by Ray P. Steiner

Ref. to 3 earlier papers in J. R. Math 2 by Madachy

& one by Finkelstein & Leybourn

Consecutive digit primes are primes formed of digits in either ascending or descending order. In particular it was shown that are no consecutive-digit primes with 9 digits in descending order, and that only 3 primes are known which begin with 1 & have consecutive digits in ascending order viz

1234567891, 12345678901234567891, and

12345678912345678912345678 9.

In this paper we complete some factorisations of numbers other than those given in the previous papers.

(1) 12345678912345678912345 67

= 2242342 51 · 55057061 3422 7317

(Both factors are prime)

For other examples.

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p. 33



1 4 7 C. W. Trigg

p. 34.

Vol. 10. No. 4 issue announcing death of Silverman

p. Article on digital products by Stewart Marchetti, Culver City, California

In Hunter - Madachy's Maths on Vacation, solns were listed for 4 types of digital product (multiplications where the factor & product 9 or 10 unrepeated digits

I A(BCDE) = FGHI — 2 solns

II AB(CDE) = FGHI — 7 "

III A(BCDE) = FGHIJ — 13 "

IV AB(CDE) = FGHIJ — 9 solns

The Editors commented that other types may have solns (but knew of none)

V A(BC)(DE) = FGHIJ (~~[illegible]~~ 9 digits, no zero)

VI A(BC)(DE) = FGHIJ (10 digits)

VII A(BC)(DEF) = GHIJ (10 digits)

Using a BASIC program executed on a Microkit 8/16 microprocessor, the solns were ~~[illegible]~~ verified for types I to IV and obtained for types through V - VII

V

7.26.19 = 3458

~~7.61.~~

7.29.18 = 3654

2.61.39 4758

1.92.53 4876

1.87.62 5394

8.36.19 5472

9.38.16 5472

8.37.19 5624

2.87.34 5916

3.71.28 5964

2.73.41 5986

2.73.59 8614

VI

5.76.39 = 14820

9.58.32 = 16704

8.64.35 = 17920

8.65.37 = 19240

8.97.31 = 24056

9.61.47 = 25803

8.91.42 = 30576

5.84.76 = 31920

8.91.45 = 32760

8.91.65 = 47320

VII

2.14.307 = 8596

1.26.345 = 8970

Amer. Math. Monthly — All ten nos. of Vol. 85 are available in the Library

No 1, p. 287 — Advanced Problems:

6107 [1976, 573]: $\sigma(n)$ = sum of all divisors of n. Let A be the set of all rational nos $\sigma(n+1)/\sigma(n)$. Determine the Closure set of A in the set of real numbers.

Prop by Shapiro of Courant Inst — Very complicated

In Editor's comments there is a reference to a paper by A. Schinzel on Generalisation of a theorem of B.S.K.R. Somayajulu on the Euler ϕ-function Ganita (Lucknow), 5, 123–28

Mathematics of Computation Vol. 32, No. 142, April 78

p. 583 — Irregular primes to 125,000 by S.S. Wagstaff

Fermat's Last Theorem has verified for all exponents up to 125,000 — Complete factorisation of the numerators of the Bernoulli nos B_{2k} for $2k \leq 60$, & of the Euler numbers E_{2k} for $2k \leq 42$ are given.

p. 586. 1473 primes below 125,000. According to Siegel's heuristic reasoning about $1473\left(1-e^{-1/2}\right)^2$ or 228 of these should be irregular primes, in fact there are 207 such primes below 125,000 & has been conjectured but never proved that there are primes of arbitrarily high index of irregularity

pp 617 - 630

Euler nos E_n $(n = 0, 1, 2 \cdots)$ can be defined by the symbolic eqn

$$(E+1)^n + (E-1)^n = 2 \text{ for } n = 0$$
$$= 0 \text{ for } n \geq 1$$

E-regular & E-irregular primes

Ibid Vol 37, No. 1&1 – 3 Jan 1978

p. 281 – Interesting article on the fluctuations of Littlewood for primes of the form $4n \pm 1$

p. 203 – Short note on Congruent numbers

a is called congruent if & only if there exists rational non-zero integers x, y, z, t such that

$$x^2 - ay^2 = z^2, \quad x^2 + ay^2 = t^2$$

p. 363 odd integers N with five distinct prime factors for which $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}$

by Masao Kishore (Author's Doctorate dissertation submitted to Princeton Univ & directed by Prof. J. Chidambaraswamy of the Univ. of Toledo).

A N is called perfect, quasi-perfect (QP) or almost perfect (AP) according as $\sigma(N) = 2N, 2N+1, N$ or $2N-1$ respectively

while 24 even perfect numbers are known, no odd perfect number (OP), no QP numbers, and no AP numbers except a power of 2 are known.

using the inequality [illegible]

17/7/79. At Inst – Found in 1st Floor number of books on recreational maths. 793.74 (N ex)

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(1) Mathematical Diversions by Hunter & Madachy

p. 3 Primes that look incredible Chap I – Friendly nos. & others

1, 111, 111, 111, 111, 111, 111

11, 111, 111, 111, 111, 111, 111, 111

909, 090, 909, 090, 909, 090, 909, 090, 909, 091.

9, 090, 909, 090, 909, 090, 909, 090, 909, 090, 909, 090, 909, 091.

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p. 4 – Fermat ~~nos~~ primes. 3; 5; 17; 257; 65,537

Robinson prime R is a factor of F_{1945}

p. 5. Euclidean constn of a regular polygon with a prime number of sides is only possible if the no. of sides be 3, 5, 17, 257, 65,537.

p. Geometrical constn for 17-gon given (quite simple) adapted from H. S. Coxeter – Intn to Geometry

p. 6 – No polynomial can generate primes exclusively

Formula devised by Malcolm H. Tallman (Recreational Math. Mag. No. 4, August 1961, p. 64)

$$N = \frac{P_n}{a_i a_k \cdots a_n} \pm a_i a_k \cdots a_n$$

where (1) P_n = product of the first n primes & $a_i, a_k, \ldots a_n$

are any of the first n primes all different, and/or unity and (2) $N <$ the square of the $(n+1)^{th}$ prime. Then every such N is a prime

Ex: $N = \dfrac{2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23}{2\cdot3\cdot11\cdot13\cdot17} - (2\cdot3\cdot11\cdot17\cdot19)$

$= 709$, a prime because $709 < 29^2$.

p.6 - Proof that no polynomial can generate a prime

p.7. Polynomial formula yielding prime pairs

$$N = 12,150 - 1710A + 60A^2$$

where A takes successive values $1, 2, 3, \ldots 20$, and $N+1$ and $N-1$ are the prime pairs. 18 pairs of $+^{ve}$ primes pairs are generated & one pair $(-29 \& -31)$ of $-^{ve}$ primes.

p.7 Bertrand's postulate - There is at least one prime between N & $2N$. where N is any integer > 1. This has been proven, but no formula has been found for the exact no. of primes in that interval; there is also a proof that, very roughly speaking, there are as many primes between N & $2N$ as between 0 & N. Applying Bertrand's postulate we know that there must be at least three primes containing exactly 100 digits - but as yet not one of them has been identified, although there must be in fact be millions in that category

[using Tchebycheff's formula for the approx. no. of primes $< x$, he finds that there are very roughly $9\cdot10^{97}$ primes between 10^{99} & 10^{100}]

P·8 – Primes in A·P. ~~– top to long known series~~

Ex. 1: 11, 17, 23, 29

Ex. 2. Starting with 199 & having C·D = 210 → 10 terms

Ex. 3: Longest known is of 12 terms only 1st term = 23,143 & C·D = 30,030.
(Golubiew) (what year?)

Unanswered qn – Is there any series of primes in A·P. of arbitrary length?
Say for eg. 50 terms?

P·9 – Lucas' method of testing Mersenne primes:

Eg: To test primality of $2^7 - 1 = 127$ by Lucas' series

1st term = 4. (misses 4, 14, 194, 37,634 in which each term is 2 less than the square of the previous term)

Second term = $4^2 - 2 = 14$

3rd term = $14^2 - 2 = 194$

Since $194 > 127$, we divide by 127 & get a remainder of 67. We then continue

$67^2 - 2 = 4487$; divide by 127 & get remainder 42

$42^2 - 2 = 1762$, divide by 127 & " 111

$111^2 - 2 = 12321$, divide by 127, get no remainder

∴ 127 is prime.

P·10 – 26th perfect number has 2663 digits

Unsolved problem: whether or not there is an infinity of Mersenne primes & therefore an infinity of perfect numbers.

odd perfect nos – none $< 10^{18}$.

Proving that there is not an infinity of Mersenne primes would not solve the question of an ∞ of perfect nos until the problem of odd perfect nos. is solved.

p. 11, Interesting ~~factors~~ facts about perfect nos

(i) Except 6, all perfect nos have digital root 1

eg. $496 \to 4+9+6 = 19 \to 1+9 = 10 \to 1+0 = 1$.

(ii) Except 6, every perfect no is the sum of consecutive odd cubes

$28 = 1^3+3^3$, $496 = 1^3+3^3+5^3+7^3$

$8128 = 1^3+3^3+5^3+7^3+9^3+11^3+13^3+15^3$

(iii) Perfect nos are sums of successive powers of 2 from 2^{p-1} to 2^{2p-2}

$6 = 2^1+2^2$

$28 = 2^2+2^3+2^4$

$496 = 2^4+2^5+2^6+2^7+2^8$.

There are many other special properties of perfect nos

Chap. II – From paradox to parastichy

All about Fibonacci series

Chap III – Mystic arrays

Magic squares & cubes

Chap IV – Topological delights

Chap V – Some inferential problems

Just 4 pages giving "problems in logic" using elementary methods of Boolean algebra – Quite interesting

Chapter 6 – Diophantus & all that

(1) Pythagorean eqn

(2) $\left.\begin{aligned} x^2 + xy &= A^2 \\ y^2 + xy &= B^2 \end{aligned}\right\}$

Finds using soln of (1) that general soln of (2) is

$$x = 4a^2b^2s, \quad y = (a^2-b^2)s$$

Ex: $a=5, b=2, s=3, x=1200, y=1323$

$$1200^2 + 1200 \cdot 1323 = 1200 \cdot 2523 = 1740^2$$

$$1323^2 + 1200 \cdot 1323 = 1323 \cdot 2523 = 1827^2$$

To find smallest integers satisfying (2)

Take $a=2, b=1 \rightarrow x=16s, y=9s$. since 16 & 9 are relatively prime

smallest value comes from $s=1$, giving $x=16, y=9$.

but $x=9, y=16$ must also be a soln. For this take $a=3, b=1, s=1/4$

(3) $X^2 + eY^2 + fZ^2 = W^2$

Simplest general soln is $X = \pm(a^2 - eb^2 - fc^2)k$

$$Y = 2abk$$

$$Z = 2ack$$

$$W = \pm(a^2 + eb^2 + fc^2)k$$

(a, b, c integers, k any common factor)

Ex: $X^2 + 2Y^2 - 3Z^2 = W^2$

Here we take $Y = \pm 2abk$, $Z = \pm 2ack$.

& with $a=2, b=2, c=1, k=1$, one particular soln is $(X, Y, Z, W) = (1, 8, 4, 9)$

(4) $x^2 + 2y^2 = z^2$.

Using (3) with $e = 2$, $f = 0$, we get $x = (a^2 - 2b^2)k$, $y = 2abk$,

$z = (a^2 + 2b^2)k$.

(5) $X^2 + Y^2 = 5Z^2$, writing this as $X^2 + Y^2 - (2Z)^2 = Z^2$; gets soln

~~as X~~ using (3) as

$$X = 4mn(10mn - m^2 - 5n^2)$$

$$Y = 4mn(2m^2 - 10n^2)$$

$$Z = 4mn(m^2 - 2mn + 5n^2).$$

We can now replace $4mn$ by an arbitrary k

(6) $\left.\begin{array}{l} x^2 + y = A^2 \\ y^2 + x = B^2 \end{array}\right\}$ Put $A = x - m$, $B = y - n$ (m, n rational nos)

leading to $y = m^2 - 2mx$, $x = n^2 - 2ny$

Combining these $2mx + y = m^2$

$$x + 2my = n^2$$

ie $4mnx + 2ny = 2m^2n$, $x + 2ny = n^2$

Hence $x = \dfrac{n(2m^2 - n)}{4mn - 1}$, $y = \dfrac{m(2n^2 - m)}{4mn - 1}$

is the reqd soln

(7) Pellian eqn

(a) $x^2 - Ny^2 = 1$ (N not a square). General soln given by

$$\left.\begin{array}{l} x_n = 2ax_{n-1} - x_{n-2} \\ y_n = 2ay_{n-1} - y_{n-2} \end{array}\right\}$$

where $x = a$ is the value of x in the first non-zero soln

This is quicker & more practical. to find the first a by trial & error

(6) $x^2 - Ny^2 = e$ (e +ve or -ve integer other than 1)

A particular case $x^2 - Ny^2 = -1$.

For any particular e, integral solns exist only for certain values of N.

Thus if $e = -1$, there will be integral solns only if $N = 2, 5, 10, 13, 17$ etc.

If $N = 3$, there will be integral solns only when $e = +1$, or $-2, -3$, or $+2$, $+6$, or -8 etc. Theoretical problem of ascertaining such compatible values of N & e is very complex.

In practice if we find an integral soln for some particular values of N & e by trial & error, say smallest values of x & y are $x = a$, $y = b$. Also let $x = m$, $y = n$ be any integral soln of $x^2 - Ny^2 = 1$, then

$$(x^2 - Ny^2) = (a^2 - Nb^2)(m^2 - Nn^2)$$
$$= (am \pm Nbn)^2 - N(an \pm bm)^2$$

} Brahmagupta's rule.

Thus upon $\pm$ being same in $x = am \pm Nbn$, $y = an \pm bm$

Ex(i) $x^2 - 3y^2 = -11$. Smallest integral soln is $x = 1$, $y = 2$

Gen soln is $x = m \pm 6n$, $y = n \pm 2m$. Since $(-x)^2 = x^2$, $(-y)^2 = y^2$ we can write more generally $x = \pm(m \pm 6n)$, $y = \pm(n \pm 2m)$

(where m, n satisfy $x^2 - 3y^2 = 1$).

Tabulating

$m = 1, 2, 2, 7, 7, 26, 26$, etc

$n = 0, 1, 1, 4, 4, 15, 15$, etc

$x = 1, 4, 8, 17, 31, 64, 116$, etc

$y = 2, 3, 5, 10, 18, 37, 67$, etc

It is possible, however, to have an eqn of this type for which the above procedure will not find all the integral solns

Illustrate: $x^2 - 2y^2 = 119$ subject to $x < 200$

By quick trial, smallest soln is $x = 11, y = 1$

A fundamental soln is $x = \pm(11m \pm 2n),\ y = \pm(m \pm 11n)$ $(m^2 - 2n^2 = 1)$

$x = 11,\ 29,\ 37,\ 163$

$y = 1,\ 19,\ 25,\ 115$

If there be other solns independent of those derived from $x = 11, y = 1$, then the smallest of such other solns must give values of x & y between the smallest and the next smallest solns ie in this case between $x = 11, y = 1$ and $x = 29, y = 19$. So we test the original eqn for successive values of y from $y = 2$ to $y = 18$. Checking in this way we find a new soln

$x = 13, y = 5$

Proceeding $x = \pm(13m \pm 10n),\ y = \pm(5m \pm 13n)$ $(m^2 - 2n^2 = 1)$

we derive $x = 13,\ 19,\ 59,\ 101$

$y = 5,\ 11,\ 41,\ 71$

The process can be repeated but it can at once be seen that there is a third "family" of solns ie Checking from $y = 6$ to $y = 10$.

In fact for $x^2 - 2y^2 = 119$ there is no third family

Tabulating both the families

$x = 11$		29	37		163
$y = 1$		19	25		115
$x =$	$13, 19$			$59, 101$	
$y =$	$5, 11$			$41, 71$	

$x^2 - Ny^2 = fN$, $x = zN$. $z^2N^2 - Ny^2 = fN$

Notice that $119 = 17 \times 7$ product of two primes. So we have a simple rule:

(i) In $x^2 - Ny^2 = e$ (N not a square) and where e & N have no common factor & if e = product of m odd prime numbers (no factor of e being a square or higher power), then there will be either m families of solns or else no integral solns at all.

(ii) If e is a multiple of N reduce the eqn to $y^2 - Nz^2 = -f$ since if $e = fN$, x must be $= zN$ (say).

(iii) If e & N have a common factor, although e is not a multiple of N is more complex & outside the scope of this book

Ex (a) $x^2 - 3y^2 = 3289$ has three families of solns since $3289 = 11 \times 13 \times 23$. Find solns for +ve values of $y < 50$.

(iv) If e is even & with squares of higher powers of primes among its factors things become rather more complex. But general principles re. finding several families of soln is concerned

The practical aspects in connection with solns of Pell's eqn have been considered in some details since such eqns arise in many recreational problems [This justifies my work on Vigdor's result etc]

The chapter covers only a small part of Diophantine eqn,

Chapter 7 — Potpourri (Much geometry)

Using one each of the digits 1, 2, 3 & 4 together with only decimal

points, minus signs & brackets a fantastically large number can be expressed

$$N = 3^{(\cdot 2)^{-(\cdot 1)^{-4}}} = 3^n$$

having approximately 6990 digits

Chapter 8 – Fun with Shapes

Chapter 9 – Alphametics & the like

p. 95 Gives the interesting problem of Rouse Ball

Chapter 10 – What are the odds

Probability problems

p. 99 – Fallacy re. no prime numbers being known as interesting

Chapter 11 – Story teasers

p. 176 Appendix – A fundamental property of the Fibonacci series

Solns to ~~several~~ problems in the several Chapters are given

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Quite a good book – specially treatment of Pell's eqn

19/7/79 at C. C. B.

Ref. Math. Mag. Vol. 51, No. 3, May, 1978.

(1) A scrap book of elementary mathematics by W. F. White (Open Court, 1908)

p. 17 – ~~A repeating~~ table.

~~9 × 1 =~~ Fourth Lucas' Recreations

~~9 × 2 =~~ 18 note entitled curious multiplication

$1 \times 9 + 2 = 11$

$12 \times 9 + 3 = 111$

$123 \times 9 + 4 = 1111$

$12345678 \times 9 + 9 = 111111111$

$9 \times 9 + 7 = 88$

$98 \times 9 + 6 = 888$

$987 \times 9 + 5 = 8888$

$98765432 \times 9 + 0 = 888888888$

$1 \times 8 + 1 = 9$

$12 \times 8 + 2 = 98$

$123 \times 8 + 3 = 987$

$123456789 \times 8 + 9 = 987654321$

$12345679 \times 8 = 98765432$

$12345679 \times 9 = 111111111$

$12345679 \times 18 = 222222222$

" $\times 27 = 333333333$

" $\times 36 = 444444444$

etc

Prop. A few numerical curiosities (due to Escott)

$11^2 = 121$, $111^2 = 12321$, $1111^2 = 1234321$, etc.

$1+2+1 = 2^2$, $1+2+3+2+1 = 3^2$

$1+2+3+4+3+2+1 = 4^2$, etc.

$121 = \frac{22 \times 22}{1+2+1}$; $12321 = \frac{333 \times 333}{1+2+3+2+1}$, etc [Mouvt, 1906, 16, 625]

Multiples of 37 & 41: Certain multiples of 37 are such multiples of 37 when their figures are permuted cyclically

$259 = 7 \times 37$, $592 = 16 \times 37$, $925 = 25 \times 37$

Same is true of 185, 518, 851

and 296, 629, 962.

41: $17589 = 41 \times 429$, $75891 = 41 \times 1851$, $58917 = 41 \times 1437$

$89175 = 41 \times 2175$, $91758 = 41 \times 2238$

1375, 1376, 1377

are probably three lowest consecutive numbers that are divisible by cubes other than 1, (divisible by cubes of 5, 2, & 3 respectively)

Powers having same digits. (consecutive nos)

p.20

$13^2 = 169$	$157^2 = 24649$	$913^2 = 833569$
$14^2 = 196$	$158^2 = 24964$	$914^2 = 835396$.

cubes containing same digits

$345^2 = 41063625$	$331^2 = 36264691$
$384^2 = 56623104$	$406^2 = 66923416$
$405^2 = 66430125$	

Square nos. containing the digits not repeated

(Ref. Math. Magazine, 1883. & completed in L'Int. des Mathématiciens, 1897)

(a) Containing 9 digits

Gives 30 cases where all numbers squared have 5 digits

(b) Containing 10 digits

Gives 10 cases also with $(abcde)^2 = ABCDEFGHIJ$

— Ref. L'Int. des Math, 1907, 14: 135)

123456789 multiplied by all digits < 9 & prime to it viz 2, 4, 5, 7, 8 gives products containing the nine digits once only.

$987654321 - 123456789 = 864197532$

To arrange the nine digits additively to make 100

15	56	$95\frac{1}{2}$
36	8	$4\frac{38}{76}$
47	4	100
98	3	
2	71	
100	29	
	100	

For other solns ref. Fourrey & Lucas

§.22 To arrange 10 digits additively

$50\frac{1}{2}$	$80\frac{27}{54}$
$49\frac{38}{76}$	$19\frac{3}{6}$
100	100

To place the 10 digits so as to produce each of the digits

$$\frac{62}{31} - \frac{970}{485} = 0$$

$$\frac{62}{31} \times \frac{485}{970} = 1$$, others as in Perelman

Ref. to Lucas (Theory de nombres)

$$\text{For } 6 = \frac{34182}{05697}$$

Says that Lucas has not been able for this case of 6, not to have a zero in first place

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9 digits arranged to form perfect cube.

$$\frac{8}{32461759} = \left(\frac{2}{319}\right)^3, \quad \frac{8}{24137569} = \left(\frac{2}{289}\right)^3$$

$$\frac{125}{438976} = \left(\frac{5}{76}\right)^3; \quad \frac{512}{438976} = \left(\frac{8}{76}\right)^3$$

10 digits: $\frac{9261}{804357} = \left(\frac{21}{93}\right)^3$

p. 23 10 digits place to give an approx. value of π

$$\pi = \frac{67389}{21450} = 3.141678+$$

pp. 218–233 – Alice in the wonderland of mathematics by Lewis Carrol – interesting – to be read

(2) Topics in recreational mathematics by J. H. Cadwell (C.U.P, 1966)

Chap 4 – Distn of prime nos

1. Sieve of Eaas thoms

Gap between primes must be an even no, smallest gap is suspected to appear ∞^{ly} often (twins)

$$10^{12} + 9649, \quad 10^{12} + 9651$$

are a very large pair of this form

Bertrand's postulate $p_{n+1} < 2p_n$

Riemann hypothesis that all complex zeros of the ζ-fn have real part 1/2 – No proof yet – Skewes proved that if the Riemann hypothesis holds, this N (ie Littlewood's proof that for large enough N, $\pi(N)$ eventually exceeds the integral approximation to it) satisfies

$$\log\log\log N < 79, \text{ or } N < 10^{10^{10^{34}}}$$ (Skewes' number)

Lord Cherwell used statistical methods to show that the interval $(N, N+\Delta N)$ contains

$$\frac{1.32\,\Delta N}{(\log N)^2}$$

twin primes — [improper conjectures]

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p. 131, ~~$r = \lim_{n\to\infty}\{1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\}$~~

Proof of ~~ratio~~ e not being rational

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots$$

If $e = p/q$ then $q!\left(e - 1 - \frac{1}{1!} - \frac{1}{2!} - \ldots\right)$

$$= \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \ldots$$

Expn on the left is an integer since $e = p/q$, while the right is +ve

and $< \frac{1}{q+1} + \frac{1}{(q+1)^2} + \ldots < 1$

ie a contradiction

p. 131 - we do not know if $\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots\frac{1}{n}-\log n\right)$ is rational or irrational or whether $\pi^{\sqrt{2}}$ is rational or irrational

p. 133 - Approx^{n} of irrationals by rationals

Dirichlet's theorem: An irrational number ξ has an infinite number of rational approximations m/n in lowest terms & such that

$$\left|\xi - \frac{m}{n}\right| < 1/n^2$$

Continued fraction method improves this to

$$\left|\xi - \frac{m}{n}\right| < \frac{1}{\sqrt{5}\,n^2}.$$

For some irrational $\sqrt{5}$ cannot be replaced by a larger no i.e. it is the best possible.

p. 135 - Algebraic nos. - form a field (the system is closed)

p. 136 - Problem of proving that a particular trans. number is transcendental is difficult - Hermite proves e transcendental in 1873.

Lindemann extended his method & used

$$e^{i\pi} + 1 = 0$$

to prove π transcendental in 1882

Gelfond & Schneider's general result (1900)

If α and β are algebraic, neither 0 nor 1, and β is irrational then α^β is transcendental. For instance we can see whether $2^{\sqrt{2}}$ is transcendental, but 2^e is still in doubt, since e is not algebraic

Mahler's work concerns polynomials $f(x)$ that take integer values when $x = 1, 2, \ldots$. As an example consider $f(x) = \frac{1}{2}x(x+1)$ taking the values $1, 3, 6, 10, \ldots$. Mahler proved that

$$0.f(1)f(2)f(3)\ldots$$

is transcendental. In the particular $\frac{1}{2}x(x+1)$

$$0.136101\ldots$$

is transcendental

Euclidean nos are those alg. nos that can be constructed by using rule & compass

Greek problems, duplication of cube, trisection of an angle & squaring the circle. The eqns to be solved are

$$x^3 - 2 = 0, \quad 4x^3 - 3x + c = 0, \quad x^2 - \pi = 0$$

1st two do not define Euclidean nos ie problems are insoluble

3rd depends on π being a tr. no & hence impossible

(Ref. Courant & Robbins for more details

p.137 — Liouville's Theorem

An alg. no defined by an eqn of degree n is not approximable to an order higher than n.

Simplest number of this type (transcendental) found by Liouville

$$\frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \ldots$$

As recently as 1955, Roth proved a more powerful result that an alg. no can be approximated to an order higher than 2.

p. 138 – Cantor's constn of transcendental nos

(To be read more carefully) – ~~All numbers~~ Almost all numbers are transcendental, since the exceptions are of measure zero

p. 139 – Normal nos (Borel)

A no. where decimal expansion contains each of the digits from 0 to 9 in just 1/10 of all possible places

Eg 0.1234567890 is simply normal

Criterion for fully normality

Number: 0.12345678910111213 ...

is both normal & transcendental

(To be read more carefully)

Chapter 14

The number π

(to be read more carefully).

[Eg. that Niven's book on irrational nos].

Math. Mag. Vol. 51, No. 3, May 1978 – Ramanujan

(20/7/79)

Notes from Madachy's Maths on Vacation

(Contd from p. 114 of Book. No. 11)

This 1966 appears to be a revised one. In fact in Chap 6 on number recreations, I find all P.D.I's of order 5, 6, 7, 8, 9 which I had proposed to solve on computer (with help of E.V.K & Miss Hyde) have been given here. Was rather greatly disappointed

Chap. 5 ~~pp. 114 – 176~~ Puzzles & problems p. 114

2. Number toughies – Problem about $2^n \times 5^n$ (given in Ogilvy & Anderson)

Soln. p. 126

$10^9 = 2^9 \cdot 5^9$ (both factors have no zero as digits)

$10^{18} = 2^{18} \cdot 5^{18}$ (")

What other powers besides 9 & 18, have 2 factors containing no zero?

Math. Tables published by the Chemical Rubber Co show that the following powers of 2 < 100 contain no zeros:

1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 15, 16, 18, 19, 24, 25, 27, 28, 31, 32, 33, 34, 35 36, 37, 39, 49, 51, 67, 72, 76, 77, 81 and 86.

Only a few of these are matched by corresponding powers of 5 containing no zeros. ie powers 1, 2, 3, 4, 5, 6, 7, 9, 18, 33.

Rudolph Ondrejka of Atlantic City, N.J. checked all the values of 2^n up to $2^{5,000}$ & found no powers of 2 beyond 86 that did not contain zeros. Fred Greenberger, a mathematician at the Rand Corporation in ~~Saint MS~~ Santa Monica, California, programmed an IBM 1620

electronic computer to produce consecutive powers of 2 and type the exponent for those that lacked zeros. The machine dutifully typed the values listed above. It went to $2^{10,535}$ before the computer had to be cut off. $2^{10,535}$ is a number with 3,172 digits. These digits pass the frequency test of randomness, so there about 317 each of 0's and 5's in the number. If a number containing n digits passes the frequency test of randomness it means that each of the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 approximates 10% of the time in the given number. Roughly half of the 0's and 5's will produce a new 0 on the next power, and the empirical results indicate that the probability of a power of 2 higher than 86 having no 0's is quite low. The matter awaits an analytical proof, but the above analysis tends to discourage further empirical searching.

For those who are interested in further empirical results, T. Charles Jones, a student at Davidson College in Davidson, North Carolina reports that up to $2^{57,134}$ (which has 17,200 digits) the results are equally negative. And I can add, even if a power of 2 greater than 86 is found which contains no 0's, the probability that the corresponding power of 5 would also contain no 0's is fantastically small.

Chapter 6 — Number recreations.

Defn of perfect, multiperfect and amicable numbers.

Sociable number - Sociable number - If after more than 2 steps of successive additions of a number, the original one is obtained the number is called sociable. Eg 12,496 with divisors 1, 2, 4, 8, 11, 16, 22, 44, 71, 88, 142, 176, 284, 368, 781, 1336, 1562, 3124 and 6248 has the sum of these divisors = 14,288. There are 19 divisors of 14,288 & their sum is 15,472; This has 9 divisors whose sum 14,536; 14,536 has 15 divisors with sum 14,264 and 14,264 has 7 divisors and their sum is 12,496 the original number. So after 5 steps the original number is reached. It is discouraging to note that only one other sociable number is known viz 14,316 which repeats itself after 28 steps.

If $N \to N_1 \to N_2 \to - \to N_n$ & N_n has only one divisor 1, then N_n is prime. If N be prime say N_1 occurs which is prime.

Also the division-addition step may go on indefinitely. For eg. if N = 138, & after 125 steps one gets 32,185,715,586. At this point the problem of factoring, finding all the divisors, the larger one becomes impossible without the aid of electronic computers.

Notation - Uses $s(N)$ instead of the usual $\sigma(N)$.

If $s(N) = N$ then N is perfect. If $s(N) = N_1$ & $s(N_1) = N$, then N & N_1 are amicable pairs or alternately as $s^2(N) = N$.

(* Example 220 & 284)

For the known sociable numbers $s^5(12,496) = 12,496$ &
$s^{28}(14,316) = 14,316$.

alternatively $s^n(N) = N$, sociable number

$s^n(N) = \infty$

$s^n(N) = 1$, prime number.

multiperfect number defined by $s(N) + N = KN$.

Perfect & multiperfect numbers

6, 28, 496, 8128, 33,550,336 — Till 1952 only 12 perfect nos were known using desk calculators. Then SWAC, the N.B.S.'s Western Automatic Computer found the next 5. More were found subsequently & by 1965, 23 perfect numbers were known.

Euler $2^{p-1}(2^p - 1)$ for even perfect nos — Mersenne primes Gives p for which Mersenne numbers are prime as 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, and 11,213. (upto 1965 ?) — Perfect nos. larger than Corresponding M_p's (23 as given above)

Giving a table of these 23 perfect nos, $V_1, V_2, \ldots V_{23}$ resp^s equal to $2^1(2^2-1)$, $2^2(2^3-1)$, $2^4(2^5-1)$, $\ldots$ $2^{11,213}(2^{11,213}-1)$ in 1st colm, no. of digits in them in 2nd col, and no of divisors in 3rd col — only first eight perfect ones (in addition to V_1, V_2, V_3, V_4, V_5) $V_6 = 2^{16}(2^{17}-1) = 8,589,869,056$ & $V_7 = 2^{18}(2^{19}-1)$ equal to

[* Orch defn of $\sigma_0(N)$ & $\sigma(N)$ appears more reasonable
ie $\sigma_0(N)$ not including N $\sigma(N)$ including N]

137, 438, 691, 328 — V_{23} has 6,751 digits and 22,425 divisors.

multiperfect numbers* — first one discovered by Mersenne is 120 for which $\sigma(120) = 240$ & hence $\sigma(120) + 120 = 360 = 3 \times 120$ ie 120 is triperfect. Largest known multiperfect number has 264 compared with the largest known perfect no. which has 6751 digits.

There are 550 multiperfect numbers known, ranging from triperfect to octoperfect — Does not list all the 550, but gives a table giving numbers of the type known

Class	No. known
Triperfect	6
Quadriperfect	36
Quinqueperfect	62
Seniperfect	222
Septiperfect	217
Octoperfect	7

The six triperfect nos are 120; 672; 523,776; 459,818,240; 1,476,304,896, and 31,001,180,160.

There is no known formula for multiperfect nos. In some respects m.p. nos have more fascination than perfect numbers. More preclude hand work, but the multiperfect appear to be amenable to such work. Alan L. Brown, one of the last of the "human computers" has

of Montclair, N.J. has devoted more time & energy to the theory of multiperfect nos. than anybody else & has tabulated all of them. A number of these were discovered without the use of electronic computers.

Prime Numbers

(i) $\sqrt{N}$ business – Since there is an ∞ of primes, computers can never complete the task of finding & listing them

(ii) No of primes $< N$ – By actual count there are 664,579 primes less than 10^7 & 5,761,455 less than 10^8. There are no tables of primes up to 10^{100} & so no count can be made.

Legendre's method to find exact no. of primes $< N$. The method though accurate is extremely cumbersome. The method is as follows: for Ex. $N = 10^9$:– All multiples of 2 (5×10^8 of them) are subtracted from 10^9, all multiples of 3 (333,333,333 of them) are subtracted, then all multiples of 5 & so on. Corrections must be made; some of the multiples of 3 are also multiples of 2, some of 5 also of 2 and 3 & so on. If the reader has thus worked for $N = 10^3$ he will soon realise & appreciate the magnitude of the problem. To determine the no. of primes $< 10^{100}$ would tax the abilities of the fastest known computer – No. $< 10^9$ is 50,847,534. [Refer to sieve method?]

Approximate no given by prime no. theorem which states that $\frac{N}{\log_e N}$ is approached as N becomes large. Percentage of error decreases as N increases. For eg for $N = 10^9$ formula gives

48,260,000 while actual number is 50,847,534. No $< 10^{100}$ is estimated to be approx $(4.82)(10^{97})$ using the prime number theorem. There are other more accurate formulae than $N/\log N$ but become more complicated as they become more accurate: what is needed is a formula as simple as $N/\log N$ and as accurate as Legendre's method.

Repunit primes - is nos of form $\frac{10^n - 1}{9}$ where n is also the number of ones. For $n = 7$, the formula gives 1, 111, 111. If such a repunit number is to be prime n must be prime, but the converse is not necessarily true. To date all possible values of n up to and including 199 have been tried. only 3 of these have yielded primes viz $n = 2$, $n = 19$ & $n = 23$. There are still 7 values of n for which have not yielded their nature to the mathematicians viz 131, 139, 157, 163, 191 and 197. It would be interesting to have a computer tackle the problem & then continue the search beyond 199.

(Bertrand's Postulate) The 9th law of prime numbers states that there must be at least one prime between N and $2N$. ($N > 1$). A consequence of this law is that there must be at least 3 primes containing any number of digits, say 100 [How? This should have been explained]. Interesting 3 such 100-digit nos were discovered in 1958 - though it is estimated that there must be about $(9)(10^{97})$ still awaiting discovery [How?]. T. Charles Jones, a

$M_{11\,213}$ has 3376 digits

Student at Davidson College, North Carolina, programmed a computer to evaluate these three primes viz

$(81)(2^{324})+1$ = ... 100 digits (five)

$(63)(2^{326})+1$ = "

$(35)(2^{327})+1$ = "

Several hundred primes are given by the formula $K \cdot 2^n + 1$ which produces primes for certain values of K and n. (The above three are of this form). Extension of these results should yield eventually a 1,000 digit prime or even a 1,000,000-digit prime might come to light. Such a large number would involve values of n of the order of 3,300,000 which could only be reached approximately by using extremely large values of K. Even if n is reduced to 100,000, then K would have to be of the order $10^{970,000}$.

Mersenne primes - Determination of primality of larger M-primes requires the use of high-speed computers not now available. For $n > 11,213$, there is no way of even determining which n may be a likely candidate. The next M-prime beyond if it exists, maybe found just a few values beyond 11,213 or several thousand values beyond 11,213 [K. Ramachandran mentions (1979) that M_{19937} is the largest M-prime so far known i.e. $19937 - 11213 = 8724$ i.e. more than 8000 values beyond 11,213] The determination of the "next" M_p (if it exists) is technically unsolved - the word "if it exists" indicates this - It is however reasonably certain that there are more Mersenne primes & more labour will be expended in the

physical search than in a theoretical proof of their existence.

A.P.'s of primes – 199, 409, ... 2089 has 10 terms with C.D = 210. The largest A.P. known containing but 12 terms begins with 23,143 and ends with 355,473 with C.D = 30,030. – Some theoretical work done – To obtain a progression of 30 terms, it would be necessary to search through the primes in the region of 187,000,000,000 & look for a C.D = 6,000,000,000. Such searches are technical problems of extreme high order of difficulty. The possible existence of such a 30-term series remains a theoretical problem; after all there may not be such a series at all.

For what values of n does n^n+1 produce primes? If $n=1, 2$ & 4, primes 2, 5, 257 result. No others are known [Note that n^n+1 is not same as $F_n = 2^{2^n}+1$ except for cases $n=2, 2, 4$ when $F_1 = 2^2+1$, $F_2 = 2^8+1 = 4^4+1$]. If $n=15$, the number has 18 digits already beyond existing prime tables. Interesting though, it has been shown that if there are any other primes of this form they must consist of more than 300,000 digits. Same for n^n+1, besides those for $n=1$, or 2, if such primes exist, such primes must have at least 1,000,000,000,000,000,000 digits

Some quite remarks about regularity in primes losing their fascination – Formulation easier than solution

Amicable numbers

There are only about 400 amicable pairs known of which some are 220 & 284 (smallest pair), 2620 & 2924; 6232 & 6368; 10744 & 10856, 9363584 & 9437056, and 111448537712 & 118853793424.

Several methods available. A common method:

Let $a = (3)(2^x) - 1$

$b = (3)(2^{x-1}) - 1$

$c = (9)(2^{2x-1}) - 1$

If $x > 1$ and a, b, c are all primes, then $2^x ab$ and $2^x c$ constitute an amicable pair. If $x = 4$, then $a = 47$, $b = 23$, $c = 1{,}151$ all primes then $(2^4)(47)(23) = 17{,}296$ and $2^4(1{,}151) = 18{,}416$ are pairs (to be verified by the reader).

Amicable triplets ie $s(N) = N_1 + N_2$ where $s(N_1) = N + N_2$ and $s(N_2) = N + N_1$. One such set is

103,340,640; 123,228,768 and 124,015,008. Another triplet is 1,945,330,728,960; 2,324,196,638,720 and 2,615,631,953,920. These are not easy to find. The numbers in this last set have 959, 959 and 479 divisors respectively.

0123456789 or digital diversions

(1) use the 9 or 10 digits to get 100 – in order 1, 2 . 9 or reverse order – using + & −1 only – using ( ), $\dot{\sqrt{}}$, Σ, !, $\sqrt{\ }$ etc. – Egs.

$-(1234) \div (5-6-7+8)(9) = -\infty$

$(1234) \times (5-6-7+8)(9) = 0$

$(1234) \div (5-6-7+8)(9) = \infty$

(2) Dudeney's smallest square (9 digits): $(11,826)^2 = 139,854,276$, and the largest: $(30,384)^2 = 923,187,456$.

Harry L. Nelson's computer results for squares pairs ~~ABCDEFGHI~~.

results containing 0, 1, -- 9

(i) Smallest & largest 10 digit cases:

$1,026,753,849 = (32,043)^2$

$9,814,072,356 = (99,066)^2$.

(ii) with each of 1, 2, -- 9 repeated twice (18 digit cases)

Smallest: $112,345,723,568,978,496 = (335,180,136)^2$

Largest: $998,781,235,573,146,624 = (999,390,432)^2$

(iii) with each 1, 2, -- 9 repeated thrice (27 digit cases)

Smallest: $111,221,338,559,598,866,946,777,344 = (10,546,200,195,312)^2$

Largest: $999,888,767,225,363,175,346,145,124 = (31,621,017,808,182)^2$

(3) Factors each with 9 digits each

$(246,913,578)(987,654,312) = (493,827,156)(493,827,156)$

Are there other such results?

[ie $(AB-I)(AB-I) = (AB-I)(AB-I)$

Study why not $(AB-J)(AB-J) = (A-J)(A-J)$.]

(4) Computer results of Nelson re-factors like (A)(BC-I) etc. (9 digits)

Table showing different types of set-ups yielding orders & their number.

No. of factors	Problem type	No. of solutions
2	(A)(BCDEFGHI)	567
2	(AB)(CDEFGHI)	342
2	(ABC)(DEFGHI)	370
2	(ABCD)(EFGHI)	346
3	(A)(B)(CDEFGHI)	87
3	(A)(BC)(DEFGHI)	210
3.	(A)(BCD)(EFGHI)	244
3	(A)(BCDE)(FGHI)	109
3	(AB)(CD)(EFGHI)	0
3	(AB)(CDE)(FGHI)	52
3	(ABC)(DEF)(GHI)	39
4.	(A)(B)(C)(DEFGHI)	16
4.	(A)(B)(CD)(EFGHI)	55
4.	(A)(B)(CDE)(FGHI)	53
4.	(A)(BC)(DE)(FGHI)	59
4.	(A)(BC)(DEF)(GHI)	66
5	(A)(B)(C)(D)(EFGHI)	0
5.	(A)(B)(C)(DE)(FGHI)	3
5	(A)(B)(C)(DEF)(GHI)	1
5.	(A)(B)(CD)(EF)(GHI)	5
6	Are zero	
7	"	
8	"	
9	"	

All the 2,624 solns are not given but for the cases of 2, 3 & 4 factor types smallest & largest are tabulated (p. 161 - not copied out here). For the case of 5 factors ~~the nine case~~ all the nine cases are given on same page.

Out of the 2624 solns the smallest is 123,459,678 being the smallest of type (A)(BC)(DEFGHI) & the largest is 839,542,176 being the largest of type (AB)(CDEFGHI).

Repetitions of products from different sets

124,367,958 = (627)(198,354) = (9)(26)(531,847)

518,423,976 = (567)(914,328) = (918)(564,732)

172,349,856 = (3)(672)(85,491) = (6)(7)(84)(92)(531)

= (8)(92)(413)(567).

169,857,324 = (36)(4,718,259) = (3,276)(51,849)

= (4)(91)(567)(823) = (6)(7)(54)(91)(823).

(5) With 10 digits - there are probably more than 25,000 or more 10-digits answers to the above problem. No computer programme was set up. Without computer, ~~inserting~~ appending 0 in some of the 9-digits cases gave correct results for eg:

(30)(41,579,862) = 1,247,395,860

(3)(415,798,620) = 1,247,395,860.

For the 1st case of 567 factors of the 9-digit case putting 0 in an internal position was the basis of getting about 1,000 additional 10-digit solns

for eg: $3(41,579,862) = 124,736,586$ yields .

$3(401,579,862) = 1204,736,586.$

Simf $(6)(27,548,913) = 165,293,478$ yields $(6)(275,489,013)$

$= 1,672,934,078$

& $(9)(25,763,841) = 231,874,569$ yields

$(9)(257,638,401) = 2,318,745,609.$

It remains now for a magnanimous company to permit one of its computers to be programmed to determine all the 10-digit solutions to this problem.

Narcissistic numbers.

Digital invariants

P.D.9's and R.D.9's.

p. 164 Gives table for these from $n=3$ to $n=10$.

order	P.D.9	R.D.9
2	none	none
3	153, 370, 371, 407	55, 136, 160, 919.
4.	1634; 8208; 9474.	1138, 2178.
5	4,150; 4,151; 54,748.	9044, 9045,
	92,727; 93,084.	24,584; 58,618
	(194,979)	
6.	548,834	239,459; 282,595.

7. 1,741,725; 4,210,818; 9,800,817
9,926,315; (14,459,929) 9,057,586.

8. 24,678,050; 24,678,051;
88,593,477.

9. 146,511,208; 472,335,975; 534,494,836
912,985,153.

10. 4,679,307,774

Note (i) - All solutions for the sixth & higher orders are from Harry L. Nelson - "More on PDI's" - Publn DC RL-7614, Univ. of California, Dec. 1, 1963.

(ii) Among the R.D.I's, we note

$136 \to 244 \to 136$
$2178 \to 6514 \to 2178$
$58,618 \to 76,438 \to 58,618$

$919 \to 1459 \to 919$
but 1459 has 4 digits

These are called Amicable digital invariants or ADI's

Qns: (1) What is connection, if any, is there between the order & number of invariants in that order, and are there any P.D.I's which are prime.

In Madachy's table above, 194,979 cannot be considered a proper 5th order P.D.I since it has 6 digits.

$1^5 + 9^5 + 4^5 + 9^5 + 7^5 + 9^5$
$= 1 + 3(59049) + 1024 + 16807 = 194979$

Similarly 14,459,929 is not a proper P.D.I of order 7 since it has 8 digits.

$1^7 + 2(4^7) + 2^7 + 5^7 + 3(9^7) = 1 + \overset{32768}{\cancel{16384}} + 128 + 78125$

$+ 14348907 = 14,459,929.$ ✓

I have made a mistake in powers 4^7, 2^8 & 2^9 & so I am afraid Miss Hegore's results for these cases will be invalid. What a fool I am? I should correct my lists given to her & ask her to try again]

V. R. Numbers

$371 = 3^3 + 7^3 + 1^3$ a P.D.I. 6-digit analogue of this

$165,033 = 16^3 + 50^3 + 33^3$ a P.D.I in the scale of 100

The following is almost all the work of J. A. London

(1) Sum of 2 squares:

$1233 = 12^2 + 33^2$

$588|2,353 = 5,882,353 = 588^2 + 2,353^2$

others are 17650|38125

88|33, 9412|2353; 82350|38125

10|100, 990|09901; 25840|43776

100|010000; 123288|328768; 116788|321168

999900|010000; 876712|328768; 883212|321168

These solns go in pairs. If $|x||.10^n + y|$ is a soln of the 2n-digit case

So is $|10^{2n}-x||10^n+y|$ for eg if $x=12$, $10^n+y=33$ ($y=23, n=1$) then $10^{2n}-x=88$ which gives the other soln $|88||33$. No great difficulty is involved in extending the list. Also the solution 5,882,353 is a prime & there many other such solns

(2) Difference of two squares:

$2661,653 = 2,661,653 = 1653^2 - 266^2$. Other nos are

4 | 8 ,	140 | 400	184 | 848
	190 | 476	530 | 901
34 | 68	216 | 513	234 | 1548
10 | 101	300 | 625	266 | 1653
16 | 128	334 | 668	1660 | 4400
34 | 188	416 | 768	3334 | 6668
		5980 | 9776.

(3) Sum of 2 Δ^r nos: $\Delta_n = \frac{1}{2}(n^2+n)$

$585,190 = \Delta_{585} + \Delta_{190}$

Other examples are

90 | 415 ; 700 | 3676 ; 3030 | 7171 ; 7804 | 9756
120 | 1545 ; 769 | 3846 ; 4774 | 8526 ; 8274 | 9850
150 | 1726 ; 1474 | 5226 ; 6154 | 9231 ; 8455 | 9880
244 | 2196 ; 2829 | 6970 ; 6324 | 9300 ;

(4) Sum of 3 cubes: 22 | 18 | 59 = 221,859 = $22^3 + 18^3 + 59^3$

Other numbers include

4|18|33 ; 34|00|67 ; 44|46|44 ; 166|500|333
16|50|33 ; 34|10|67 ; 48|72|15 ; 333|667|000
22|18|59 ; 40|70|00 ; 98|28|27 ; 333|667|001
33|67|00 ; 40|70|01 ; 98|32|21 ; 334|000|667
33|67|01 ;

(5) Amicable pairs: $3869 = 62^2 + 05^2$; $6205 = 38^2 + 69^2$
$5965 = 77^2 + 06^2$; $7706 = 59^2 + 65^2$.

(6) Sum of factorials

$1 = 1!$

$2 = 2!$

$145 = 1! + 4! + 5!$

$40,585 = 4! + 0! + 5! + 8! + 5!$ (note $0! = 1$)

First 3 nos are well known, but the last example was found in 1964. Later Ron. S. Dougherty, a student at Davidson College, North Carolina made an exhaustive search with a computer, and found that the above factorial nos are the only ones that exist.

(7) Sum of subfactorials viz $!n$ defined as

$$!n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right\}$$

$!0 = 0$, $!1 = 0$, $!2 = 1$, $!3 = 2$, $!4 = 9$, $!5 = 44$, $!6 = 265$,
$!7 = 1854$; $!8 = 14,833$, $!9 = 133,496$.

After his work on factorial nos Dougherty was asked to find number results for subfactorials. There is only one soln to this problem viz

$$148,349 = !1 + !4 + !8 + !3 + !4 + !9.$$

(8) Sums & powers: Nos. equal to the nth power of the sums of their digits, such as

$$81 = (8+1)^2 = 9^2$$

$$4,913 = (4+9+1+3)^3 = 17^3$$

$$34,012,224 = (3+4+0+1+2+2+2+4)^6 = 18^6.$$

Table giving N & P^n for a "few typical examples" given [with certainty N, the P^n are 9^2; 8^3, 17^3, 18^3, 26^3, 27^3; 7^4, 22^4, 25^4, 28^4, 36^4; 28^5, 35^5, 46^5; 18^6, 45^6, 54^6, 64^6; 18^7, 27^7, 31^7, 34^7, 43^7, 53^7, 58^7, 68^7; 46^8, 54^8, 63^8. and finally 207^{20} with 47 digits

$$N = abcd\ldots = (a+b+c+d+\ldots)^n = P^n$$

Special Example of 2^{70} where sum of digits equals the exponent 70.

Gives outline of method of obtaining the search for such nos applies to the search of all $N = P^4$ examples, depending on notion of digital roots — Gives of table of digital roots of d^4 for $d = 1, 2, \ldots 9$ & remainders when digital root is divided by 9.

Methods used by T. Charles Jones of Davidson College to determine all such numbers up to & including P^{101}. All values of n in P^n are represented in the range $n = 2$ to $n = 101$, and there is a total of 432 examples of these numbers in this range. All cannot be given here, but gives one example of P^{101} viz:

1468^{101} = a 320 digit number the sum of digits of which = 1468 equal to P. This is the largest known number of this type.

(9) Miscellaneous: Gives the digital invariants

$43 = 4^2 + 3^3$; $1306 = 1^1 + 3^2 + 0^3 + 6^4$

$63 = 6^2 + 3^3$; $1676 = 1^1 + 6^2 + 7^3 + 6^4$

$135 = 1^1 + 3^2 + 5^3$; $2427 = 2^1 + 4^2 + 2^3 + 7^4$

$175 = 1^1 + 7^2 + 5^3$; $3435 = 3^3 + 4^4 + 3^3 + 5^5$

$518 = 5^1 + 1^2 + 8^3$; (last one key $abcd = a^a + b^b + c^c + d^d$)

$598 = 5^1 + 9^2 + 8^3$, Are there others of the last type?

Multigrades

Ex:

$$1^n + 6^n + 8^n = 2^n + 4^n + 9^n \quad (n = 1, 2)$$

$$1^n + 5^n + 8^n + 12^n = 2^n + 3^n + 10^n + 11^n \quad (n = 1, 2, 3)$$

$$1^n + 5^n + 8^n + 12^n + 18^n + 19^n = 2^n + 3^n + 9^n + 13^n + 16^n + 20^n \quad (n = 1, 2, 3, 4)$$

In short hand

$1, 6, 8 \overset{2}{=} 2, 4, 9$	$S_1 = 15, S_2 = 101$
$1, 5, 8, 12 \overset{3}{=} 2, 3, 10, 11$	$S_1 = 26, S_2 = 324, S_3 = 2366$
$1, 5, 8, 12, 18, 19 \overset{4}{=} 2, 3, 9, 13, 16, 20$	$S_1 = 63, S_2 = 919, S_3 = 15{,}057, S_4 = 260{,}755$

The property of a multigrade is that the same quantity can be added to each term without affecting the relationship. The first example can be changed by adding 7 to each term to form

$$8, 13, 15 \overset{2}{=} 9, 11, 16 \qquad S_1 = 36, S_2 = 458.$$

Forming a multigrade is easy. Start with a simple equality such as $1+4 = 2+3$. Now add 4 to each term getting $5+8 = 6+7$.

A second order multigrade can be obtained by "switching sides" and combining, yielding

$$1^n+4^n+6^n+7^n = 2^n+3^n+5^n+8^n \qquad (n=1,2).$$

Four was added since that increase is the smallest that would result in the second order multigrade with all the terms different.

A third order multigrade can be built up by using the newly formed 2nd-order one & one formed by adding 8 to each of its terms

$$9^n+12^n+14^n+15^n = 10^n+11^n+13^n+16^n \qquad (n=1,2)$$

Switching sides in these two eqns

$$1^n+4^n+6^n+7^n+10^n+11^n+13^n+16^n = 2^n+3^n+5^n+8^n+9^n+12^n+14^n+15^n \qquad (n=1,2,3)$$

Here is a 5th-order multigrade derived by another method (?)

$$1,9,18,38,47,55 \overset{5}{=} 3,5,22,34,51,53. \qquad (S_1 \text{--} S_5 \text{ given}).$$

The following 5th order multigrade is of considerably lower magnitude

$$1,6,7,17,18,23 \overset{5}{=} 2,3,11,13,21,22 \qquad (S_1 \text{---} S_5 \text{ given})$$

The late Royal V. Heath derived a large number of unusual multigrades

$$12,43,65,78 \overset{2}{=} 87,56,34,21 \qquad (S_1 = 198,\ S_2 = 12,302)$$

Note that the digits from 1 to 8 are used and that the two sides are reversed [This is my multigrade palindrome].

Another quite unusual Heath multigrade:

$1,118; 1,881; 8,181; 8,818 \overset{2}{=} 1,181; 1,818; 8,118; 8,881$

$(S_1 = 19,998,\ S_2 = 149,473,970)$.

If this multigrade is reversed, another is formed

$1,888; 8,118; 8,181; 1,811 \overset{2}{=} 8,188; 1,818; 1,881; 8,111$

$(S_1 = 19,998,\ S_2 = 139,674,950)$

Miscellaneous

To find $N^3 = a^3 + b^3 + c^3$ $(N < 1,000)$ in 5 distinct ways (Amer. Math. Monthly, Jan 57) proves. Three solns to the problem which have gone far beyond the proposal are shown in table below. David A. Klarner of Edmonton, Alberta supplied those in the first two cols and Leon Bankoff of Los Angeles, California, furnished that in the last column

$N = 492$	$N = 792$	$N = 870$
a, b, c	a, b, c	a, b, c
24, 204, 480	30, 436, 738	17, 687, 694
↓	↓	↓
(10 solns pairs)	(11 solns)	10 solns

Factorial products.

Factorial products of nos in A.P.

$(0!)(1!) = 1!$ $\quad (1!)(3!)(5!) = 6!$

$(1!)(2!) = 2!$ $\quad (1)(3!)(5!)(7!) = 10!$

$(6!)(7!) = 10!$

If A.P. condn be dropped any no of examples can be found

$$(4!)(23!) = 24!$$

$$(2!)(4!)(47!) = 48!$$

$$(2!)(3!)(4!)(287!) = 288!$$

Printers errors

(1) Dudeney's $2^5 9^2 = 2592$

(2) Donald L. Vanderpol's cases

$$2^5 \cdot \frac{25}{31} = 25\frac{25}{31}$$

$$11^2 \cdot 9\frac{1}{3} = 1{,}129\frac{1}{3}$$

$$21^2 \cdot 4\frac{9}{11} = 2{,}124\frac{9}{11}$$

(3) $13^2 \cdot 7\frac{6}{7} = 1{,}327\,6/7$

$$13^2 \cdot 7{,}857{,}142\frac{6}{7} = 1{,}327{,}857{,}142\frac{6}{7}$$

$$13^2 \cdot 7{,}857{,}142{,}857{,}142\frac{6}{7} = 1{,}327{,}857{,}142{,}857{,}142\frac{6}{7}$$

— — — — —

(4) $3^4 \cdot 425 = 34{,}425$

$$3^4 \cdot 4250 = 344{,}250$$

$$3^4 \cdot 42500 = 3{,}442{,}500$$

— — —

(5) $31^2 \cdot 325 = 312{,}325$

$$31^2 \cdot 3{,}250 = 3{,}123{,}250$$

— —

(6) without exponents & fractions (Madachy)

$$73 \cdot 9 \cdot 42 = 7 \cdot 3{,}942$$

$$73 \cdot 9 \cdot 420 = 7 \cdot 39{,}420$$

— — — —

Automorphic numbers

ie a number ~~which~~ whose square ends with the given number

eg: $5^2 = 25$, $76^2 = 5,776$; $625^2 = 390,625$. Thus 5, 76 and 625 are automorphic numbers.

Only two automorphic numbers have any given number of digits. Except for the final digits, the sum of the corresponding digits in the pair sums to 9.

[Ex. 376 & 625 constitute such a pair with 3 digits
$3+6=9$, $7+2=9$]

J. A. H. Hunter found a pair of 17-digit automorphic numbers early in 1964 which was the record for that time. R. A. Fairbairn of Toronto, Ontario & a friend of Mr. Hunter, was determined to do a bit better. Eventually Mr. Fairbairn went overboard & produced a pair of 100-digit automorphic numbers! Here they are in full

3,953,007,319,108,169,802,938,509,890,062,166,509,
580,863,811,000,557,423,423,230,896,109,004,106,619,977,
392,256,259,918,212,890,625 and

~~6,046,992,680~~, and

6,046,992,680,891,830,197,061,490,109,937,833,490,419,
136,188,999,442,576,576,769,103,890,995,893,380,022,
607,743,740,081,787,109,376

[It is easily verified that sum of corresp. digits = 9 in the two numbers except 6 & 5]

The devoted number enthusiast might check these out by squaring either one of them to see if the product terminates with the original number!

These numbers actually constitute a table of all known automorphic numbers. Dropping digits successively from the left of each of these numbers forms new automorphic ones, though with fewer digits. Excluding those that begin with 0, there are 167 different automorphic numbers known at present.

Two rather large numbers

(1) $9^{9^{9}} = 9^{387,420,489}$

Contains 369,693,100 digits - can never be calculated

The last 2000 digits & the first 1200 digits were calculated recently by Fred Greenleaf a mathematician of the RAND corporation: To satisfy readers curiosity first & last 10 digits are given

4,281,247,731 2,627,177,289

Although the last 9 or 10 digits & the first 1200 digits are in doubt, there is still a total of 3,190 digits (2000 + 1200 − 10) known. Even if the remaining spaces were filled in by a random array the error involved would be infinitesimal, something of the order of 1 part in $10^{1,190}$.

$9^{9^{99}}$ has last ten digits . . . . 1,045,865,289.

[Anyway a colossal lot of very interesting information re-number recreations in this Chapter. It is impossible to excel the comprehensive treatment of the topics dealt with and their accuracy.

Only lacuna is want of explanation as to how the results were derived. Perhaps this would have doubled or trebled the length of the Chapter. At least some inkling might have been given.

This reminds one of ancient mystic magic]

Notes from: Math. Games & Pastimes by A. P. Domoryad.

(must have already made some notes, but I could not trace it immediately)

p. 13: How many zeros at the end of 100!

Let $100! = 2^{\alpha} 3^{\beta} 5^{\gamma} \ldots 97^{\sigma}$, then

$$\alpha = \left[\frac{100}{2}\right] + \left[\frac{100}{4}\right] + \ldots + \left[\frac{100}{128}\right] + \ldots = 50 + 25 + 12 + 6 + 3 + 1 = 97$$

and $\gamma = \left[\frac{100}{5}\right] + \left[\frac{100}{25}\right] + \ldots = 20 + 4 = 24$

$100! = 2^{97} 3^{\beta} 5^{24} \ldots$ Hence 100! ends with $2^{24} \cdot 5^{24}$ ie 24 zeros.

p. 28 Ref. to Lierpinski's smallest solving $x^2 - 991y^2 = 1$, having

$x = 379,516,400,906,811,930,638,014,896,080$

$y = 12,055,735,790,331,359,447,442,538,767.$

p. 30 mentions a cattle herd problem leading to the Pellian eqn

$$w^2 - 410,286,423,278,424\, s^2 = 1.$$

& the no. of cattle would be $N \cong 77 \times 10^{206,543}$.

p. 35 Calls illegal cancellation as amusing cancellation

$$\frac{143185}{170185\cancel{6}60} = \frac{1435}{170560}; \quad \frac{4251935346}{91819355185} = \frac{425345}{9185185} \text{ etc}$$

(which is the wrong one here?)

p. 236 The identity $\dfrac{a^3+b^3}{a^3+(a-b)^3} = \dfrac{a+b}{a+(a-b)}$ can be used for

"inadmissible cancellations" leading to correct results, for eg.

$$\frac{37^3+13^3}{37^3+24^3} = \frac{37+13}{37+24} = \frac{50}{61}$$

identity $\dfrac{\log\left(\frac{m+1}{m}\right)^m}{\log\left(\frac{m+1}{m}\right)^{m+1}} = \dfrac{\left(\frac{m+1}{m}\right)}{\left(\frac{m+1}{m}\right)^{m+1}}$ ⟶ must be $\left(\frac{m+1}{m}\right)^m$.

leads to illegal cancellation of log for eg putting $m=2$

$$\frac{\log\frac{9}{4}}{\log\frac{27}{8}} = \frac{9/4}{27/8}$$

p.236

Identity $\sqrt[n]{a+\frac{a}{a^n-1}} = a\sqrt[n]{\frac{a}{a^n-1}}$ leads to a number of curious eqns

For eg: $\sqrt[3]{2\frac{2}{7}} = 2\sqrt[3]{\frac{2}{7}}$; $\sqrt[4]{5\frac{5}{624}} = 5\sqrt[4]{\frac{5}{625}}$, $\sqrt[5]{2\frac{2}{31}} = 2\sqrt[5]{\frac{2}{31}}$.

(Printer's error)

Cancelling symbol Sin in the identity

$$\sin\alpha + \sin 2\alpha + \cdots + \sin n\alpha = \frac{\sin\frac{(n+1)}{2}\alpha \sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$$

we get $\alpha + 2\alpha + \cdots + n\alpha = \frac{\frac{(n+1)\alpha}{2}\cdot\frac{n\alpha}{2}}{\alpha/2}$ which is correct?

(illegal cancellation)

The identity: $\left[3\left(10^k+10^{k-1}+\cdots+10^2+10+1\right)\right]^2 = n^2\left(10^{2k+1}+10^{2k}+\cdots+10^{k+1}\right)$
$+(6n-n^2)\left(10^k+10^{k-1}+\cdots+10+1\right)+1$

leads to two series of interesting numerical eqns

For $n=1$, $k=1,2,3,4,\ldots$ we have

$34^2 = 1156$, $334^2 = 111556$, $3334^2 = 11115556$;

$33334^2 = 1111155556$, etc.

For $n=2$, $k=1,2,3,4,\ldots$ also we have

$667^2 = 444889$, $6667^2 = 44448889$,

$66667^2 = 4444488889$, etc.

(30|7|79) — M.Sc. — Gave P.D.E's for $n=10, 11, 12$

to Miss Hyde — She got results for $n=7$ and thereby h

my data were wrong – to look this up. She also gave me the list of R.S. Invariants for 5 & 6 – E.C.G.S. came home in the morning along with Rajaraman & Nilloor – Later met E.C.G.S. at C.T.S along with Pashupathy – Talked to T.K.S. Iyengar about publication of results relating to recreational mathematics & he agreed.

Mathematical Bafflers – Compiler & Editor by Angela Dunn

Chapter 3 – Diophantine diversions:

(1) Ramanujan's problem of 1729 = $12^3+1^3 = 10^3+9^3$.

with –ve nos, $(-12)^3+(-1)^3 = (-10)^3+(-9)^3 = -1729$ Stopped

✓ Somebody gave $8^3+6^3 = 9^3+(-1)^3 = 728$ is smaller than 1729.

(2) Product of 4 consecutive +ve integers cannot be a perfect square

Soln $N(N+1)(N+2)(N+3) = (N^2+3N)(N^2+3N+2)$

$= (N^2+3N+1)^2 - 1$ which $\neq \square$.

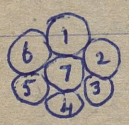
(3) $\dfrac{ABCDE}{FGHIJ} = 9$ – only 6 solns exist.

(4) creating palindromes – Adding a number to its reversal business for 2 digits

✓ Any no. with two different digits, having the sum of its digits less than 10 will produce a palindrome in one operation. We need consider

~~Therefore only~~ two-digit numbers the sum of whose digits is 10 or greater. If the sum of the digits be 10, 11, 12, 13, 14, 15 or 16, the number of fractions required will be 2, 1, 2, 2, 3, 4, 6 respectively. only when sum of digits is 17 ~~or more~~ are more than 10 fractions required. But the only number having 17 as the sum of its digits is 89 or 98. [710, but how many?]

(5) The 7-digit problem



There are 120 seven-digit numbers which can be formed by starting from any number in the above diagram and proceeding to any neighbouring number, using each number once & only once. Of these how many are divisible by 11

~~Second~~ Solution: Since the sum of the seven digits is 28, we seek partitions of 14 into three and four parts, using the numbers 1 to 7 once & once only. There are exactly 10 such numbers subject to the given conditions

1237456 (1,3,4 & 2,7,5 are the partitions of 14 here), 1237654, 1695432, 2345761, 3217654, 4567123, 4567321, 6547123 and 6547321 [only 9 which is the 10th!]

(6) Herbault's problem (Analogue)

Soln: If $N \geq 22$, N^3 will contain at least 5 digits & N^4 at least 6, so that total will be 11 digits. Likewise if $N \leq 17$, N^3 will not contain and N^4 will at most contain 6 digits together. So trying $N = 18, 19, 20, 21$ in turn we find $18^3 = 5832$, $18^4 = 104976$.

[This is not Herbault's problem which is $69^2 = 4761$, $69^3 = 328509$ i.e. N^2 & N^3, Here it is N^3 & N^4.

(*) There is nothing like this in Madachy

(~~Madachy's R. no.~~)

(7) $abc, def = (abc + def)^2$ (*) has the only two solns

$998{,}001 = (998 + 001)^2$ & $494{,}209 = (494 + 209)^2$.

Proof by Dr de Buda from Canada:

Let the number be $1000a + b$, with $c = a + b$, Then $1000a + b = c^2$ gives $a = c(c-1)/999$. Hence between them c & $c-1$ contain the factors of 999 ie 37 & 27, $c - 1 \neq 999$ since a has 3 digits. But $c = 999$ provides one soln viz 998,001. otherwise c & $c-1$ each contain one of the factors. Either $c = 37p$ & $c - 1 = 27q$ or $c = 27q$ & $c - 1 = 37p$. Solving these linear diophantine eqns, $c = 703 + 999k$ or $c = 297 + 999m$. only the former with $k = 0$ is the square root of six digit giving rise to the other soln 494,209. [See Bk. No. 24 (a) for other results]

(8) Rightmost digit of 7^{7^7}

Rightmost digit of 7^N is 1, 7, 9 or 3 as $N = 0, 1, 2$ or $3 \pmod 4$

Since $7^7 \equiv 3^7 \equiv 3 \pmod 4$, the rightmost digit is 3.

(9) Can 19^{19} be represented as the sum of a cube & a 4th power

From multiplication table (mod 13) one sees that

$X^3 \equiv 0, 1, 5, 8$ or 12 and $Y^4 \equiv 0, 1, 3$ or 9. Thus $X^3 + Y^4$ can be congruent to anything except 7 (mod 13). But

$19^{19} \equiv 6^{19} \equiv 7 \pmod{13}$ — So the answer is no.

(10) 0, 3, 6, 8, 9 ~~found by~~ written with only curves, 1, 4, 7, 11, 14, with only st. lines & 2, 5, 10, 12, 13, written with both

Both both curves & st: lines

Make a number out of this:

Q. what is the difference between 6 and 4?

Ans: The first number is formed by curves & the second by st: lines

A profound example of the geometry of numbers!

(11) Find an A.P. of three numbers whose product is a positive prime.

Ans: $-3, -1, 1$.

(12) Find the smallest integer which is such that if the digit on the extreme left is transferred to the extreme right, the new number is three and half times the original number

Soln. Let us be $abc\ldots z$. Then

$$2bc\ldots za = 7abc\ldots z \text{ and}$$

$$b(2\cdot 10^k - 7\cdot 10^{k-1}) + c(2\cdot 10^{k+1} - 7\cdot 10^{k-2}) + \ldots + z(20-7)$$

$$= a(7\cdot 10^k - 2) + b(13\cdot 10^{k-1}) + c(13\cdot 10^{k-2}) + \ldots + z\cdot 13$$

$$= a(7\cdot 10^k - 2)$$

Since the prime 13 divides the left member it must also divide the right: Trying in turn the values 68, 698, 6998, 69998 & 699998, the latter is the first to be divisible by 13. Thus $k = 5$ & our number has 6 digits $abcdef$. Also $130,000b + 13,000c + 1300d + 130e + 13f$

$= 699,998$ or $bcdef = 53,846a$. Hence $a = 1$ & the desired number is ~~[illegible]~~ 153,846

[Cf. with problems in Frohnstein etc]

(13) How many 9 digit numbers are divisible by 11 no digit equal to 0 & no two digits alike

Soln: Sum of all nine digits $= 45$. For divisibility by 11 sum of 1st, 3rd, 5th, 7th & 9th digits differs from a multiple of 11 from the sums of the 2nd, 4th, & 6th & 8th digits. These sums are therefore 28 and 17 or 39 and 6 & the latter may be quickly ruled out [$\because$ sum of 5 digits $= 6$ or sum of 4 digits $= 6$ is not possible]. By trial & error one finds that there are 9 ways in which 4 digits can sum to 17 & ~~two ways~~ 2 ways they can sum to 28. Thus there are $9 + 2 = 11$ ways ~~in which~~ for the even-place digits. These may be permuted in $4! = 24$ ways & for each choice the odd-place digits can be permuted in $5! = 120$ ways. Hence the total number of ways equals

$$11 \times 24 \times 120 = 31,680$$

[Note that this is the number of ABCDEFGHI which are divisible by 11]

(14) Find a 2-digit number which is a factor of the sum of the

cubes of its digits, while the reverse of the number is a factor of the sum of the fourth powers of the digits

Soln: Cases where $A^3 + B^3$ has a factor $10A + B$ are given by

$$2^3 + 4^3 = 3 \times 24$$
$$2^3 + 7^3 = 13 \times 27$$
$$3^3 + 7^3 = 10 \times 37$$
$$4^3 + 8^3 = 12 \times 48$$

of these only $3^4 + 7^4 = 34 \times 73$. Hence $A = 3$, $B = 7$.

✓ [Requires some further investigation]

(15) What are the last three digits of 7^{9999}?

[$7^{9999} = 7^{10,000} / 7 = 7^{4 \times 2500} / 7$; then introducing a suitable variable K, one ends with a two-term number. The first term is equal to $10^3(10^K - 7)$ divided by 7 & the second term, $7^3 = 343$ is the sum to the problem.

Soln not at all clear]

Further notes from Caldwell's Topics in recreational mathematics:

Chaps 4, 12 & 14 devoted to number theory.

(continued from p. 91 of this book)

[This chapter appears too sophisticated]

Chap. 12 - ~~The reason~~ more reasonable

Since Chap. 14 on π is also more elementary

Institute (20/8/79)

Math. Comp. 32, No. 143, July 78

p. 936 - Baker & Davenport's result that if the simultaneous diophantine eqn $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$ have a solution it had the property $x < 4^{10^{487}}$. Further all solns in this region ruled out except for the two known solns $x = 1$ and $x = 11$.

p. 941 - using I.B.M (D.E. Shiffee) finds all previously found factors ~~within the~~ & in addition

$697 \cdot 2^{64} + 1$ divides F_{62}

$7551 \cdot 2^{69} + 1$ " F_{66}

$683 \cdot 2^{73} + 1$ " F_{71}

$1421 \cdot 2^{93} + 1$ " F_{91}.

Amer. Math. Monthly 86, No. 2, Feb. 79

Article on Ramanujan's "lost" note books by George E. Andrews

Watson's remarks at the end

"The study of Ramanujan's work and of the problems to which it gives rise inevitably recalls to mind Lamé's remarks that when reading Hermite's papers on modular functions "on a la chair de poule", I would express my own attitude with more

prolixity by saying that such a formula as

$$\int_0^\infty e^{-3\pi x^2} \frac{\sinh \pi x}{\sinh 3\pi x} = \frac{1}{e^{2\pi/3\sqrt{3}}} \sum_{n=0}^{\infty} \frac{e^{-2n(n+1)\pi}}{(1+e^{-\pi})^2(1+e^{-3\pi})^2 \cdots (1+e^{-2n+1\pi})^2}$$

gives me a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Capelle Medicee & see before me the austere beauty of the four statues representing Day, Night, Evening & Dawn which Michelangelo has set over the tombs of Guiliano de' Medici and Lorenzo de' Medici "

Andrews' remark: As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance

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Mathematical Magic Show by Martin Gardner (1977) – Collections from articles in Sc. Amer.

Intn: Defn of ~~math~~ simple math. terms

Under primes mentions two interesting primes

(i) 1, 234, 567, 891

and (ii) ~~repeat~~ repunit with 23 digits

Chap. 4 – Factorial oddities

p. 52 – Fig. 8 – Gives factors upto from 0 to 20.

mention $7! = 5040$ as Plato's number having 59 divisors (including 1 but not 5040) [Ref to Plato's number in Gardner's book "The incredible Dr. Matrix", Scribners 1976] – Plato was not aware that 7,560 and 9,240 each have 63 proper divisors, the maximum possible for a number of four or fewer digits [Ref. to above book]

p. 53 Stirling's approximation for $n!$

$$n! = n^n e^{-n} \sqrt{2\pi n}$$

(involving e & π)

Formula's absolute error (ie difference between a factorial's true value & the approx^n) increases as n increases, but the "percentage of error" (ie absolute error divided by the true value) steadily decreases.

pp. 53-57 – Tree print-out of digits of a factorial

(i) $12!$ has nine digits ($9 = 1+3+5$ square) & has the tree formation

4
7 9 0
0 1 6 0 0

Robert E. Smith, director of computer applns at the Control Data Institute in Minneapolis programmed a computer to search for large factorials that have a square number of

digits so as print ~~them~~ them in rows with ~~1+3+5~~ 1, 3, 5, 7 digits. This was done up to factorials approx up to 1000! (& a few larger ones) and found exactly 20 factorials less than 1000! as in Fig. 9 (p. 55)

Factorials	No of digits
7	4
12	9
18	16
32	36
59	81
81	121
105	169
~~132~~ 132	225
228	441
265	529
284	576
304	625
367	784
389	841
435	961
483	1,089
508	1,156
697	1,681
726	1,764
944	2,401

Fig 9 (p. 55)

Fig. 10 on p. 55 gives Smith's tree print out of 169 digits of 105! The last row has 25 zeros (25 is the last in the series 1+3+5+7 + . + 25 = 169)

Fig. 11 on p. 57 gives Tree printout of the 1,156 digits of 508!

Smith's remarks re. computer work on p. 56 "It may surprise the reader to know that computers cannot calculate results this large in one piece. That is, the capacities of individual computer cells are soon exceeded, so normal computer arithmetic cannot be used" The trick, he explains, is to use several internal "bins" each of which is allowed to hold t digits of a result. "After

each multiplication is performed, excess digits beyond the t digits, in each bin, are removed and added to the bin on their immediate left. Then the digits in all the bins, which form the factorial result, are printed. A fuller explanation of this program will be found in Smith's book "The bases of FORTRAN, published in 1967 by the Control Data Institute

[I think this is type of problem encounter by Miss Hegde in the ~~proble~~ work I gave her on P.D. I's of orders 10, 11 & 12. Should bring this programming of Smith to her notice]

Forms ~~with~~ other than tree formations

p. 56. – When no. digits = sum of two consecutive squares for eg 35! which has $4^2+5^2 = 41$ digits, it can be printed as a diamond. One simply turns the smaller tree of 4^2 digits upside down and fits it against the base of the 5^2-tree

```
        1
      0 3 3
    3 1 4 7 9
  6 6 3 8 6 1 4
4 9 2 9 . 6 6 6 5
  1 3 3 7 5 2 3
    2 0 0 0 0
      0 0 0
        0
```

[what he means is "~~Each the~~ write the first 25 digits of 35! in rows of 1+3+5+7+9 and then the next 16 digits in rows of 7+5+3+1 below the row of 9]

p. 58 – Fig 12 – Henagon printout of 477! (1,073 digits) using Smith's computer

p. 59 – Fig. 13 – Octagon printout one of Smith's computer's curiosities

If some one had predicted 50 years ago that before the end of the century ended this factorial would be written out in full, digit by digit, most mathematicians would have laughed at preposterous a prophecy

p. 60 – Wilson's theorem

Problems about factorials unsolved! – (i) No one knows if a finite or infinite number of factorials become primes by addition of 1; (ii) How many become squares by the addition of 1. In 1876 Brocard conjectured that only three factorials 4!, 5! & 7! become squares when they are increased by 1. Albert H. Beiler in Recreations in the Theory of numbers says that this has been investigated by computers up to 1,020! without finding any other soln, but Brocard's conjecture remains unproved (iii) Products of consecutive factorials being factorials, only 4 examples known $0! \times 1! = 1!$, $0! \times 1! \times 2! = 2!$, $1! \times 2! = 2!$, $6! \times 7! = 10!$

(iv) No. equal to the sum of the factorials of their digits. There are four solns two trivial viz $1 = !$, $2 = 2!$. Largest example found in 1964 of Leigh Jones of Houston using a computer: $40,585 = 4! + 0! + 5! + 8! + 5!$. The other one given on p. 64 (in answers) is $145 = 1! + 4! + 5!$ – Refers to a proof that 1, 2, 145 & 40,585 are the only such four integers by George D. Poole – Math Magazine, Vol-44, Nov. 1971, pp. 278–79

[There four other also given by Madachy, (p. 109 of this note book) but he refers to an exhaustive search by a student Daugherty using a computer showing (not proving) that there are the only four sets. Also Madachy gives the products of factorials in A.P viz $1! \times 3! \times 5! = 6!$ & $1! \times 3! \times 5! \times 7! = 10!$ (p. 113 of this note book. These are not given in Gardner).]

(v) (p. 631 - oddity involving factorials & the calculus of finite differences - Starting with $1^n 2^n 3^n 4^n 5^n 6^n \ldots$ the nth row of differences is an endless repetition of $n!$

(vi) Calculating no of zeros in the tail of a factorial - The no. of terminal zeros of $n!$ can be found by dividing $n!$ by 5, discarding the remainder, dividing the quotient by 5 & discarding the remainder, and repeating this process until the quotient is less than 5. The sum of all the quotients is the number of zeros.

(should be n) (misprint)

Eg: 2206! as shown in Fig ~~[illegible]~~ 13 (Octagonal printout of Smith) has 549 zeros. The successive quotients, after 2206 has been repeatedly divided by 5 (remainders discarded) are 441, 88, 17 and 3. These sum to 549

Chapter V

p. 77 (in answers) - Langford's problem

Arranging the six digits 1, 1, 2, 2, 3 3 so that there is one digit between the 1's, two digits between the 2's, and three digits between

Ans. 3 1 2 1 3 2. (unique). For 4 pairs unique soln to Langford's problem is 4 1 3 1 2 4 3 2 (can be reversed but not counted a diff. soln). If n the no. of pairs there is a soln only if n is a multiple of 4 or one less than such a multiple - this problem solved by Dudley Langford in (Math. Gaz. Vol. 42, Oct, 1958, p. 228). Subsequent discussion by Priday & Davies in ibid. Vol. 43, Dec. 1959, pp. 250 - 55).

The 26 solns for $n = 7$ are given Math. Gaz (Vol. 55, Feb. 71, p. 73). Numerous computer programmes have confirmed this list & found 150 solns for $n = 8$. Groth & Miller independently ran programs which agreed on 17,792 sequences for $n = 11$ & 108,144 for $n = 12$

For more literature on this problem on p. 78 & its extension p. 70. There are no solns to "Langford's problem" as it now ~~stands~~ called, with five or six pairs. There are 26 ordered solns for $n = 7$ pairs. No one knows how to determine the no. of ordered solns for given n pairs except by exhaustive trial & error methods.

p. 70 - No. 6 - In 1950 when Charles W. Trigg dean emeritus of Los Angeles City College was editing the problems dept of Math. Magazine, he introduced a popular section headed "Quickies". In 1967, McGraw Hill published Trigg's first book Mathematical Quickies, a splendid collection of 270 of the best quickies he encountered or invented in his distinguished career as a problem expert.

p. 79 - To prove (Solomon W. Golomb) that Oct 31 = Dec 25

Answer p. 79: - If Oct is taken as abbreviation for "Octal" & Dec for "decimal", then 31 (in base 8) = 25 (in base 10). This remarkable

coincidence is the basic clue in "A curious case of Income Tax fraud" one of Isaac Asimov's tales about a club called The Black Widowers (See Ellery Queen's Mystery Magazine, Nov. 1976)

Suzanne L. Hanauer using modulo arithmetic - Oct. 31 = 10/31 or 1,031, Dec 25 is ~~25~~ 12/25 or ~~12~~ 1,225 and $1031 \equiv 1,225 \pmod{194}$ [Where does 194 come from?]

Donald E. Scott & Jay Beattie - Let the 5 letters in Oct & Dec. stand for digits as follows:- O=6, C=7, T=5, D=8, E=3 We then decode Oct. 31 = Dec. 25 as

$$675 \times 31 = 837 \times 25 = 20,925$$

Beattie's computer program - Assuming O & D $\neq 0$ & the next three letters any digit including 0. 24,192 such possibilities were all programmed. The program proved (what Scott had conjectured) that the above equation is unique!

Chapter 12 - Perfect, Amicable, Sociable

Perfect no. properties mentioned

(1) They are Δ^r

(2) Except 6, ~~the~~ each is partial sum of the series of cubes $1^3 + 3^3 + 5^3 + \ldots$

(3) Except 6, all have digital roots 1 (i.e. remainder after dividing by 9 is 1.

(4) Except 6, all evenly divisible by 4

(5) " all equal to 4 (mod 12)

(6) Sum of the reciprocals of all the divisors (including N) = 2

If $a, b, c, \ldots$ be divisors, $\frac{N}{a}, \frac{N}{b}, \frac{N}{c}$ — are also proper divisors

$$\text{i.e. } \frac{N}{a} + \frac{N}{b} + \frac{N}{c} + \ldots = 2N \quad (N/N = 1).$$

$$\text{i.e. } \frac{1}{a} + \frac{1}{b} + \ldots = 2.$$

Converse also true.

Unanswered questions

(1) Is there an odd perfect no?

(2) Is there an infinity of even perfect nos?

Re(1) Tuckerman showed in 1967 that if an odd p.n. exists, it must be $> 10^{36}$

Re(2) the question hinges on whether there is an ∞ of Mersenne primes $M_n = M_{127}$ too

The largest M_n found without using computers (by Lucas)

p. 165 – Eq. 71 – gives the 24 known perfect nos & the nos. of digits in them

These correspond to M_n's for $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107,$

$127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689,$

$9941, 11,213$ and $19,937$.

The 23rd perfect no. $2^{11,212}(2^{11,213}-1)$ with 6751 digits was discovered by a computer at the Univ. of Illinois & a postage stamp issued in honor of this in 1963. In 1971 the 24th one $2^{19,936}(2^{19,937}-1)$ was found by Tuckerman at the I.B.M's research center in Yorktown Heights, N.Y. $M_{19,937}$ with 6002 digits is the largest prime known & gives the largest perfect no. with 12,003 digits

Binary form of nth perfect number – Rule is: Put down n ones followed by $(n-1)$ zeroes.

Rule for finding the final digit of a perfect no – Except 6, if the first exponent of $(n-1)$ is a multiple of 4, P_n ends in 6; otherwise it ends in 28.

Series of terminal digits for the 24 known ones are

6, 8, 6, 8, 6, 6, 8, 8, 6, 6, 8, 8

6, 8, 8, 8, 6, 6, 6, 8, 6, 6, 6, 6

No one has found a rule for predicting the last digit of the next undiscovered perfect number.

p. 167 Almost perfect nos – Nos having that are 1 more or 1 less than the sum of their proper divisors have been called "almost perfect". All powers of 2 are almost perfect of the +1 type. No other +1 almost perfect nos. are known, nor have they been proved impossible. No −1 almost perfect nos have been found, and they too have not yet been proved not to exist.

Amicable nos – Each equal to the sum of the divisors of the other (ie chain of two links)

Earliest eg. 220 & 284

Next example in 1636 were discovered by Fermat: they are 17,296 & 18,416 – Rule for certain types had been given by an 9th century Arabian mathematician. Using this rule Descartes found a third pair: 9,363,584 & 9,437,056 – Euler drew up a list of 64

(of which two were later shown to be unfriendly). Legendre found another pair in 1830. Then in 1867 a sixteen-year-old Italian Paganini startled the mathematical world by announcing that 1,184 and 1,210 were friendly. It was the second lowest pair and had been completely overlooked until then! Although the boy found it by trial & error, the discovery put his name permanently into the history of number theory.

More than 1,000 amicable pairs are now known (Fig. 73 lists all 13 pairs smaller than 100,000). The most complete listing is in a three-part monograph "The History & Discovery of Amicable Numbers" by Elvin J. Lee and Joseph Madachy (J. R. M., Vol 5, Nos 2, 3, and 4, 1972). The largest pair on this list of 1,095 pairs has 25 digits each. Discovered too late to be included are several pairs found in 1972 by H. J. J. te Riele of Amsterdam, the largest of which 152 digits in each number. So far as I know, this is the largest known pair.

All known amicable pairs have the same parity: two even nos or (more rarely) two odd numbers. No one has ~~so far~~ yet proved that a pair of mixed parity is impossible. Every odd amicable pair so far discovered is a multiple of 3. It has been conjectured that this holds for all odd amicables. There is no known formula for generating all amicable pairs, nor is it known whether their number is infinite or finite.

In 1968, I conjectured that all even amicable pairs have a sum that is a multiple of 9. Lee shot the conjecture down by finding

three counterexamples among the known amicables, and later finding eight more and later finding eight more among new pairs of his own discovery (See "On Division by Nine of the sums of even amicable pairs by Elvin J. Lee, Math. Comp Vol. 22, July 1969, pp. 545-48). Because all these counter-instances are numbers with digital roots of 7, I modified my conjecture to: Except for even amicable pairs of numbers equal to 7 (modulo 9), the sums of all pairs of even amicable numbers equal 0 (mod. 9).

?

If the chain leading back to the original number has more than two links, the number is called "Sociable". Until 1969 only two such chains were known. Both were announced in 1918 by P. Poulet a French mathematician. one is an order-5 chain: 12496, 14288, 15472, 14536, 14264. The other is a truly astonishing chain of 28 links that starts with 14,316. This is the largest known chain (note that the no. 28 of links is a perfect number & if the 3 in 14,316 is moved to the front you have π to four decimals)

Suddenly in 1969, Henri Cohen of Paris, discovered 7 sociables of order 4 (Math. Comp. Vol. 22, 1970, pp. 423-29). 6 more were later found by Steve Root, at M.I.T. whose program considered all numbers less than 6,600,000,000. The smallest numbers in these 13 chains are,

(Table given on p. 170)

The biggest unsolved problem in sociable number theory is whether a ~~3-chain~~ 3-link chain, known as a "crowd" exists. No one has come up with any reason why such chains are impossible, but neither has any found an example. Computer sweeps of numbers higher than Root's upper bound have been made without success. Useless they crowds may be, such searches are likely to continue until a triple chain is encountered or until ~~some~~ some clever number theorist proves their impossibility.

Chapter 14 — Knights of the square table

Should be useful in connection with Magic squares

Chap. 15

p. 210 - The integers 1, 3, 8, and 120 form a set with the remarkable property that the product of any two integers is one less than a perfect square. Is there a fifth positive integer (other than 1, 3, 8, 120) which can be added to the set so that the set retains the property?

For this unusually difficult disheartening problem see Ref. Dickson's History, Vol. 2, pages 517 ff. In 1968 Lin showed that if 120 could be replaced, without destroying the set property, the number would have to have more than 1,700,000 digits. Alan Baker of Cambridge (Qrly. J. Math.

2nd Series, Vol. 78, 1969, pp. 129-38) proved finally that there is no replacement for 120 & of course it follows that there can be no fifth member of the set.

It is known that there is an infinity of sets of four positive integers with this property. The general question of existence such five number sets has so far not been answered.

p. 211 – The nos. 2178 and 1089 are the only two smaller than 10,000 with multiples that are reversals of – generalisation to 21999978 × 4 = 87999912 & sim for 1089 – Ref. Alan Sutcliffe Math. Magazine Vol. 39, No. 5, Nov. 1966, pp. 282-87.

By fabrication of them

2178 2178 2178 × 4 = 871287128712

1089 1089 1089 × 9 = 980198019801

Also nos. like 21999 .. 978 might also be fabricated

In Math. Mag. 42, Sept. 69, pp. 208-10, Klosinski & Smolarski show that 4 & 9 are the only numbers which can serve as multipliers for reversing non-palindromic nos.

Hardy in Mathematician's Apology calling this non-serious math as.

Gaiennie's chart

1089	6534	The nine nos. are the first nine multiples of 1089
2178	7623	First digits in the cols go from 1 to 9
3267	8712	& last digits " " 9 to 1
4356	9801	
5445		

If the first five nos. are multiplied by 9, 4, 7/3 and 3/2, the products give the last five numbers in reverse order. What 1, 4, 9 are the first three square numbers, but those other two multipliers $2\frac{1}{3}$ and $1\frac{1}{2}$ seem to come out of left field!

Chap. 16.

p. 234 - The diophantine eqn

$$\frac{n(n^2+2)}{3} = m^2$$

has a finite no. of soln's of which the smallest are $n = 1, 2$ and 24 & no other solns for $n < 5000$. (J. R. M. 4, 1971, 137 & 5, 1972, 72-73)

$$\frac{n(n^2+2)}{3} = 6m^2$$

has an integral soln only when $n = 4$ (Littlewood's proof)

Chap. 17 - Trees.

p. 241: Fig 119 finds topologically distinct trees of two to six points

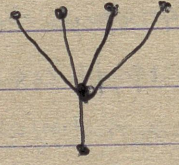
The last one corresp. to 6 pts  resembles my fig. for L-process R. S. constants

Fig. 120, p. 242 gives 12 seven-point trees two of which are hers - last one & the one on the left

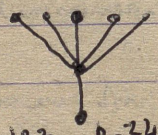


Fig. 122, p. 246. gives a similar figure which I can use for (1,2) & (2,6) trees of KAM-process as in fig on right

Ref. to work by D. E. Knuth on "Fundamental Algorithms"

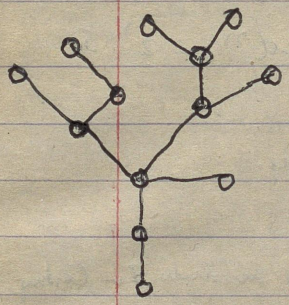


Fig. 122, p. 246

Chapter 18 – Dice. (About probability)

Chapter 19 – Everything (Miscellany – Quant. mech. also) Cosmology, Blackholes, Gravity etc)

Bibliography – pp. 279–283

For Chap. 12 on perfect, amicable & sociable nos

(1) "Amicable nos" E. B. Escott – Scripta Mathematica, 12, 1946 (March), 61–72

(2) "Excursions into maths" Anatole Beck etc – 1962, p. Chap. 2

(3) "Friendly nos" Shoemaker, National Council of Teachers of Maths, 1973

(4) "Almost perfect nos" Gerrard & Temperley – Math. Mag. 46, March 1973 pp. 84–87

(5) "Search procedure for odd p. nos" Math. Comp. 27, Oct 73, pp. 943–49

(6) "Note on almost perfect nos" J. T. Cross, Math. Mag. 47, Sept. 74, 230–31

Quite a beautiful book

Mathematical Puzzles & Diversions by Martin Gardner (From Sc. Amer. - 1961 Edn - 2nd Sc. Amer. Book.

Chap. 3 - p. 32 - Albert H. E. Dudeney.

Chap 4 - Digital roots - p. 43

Digital root = remainder modulo 9 - Casting out nines

All square nos. have digital roots 1, 4, 7 or 9 & the last digit of the number can be 2, 3, 7 or 8. A cube may end with any digit, but its digital root must be 1, 8 or 9. - All perfect nos (ending with 6 or 8) have a digital root 1 (except 6)

Chap 5

p. 52 - Trick based on $1001 = 7 \times 11 \times 13$ / Ref. to Perelman - Tips for fun

Chap. 8, p. 89 - Phi : The Golden Ratio - Fibonacci series

Ref. to literature on p. 267 (5 references)

Chap. 9 - Monkey & the Coconuts

Chap. 18 - Mechanical Puzzles

p. 212 - Chinese tangrams - Very interesting Tip

This book is relatively less interesting than the first book on Math. Magic Show

In Intuition 23/10/79

Science Today - October 1979. p. 59

Some palindromic expressions by A. R. Kaye

(1) $111^2 = 12321$

$121^2 = 14641$

$212^2 = 44944$

(2) $10101^2 = 102030201$

$1002001^2 = 1004006004001$

$2000100002^2 = 4000400090004 0004$

(3) $22^2 = 484$, $26^2 = 676$

(4) $11^2 + 26^2 = 797$

$81^2 + 100^2 = 16561$

$91^2 + 108^2 = 19281$

$146^2 - 1^2 = 21312$

$161^2 - 13^2 = 25752$

$2112^2 - 100^2 = 4458544$

$(13^2 + 31^2) - (12^2 + 21^2) = 5115$

(5) $7^3, 11^3$ are palindromes

$183 - 173 = 919$

$(123 - 153) + (73 - 63) = 848$

$(133 - 123) + (93 - 83) + (63 - 53) = 777$

(6) $\frac{1}{7}\{(233 - 223) + (203 - 193) + (183 - 173) + (153 - 123) + (93 - 83)$

$+ (63 - 53) + (23 - 13)\} = 595$

(7) $\frac{100^3+101^3}{100+101} = 10101$

$\frac{1000^3+1001^3}{1000+1001} = 1001001$

$\frac{2^3+3^3+4^3}{2+3+4} = 11$

$\frac{84^3+85^3+86^3}{84+85+86} = 7227$

(8) $\frac{34^3+35^3+36^3}{34+35+36} - \frac{1^3+2^3+3^3}{1+2+3} = 1221$

$(8^3-7^3)-(7^3-6^3)+(5^3-4^3)-(4^3-3^3) = 66$

Palindromes from primes nos

(9) $17 \times 16 = 272$, $17 \times 19 = 323$ — Adding $17 \times 35 = 595$

$19 \times 9 = 171$, $19 \times 17 = 323$ — $19 \times 26 = 494$

$23 \times 7 = 161$, $23 \times 18 = 414$ — $23 \times 25 = 575$

(10) Palindromes from power series

$A = 1^b + 2^b + \cdots + n^b$

with $b=4$, $n=5$, $A = 979$

$b=5$, $n=13$, $A = 1066206$

This no. divided by 13 gives 77077 & when it is divided by $1+2+3+\cdots+13$ it gives 11011.

(11) $A = K^0 + K^1 + K^2 + \cdots + K^n$

$K=3,\ n=2,\ A=121$

$K=18,\ n=2,\ A=343$

$K=10,\ n=3,\ A=1111$

(12) $2^0 + 2^3 + 2^6 + 2^9 = 585$

(13) $1^4 + 5^4 = 626$

$2^4 + 3^4 + 4^4 = 353$

Ans. $1^4 + 2^4 + 3^4 + 4^4 + 5^4 = 979$

(14) Palamonus "caterpillars"

$$101 = \frac{1^2 + 2^2 + 3^2 + \cdots + 15^2}{1 + 2 + 3 + \cdots + 15}$$

$$121 = \frac{1^2 + 2^2 + 3^2 + \cdots + 18^2}{1 + 2 + 3 + \cdots + 18}$$

$$30103 = \frac{1^2 + 2^2 + 3^2 + \cdots + 45154^2}{1 + 2 + 3 + \cdots + 45154}$$

(15) Interesting relations

$$\frac{3^5 + 4^5}{3+4} = 181, \quad \frac{1^5 + 6^5}{1+6} = 1111, \quad \frac{5^5 + 7^5}{5+7} = 1661$$

$$\frac{4^7 + 5^7}{4+5} = 10501, \quad \frac{2^7 + 7^7}{2+7} = 91319, \quad \frac{1^9 + 2^9}{1+2} = 171$$

(16) $666 = \left\{\frac{1^7 + 2^7 + 3^7}{1+2+3} + \frac{1^7 + 2^7}{1+2} + \frac{1^7}{1}\right\} + \left\{\frac{1^6 + 2^6 + 3^6}{1+2+3} + \frac{1^6 + 2^6}{1+2} + \frac{1^6}{1}\right\}$

$$+ \left\{\cdots + \cdots + \frac{15^5}{1}\right\} + \left\{\cdots + \cdots + \frac{14^4}{1}\right\}$$

(17) Palmorus & Diophantine eqns

$\cancel{28}\ 2202022^2 + 3303033^2 + 6606066^2 = 7707077$

$303^3 + 404^3 + 505^3 = 606^3$

$1001^3 + 6006^3 + 8008^3 = 9009^3$.

(18) $13131 = 1^2 + 2^2 + 3^2 + \cdot + 26^2 + 27^2 + 26^2 + \cdot\cdot + 3^2 + 2^2 + 1^2$.

Selected Topics in Mathematics by Edward L. Spitznagel Jr

p. 28 – "Artificial formula giving primes: The artificiality lies in the fact that each such formula contains some constant that cannot be completely known unless one already knows all the primes. It is as though the primes are recorded on this single constant as music is recorded on magnetic tape. The substitution of the numbers 1, 2, 3, .. into the formula causes the primes to be "played" back one after the other

One such formula, which we are writing down only so you can see what it looks like, is

$$\left[10^{2^n} c\right] - 10^{2^{n-1}} \left[10^{2^{n-1}} c\right]$$

The constant c is the number

$$c = .01020003000000050000000000000000700\ldots$$

The rule for forming c is that the nth prime number is inserted into the decimal expansion of c so that the rightmost digit is ~~the~~ in the 2^n place

following the decimal point. [ ] = integral part

Wiley Interscience Publn of Texts, monographs & tracts

E. A. Melzak – Companion to Concrete mathematics – Mathematical techniques & various applications

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Preface – Conviction that separating calculus from geometry, combinatorics, probability etc, is decidedly unhealthy, and that it is downright sinful to teach the abstract before the concrete. It grew further on noticing the current low ebb of mathematical ingenuity and the corresponding glorification of mathematical machinery. Finally, it was staggering to discover how ignorant some brilliant mathematicians were concerning certain simple things. So, for instance, a world-famous analyst was ignorant of Euler's product for $(1-x)^{-1}$, and a world-famous topologist could not prove that conic sections are sections of a cone. These two men were so obviously brilliant and such powerful mathematicians that they could well afford to be ignorant of lesser stuff. But one suspects that these students may be less brilliant & powerful and these students might pass on their ignorance and the associated contempt to their students. In justice to the two men it ought to be added that they at least were not contemptuous of what they did not know.

Chap 2 – Iterations

p. 63. – To extend factorials to real & complex values of n, we use the Γ-function $\Gamma(x)$ defined by the following equivalent expressions:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \quad \text{(Euler integral } \mathrm{Re}(x) > 0\text{)}$$

$$\frac{1}{\Gamma(x)} = x e^{\gamma x} \prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n} \quad \text{(Weierstrass product, } x \text{ arbitrary complex no.)}$$

$$\Gamma(x) = \lim_{n \to \infty} \frac{n!\, n^x}{x(x+1)\cdots(x+n)} \quad \text{(Euler product, } x \text{ complex} \neq 0, -1, -2, \ldots\text{)}$$

$$\Gamma(x+1) = x\Gamma(x)$$

Property of logarithmic convexity makes $\Gamma(x)$ a unique generalization of factorials

Analogous to $f^{(k)}(x) = \dfrac{n!}{(n-k)!} x^{n-k}$ $(f^{(k)}(x) = x^n)$

we have a kth derivative

$$g^{(a)}(x) = \frac{\Gamma(b+1)}{\Gamma(b-a+1)} x^{b-a}$$

$$I_a(x) = \frac{1}{\Gamma(a+1)} \int_{x_0}^{x} (x-y)^{a-1} f(y)\, dy$$

p. 134 – To compute $2^{1/2}$

Start with any +ve integers p, q and use

define p_n

$$p_{n+1} = p_n + 2q_n, \quad q_{n+1} = p_n + q_n \quad (n = 0, 1, 2, -$$

then $p_n/q_n \to 2^{1/2}$. For it may be verified that

$$p_{n+1}^2 - 2q_{n+1}^2 = -(p_n^2 - 2q_n^2) = \cdots = (-1)^{n+1}(p_0^2 - 2q_0^2)$$

& hence $$\frac{p_{n+1}}{q_{n+1}} - 2^{1/2} = \frac{(-1)^{n+1}(p_0^2 - 2q_0^2)}{q_{n+1}(p_{n+1} + 2^{1/2}q_{n+1})}$$

Therefore the sequence $p_0/q_0, p_1/q_1, \ldots$ converges to $2^{1/2}$ steadily & the successive terms are alternately below & above $2^{1/2}$. Thus

$$\left| \frac{p_n}{q_n} - 2^{1/2} \right| < \frac{1}{2} \left| \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right|$$

which gives a convenient stopping rule when desired accuracy is ~~obtained~~ achieved: successive approximations p_n/q_n are formed step by step by means of two additions

If a real irrational x exists such that

$$x = \lim_{n \to \infty} p_{1n}/p_{2n}$$

we call x computable by additions alone. It turns out that x is computable by additions alone if and only if it is algebraic

But additions alone are not sufficient to compute transcendental numbers. we want now additions & multiplications

Instead of $p_{i,n+1} = \sum_{j=1}^{K} c_{ij} p_{jk}$ $(i = 1, 2, -k)$, $n = 0, 1, 2, -$ ——(1)

where c_{ij} are finite integers

If instead of the above we have

$$p_{i,n+1} = P_i(p_{1n}, \cdots p_{kn}) \quad [i = 1, 2, -K, \; n = 0, 1, \cdots \quad -(2)$$

where the P_i are K-polynomials each in K variables & with integer coefficients. If (2) holds we call the real irrational number x A M-Computable (computable by additions & multiplications). Since π is irrational $K \geq 2$ for we need one sequence for numerators & one for denominators. It turns out that a real irrational x is algebraic if & only if $K = 2$ ie minimum number of sequences required is 2.

The reader may wish to verify that the transcendental numbers

$$\pi, e, \log 2, \sum_{n=0}^{\infty} 2^{-2^n}$$

are all of them three-sequence numbers. An example of a number which requires minimally more than 3 sequences appears to be known. It may be conjectured that if $p_1, p_2, -$ are the primes $2, 3, 5, -$ then the number

$$\sum_{n=0}^{\infty} \left(\sum_{i=1}^{K} p_i^{p_i^n} \right)^{-1}$$

requires exactly $K+2$ sequences

[This sequence business is not at all clear]

p. 165 computing π

The first exact formula for π, which also happens to be first infinite product ever used, was given by Viète (1540–1603)

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\,\frac{\sqrt{2+\sqrt{2}}}{2}\,\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots$$

$$\pi/4 = 1 - 1/3 + 1/5 - - \quad \text{(Leibniz)}$$

$$\pi/2 = \prod_{n=1}^{\infty} \frac{(2n)(2n)}{(2n-1)(2n+1)} \quad \text{(Wallis)}$$

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cdots}}} \quad \text{(Brouncker)}$$

$$\arctan\frac{m}{n} = \frac{mn}{m^2+n^2}\left[1 + \frac{2}{3}\cdot\frac{m^2}{m^2+n^2} + \frac{2\cdot 4}{3\cdot 5}\left(\frac{m^2}{m^2+n^2}\right)^2 + \cdots\right]$$

(~~Alice~~ Lewis Carroll Alice in Wonderland)

~~$\frac{\pi}{4}$ = using~~

Using $\pi/4 = 5\arctan 1/7 + 2\arctan 3/79$ & noting that $2(1^2+7^2) = 10^2$, $12(3^2+79^2) = 10^5$ we have

$$\frac{\pi}{4} = \frac{7}{10}\left[1 + \frac{2}{3}\left(\frac{2}{100}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{2}{100}\right)^2 + \cdots\right]$$

$$+ \frac{7584}{100000}\left[1 + \frac{2}{3}\left(\frac{144}{100000}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{144}{100000}\right)^2 + \cdots\right]$$

No hyperexponentially fast procedure appears to be known for computing π. as in case of e.

Two formulas allowing computation of $\pi \triangleq e^{-\pi}$ by means of very fast converging series using theory of elliptic fns
[Ref. to Whittaker & Watson]

Inverting $x = \int_0^y \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}$ $\quad 0 \le k \le 1$

we get $y = Sn(x,k)$ which is the Jacobian elliptic function Sn with modulus k. The complementary modulus k_1 is given by

$$k^2 + k_1^2 = 1, \quad 0 \le k_1 \le 1.$$

Let $\underline{K} \triangleq K_1$ be the complete elliptic integrals

$$K = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1-k^2\sin^2\phi)}}$$

$$K_1 = \int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k_1^2u^2)}} = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1-k_1^2\sin^2\phi)}}$$

Then it turns out $sn(x,k)$ is doubly periodic

$$sn(x + 4\underline{K}, k) = sn(x + 2i\underline{K}_1, k) = sn(x,k)$$

From the theory of elliptic theta fns [Whittaker] we get the following relation: if

$$k = \frac{1}{2}\,\frac{1-\sqrt{k_1}}{\sqrt{1+k_1}} \quad \text{and} \quad q = e^{-\pi \underline{K}/\underline{K}_1}$$

then

$$\left(1 + 2q^4 + 2q^{16} + 2q^{36} + \cdots\right) k = q + q^9 + q^{25} + q^{49} + \cdots \qquad (4)$$

Substituting a Maclaurin series

$$q = \sum_{n=0}^{\infty} a_n k^n \text{ into (4)}$$

and comparing coeffs, we get

$$q = b + 2b^5 + 15b^9 + 150b^{13} + 1707b^{17} + \cdots \quad (5)$$

Next we form the series

$$q/b = 1 + 2b^4 + 15b^8 + 150b^{12} + 1707b^{16} + \cdots$$
$$= 1 + X, \text{ say}$$

Taking logarithms & using power series for $\log(1+x)$, we get

$$\log q = \log b + 2b^4 + 13b^8 + \frac{368}{3}b^{12} + \frac{2701}{2}b^{16} + \cdots \quad (6)$$

Suppose now $R = R_1 = 1/\sqrt{2}$ so that $K = K_1$

and $q = e^{-\pi}$, and

$$b = \frac{1}{2}\frac{\sqrt[4]{2} - 1}{\sqrt[4]{2} + 1} \quad (7)$$

Then (5) & (6) give the following series for $e^{-\pi}$ and for π

$$e^{-\pi} = b + 2b^5 + 15b^9 + 150b^{13} + 1707b^{17} + \cdots \quad (8)$$

$$\pi = \log\frac{1}{b} - 2b^4 - 13b^8 - \frac{368}{3}b^{12} - \frac{2701}{2}b^{16} \quad (9)$$

Since the value of b given by (7) is a small number

$$b = 0.043213616862944896021937\ldots$$

our series have terms which fall off very rapidly. The first term alone $\log 1/b$ in (9) gives π correctly to better than five decimal places. The first four terms in (9) give

$$\pi = 3.14159265358979323846265.$$

Correct to 23 decimal places.

By (8) $e^{-\pi}$ is approximately equal to $\underline{b}$, to better than six digits accuracy. In view of (7) this may also be expressed by saying that $e^{-\pi}$ very nearly satisfies the eqn

$$16x^4 - 96x^3 + 24x^2 - 24x + 1 = 0$$

p. 249 PRF (polynomially recursive function)

APRF (asymptotically polynomially recursive function)

AM computable (p. 151)

Which transcendental fns are APRF's & which are not. We conjecture for instance, that the function $\log n$ is not an APRF & we note that the truth of this conjecture implies not only that the Euler constant γ is transcendental, but further that it is algebraically independent of $1, \pi, e, \log 2, \zeta(3)$, and $J_0(1)$ simultaneously. This implication is not hard to show since

(a) $\gamma = \lim_{n \to \infty} \left[\sum_{j=1}^{n} 1/j - \log n \right]$

(b) $\sum_{j=1}^{n} (1/j)$ is an APRF, in fact a quotient of two PRF's

(c) the transcendental constants above, as well as $\zeta(3)$ are APRF's being AM-numbers

(d) Proof that $k=2$ for irrational alg. no can be extended via: if $\underline{x}$ & $\underline{y}$ are real irrational nos of orders p & q, where $2 \leq p < q-2$ then x & y are algebraically independent

p. 92 - Euler's product & Cantor's theorem

From $\frac{1-y^2}{1-y} = 1+y$

replacing y successively by $x, x^2, x^{2^2}, \ldots x^{2^{n-1}}$, & multiplying & cancelling we get

$$\frac{1-x^{2^n}}{1-x} = \prod_{\nu=0}^{n-1} (1+x^{2^\nu})$$

If $|x|<1$ we may pass to limit $n\to\infty$ & we obtain the Euler product

$$\frac{1}{1-x} = \prod_{n=0}^{\infty} (1+x^{2^n}) \qquad (1)$$

From (1) we may deduce Cantor's theorem: Every real number $A>1$ may be uniquely represented by as the product

$$A = \prod_{n=1}^{\infty} \left(1+\frac{1}{a_n}\right)$$

where $a_1, a_2, \ldots$ are +ve integers satisfying the conn $a_{n+1} > a_n^2$. Moreover A is rational if & only if n_0 exists such that $a_{n+1} = a_n^2$ for $n \geq n_0$

From Cantor's theorem it follows that for any integer $a \geq 1$ the value of $\prod_{n=0}^{\infty} (1-a^{-2^n})$ is irrational such

reciprocal $\prod_{n=0}^{\infty}\left(1+\frac{1}{a^{2^n}-1}\right)$

is irrational by Cantor's theorem.

Two related results also due to Cantor

(a) If $0<x\le 1$ there exist unique integers $n_1, n_2, n_3, \ldots$ such that $1<n_1\le n_2\le \cdots$, and

$$x=\frac{1}{n_1}+\frac{1}{n_1 n_2}+\frac{1}{n_1 n_2 n_3}+\cdots$$

Moreover x is rational if and only if all n_k's are eventually equal.

(b) Let $a_1, a_2, a_3, \ldots$ be a sequence of $+^{ve}$ integers such that for every n, $n \mid a_1, a_2, \ldots a_k$ for a suitable k; let $c_0, c_1, c_2, \ldots$ be non-negative integers satisfying $0\le c_n < a_n$ and $c_n \neq 0$ infinitely often; let

$$x=c_0+\frac{c_1}{a_1}+\frac{c_2}{a_1 a_2}+\frac{c_3}{a_1 a_2 a_3}+\cdots$$

be a convergent series; then x is rational if & only if $c_n = a_n - 1$ for all sufficiently large n

By taking $n_i = i+1$ in (a) we see that e is irrational using (b) we can prove the irrationality of such numbers as

$$\sum_{n=0}^{\infty}\frac{[n!]}{n!}, \quad I_0(1)=1+\frac{1}{2^2}+\frac{1}{2^2 4^2}+\frac{1}{2^2 4^2 6^2}+\cdots$$

If $a_0, a_1, a_2, \ldots$ are non-negative integers such that

$$f(x)=\sum_{n=0}^{\infty} a_n x^n$$

is a non-polynomial rational function, then (b) shows that

$$a_0 + \frac{a_1}{1} + \frac{a_2}{1\cdot 2^2} + \frac{a_3}{1\cdot 2^2\cdot 3^3} + \cdots$$

is irrational.

p. 107 – ~~to~~ series for log 2

$$\log 2 = \sum_{k=1}^{\infty} \frac{1}{2^k k}$$

shows spectacularly faster convergence than

$$\log 2 = 1 - (1/2) + (1/3) - \cdots$$

p. 85 – It is still unknown whether $\zeta(3)$ is irrational or not.

p. 217

$$e^{e^t - 1} = 1 + \sum_{n=1}^{\infty} B_n t^n / n!$$

B_n are known as ~~Bell nu~~ Bell numbers

Above formula written symbolically is

$$e^{Bt} = e^{e^t - 1}.$$

p. 119

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{b_k}{k!} x^k$$

coeffs of b_k ~~are~~ is called the kth Bernoulli number, although sometimes ~~sometim~~ $B_k = (-1)^{k-1} b_{2k}$ $(k = 1, 2.$

p. 164. $\pi = \sqrt{10}$ (approximation ascribed to Brahmagupta a 15th century Hindu mathematician)

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p. 86 – $\gamma = 0.577215665\ldots$ Euler's constant

p. 198 $\int_0^\infty e^{-x^2} \log^2 x \, dx = \frac{\pi^2}{6} - \gamma^2$.

p. [illegible] (140-21) – Remarks about Napoleon Bonaparte & mathematicians

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p. 263 – For transcendency measures refers to LeVeque's Vol 2.

Quite an imposing collection miscellaneous concrete problems. – The author must have put a colossal effort in collecting the material

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p. 197. $\int_0^\infty e^{-x} \log x \, dx = -\gamma$

p. 198. $\int_0^\infty e^{-x} \log^2 x \, dx = \frac{\pi^2}{6} - \gamma^2$.

p. 208 – Elementary fns: trig. functions, hyperbolic & their inverses are obtained out of $\log x$, e^x & algebraic fns. – standard elementary fns of analysis

p. 209. elliptic integral

$$I(x) = \int \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \qquad k \neq 0, 1 \text{ is not elementary}$$

$\int e^{-x^2} dx$, $\int \frac{e^x}{x} dx$ are not elementary

$$\int x^p (a + bx^r)^q dx$$

is elementary if & only if at least one of the following three numbers is an integer

$(p+1)/r$, q, $q + (p+1)/r$

p. 211. An alg. diff. eqn $P(x, y, y', \ldots y^{(n)}) = 0$

where P is a polynomial with constant coeffts.

Holder's result that the Γ-fn does not satisfy any alg. diff eqn

$\Gamma(x)$ is an elementary fn

p. 212: The Riemann-zeta fn is not elementary & does not satisfy an alg. diff. eqn

The Bessel diff eqn

$$x^2 y'' + xy' + (x^2 - n^2) y = 0$$

has elementary solns if & only if $2n$ is an odd integer

p. 218. $\tan(p/2q)$ $(p, q$ integers) is irrational

$e^{p/q}$ is irrational

$\frac{\tan q}{q}$ is irrational if q^2 is rational & $\neq 0$. Suppose now that π is rational, then so is $\pi^2/16$ & therefore the foregoing number

$(\tan \pi/4) / (\pi/4)$ is irrational; but this number is $4/\pi$, hence π is ~~rational~~ irrational – a contradiction

p. 219 $J_0(1) = 1 - \frac{1}{2^2} + \frac{1}{2^2 \cdot 4^2} - \frac{1}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

and $\sum_{n=0}^{\infty} b_n/n!$ (with $b_n = \pm 1$)

are irrational

p. 220 – one sentence proof of irrationality of $\sqrt{2}$.

$\sqrt{2}$ is irrational since the eqn $p^2 = 2q^2$ is impossible as can be seen by examining the last digits of p & q when these two coprime positive integers are written in the ternary scale

p. 91 – Euler identity

$$\frac{\sin x}{x} = \prod_{j=1}^{\infty} \cos \frac{x}{2^j} \qquad \text{—(1)}$$

Analogous extension

$$\frac{\sin x}{x} = \prod_{j=1}^{\infty} \frac{4\cos^2 \frac{x}{3^j} - 1}{3}$$

Starting with (1), putting $x = \pi/2$ & evaluating the ∞-product we have Vieta's formula (see p. 152)

5/11/79 At Inst.

(1) The Thue-Siegel-Roth theorem.

Let α be an algebraic number & let $\delta > 0$, then the inequality

$$0 < \left|\alpha - (x/y)\right| < y^{-2-\delta}$$

has only a finite number of solutions in rational integers x & y $(y \geq 1)$

(2) Topics in number theory edited by P. Turan

Lots of interesting advanced papers.

p. 417 – Some topics in transcendental number theory by M. Waldschmidt

The only understandable thing is Liouville's sufficient cond^{n} for a number to be transcendental viz

Let Θ be a number, assume that there exists a sequence (p_n/q_n) of distinct rational numbers such that

$$0 < \left|\Theta - \frac{p_n}{q_n}\right| < 1 \quad \text{for all } n$$

and $$\lim_{n\to\infty} \frac{\log\left|\Theta - p_n/q_n\right|}{\log|q_n|} \to -\infty$$

then Θ is a transcendental number

J. R. Math – Vol 11

No. 1

p. 9 Some interesting reversible primes by L. E. Carl

The prime 1193 is reversible to 3911. For either form it can be converted into a 5-digit reversible prime by prefixing 9 or adding 3 to make it read 91193, 93911, 11933, or 39113. Prefixing 3 gives the prime 31193 & adding 9 gives another prime 39119, all of the reversible

There are 24 other 4-digit reversible primes with similar properties but the simple digits vary & in most cases there are only two possible 5-digit reversible primes — Given a full list

In addition to the foregoing there are 98 other 4-digit reversible primes that can be converted into 5-digit reversibles in only one way, by prefixing or adding one of the digits 1, 3, 7, 9

p. 17 — Isolated primes by Richard L. Francis — Quite a good article — To be read later.

prime is isolated provided it is not an element of a set of twin primes

Table I — Isolated primes less than 500

23	113	211	307	401
37	127	223	317	409
47	131	233	331	419
53	157	251	337	443
67	163	257	353	449
79	167	263	359	457
83	173	277	367	467
89		293	373	479
97			379	487
			383	491
			389	499
			397	

Listed prime number sets are not included entirely in the sets of isolated primes

Consider M (Mersenne primes $2^n - 1$

F (set of Fermat primes $2^{2^n}+1$

Pa (set of palindromic primes)

Pr (set of prime primes)

A (set of absolute primes viz those which are prime for all

11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

permutations of digits in their decimal repn

The Mersenne prime 127 belongs to S, but 31 does not, that the F-prime 257 belongs to S but 17 does not, the A-prime 79 is in S but 17 is not, the P_a prime 373 is in S but 101 is not, and that the P_r-prime 67 belongs to S but 31 does not

— order of isolation

defn = 53, All the integers in the interval $53-4 \le x \le 53+4$ are composite $\Leftrightarrow 2k = 4$, or failing that 53 is an isolated prime of order 2.

Smaj 211 is a prime of order 5.

other results

p. 36 — Reversible primes

n digits	No of primes (P_n)	No of palindromic primes	No of reversible prime pairs	Total no of reversible primes
1	4	4	0	4
2	21	1	4	9
3	143	15	14	43
4	1061	0	102	204
5	8363	93	703	1499
6	68906	0	4769	9538
7	586081	668	35237	71142
8	5096876	0	287789	535578
9	45086079	?	?	?

p.36 – Power chains by Harry L. Nelson.

For eg: if $n = 7$ or $n = 8$ we have chains (adjacent nos adding to a power)
6 – 2 – 7 – 1 – 3 – 5 – 4 and 8 – 1 – 7 – 2 – 6 – 3 – 5 – 4.

(a) Find two other power chains

(b) Prove or disprove: There exist only four power chains

Problem 712 by Problem Editor

The largest known A.P. of primes has 16 terms. Without knowing the C.D, is it possible to infer what some of its factors might be?

p.42 – An article on multigrades

Author will be happy to supply further details to any reader of the J.R.M – Address:–

Donald Cross, St. Luke's College, Exeter, England

p.46 – Eqns whose roots are close to π

$x^3 - x^2 - 8x + 4 = 0$, zero of which is 3.141336

Best eqn of 2nd degree is $2x^2 - 5x - 4 = 0$, root being 3.137459.

$11x^5 - 17x^4 - 42x^3 - 37x^2 - 27x + 42 = 0$

has a root 3.14159265 35897, which differs from the correct value of π by approximately 10^{-13}. (due to Harry L. Nelson, Livermore CA. Editor)

Editor's note – other interesting eqns

$19x^2 - 39x - 65 = 0 \longrightarrow 3.14158805$

$99532x = 312689 \longrightarrow 3.14149265361$

$x^4 = 81 + 361/22 \longrightarrow 3.1415926526.$

[Can we have one using 0 to 9 or 1 to 9 only once?]

$56 + 16 + 9 = 81$

Science Today – November 1979 p. 57

(1) Kaprekar again on his Harshad numbers.

(2) Some hidden properties of numbers by S. Srinivasan of T.I.F.R (Ramachandra's student?)

Take ~~98~~ arbitrarily 98787977 8987 (even no of digits

$98 + 78 + 79 + 77 + 89 + 87 = 508$

$5 + 08 = 13$

Starting with the reverse of original no

$78 + 98 + 77 + 97 + 87 + 89 = 526$

$5 + 26 = 31$

Note that 31 is reverse of 13.

Take a no of odd no. of digits say 8787977 8987

$8 + 78 + 79 + 77 + 89 + 87 = 418$

$\cancel{1 + 35} = 4 + 18 = 22$

Reverse: $7 + 89 + 87 + 79 + 78 + 78 = 418$

$4 + 18 = 22$

Take some no. say 237

$237^2 = 56169$, $5 + 61 + 69 = 135$

$1 + 35 = 36$

$732^2 = 535824$, $53 + 58 + 24 = 135$, $1 + 35 = 36$

Again $8134 \times 27 = 219618$, $21 + 96 + 18 = 135$, $1 + 35 = 36$

$4318 \times 72 = 310896$, $31 + 08 + 96 = 135$, $1 + 35 = 36$

[Again mystic like Kaprekar, that too a man of the T.I.F.R. School of Mathematics.]

$$n^2 \cdot 10^{2k+1} + n^2 \cdot 10^{2k} + \cdots + n^2 \cdot 10^{k+1} + (6n-n^2)10^k + (6n-n^2)10^{k-1} + \cdots + (6n-n^2)10 + (6n-n^2+1)$$

$n^2 = 6n - n^2 + 1.$

$\cancel{2n^2 - 6n - 1 = 0}$

$\cancel{n = \frac{6 \pm \sqrt{36+8}}{4}}$

123
321
444

135
531
666

1233
3321
4554

89
98
187
781
968
869
1837
7381
9218
8129
17347
74371
91718
81719
173437
734371
907808
808709
1716517

1716517
7156171
8872688
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17735476
67453771
~~85188677~~
~~74688158~~
~~159876805~~
~~508678951~~
~~668555756~~
~~657555866~~
1326111622

85189247
74298158
159487405
504784951
664272356
653272466
~~1317544822~~
1327544822
2284457231
5612002053
3502002165
9114004218
8124004119
18238008337
73380083281
91618091618
81619081619
173237174237

173237174237
732471732371
905708906608
806609807509
1712318714117
7114178132171
8826496846288
8826486946288
17652983792576
67529738925671
85182722718247
74281722728158
159464445446405
504644544464951

56
56
65
121

74
47
121

92
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121

Amoxylin
Idozol

What are these?

Actifide for all,

UNIVERSITY OF MYSORE

CENTRAL COLLEGE
BANGALORE

Date.....195 .

103 340 640

10, 4, 4, 3, 13, 16561.

103 340 64.
2583516
645879
215293
16561

126
136
148
281.

165
141
~~366~~ 216
328138
~~387~~ ~~281~~

8 (127)

1 | 16561 (129
22 | 65
44
249 | 2161
2241

53) 16561 (31
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59) 16561 (28
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472

61) 16561 (27
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436
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91

67) 16561 (24
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481

71) 16561 (23
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73) 16561 (22
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89) 16561 (18
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107) 16561 (15
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127) 16561 (13
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terrible

10, 4, 4, 3, 13, 16561.

1, 2, 3, 4, 5, 6, 8, 10, 12, 13, 15, 16, 20, 24, 26, 30, 32, 39, 40
48, 52, 60, 65, 80, 96, 78, 104, 120,