Syllabus

(1) Physical basis – Old quantum theory; uncertainty and complementarity; wave formalism.

(2) Schrödinger wave equation – Development and interpretation; General solution when the potential energy is independent of time.

(3) Eigenfunctions and Eigen-values – General properties of momentum and energy eigenfunctions, and eigen values.

(4) Discrete eigenvalues – Linear Harmonic Oscillator; spherically symmetric potential; the hydrogen atom.

~~(5) Continuous eigenvalues – one dimensional square potential.~~ omitted

(5) Matrix formulation – Matrices in quantum mechanics; equations of motion in matrix form; angular momentum.

(6) Identical particles and spin – Symmetric and antisymmetric wave functions; spin angular momentum.

(7) Relativistic quantum mechanics – Schrodinger's relativistic equation, Dirac's equation for a free particle, and for a central field.

(8) Quantisation of wave fields – Classical and quantum field equations; Quantisation of the Dirac equation

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Text Book – Schiff's Quantum Mechanics

Reference – Pauli's Article in the Encyc. Math. Wiss.

Lecture Notes on Quantum Mechanics

delivered

during 1963-64

Also during 1964-65.

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† Ex: $\ddot{q} = -q$ has two integrals $q^2 + \dot{q}^2 = a_1$

$$\left.\begin{aligned} q^2 + \dot{q}^2 &= a_1 \\ \tan^{-1}\left(q/\dot{q}\right) - t &= a_2 \end{aligned}\right\}$$

* L may not involve t but still be non-natural, for eg T may a function of $\dot{q}_1^2$, $\dot{q}_1\dot{q}_2$, $\dot{q}_2^2$ and $\dot{q}_1$, $\dot{q}_2$. This happens for eg. if

x_i is a fn of q_1, q_2 and contains t linearly with constant coefficients: Thus $x_i = \frac{1}{2}q_1^2 + \frac{2}{2}q_2^2 + pct$

gives $\dot{x}_i = q_1\dot{q}_1 + q_2\dot{q}_2 + c$

& $\dot{x}_i^2$ contains $2cq_1\dot{q}_1$ & $2cq_2\dot{q}_2$

* Ex: $Q = (2q)^{\frac{1}{2}} K^{-\frac{1}{2}} \cos p$; $P = (2q)^{\frac{1}{2}} K^{\frac{1}{2}} \sin p$.

$$H = \frac{1}{2}(p^2 + K^2 q^2) \qquad K = KQ$$

~~(*) Ex (i) $(f,(\phi,\psi)) + (\phi((f,\phi),\psi) + ((\phi,\psi),f) + ((\psi,f),\phi) = 0$~~

~~(ii) $(F,\phi) = \left(\frac{\partial F}{\partial f_r}\frac{\partial \phi}{\partial f_s} - \frac{\partial F}{\partial f_s}\cdot\frac{\partial \phi}{\partial f_r}\right)(f_r, f_s)$ $(r, s = 1, \ldots K)$~~

8/7/63 – 2-3 P.M

1st Lecture – Introductory – Make some remarks about classical mechanics, relativity and quantum mechanics – Role of mathematics in quantum mechanics – Suggested Schiff & Pauli – mentioned introductory lectures on classical mechanics, electrodynamics, and relativity (special)

2nd Lecture – 9/7/63 – Starting classical mechanics – Dynamical systems – generalized coordinates (Examples, simple point, S, Eulerian angles) – holonomic and non-holonomic systems (Ex. of latter of a sphere on a rough plane) – generalized velocities – Lagrange's equations of motion $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r$ (with T & Q_r defined) for holonomic systems → conservative systems $Q_r = -\partial V/\partial q_r$ & form of L-eqns

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0 \qquad (L = T - V)$$

– [meaning of an integral of the system is $f(q_r, \dot{q}_r, t) = \text{const}$ is an integral if $\frac{df}{dt} = 0$ in virtue of diff. eqns]† for conservative holonomic systems $\sum \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L = h$, is an integral, integral of energy. For natural systems, this reduces to $T + V = h$. (not L not involving t)*

– Hamilton's Principle – Principle of Least Action.

3rd Lecture – 10/7/63 – Summary of essential points of last lecture – Remaining portions of last lecture up to ~~Hamilton's Principle~~ up to proving integral of energy (without proof)

4th Lecture – 11/7/63 – Proof of integral of energy – natural systems – non-natural systems involving t and not involving t – integral of energy for natural systems as $T + V = h$ – Hamilton's Principle – Principle of Least Action (both for conservative holonomic systems† and in the latter case where the integral of energy exists); Principle of least action for natural systems – Advanced theory in terms of Hamiltonians for conservative holonomic systems – generalized momenta – Hamiltonian H – Hamiltonian eqns of motion – If L does not contain t, nor does H & integral of energy is $H = h$ & for natural systems $H = T + V = h$ – Example of simple pendulum in H-form.

5th Lecture – 16/7/63 – Portions left over from previous lecture – Contact transformations; Defn & possession of the group property – ~~Example~~ $Q = (2q)^{1/2} e^{k} \cos p$; $P = (2q)^{\frac{1}{2}} e^{-k} \sin p$; ~~(1) (2)~~ –

6th Lecture – 18/7/63 – Portions left over from previous lecture – Bilinear covariant of a differential form and expression for a C.T. in terms of bilinear covariant* – Conds for C.T. in terms of L-B's – P.B's and conds for a C.T. in terms of P.B's.

7th Lecture – 23/7/63 – Remaining parts of previous lecture – Infinitesimal C.T's & resulting view of dynamics – ~~Conservation of form of H-eqns under a C.T, Ex. $q = (2q)^{1/2} k^{-\frac{1}{2}} \cos P$; $p = (2q)^{1/2} k^{\frac{1}{2}} \sin P$~~ Hamiltonian eqns as the first Pfaffian system of eqns of diff. form $\sum p_r dq_r - H dt$ – Conservation of Hamiltonian form under a C.T, Ex. p. 307 Whittaker – Integration problem as a transformation into a diff form with last term a ~~perfect~~ perfect diff.

Ex. 1, W. p. 301] Denote ~~Call~~ $Q_1 \cdots Q_n, P_1 \cdots P_n$ by $u_1 \cdots u_{2n}$

$$(\phi, \psi) = \left(\frac{\partial \phi}{\partial u_i}\cdot\frac{\partial \phi}{\partial u_j} - \frac{\partial \phi}{\partial u_j}\cdot\frac{\partial \phi}{\partial u_i}\right)(u_i, u_j), \text{ using the previous example (Ex. 2, p. 300 W).}$$

$= 0$ for $i, j = 1 \cdots n$

$= 0$ for $i, j = n+1, \cdots n$

$= \delta_{ij}$ for $u_i = Q_i$, $u_j = P_j$. Putting $i = j$ result follows

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Alternatively: If $\phi = \text{const}$, $\psi = \text{const}$ be two integrals ~~$(\phi, H) = 0$, $(\psi, H) = 0$~~, then by Poisson-Jacobi theorem

$(\phi, \psi) = \text{const}$ is also an integral.

Under a C.T., $(\phi, H) = 0 \rightarrow (\phi', K) = 0$

$(\psi, H) = 0 \rightarrow (\psi', K) = 0$

$((\phi, \psi), H) = 0 \rightarrow ((\phi', \psi'), K) = 0.$

and $H \rightarrow K$

$\therefore (\phi, \psi) \rightarrow (\phi', \psi')$.

* made of electric dipoles.

24/7/63

8th Lecture – Remaining parts of previous lecture – ~~Hamilton-Jacobi partial differential equation~~ – In particular

g.c.T given by $\delta q_r = \epsilon \frac{\partial \phi}{\partial p_r}$; $\delta p_r = -\epsilon \frac{\partial \phi}{\partial q_r}$, where ϕ = const is an integral, changes system into itself –

Poisson's theorem – Hamilton Jacobi equation

9th Lecture – 25/7/63 – Remaining portions of 7th Lecture – whole of 8th Lecture.

10th Lecture – 29/7/63 – Ex. on Poisson's theorem – ~~Hamilton-Jacobi equation~~ Ex. 2, W, p. 336 – Hamilton-Jacobi eqn.

11th Lecture – 30/7/63 – General one on electromagnetic theory based on Slater & Frank's ~~Preface & Chap I~~ Introduction; Chap I.

12th Lecture – 5/8/63 – Electrostatics : $\vec{E} = \frac{e}{4\pi\epsilon_0 r^2}$ & sim. anwar, $F = \frac{ee'}{4\pi\epsilon_0 r^2}$; $\varphi = \frac{e}{4\pi\epsilon_0 r}$, $\vec{E} = -\text{grad}\,\varphi$, $\text{rot}\,E = 0$

Above true for System of charges also – Gauss' theorem $\int \vec{E}\cdot\vec{n}\cdot d\sigma = \sum \frac{e_i}{\epsilon_0}$ – for continuous distn Gauss' theorem becomes $\int_S \vec{E}\cdot\vec{n}\cdot d\sigma = \frac{1}{\epsilon_0}\int_V \rho\, dv$ ie $\int_V \text{div}\,E\cdot dv = \frac{1}{\epsilon_0}\int \rho\, dv$ ie $\text{div}\,\vec{E} = \rho/\epsilon_0$ leading to Poisson's eqn

$\nabla^2\varphi = -\rho/\epsilon_0$ & for no volume charge ie $\rho = 0$, $\nabla^2\varphi = 0$ ie Laplace's eqn – Green's solⁿ of Poisson's eqn

Green's theorem : $\int_S (\varphi\,\text{grad}\,\psi\cdot\vec{n} - \psi\,\text{grad}\,\varphi\cdot\vec{n})\,d\sigma = \int_V (\varphi\nabla^2\psi - \psi\nabla^2\varphi)\,dV$. using this, Green's solⁿ is

$$4\pi\varphi = -\int \frac{\nabla^2\varphi}{r}\,dv \quad (\text{inf. sphere surrounding point } P)$$

ie $$\varphi = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{r}\,dv\,,$$ using Poisson's theorem.

Laplace's eqn $\nabla^2\varphi = 0$ in sph. polar coords (r, θ, Φ), $\psi = R\cdot\Theta\Phi$

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0.$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{l(l+1)}{r^2}R = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0.$$

Genl. soln in sph. harmonics $P_l^m(\cos\theta)\sin m\varphi$ or $P_l^m(\cos\theta)\cos m\varphi$

$$\psi = \left(ar^l + \frac{b}{r^{l+1}}\right)P_l^m(\cos\theta)(C\sin m\varphi + D\cos m\varphi) \qquad [l \text{ an integer}]$$

13th Lecture – 6/8/63 (1) Dielectrics* – $\vec{E}$ gives rise to polarisation vector $\vec{P} \propto \vec{E}$, and a polarisation charge ρ' with $\text{div}\,P = -\rho'$ – Poisson's theorem to be replaced by $\epsilon_0\,\text{div}\,E = \rho + \rho' = \rho - \text{div}\,P$ or $\text{div}(\epsilon_0\vec{E} + \vec{P}) = \rho$

$\epsilon_0\vec{E} + \vec{P}$ called electric displacement $\vec{D}$ ie $\text{div}\,D = \rho$ ~~with~~ & since $P \propto E$, $D \propto E$ and $D = \epsilon\vec{E}$ (ϵ is dielectric constant) – Surface of separation between two homogeneous dielectrics from $\text{div}\,\vec{D} = \rho$, normal component of D is continuous & from $\text{curl}\,\vec{E} = 0$, tangential component of $\vec{E}$ ($\vec{P}$) is continuous.

(2) Magnetic field of currents – Just ~~Just~~ An electric field is characterised in a vacuum by $\vec{E}$, a magnetic field is characterised by the magnetic induction $\vec{B}$ [~~mag~~ magnetic force ∝ velocity of charge ie current carried by charge]

ie $\vec{B}$ appears in electrodynamics, and $\vec{F} = e\{\vec{E} + \vec{v}\times\vec{B}\}$ is the force on a charge e – Magnetic field produced by magnetic dipoles or by currents – Consideration of the latter .. Biot-Savart's law; $B = \frac{\mu_0}{4\pi} \cdot e(\vec{v}\times\vec{r})/|\vec{r}|^3$ or in infinitesimal form with $e\vec{v} = i\cdot d\vec{s}$ (i = current), $d\vec{B} = \frac{\mu_0}{4\pi} \frac{i(d\vec{s}\times\vec{r})}{|\vec{r}|^3}$ – from either of these expressions it is easy to show that div $\vec{B} = 0$ ~~(in vac)~~ in empty space curl $\vec{B} = 0$, but inside a conductor carrying a current, we have Ampères law curl $\vec{B} = \mu_0 \vec{J}$ (J = current density vector ie no. of amperes per sq. m) – from div $\vec{B} = 0$, using div curl $\vec{A} = 0$ for any $\vec{A}$, write $\vec{B}$ = curl $\vec{A}$ ($\vec{A}$ arbitrary & vector pot.) & Ampere's law becomes curl curl $\vec{A} = \mu_0 J$ ie grad div $\vec{A} - \nabla^2 A = \mu_0 J$ & assuming div $\vec{A} = 0$, $\nabla^2 \vec{A} = -\mu_0 J$ ie $\vec{A}$ satisfies Poisson eqn like ϕ in Electrostatics & Green's soln is $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{r} dV$

(3) Magnetic materials – analogous to dielectrics – diamagnetic, paramagnetic & ferromagnetic media – Consider only first two – Magnetisation vector $\vec{M}$ (vector sum of dipole moments in unit volume), $\vec{M} \propto \vec{B}$ – M gives rise to (induced) current J', curl $\vec{M} = J'$ – curl $B = \mu_0(J+J') = \mu_0(J + \text{curl } \vec{M})$ or curl $\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = J$, $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$ or $\vec{B} = \mu_0(\vec{H} + \vec{M})$, $\vec{H}$ called the magnetic field, curl $\vec{H} = J$, $\vec{B} = \mu\vec{H}$ (steady closed currents)

(4) Electromagnetic induction & Maxwell's eqns – Faraday's law of e.m. induction ie induced current not due to magnetic flux itself, but its time rate of change – Faraday's law in math. form is $\int_L E\cdot ds = -\frac{d}{dt}\int_S \vec{B}\cdot\vec{n}\cdot d\sigma$ By Stokes theorem L.H.S = $\int$ curl $\vec{E}\cdot\vec{n}\cdot d\sigma$ & with circuit fixed R.H.S = $-\int \frac{\partial\vec{B}}{\partial t}\cdot\vec{n}\cdot d\sigma$ leading to

$$\text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Modification of the eqn curl $\vec{H} = J$ for non-closed (non-steady) currents, say cases of sources & sinks (like condenser plates) from div curl $H = 0$, it follows that $\text{div } J$ is always $= 0$ ie currents are always closed ie a contradiction. So Maxwell replaced it by div $J = -\frac{\partial \rho}{\partial t}$ (for non-steady currents also) really an eqn of continuity & using div $\vec{D} = \rho$, this can be written as div $\left(J + \frac{\partial \vec{D}}{\partial t}\right) = 0$. Thus he replaced curl $\vec{H} = \vec{J}$ by curl $H = J + \frac{\partial \vec{D}}{\partial t}$ (second term called displacement current) – Time rate of change of displacement can be obtained in rapidly varying fields ie cases of e.m. waves & in these cases exptl. verification has been obtained. Thus Maxwell's eqns are

$$\text{curl } \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \text{div } \vec{B} = 0$$

$$\text{curl } \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \quad \text{div } \vec{D} = \rho$$

with constitutive eqns $\vec{D} = \epsilon\vec{E}$ & $B = \mu\vec{H}$. $\left[\frac{\epsilon}{\epsilon_0} = \text{dielectric const}; \frac{\mu}{\mu_0} = \text{permeability}\right]$.

14th Lecture – 7/8/63 – If J obeys Ohm's law (current in a conductor $\propto$ electric field in it) ie $\vec{J} = \sigma\vec{E}$ (σ = Sp. conductivity), this is also added as a constitutive equation

(1) Separation of Maxwell's eqns into pure, and connected types

(2) Div. eqns follow from the curl. eqns by taking divergence, eqn of continuity & integrating w.r.t. time

(3) M. eqns ~~determine with~~ constitutive eqns determine the fields when charges & currents are given

(4) Also the force eqns to be added viz: $F = e\{\vec{E} + (\vec{v}\times\vec{B})\}$ or per unit volume, ~~$F = \rho\{\vec{E} + (\vec{J}\times\vec{B})\}$~~

$$F = \rho\vec{E} + (\vec{J}\times\vec{B}).$$

(5) M-eqns, constitutive eqns & the force eqn form the whole foundation of e-m. Theory and are exactly correct except for (i) corrections of quantum theory really changing the whole formulation of the theory (ii) use of relativistic mechanics for particles moving with velocities not nearly equal to c.

Success of Maxwell's theory in predicting e.m. waves travelling with vel c; also Herzian waves of longer wave length were shown to satisfy Maxwell's eqns - Solng Maxwell's eqn in a uniform material (constant ϵ & μ) and no charge & current except given by Ohm's law (σ = cond), Maxwell's eqns become

$$\left.\begin{aligned} \operatorname{curl}\vec{E} &= -\mu\frac{\partial \vec{H}}{\partial t} \quad ; \ \operatorname{div}\vec{H} = 0 \\ \operatorname{curl}\vec{H} &= \sigma\vec{E} + \epsilon\frac{\partial \vec{E}}{\partial t} \quad ; \ \operatorname{div}\vec{E} = 0. \end{aligned}\right\} \ (\rho = 0)$$

Taking curl of the curl eqns & using curl curl = grad div $-\nabla^2$ & div = 0, we get wave eqns

$$\nabla^2\vec{E} - \sigma\mu\frac{\partial \vec{E}}{\partial t} - \epsilon\mu\frac{\partial^2\vec{E}}{\partial t^2} = 0 \text{ & simily for } \vec{H} \text{ also}$$

In case of non-conducting medium $\sigma = 0$ & eqns become $\nabla^2\vec{E} = \epsilon\mu\frac{\partial^2\vec{E}}{\partial t^2}$, giving vel. of propagating waves

$$v = \frac{1}{\sqrt{\epsilon\mu}} \text{ & in empty space } v = \frac{1}{\sqrt{\epsilon_0\mu_0}} \simeq 3\times10^{10}\ \text{cm/sec} = c.$$

Poynting's theorem & Poynting's vector — from $\operatorname{div}(E\times H) = H\cdot\operatorname{curl}E - E\cdot\operatorname{curl}H$

$$= -E\frac{\partial D}{\partial t} - H\frac{\partial B}{\partial t} - E\cdot J \ \text{@}$$

$$\operatorname{div}(E\times H) + \frac{\partial}{\partial t}\tfrac{1}{2}(\epsilon E^2 + \mu H^2) = -E\cdot J \quad \text{a continuity eqn for a quantity}$$

& $S = (E\times H)$ is the flux of the quantity & the quantity $= \tfrac{1}{2}(\epsilon E^2 + \mu H^2)$ = energy density

S is Poynting's vector.

Maxwell's eqns in Gaussian units : $\operatorname{curl}\vec{E} = -\frac{1}{c}\cdot\frac{\partial \vec{B}}{\partial t}$, $\operatorname{div}\vec{B} = 0$

$\operatorname{curl}\vec{H} = 4\pi\frac{J}{c} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}$, $\operatorname{div}\vec{D} = 4\pi\rho$

$\vec{B} \propto \vec{H}$ $\quad B = K_m H$

$\vec{D} \propto \vec{E}$ $\quad D = K_e E$

with $\rho = 0$, & $J = 0$, $\quad \operatorname{curl}\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}$; $\operatorname{div}\vec{H} = 0$

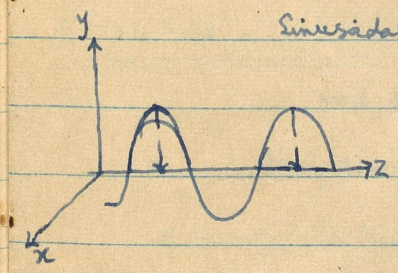
$\operatorname{curl}H = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$; $\operatorname{div}\vec{E} = 0$.

15th lecture — 8/8/63 — Poynting's theorem - Solns of the wave eqns

(1) $\nabla^2\phi - \epsilon\mu\frac{\partial^2\phi}{\partial t^2} = 0$, ϕ = a scalar — Plane wave $\phi = \phi(z,t)$. Eqn becomes $\frac{\partial^2\phi}{\partial z^2} - \epsilon\mu\frac{\partial^2\phi}{\partial t^2}$

Assume $\phi = \phi_0 e^{i\omega t - \gamma z}$ (ω, γ const) as a soln. This will be a soln if $\gamma^2 + \epsilon\mu\omega^2 = 0$

or $\gamma = \pm i\omega\sqrt{\epsilon\mu}$ i.e $\phi = \phi_0 e^{i\omega(t \mp z\sqrt{\epsilon\mu})}$ [Assumption justified if a soln is obtained].



Sinusoidal waves in t as also in z - Period $= 2\pi/\omega$, wave length $= 2\pi/\omega\sqrt{\epsilon\mu}$

$\therefore$ if velocity = v, $v\cdot\frac{2\pi}{\omega} = \frac{2\pi}{\omega\sqrt{\epsilon\mu}}$ or $v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{1}{\sqrt{\epsilon_0\mu_0}\cdot\sqrt{K_e K_m}}$

if $c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$ & $n = \sqrt{K_e K_m}$, $v = \frac{c}{n}$ (n = index of refraction)

In empty space $n = 1$, & $v = c$//.

(2) Light waves in a rectangular box of sides a, b, d in x, y, z dirns, with light entering only through a pin hole, box being light-tight except for this hole is black body radiation

* distance $r = \sqrt{\Sigma (x_i' - x_i^{\bullet})^2}$ between two points remains invariant

since $x_i'' - x_i^{\bullet\prime} = x_i' - x_i^{\bullet}$ ($\pm$ being the same)

and hence V depending on mutual distances of points of the system. Also $\frac{\partial V}{\partial x_i} = \frac{\partial V}{\partial x_i'}$

since $\frac{\partial x_i'}{\partial x_i} = 1$.

$$\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}.$$

Consider E_x and try soln $E_x = X(x)\,Y(y)\,Z(z)\,T(t) \rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \dots + \dots = \frac{1}{c^2}\frac{\partial^2 T}{\partial t^2}$ i.e. each $= \text{const} = -\omega^2/c^2$

$$\frac{\partial^2 T}{\partial t^2} = -\omega^2 T \qquad T = D_1 e^{i\omega t} + D_2 e^{-i\omega t} := D e^{i\omega t}$$

Sim $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k'^2$, $\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -l'^2$, $\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -m'^2$

$X = A_{1k'x}\sin k'x + A_{2k'x}\cos k'x$, $Y = \dots$, $Z = \dots$

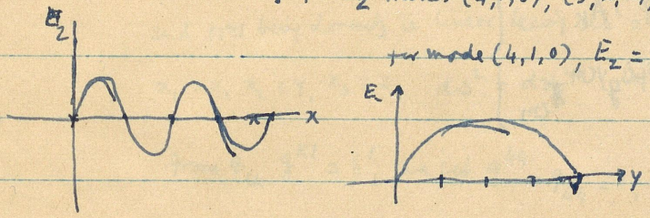
Similarly for E_y and E_z. If walls of box be infinitely conducting, $E_x = 0$ on four walls $\perp^r$ to planes of y-axis & z-axis & similarly for E_y and E_z. Using these the soln can be obtained as ~~other space part of E_x~~

$$\left.\begin{aligned} E_x &= A_{klm}\cos\frac{k\pi x}{a}\sin\frac{l\pi y}{b}\sin\frac{m\pi z}{d}\cdot T \\ E_y &= B_{klm}\sin\frac{k\pi x}{a}\cos\frac{l\pi y}{b}\cdot\sin\frac{m\pi z}{d}\cdot T \\ E_z &= C_{klm}\sin\frac{k\pi x}{a}\sin\frac{l\pi y}{b}\cos\frac{m\pi z}{d}\cdot T \end{aligned}\right\}$$

Substituting in the wave eqn $\left(\frac{k\pi}{a}\right)^2 + \left(\frac{l\pi}{b}\right)^2 + \left(\frac{m\pi}{d}\right)^2 = \frac{\omega^2}{c^2}$ integers

Ex: for a cube $a = b = d$, $k^2 + l^2 + m^2 = \frac{\omega^2 a^2}{\pi^2 c^2}$ & for given ω, all k, l, m satisfying above eqn are possible

for eg: for E_z modes $(4,1,0)$, $(3,2,2)$, $(2,3,2)$, $(2,2,3)$ with $\omega = \sqrt{17\pi}\,\frac{c}{a}$



for mode $(4,1,0)$, $E_z = C_{410}\sin\frac{4\pi x}{a}\sin\frac{\pi y}{a}$.

& graphs of E_z are as shown.

(3) Spherical electromagnetic waves :- waves propagated along the radius r - two types TE ($E_r = 0$), TM ($H_r = 0$)

writing $u = rH_r$, $v = rE_r$, u and v satisfy the wave eqn in sph. polar coords

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad (\text{same for } v)$$

Putting $u = R(r)\,\Theta(\theta)\,\Phi(\varphi)\,e^{i\omega t}$ ~~we can solve~~

we can solve & find the space part of $u = J_l\left(\frac{\omega r}{c}\right)P_l^m(\cos\theta)\sin m\varphi$

$$v = N_l\left(\frac{\omega r}{c}\right)P_l^m(\cos\theta)\cos m\varphi$$

These give H_r & E_r & ~~solving Maxwell's eqns in sph~~ we can find other components $H_\theta, H_\varphi, E_\theta, E_\varphi$ and ~~E_θ, E_φ etc~~ by using Maxwell's eqns in sph. coords.

16th Lecture - 12/8/63 - Galilean transformations (frames of reference alternatively) : $x_i' = c_{ij}x_j + x_{0i}$ and $x_i' = x_i - v_i t$ ($i, j = 1, 2, 3$) and v_i are components of vel. of frame S' relative to S - Mechanical systems invariant (or covariant) under Galilean transformations are called inertial - covariance specially under the second trn. - covariance of the Newtonian eqns of motion for a single particle, for a system of particles* - covariance of the Lagrangian & Hamiltonian eqns - Non-covariance of Maxwell's eqns under Galilean transformations established by $\sum u_i'^2 = c^2 + v^2 - 2c\sum v_i\cos\alpha_i$ ($u_i = c\cos\alpha_i$) and using speed of light is $c - v$ or $c + v$ according as direction is that of v or $-v$ — Derivation of the L.T:

* Minkowski world.

* use of metric tensor to raise & lower indices ie change covariant $\rightleftarrows$ contravariant

$$\left.\begin{aligned} A^i &= g^{ik} A_k \\ T^{ik} &= g^{il} g^{km} T_{lm} \end{aligned}\right\}$$

need to retain principle of relativity, generalising Galilei transformation, specially $t' = t$, using (1) linearity of eqns of trn, following from homogeneity of space & time, (2) c is a universal const. independent of the velocity of its source — L.T also characterised as the linear transformation leaving $ds^2 = c^2dt^2 - (dx^2 + \dots)$ invariant* - Law of addition of velocities $\left(\frac{v+w}{1+\frac{vw}{c^2}}\right)$ - Relativistic mechanics: $\frac{dE}{dt} = u \cdot \frac{d}{dt}\left(\frac{mu}{\sqrt{1-u^2/c^2}}\right)$

$= \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right)$

17th Lecture — 19/8/63 — $E = \frac{mc^2}{\sqrt{1-u^2/c^2}}$; $\vec{P} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$. ($m$ = rest mass)

(not taken)

For $u = 0$, $P = 0$, but $E \neq 0$ & $E_0 = m_0c^2$ & if $m = \frac{m_0}{\sqrt{1-u^2/c^2}}$, $E = mc^2$ (~~rest mass~~).

17th Lecture — 20/8/63 — ~~Electrons~~ Relativistic electrodynamics — Maxwell's eqns are invariant under Lorentz transformations. To prove this we must write Maxwell's eqns in tensor form — General tensor analysis specified by invariance of $ds^2 = g_{ik}\,dx^i dx^k$ ($i, k = 1, 2, 3, 4$), (summation convention) where g_{ik} are genl fns of x^i — Gen. coord. trns $\Rightarrow$ covariant & contravariant vectors and tensors — g_{ik} metric tensor — (covariant tensor)

$g_{ik} = g_{ki}$; g^{ik} contravariant metric tensor, $g_{ik}\,g^{kl} = \delta_i^l$ * Special relativity with g_{ik} constants, and trns being Lorentz ie linear leaving $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ invariant. Putting $x_4 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$, $ds^2 = dx_4^2 - (dx_1^2 + dx_2^2 + dx_3^2) = g_{ik}\,dx_i\,dx_k$ ($g_{44} = 1$, $g_{11} = g_{22} = g_{33} = -1$ & $g_{ik} = 0$ for $i \neq k$).

From $g_{ik}\,g^{kl} = \delta_i^l$ we get $g^{44} = g_{44} = 1$, $g^{11} = g_{11} = -1$, ..

$$A^i = g^{ik}A_k \rightarrow A^i = g^{ii}A_i \quad \text{(no summation)}$$

$$T^{ik} = g^{il}g^{km}T_{lm} \rightarrow T^{ik} = g^{ii}g^{kk}T_{ik} \quad \text{(no summation)}.$$

Maxwell's eqns in free space. $\operatorname{div}\vec{H} = 0$, $\operatorname{div}\vec{E} = 0$, $\operatorname{rot}\vec{H} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$, $\operatorname{rot}\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}$

[in em. field, $\operatorname{div}\vec{H} = 0$, $\operatorname{div}\vec{E} = 4\pi\rho$, $\operatorname{rot}\vec{H} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}$, $\operatorname{rot}\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}$.]

— vector & scalar potentials $\vec{A}$ & ϕ, $\vec{H} = \operatorname{rot}\vec{A}$, $\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \operatorname{grad}\phi$

$(\vec{A}, \phi)$ a 4-vector with $A_1 = A_x, \dots, A_4 = -\phi$

F_{ik} an antisymmetric tensor (ie $F_{11} = F_{22} = F_{33} = F_{44} = 0$) with $F_{23} = H_x$, $F_{31} = H_y$, $F_{12} = H_z$, $F_{14} = E_x$, ...

(ie 6 distinct components)

cmnor $F_{ikl} = \frac{\partial F_{ik}}{\partial x_l} + \frac{\partial F_{kl}}{\partial x_i} + \frac{\partial F_{li}}{\partial x_k}$, also antisymmetric for every pair of indices

(i) $F_{ikl} = 0 \rightarrow$ ~~$\operatorname{div}\vec{H} = 0$, $\operatorname{rot}\vec{E}$~~ $\operatorname{div}\vec{H} = 0$, $\operatorname{rot}\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}$.

(ii) $\frac{\partial F^{ik}}{\partial x_k} = 0 \rightarrow \operatorname{div}\vec{E} = 0$, $\operatorname{rot}\vec{H} = \frac{1}{c}\cdot\frac{\partial\vec{E}}{\partial t}$.

(iii) introducing $S^i = \left(\frac{4\pi}{c}\vec{j}, 4\pi\rho\right)$ as current four vector, the general case of Maxwell's eqns are given by $\frac{\partial F^{ik}}{\partial x_k} = S^i$

18th Lecture — 21/8/63 — Remaining portions of above lecture — From the identity $\frac{\partial^2 F_{ik}}{\partial x_i \partial x_k} = 0$ which follows because of the antisymmetry of F_{ik}, (iii) gives $\frac{\partial S^i}{\partial x_i^*} = 0$, ie $\frac{4\pi}{c}\cdot\operatorname{div}\vec{j} + 4\pi\frac{\partial\rho}{\partial x_4} = 0$

ie $\operatorname{div}\vec{j} + \frac{\partial\rho}{\partial t} = 0$.

which is the eqn for the law of conservation of charge — eqn of continuity.

– Transformations of field quantities under L.T's : for the special L.T with $V_x = V$, $V_y = V_z = 0$ ie

$$x' = (x - Vt)/\sqrt{1-V^2/c^2}\,,\quad y' = y,\ z' = z,\ t' = (t - Vx/c^2)/\sqrt{(1-V^2/c^2)}$$

ie $x_1' = (x_1 - Vx_4/c)/\beta$, $x_2' = x_2$, $x_3' = x_3$, $x_4' = (x_4 - Vx_1/c)/\beta$ $\quad [\beta = \sqrt{(1-V^2/c^2)}$.

Using $F'^{ik} = \dfrac{\partial x_i'}{\partial x_l}\cdot\dfrac{\partial x_k'}{\partial x_m} F^{lm}$ and the above L.T's & the fact that F^{ik} is antisymmetric and that $F^{23} = H_1, \ldots$; and $F^{14} = -E_1, \ldots$, we can show that,

$$\left.\begin{aligned} H_x' &= H_x\,,\quad E_x' = E_x \\ H_y' &= (H_y + VE_z/c)/\beta\,;\quad E_y' = (E_y - VH_z/c)/\beta \\ H_z' &= (H_z - VE_y/c)/\beta \quad E_z' = (E_z + VH_y/c)/\beta \end{aligned}\right\}$$

From these we have $\vec{E}'^2 - \vec{H}'^2 = \vec{E}^2 - \vec{H}^2$

and $\vec{E}'\cdot\vec{H}' = \vec{E}\cdot\vec{H}$

ie two expns invariant under L.T's. This can also be seen from the fact that these can be put in tensor form viz.

$$\vec{E}^2 - \vec{H}^2 = -\frac{1}{2} F_{ik} F^{ik}$$

and $\vec{E}\cdot\vec{H} = \dfrac{1}{4} F_{ik} \overset{*}{F}{}^{ik}$. $\overset{*}{F}{}^{ik}$ is dual to F_{ik} ie defined by $\overset{*}{F}{}^{23} = F_{14}, \ldots$ and $\overset{*}{F}{}^{14} = F_{23}, \ldots$

– motion of a charged mass-point in an external field : Velocity four-vector u^i defined by

$$u^i = dx^i/d\tau \quad \text{where } \tau \text{ is the proper time defined by } d\tau = \sqrt{1-v^2/c^2}\,dt$$

ie $c^2 d\tau^2 = (c^2 - v^2)dt^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = ds^2$ ie $c\,d\tau = ds$, so that τ is an invariant

$$u' = dx'/d\tau = \frac{dx_1}{d\tau} = \frac{1}{\sqrt{1-v^2/c^2}}\frac{dx_1}{dt} = \frac{v_1}{\sqrt{1-v^2/c^2}}\,,\ \text{sim}^{ly}\ u^2 \text{ and } u^3$$

$$u^4 = \frac{dx_4}{d\tau} = \frac{c\,dt}{d\tau} = \frac{c}{\sqrt{1-v^2/c^2}}$$

Consider the covariant vector $B_i = -\dfrac{e}{c} u^k F_{ik}$

$$B_1 = -\frac{e}{c} u^k F_{1k} = -\frac{e}{c}\left\{u^2 F_{12} + u^3 F_{13} + u^4 F_{14}\right\}$$

$$= -\frac{e}{c}\left\{\frac{+v_y}{\sqrt{1-v^2/c^2}} H_z + \frac{v_z}{\sqrt{1-v^2/c^2}} H_y + \frac{c}{\sqrt{1-v^2/c^2}} E_x\right\}$$

$$= -\frac{e}{\sqrt{1-v^2/c^2}}\left[\frac{1}{c}(\vec{v}\times\vec{H}) + \vec{E}\right]_x \quad ;\ \text{sim}^{ly}\ B_2 \text{ and } B_3$$

Noting that $e\left\{\vec{E} + \frac{1}{c}(\vec{v}\times\vec{H})\right\}$ is the (covariant) force on the moving charge, the vector B_i is taken as the covariant force four-vector, and the eqns of motions are taken as

$$m w_i = -\frac{e}{c} u^k F_{ik}\,,\ \text{where } w_i \text{ is the acceleration four-vector.}$$

given by $w_i = du_i/d\tau$. ie $w_1 = du_1/d\tau = -du_1'/d\tau = -du_1'/\sqrt{1-v^2/c^2}\,dt$

$$w_1 = -\frac{1}{\sqrt{(1-v^2/c^2)}}\frac{d}{dt}\left(\frac{v_1'}{\sqrt{1-v^2/c^2}}\right),\ \text{sim}^{ly}\ w_2 \text{ & } w_3 \text{ ie } w' = \frac{1}{\sqrt{1-v^2/c^2}}\frac{d}{dt}\left(\frac{v'}{\sqrt{1-v^2/c^2}}\right)$$

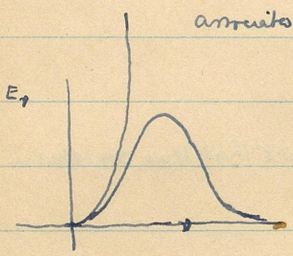
$$w_4 = \frac{1}{\sqrt{1-v^2/c^2}}\cdot\frac{d}{dt}\left(\frac{c}{\sqrt{1-v^2/c^2}}\right) = w^4$$

$$B_4 = -\frac{e}{c} u^k F_{4k} = -\frac{e}{c}\left\{u^1 F_{41} + u^2 F_{42} + u^3 F_{43}\right\} = \frac{e}{c}\cdot\frac{(\vec{v}\cdot\vec{E})}{\sqrt{1-v^2/c^2}}$$

* $k^2+l^2+m^2 = \frac{\omega^2 a^2}{\pi^2 c^2}$. No. of modes having a frequency $\leq \omega$ is given by all possible combinations of k, l, m, such that $k^2+l^2+m^2 \leq \frac{\omega^2 a^2}{\pi^2 c^2}$. Noting that $x^2+y^2+z^2 = \frac{\omega^2 a^2}{\pi^2 c^2}$ is a sphere of radius $\omega a/\pi c$, the no. of modes $= \frac{1}{8}$ vol. of this sphere $i.e. = \frac{1}{8}\cdot\frac{4}{3}\pi\left(\frac{\omega a}{\pi c}\right)^3$ Noting that for each set k, l, m there are two independent vibrations, the no. of modes $= \frac{\pi}{3}\cdot\frac{\omega^3 a^3}{\pi^3 c^3} = \frac{\omega^3 a^3}{3\pi^2 c^3}$ $i.e.$ no. of modes per unit volume of the cube $= \frac{\omega^3}{3\pi^2 c^3} = \frac{8\nu^3\pi^3}{3\pi^2 c^3} = \frac{8\pi\nu^3}{3c^3}$. Hence no. of modes of frequency between ν & $\nu+d\nu$ per unit vol $= N_\nu\, d\nu = \frac{8\pi\nu^2}{c^3}\, d\nu$. Using equipartition law of energy $i.e.$ for each mode is

$(k, l, m + ve)$

associate kT ergs of energy, $E_\nu\, d\nu = \frac{8\pi\nu^2}{c^3} kT\, d\nu$ – Rayleigh–Jeans formula



As $\nu \to \infty$, $E_\nu \to \infty$, ultraviolet catastrophe

Principle of equipartition is a consequence of Boltzmann's law that the number of molecules having energy E_n is given by $N(E_n) = C e^{-E_n/kT}$, so that mean energy $= \frac{\sum E_n e^{-E_n/kT}}{\sum e^{-E_n/kT}}$

Planck (1899 Christmas week) took $E_n = nh\nu$ (h a const), so that mean energy

$$= h\nu\,\frac{\sum_0^\infty n e^{-nh\nu/kT}}{\sum_0^\infty e^{-nh\nu/kT}} = h\nu\cdot\frac{\sum_0^\infty n e^{-\alpha n}}{\sum_0^\infty e^{-\alpha n}} \qquad (\alpha = h\nu/kT)$$

$$= h\nu\,\frac{e^{-\alpha}/(1-e^{-\alpha})^2}{1/(1-e^{-\alpha})} = h\nu\cdot\frac{e^{-\alpha}}{1-e^{-\alpha}} = \frac{h\nu}{e^{\alpha}-1} = \frac{h\nu}{e^{h\nu/kT}-1}$$ (instead of kT)

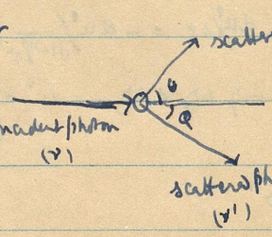
so that $E_\nu\, d\nu = \frac{8\pi\nu^2}{c^3}\cdot\frac{h\nu}{e^{h\nu/kT}-1}\, d\nu$, so that as $\nu \to 0$, $E_\nu \to 0$.

As $\nu \to 0$, $\frac{h\nu}{e^{h\nu/kT}-1} \to kT$ $i.e.$ ~~Rayleigh–Jeans~~ equipartition.

$E = h\nu$ → notion of light quanta – birth of quantum theory – h Planck's constant.

Photo-electric effect – Ultraviolet light incident on metallic surfaces causes emission of electrons. Energy of electrons independent on intensity, but only on frequency (increase in intensity means only more electrons emitted), energy of electrons $\propto$ frequency and ~~for some~~ less than critical value of ν, no electrons are emitted. Einstein's eqn $E = h\nu - W$ (W = energy req'd to ~~pull~~ free an electron from attractive force field of the metal. Millikan confirms that if $\nu < W/h$, no electrons are emitted ($\frac{1}{2}mv^2 = h\nu - W$).

Compton effect



$\frac{c}{\nu} - \frac{c}{\nu'}$ or $= K(1-\cos\varphi)$ exactly observed.

Calculations using conservation of energy & momentum & using energy of quantum $= h\nu$ & momentum $= h\nu/c$ confirm this. ($\frac{E^2}{c^2} - p^2 = m_0^2 c^2$ with $m = 0$)

(Raman effect).

Hence eqns $mw^i = B^i$ become

$$\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-v^2/c^2}}\right) = e\left\{\vec{E} + \frac{1}{c}(\vec{v}\times\vec{H})\right\}$$

$$\frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-v^2/c^2}}\right) = e(\vec{v}\cdot\vec{E}).$$

Hence interpretation of $\vec{P} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$ and $W = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ as momentum and K.E. of the particle

$(\vec{P}, W/c)$ are components of a 4-vector called the energy momentum tensor, since these are nothing but mu^i and invariant of this vector $= \frac{W^2}{c^2} - \vec{P}^2 = m^2c^2$, defining rest mass in terms of this vector.

For $v = 0$, $W_0 = mc^2$ & for any v, $W = Mc^2$ if $M = \frac{m}{\sqrt{1-v^2/c^2}}$ & as $v \to c$, $M \to \infty$, showing c as a limit for a ~~body~~ particle $m \neq 0$.

18th Lecture – 26/8/63 – Remaining portion of last lecture viz. motion of a charged mass point in an external field –

19th Lecture – 27/8/63 – General type of lecture based on Gamow up to Radiation Theory.

20th Lecture – 28/8/63 – Use of results of 15th Lecture to obtain Jeans energy distribution, Planck's hypothesis and derivation of mean energy as $h\nu/(e^{h\nu/kT} - 1)$ using Boltzmann's law $N(E_n) = Ce^{-E_n/kT}$, Jeans law & ~~Wien's law~~ as approximations* – Photoelectric effect – Compton effect – Bohr atom – Pauli principle – Periodic Law & valency explained – Wave nature of particles; de Broglie waves, electron diffraction – Schrodinger's wave equation – Heisenberg's uncertainty principle – Dirac's relativistic theory of anti-particles (anti-protons & anti-neutrons)

[Bohr atom – The three rules & satisfactory explanation of ~~spectra~~ atomic spectra – ~~Electron~~

Pauli principle – Atomic electrons have a tendency to jump from higher to lower orbits & atomic volume would be less for elements of higher atomic number – not more than two electrons can be accommodated in an orbit, (spinning in opp. dirns)

Explanation of periodicity – He having 2 in first orbit (first shell closed), Ne 10 electrons 2 each in 5 orbits (2nd shell closed) and so on – explaining periodic properties

Valence – Atoms with almost finished shell have tendency to take in extra electrons to finish the shell and those just beginning a new shell have tendency to get rid of this eg. Cl (17 = 2 + 8 + 7) Na (11 = 2 + 8 + 1) & formation of NaCl.

($\lambda = h/p$) de Broglie's matter waves : $\lambda = \frac{h}{mv}$ ($p = h/\lambda$) – diffraction of an electron beam – light & electron having both as waves & as particles – no meaning in asking which model is really correct.

Schrodinger's wave equation : In non-relativistic case $E = p^2/2m$, and the wave eqn is obtained by the correspondence $p \to -ik\frac{\partial}{\partial x}$, $E \to ik\frac{\partial}{\partial t}$ for $E = p^2/m + V$ leads to the Schrodinger wave eqn $-\frac{k^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = ik\frac{\partial\psi}{\partial t}$.

Natural radioactivity — Becquerel's discovery — Madame Curie's discovery of radium and polonium from pitchblende an uranium ore — uranium family of natural radioactive disintegration eg:

$$_{92}U^{238} = {}_2He^4 + {}_{90}Th^{234} \longrightarrow {}_{82}Pb^{206}$$

is a nuclear transformation — Uranium-Lead dating by uranium-lead ratio in igneous rocks — using half-times (4.7×10^{12} years for U) — Age of the Jurassic period in

Artificial nuclear transformations: use of high energy particles from unstable heavy nuclei to bombard light nuclei — eg. (by α-particles): $_7N^{14} + {}_2He^4 = {}_8O^{17} + {}_1H^1$ — Wilson's cloud chamber — use of protons (instead of α-particles) by Cockcroft (aided by Rutherford) accelerated to 10^6 e.v.: $_3Li^7 + {}_1H^1 = {}_2 2{}_2He^4$ and

$$_5B^{11} + {}_1H^1 = 3\,{}_2He^4$$

— van de Graaff generator — Lawrence's cyclotron (40 Mev = 4.5 times that of α-particles — all (10^7 e.v.) elements up to the heaviest ones).

Structure of atomic nucleus: — Isotopes having integral atomic weights suggest nuclei made of protons (H-nuclei) but considerations of electric charge neutrality requires protons + neutrons, for eg

$$_8O^{16} = 8 \text{ protons} + 8 \text{ neutrons}$$

$$_{92}U^{238} = 92 \text{ protons} + 146 \text{ neutrons}.$$

— nuclear droplet theory & nuclear fluid, nuclear radii $\propto \sqrt[3]{m}$ — high density of nuclear fluid & high surface tension ($\rho = 2.4 \times 10^{14}$ gm/cm^3, $\tau = 93 \times 10^{18}$ dynes/cm i.e. 1 cc. weighs 240 million tons, nuclear film attached to 1 cm wire would support weight of 10^{12} tons) — about 5×10^6 e.v. req'd to remove a nucleon from the surface of nucleus against forces of cohesion.

— Mass defect — Mass defect & nuclear reactions — nuclear shell model: no of protons or neutrons being 2, 8, 14, 20, 28, 50, 82, 126 (two independent sets of shells interpenetrating)

— Fusion & fission (electrostatic & surface tension forces acting in same or opp. directions)

— Nuclear potential barriers lower than those calculated by classical mechanics; α particles entering & leaving nucleus

Large-scale nuclear reactions — fission by neutron bombardment (U → Ba + Kr by Hahn)

$$_{92}U^{238} + {}_0n^1 \rightarrow {}_{56}Ba^{145} + {}_{36}Kr^{94}$$

— Bohr & Wheeler (Sept. 1939 issue of Phys. Rev.) first open article on nuclear fission

— fission neutrons in addition to fission, for U^{235} about 2.5 neutrons/nucleus leading to a chain reaction — Setting more easily fissionable U^{235} (rather than U^{238}) (Manhattan project) — fission of U^{235} increases when fission neutrons are slowed down & then U^{238} does not matter; Fermi did this slowing down & made chain reaction possible — Plutonium (2.7 neutrons/nucleus) produced & thermo-nuclear reactions of light elements.

[Heisenberg's uncertainty principle – Motion of atomic particles cannot be done in the conventional way by giving their successive positions & velocities, both cannot be simultaneously measured accurately

$$\Delta x \cdot \Delta v = h/m.$$

Anti-particles – Electrons & protons – positrons as holes in the Dirac's ocean – Formation in pairs – Annihilation – Theory built in combining relativity & quantum mechanics – Negative protons – Neutrons & anti-neutrons

21st Lecture – Remaining portions of previous lecture – Natural radioactivity – Artificial nuclear transformations – Structure of the atomic nucleus – Large scale nuclear reactions – Mystery particles

[Mystery particles – Energy balance in β-particle emission suggested the neutrino ($e=0$, $m \cong 0$). To stop a beam of neutrino requires a shield several light years thick – $p + \nu \rightarrow n + e^+$ verifying ν as an actual particle – Mesons exchanged between two nucleons & explaining binding of protons & neutrons – Hideki Yukawa (Headache) (hungry dogs being locked in biting a juicy bone); Mesons observed in cosmic rays (extremely high energy protons & heavier +ve charged nucleii) mass about 200 m_e – pions (upper fringes of atmosphere) & muons (π-mesons & μ-mesons), $\pi^\pm \rightarrow \mu^\pm$ + neutrino {μ taking 10^{-3} secs. to reach earth's surface & lifetime 10^{-6} sec ($\mu^\pm \rightarrow e^\pm + 2\nu$) – our time & ~~mesons time~~ muon's time $10^{-3} < 10^{-6}$ of muon's time – Neutral pions π^0 (lifetime 10^{-16} sec, but still break up into 2 γ-ray quanta) – K-mesons – hyperons (Λ, Σ, Ξ) – Spin & Statistics – Parity – CPT-theorem – Strangeness.

22nd Lecture – 3/9/63 – Remaining portions of previous lecture.

23rd Lecture – 4/9/63 – Gave syllabus to class and did the whole of Chap I of Schiff

24th Lecture – 11/9/63 – Particle as a wave packet & derivation of wave eqn from $E \rightarrow i\hbar\partial/\partial t$, $p_x = -i\hbar\frac{\partial}{\partial x}$ from the classical equation of motion with the aid of a wave function ψ – Schrodinger's wave equation – Considerations of probability $P = |\psi|^2$, $\int|\psi|^2 = 1 - \frac{\partial P}{\partial t} + \operatorname{div} S = 0$. ($S$ = probability current density) – Expectation value – Ehrenfest's theorems

25th Lecture – 12/9/63 – $\frac{\partial P}{\partial t} + \operatorname{div}\vec{S} = 0$ (S = probability current density) – Reason for taking $\langle r\rangle = \int \bar{\psi} r \psi \, d\tau$ from $\langle i\hbar\frac{\partial}{\partial t}\rangle = \langle -\frac{\hbar^2}{2m}\nabla^2\rangle + \langle V\rangle$ where ψ should be on the right to be consistent with Schrodinger's wave equation – Proof of $\int S_n \, d\sigma = 0$ by considering a cubical box of side L such that ψ & $(\operatorname{grad}\psi)_n$ have same values at opposite ends of the box – Ehrenfest's theorem – An example of the correspondence Principle – Energy eigenfunctions (just up to mere definitions)

26th Lecture – 16/9/63 – The second of Ehrenfest's theorems viz. $\frac{d}{dt}\langle p_x\rangle = \langle -\partial V/\partial x\rangle$

$$\frac{d}{dt}\langle p_x\rangle = -i\hbar\frac{d}{dt}\int\bar{\psi}\frac{\partial\psi}{\partial x}d\tau = -i\hbar\int\left(\bar{\psi}\frac{\partial}{\partial x}\frac{\partial\psi}{\partial t} + \frac{\partial\bar{\psi}}{\partial t}\cdot\frac{\partial\psi}{\partial x}\right)d\tau$$

$$= -\int\bar{\psi}\frac{\partial}{\partial x}\left(-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi\right)d\tau + \int\left(-\frac{\hbar^2}{2m}\nabla^2\bar{\psi} + V\bar{\psi}\right)\frac{\partial\psi}{\partial x}d\tau$$

Consider $\int\left[\bar{\psi}\frac{\partial}{\partial x}(\nabla^2\psi) - \nabla^2\bar{\psi}\cdot\frac{\partial\psi}{\partial x}\right)d\tau$

27th

*(a) & (b) together stated as ψ & prod ψ continuous, uniform & bounded for over all space.

28th

~~* to be taken up later~~

29th

$$\text{This} = \int \left[\bar{\psi} \operatorname{div} \operatorname{grad}\left(\frac{\partial \psi}{\partial x}\right) - \operatorname{div} \operatorname{grad} \bar{\psi} . \frac{\partial \psi}{\partial x} \right] d\tau$$

Integrating by parts, and putting surface integrals $= 0$,

$$\text{This} = \int \left\{ \operatorname{grad}\left(\frac{\partial \psi}{\partial x}\right) . \operatorname{grad} \bar{\psi} - \operatorname{grad} \bar{\psi} . \operatorname{grad}\left(\frac{\partial \psi}{\partial x}\right) \right\} d\tau = 0$$

$$\therefore \quad \frac{d}{dt} \langle p_x \rangle = - \int \bar{\psi} \frac{\partial}{\partial x} (V\psi)\, d\tau + \int V \bar{\psi} \frac{\partial \psi}{\partial x} d\tau$$

$$= - \int \bar{\psi} \frac{\partial V}{\partial x} \psi \cdot d\tau = \langle (-\partial V/\partial x) \rangle$$

– Case where V is independent of t, Energy eigen functions, $i\hbar \frac{\partial \psi}{\partial t} = E\psi$, eigen value equation, eigenfunction ψ, eigen value E – Energy eigenfuncton $\psi = u(\vec{r})\, e^{-iEt/\hbar}$.

27th Lecture – 18/9/63. – Stationary state – Eigen eqn for u; also eigen equation for ψ with same eigen value – for time independent eigen eqn for u, cases (i) $E < 0$, (ii) $E > 0$; eg $u = e^{i\vec{k}.\vec{r}}$ (without proof)

(iii) energy eigen values of E being possible results of precise measurements of total energy of particle.

(iv) Conditions for soln of $u(\vec{r})$ to exist for real E [for $V(r)$ finite]: (a) ψ bounded, (b) ψ & grad ψ finite, continuous & single valued at every point ie P & S finite & continuous everywhere.*

– Case of $V(\vec{r})$ becoming infinite reduces to a one-dimensional problem – one dimensional quantised systems; Example of infinitely high potential barrier as in Schiff.

28th Lecture – 19/9/63 – For above Example $d\psi/dx$ being indeterminate on left of $x = 0$ & $= 0$ on right of $x = 0$ makes it discontinuous, but $P = A^2 \sin^2 \alpha x = 0$ on left of & right of $x = 0$ & so also $S_x \equiv 0$ ie P & S_x are continuous at $x = 0$ – Infinitely deep square potential well – Parity for this case & for general one-dimensional wave equation.

29th Lecture – 23/9/63 – Ex. 5, p. 40 of Schiff

$$\frac{d}{dt} \langle x^2 \rangle = \frac{d}{dt} \int \bar{\psi} x^2 \psi \cdot d\tau = \int \bar{\psi} x^2 \frac{\partial \psi}{\partial t} + \int \frac{\partial \bar{\psi}}{\partial t} x^2 \psi \, d\tau$$

$$= - \frac{i}{\hbar} \left[\int \bar{\psi} x^2 \left(-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi \right) d\tau - \int \left(-\frac{\hbar^2}{2m} \nabla^2 \bar{\psi} + V\bar{\psi} \right) x^2 \psi \cdot d\tau \right]$$

$$= \frac{Q}{\hbar} = \frac{i\hbar}{2m} \int \left\{ \bar{\psi} x^2 (\nabla^2 \psi) - (\nabla^2 \bar{\psi}) x^2 \psi \right\} d\tau$$

$$\int \nabla^2 \bar{\psi}\, x^2 \psi \cdot d\tau = - \int \operatorname{grad} \bar{\psi} \operatorname{grad} (x^2 \psi)\, d\tau = \int \bar{\psi} \nabla^2 (x^2 \psi)\, d\tau$$

$$\therefore \quad \frac{d}{dt} \langle x^2 \rangle = \frac{i\hbar}{2m} \int \bar{\psi} \left\{ x^2 \nabla^2 \psi - \nabla^2 (x^2 \psi) \right\} d\tau$$

$$= - \frac{i\hbar}{m} \int \bar{\psi} \left(\psi + 2x \nabla \psi \right) d\tau .$$

$$\nabla^2 (x^2 \psi) = 2x\psi + x^2 \nabla \psi$$

$$\nabla^2 (x^2 \psi) = 2\psi + 4x \nabla \psi + x^2 \nabla^2 \psi$$

$$\langle x p_x \rangle = \int \bar{\psi}\, x p_x \psi \, d\tau = -i\hbar \int \bar{\psi}\, x \frac{\partial \psi}{\partial x} d\tau$$

$$\langle p_x x \rangle = \int \bar{\psi}\, p_x x \psi \, d\tau = -i\hbar \int \bar{\psi} \frac{\partial}{\partial x} (x\psi) d\tau$$

$$\langle x p_x \rangle + \langle p_x x \rangle = - i\hbar \int \left(\bar{\psi} \psi \, dx + 2 \bar{\psi} x \frac{\partial \psi}{\partial x} \right) d\tau .$$ Hence the result since for a wave packet along x-dir $\nabla \psi = \frac{\partial \psi}{\partial x}$.

$$\langle \hat{p}_x \rangle = \int \psi^* \left(-i\hbar \frac{\partial}{\partial x} \right) \psi \, d\tau = -i\hbar$$

Ex. 4, p. 40. Schiff

$$\langle p_x \rangle = \int \bar{\psi} (-i\hbar) \frac{\partial \psi}{\partial x} d\tau \quad \& \text{ with } \psi = e^{ikx}, \ \bar{\psi} = e^{-ikx}$$

$$\langle p_x \rangle = \int e^{-ikx} (-i\hbar)(ik) e^{ikx}, \ d\tau = \int \hbar k \cdot d\tau = \text{real}.$$

Ex. 8, p. 40. Schiff. If $V(r) \to V(r) + V_0$, then $E \to E + V_0$ and the time-independent Schrodinger eqn becomes $\left(-\frac{\hbar^2}{2m}\nabla^2 + V + V_0\right) u'(r) = (E + V_0) u'(r)$ ie $u(r) = u'(r)$

The energy eigen-values are also changed by the constant amount V_0.

30th Lecture — 26/9/63. Ex. 2, p. 40 of Schiff about the non-relativistic Schrödinger equation being invariant against Galilean transformations. This Ex. is wrongly worded and is asked to be proved only for the case of the free particle. It can also be done for the case where the potential V also exists [Vide pp. 163–165 of Problems in Q.M. by Gol'dman etc]

Case I - Free particle - The Schrodinger eqn is $\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2}$

If $\psi'(x', t)$ is the wave ~~equation~~ function under the transformation $x = x' + vt'$, $t' = t$ & $\psi(x, t)$ the original wave function, the interpretation of probability density, a notion which is independent of change of coordinates, requires that $\overline{\psi'}\psi' = \overline{\psi}\psi$ ie we have in general

$$\psi' = e^{if}\psi, \text{ where } f = f(x, t).$$

To find f, we shall use the condition that the Schrödinger eqn $\frac{\partial \psi'}{\partial t'} = \frac{i\hbar}{2m}\frac{\partial^2 \psi'}{\partial x'^2}$ be the same as $\frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2}$ under the transformation. [$x' = x - vt$, $t' = t$] or [$x = x' + vt'$, $t = t'$]

$$\frac{\partial \psi'}{\partial t'} = \frac{\partial \psi'}{\partial x}\cdot\frac{\partial x}{\partial t'} + \frac{\partial \psi'}{\partial t}\cdot\frac{\partial t}{\partial t'} = v\frac{\partial \psi'}{\partial x} + \frac{\partial \psi'}{\partial t}$$

$$= v\left(\frac{\partial \psi}{\partial x}e^{if} + i\psi e^{if}\frac{\partial f}{\partial x}\right) + \left(\frac{\partial \psi}{\partial t}e^{if} + i\psi e^{if}\frac{\partial f}{\partial t}\right)$$

$$= e^{if}\left\{v\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial t} + i\psi\left(v\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}\right)\right\} \quad \text{---} \ (1)$$

$$\frac{\partial \psi'}{\partial x'} = \frac{\partial \psi'}{\partial x}\cdot\frac{\partial x}{\partial x'} + \frac{\partial \psi'}{\partial t}\cdot\frac{\partial t}{\partial x'} = \frac{\partial \psi'}{\partial x} = \frac{\partial \psi}{\partial x}e^{if} + i\psi e^{if}\frac{\partial f}{\partial x}.$$

$$\text{& similarly } \frac{\partial^2 \psi'}{\partial x'^2} = \frac{\partial}{\partial x}\left(\frac{\partial \psi'}{\partial x'}\right) = \frac{\partial^2 \psi}{\partial x^2}e^{if} + i\frac{\partial \psi}{\partial x}e^{if}\frac{\partial f}{\partial x} + i\frac{\partial \psi}{\partial x}e^{if}\frac{\partial f}{\partial x} - \psi e^{if}\left(\frac{\partial f}{\partial x}\right)^2 + i\psi e^{if}\frac{\partial^2 f}{\partial x^2}$$

$$= e^{if}\left[\frac{\partial^2 \psi}{\partial x^2} + 2i\frac{\partial \psi}{\partial x}\frac{\partial f}{\partial x} + i\psi\left\{\frac{\partial^2 f}{\partial x^2} + i\left(\frac{\partial f}{\partial x}\right)^2\right\}\right] \quad \text{--(2)}$$

In order that $\frac{\partial \psi'}{\partial t'} = \frac{i\hbar}{2m}\frac{\partial^2 \psi'}{\partial x'^2} \to \frac{\partial \psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2}$, we have using (1) and (2), and equating the coefficients of $\frac{\partial \psi}{\partial x}$ and $i\psi$ to zero (e^{if} cancels on both sides)

$$v = \frac{i\hbar}{2m}\cdot 2i\frac{\partial f}{\partial x} \quad \text{~~(2)~~} = -\frac{\hbar}{m}\frac{\partial f}{\partial x} \text{ or } \frac{\partial f}{\partial x} = -\frac{mv}{\hbar} \quad \text{--(3)}$$

$$v\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} = \frac{i\hbar}{2m}\left\{\frac{\partial^2 f}{\partial x^2} + i\left(\frac{\partial f}{\partial x}\right)^2\right\} = \frac{i\hbar}{2m}\frac{\partial^2 f}{\partial x^2} - \frac{\hbar}{2m}\left(\frac{\partial f}{\partial x}\right)^2 \quad \text{---(4)}$$

or ~~$\frac{\partial f}{\partial x}$~~ from (3) $f = -\frac{mv}{\hbar}x + \varphi(t)$ & $\varphi(t)$ can be obtained using (4) viz

$$-\frac{mv^2}{\hbar} + \varphi'(t) = 0 - \frac{\hbar}{2m}\cdot\frac{m^2v^2}{\hbar^2} = -\frac{mv^2}{2\hbar} \text{ ie } \varphi'(t) = \frac{mv^2}{2\hbar}$$

* i.e. u's vanish at ∞ and v's do not vanish at ∞

* $\operatorname{div}(u \operatorname{grad} v) = u \nabla^2 v + \operatorname{grad} u \cdot \operatorname{grad} v$.

$\therefore \operatorname{div}(u \operatorname{grad} v - v \operatorname{grad} u) = u \nabla^2 v - v \nabla^2 u$.

ie $\Phi(t) = \frac{mv^2}{2\hbar} t$, since an additive const. can be neglected.

Hence $f = -\frac{mv}{\hbar} x + \frac{mv^2}{2\hbar} t$

and $\psi' = e^{-i\left(\frac{mv}{\hbar} x - \frac{mv^2}{2\hbar} t\right)} \cdot \psi$. which is the reqd. soln

For the special wave eqn mentioned in Schiff viz $\psi = Ae^{i(kx-\omega t)}$

$$\psi' = Ae^{i(kx'-\omega t')} = Ae^{i\{k(x-vt)-\omega t\}} = Ae^{i(kx-\omega t)} \cdot e^{-ikvt}.$$

$$= e^{if}\psi, \text{ where } f = -kvt.$$

Case 2 – V (potential) present – Noting that $V'(x',t') = V'(x-vt,t) \equiv V(x,t)$, the same result follows as in Case 1, since $\nabla'\psi'$ cancels with $\nabla\psi$ in the Schrodinger eqn. –

General postulates re. eigenfunctions and eigenvalues:

(i) Each dynamical variable relating to the motion of a particle can be represented by a linear operator (Eg. $\vec{r}$, $-i\hbar\,\mathrm{grad}$), say Ω, and with Ω can be associated a linear eigenvalue equation $\Omega u_\omega = \omega \cdot u_\omega$ (reality of eigenvalues).

(ii) One or another of the eigen-values ω is the only possible result of a precise measurement of the dynamical variable represented by Ω. (reality of eigenvalues)

(iii) Any wave function ψ can be expanded in terms of u_ω, and the number of measurements that result in the eigenvalue ω is $\propto$ to the square of the magnitude of the coefficient of u_ω in the expansion of ψ – A Corollary is that if $\psi = u_\omega$, we are certain to measure ω. (Probability fn)

– Consideration of two operators viz. total energy operator, and momentum operator.

(1) Total energy operator given by $i\hbar\partial/\partial t$ or $-\frac{\hbar^2}{2m}\nabla^2 + V(r)$, with V not being a fn of t in view of $[\psi = u(r)f(t)]$. $\Delta t \cdot \Delta E \simeq \hbar$ (ie a precise measurement of total energy of a particle cannot be made in a finite length of time) – Case of discrete & continuous eigen values: $E<0$, $E>0$.*

(a) orthonormality of energy eigenfunctions – ie eigenfunctions corresponding to two different eigen values (discrete case) are orthogonal ie $\int \bar{u}_{E'} u_E \cdot d\tau = 0$.

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right] u_E(\vec{r}) = E u_E(\vec{r})$$

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right] \bar{u}_{E'}(\vec{r}) = E' \bar{u}_{E'}(\vec{r})$$

Multiply resply by $\bar{u}_{E'}(\vec{r})$ and $u_E(\vec{r})$, subtract and integrate over all space

$$-\frac{\hbar^2}{2m}\int\left(\bar{u}_{E'}\nabla^2 u_E - u_E \nabla^2 \bar{u}_{E'}\right) d\tau = (E-E')\int \bar{u}_{E'} u_E \cdot d\tau$$

By Green's theorem L.H.S integral $= \int \mathrm{div}\left(\bar{u}_{E'}\,\mathrm{grad}\,u_E - u_E\,\mathrm{grad}\,\bar{u}_{E'}\right) d\tau$ *

$$= \int\left(\bar{u}_{E'}\,\mathrm{grad}\,u_E - u_E\,\mathrm{grad}\,\bar{u}_{E'}\right)_n d\sigma = 0$$

* (2) can be interpreted as $E = \frac{1}{2m}\langle p^2 \rangle + \langle V \rangle$

↳ the reality proof shows that $\langle p^2 \rangle$ cannot be negative.

$\therefore$ If $E \neq E'$, $\int \bar{u}_{E'} u_E \, d\tau = 0$.

~~Degenerate energy eigen-value~~

31st Lecture – 30/9/63 – Remaining portion of previous lecture.

~~Degenerate eigenvalues eigen~~ – orthonormal set of eigen functions –

~~(a) Reality of the energy eigenvalues – Assumed in~~ (ii)

(b) Expansion in energy eigen-functions – completeness – closure property

(c) Probability & expectation value –

32nd Lecture – (i) Closure property using Dirac's delta function ie $\sum_E \bar{u}_E(\vec{r}') \, u_E(\vec{r}) = \delta(\vec{r}-\vec{r}')$

so that $\sum_E \quad = 0$ if $r \neq r'$

and $\int \left[\sum \quad\right] . d\tau' = \int \delta(\vec{r}'-\vec{r}) \, d\tau' = \int \delta(x'-x) dx' \int \delta(y'-y) dy' \int \delta(z'-z) dz'$

$= 1$. (if volume of integration includes $\vec{r}$)

(ii) – Reality of eigen-values of energy verified :

we have using $\operatorname{div}(\bar{u}_E \operatorname{grad} u_E) = \bar{u}_E \nabla^2 u_E + \operatorname{grad} \bar{u}_E \cdot \operatorname{grad} u_E$ – (1)

we can obtain from $\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right] \bar{u}_E(r) = E u_E(\vec{r})$

on multiplying both sides by $\bar{u}_E(r)$ & integrating & using fact that u_E is normalised

$$-\frac{\hbar^2}{2m}\int \bar{u}_E \nabla^2 u_E \, d\tau + \int V(\vec{r}) \left|u_E\right|^2 d\tau = \int E \left|u_E\right|^2 d\tau = E \quad \text{– (2)}^*$$

In (2) the second integral is real, and from (1) the first integral

$$\int \bar{u}_E \nabla^2 u_E \cdot d\tau = \int \operatorname{div}(\bar{u}_E \operatorname{grad} u_E) d\tau - \int \operatorname{grad} \bar{u}_E \cdot \operatorname{grad} u_E \cdot d\tau \quad \text{– (3)}$$

The second integral in (3) is by Green's theorem $= \int u_E (\operatorname{grad} u_E)_n \, d\sigma = 0$

& the third integral is obviously real & hence E is real.

(iii) – Probability for & expectation value –

The third postulate states that the probability of finding a particular value for E when wave fn is described by $\psi(\vec{r})$ is $\propto |A_E|^2$. If $P(E) = K \cdot |A_E|^2$

$$\sum_E P(E) = K \sum_E |A_E|^2 = K \sum_E \int \bar{u}_E(r) \bar{\psi}(r) d\tau \int u_E(r') \bar{\psi}(r') d\tau'$$

$$= K \iint \bar{\psi}(r') \bar{\psi}(r) \left[\sum_E \bar{u}_E(r) u_E(r')\right] d\tau . d\tau'$$

$$= K \iint \bar{\psi}(r') \bar{\psi}(r) \, \delta(r-r') \, d\tau \, d\tau'$$

$$= K \int \bar{\psi}(r) \psi(r) \, d\tau = K \int |\psi(r)|^2 d\tau = K \quad [\psi \text{ normalised}]$$

but $\sum_E P(E) = 1$; hence $K = 1$

To find the expectation value $\langle E \rangle$, $\langle E \rangle = \sum_E E P(E)$

$$= \sum_E \int E \bar{u}_E(r) \psi(r) d\tau \int u_E(r') \bar{\psi}(r') \cdot d\tau'$$

$$\int E \bar{u}_E(r) \psi(r) \cdot d\tau = \int \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right) \bar{u}_E(r) \psi(r)$$

Using $\operatorname{div}(\psi \operatorname{grad} \bar{u}_E) = \psi \nabla^2 \bar{u}_E + \operatorname{grad}\psi \cdot \operatorname{grad} \bar{u}_E$

the integral $\int \nabla^2 u_E \psi \, d\tau - \int \operatorname{grad}\psi \cdot \operatorname{grad} \bar{u}_E \cdot d\tau$, the surface integral vanishing

Again $\operatorname{div}(\bar{u}_E \operatorname{grad}\psi) = \bar{u}_E \nabla^2 \psi + \operatorname{grad} \bar{u}_E \cdot \operatorname{grad}\psi$,

$$\int \nabla^2 \bar{u}_E \psi \cdot d\tau = \int \bar{u}_E \nabla^2 \psi \cdot d\tau, \text{ the surface integral vanishing}$$

Hence $\int E \bar{u}_E \psi(r) d\tau = \int \bar{u}_E \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right) \psi(r) d\tau$.

$$\therefore \langle E \rangle = \sum_E \int \bar{u}_E(r) \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right) \psi(r) d\tau \int u_E(r') \bar{\psi}(r') d\tau'$$

$$= \iint \bar{\psi}(r') \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right) \psi(r) \left[\sum_E \bar{u}_E(r) u_E(r')\right] d\tau \cdot d\tau'$$

$$= \iint \bar{\psi}(r') \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right) \psi(r) \cdot \delta(r - r') \, d\tau \cdot d\tau.$$

$$= \int \bar{\psi}(r) \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right) \psi(r) \, d\tau. \quad = \int \bar{\psi}(r) E \psi(r) d\tau.$$

Verification of correctness of Ehrenfest's theorem.

~~(iv) Properties of Dirac δ-function~~:

33rd lecture – 4/10/63 – Momentum eigen-functions & eigen-values.

Eigen value eqns are $-i\hbar \operatorname{grad} u_p(\vec{r}) = \vec{p} \cdot u_p(\vec{r})$ $\quad [\vec{k} = \vec{p}/\hbar$

or $-i\hbar \operatorname{grad} u_k(\vec{r}) = \hbar \vec{k} u_k(\vec{r})$

or $\operatorname{grad} u_k(\vec{r}) = i\vec{k} u_k(\vec{r})$

leading to $\int \left[\bar{u}_l(\vec{r}) \operatorname{grad} u_k(\vec{r}) - u_k(\vec{r}) \operatorname{grad} \bar{u}_l(\vec{r})\right] d\tau = i(k-l) \int \bar{u}_l(\vec{r}) u_k(\vec{r}) d\tau$

L.H.S $= \int \operatorname{div}\{\bar{u}_l(\vec{r}) \operatorname{grad} u_k(\vec{r}) - u_k(\vec{r}) \operatorname{grad} \bar{u}_l(\vec{r})\} d\tau = 0$

$\therefore (k-l) \int \bar{u}_l(\vec{r}) u_k(\vec{r}) d\tau = 0$

& since k & l are continuous variables,

(i) the orthonormality conditions are $\int \bar{u}_l(\vec{r}) u_k(\vec{r}) d\tau = \delta(\vec{k} - \vec{l})$

(ii) ~~Closure property~~ Completeness relation: Unlike as in the case of energy, we have to replace Σ by $\int$ and write $\psi(r) = \int f(k) u_k(\vec{r}) \cdot d\vec{k}$ as the completeness relation.

Multiplying both sides by $\bar{u}_l(\vec{r})$ and integrating over whole space, we get

$$\int \bar{u}_l(\vec{r}) \psi(\vec{r}) d\tau = \iint f(\vec{k}) \bar{u}_l(\vec{r}) u_k(r) \, d\tau \cdot d\vec{k}$$

$$= \int f(\vec{k}) \delta(\vec{k} - \vec{l}) \cdot d\vec{k}, \text{ using orthonormality relation}$$

$$= f(\vec{l}), \text{ using property of the Dirac } \delta\text{-fn}$$

(iii) Closure property : The completeness relation can now be written as

$$\psi(r) = \iint \bar{u}_K(r')\,\psi(r')\,u_K(\vec{r})\,dK\,d\tau$$
$$= \int \psi(r')\,d\tau \int \bar{u}_K(\vec{r}')\,u_K(\vec{r})\cdot dK$$

and this is satisfied by $\int \bar{u}_K(r')\,u_K(\vec{r})\,dK = \delta(\vec{r}-r')$, which is the closure property.

(iv) Probability function and expectation value :

Here $P(\vec{k}) \propto |f(\vec{k})|^2$ where $\psi(\vec{r}) = \int f(\vec{k})\,u_K(\vec{r})\cdot d\vec{k}$.

and $\int P(\vec{k})\cdot d\vec{k} = 1$, and taking constant of proportionality as K_0 we have

$$1 = K_0 \int |f(\vec{k})^2|\cdot dK = K_0 \int \bar{u}_K(\vec{r})\,\psi(r)\,d\tau \int u_K(r')\,\bar{\psi}(r')\,d\tau'$$
$$= K_0 \iint \bar{\psi}(r')\,\psi(r)\,\delta(\vec{r}-\vec{r}')\,d\tau'\,d\tau$$
$$= K_0 \int \bar{\psi}(r)\,\psi(r)\,d\tau = K_0 \quad \text{ie } K_0 = 1.$$

Re. expectation value

$$\langle p\rangle = \hbar \int \vec{K}\,P(\vec{K})\,dK$$

$$= \hbar \iint \vec{K}\,\bar{u}_K(r)\,\psi(r)\,d\tau\,dK \int u_K(r')\,\bar{\psi}(r')\,d\tau'$$

$$= i\hbar \iint \operatorname{grad}\bar{u}_K(r)\,\psi(r)\,d\tau\cdot dK \int u_K(r')\,\bar{\psi}(r')\,d\tau'$$

$$= -i\hbar \iint \bar{u}_K(r)\operatorname{grad}\psi(r)\,dK \int u_K(r')\,\bar{\psi}(r')\,d\tau'$$

$$= -i\hbar \iint \bar{\psi}(r')\operatorname{grad}\psi(r)\,\delta(r-r')\,d\tau\,d\tau'$$

$$= -i\hbar \int \bar{\psi}(r)\operatorname{grad}\psi(r)\,d\tau$$

$-i\hbar \operatorname{grad} u_p(r) = p\cdot u_p(r)$
$= \hbar k\,u_p(r)$
ie $-i\operatorname{grad} u_K(r) = K\,u_K(r)$
$i\operatorname{grad}\bar{u}_K(r) = K\,\bar{u}_K(r)$

12/11/63 – After the Dasara & Diwali vacation – 34th Lecture.

Linear harmonic oscillator – eqn $-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + \frac{1}{2}Kx^2u = Eu$, with $F = -Kx$ & $K = -\frac{\partial F}{\partial}$ $F = -\frac{\partial V}{\partial x}$ –

Derivation of a dimensionless eqn with $\xi = \alpha x$ & $\alpha = L^{-1}$ is possible with $\alpha^4 = \frac{mK}{\hbar^2}$, $\lambda = \frac{2E}{\hbar}\left(\frac{m}{K}\right)^{\frac{1}{2}} = \frac{2E}{\hbar\omega_c}$,

as could be dimensionally verified from $\frac{d^2u}{d\xi^2} - \frac{mK}{\hbar^2\alpha^4}\xi^2 u = -\frac{2mE}{\hbar^2\alpha^2}u$ — for large ξ, considering leading terms $u = \xi^n e^{\pm\frac{1}{2}\xi^2}$ is a soln ie retaining only the term $\xi^2 u$ [ie $\frac{d^2u}{d\xi^2} + (\lambda - \xi^2)u = 0$] since highest order terms in $\frac{d^2u}{d\xi^2}$ & $\xi^2 u$ are ξ^{n+2} – Boundary cond^n requires $u \to 0$ as $\xi \to \infty$ ie $e^{-\frac{1}{2}\xi^2}$ is to be chosen

Hence suggestion of $u = H(\xi)e^{-\frac{1}{2}\xi^2}$ as a possible exact soln with $H(\xi)$ a finite polynomial – diffl eqn for H ~~recurrence relations~~ – Defn of Hermite polynomials by $e^{-t^2+2t\xi} = \sum_{n=0}^{\infty}\frac{H_n(\xi)}{n!}t^n$ or by $H_n(\xi) = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}(e^{-\frac{1}{2}x^2})$ satisfying diffl eqn $H_n'' - 2\xi H_n' + 2nH_n = 0$ & hence $\lambda = 2n+1$ since diffl eqn for H is $H'' - 2\xi H' + (\lambda-1)H$ leading to eigenvalue $E_n = (n+\frac{1}{2})\hbar\omega_c$ $(n = 0, 1, 2, \ldots)$ –

Eigen-functions $u_n(\xi) = H_n(\xi)e^{-\frac{1}{2}\xi^2} \times$ constant $= N_n H_n(\xi)e^{-\frac{1}{2}\xi^2}$ – Integ relatn $\int_{-\infty}^{\infty} H_n^2(\xi)e^{-\xi^2}d\xi = \pi^{\frac{1}{2}}2^n n!$

$\int_{-\infty}^{\infty} H_n(\xi)H_m(\xi)e^{-\xi^2}d\xi = 0$ $(m \neq n)$ $\to N_n = (\alpha/\pi^{\frac{1}{2}}2^n n!)^{1/2}$ from $\int_{-\infty}^{\infty}|u_n(x)|^2dx = 1$ (normalisation)

35th Lecture – 13/11/63 – Remaining portions of last lecture – Derivation of properties of $H_n(\xi)$ from the expression

(*) In particular, $H_0(\xi) = 1$, $H_1(\xi) = -e^{\xi^2}\frac{d}{d\xi}(e^{-\xi^2}) = -e^{\xi^2}\cdot e^{-\xi^2}\cdot(-2\xi) = 2\xi$

$$H_2(\xi) = e^{\xi^2}\frac{d^2}{d\xi^2}\left(e^{-\xi^2}\right) = e^{\xi^2}\left\{-2e^{-\xi^2} + 4\xi^2 e^{-\xi^2}\right\} = 4\xi^2 - 2$$

$H_3(\xi) = 8z^3 - 12z$, $H_4(z) = 16z^4 - 48z^2 + 12$, $H_5(z) = 32z^5 - 160z^3 + 120z$

defn viz $e^{-t^2+2t\xi} = \sum_{n=0}^{\infty} \frac{H_n(\xi)}{n!} t^n$. Calling L.H.S as $S(\xi, t)$

$\frac{\partial S}{\partial \xi} = 2tS = \sum \frac{H_n(\xi)}{n!} 2t^{n+1}$ and also $= \sum \frac{H_n'(\xi)}{n!} t^n$

& equating powers of t^n in the two expansions for $\frac{\partial S}{\partial \xi}$ gives $\frac{2H_{n-1}(\xi)}{(n-1)!} = \frac{H_n'(\xi)}{n!}$ ie $H_n' = 2n H_{n-1}$ — (A)

Using $\frac{\partial S}{\partial t} = (-2t+2\xi) S = \sum (-2t+2\xi) \frac{H_n(\xi)}{n!} t^n$ and also $= \sum \frac{H_n(\xi)}{n!} n t^{n-1} = \sum \frac{H_n(\xi)}{(n-1)!} t^{n-1}$

& equating powers of t^n in the two expansions for $\partial S/\partial t$ gives

$$-\frac{2H_{n-1}(\xi)}{(n-1)!} + \frac{2\xi H_n(\xi)}{n!} = \frac{H_{n+1}(\xi)}{n!} \text{ or } H_{n+1} = 2\xi H_n - 2n H_{n-1} \quad \text{— (B)}$$

~~From (B) by differentiation $H_{n+1}' = 2\xi H_n' + 2H_n - 2n$~~ from (A) & (B), $H_{n+1}' = 2\xi H_n - H_n'$ — (C)

Differentiating (C) gives $H_{n+1}' = 2\xi H_n' + 2H_n - H_n''$

or using (A), ~~$2(n+1)H_n = 2\xi H_n' + 2H_n - H_n''$~~ $2(n+1)H_n = 2\xi H_n' + 2H_n - H_n''$

or $H_n'' - 2\xi H_n' + 2n H_n = 0$ — (D) the diff eqn for H_n.

To derive the second explicit defn of H_n, ~~explicit~~, we note $\left(\frac{\partial^n S}{\partial t^n}\right)_{t=0} = H_n(\xi)$, since terms in $t^{n+1}, t^{n+2}, \ldots$ after diffn n times leave terms in $t, t^2, \ldots$ which vanish when $t=0$. Writing $S = e^{\xi^2} \cdot e^{-(t-\xi)^2}$, we have

$$\frac{\partial^n S}{\partial t^n} = e^{\xi^2} \cdot \frac{\partial^n}{\partial t^n} \left\{ e^{-(t-\xi)^2} \right\} = e^{\xi^2} \cdot (-1)^n \frac{\partial^n}{\partial \xi^n} \left\{ e^{-(t-\xi)^2} \right\}$$ since for a function $f(t-\xi)$,

one has $\partial f/\partial t = -\partial f/\partial \xi$. Hence $\left(\frac{\partial^n S}{\partial t^n}\right)_0 = (-1)^n e^{\xi^2} \frac{\partial^n}{\partial \xi^n} (e^{-\xi^2}) = H_n(\xi)$ (*).

To derive the orthogonality relations, take the product of two series for $e^{-s^2+2s\xi}$ & $e^{-t^2+2t\xi}$ & integrate after multiplying by $e^{-\xi^2}$ getting

$$\int_{-\infty}^{\infty} e^{-s^2+2s\xi} \cdot e^{-t^2+2t\xi} \cdot e^{-\xi^2} \cdot d\xi = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{s^n t^m}{n!\, m!} \int_{-\infty}^{\infty} H_n(\xi) H_m(\xi) e^{-\xi^2} \cdot d\xi.$$

L.H.S $= \int_{-\infty}^{\infty} e^{2st} \cdot e^{-(s+t)^2} \cdot e^{2(s+t)\xi} \cdot e^{-\xi^2} \cdot d\xi = e^{2st} \int_{-\infty}^{\infty} e^{-\{\xi-(s+t)\}^2} \cdot d\xi = e^{2st} \int_{-\infty}^{\infty} e^{-\eta^2} d\eta$

$= \sqrt{\pi} \cdot e^{2st} = \sqrt{\pi} \cdot \sum_{n=0}^{\infty} \frac{(2st)^n}{n!}$

equating equal powers of s and t, we have $\int_{-\infty}^{\infty} H_n(\xi) H_m(\xi) e^{-\xi^2} d\xi = 0$, if $n \neq m$

~~and~~ $\frac{1}{(n!)^2} \int_{-\infty}^{\infty} H_n^2(\xi) e^{-\xi^2} d\xi = \sqrt{\pi} \cdot 2^n \cdot \frac{1}{n!}$

or $\int H_n^2(\xi) e^{-\xi^2} d\xi = \sqrt{\pi} \cdot 2^n \cdot n!$

~~&~~ For the case $n \neq m$, orthogonality of non-degenerate energy eigenfunctions follows, since here there is no degeneracy @ as seen from $E_n = (n+\tfrac{1}{2})\hbar\omega_c$ ie $E_n \neq E_m$ if $n \neq m$.

~~Expectation value of the coordinate $x = \int_{-\infty}^{\infty}$~~

36th lecture — 14/11/63 — Remaining portions of previous lecture — question raised by a student re. 1st defn being derived from second defn being obvious since coeff of $\frac{t^n}{n!}$ in expn of $S(t, \xi)$ in powers of $t = \left(\frac{\partial^n S}{\partial t^n}\right)_{t=0}$ and this is shown above $= (-1)^n e^{\xi^2} \frac{\partial^n}{\partial \xi^n} (e^{-\xi^2}) = H_n(\xi)$ from the second defn — ~~Calculation of expectation~~

~~redraw~~ — Polynomial method to prove that $H(\xi)$ in $u = H(\xi)e^{-\frac{1}{2}\xi^2}$ must be the Hermite $H_n(\xi)$.

Assuming $H(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots + a_n\xi^n$, so that

$$H'(\xi) = a_1 + 2a_2\xi + \cdots + na_n\xi^{n-1}$$

$$H''(\xi) = 2a_2 + 2\cdot 3a_3\xi + \cdots + n(n-1)a_n\xi^{n-2}$$

and substituting in $H'' - 2\xi H' + (\lambda-1)H = 0$ & equating coeffts of all powers of ξ to 0, since this must be true for all values of ξ, we have

$$2a_2 + (\lambda-1)a_0 = 0$$

$$6a_3 - 2a_1 + (\lambda-1)a_1 = 0$$

$$(n-1)(n-2)a_{n-1} - 2(n-3)a_{n-3} + (\lambda-1)a_{n-3} = 0$$

$$n(n-1)a_n - 2(n-2)a_{n-2} + (\lambda-1)a_{n-2} = 0$$

$$-2(n-1)a_{n-1} + 2(n-1)a_{n-1} + (\lambda-1)a_{n-1} = 0$$

$$-2na_n + (\lambda-1)a_n = 0$$

If $a_n \neq 0$, the last eqn gives $\lambda - 1 = 2n$ or $\lambda = 2n+1$ & hence the diffl eqn shows $H(\xi) = H_n(\xi)$ & now the last but one eqn gives $a_{n-1} = 0$ since $\lambda \neq 2n-1$ & it can be shown that $a_{n-3} = a_{n-5} = \cdots = 0$. and this merely verifies the correct form of $H_n(\xi)$.

— Notion of parity. For the general case of wave eqn with $V(x)$ symmetric about $x=0$ ie $V(-x) = V(x)$,

$$-\frac{\hbar^2}{2m}\frac{d^2u(x)}{dx^2} + V(x)u(x) = Eu(x)$$

changing $x \to -x$, $\quad -\frac{\hbar^2}{2m}\frac{d^2u(-x)}{dx^2} + V(x)u(-x) = Eu(-x)$ since $V(-x) = V(x)$

ie $u(x)$ & $u(-x)$ satisfy the same wave eqn & correspond to same eigenvalue E. Hence if there be no degeneracy, $u(x) = \epsilon u(-x)$ where ϵ is a const.

ie $u(-x) = \epsilon u(x)$ giving $\epsilon^2 = 1$ on multiplication ie $\epsilon = \pm 1$

ie $u(x) = u(-x)$ or $u(x) = -u(-x)$ ie all eigenfns of symmetric V are either even or odd for changing sign in x ie they are of even or odd parity.

In the case of L.H.O, $e^{-\xi^2/2}$ is always even & $H_n(\xi)$ is odd or even according as n is odd or even ie $u_n(x)$ is of odd or even parity according as n is odd or even, since there is no degeneracy in this case.

— Calculation of expectation values. (i) $\langle x\rangle = \int \bar{\psi}\, x\, \psi\, d\tau$ and here $= \int_{-\infty}^{\infty} \bar{u}_n(x)\, x\, u_n(x)\cdot dx$.

First calculate $\int_{-\infty}^{\infty} u_n(x)\, x\, u_m(x)\cdot dx = \frac{C_nC_m}{a^2}\int \xi\, H_n(\xi)\, H_m(\xi)\, e^{-\xi^2}\cdot d\xi$.

Using generating function for $H_n(\xi)$, $\int_{-\infty}^{\infty} e^{-s^2+2s\xi}\cdot e^{-t^2+2t\xi}\cdot \xi e^{-\xi^2} d\xi = \sum\sum \frac{s^n t^m}{n!\,m!}\int_{-\infty}^{\infty} \xi\, H_n(\xi)\, H_m(\xi)\, e^{-\xi^2} d\xi$

$$\text{L.H.S} = e^{2st}\int_{-\infty}^{\infty} \xi\, e^{-\{\xi-(s+t)\}^2}\cdot d\xi = e^{2st}\int_{-\infty}^{\infty}\{\eta + (s+t)\}e^{-\eta^2} d\eta, \text{ putting } \xi-(s+t) = \eta$$

$$\int_{-\infty}^{\infty} \eta\, e^{-\eta^2} d\eta = -\tfrac{1}{2}e^{-\eta^2}\Big]_{-\infty}^{\infty} = 0. \quad \therefore \text{L.H.S} = (s+t)e^{2st}\int e^{-\eta^2} d\eta = \pi^{\frac{1}{2}}(s+t)e^{2st}$$

$$= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{2^n (s^{n+1}t^n + s^n t^{n+1})}{n!}$$

Equating coefficients of $s^n t^m$ on both the L.H.S & R.H.S. shows that $\int_{-\infty}^{\infty} \xi H_n(\xi) H_m(\xi) e^{-\xi^2} d\xi = 0$, except for the cases $m = n+1$ or $m = n-1$. Hence it follows immediately that $\langle x \rangle = 0$. We can now evaluate $\int_{-\infty}^{\infty} u_n(x) \cdot x\, u_m(x)\, dx$ for the cases $m = n+1$ and $m = n-1$

(i) Equating coeffts of $s^n t^{n+1}$, $\frac{\sqrt{\pi} \cdot 2^n}{n!} = \frac{1}{n!(n+1)!} \int_{-\infty}^{\infty} \xi\, H_n(\xi) H_{n+1}(\xi) e^{-\xi^2} d\xi$ or $\int_{-\infty}^{\infty} = \sqrt{\pi} \cdot 2^n (n+1)!$

(ii) Equating coeffts of $s^n t^{n-1}$, $\frac{\sqrt{\pi} \cdot 2^{n-1}}{(n-1)!} = \frac{1}{n!(n-1)!} \int_{-\infty}^{\infty} \xi\, H_n(\xi) H_{n-1}(\xi) e^{-\xi^2} d\xi$ or $\int_{-\infty}^{\infty} = \sqrt{\pi} \cdot 2^{n-1} (n!)$.

Hence, Now $\int_{-\infty}^{\infty} u_n(x)\, x\, u_{n+1}(x)\, dx = \frac{C_n C_{n+1}}{\alpha^2} \cdot \sqrt{\pi} \cdot 2^n (n+1)! = \frac{1}{\alpha} \left(\frac{n+1}{2}\right)^{1/2}$, using $C_n = \left(\frac{\alpha}{\sqrt{\pi} \cdot 2^n \cdot n!}\right)^{1/2}$

and $\int_{-\infty}^{\infty} u_n(x)\, x\, u_{n-1}(x)\, dx = \frac{C_n C_{n-1}}{\alpha^2} \sqrt{\pi} \cdot 2^{n-1} \cdot n! = \frac{1}{\alpha} \left(\frac{n}{2}\right)^{1/2}$, "

and $\int_{-\infty}^{\infty} u_n(x)\, x\, u_m(x)\, dx = 0$ for all other cases.

(ii) $\langle V \rangle = \int_{-\infty}^{\infty} u_n(x) \tfrac{1}{2} K x^2 u_n(x)\, dx$

$$\int_{-\infty}^{\infty} e^{-s^2 + 2s\xi} \cdot e^{-t^2 + 2t\xi} \cdot \xi^2 e^{-\xi^2} d\xi = \sum\sum \frac{s^n t^m}{n!\, m!} \int_{-\infty}^{\infty} \xi^2 H_n(\xi) H_m(\xi) e^{-\xi^2} d\xi$$

$$\text{L.H.S} = \int_{-\infty}^{\infty} e^{2st} \cdot \xi^2 \cdot e^{-\{\xi - (s+t)\}^2} d\xi = e^{2st} \int_{-\infty}^{\infty} \{\eta + (s+t)\}^2 e^{-\eta^2} d\eta, \quad \eta = \xi - (s+t)$$

$$= e^{2st} \int_{-\infty}^{\infty} \{\eta^2 + 2(s+t)\eta + (s+t)^2\} e^{-\eta^2} d\eta$$

$$\int_{-\infty}^{\infty} \eta^2 e^{-\eta^2} d\eta = \int_{-\infty}^{\infty} (-\tfrac{1}{2}\eta)(-2\eta e^{-\eta^2})\, d\eta = -\tfrac{1}{2}\eta e^{-\eta^2} \Big]_{-\infty}^{\infty} + \int \tfrac{1}{2} e^{-\eta^2} d\eta = \tfrac{1}{2} \int e^{-\eta^2} d\eta = \frac{\sqrt{\pi}}{2}$$

& $\int \eta e^{-\eta^2} d\eta = 0$.

$$\therefore \text{L.H.S} = e^{2st} \left\{ \frac{\sqrt{\pi}}{2} + (s+t)^2 \sqrt{\pi} \right\} = \frac{\sqrt{\pi}}{2} \sum \frac{2^n s^n t^n}{n!} + \sqrt{\pi} \sum \frac{2^n (s^{n+2} t^n + 2 s^{n+1} t^{n+1} + s^n t^{n+2})}{n!}$$

Equating powers of $s^n t^n$ on both L.H.S and R.H.S.

$$\frac{1}{n!} \cdot \frac{1}{n!} \int_{-\infty}^{\infty} \xi^2 H_n^2(\xi) e^{-\xi^2} d\xi = \frac{\sqrt{\pi}}{2} \cdot \frac{2^n}{n!} + \sqrt{\pi} \cdot 2^{n-1} \frac{2}{(n-1)!}$$

$$\text{or} \int \xi^2 H_n^2(\xi) e^{-\xi^2} d\xi = \frac{\sqrt{\pi}}{2} \cdot 2^n \cdot n! + \sqrt{\pi} \cdot 2^n \cdot n!\, n = \frac{\sqrt{\pi}}{2} \cdot 2^n \cdot n! (2n+1)$$

$$\therefore \int u_n(x)\, x^2 u_n(x)\, dx = \frac{C_n^2}{\alpha^3} \int \xi^2 H_n^2(\xi) e^{-\xi^2} d\xi = \frac{\sqrt{\pi}}{2} \cdot 2^n \cdot n! (2n+1) \cdot \frac{1}{\alpha^3} \cdot \frac{\alpha}{\sqrt{\pi} \cdot 2^n \cdot n!}$$

$$= \frac{2n+1}{2\alpha^2}$$

$$\therefore \langle V \rangle_n = \tfrac{1}{2} K \left(\frac{2n+1}{2\alpha^2}\right) = \tfrac{1}{2}(n+\tfrac{1}{2}) \hbar \omega_c, \text{ because } \frac{K}{\alpha^2} = \hbar \left(\frac{K}{m}\right)^{\frac{1}{2}} = \hbar \omega_c \text{ from } \alpha^4 = \frac{mK}{\hbar^2}.$$

$$= \tfrac{1}{2} E_n$$

[from Kompaneyets, p. 61]

* i.e. $\ddot{x} = -\omega_c^2 x$, $x = a\cos\omega_c t$, $\dot{x} = -a\omega_c \sin\omega_c t$

Average values of K.E & P.E ~~are given by~~ during a period are given by

$$\langle K.E\rangle = \frac{1}{2} m a^2 \omega_c^2 \cdot \overline{\sin^2 \omega_c t} = \frac{1}{2} m a^2 \omega_c^2 \cdot \frac{1}{2}$$

$$\langle P.E\rangle = \frac{1}{2} K a^2 \overline{\cos^2 \omega_c t} = \frac{1}{2} K a^2 \cdot \frac{1}{2}$$

i.e. $\langle K.E\rangle = \langle P.E\rangle$ since $K = m\omega_c^2$.

$$\overline{\sin^2 \omega_c t} = \frac{\omega_c}{2\pi} \int_0^{2\pi/\omega_c} \sin^2 \omega_c t \cdot dt = \frac{1}{2\pi} \int_0^{2\pi} \sin^2\theta \cdot d\theta = \frac{1}{2\pi} \cdot 4 \int_0^{\pi/2} \sin^2\theta \cdot d\theta$$

$$= \frac{1}{2\pi} \cdot 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{1}{2}$$

Simly $\overline{\cos^2 \omega_c t} = 1/2$.

* Did not take the class today

Just as in the classical case viz $m\ddot{x} = -Kx$*, ~~whose first integration from $m\ddot{x}\dot{x} = -Kx\dot{x}$~~,

~~$\frac{1}{2}m\dot{x}^2 = -\frac{1}{2}Kx^2$ i.e. K.E = P.E = $\frac{1}{2}$ total energy~~,

here also the same result holds for expectation values.

(iii) $\langle p \rangle$ – from the general relation $\langle p \rangle = -i\hbar \int \bar{\psi} \frac{\partial \psi}{\partial x} d\tau$, we have here $\langle p \rangle = -i\hbar \int u_n(x) \frac{du_n(x)}{dx} \cdot dx$

which leads to calculation of $\int H_n(\xi) H_n'(\xi) e^{-\xi^2} d\xi = \int \xi H_n^2(\xi) e^{-\xi^2} d\xi$ & second term $= 0 \therefore \langle x \rangle = 0$

and 1st term ~~has this~~ $= 2n \int H_n(\xi) H_{n-1}(\xi) e^{-\xi^2} d\xi = 0$ from orthonormality condn

i.e $\langle p \rangle = 0$.

(iv) Uncertainty relation between x & p: Uncertainty principle states $\Delta x \cdot \Delta p \gtrsim \hbar$ i.e product is of order of $\hbar$. This can be made more precise by defining $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$ and $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2$ and leads in general to the minimum uncertainty product or a precise Heisenberg relation $\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar$. (to be assumed without proof). In the case of the L.H.O, $(\Delta x)^2 = \langle x^2 \rangle$ & $(\Delta p)^2 = \langle p^2 \rangle$

To calculate $\langle p^2 \rangle$, we have ~~$\langle p^2 \rangle$~~ = from $p = -i\hbar \frac{\partial}{\partial x}$, $\langle p^2 \rangle = -\hbar^2 \int_{-\infty}^{\infty} u_n(x) \frac{d^2 u_n(x)}{dx^2} \cdot dx$

$= -\hbar^2 \alpha \int_{-\infty}^{\infty} u_n(\xi) \frac{d^2 u_n(\xi)}{d\xi^2} d\xi$

$= \hbar^2 \alpha \cdot \int_{-\infty}^{\infty} u_n (\lambda - \xi^2) u_n \cdot d\xi$, substituting from the diffl eqn $\frac{d^2u}{d\xi^2} + (\lambda - \xi^2) u = 0$

$= \hbar^2 \alpha C_n^2 \int_{-\infty}^{\infty} \{(2n+1) H_n^2 e^{-\xi^2} - \xi^2 H_n^2 e^{-\xi^2}\} d\xi$, putting $\lambda = 2n+1$.

$= \hbar^2 \alpha C_n^2 \left\{ (2n+1)\sqrt{\pi} \cdot 2^n \cdot n! - \frac{\sqrt{\pi}}{2} 2^n \cdot n! (2n+1) \right\}$

$= \hbar^2 \alpha C_n^2 \cdot \frac{\sqrt{\pi}}{2} \cdot 2^n \cdot n! (2n+1) = \hbar^2 \alpha^2 \frac{1}{2}(2n+1)$, using $C_n^2 = \frac{\alpha}{\sqrt{\pi} \cdot 2^n \cdot n!}$

$\therefore (\Delta x)^2 (\Delta p)^2 = \langle x^2 \rangle \langle p^2 \rangle = \frac{2n+1}{2\alpha^2} \cdot \hbar^2 \alpha^2 \cdot \frac{(2n+1)}{2} = (2n+1)^2 \cdot \frac{\hbar^2}{4}$

i.e $\Delta x \cdot \Delta p = (n + \frac{1}{2})\hbar$, giving the equality sign in the Heisenberg relation.

37th lecture – 18/11/63 – Remaining portions of previous lecture not completed.

38th lecture – 19/11/63 – Remaining portions starting from $\langle p \rangle$ – Preliminary results re. wave eqn in a central field; Separation of the variables in the wave eqn $\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)u = Eu$ in polar coordinates, where $V = V(r)$.

39th lecture – 20/11/63.* – Separation into radial and angular equations – further separation of the angular eqn –

radial eqn: $\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \left\{\frac{2m}{\hbar^2}(E-V) - \frac{\lambda}{r^2}\right\} R = 0$. – – (1)

angular eqn is $\frac{1}{\Theta \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi \sin^2\theta} \cdot \frac{d^2\Phi}{d\phi^2} = -\lambda$

$\frac{\sin\theta}{\Theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \lambda \sin^2\theta = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \nu$, say

giving $\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left(\lambda - \frac{\nu}{\sin^2\theta}\right)\Theta = 0$ – – (2)

$\frac{d^2\Phi}{d\phi^2} + \nu\Phi = 0$ – – – (3)

* Probably on the ground that value of Φ should remain unchanged when $\phi \to \phi + 2s\pi$ (s an integer) requires soln of (3) containing $\cos n\phi$ & $\sin n\phi$ (n an integer); hence $\phi = n^2$.

(*) ~~This is taken~~ From here onwards treatment of Legendre polynomials is based on Tricomi, Diff. Eqns p. 191 onwards.

* It can be shown that when $\lambda \neq l(l+1)$, $F(x^2)$ & $G(x^2)$ are of the order $\log \frac{1+x}{1-x}$.

or $> C x^{2r+1} \log \frac{1+x}{1-x}$ $(2r > \lambda/2)$

& $\to \infty$ for $|x| = 1$. (even for λ)

(3) requires on boundedness & continuity condns* on Φ & $\frac{\partial \Phi}{\partial \phi}$ that $\nu = m^2$ (square of an integer) ie $\Phi = \sqrt{2\pi}\cdot e^{im\phi}$ very normalised. (Domain of ϕ is 0 to 2π)

Next from (2), put $x = \cos\theta$, $\Theta(\theta) = R(y)$ & with $\nu = m^2$, (2) becomes

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0 \quad \text{------} \quad (4)$$

Domain of θ is π & so domain of x is 1 to -1.

39th Lecture — 21/11/63 — (4) Can be written as $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0$ -- (5)

(1–2 P.M & also 3 to 4 P.M)

Consider the equation $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0$ ------ (6) (ie $m = 0$)

Differentiating m times & putting $u = d^m y/dx^m$, this becomes

$$(1-x^2)\frac{d^2u}{dx^2} - 2x(m+1)\frac{du}{dx} + \{\lambda - m(m+1)\}u = 0$$

Substituting $v = (1-x^2)^{\frac{1}{2}m}u$ or $u = (1-x^2)^{-\frac{1}{2}m}v$, this reduces to

$$(1-x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + \left(\lambda - \frac{m^2}{1-x^2}\right)v = 0 \text{ ie } \ldots (7) \text{ ie same as (5)}$$

(*) The singular points of (6) or (7) are $x = \pm 1$, and if we can find value of λ for which (6) has a solution which is bounded as $x \to \pm 1$, we have also a solution of (7) satisfying the same condition.

For (6), assume a soln $y = c_0 + c_1 x + c_2 x^2 + \cdots + c_l x^l + \cdots$

$$\frac{dy}{dx} = c_1 + 2c_2 x + \cdots + l c_l x^{l+1} + \cdots$$

$$\frac{d^2y}{dx^2} = 2c_2 + \cdots + l(l-1)c_l x^{l-2} + (l+1)l c_{l+1} x^{l-1} + (l+2)(l+1)c_{l+2}x^l + \cdots$$

Substituting in (6) we get

$$\sum_{l=0}^{\infty}\left[(l+1)(l+2)c_{l+2} - (l-1)l c_l - 2l c_l + \lambda c_l\right]x^l = 0$$

Since this is to be true for all values of x, equating coeffts of x^l to zero for all l, we have

$$(l+1)(l+2)c_{l+2} - l(l+1)c_l + \lambda c_l = 0$$

ie we have the recurrence relation between the coefficients viz.

$$c_{l+2} = \frac{l(l+1) - \lambda}{(l+1)(l+2)}c_l.$$

Separating $y = c_0 + c_1 x + c_2 x^2 + \cdots$ into odd & even parts, we can write

$$y = AF(x^2) + Bx\,G(x^2) \quad \text{or generally } y = AF(x^2) + Bx\,G(x^2).$$

where $F(x^2) = \sum_{l=0}^{\infty} c_{2l}x^{2l}$; $G(x^2) = \sum_{l=0}^{\infty} c_{2l+1}x^{2l}$

Since $\lim_{l\to\infty}\frac{c_l}{c_{l+2}} = 1$, it follows that both series are convergent & have a radius of convergence $= 1$ ie convergent for $|x| < 1$.

We see that if $\lambda = l(l+1)$, where l is a +ve integer one or other of F & G reduces to a polynomial. It can be shown that if $\lambda \neq l(l+1)$, both F & G tend to ∞ as $x^2 \to 1$* ie physically acceptable soln is given by the case where y reduces to a polynomial (odd or even) ie is a solution of the eqn $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0$, which is the Legendre equation.

(*) This should be $\int_0^{2\pi} e^{(-im+im')\phi} \cdot d\phi$ & is $= 0$ if $m \neq m'$

$= 2\pi$ if $m = m'$.

and the polynomial solns are the Legendre polynomials $P_l(x)$. (of parity l)

Properties of Legendre polynomials $P_n(x)$ (i) Defn in terms of a generating function:

$$(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} h^n P_n(x) = \sum_{n=0}^{\infty} h^n P_n(x). \quad \text{(convergence assumed)}$$

$$P_n = \frac{(2n)!}{2^n \cdot (n!)^2}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\,(2n-1)(2n-3)}x^{n-4} - \cdots\right\}$$

(ii) Rodrigues formula, $P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2-1)^n$.

(iii) Recurrence formulae: (a) $(1-x^2)P_n' = -nxP_n + nP_{n-1}$

(b) $(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$

These can be obtained from the generating function definition.

(iv) Integral properties (orthonormality): $\int_{-1}^{1} P_m(x)P_n(x) = 0$ if $m \neq n$

$= \frac{2}{2n+1}$ if $m = n$

Associated Legendre polynomials — Polynomial solns of (7) where $m \neq 0$. These are given by

$$P_l^m = (1-x^2)^{\frac{1}{2}m}\frac{d^m}{dx^m}P_l(x) \quad (m \leq l)$$

P_l^m is of parity $l-m$ since $(1-x^2)^{\frac{1}{2}m}$ is ~~always~~ of even parity whatever m.

These also have properties similar to (i)–(iv). The orthonormal relations corresponding to (iv) are

$$\int_{-1}^{1} P_n^m(x)P_r^m(x)\,dx = 0 \quad (r \neq n$$

$$= \frac{2}{2n+1}\cdot\frac{(n+m)!}{(n-m)!} \quad \text{if } r = n.$$

Spherical Harmonics — If $Y = \Theta\Phi$, the angular eqn is

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \lambda Y = 0$$

Solutions of this are spherical harmonics, when $\lambda = l(l+1)$. and are given by

$$Y_{lm}(\theta,\phi) = N_{lm}P_l^m(\cos\theta)(2\pi)^{-\frac{1}{2}}e^{im\phi} \quad (N_{lm} \text{ is normalisation const for associated Legendre polynomials}).$$

& $N_{lm} = \left\{\frac{2l+1}{2}\cdot\frac{(l-m)!}{(l+m)!}\right\}^{1/2}$ $\qquad Y_{lm}(\theta,\phi) = \left\{\frac{2l+1}{4\pi}\cdot\frac{(l-m)!}{(l+m)!}\right\}^{1/2}P_l^m(\cos\theta)e^{im\phi}$

Y_{lm}

40th Lecture — ~~25/11/63~~ — 26/11/63 — Relations of orthonormality for spherical harmonics are

$$\int_0^{\pi}\int_0^{2\pi} Y_{lm}(\theta,\phi)\,Y_{l'm'}(\theta,\phi)\sin\theta\,d\theta\,d\phi = 0, \text{ unless } l = l' \text{ and } m = m'$$

the ϕ part part vanishes when $m \neq m'$, whatever l, since $\int_0^{2\pi} e^{i(m+m')\phi}\,d\phi = 0$, whatever l (*)

and the θ part vanishes when $l \neq l'$ and $m = m'$, in view of orthonormal relations for P_l^m.

$$Y_{lm}(\theta,\phi) = \left\{\frac{2l+1}{4\pi}\cdot\frac{(l-m)!}{(l+m)!}\right\}^{1/2}P_l^m(\cos\theta)e^{im\phi}.$$

~~Radial wave equation~~: Interpretation of l and m :— Putting $R(r) = \chi(r)/r$, the radial eqn can be written as $-\frac{\hbar^2}{2m}\frac{d^2\chi}{dr^2} + \left[V(r) + \frac{l(l+1)\hbar^2}{2mr^2}\right]\chi = E\chi$

i.e same as one-dimensional motion of a particle in a potential V + extra term.

(*) These are obtained from ~~$r^2 = x^2+y^2+z^2$~~; $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

& inverse relns $r^2 = x^2+y^2+z^2$, $\cos\theta = z/r$, $\tan\phi = y/x$

so that $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial\theta}\cdot\frac{\partial\theta}{\partial x} + \frac{\partial f}{\partial\phi}\cdot\frac{\partial\phi}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{x}{r} + \frac{\partial f}{\partial\theta}\left(\frac{\cos\theta\cos\phi}{r}\right) + \frac{\partial f}{\partial\phi}\left(-\frac{\operatorname{cosec}\theta\sin\phi}{r}\right)$

$= \frac{\partial f}{\partial r}(\sin\theta\cos\phi) + \;''\; + \;''$

$-\sin\theta\cdot\frac{\partial\theta}{\partial x} = -\frac{x}{r^2}\cdot\frac{\partial r}{\partial x} = -\frac{zx}{r^3}$

$-\sin\theta\,\frac{\partial\theta}{\partial x} = -\frac{\cos\theta\sin\theta\cos\phi}{r}$

$\sec^2\phi\cdot\frac{\partial\phi}{\partial x} = -\frac{y}{x^2} = -\frac{\sin\phi}{r\cos^2\phi\sin\theta}$

Simily $\frac{\partial f}{\partial y}$ & $\frac{\partial f}{\partial z}$ may $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in terms of $\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}$ as in (*)

(†) Result already obtained for general $V(r)$ as equivalent to one dimensional motion of a particle in a potential $V(r) + \frac{l(l+1)\hbar^2}{2mr^2}$.

To interpret second term, consider A.M, $\vec{M} = \vec{r} \times \vec{p}$ ie $M_x = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$, etc. Transforming into polar coords

$$\left.\begin{aligned} M_x &= i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \\ M_y &= i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \\ M_z &= -i\hbar\,\partial/\partial\phi. \end{aligned}\right\} \text{(*)}$$

41st Lecture — 28/11/63 — Calculating of $\vec{M}^2 = M_x^2 + M_y^2 + M_z^2$ as an operator

$$M_x^2 = -\hbar^2\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

$$= -\hbar^2\left\{\sin\phi\frac{\partial}{\partial\theta}\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\right\}$$

$$= -\hbar^2\left[\left\{\sin^2\phi\frac{\partial^2}{\partial\theta^2} + \sin\phi\cos\phi\left(-\operatorname{cosec}^2\theta\frac{\partial}{\partial\phi} + \cot\theta\frac{\partial^2}{\partial\theta\partial\phi}\right)\right\} + \left\{\cot\theta\cos\phi\left(\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\frac{\partial^2}{\partial\theta\partial\phi}\right)\right.\right.$$
$$\left.\left. + \cot^2\theta\cos\phi\left(-\sin\phi\frac{\partial}{\partial\phi} + \cos\phi\frac{\partial^2}{\partial\phi^2}\right)\right\}\right]$$

$$= -\hbar^2\left\{\sin^2\phi\frac{\partial^2}{\partial\theta^2} + 2\cot\theta\sin\phi\cos\phi\frac{\partial^2}{\partial\theta\partial\phi} + \cot^2\theta\cos^2\phi\frac{\partial^2}{\partial\phi^2} + \cot\theta\cos^2\phi\frac{\partial}{\partial\theta} - \sin\phi\cos\phi\left(\cot^2\theta + \operatorname{cosec}^2\theta\right)\frac{\partial}{\partial\phi}\right.$$

Similarly $M_y^2 = -\hbar^2\left\{\cos^2\phi\frac{\partial^2}{\partial\theta^2} - 2\cot\theta\sin\phi\cos\phi\frac{\partial^2}{\partial\theta\partial\phi} + \cot^2\theta\sin^2\phi\frac{\partial^2}{\partial\phi^2} + \cot\theta\sin^2\phi\frac{\partial}{\partial\theta} + \sin\phi\cos\phi\left(\cot^2\theta + \operatorname{cosec}^2\theta\right)\frac{\partial}{\partial\phi}\right.$

$$\therefore M_x^2 + M_y^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \cot\theta\frac{\partial}{\partial\theta}\right)$$

$$\therefore \text{Q.M } M^2 = M_x^2 + M_y^2 + M_z^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \operatorname{cosec}^2\theta\frac{\partial^2}{\partial\phi^2}\right)$$

$$= -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

But $Y_{lm} = Y$ satisfies $\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \lambda Y = 0$ with $\lambda = l(l+1)$

Hence $M^2 Y_{lm}(\theta,\phi) = l(l+1)\hbar^2 Y_{lm}(\theta,\phi)$ — — (A)

Also $M_z\left\{(2\pi)^{-\frac{1}{2}} e^{im\phi}\right\} = -i\hbar\cdot(2\pi)^{-\frac{1}{2}}\frac{\partial}{\partial\phi}\left(e^{im\phi}\right) = m\hbar\,(2\pi)^{-\frac{1}{2}} e^{im\phi}$

$$\therefore M_z Y_{lm}(\theta,\phi) = m\hbar Y_{lm}(\theta,\phi) \quad \text{— —} \quad \text{(B)}$$

(A) & (B) show that Y_{lm} is an eigenfunction of $\vec{M}^2$ with the eigen-value $l(l+1)\hbar^2$, and also of M_z with the eigen-value $m\hbar$, l is called the azimuthal quantum number & m the magnetic quantum number. In quantum theory only M^2 & M_z can be specified simultaneously, while in the classical theory all three M_x, M_y & M_z can be specified simultaneously. The above result is true only for a $V(r)$ ie sph. sym. field. From the property of Legendre polynomials, it is seen that l should be a $+^{ve}$ integer or 0 ie $l = 0, 1, 2, \ldots$, while $m \leq l$, and m can take also negative values from $-l$ to $\oplus$ ie $|m| \leq l$ [m appears as a square in the eqn for $\Theta(\theta)$ where $Y = \Theta(\theta)\Phi(\phi)$].

Radial equation ⊕ — Solving of interest for special form of $V(r)$ & hence consider H-atom with $V(r) = -ze^2/r$ (ie nucleus-Coulomb attraction between atomic nucleus of charge ze & an electron of charge $-e$; for H, $z = 1$, for He, $z = 2$, etc)

Hydrogen atom — Case of two particles moving under mutual attraction depending on distance between them —

* put it as $-\frac{\hbar^2}{2M}\cdot\frac{1}{U}\left(\frac{\partial^2 U}{\partial x^2}+\cdots\right) = \frac{\hbar^2}{2\mu}\cdot\frac{1}{u}\left(\frac{\partial^2 u}{\partial x^2}+\cdots\right) - V + E_0 = E'$

giving $-\frac{\hbar^2}{2M}\frac{1}{U}\left(\frac{\partial^2 U}{\partial x^2}+\cdots\right) = E'$, $-\frac{\hbar^2}{2\mu}\cdot\frac{1}{u}\left(\frac{\partial^2 u}{\partial x^2}+\cdots\right) + V = E_0 - E' = E$, say

ie $E_0 = E + E'$

Just as Schrödinger eqn in one dimension $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2}$ was generalised to three dimensions viz $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$ or to $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{r},t)\psi$, for a force field, the case of two particles of masses m_1 & m_2 has its Schrödinger eqn

$$i\hbar \frac{\partial \psi(x_1,y_1,z_1,x_2,y_2,z_2,t)}{\partial t} = \left[-\frac{\hbar^2}{2m_1}\left(\frac{\partial^2}{\partial x_1^2}+\cdots\right) - \frac{\hbar^2}{2m_2}\left(\frac{\partial^2}{\partial x_2^2}+\cdots\right) + V(x_1,\cdots,x_2,\cdots,t)\right]\psi$$

If $V = V(x_1-x_2, y_1-y_2, z_1-z_2)$, a simplification is possible. If (x,y,z) be rel. coords & (X,Y,Z) of the C.G

$x = x_1 - x_2, \cdots$, $MX = m_1x_1 + m_2x_2, \cdots$, where $M = m_1+m_2$.

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial x_1} + \frac{\partial}{\partial X}\cdot\frac{\partial X}{\partial x_1} = \frac{\partial}{\partial x} + \frac{m_1}{M}\frac{\partial}{\partial X}$$

$$\frac{\partial}{\partial x_2} = -\frac{\partial}{\partial x} + \frac{m_2}{M}\frac{\partial}{\partial X}$$

& since coeffts are const. $\frac{\partial^2}{\partial x_1^2} = \frac{\partial^2}{\partial x^2} + \frac{2m_1}{M}\frac{\partial^2}{\partial x \partial X} + \frac{m_1^2}{M^2}\frac{\partial^2}{\partial X^2}$ & $\frac{\partial^2}{\partial x_2^2} = \frac{\partial^2}{\partial x^2} - \frac{2m_2}{M}\frac{\partial^2}{\partial x\partial X} + \frac{m_2^2}{M^2}\frac{\partial^2}{\partial X^2}$

$$-\frac{\hbar^2}{2m_1}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m_2}\frac{\partial^2}{\partial x_2^2} = -\frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right) - \frac{\hbar^2}{2M}\frac{\partial^2}{\partial X^2} = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2M}\frac{\partial^2}{\partial X^2} \quad \left[\mu = \frac{m_1m_2}{m_1+m_2}\right.$$

& the S-eqn becomes $i\hbar\frac{\partial \psi}{\partial t} = \left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) - \frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial Y^2}+\frac{\partial^2}{\partial Z^2}\right) + V(x,y,z)\right]\psi$

Putting $\psi = v(x,y,z,X,Y,Z)\,f(t)$

$$i\hbar\, v\frac{df}{dt} = [\quad]\,vf$$

leading to $i\hbar\frac{1}{f}\frac{df}{dt} = \frac{1}{v}\left[-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2 v}{\partial x^2}+\cdots\right) - \frac{\hbar^2}{2M}\left(\frac{\partial^2 v}{\partial X^2}+\cdots\right) + Vv\right] = E_0$, say

ie $f = Ce^{-iE_0t/\hbar}$, and

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2 v}{\partial X^2}+\cdots\right) - \frac{\hbar^2}{2\mu}\left(\frac{\partial^2 v}{\partial x^2}+\cdots\right) + Vv = E_0 v$$

Putting $v = u(x,y,z)\,U(X,Y,Z)$, this further leads to

$$-\frac{\hbar^2}{2M}u\left(\frac{\partial^2 U}{\partial X^2}+\cdots\right) - \frac{\hbar^2}{2\mu}U\left(\frac{\partial^2 u}{\partial x^2}+\cdots\right) + V\cdot u\cdot U = E_0\cdot u\cdot U$$

or $$-\frac{\hbar^2}{2M}\cdot\frac{1}{U}\left(\frac{\partial^2 U}{\partial X^2}+\cdots\right) - \frac{\hbar^2}{2\mu}\frac{1}{u}\left(\frac{\partial^2 u}{\partial x^2}+\cdots\right) + V = E_0 \quad *$$

or $$-\frac{\hbar^2}{2M}\cdot\frac{1}{U}\left(\frac{\partial^2 U}{\partial X^2}+\cdots\right) = E',$$

$$-\frac{\hbar^2}{2\mu}\frac{1}{u}\left(\frac{\partial^2 u}{\partial x^2}+\cdots\right) + V = E, \quad \text{where } E_0 = E'+E.$$

ie original Schrödinger eqn is completely separated into, $f = e^{-i(E+E')t/\hbar}$

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2 U}{\partial X^2}+\cdots\right) = E'U$$

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2 u}{\partial x^2}+\cdots\right) + Vu = Eu.$$

ie motion of C.G corresponds to motion of a free particle & relative motion to that of a particle in a central field $V(r)$. The latter eqn is of interest in the hydrogen-atom problem ie relative motion of electron round nucleus & since $\mu = \frac{m_1+m_2}{m_1+m_2} = \frac{m_1m_2}{m_1+m_2} = \frac{m_1}{1+m_1/m_2} \simeq m_1$, since $\frac{\text{mass of electron}}{\text{mass of proton}}$ is small.

The motion is very nearly correct. For the relative motion Y_{lm} are eigen-functions of $\vec{M}^2$ & M_z and we

⊛ Can be written as

$$\frac{d^2R}{d\rho^2} + \frac{2}{\rho}\frac{dR}{d\rho} + \left\{ \quad \right\} R = 0.$$

have to consider only radial motion with $V = -Ze^2/r$. the radial eqn of wr

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2mr^2}{\hbar^2}(E-V) = \lambda = l(l+1), \text{ corresp- to azimuthal quantum number } l$$

$$\text{or } \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2m}{\hbar^2}(E-V)R = \frac{l(l+1)R}{r^2}.$$

$$\text{or } -\frac{\hbar^2}{2m}\cdot\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{ze^2}{r}R + \frac{l(l+1)\hbar^2}{2mr^2}R = ER$$

is now to be written with μ instead of m

$$-\frac{\hbar^2}{2\mu}\cdot\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{ze^2}{r}R + \frac{l(l+1)\hbar^2}{2\mu r^2}R = ER. \qquad \left(\begin{array}{l}E<0, \text{ discrete} \\ \text{eigenvalues} \\ E>0, \text{ Continuous}\end{array}\right)$$

42nd Lecture – 2/12/63 – Remaining portions of previous lecture – Rewriting the radial eqn in a non-dimensional form with $\rho = \alpha r$, where $\alpha^2 = \frac{8\mu|E|}{\hbar^2}$, $\lambda = \frac{2\mu z e^2}{\alpha\hbar^2}$.

43rd Lecture – 3/12/63 – The dimension of α^2 ~~$= \frac{M}{E L^2} = M$~~ is given by $[E] = ML^2T^{-2}$ & $[\hbar] = [ET]$ ($E = \hbar\nu$)

i.e. as $\mu = [M]$ or $[\alpha^2] = \frac{M^2L^2T^{-2}}{M^2L^4T^{-2}} = L^{-2}$ $\quad = ML^2T^{-1}$

or $[\alpha] = L^{-1}$ & hence ρ is dimensionless.

The radial eqn becomes, with $\alpha^2 = 8\mu E/\hbar^2$ and $-E$ in the R.H.S,

$$-\frac{\hbar^2}{2\mu}\cdot\frac{\alpha^2}{\rho^2}\cdot\frac{\alpha}{\alpha}\frac{d}{d\rho}\left(\frac{\rho^2}{\alpha^2}\cdot\frac{\alpha}{\rho}\cdot\frac{dR}{d\rho}\right) - \frac{\lambda\alpha^2\hbar^2}{2\mu\rho}R + \frac{l(l+1)\hbar^2}{2\mu\rho^2}\alpha^2R = -ER.$$

$$-\frac{4E}{\rho^2}\cdot\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) - \frac{\lambda}{\rho}\cdot 4ER + \frac{l(l+1)}{\rho^2}\cdot 4ER = -ER.$$

$$\text{i.e. } \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \left\{\frac{\lambda}{\rho} - \frac{1}{4} - \frac{l(l+1)}{\rho^2}\right\}R = 0. \quad (*)$$

for large ρ, $R = \rho^n e^{-\frac{1}{2}\rho}$ satisfies the eqn so far as leading terms are concerned, since the leading term in $\frac{d^2R}{d\rho^2}$ is $\frac{1}{4}\rho^n e^{-\frac{1}{2}\rho}$ & this cancels with $-\frac{1}{4}R = -\frac{1}{4}\rho^n e^{-\frac{1}{2}\rho}$. Hence we seek a solution

$$R(\rho) = F(\rho)e^{-\frac{1}{2}\rho} \text{ where } F(\rho) \text{ is a polynomial of order } n \text{ in } \rho.$$

$$\frac{dR}{d\rho} = F'e^{-\frac{1}{2}\rho} - \tfrac{1}{2}Fe^{-\frac{1}{2}\rho}; \quad \frac{d^2R}{d\rho^2} = F''e^{-\frac{1}{2}\rho} - \tfrac{1}{2}F'e^{-\frac{1}{2}\rho} - \tfrac{1}{2}F'e^{-\frac{1}{2}\rho} + \tfrac{1}{4}Fe^{-\frac{1}{2}\rho}$$

$$\text{i.e. } (F'' - F' + \tfrac{1}{4}F)e^{-\frac{1}{2}\rho} + \frac{1}{\rho}(2F' - F)e^{-\frac{1}{2}\rho} + \left\{\frac{\lambda}{\rho} - \frac{1}{4} - \frac{l(l+1)}{\rho^2}\right\}Fe^{-\frac{1}{2}\rho} = 0$$

$$F'' + \left(\frac{2}{\rho} - 1\right)F' + \left\{\frac{\lambda-1}{\rho} - \frac{l(l+1)}{\rho^2}\right\}F = 0.$$

If $F = a_0 + a_1\rho + a_2\rho^2 + \cdots$, then the eqn is not satisfied since for $\rho = 0$ since there are terms in L.H.S

in $\frac{1}{\rho}$, $\frac{1}{\rho^2}$, which $\to \infty$ as $\rho \to 0$. Hence we try $F = \rho^s L(\rho)$ $(s > 0)$, which gives

$$\rho^2 L'' + \rho\{2(s+1) - \rho\}L' + \{\rho(\lambda - s - 1) + s(s+1) - l(l+1)\}L = 0$$

The terms without ρ should $= 0$, for eqn to be satisfied by $\rho = 0$. i.e. $s(s+1) = l(l+1)$

i.e. $s = l$ & $s = -(l+1)$, $R(\rho)$ being finite for $\rho = 0$ requires choosing $s = l$.

i.e. $F = \rho^l L(\rho)$ & eqn becomes $\rho L'' + \{2(l+1) - \rho\}L' + (\lambda - l - 1)L = 0$ ——(1)

Re – Laguerre polynomials & associated Laguerre polynomials $L_q(\rho)$ & $L_q^p(\rho) = \frac{d^p}{d\rho^p}L_q(\rho)$

$L_q^p(\rho)$ satisfies diff eqn: $\rho L_q^{p\,''} + (p+1-\rho)L_q^{p\,'} + (q-p)L_q^p = 0$ ——(2)

Comparison of (1) & (2) gives $p+1 = 2(l+1)$ i.e. $p = 2l+1$, $q = \lambda + l$ i.e. λ in (1) must be an integer $= n$, say

44th Lecture – 4/12/63 – Laguerre & associated Laguerre polynomials – Hermite polynomials used in L.H.O. were defined by $G = e^{-s^2+2sp} = \sum_{q=0}^{\infty} \frac{H_q(p)}{q!} s^q$ – Legendre polynomials: $(1-2sp+s^2)^{-\frac{1}{2}} = G = \sum_{q=0}^{\infty} P_q(p) s^q \quad (s<1)$.

[Laguerre polynomials are similarly defined by

$$G = \frac{e^{-ps/1-s}}{1-s} = \sum_{q=0}^{\infty} \frac{L_q(p)}{q!} s^q \qquad (s<1).$$

$$\frac{\partial G}{\partial p} = -pG = \sum -\frac{pL_q(p)}{q!} s^q = \sum \frac{L'_q(p)}{q!} s^q$$

$$\frac{\partial G}{\partial p} = -\frac{s}{1-s} G = -\sum \frac{L_q(p)}{q!} s^q \frac{s}{1-s} = \sum_{q=0}^{\infty} -\frac{L_q}{q!} s^{q+1}(1+s+s^2+\dots) = \sum \frac{L'_q}{q!} s^q$$

Comparing coeffts of s^q on both sides $\quad \frac{L'_q}{q!} = -\frac{L_{q-1}}{(q-1)!} - \frac{L_{q-2}}{(q-2)!} - \dots$

" s^{q-1} " $\quad \frac{L'_{q-1}}{(q-1)!} = -\frac{L_{q-2}}{(q-2)!} - \dots$

ie $\quad \frac{L'_q}{q!} = -\frac{L_{q-1}}{(q-1)!} + \frac{L'_{q-1}}{(q-1)!}$

ie $\quad L'_q - q L'_{q-1} = -q L_{q-1}.$

$$\frac{\partial G}{\partial s} = \frac{1}{(1-s)^2} e^{-ps/(1-s)} + \frac{1}{1-s} e^{-ps/1-s} \left\{ -p \cdot \frac{1}{(1-s)^2} \right\}$$

$$= \sum \frac{L_q}{q!} s^{q+1}(1+s+s^2+\dots) - \sum \frac{pL_q}{q!}(1+2s+3s^2+\dots) s^q$$

$$= \sum \frac{qL_q}{q!} s^{q-1} = \sum \frac{L_q}{(q-1)!} s^{q-1} \Big]$$

Better to define Laguerre polynomials by $L_q(p) = e^p \frac{d^q}{dp^q}(p^q e^{-p})$.

$$L_q^p = \frac{d^p}{dp^p} L_q(p).$$

$$\frac{d^{p+2}}{dp^{p+2}} L_q(p) + \left\{ \frac{p+1}{p} - 1 \right\} \frac{d^{p+1}}{dp^{p+1}} L_q(p) + \frac{q-p}{p} \frac{d^p}{dp^p} L_q(p) = 0.$$

45th Lecture – 10/12/63. – Differential equation satisfied by $L_q^p(p)$, the associated Laguerre polynomial is

$$p L_q^{p''} + (p+1-p) L_q^{p'} + (q-p) L_q^p = 0$$

Comparing this with the differential eqn for $L(p)$ used in the radial eqn

$$p L'' + \{2(l+1) - p\} L' + (\lambda - l - 1) L = 0$$

shows that $L(p)$ is an associated Laguerre polynomial L_q^p [ie series for $L(p)$ must terminate]

with $p = 2l+1$, $q-p = \lambda - l - 1$ ie λ must be an integer $= n$, say.

so that $q = p + n - l - 1 = n + l$ ie soln of radial eqn is $L_{n+l}^{2l+1}(p)$.

n is called the total quantum number. & $n' = n - l - 1$ is called the radial quantum number

Using $\alpha^2 = \frac{8\mu E}{\hbar^2}$ & $\lambda = \frac{2\mu Z e^2}{\alpha \hbar^2}$ $\quad E = \frac{\alpha^2 \hbar^2}{8\mu} = \frac{4\mu^2 Z^2 e^4}{8\mu \lambda^2 \hbar^4} \cdot \hbar^2 = \frac{\mu Z^2 e^4}{2\hbar^2 \lambda^2}$

$$E_n = -\frac{R}{n^2} \quad (\lambda = n). \qquad E_n - E_m = -R\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

- Balmer series formula using $E = h\nu$ - Hydrogen atom wave functions $u_{n\ell m}(r, \theta, \phi)$

46th Lecture – 17/12/63 – Recapitulation of the earlier results re. the Hydrogen atom – Degeneracy; E_n-eigenvalues depend only on n & order of degeneracy $= \sum_{\ell=0}^{n-1}(2\ell+1)$ [for each n, ℓ can vary from 0 to $n-1$, & for each ℓ, m can vary from $-\ell$ to $+\ell$] $= n^2$ –

Wave eqn of H-atom in parabolic coordinates – Defn of parabolic coords (ξ, η, ϕ) & surfaces ξ = const & η = const being paraboloids of revolution about ϕ = const being planes thro' OZ – Wave eqn in (ξ, η, ϕ) – Transformation of $\nabla^2 u$ into parabolic coordinates

[Start with the polar eqn $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2 u}{\partial\phi^2}$

From $\xi = r - z = r(1-\cos\theta)$; $\eta = r + z = r(1+\cos\theta)$, $\xi\eta = r^2\sin^2\theta$. Hence the last term becomes $\frac{1}{\xi\eta}\frac{\partial^2 u}{\partial\phi^2}$, and it remains only to transform the 1st two terms.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial\xi}\cdot\frac{\partial\xi}{\partial r} + \frac{\partial u}{\partial\eta}\cdot\frac{\partial\eta}{\partial r}, \text{ since } \frac{\partial\phi}{\partial r} = 0$$

$$= (1-\cos\theta)\frac{\partial u}{\partial\xi} + (1+\cos\theta)\frac{\partial u}{\partial\eta}.$$

$$\text{Also } \frac{\partial u}{\partial\theta} = \frac{\partial u}{\partial\xi}\cdot\frac{\partial\xi}{\partial\theta} + \frac{\partial u}{\partial\eta}\cdot\frac{\partial\eta}{\partial\theta}, \text{ since } \frac{\partial\phi}{\partial\theta} = 0$$

$$= r\sin\theta\frac{\partial u}{\partial\xi} - r\sin\theta\frac{\partial u}{\partial\eta}.$$

Using these, and performing the further differentiations & adding, it can be shown that the terms in $\partial^2 u/\partial\xi\partial\eta$ vanish & the final result will be

$$2(1-\cos\theta)\frac{\partial^2 u}{\partial\xi^2} + 2(1+\cos\theta)\frac{\partial^2 u}{\partial\eta^2} + \frac{2}{r}\frac{\partial u}{\partial\xi} + \frac{2}{r}\frac{\partial u}{\partial\eta}$$

$$= \frac{2\xi}{r}\frac{\partial^2 u}{\partial\xi^2} + \frac{2\eta}{r}\frac{\partial^2 u}{\partial\eta^2} + \frac{2}{r}\cdot\frac{\partial u}{\partial\xi} + \frac{2}{r}\frac{\partial u}{\partial\eta}$$

$$= \frac{2}{r}\left\{\left(\xi\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial u}{\partial\xi}\right) + \left(\eta\frac{\partial^2 u}{\partial\eta^2} + \frac{\partial u}{\partial\eta}\right)\right\} = \frac{2}{r}\left\{\frac{\partial}{\partial\xi}\left(\xi\frac{\partial u}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\eta\frac{\partial u}{\partial\eta}\right)\right\}$$

$$= \frac{4}{\xi+\eta}\left\{\frac{\partial}{\partial\xi}\left(\xi\frac{\partial u}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\eta\frac{\partial u}{\partial\eta}\right)\right\} \text{ since } 2r = \xi+\eta.\Big]$$

47th Lecture – 18/12/63 – Using Lame's results viz. in general orthogonal coordinates λ, μ, ν if $ds^2 = dx^2+dy^2+dz^2$ $= H_1^2 d\lambda^2 + H_2^2 d\mu^2 + H_3^2 d\nu^2$, then $\nabla^2 u = \frac{1}{H_1H_2H_3}\left\{\frac{\partial}{\partial\lambda}\left(\frac{H_2H_3}{H_1}\frac{\partial u}{\partial\lambda}\right) + \dots + \dots\right\}$

we might deduce $\nabla^2 u$ in parabolic coords:

In polar coords, $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\cdot d\phi^2$

$dr = \frac{1}{2}(d\xi + d\eta)$, from $\xi = r(1-\cos\theta)$, $\eta = r(1+\cos\theta)$, $d\xi = dr(1-\cos\theta) + r\sin\theta\, d\theta$, &

and $d\eta = dr(1+\cos\theta) - r\sin\theta\cdot d\theta$

Eliminating dr, $d\xi(1+\cos\theta) - d\eta(1-\cos\theta) = 2r\sin\theta\cdot d\theta$

i.e. $\frac{\eta\, d\xi}{r} - \frac{\xi\, d\eta}{r} = 2r\sin\theta\cdot d\theta$ or $\frac{\eta\, d\xi - \xi\, d\eta}{\frac{1}{2}(\xi+\eta)} = 2\sqrt{\xi\eta}\cdot d\theta$

or $d\theta = \frac{\eta\, d\xi - \xi\, d\eta}{(\xi+\eta)\sqrt{\xi\eta}}$, $d\phi = d\phi$.

$$\therefore ds^2 = \tfrac{1}{4}(d\xi+d\eta)^2 + \tfrac{1}{4}(\xi+\eta)^2\frac{(\eta\, d\xi - \xi\, d\eta)^2}{(\xi+\eta)^2\cdot\xi\eta} + \xi\eta\cdot d\phi^2.$$

ie $ds^2 = \frac{(\xi+\eta)}{4\xi} d\xi^2 + \frac{(\xi+\eta)}{4\eta} d\eta^2 + \xi\eta \cdot d\phi^2$ — (1)

This can also be derived from $ds^2 = dx^2 + dy^2 + dz^2$

$x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$

$x = \sqrt{\xi\eta}\cos\phi$, $y = \sqrt{\xi\eta}\sin\phi$, $z = \frac{1}{2}(\eta - \xi)$

$dx = \frac{1}{2}\sqrt{\eta/\xi}\cos\phi \cdot d\xi + \frac{1}{2}\sqrt{\xi/\eta}\cos\phi \cdot d\eta - \sqrt{\xi\eta}\sin\phi \cdot d\phi$

$dy = \frac{1}{2}\sqrt{\eta/\xi}\sin\phi \cdot d\xi + \frac{1}{2}\sqrt{\xi/\eta}\sin\phi \cdot d\eta + \sqrt{\xi\eta}\cos\phi \cdot d\phi$

$dz = \frac{1}{2} d\eta - \frac{1}{2} d\xi$

Giving $ds^2 = dx^2 + dy^2 + dz^2 = \frac{1}{4}\left(\frac{\eta}{\xi} + 1\right) d\xi^2 + \frac{1}{4}\left(\frac{\xi}{\eta} + 1\right) d\eta^2 + \xi\eta\, d\phi^2$ ie same as (1)

In (1) $H_1 = \frac{1}{2}\sqrt{\frac{\xi+\eta}{\xi}}$, $H_2 = \frac{1}{2}\sqrt{\frac{\xi+\eta}{\eta}}$, $H_3 = \sqrt{\xi\eta}$

ie $\nabla^2 u = \frac{4}{\xi+\eta}\left\{\frac{\partial}{\partial\xi}\left(\xi\frac{\partial u}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\eta\frac{\partial u}{\partial\eta}\right) + \frac{(\xi+\eta)}{4\xi\eta}\frac{d^2u}{d\phi^2}\right\}$

$= \frac{4}{\xi+\eta}\left\{\frac{\partial}{\partial\xi}\left(\xi\frac{\partial u}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\eta\frac{\partial u}{\partial\eta}\right)\right\} + \frac{1}{\xi\eta}\frac{d^2u}{d\phi^2}\Big\}$

— Solution by separation of variables, with $u = f(\xi)\, g(\eta)\, \Phi(\phi)$. If the Schrödinger eqn

$-\frac{\hbar^2}{2\mu}\Big[\qquad\qquad\Big] - \frac{2Ze^2}{\xi+\eta}u = Eu \quad (E<0)$.

$= -Eu$, say.

Setting $\Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} \cdot e^{im\phi}$ $(m = 0, \pm 1, \pm 2, \cdots)$.

and the eqns for f and g, after introducing dimensionless parameter $\rho = \alpha\xi$, become

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) - \left(\frac{m^2}{4\rho^2} + \frac{1}{4} - \frac{\lambda_1}{\rho}\right)f = 0 \qquad \alpha^2 = \frac{2\mu E}{\hbar^2},\ \lambda_1 = \frac{1}{\alpha}\left(\frac{\mu Ze^2}{\hbar^2} - \nu\right)$$

(ν = separation const)

and $$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg}{d\rho}\right) - \left(\frac{m^2}{4\rho^2} + \frac{1}{4} - \frac{\lambda_2}{\rho}\right)g = 0, \qquad \lambda_2 = \frac{\nu}{\alpha}.$$

Substituting $f(\rho) = e^{-\frac{1}{2}\rho}\rho^{\frac{1}{2}m}L(\rho)$ & similarly for $g(\rho)$, the eqns become

$$\rho L''(\rho) + (m+1-\rho)L'(\rho) + (\lambda_1 - \tfrac{1}{2}m - \tfrac{1}{2})L(\rho) = 0$$

and $$\rho L''(\rho) + (m+1-\rho)L'(\rho) + (\lambda_2 - \tfrac{1}{2}m - \tfrac{1}{2})L(\rho) = 0.$$

48th lecture — 23/12/63 — Recapitulation of the old results — Comparing the above eqns with the diff. eqn for the associated Laguerre polynomials L_q^p viz $\left[L_q^p(\rho) = \frac{d}{d\rho^p}\{L_q(\rho)\}\right]$

$$\rho L_q^{p\,\prime\prime} + (p+1-\rho)L_q^{p\,\prime} + (q-p)L_q^p = 0,$$ we have

$p = m$ in both cases, and $q = n_1 + m$ or $n_2 + m$ in the two cases

with $n_1 = \lambda_1 - \frac{1}{2}(m+1)$, $n_2 = \lambda_2 - \frac{1}{2}(m+1)$. ie $L^m_{n_1+m}$ or $L^m_{n_2+m}$ (n_1 & n_2 integers)

$\lambda_1 + \lambda_2 = n_1 + n_2 + m + 1 \equiv n$ (defn).

which gives $\frac{\mu Ze^2}{\alpha\hbar^2} = n$ or $\alpha = \frac{\mu Ze^2}{\hbar^2 n}$

Energy levels are given $\alpha^2 = \frac{2\mu E}{\hbar^2}$ or $E_n = -\frac{\alpha^2\hbar^2}{2\mu} = -\frac{\mu^2 Z^2 e^4 \hbar^2}{2\mu\hbar^4 n^2} = -\frac{\mu Z^2 e^4}{2\hbar^2}\cdot\frac{1}{n^2}$

same as obtained by the use of the polar coords. (n is the principal quantum)

Degeneracy of energy level E_n : from $n_1 + n_2 + m + 1 = n$, we have for $m = 0$, n ways of choosing n_1 & n_2

(*) Since $AA^{-1} = 1 = A^{-1}A$, and AB has as many ~~rows~~ as A & cols. as B & 1 is square

~~no. of cols of A = no. of rows of A^{-1}~~ ; no. of rows of A = no. of cols of A^{-1} = no. of rows or cols of 1

& ~~no. of cols of A^{-1} = no. of rows of A~~ ; no. of rows of A^{-1} = no. of cols of A = "

ie no. of cols of A = no. of rows of A = no. of rows of A^{-1} = no. of cols of A^{-1} = "

ie both A & A^{-1} are square.

use of matrices in the form $A(x, x')$, {x & x' in the interval (x_1, x_2)} for continuous case

Product $C(x, x') = \int_{x_1}^{x_2} A(x, x'')\, B(x'', x'')\, dx''$

Diagonal $D(x, x') = d(x)\, \delta(x - x')$ [Since δ-function $\delta(x)$]

and for $m \neq 0$, 2 ways of choosing m and $n-m$ ways of choosing n_1 and n_2. Hence

degeneracy $= n + 2\sum_{m=1}^{n-1}(n-m)$ $\quad$ [$m = n-1-(n_1+n_2)$ ie max of $m = n-1$]

$= n + 2\left\{n(n-1) - \frac{1}{2}(n-1)n\right\} = n + n(n-1) = n^2$, as in previous case.

Wave fns $u_{n_1 n_2 m}(\xi,\eta,\phi) = e^{-\frac{\alpha}{2}(\xi+\eta)}(\xi\eta)^{\frac{1}{2}m} L^m_{n_1+m}(\alpha\xi)\, L^m_{n_2+m}(\alpha\eta)\, e^{im\phi}$

where $\alpha = \mu z e^2/\hbar^2 n = \dfrac{\mu z e^2}{h^2(n_1+n_2+m+1)}$.

For given E_n (ie n) & m ($<n$), n_1 & n_2 can be chosen in $n-m$ different ways, just as in the spherical coords case l can be chosen in $n-m$ ways. Thus $(n-m)$ products of the ξ & η fns are linear combinations of $(n-m)$ products of the r & θ fns.

Ground state – $n=1$, ie $n_1 = n_2 = m = 0$, the parabolic soln is $e^{-\frac{1}{2}\alpha(\xi+\eta)} = e^{-\mu z e^2(\xi+\eta)/2\hbar^2}$

and the spherical soln is $e^{-\mu z e^2 r/\hbar^2}$ (for $n=1$, $l=m=0$) as can be seen from the

general wave eqn $u_{nlm}(r,\theta,\phi) = R_{nl}(r)\, Y_{lm}(\theta,\phi)$

$= e^{-\frac{1}{2}\rho}\rho^l L^{2l+1}_{n+l}(\rho)\, Y_{lm}(\theta,\phi)$

and these two solutions are identical.

49th Lecture – 13/1/64. – Matrices – Sum of matrices of same rank – Defn of product AB – $A(B+C) = AB + AC$ – $(AB)C = A(BC)$ – AB not necessarily equal to BA – Zero matrix 0 such that $0\cdot A = 0$, $A\cdot 0 = 0$ – unit matrix $1\cdot A = A$ and $B\cdot 1 = B \to 1$ a square matrix δ_{kl} – inverse A^{-1} given by $AA^{-1} = A^{-1}A = 1$, singular & non-singular matrices accg. as inverse does not exists or exists – Spur of a square matrix – Determinant of a square matrix viz $|A|$ – Non-singular matrix of finite rank is square (*) – $(A^{-1})_{kl} = \dfrac{\text{cofactor of } A_{lk}}{|A|}$ & hence matrix is singular if $|A| = 0$ – For non-singular matrices $(AB)^{-1} = B^{-1}A^{-1}$

50th Lecture – 14/1/64 – (1) Hermitian adjoint A^* ie if $B = A^*$, $B_{kl} = \bar{A}_{lk}$ – $(AB)^* = B^*A^*$ for $(AB)^*_{kl} = \overline{A_{lm}B_{mk}}$

$= \bar{A}_{lm}\bar{B}_{mk}$ and $(B^*A^*)_{kl} = B^*_{km}A^*_{ml} = \bar{B}_{mk}\bar{A}_{lm} = \bar{A}_{lm}\bar{B}_{mk}$ –

(2) Self adjoint or Hermitian matrix if $A = A^*$ ie $A_{kl} = \bar{A}_{lk}$ ie Hermitian matrix is square

(3) Unitary matrix if $A^* = A^{-1}$ or $AA^* = A^*A = 1$ ie $A_{km}\bar{A}_{lm} = \bar{A}_{mk}A_{ml} = \delta_{kl}$; unitary matrix is non-singular & so if of finite rank it is square.

(4) Transformation & diagonalisation matrices – Transformation A' of A by non-singular S is defined by $SAS^{-1} = A'$ so that $S^{-1}A'S = S^{-1}SAS^{-1}S = A$ ie $A' \to A$ under S^{-1} – Matrix equation is invariant against transformations – Diagonal square matrix & its eigenvalues; if A is diagonal, A^n is also diagonal and eigenvalues of A^n are n^{th} powers of corresp. eigenvalues of A – Diagonalisation of A by S means that $A' = SAS^{-1}$ is diagonal ie $A'_{kl} = A'_k\delta_{kl}$ – Secular eqn, eigenvalues of A' independent of S also eigen-values of A, degenerate when two or more eigen values are equal

(5) Matrices of infinite rank – Condn for $\sum A_{km}B_{ml}$ being convergent – Denumerably & non-denumerably infinite matrices. Summation replaced by integration in latter case – Square if $A = (A_{kl})$ Hermitian, non-square if $A = (A_{xl})$ eg. unitary matrix – Matrices is Q.M. are mostly infinite –

Proof of (2) $SAS^* = A'$, $SBS^* = B'$

$SABS^* = SAS^*SBS^* = SABS^* = A'B'$

$SBS^*SAS^* = SBAS^* = B'A'$ & $B'A' = A'B'$ (diagonal)

$SABS^* = SBAS^*$

i.e. $S^*(SABS^*)S = S^*(SBAS^*)S$ i.e. $AB = BA$.

Proof of (3) $SAS^* = A'$ i.e. $A'^* = SA^*S^* = SAS^*$ ($\because A^* = A$)

i.e. $A'^* = A'$ & being diagonal matrix, it follows eigenvalues are real.

$AS^* = S^{-1}A'$ & hence $SA^* = A'^*(S^{-1})^*$ i.e. $SA^*S^* = A'^*(S^{-1})^*S^* = A'^*(SS^{-1})^* = A'^*$

Theorems assumed without proof :- (1) Every Hermitian matrix can be diagonalised by a unitary matrix & resulting eigen-values are unique

To be proved → (2) necessary & sufficient condn that two Hermitian matrices can be diagonalised by the same unitary matrix is that $AB = BA$.

To be proved → (3) Eigen values of a Hermitian matrix are all real

51st Lecture – 15/1/64 – Remaining parts of previous lecture –

52nd Lecture – 18/1/64 – Matrix representation of an operator A [for eg. $-\frac{\hbar^2}{2m}\nabla^2 + V(r)$] – Matrix repⁿ of AB – Matrix operator as a linear transformation $\Sigma C_n \psi_n \to \Sigma C'_m \psi_m$ – $\int \psi^* \psi \cdot dx = \Sigma C_m^* C_m$ (normalisation) – $0 = \int \psi_a^* \psi_b \cdot dx = \sum_{m,n} C_{am}^* C_{bn}$ (orthogonality of vectors) – Average value $\bar{A} = \Sigma C_j^* a_{ji} C_i$ – Change of representation from one set to another of orthonormal functions viz $A = \alpha^\dagger A' \alpha$

53rd Lecture – 20/1/64 – In the last lecture, I have changed the notation by using (*) for complex conjugate and (†) for Hermitean adjoint as in Bohm – Return to the old notation – Recapitulation

(1) $A\psi_m = \Sigma a_{nm} \psi_n$; (2) $A\psi = \Sigma C_m A\psi_m = \Sigma C_m a_{nm} \psi_n$

(3) $\int \bar{\psi}_r A \psi_m \cdot dx = \int \bar{\psi}_r \Sigma a_{nm} \psi_n \cdot dx = \Sigma a_{nm} \delta_{rn} = a_{rm}$.

(4) $(AB)_{ij} = \int \bar{\psi}_i AB \psi_j dx = \int \bar{\psi}_i A \Sigma b_{kj} \psi_k \cdot dx = \int \Sigma b_{kj} \bar{\psi}_i \Sigma a_{lk} \psi_l \cdot dx$

$= \Sigma\Sigma a_{lk} b_{kj} \int \bar{\psi}_i \psi_l \cdot dx = \Sigma\Sigma a_{lk} b_{kj} \delta_{il} = \Sigma a_{ik} b_{kj}$.

(5) Change of representation from $\psi_m(x) \to \varphi_p(x)$ given by $\psi_m = \Sigma \alpha_{pm} \varphi_p$ ~~(α matrix of change of representation)~~

$(A)_{mn} = \int \bar{\psi}_m A \psi_n \cdot dx = \int \Sigma \bar{\alpha}_{pm} \varphi_p A \Sigma \alpha_{qn} \varphi_q \cdot dx$

$= \Sigma\Sigma \bar{\alpha}_{pm} (A')_{pq} \alpha_{qn} = \Sigma\Sigma \alpha^*_{mp} (A')_{pq} \alpha_{qn}$

$(A)_{mn} = (\alpha^* A' \alpha)_{mn}$ or $A = \alpha^* A' \alpha$.

(5) Change of representation from $\psi_m \to \varphi_p$. given by $\psi_m = \Sigma S_{pm} \varphi_p$

~~$(A)_{mn} = \int \bar{\psi}_m A \psi_n \cdot dx = \int \bar{\psi}_m \Sigma a_{in} \psi_i \, dx$~~

~~$= \int \Sigma \bar{S}_{pm} \bar{\varphi}_p \Sigma$~~

$(A)_{mn} = \int \bar{\psi}_m A \psi_n \cdot dx = \sum_{p,q} \int \bar{S}_{pm} \bar{\varphi}_p A S_{qn} \varphi_q \cdot dx$

$= \sum_{p,q} \bar{S}_{pm} (A')_{pq} S_{qn} = \sum_{p,q} S^*_{mp} (A')_{pq} S_{qn}$

$A = S^* A' S$.

– unitary character of the matrix S. we have $\psi_m = \Sigma S_{pm} \varphi_p$ & $\int \psi_m \bar{\varphi}_q dx = \int \Sigma S_{pm} \bar{\varphi}_q \varphi_p \bar{\varphi}_q \cdot dx$

$S_{Kn} = \int \bar{\varphi}_K \psi_n \cdot dx$ $= \Sigma S_{pm} \delta_{pq} = S_{qm}$

$\sum_n \bar{S}_{Kn} \psi_n = \int \Sigma \varphi_K \psi_n \psi_n \cdot dx = \varphi_K$

$\delta_{rs} = \int \bar{\psi}_r \psi_s \cdot dx \to S^* S = 1$, using $\psi_r = S_{pr} \varphi_p$

& $\delta_{rs} = \int \bar{\varphi}_r \varphi_s dx \to S S^* = 1$, using $\varphi_r = \bar{S}_{rp} \psi_p$

(†) $(SHS^*)_{KL} = \sum_m \sum_n S_{Km} H_{mn} S^*_{nL} = \sum\sum S_{Km} H_{mn} \bar{S}_{Ln}$

$= \sum_m \sum_n \int \bar{u}_K v_m \, d\tau \int \bar{v}_m' H' v_n' \, d\tau' \int u_L'' \bar{v}_n'' \, d\tau''$

$= \sum_m \int \bar{u}_K v_m \, d\tau \iint \sum_n \bar{v}_m' H' u_L'' v_n' \bar{v}_n'' \, d\tau' d\tau''$

$= \sum_m \int \bar{u}_K v_m \, d\tau \iint \bar{v}_m' H' \delta(r'-r'') u_L'' \, d\tau'' d\tau''$ (Closure property)

~~$= \sum_m \int \bar{u}_K v_m \, d\tau \int \bar{v}_m' H' u_L' = \sum_m \int \bar{u}_K v_m \, d\tau \cdot H' \int \delta(r'-r'') u_L'' \, d\tau''$~~

~~$= \sum_m \int \bar{u}_K v_m \, d\tau$~~

$= \sum_m \int \bar{u}_K v_m \, d\tau \int \bar{v}_m' H' \, d\tau' \int u_L'' \delta(r'-r'') \, d\tau''$

$= \sum_m \int \bar{u}_K v_m \, d\tau \int v_m' H' u_L' \cdot d\tau'$ [from $\sum_K \bar{u}_K(r') u_K(r) = \delta(r-r')$, we get $\int u_L' \delta(r-r') \, d\tau = \int \sum_K \bar{u}_K' u_L' u_K \, d\tau' = \sum_K u_K \delta_{KL} = u_L$]

$= \iint \bar{u}_K H' u_L' \sum_m v_m \bar{v}_m' \, d\tau d\tau'$

$= \iint \bar{u}_K H' u_L' \delta(r-r') \, d\tau d\tau'$

$= \int \bar{u}_K H u_L \cdot d\tau = H_{KL}$ in the u_K-repⁿ $= E_K \delta_{KL}$.

(††) $u_K(\vec{r})$ labelled by $\vec{r}$ & K can be considered a matrix, say U_{rK}

The closure property $\sum_K u_K(\vec{r}) \bar{u}_K(\vec{r}') = \delta(r-r')$

and the orthonormal property $\int \bar{u}_K(\vec{r}) u_L(\vec{r}) \, d\tau = \delta_{KL}$, of the eigenfns

show that $(U U^*)_{rr'} = \sum_K U_{rK} U^*_{Kr'} = \sum_K U_{rK} \bar{U}_{r'K} = \delta(r-r') = (1)_{rr'}$

$(U^* U)_{KL} = \int U^*_{Kr} U_{rL} \cdot d\tau = \int \bar{U}_{rK} U_{rL} \cdot d\tau = \delta_{KL} = (1)_{KL}$

ie U is unitary.

<u>An useful identity</u>. If $\Omega(r,r')$ be an operator, and $\Omega f(r) = \int \Omega(r,r') f(r') \, d\tau'$, we can show

$$\int \bar{g}(r) [\Omega f(r)] \, d\tau = \int \overline{[\Omega^* g(r)]} f(r) \, d\tau$$

for, representing $\Omega(r,r')$ as a matrix in the r-repⁿ, $\Omega^* g(r) = \int \overline{\Omega(r',r)} g(r') \, d\tau'$

$\therefore \int \overline{[\Omega^* g(r)]} f(r) \, d\tau = \iint \Omega(r',r) \bar{g}(r') f(r) \, d\tau d\tau'$

$= \int \bar{g}(r) \int \Omega(r,r') f(r') \, d\tau' \cdot d\tau$

$= \int \bar{g}(r) [\Omega f(r)] \, d\tau.$

54th Lecture — 22/1/64 — (A) Hamiltonian operator $H = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})$ — $H u_k(\vec{r}) = E_k u_k(\vec{r})$ being Schrodinger's eqn with u_k as energy eigenfunctions & E_k energy eigenvalues. Transformation of u_k by the unitary transformation S to v_n, operating on H gives

(+)

$$(SHS^*)_{kl} = E_l \delta_{kl} \quad \text{ie in diagonal form with eigenvalues } E_k$$

Hence (i) Problem of solving Schrodinger eqn is completely equivalent to diagonalising matrix H in any arbitrary repn v_n — Eigenvalues of matrix H are energy eigenvalues of the Schrodinger eqn — the unitary S diagonalising H gives u_k in terms of v_n.

(ii) Eigenvalues of H are real being real energy eigenvalues & hence H is a Hermitian matrix in any representation (proved by taking Hermitian adjoint of $SHS^* = H'$; H' real & $SH^*S^* = H'^* = H'$ ie $H = H^*$)

(iii) What is true of H is true of any dynamical variable whose eigenvalues are real. Thus any physically measurable dynamical variable is Hermitian.

55th Lecture — 25/1/64

(++) (B) Equations of motion in matrix form — $i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t} = H\psi(\vec{r},t)$ — $\int \bar{\phi} F \psi \, d\tau = \iint \bar{\phi}(r,t) F(r,r',t) \psi(r',t) d\tau d\tau'$

[Instead of $\bar{u}_k$ & u_k use ϕ & ψ]

$$\frac{d}{dt}\int \bar{\phi} F \psi \, d\tau = \iint \bar{\phi}(\vec{r},t)\left[\frac{\partial}{\partial t}F(r,r',t)\right]\psi(r',t)\, d\tau d\tau' + \iint \bar{\phi}(r,t) F(r,r',t)\left[\frac{\partial}{\partial t}\psi(r',t)\right] d\tau d\tau'$$

$$+ \iint \left[\frac{\partial}{\partial t}\bar{\phi}(r,t)\right] F(r,r',t)\, \psi(r',t)\, d\tau d\tau'$$

$$= \int \bar{\phi}\frac{\partial F}{\partial t}\psi \, d\tau + \frac{1}{i\hbar}\iint \bar{\phi}(r,t) F(r,r',t)\left[H'\psi(r',t)\right] d\tau d\tau'$$

$$- \frac{1}{i\hbar}\iint \left[H\bar{\phi}(r,t)\right] F(r,r',t)\, \psi(r',t)\, d\tau d\tau'$$

Now $\iint H\bar{\phi}(r,t) F(r,r',t)\psi(r',t)\, d\tau d\tau' = \iint \overline{H^*\phi(r,t)}\, F(r,r',t)\psi(r',t)\, d\tau d\tau'$ $[H = \bar{H} = H^*]$

If effect of operator H on any fn $f(r)$ be given by $Hf(r) = \int H(r,r') f(r') d\tau'$

then $H^* g(r) = \int \overline{H(r',r)}\, g(r')\, d\tau'$ [Considering H as a matrix in r & r']

$$\therefore \iint \overline{H^* g(r)}\, f(r)\, d\tau = \iint H(r',r)\, \bar{g}(r') f(r)\, d\tau d\tau'$$

$$= \iint H(r,r') f(r') \bar{g}(r)\, d\tau d\tau'$$

$$= \int \bar{g}(r) H f(r)\, d\tau$$

Hence the third integral $= \iint \bar{\phi}(r,t) H F(r,r',t)\psi(r',t)\, d\tau d\tau'$

$$\therefore \frac{d}{dt}\int \bar{\phi} F \psi \, d\tau = \int \bar{\phi}\frac{\partial F}{\partial t}\psi + \frac{1}{i\hbar}\int \bar{\phi}(FH - HF)\psi \, d\tau$$

ϕ & ψ being arbitrary $\rightarrow \frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{1}{i\hbar}(FH - HF)$ — Heisenberg's form of eqn of motion of F.

Classical eqns of motions in terms of P.B's are $\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}$

Hence suggestion of finding quantum analogues using $\{AB\} = \frac{1}{i\hbar}[AB] = \frac{1}{i\hbar}(AB - BA)$

Support for this suggestion: (i) For C.T's condns in classical mechanics are $\{Q_i, P_j\} = \delta_{ij}$, $\{Q_i, Q_j\} = \{P_i, P_j\} = 0$

and quantum analogues are $[x, p_x] = [y, p_y] = [z, p_z] = i\hbar$

$[x, p_y] = [y, p_x] = \text{etc} = \cdot = 0$

$[x, y] = [p_x, p_y] = \text{etc} = \cdot = 0$

obtained from $p_x = -i\hbar\frac{\partial}{\partial x}$ used to operate on any $g(\vec{r})$

$$(xp_x - p_x x)\, g(\vec{r}) = -i\hbar x \frac{\partial g}{\partial x} + i\hbar\frac{\partial}{\partial x}(xg) = i\hbar g(\vec{r})$$

$$[AB] = AB - BA$$

⊛ $$[q_i, p_j] = i\hbar \{q_i, p_j\} = i\hbar \sum_m \left(\frac{\partial q_i}{\partial q_m} \cdot \frac{\partial p_j}{\partial p_m} - \frac{\partial q_i}{\partial p_m} \cdot \frac{\partial p_j}{\partial q_m} \right)$$

$$= i\hbar \sum_m \frac{\partial q_i}{\partial q_m} \cdot \frac{\partial p_j}{\partial p_m} = i\hbar \delta_{ij}$$

$$[q_i, q_j] = i\hbar \{q_i, q_j\} = i\hbar \sum_m \left(\frac{\partial q_i}{\partial q_m} \frac{\partial q_j}{\partial p_m} - \frac{\partial q_i}{\partial p_m} \frac{\partial q_j}{\partial q_m} \right) = 0$$

and similarly $[p_i, p_j] = 0$

⊕ From $\vec{M} = \vec{r} \times \vec{p}$, since $\vec{r}$ & $\vec{p}$ are real physical quantities & hence hermitian, so also is $\vec{M}$ & hence M_x, M_y & M_z & $\vec{M}^2$ also. Since this can be taken over as true for the general defns of M_x, M_y, M_z by $[M_y, M_z] = i\hbar M_x$, etc..

$$M_y M_z - M_z M_y = i\hbar M_x$$

Taking hermitian conjugates $M_z^* M_y^* - M_y^* M_z^* = -i\hbar M_x^*$

i.e. $M_z M_y - M_y M_z = -i\hbar M_x$

i.e. $M_y M_z - M_z M_y = i\hbar M_x$. $[M_y, M_z) = i\hbar M_x$

(ii) algebraic properties of commutators are same as for P.B's viz

$$\{A, B\} = -\{B, A\}, \quad \{A, C\} = 0 \quad (C, \text{a number})$$

$$\{(A_1 + A_2), B\} = \{A_1, B\} + \{A_2, B\}$$

$$\{A, A_2, B\} = \{A_1, B\} A_2 + A_1 \{A_2, B\}$$

$$\{A, \{B, C\}\} + \{B\{C, A\}\} + \{C, \{A, B\}\} = 0$$

– quantisation technique to be applies to classical systems viz. taking classical eqns of motion in terms of classical H and q_i, p_i & replacing P.B's by commutators. The canonical variables obey $[q_i, p_j] = i\hbar \delta_{ij}$; $[q_i, q_j] = [p_i, p_j] = 0$. (*)

(C) Angular momentum of a point $\vec{M} = \vec{r} \times \vec{p}$. From the above $[x, p_x] = [y, p_y] = [z, p_z] = i\hbar$, while all other commutators are zero. Therefore $[M_x, M_y] = (yp_z - zp_y)(zp_x - xp_z) - (zp_x - xp_z)(yp_z - zp_y)$

[$\vec{M}$ hermitian]

$$= yp_z zp_x - yp_z \cancel{xp_z} - zp_y \cancel{zp_x} + zp_y xp_z$$
$$- zp_x yp_z + xp_z \cancel{yp_z} + zp_x \cancel{zp_y} - xp_z zp_y$$
$$= yp_x (p_z z - zp_z) + xp_y (zp_z - p_z z) = i\hbar (xp_y - yp_x)$$

ie $[M_x, M_y] = i\hbar M_z$ & similarly $[M_y, M_z] = i\hbar M_x$ and $[M_z, M_x] = i\hbar M_y$

These hold also for total A.M of a system of particles & are taken as more fundamental than $\vec{M} = \vec{r} \times \vec{p}$.

[Earlier results $\vec{M}^2 Y_{lm}(\theta, \phi) = l(l+1)\hbar^2 Y_{lm}(\theta, \phi)$ (Sph. harmonics)

$M_z Y_{lm}(\theta, \phi) = m\hbar Y_{lm}(\theta, \phi)$

ie $Y_l(\theta, \phi)$ are eigen-fns of both $\vec{M}^2$ and M_z, l being orbital or azimuthal quantum number, and m is the magnetic quantum number. ie $\vec{M}^2$ and M_z can be specified together in the quantum theory]

From $[M_y, M_z] = i\hbar M_x$, etc, it is clear that no two of M_x, M_y, M_z commute with each other & hence it is not possible to find a representation that diagonalises more than one of them. However all three components commute with M^2, for eg $[M_z, \vec{M}^2] = M_z M_x^2 - M_x^2 M_z + M_z M_y^2 - M_y^2 M_z$

$$= [M_z, M_x^2] + [M_z, M_y^2] = [M_z, M_x] M_x + M_x [M_z, M_x]$$
$$+ [M_z M_y] M_y + M_y [M_z, M_y]$$

$$= i\hbar (M_y M_x + M_x M_y) - i\hbar (M_x M_y + M_y M_x) = 0.$$

ie both M_z and $\vec{M}^2$ can be diagonalised at the same time as in the earlier results

56th Lecture – 27/1/64 – To find simultaneous diagonal repn of M_z and $\vec{M}^2$, and the corresp. repns of M_x & M_y, introduce $L = M_x + iM_y$ which is non-Hermitian. Note that M_x, M_y, M_z & $\vec{M}^2$ are all Hermitian (†), so that $L^* = M_x^* - iM_y^* = M_x - iM_y$ ie $M_x = \frac{1}{2}(L^* + L)$, $M_y = \frac{1}{2} i (L^* - L)$

Also, $[\vec{M}^2, L] = \cancel{[M_x^2, L]} + \cancel{[M_y^2, L]} + \cancel{[M_z^2, L]} = \cancel{[M_x^2, iM_y]} + \cancel{[M_y^2, M_x]} + \cancel{[M_z^2, M_x]} + \cancel{[M_z^2, iM_y]}$

$= [\vec{M}^2, M_x] + i[\vec{M}^2, M_y] = 0$, since $\vec{M}^2$ commutes with all three components.

$[M_z, L] = [M_z, M_x] + i[M_z, M_y] = [M_z, M_x] - i[M_y, M_z] = i\hbar M_y + \hbar M_x = \hbar L$

and $[L, L^*] = -2i[M_x, M_y] = 2\hbar M_z$. – Also $\vec{M}^2 = M_z^2 + \frac{1}{2}(LL^* + L^* L)$

* The eqns in terms of $Y_{lm}(\theta,\phi)$ for $\vec{M}^2$ & M_z show that the matrices derived from $\vec{M} = \vec{r} \times \vec{p}$, have the same form constitute another representation of the matrices derived here, for integral values. All integral j representations derived here taken together constitute a single rep^{n} of ∞-rank with the $\vec{M}$ matrices satisfying $\vec{M} = \vec{r} \times \vec{p}$.

In a repⁿ in which M_z is a diagonal matrix, any diagonal element $= m\hbar$ (where $\underline{m}$ is a real number, not necessarily an integer), and we can denote the diagonal element by the symbol $\underline{m}$. From eqⁿˢ $[M_z, L] = \hbar L$, and $[L, L^*] = 2\hbar M_z$ we see that L also can be represented by only the same symbol m.

$$[M_z, L] = \hbar L \text{ gives } (M_z L)_{m', m''} - (L M_z)_{m' m''} = \hbar L_{m', m''}$$

$$\text{ie } \sum_m \left\{ (M_z)_{m'm} L_{mm''} - L_{m'm} (M_z)_{mm''} \right\} = \hbar L_{m', m'}$$

$$\text{ie } \quad m'\hbar \cdot L_{m', m''} - L_{m', m''} \cdot m''\hbar = \quad ''$$

$$(m' - m'' - 1) L_{m', m''} = 0 \quad \text{ie } L_{m', m''} \neq 0 \text{ only if } m' = m'' + 1. \text{ \& say } L_{m+1, m} = \lambda_m \hbar.$$

(λ_m may be a complex no)

Taking diagonal element of $[L, L^*] = 2\hbar M_z$, corresponding to $m\hbar$ of M_z gives

$$(LL^*)_m - (L^*L)_m = 2\hbar^2 m$$

$$\sum_{m'} \left\{ L_{mm'} L^*_{m'm} - L^*_{mm'} L_{m'm} \right\} = 2\hbar^2 m$$

$$\sum_{m'} \left\{ L_{mm'} \bar{L}_{mm'} - \bar{L}_{m'm} L_{m'm} \right\} = 2\hbar^2 m$$

$$\text{ie } \quad L_{m, m-1} \bar{L}_{m, m-1} - L_{m+1, m} L_{m+1, m} = 2\hbar^2 m$$

$$\text{ie } \quad |\lambda_{m-1}|^2 - |\lambda_m|^2 = 2m$$

which is a difference eqⁿ whose solⁿ is $|\lambda_m|^2 = C - m(m+1)$ ($\underline{C}$ a const)

$|\lambda_m|^2 \geq 0$ & so $\underline{m}$ cannot take arbitrarily large +ve or −ve values

ie $\underline{m}$ must lie between certain limits & have a finite range. If $\lambda_m = 0$ for m_1 & m_2 ($m_1 > m_2$)

ie if m_1 & m_2 be solns of $C - m(m+1) = 0$, then these m_1 & m_2 determine the range of $\underline{m}$

and being solns of the difference eqⁿ, the several values of $\underline{m}$ from m_1 downwards must each diminish by unity

ie $(m_1 - m_2)$ is an integer. From $(m' - m'' - 1) L_{m', m''} = 0$, we have if $m' = m_1$, $L_{m_1, m_1 - 1} = 0$

57th Lecture – 28/1/64. ~~& if m_2 being smallest values of~~ $L_{m_2+1, m_2} = 0$ gives $m' = m_2 + 1$ ie the range is from m_1 to $m_2 + 1$ and not m_2 (in fact $m'' = m_2$).

Solving $C - m(m+1) = 0$, $m_1 = -\frac{1}{2} + \frac{1}{2}(1+4C)^{\frac{1}{2}}$, $m_2 = -\frac{1}{2} - \frac{1}{2}(1+4C)^{\frac{1}{2}}$

Calling $m_1 = j$, $C = j(j+1)$, & $m_2 = -1 - j$ ie $m_2 + 1 = -j$

ie $\underline{m}$ varies from j to $-j$ by unit steps, which shows that $2j$ = integer ie j = integer or half-odd integer.

From $\vec{M}^2 = M_z^2 + \frac{1}{2}(LL^* + L^*L)$,

$= M_z^2 + \frac{1}{2}\left\{ |\lambda_{m-1}|^2 + |\lambda_m|^2 \right\} \hbar^2$ and using $|\lambda_m|^2 = j(j+1) - m(m+1) = (j-m)(j+m+1)$

also diagonal and value of element corresponding to $m\hbar$ of M_z

$= \left[m^2 + \frac{1}{2}\left\{ j(j+1) - (m-1)m + j(j+1) - m(m+1) \right\} \right] \hbar^2 = j(j+1)\hbar^2.$

~~Matrices for $j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$~~ $L_{m+1, m} = \lambda_m \hbar$ gives $L_{m+1, m} = \{(j-m)(j+m+1)\}^{\frac{1}{2}} \hbar$

Put the non-vanishing elements below the zero in the first col, above the zeros in the diagonal in the other columns so as to get L – Get the matrices for the several cases of

~~(0)~~ $j = 0, \frac{1}{2}, 1, \frac{3}{2}$ — (i) For integral values*, the $\vec{M}$ matrices can be expressed in terms of $\vec{r}$ and $\vec{p}$ matrices

58th Lecture – 29/1/64 using $\vec{M} = \vec{r} \times \vec{p}$ since $p_x = -i\hbar \frac{\partial}{\partial x}$, .. lead for eg. to a factor $\hbar$ in M_z, but this is not true

* $\vec{M}^2$ commutes with H since it commutes with the coordinates & momenta & $\therefore \frac{d\vec{M}^2}{dt} = 0$.

† Largest value of $m = m_1 + m_2$ and corresponding $j = j_1 + j_2$. The next smaller values of m and j are resp'ly $m_1 + m_2 - 1$ & $j_1 + j_2 - 1$, and so on. The ~~least~~ least value of $j = j_1 - j_2$ ~~corresponding value of $m = m_1 - m_2$ since~~ ~~since $(\vec{M}_1 + \vec{M}_2)^2$~~

$(j_1 > j_2$, say)

$= \cancel{\vec{M}_1^2 + \vec{M}_2^2 + 2\vec{M}_1 \cdot \vec{M}_2} = \cancel{j_1^2 + j_2^2 + 2j_1 j_2 \cos\theta}$ ~~and so the greatest & least j^2 values are given $(j_1 + j_2)^2$ and $(j_1 - j_2)^2$. For each j~~ there are $2j+1$ independent states.

not correct ?

BSM 8/2/65

See p. 56.

Hence the no. of rows or cols of the 2nd representation

$$= \sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) \quad \text{ie} \quad (j_1+j_2) - (j_1-j_2) + 1 \text{ terms ie } 2j_2+1 \text{ terms in the summation}$$

$$= \sum_{j_1-j_2}^{j_1+j_2} 2j + (2j_2+1)$$

$$= 2(j_1-j_2) + 2(j_1-j_2+1) + \cdots \text{ to } 2j_2+1 \text{ terms} + (2j_2+1)$$

$$= 2(j_1-j_2)(2j_2+1) + 2(1+2+\cdots \text{ to } 2j_2 \text{ terms}) + (2j_2+1)$$

$$= 2(j_1-j_2)(2j_2+1) + 2\cdot\tfrac{1}{2}\cdot 2j_2(2j_2+1) + (2j_2+1)$$

$$= 4j_1j_2 + 2j_1 + 2j_2 + 1 = (2j_1+1)(2j_2+1).$$

of matrices corresponding to half-odd integral j, for eg. for $j = \frac{1}{2}$, M_z has the factor $\frac{1}{2}\hbar$ & also for $j = 3/2$. Thus for orbital A.M., the eigenvalues of matrices of components of $\vec{M}$ must be integral multiples of $\hbar$.

(ii) An intrinsic A.M. defined by $[M_y, M_z] = i\hbar M_x$, etc & not satisfying $\vec{M} = \vec{r} \times \vec{p}$ is more general, and eigenvalues of such $\vec{M}$ may be integers or half-odd integers. Also $\vec{M}^2$ has a single eigenvalue (for a single value of j) and it might commute with the matrices representing $\vec{r}$ & $\vec{p}$. Then it will commute with all dynamical variables describing the particle & will be a constant of the motion*. This is not possible with orbital A.M. since in this case $\vec{M}^2$ does not commute with $\vec{r}$ and $\vec{p}$ [Consider for eg. $[\vec{M}^2, y] \to [M_z^2, y] \to [M_z, y] \to [p_y, y] \neq 0$, but $= -i\hbar$]

(iii) Intrinsic A.M. called spin A.M, electrons, positrons, protons, anti-protons, neutrons & μ-mesons have $j = \frac{1}{2}$; π-mesons have $j = 0$; other mesons with $j = 1$; but none observed so far with $j > 1$ ie $3/2$, 2, etc (j = spin)

(iv) Addition of angular momenta – $\vec{M} = \vec{M}_1 + \vec{M}_2$ with $[\vec{M}_1, \vec{M}_2] = 0$ and both $\vec{M}_1$ & $\vec{M}_2$ satisfying the spin commutation relations. $\vec{M}_1$ & $\vec{M}_2$ may be A.M of two separate particles or spin & orbital A.M. of same particle [for eg. spin A.M. of a particle of spin $1/2$ & orbital A.M. of same particle with the expn of A.M for $j = 1$]. $\vec{M}$ obeys same spin commutation relations as $\vec{M}_1$ and $\vec{M}_2$ for eg $[M_x, M_y] = [M_{1x} + M_{2x}, M_{1y} + M_{2y}]$

$= [M_{1x}, M_{1y}] + [M_{2x}, M_{2y}] + [M_{1x}, M_{2y}] + [M_{2x}, M_{1y}]$; the last two brackets are zero since $[\vec{M}_1, \vec{M}_2] = 0$

$= i\hbar(M_{1z} + M_{2z}) = i\hbar M_z$ – Two representations (a) $\vec{M}_1^2$, $\vec{M}_2^2$, M_{1z} and M_{2z} are diagonal specified by j_1, j_2 and m_1, m_2. For fixed j_1 and j_2 the additive repn. has $(2j_1+1)(2j_2+1)$ rows and columns, for eg considering M_z for $j_1 = \frac{1}{2}$ & $j_2 = 1$, ~~[$M_{1z} + M_{2z}$ has six terms in the diagonal given by adding each of $1/2$ & $-\frac{1}{2}$ with 1, 0, −1]~~ (b) $\vec{M}_1^2$, $\vec{M}_2^2$, $\vec{M}^2$ and M_z are diagonal, specified by j_1, j_2, j & m. Here also the matrices of the repn has $(2j_1+1)(2j_2+1)$ rows & columns.†

Correct BSM 4/2/65 ~~not correct BSM 4/2/65~~

59th Lecture – (†) ~~and Examples~~

60th – 4/2/64 Identical particles constitute a system in which particles can be substituted for each other under the most general possible circumstances with no change in the physical situation – Schrödinger eqn for n identical particles is: $i\hbar \frac{\partial}{\partial t}\psi(1, 2, \cdots n; t) = H(1, 2, \cdots n)\psi(1, 2, \cdots n; t)$, where any index i denotes positional & spin coordinates & H is symmetric in ~~the~~ all the arguments – Symmetric and antisymmetric wave fns are both solns of the above Schrödinger equation since H is symmetric – Symmetry or antisymmetry ~~does~~ does not change with time – Experimental evidence re. electrons, protons & neutrons being described by antisymmetric wave functions – Construction from unsymmetrised wave functions (i) Sum of all the $n!$ wave fns symmetric, (ii) sum of permuted fns arising from even no. of permutations – Sum of fns arising from odd no. of permutations is antisymmetric, for, a further ~~change~~ interchange changes odd $\rightleftarrows$ even & sign of ψ is changed. If original fn be symmetric, an antisymmetric soln cannot be formed

H not involving time, $\psi = u(1, 2, \cdots n)e^{-iEt/\hbar}$ with $Hu = Eu$ – Energy eigenvalue degenerate when two or more independent eigenfunctions correspond to it; here we have exchange degeneracy – symmetric & antisymmetric combinations for $n = 2$ & $n = 3$ – Exclusion Principle (Pauli's) – appln to periodic systems.

61st

62nd

* $E^2 = c^2p^2 + m_0^2c^4$ follows from $E = mc^2 = \dfrac{m_0c^2}{\sqrt{1-v^2/c^2}}$, which gives

for $E^2(1 - v^2/c^2) = m_0^2c^4$ ie $E^2 = \dfrac{E^2v^2}{c^2} + m_0^2c^4 = \dfrac{m^2c^4v^2}{c^2} + m_0^2c^4$

$= c^2p^2 + m_0^2c^4$.

63rd

Pauli spin matrices:

$$s = \tfrac{1}{2}, \quad M_x = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_y = \tfrac{1}{2}\hbar \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$M_z = \tfrac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M^2 = \tfrac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

64th

(†) $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$ & two other relations.

and $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1$ $\quad (\sigma_1 = \sigma_x, \sigma_2 = \sigma_y, \sigma_3 = \sigma_z)$

(#) $\gamma_1 = \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$; $\gamma_2 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

$\gamma_3 = \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$; $\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

65th

With the notation in Schiff viz $\alpha_K = i\gamma_4\gamma_K$, $\beta = \gamma_4$

we have $\alpha_1 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} = i\begin{pmatrix} 0 & -i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$\alpha_2 = i\gamma_4\gamma_2 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$

$\alpha_3 = i\gamma_4\gamma_3 = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$

& $\beta = \gamma_4$ & These matrices are as in Schiff, p. 326.

(↓

61st Lecture – 5/2/64 – Pauli's Exclusion Principle – Atomic periodic systems.

62nd Lecture – 8/2/64 – Some more periodicities in the ground-state electron configurations of the elements.

(1) Alkalis – Li, Na, K, Rb, Cs, Fr (3, 11, 19, 37, 55, 87) – The first electron to go into each ~~shell~~ s shell (beyond 1s) gives an alkali.

Enhancer for Expressions for the configurations of all these elements.

(2) Rare gases – Elements just before each of the alkalis with full 1s shell or a full p-shell – He, Ne, A, Kr, Xe, Rn (2, 10, 18, 36, 54, 86) – Indication of expressions for these from (1) by deleting the last s-electron.

(3) Elements corresponding to full bracketed shells – Zn, Cd, Hg (30, 48, 80).

(4) Noble metals – Cu, Ag, Au – with one ~~e~~ electron missing from the full bracketed shells (29, 47, 79).

– Bose-Einstein & Fermi-Dirac Statistics – Connection between spin & statistics. ✓

63rd Lecture – 25/2/64 – Relativistic wave eqns (as from Mandl) – Klein-Gordan eqn & its relativistic invariance* – first order Dirac eqn $\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + M\right)\psi(x_\nu) = 0$ with $\frac{1}{2}(\gamma_\lambda \gamma_\mu + \gamma_\mu \gamma_\lambda) = \delta_{\lambda\mu}$; ψ a column-matrix & γ_μ a ~~4×4~~ matrix & matrix multiplication used – components of wave fns called spinors – γ_μ as 4×4 matrices – use of $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, $\gamma_5^2 = 1$, $\gamma_\lambda\gamma_5 + \gamma_5\gamma_\lambda = 0$. ✓ & $\gamma_\lambda^2 = 1$

64th Lecture – 26/2/64 – $1, \gamma_\lambda, i\gamma_\lambda\gamma_\mu, i\gamma_5\gamma_\lambda, \gamma_5$ as the 16 basic quantities, which agree with $1, \gamma_\lambda, \gamma_\lambda\partial_\mu$, & $\gamma_\lambda\gamma_\mu\gamma_\nu$, $\gamma_1\gamma_2\gamma_3\gamma_4$ except for $\pm i$, which are introduced so as to make the square of each of the basic quantities equal to 1 – Introduction of the Hermitian form of the γ_λ by use of the Pauli spin matrices ⊕ viz $\gamma_K = \begin{pmatrix} 0 & -i\sigma_K \\ i\sigma_K & 0 \end{pmatrix}$, (K = 1, 2, 3), $\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with $\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ – γ_λ as 4×4 matrices (explicit obtained) ⊕ – $\gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ – Verification of the commutation relations in the forms $\frac{1}{2}(\gamma_\lambda\gamma_\mu + \gamma_\mu\gamma_\lambda) = \delta_{\lambda\mu}$ and $\gamma_5^2 = 1$, $\gamma_5\gamma_\lambda + \gamma_\lambda\gamma_5 = 0$ by applying multiplication of square matrices of same order, with elements themselves being matrices of the same order (ie 4×4 matrices, whose elements are 2×2 matrices), and using the properties ⊕. ✓ – Relativistic Pauli spin matrices (4×4) given by $\hat{\sigma}_K = \begin{pmatrix} \sigma_K & 0 \\ 0 & \sigma_K \end{pmatrix}$, (K = 1, 2, 3), which can be shown equal to $i\gamma_4\gamma_5\gamma_K$ by actual multiplication of matrices for γ_4, γ_5 & γ_K ; also $\hat{\sigma}_K = -i\gamma_L\gamma_M$ (K, L, M in cyclic order), for eg $\hat{\sigma}_1 = i\gamma_4\gamma_5\gamma_1 = i\gamma_4\gamma_1\gamma_2\gamma_3\gamma_4\gamma_1 = -i\gamma_2\gamma_3$; etc – The Dirac algebra (semi-simple) & unique representation of γ_λ as 4×4 matrices ($4^2 = 16$ being the rank relation) but for equivalence. ✓

65th Lecture – (Special) – 27/2/64 – (a) Particle density & current density in terms of spinors – If ψ^* be Hermitian adjoint spinor of ψ, but $\bar{\psi} = \psi^*\gamma_4$ (ψ^* a one-row matrix) a Hermitian adjoint of the Dirac eqn $\gamma_\mu\frac{\partial\psi}{\partial x_\mu} + M\psi = 0$ is $\frac{\partial\psi^*}{\partial x_\mu}\gamma_\mu + M\psi^*$ (since $\gamma_\mu^* = \gamma_\mu$) and $\bar{\psi}\gamma_4 = \psi^*\gamma_4^2 = \psi^*$ & the Hermitian adjoint eqn becomes ($\psi^* = \bar{\psi}\gamma_4$) $\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_4\gamma_\mu + M\bar{\psi}\gamma_4 = 0$ ie $-\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\mu\gamma_4 + M\bar{\psi}\gamma_4 = 0$ or $\frac{\partial\bar{\psi}}{\partial x_\mu}\gamma_\mu - M\bar{\psi} = 0$ – Particle

* $\frac{\partial(\bar\psi \gamma_\mu \psi)}{\partial x_\mu} = \frac{\partial \bar\psi}{\partial x_\mu}\gamma_\mu\psi + \bar\psi\gamma_\mu\frac{\partial\psi}{\partial x_\mu}$

$= M\bar\psi\psi - M\bar\psi\psi = 0$, from Dirac eqn for ψ & $\bar\psi$.

(+) $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ becomes (since $x_4 = ict$, and $p_k = i\hbar\frac{\partial}{\partial x_k}$)

$-\hbar c\frac{\partial\psi}{\partial x_4} = ic\gamma_4\gamma_k\left(-i\hbar\frac{\partial\psi}{\partial x_k}\right) + mc^2\gamma_4\psi$

$i\gamma_4\gamma_k\frac{\partial\psi}{\partial x_k} + \frac{\partial\psi}{\partial x_4} + \frac{mc}{\hbar}\gamma_4\psi = 0$

& multiplying by γ_4 on the left this reduces to

$\gamma_k\frac{\partial\psi}{\partial x_k} + \gamma_4\frac{\partial\psi}{\partial x_4} + M\psi = 0$ i.e. $\gamma_\mu\frac{\partial\psi}{\partial x_\mu} + M\psi = 0$, the Dirac eqn

(* *) $\vec E = -\frac{1}{c}\frac{\partial\vec A}{\partial t} - \text{grad}\,\phi = -\frac{1}{c}\frac{\partial\vec A}{\partial t} + i\,\text{grad}\,\phi$ (since $A_4 = i\phi$)

$\vec H = \text{curl}\,\vec A$

for a pure magnetic field constant in time requires $\vec A$ to be independent of time & $E = 0$

i.e. $\text{grad}\,\phi = 0$ or $\phi = \text{const}$. But because of gauge invariance $\vec A' = \vec A + \text{grad}\,f$,

$\phi' = \phi - \frac{1}{c}\frac{\partial f}{\partial t}$, we have for an arbitrary f, we can choose f so as to make $\phi = 0$ i.e. $A_4 = 0$.

Note :- f cannot be chosen to make $\vec A = 0$ since this means three auxiliary conditions i.e

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(+ +) That this is true for all μ, we note that for k, $p_k \to p_k + \frac{e}{c}A_k$ means

$$-i\hbar\frac{\partial}{\partial x_k} \to -i\hbar\frac{\partial}{\partial x_k} + \frac{e}{c}A_k$$

$$\text{or}\quad \frac{\partial}{\partial x_k} \to \frac{\partial}{\partial x_k} + \frac{ie}{\hbar c}A_k$$

To show that this also holds for $\mu = 4$, we have to show that $p_4 = -i\hbar\frac{\partial}{\partial x_4}$ i.e $p_4 = -\frac{\hbar}{c}\frac{\partial}{\partial t} = \frac{iE}{c}$

which is the correct expression for the fourth component of the momentum. This can be seen in

two ways (i) from $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ we have the form $ds = cdt\sqrt{1 - v^2/c^2}$

and the four-velocity $v_\mu = \frac{dx_\mu}{ds}$ i.e $v_4 = \frac{dx_4}{ds} = \frac{icdt}{cdt\sqrt{1-v^2/c^2}} = \frac{i}{\sqrt{1-v^2/c^2}}$

and the four-momentum $p_\mu = mcv_\mu$ i.e $p_4 = \frac{imc}{\sqrt{1-v^2/c^2}} = \frac{i}{c}\cdot\frac{mc^2}{\sqrt{1-v^2/c^2}} = \frac{iE}{c}$.

or (ii) Just as for the four-vector $(x_1, x_2, x_3, x_4 = ict)$, the invariant is given by ds^2, we have in

$E^2 = m^2c^4 + \vec p^2c^2$ $\quad \frac{E^2}{c^2} - p^2 = m^2c^2$ i.e four momentum is $\left(\vec p, \frac{iE}{c}\right)$.

density defined as $\rho(x_\rho) = \psi^* \psi = \bar\psi \gamma_4 \psi$, supporting definition of current density by $s_K = ic\,\bar\psi \gamma_K \psi$. With the four-vector of density defined by $s_\mu = i\,\bar\psi \gamma_\mu \psi$ (i.e. $s_K = s_K/c$ and $s_4 = i\rho$), it can be shown that the eqn of continuity $\partial s_\mu/\partial x_\mu = 0$ is satisfied * and justifies choice of s_K as current density.

(b) Spin ~~& magnetic moment~~ of the electron. – Consider the operator $\vec J = \vec M + \frac{1}{2}\hbar\vec\sigma$ i.e. $J_K = M_K + \frac{1}{2}\hbar\sigma_K$, where $\vec M = (\vec r \times \vec p)$ is the orbital A.M. The Dirac eqn in Hamiltonian form is $i\hbar\frac{\partial\psi}{\partial t} = H\psi$, where $H = ic\gamma_4\gamma_K p_K + mc^2\gamma_4$ ⊕ – Proof that $\vec M$ does not commute with H & hence is not a constant of the motion follows from $[\vec M, H] = -\hbar c\gamma_4(\vec r \times \vec p_\theta)$ – Proof that $\frac{1}{2}\hbar\vec\sigma$ also does not commute with H follows from $[\frac{1}{2}\hbar\vec\sigma, H] = \hbar c\gamma_4(\vec\pi \times \vec p_\theta)$ – From these two it follows that $[\vec J, H] = 0$ i.e. J is a constant of the motion i.e. the total A.M is constant, giving interpretation of $\frac{1}{2}\hbar\vec\sigma$ as the spin A.M i.e. spin of the particle defined by the Dirac eqn is ½ i.e. electron spin ½.

$$[M_1, H] = [M_1, ic\gamma_4\gamma_K p_K] = ic\gamma_4\gamma_k[M_1, p_k] = -\hbar c\gamma_4(\gamma_2 p_3 - \gamma_3 p_2)$$

using $[M_1, p_1] = 0$, $[M_1, p_2] = i\hbar p_3$, $[M_1, p_3] = -i\hbar p_2$, which follow from $[q_i, p_j] = i\hbar\delta_{ij}$, $[q_i, q_j] = 0$ and $[p_i, p_j] = 0$.

For ~~Also~~ $[i\hbar\vec\sigma, H]$, consider $[i\hbar\hat\sigma_1, H] = \hbar[\gamma_2\gamma_3, H]$ & since $[\gamma_2\gamma_3, \gamma_4] = 0$

$$[i\hbar\hat\sigma_1, H] = \hbar[\gamma_2\gamma_3, ic\gamma_4\gamma_K p_K] = ic\hbar[\gamma_2\gamma_3, \gamma_4\gamma_K]p_K$$
$$= ic\hbar\{[\gamma_2\gamma_3, \gamma_4\gamma_1]p_1 + [\gamma_2\gamma_3, \gamma_4\gamma_2]p_2 + [\gamma_2\gamma_3, \gamma_4\gamma_3]p_3\}$$
$$= ic\hbar\{\quad 0 \quad - 2\gamma_4\gamma_3 p_2 + 2\gamma_4\gamma_2 p_3\} =$$

For $[\hbar\vec\sigma, H]$, consider $[\hbar\hat\sigma_1, H] = -i\hbar[\gamma_2\gamma_3, H] = -i[\gamma_2\gamma_3, ic\gamma_4\gamma_K p_K]$ since $[\gamma_2\gamma_3, \gamma_4\gamma_4] = 0$.

$$= \hbar c\{[\gamma_2\gamma_3, \gamma_4\gamma_1]p_1 + [\gamma_2\gamma_3, \gamma_4\gamma_2]p_2 + [\gamma_2\gamma_3, \gamma_4\gamma_3]p_3\}$$
$$= \hbar c\{\quad 0 \quad - 2\gamma_4\gamma_3 p_2 + 2\gamma_4\gamma_2 p_3\} = \hbar c\gamma_4(\vec r \times \vec p)$$

i.e. $A_4 = 0$.

66th lecture – (c) magnetic moment of the electron. – 2/3/64.

Motion of an electron in an e.m. field, which we restrict to a pure magnetic field H given by $\vec H = \operatorname{curl}\vec A$, and also constant in time & i.e. $\vec A$ is a 4-vector with $A_4 = 0$ (**) & also $\partial A_K/\partial x_4 = 0$ – The effect of the e.m. field is correctly described by replacing p_μ by $p_\mu + \frac{e}{c}A_\mu$ – Working Dirac's eqn with this in: i.e. $p_\mu \to p_\mu + \frac{e}{c}A_\mu$ or $-i\hbar\frac{\partial}{\partial x_\mu} \to -i\hbar\frac{\partial}{\partial x_\mu} + \frac{e}{c}A_\mu$ or $\frac{\partial}{\partial x_\mu} \to \frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}A_\mu$ (††), the eqn becomes

$$\gamma_\mu\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}A_\mu\right) + M\psi = 0.$$

Operating with $\gamma_\lambda\left(\frac{\partial}{\partial x_\lambda} + \frac{ie}{\hbar c}A_\lambda\right) - M$

and taking terms in γ_λ^2 & $\gamma_\lambda\gamma_\mu$, this eqn becomes $\left[\sum_\lambda\left(\frac{\partial}{\partial x_\lambda} + \frac{ie}{\hbar c}A_\lambda\right)^2 - \frac{e}{\hbar c}\vec\sigma\cdot\vec H - M^2\right]\psi$

Going to the non-relativistic limit, this → Schrodinger eqn — From $E^2 = m^2v^2 + m^2c^4$, these have the limit $E = mc^2$ & $i\hbar\frac{\partial\psi}{\partial t} = mc^2\psi$ $i\hbar\frac{\partial}{\partial t} = mc^2$ makes $\frac{\partial}{\partial x_4} = -M$ & so the limiting Schrodinger eqn becomes

$$i\hbar\frac{\partial\psi}{\partial t} = \left[mc^2 - \frac{\hbar^2}{2m}\left\{\sum\left(\frac{\partial}{\partial x_K} + \frac{ie}{\hbar c}A_K\right)^2 + \frac{e\hbar}{2mc}\vec\sigma\cdot\vec H\right\}\right]\psi$$

(*) In $i\hbar\frac{\partial\psi}{\partial t} = E\psi$, writing $E = E' + mc^2$, the term $i\hbar\frac{\partial\psi}{\partial t} = mc^2$ makes the term in this second order time derivative in the K-G. eqn zero, since $i\hbar\frac{\partial}{\partial t} = mc^2$ is equivalent to $\frac{\partial}{\partial x_4} = -M$

$$\left[i\hbar\frac{\partial}{\partial t} = i\hbar\frac{\partial}{\partial x_4}\cdot ic = -\hbar c\frac{\partial}{\partial x_4} = mc^2 \rightarrow \frac{\partial}{\partial x_4} = -\frac{mc}{\hbar} = -M \right]$$

& so the remaining part of the K.G. eqn is equivalent to $\left(i\hbar\frac{\partial}{\partial t} - mc^2\right)\psi = E'\psi$

& the corresponding Schrodinger eqn is

$$\left(i\hbar\frac{\partial}{\partial t} - mc^2\right)\psi = -\frac{\hbar^2}{2m}\left\{\sum_k \left(p_k + \frac{e}{c}A_k\right)^2 - \frac{e}{\hbar c}\vec{\sigma}.\vec{H}\right\}\psi$$

showing that there is an additional energy term $\frac{e}{mc}\left(\frac{1}{2}\hbar\vec{\sigma}.\vec{H}\right)$ ie the electron can be considered a magnet of moment magnetic moment $-\frac{e}{mc}\left(\frac{1}{2}\hbar\vec{\sigma}\right)$.

(†) Using $\gamma_1 = \begin{pmatrix} 0&0&0&-i\\ 0&0&-i&0\\ 0&i&0&0\\ i&0&0&0\end{pmatrix}$, $\gamma_2 = \begin{pmatrix} 0&0&0&-1\\ 0&0&1&0\\ 0&1&0&0\\ -1&0&0&0\end{pmatrix}$, $\gamma_3 = \begin{pmatrix} 0&0&-i&0\\ 0&0&0&i\\ i&0&0&0\\ 0&-i&0&0\end{pmatrix}$, $\gamma_4 = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1\end{pmatrix}$

and $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$

The eqn $\gamma_\mu\frac{\partial\psi}{\partial x_\mu} + M\psi = 0$ reduces to

$$\begin{pmatrix} -i\partial\psi_4/\partial x_1\\ -i\partial\psi_3/\partial x_1\\ i\partial\psi_2/\partial x_1\\ i\partial\psi_1/\partial x_1\end{pmatrix} + \begin{pmatrix} -\partial\psi_4/\partial x_2\\ \partial\psi_3/\partial x_2\\ \partial\psi_2/\partial x_2\\ -\partial\psi_1/\partial x_2\end{pmatrix} + \begin{pmatrix} -i\partial\psi_3/\partial x_3\\ i\partial\psi_4/\partial x_3\\ i\partial\psi_1/\partial x_3\\ -i\partial\psi_2/\partial x_3\end{pmatrix} + \begin{pmatrix} \partial\psi_1/\partial x_4\\ \partial\psi_2/\partial x_4\\ -\partial\psi_3/\partial x_4\\ -\partial\psi_4/\partial x_4\end{pmatrix} + M\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = 0$$

ie
$$\left.\begin{aligned}
&-i\frac{\partial\psi_4}{\partial x_1} - \frac{\partial\psi_4}{\partial x_2} - i\frac{\partial\psi_3}{\partial x_3} + \frac{\partial\psi_1}{\partial x_4} + M\psi_1 = 0\\
&-i\frac{\partial\psi_3}{\partial x_1} + \frac{\partial\psi_3}{\partial x_2} + i\frac{\partial\psi_4}{\partial x_3} + \frac{\partial\psi_2}{\partial x_4} + M\psi_2 = 0\\
&i\frac{\partial\psi_2}{\partial x_1} + \frac{\partial\psi_2}{\partial x_2} + i\frac{\partial\psi_1}{\partial x_3} - \frac{\partial\psi_3}{\partial x_4} + M\psi_3 = 0\\
&i\frac{\partial\psi_1}{\partial x_1} - \frac{\partial\psi_1}{\partial x_2} - i\frac{\partial\psi_2}{\partial x_3} - \frac{\partial\psi_4}{\partial x_4} + M\psi_4 = 0
\end{aligned}\right\} \quad ---(1)$$

Putting $\psi_\nu = u_\nu e^{ip_\nu x_\nu/\hbar}$ to find the plane wave solutions, noticing $\frac{\partial\psi_\nu}{\partial x_\nu} = \frac{ip_\nu}{\hbar}\psi_\nu$, that $p_\nu = (p_1, p_2, p_3, iE/c)$ ie $p_4 = iE/c$, and $mc/\hbar = M$, we can write the solns of (1) in the u_ν's only after removing the exponential factor, and multiplying each eqn by $\hbar c$, in the form

$$\left.\begin{aligned}
&-(E - mc^2)u_1 + cp_3u_3 + c(p_1 - ip_2)u_4 = 0\\
&-(E - mc^2)u_2 + c(p_1 + ip_2)u_3 - cp_3u_4 = 0\\
&-cp_3u_1 - c(p_1 - ip_2)u_2 + (E + mc^2)u_3 = 0\\
&-c(p_1 + ip_2)u_1 + cp_3u_2 + (E + mc^2)u_4 = 0
\end{aligned}\right\} \quad ---(2)$$

These have a solution if the determinant

$$\begin{vmatrix} -(E+mc^2) & 0 & cp_3 & c(p_1 - ip_2)\\ 0 & -(E-mc^2) & c(p_1+ip_2) & -cp_3\\ -cp_3 & -c(p_1-ip_2) & E+mc^2 & 0\\ -c(p_1+ip_2) & cp_3 & 0 & E+mc^2\end{vmatrix} = 0$$

For a magnet of moment $\vec{M}$ in a magnetic field $\vec{H}$, potential energy $= -\vec{M}\cdot\vec{H}$ ie electron is a magnet of moment $\left(-\frac{e}{2mc}\hbar\vec{\hat{\sigma}}\right)$ - experimentally confirmed & a brilliant success of the Dirac theory

67th Lecture - 3/3/64 - Remaining portions of the previous lecture.* - Plane wave solutions of the Dirac equation (ie free particle solns). $(\gamma_\mu \frac{\partial}{\partial x_\mu} + M)\psi = 0$; $\psi_\nu = u_\nu e^{ip_\nu x_\nu/\hbar}$ give, using the matrices for γ_μ viz $\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix}$, $\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the ψ's as 1-col matrices, four equations homogeneous in u_ν's ⊕ condn for soln existing reduces to $(E^2 - m^2c^4 - c^2\vec{p}^2)^2 = 0$ ie has double roots $E = +\sqrt{m^2c^4 + c^2\vec{p}^2}$, $E = -\sqrt{m^2c^4 + c^2p^2}$.

$p_4 = \frac{iE}{c}$

Hence we can choose for each case E_+ & E_- two of the u_ν arbitrarily & determine the other two. It is usual choose for the case E_+, two solns $u_3 = 1, u_4 = 0$ & $u_3 = 0, u_4 = 1$ & for E_-, the two solns $u_1 = 1, u_2 = 0$ & $u_1 = 0, u_2 = 1$ - Interpretation of the negative energy solns; from Pauli's exclusion principle there is a state with all negative energy levels filled each by one electron & this is to be considered a vacuum & the negative energy sea of electrons are not to produce any physical effects. A hole in the sea of negative energy electrons (ie an unoccupied state) corresponds to a particle of energy $+E_p$ & momentum $\vec{p}$, but also of opt. charge ie a positron - Exptl. confirmation: pair creation & pair annihilation, the former being due to the hole caused by a high energy photon

68th Lecture - 4/3/64 - Remaining portions of last lecture with further comments on hole theory viz: (1) assumptions of the hole theory: (a) In the physical world, all the states of negative energy are occupied, one by each electron according to the Pauli exclusion principle. (b) when all the negative energy states are occupied, the distn does not cause an external field & hence not observed ie the distn in which all negative energy states are occupied and all the +ve energy states are empty must be taken as zero point for the measurement of e, $\vec{p}$ and E - If an external field can remove an electron in a negative energy state, the hole is equivalent to a particle of electronic rest mass m_0, $E = -(-E) = E_+$, $\vec{p} = -\vec{p}$ and $e = -(-e) = e_+$ ie positively charged ie a positron observed by Anderson (1932) in studying tracks produced in a Wilson chamber by cosmic ray particles in the presence of a strong magnetic field - Pair creation or pair production is creation of a hole by an external field. Examples (i) hole caused by a high r-ray of frequency ν, condn being $h\nu > 2mc^2$ (ii) By a proton of K.E $> 2mc^2$ (iii) by an electron through the field of a nucleus - Pair annihilation corresponds to the transition of an electron from a state of +ve energy to a hole in the sea of negative energy states, and the annihilation energy is given in the form of radiation of frequency ($h\nu < 2mc^2$) ie; no of examples of annihilation ($2mc^2 = 1.02$ MeV) & corresponding wavelength of radiation is 0.012 A

- Relativistic invariance of the Dirac equation - That this eqn is relativistic in form has been obtained indirectly by transforming it to the relativistic 2nd order Klein-Gordon equation. We can also establish

This is of the form

$$\Delta = \begin{vmatrix} a & a & h & g \\ 0 & a & \bar{g} & -h \\ -h & -g & b & 0 \\ -\bar{g} & h & 0 & b \end{vmatrix}$$

Interchanging the first and second columns, and the third and columns, and multiplying the resulting 2nd & 4th columns by -1, the determinant becomes

$$\Delta = \begin{vmatrix} 0 & -a & g & -h \\ a & 0 & -h & -\bar{g} \\ -g & h & 0 & -b \\ h & \bar{g} & b & 0 \end{vmatrix}$$

which is obviously a 4×4 skew symmetric determinant, and hence is a perfect square and using the usual formula for such a $|a_{ij}|$ viz $\Delta = (a_{23}a_{14} + a_{31}a_{24} + a_{12}a_{34})^2$, we have

$\Delta = (ab + h^2 + g\bar{g})^2$, and putting in the values of for a, b, h & g, we have

$$\Delta = \left\{-(E^2 - m^2c^4) + c^2p_3^2 + c^2(p_1^2+p_2^2)\right\}^2 = (E^2 - m^2c^4 - c^2\vec{p}^2)^2 = 0$$

ie Δ has two double roots $E_+ = +\sqrt{m^2c^4 + c^2\vec{p}^2}$, $E_- = -\sqrt{m^2c^4 + c^2\vec{p}^2}$. Hence there will be two sets of two roots each corresponding to each the two double roots. For each set any two of the u_s may be chosen arbitrarily & the other two determined from any two equations of the set. For the double root E_+ it is usual to get the two roots by taking $u_3 = 1$, $u_4 = 0$ & $u_3 = 0$, $u_4 = 1$, and for the double root E_-, by taking $u_1 = 1$, $u_2 = 0$ & $u_1 = 0$, $u_2 = 1$. Thus for E_+,

(i) (a) $u_3 = 1$, $u_4 = 0$ give from (2), $u_1 = \dfrac{cp_3}{E_+ - mc^2}$, $u_2 = \dfrac{c(p_1 + ip_2)}{E_+ - mc^2}$. from the first two eqns of (2)

and these obviously satisfy the last two eqns

(b) $u_3 = 0$, $u_4 = 1$ give from the first two eqns of (2) $u_1 = \dfrac{c(p_1 - ip_2)}{E_+ - mc^2}$, $u_2 = -\dfrac{cp_3}{E_+ - mc^2}$

which also satisfy the last two eqns

(ii) (a) Putting $u_1 = 1$, $u_2 = 0$ in the last two eqns of (2) gives $u_3 = \dfrac{cp_3}{E_- + mc^2}$, $u_4 = \dfrac{c(p_1 + ip_2)}{E_- + mc^2}$

(b) ~~Putting $u_4 = 0$, $u_3 = 0$ in the last two~~ and these satisfy the first two eqns

(b) Putting $u_1 = 0$, $u_2 = 1$ in the last two eqns of (2) gives, $u_3 = \dfrac{c(p_1 - ip_2)}{E_- + mc^2}$, $u_4 = -\dfrac{cp_3}{E_- + mc^2}$

and these satisfy the first two eqns of (2)

Thus there are roots corresponding to E_+ and E_- & the negative energy states cannot be omitted as in classical theory, since in quantum jumps from $+^{ve}$ to $-^{ve}$ & $-^{ve}$ to $+^{ve}$ energy states ~~are~~ cannot be disregarded. ~~permissible~~. Hence the explanation by the "hole" theory.

the relativistic invariance of the Dirac eqn is its invariance under Lorentz transformations.

(1) Lorentz transformations defined by $x_\mu' = a_{\mu\nu} x_\nu$ -- (1)

where $\sum_{\nu=1}^{4} x_\nu'^2 = \sum_{\nu=1}^{4} x_\nu^2$ (ie const^n that c is the same in all reference frames) -- (2)

Write L.H.S of (2) as $\sum_{\mu=1}^{4} x_\mu'^2$ and using (1) const^n for L.T becomes

$$a_{\mu\nu} a_{\mu\sigma} = \delta_{\nu\sigma} \quad \text{--- (3)}$$

using (3), $a_{\mu\nu} x_\mu' = a_{\mu\nu} a_{\mu\sigma} x_\sigma = \delta_{\nu\sigma} x_\sigma = x_\nu$ ie the inverse transformation

$$x_\nu = a_{\mu\nu} x_\mu' \quad \text{-- (1,a)}$$

using (1a) in (2), we also have

$$a_{\rho\mu} a_{\sigma\mu} = \delta_{\nu\sigma} \quad \text{-- (3,a)}$$ which are not new, but follow from (3) since a det. is unaltered by changing rows $\rightarrow$ cols

the determinant of a L.T is ± 1, for $|a_{\alpha\mu}||a_{\beta\mu}| = |a_{\alpha\mu} a_{\mu\beta}| = |\delta_{\alpha\beta}| = 1$ (take a 2×2 determinant for eg.)

ie square of det. of L.T = 1 ie $|a_{\nu\mu}| = \pm 1$ ie proper or improper L.T's.

Taking $a_{44} > 0$ excludes time reversals, spatial inversions $\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$ are included in improper L.T's

The proper L.T's form a group & hence only infinitesimal L.T's may be considered. for x_k real & x_4 imaginary in all systems, we should have a_{kl}, a_{44} real and a_{k4}, a_{4k} pure imaginary (as can be seen from $x_k' = a_{k\mu} x_\mu$ and $x_4' = a_{4\mu} x_\mu$), Also [ ] below.

(2) $$(\gamma_\mu \frac{\partial}{\partial x_\mu} + M)\psi(x_\nu) = 0 \quad \text{-- (4)}$$

is invariant under L.T's, if (4) transforms to

$$(\gamma_\mu \frac{\partial}{\partial x_\mu'} + M)\psi'(x_\nu') = 0 \quad \text{-- (5)}$$

under this is the case provides $\psi'(x_\nu') = S\psi(x_\nu)$ (where S is a constant 4×4 matrix -- (6) determined by the L.T. $a_{\mu\nu}$; b denoting the inverse S^{-1}.)

69th Lecture -- 5/3/64 -- Relativistic invariance of the Dirac eqn -- [Proof that the proper L.T's form a group (i) existence of identity is obvious (ii) existence of inverse given by (1, a) from (1), (iii) if $x_\mu' = a_{\mu\nu} x_\nu$, and $x_\lambda'' = b_{\lambda\rho} x_\rho'$ then $x_\lambda'' = b_{\lambda\rho} a_{\rho\nu} x_\nu = c_{\lambda\nu} x_\nu$, say. Then $c_{\mu\nu} c_{\mu\sigma} = b_{\mu\alpha} a_{\alpha\nu} b_{\mu\beta} a_{\beta\sigma} = b_{\mu\alpha} b_{\mu\beta} a_{\alpha\nu} a_{\beta\sigma}$

$= \delta_{\alpha\beta} a_{\alpha\nu} a_{\beta\sigma} = a_{\alpha\nu} a_{\alpha\sigma} = \delta_{\nu\sigma}$ ie $c_{\mu\nu}$ represent an L.T.

Also $|c_{\lambda\nu}| = |b_{\lambda\rho}||a_{\rho\nu}| = 1$. from $c_{\lambda\nu} = b_{\lambda\rho} a_{\rho\nu}$ and using the fact that a_{kl}, b_{kl}, a_{44}, b_{44} are real, and a_{k4}, a_{4k}, b_{k4}, b_{4k} are pure imaginary it follows that c_{kl}, c_{44} are real & c_{4k} & c_{k4} are pure imaginary] -- (6) holds if

$$S^{-1} \gamma_\lambda S = a_{\lambda\mu} \gamma_\mu \quad \text{--- (7)}$$

Transformation corresponding to (6) for $\bar\psi$ is obtained by use the current vector s_μ, and is given by

$$\bar\psi'(x_\lambda') = \bar\psi(x_\lambda) S^{-1} \quad \text{-- (8)}$$

just as (6) leads to (7), (8) leads to

$$\gamma_4 S^* \gamma_4 = S^{-1} \quad \text{-- (9)} \qquad (S^* = \text{adjoint of } S \text{ ie complex conjugate of transpose})$$

(3) from the group property of proper L.T's, finite L.T's can be generated from

Proof of (8): ~~from (7) $\bar{\psi}' s_\mu$ being the~~ $s_\mu = i\bar{\psi}\gamma_\mu\psi$ being the current 4-vector

$$s'_\mu = a_{\mu\nu} s_\nu$$

ie $\bar{\psi}'\gamma_\mu\psi' = a_{\mu\nu}\bar{\psi}\gamma_\nu\psi$

from (7) $\gamma_\mu = a_{\mu\nu} S\gamma_\nu S^{-1}$, substituting this and $\psi' = S\psi$ on the L.H.S. we get

$$\bar{\psi}' a_{\mu\nu} S\gamma_\nu\psi = a_{\mu\nu}\bar{\psi}\gamma_\nu\psi$$

ie $\bar{\psi}' S = \bar{\psi}$ or $\bar{\psi}' = \bar{\psi} S^{-1}$ which is (8)

Proof of (9): Noting that $\bar{\psi} = \psi^*\gamma_4$, we have $\psi^{*\prime}\gamma_4 = \bar{\psi} S^{-1}$, and taking adjoint of $\psi' = S\psi$

ie $\psi'^* = \psi^* S^*$, the L.H.S. becomes ~~$\psi^* S^*\gamma_4 S$ and $\psi^* = \psi$~~

$$\psi^* S^*\gamma_4 = \bar{\psi}\gamma_4 S^*\gamma_4$$ & equating with the right hand side

$$\gamma_4 S^*\gamma_4 = S^{-1}. \quad --(10)$$

infinitesimal L.T's. Hence to prove rel. invariance of Dirac eqn, it is enough to prove that for inf. L.T's, a matrix S exists satisfying (7) & (9). This is done by Pauli: ~~Enzy. Math~~ Holbuch d. Phys, Vol 24, Part I.

(4) The five covariants constructed from ψ, $\bar{\psi}$ and γ's – Derivation of their transformation properties using (6), (8) and (7). (These covariants are constructed from $1, i\gamma_\mu, i\gamma_\lambda\gamma_\mu, i\gamma_5\gamma_\mu$ and $i\gamma_5$, the 16 basic quantities)

(a) Scalar $I = \bar{\psi}\psi$ – $I' = \bar{\psi}'\psi' = \bar{\psi}S^{-1}S\psi = \bar{\psi}\psi = I$.

(b) Four-vector: $s_\mu = i\bar{\psi}\gamma_\mu\bar{\psi}$ ie the four current-vector — $s'_\mu = a_{\mu\nu}s_\nu$.

(c) Antisymmetric tensor or six-vector: $M_{\mu\nu} = i\bar{\psi}\gamma_\mu\gamma_\mu\psi$ $(\mu \neq \nu)$ & $= 0$ if $\mu = \nu$ – $M'_{\mu\nu} = a_{\mu\sigma}a_{\nu\tau}M_{\sigma\tau}$

(d) Pseudo-four vector or axial vector: $K_\mu = i\bar{\psi}\gamma_5\gamma_\mu\psi$ – $K'_\mu = |a_{\sigma\tau}|a_{\mu\nu}K_\nu$

(e) Pseudo-scalar: $N = i\bar{\psi}\gamma_5\psi$: $N' = |a_{\mu\nu}|N$.

From (e), we note that for proper L.T's, $N' = N$ and for space-inversions $N' = -N$ ie defn of a pseudoscalar

From (d) we note that under space inversions, $K'_l = -a_{l\nu}K_\nu$ and using $a_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}$, we have $K'_l = K_l$ unlike an ordinary vector $\vec{r}$ which goes over to $-\vec{r}$.

Also from $\hat{\sigma}_l = i\gamma_4\gamma_5\gamma_l$, we have using $\bar{\psi} = \psi^*\gamma_4$ or $\bar{\psi}\gamma_4 = \psi^*$

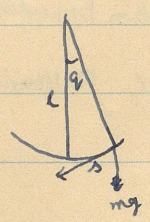
$$\cancel{\bar{\psi}\hat{\sigma}_l\psi = i} \quad \psi^*\hat{\sigma}_l\psi = i\bar{\psi}\gamma_4\gamma_4\gamma_5\gamma_l\psi = i\bar{\psi}\gamma_5\gamma_l\psi = K_l$$

~~ie $\hat{\sigma}_l$ transforms like a pseudo-vector~~

70th Lecture – 6/3/64 – Remaining portions of the previous lecture

P.T.O

(†) For simple pendulum, $T = \frac{1}{2} s^2 \ (m=1) = \frac{1}{2} l^2 \dot{q}^2$



$\therefore \ p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial T}{\partial \dot{q}} = l^2 \dot{q}$ ... (i)

& eqn of motion is $m \frac{d^2 s}{dt^2} = -mg \sin q$ or $\frac{d^2 q}{dt^2} = -\frac{g}{l} \sin q$ —(ii)

(i) & (ii) are given by $\frac{\partial H}{\partial p} \frac{dq}{dt} = \frac{\partial H}{\partial p}$ & $\frac{dp}{dt} = -\frac{\partial H}{\partial q}$

with $H = \frac{p^2}{2l^2} - gl \cos q$

* on corpuscular theory of light, Fermat's Principle states that the actual path of time the ray is given by $\delta \int \mu(x,y,z)\, ds = 0 \qquad (\mu = \text{refractive index})$

For a single particle in dynamics (say of unit mass), $T = \frac{1}{2} \dot{s}^2$ & Principle of least action is

$$\delta \int T\, dt = 0 \ \text{ie} \ \delta \int \dot{s}^2 dt = 0 \ \text{or} \ \delta \int \dot{s}\, ds = 0 \ \text{or} \ \delta \int \sqrt{T} \cdot ds = 0$$

$$\text{or} \ \delta \int \{h - V(x,y,z)\}^{\frac{1}{2}} ds = 0.$$

ie trajectories of the particle in the dynamical problem are the same as the paths of the rays in the optical problem provided a suitable correspondence $\mu = (h-V)^{\frac{1}{2}}$ is set up between μ & V.

(**) Verify with $Q = (2q)^{\frac{1}{2}} e^{K} \cos p$; $P = (2q)^{\frac{1}{2}} e^{-K} \sin p$.

& also $Q = \log\left(\frac{1}{q} \sin p\right)$; $P = q \cot p$.

(‡) $\left.\begin{aligned} ab \cdot c &= a \cdot bc \\ ea &= a \\ a^{-1} a &= e \end{aligned}\right\}$

(††) Including the examples (i) $((f_2, f_3), f_1) + \dots + \dots = 0$

(ii) $(\phi, \psi) = \left(\frac{\partial \phi}{\partial f_r} \cdot \frac{\partial \psi}{\partial f_s} - \frac{\partial \phi}{\partial f_s} \frac{\partial \psi}{\partial f_r}\right)(f_r, f_s)$

(iii) $(\phi, \psi)_{Q,P} = (\phi, \psi)_{q,p}$ for a C.T.

Session 1964 – 65

1st Lecture – 15/7/64 – Dynamical systems – generalised coords (Exx. single point, s, Eulerian angles); derivation of relations relating to Eulerian angles – holonomic & non-holonomic systems illustrated by motion of a sphere on smooth or rough plane – Lagrange's equations of motion (no proof) (generalised forces) – Integrals of the system $f(q_r, \dot{q}_r, t) = \text{const}$ if $\frac{df}{dt} = 0$ (Ex. of $\ddot{q} = -q$ having two integrals) – Solns given by $f_r(q_1 \ldots q_n, t) = \alpha_r$ or $q_r = q_r(t, \alpha_1 \ldots \alpha_n)$ – Conservative forces $Q_r = -\frac{\partial V}{\partial q_r}$ & Lagrange's eqns with $L = T - V$ – Integral of energy for conservative, holonomic systems with L not involving t viz $\Sigma \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L = h$ – Natural systems (T quadratic homogeneous and V fn of coords only), this integral is $T + V = h$ – Ignorable coords & generalised momenta.

2nd Lecture – 16/7/64 – Remaining portions – Hamilton's Principle – Principle of Least Action, both for conservative holonomic systems, one deduced from the other using $\Sigma \dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L = h$ – For natural systems Principle of Least Action is $\delta \int T dt = 0$ – Hamiltonian form of eqns of motion – Ex. of H-eqns for a simple pendulum (†) – Contact transformations ~~– group property~~ – Exx: $Q = (2q)^{\frac{1}{2}} e^{K} \cos p$, $P = (2q)^{1/2} e^{-K} \sin p$ and $Q = \log\left(\frac{1}{q} \sin p\right)$, $P = q \cot p$.

3rd Lecture – 22/7/64. Connection between Fermat's Principle and Principle of Least Action* – ~~Remaining portions of previous lecture~~ Corpuscular & wave theory of light; Derivation of eqns of H-type using wave theory – contact trns as transformation from wave front σ to wave front Σ & of the form given by defn $\Sigma P dQ - \Sigma p dq = dW(q, p)$ – Remaining portions of previous lecture – Bilinear covariant of a general differential form (Pfaffian) – condns for C. T. by means of bilinear covariant

4th Lecture – 23/7/64 – Remaining portion (**) – L. B's & P. B's and conditions for a C. T. in terms of L. B's & P. B's – Infinitesimal C. T's & resulting new view of Dynamics

5th Lecture – 24/7/64 – Remaining portions (dets. of L. B's & P. B's) – Verification with P. B's of the two examples of the 2nd Lecture (one 1st Ex. from W. p. 300) – Group property of C. T's (?) – Symbol of inf. C. T's & new view of Dynamics with symbol (f, H).

6th Lecture – 25/7/64 – Dern of condn for C. T in terms of P. B's – Other remaining portions of 4th & 5th lectures.

7th Lecture – 29/7/64 – Remaining portions of previous lectures (††) – Hamiltonian eqns as the first Pfaffian system of eqns of the differential form $\Sigma p_r dq_r - H dt$ – Conservation of H-form under a C. T. (with Exx) – Integration problem as a transformation into a differential form with last term a perfect differential.

8th Lecture – 30/7/64 – Finishing the result that the 1st Pfaffian system of eqns of $\Sigma p_r dq_r - H dt$ are the H-eqns – Remaining portions in part.

9th Lecture – 31/7/64 – Remaining portions in full.

* In dielectrics, taking polarised charge & polarisation vector $\vec{P}$ into consideration, ~~$\epsilon_0 \vec{E} + \vec{P} = \vec{D}$~~, $\epsilon_0 \operatorname{div} \vec{E} = \rho$ becomes $\operatorname{div}(\epsilon_0 \vec{E} + P) = \rho$ and defining $D = \epsilon_0 \vec{E} + P$, this becomes $\operatorname{div} \vec{D} = \rho$. $\vec{D} = \epsilon \vec{E}$ still holds

** In magnetic materials, taking polarised current density & magnetisation vector $\vec{M}$ into consideration, $\operatorname{curl} \vec{B} = \mu_0 \vec{J}$ becomes $\operatorname{curl}\left(\frac{B}{\mu_0} - M\right) = J$, and defining $H = \frac{B}{\mu_0} - M$, this becomes $\operatorname{curl} \vec{H} = \vec{J}$. $\vec{B} = \mu \vec{H}$ still holds.

10th Lecture – 5/8/64 – Hamilton-Jacobi partial diff. eqn – Connection of integrals with D.C.T's and Poisson's theorem – Ex. 2, p. 336, W. & ~~Berry~~ a particle over ~~Capital~~ Ex. p. 321.

11th Lecture – 6/8/64 – Remaining portions of proof of Poisson's Theorem – Ex. p. 321 – Ex. 2, p. 336 – Ex. 3, p. 337 – Involution systems – Soln when half the integrals are known.

12th Lecture – 7/8/64 – Remaining portions of last lecture.

13th Lecture – 12/8/64 – no class because of "Maharashtra Bandh"

13th Lecture – 13/8/64 – General one on electromagnetic theory based on Intro in Chap I of Slater & Frank's Electromagnetism –

14th Lecture – 14/8/64 – Electrostatics – $\vec{E}$ independent of t – $F = e\vec{E}$ defines $\vec{E}$ – Charges at rest; for a single charge e

$E = \frac{e}{4\pi\epsilon_0 r^2}$ away from e – Coulomb's Law – Potential $\varphi\left(\frac{e}{4\pi\epsilon_0 r}\right)$ – $E = -\text{grad}\,\varphi$; ~~div~~ rot $\vec{E} = 0$ – $\varphi = \int \frac{\rho\, dv}{r}$

Use of φ in System of charges – Equipotentials, normal to lines of force – Conductors – Problems re. charges in presence of Conductors; particular case of method of images – Gauss's Theorem $\int_S \vec{E}\cdot\vec{n}\cdot d\sigma = \Sigma \frac{e_i}{\epsilon_0}$ or $= \int_V \frac{\rho\, dv}{\epsilon_0}$. Leading to $\text{div}\,\vec{E} = \frac{\rho}{\epsilon_0}$ or $\nabla^2\varphi = -\frac{\rho}{\epsilon_0}$ (Poisson's eqn) or $\nabla^2\varphi = 0$ (Laplace's eqn) – Soln of L-eqn in (r, θ, ϕ) coords given in general by $\varphi = \left(ar^l + \frac{b}{r^{l+1}}\right) P_l^m(\cos\theta)\left(C\cos m\varphi + D\sin m\varphi\right)$ – Case of dielectrics, replace ~~the~~ $\vec{E}$ by $\vec{D}$ given by $\vec{D} = \epsilon\vec{E}$, where ϵ/ϵ_0 = dielectric const or sp. ind. capacity.

Magnetostatics – Permanent magnets or case of steady currents in conductors – using reasoning corresponding to $\vec{E}$ we have $\vec{B}$ (mag. induction) given by Biot-Savart's law, $\vec{B} = \frac{\mu_0 e(\vec{v}\times\vec{r})}{4\pi |r|^3}$ for a charge e moving with vel. $\vec{v}$. ~~Eq $e(\vec{v}\times\vec{B})$ defining~~ $F = e\{\vec{E} + (\vec{v}\times\vec{B})\}$, second part defining $\vec{B}$. – $\text{div}\,\vec{B} = 0$. ~~in empty space & inside a conductor carrying current~~ rot $\vec{B} = 0$ in empty space & inside a conductor carrying a current $\text{curl}\,\vec{B} = \mu_0\vec{J}$ ($\vec{J}$ = current density vector) (Ampere's law) – from $\text{div}\,\vec{B} = 0$, we write $\vec{B} = $ ~~curl~~ rot $\vec{A}$ (vector potential) and Ampere's law becomes rot rot $\vec{A} = \mu_0 J$ i.e. grad div $\vec{A} - \nabla^2\vec{A} = \mu_0 J$, and assuming div $A = 0$, $\nabla^2\vec{A} = -\mu_0\vec{J}$ i.e. Poisson's eqn as in electrostatics – from notion of permanent magnets analogous to dielectrics, we have $\vec{B} \to \vec{H}$ by $\vec{B} = \mu\vec{H}$ (μ/μ_0 = permeability) and $\text{curl}\,\vec{H} = \vec{J}$.

15th Lecture – 18/8/64 – correct derivation* in electrostatics of $\text{div}\,\vec{D} = \rho$ & mention of the two fundamental eqns of electrostatics viz $\text{div}\,\vec{D} = \rho$, $\text{curl}\,\vec{E} = 0$ – correct derivation** in magnetostatics of $\text{curl}\,\vec{H} = \vec{J}$ & mention of the two fundamental eqns of magnetostatics viz. $\text{div}\,\vec{B} = 0$, $\text{curl}\,\vec{H} = \vec{J}$ – Electromagnetic induction: If B through a conducting current circuit changes with time, a current flows through the circuit; this is expressed by $\text{curl}\,\vec{E} = -\partial\vec{B}/\partial t$ – Modification of $\text{curl}\,\vec{H} = \vec{J}$ which is derived for steady currents only by Ampere's Law. Taking div, we have $\text{div curl}\,\vec{H} = \text{div}\,\vec{J}$ i.e. $\text{div}\,\vec{J} = 0$ is true only for special types of currents as in closed circuits. From eqn of continuity $\text{div}\,\vec{J} + \frac{\partial\rho}{\partial t} = 0$ and so replacing $\vec{J}$ by $\vec{J} + \frac{\partial D}{\partial t}$, then $\text{div}\left(\vec{J} + \frac{\partial D}{\partial t}\right) = 0$ in view of $\text{div}\,D = \rho$. So Maxwell generalises $\vec{J}$ by $\vec{J} + \partial\vec{D}/\partial t$, the second term being called the "displacement current" so that $\text{curl}\,\vec{H} = \vec{J}$ is replaced by $\text{curl}\,\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$ – Maxwell's eqns are: $\text{div}\,\vec{D} = \rho$, $\text{div}\,\vec{B} = 0$, $\text{curl}\,\vec{E} = -\frac{\partial\vec{B}}{\partial t}$, $\text{curl}\,\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$ with $\vec{D} = \epsilon\vec{E}$ & $\vec{B} = \mu\vec{H}$ as constitutive eqns & also Ohm's law ~~$\vec{E} = \sigma\vec{J}$~~ $\vec{J} = \sigma\vec{E}$ along with $\vec{F} = e(\vec{E} + \vec{v}\times\vec{B})$ or $\vec{F} = \rho(\vec{E} + \vec{J}\times\rho\vec{E} + (\vec{J}\times\vec{B})$

* Let $x' = \alpha x + \beta t$, $t' = \gamma x + \delta t$; ~~for x'~~ then $x = \dfrac{\delta x' - \beta t'}{\alpha\delta - \beta\gamma}$; ~~$t = \dfrac{-\gamma x' + \alpha t'}{\alpha\delta - \beta\gamma}$~~ ; $t = \dfrac{-\gamma x' + \alpha t'}{\alpha\delta - \beta\gamma}$

For $x' = 0$, $x = vt$ ie $\alpha v + \beta = 0$, and for $x = 0$, $x' = -vt'$ ie ~~$\dfrac{\delta v + \beta}{\alpha\delta - \beta\gamma} = 0$ ie $\alpha = \delta$, $\alpha\delta - \beta\gamma \neq 0$~~

~~and $-\dfrac{\beta}{\alpha} =$ ie $\alpha = \delta$, $\alpha\delta - \beta\gamma \neq 0$~~ should give same value of v ie we must have

$x = \alpha x' - \beta t'$. Comparing with the soln for x (inverse), $\alpha = \delta$, $\alpha\delta - \beta\gamma = 1$. ie $t = -\gamma x' + \alpha t'$

$\dfrac{dx}{dt} = c \rightarrow \dfrac{dx'}{dt'} = c$ gives using $\dfrac{dx'}{dt'} = \dfrac{dx'}{dt} \Big/ \dfrac{dt'}{dt} = \dfrac{\alpha \frac{dx}{dt} + \beta}{\gamma \frac{dx}{dt} + \delta}$ ie $c = \dfrac{\alpha c + \beta}{\gamma c + \delta}$ or $c = \dfrac{\alpha c + \beta}{\gamma c + \alpha}$

$$\left.\begin{array}{l} \text{or } c = \dfrac{\alpha(c - v)}{\gamma c + \alpha} \\ \text{or } \gamma c^2 = -\alpha v \end{array}\right\} \begin{array}{l} \text{a } \alpha\delta - \beta\gamma = 1 \text{ gives } \alpha^2 + \alpha v \gamma = 1 \quad \text{or } \gamma = \dfrac{1 - \alpha^2}{\alpha v} \\ \text{ie } \dfrac{(1 - \alpha^2)c^2}{\alpha v} = -\alpha v \quad \text{ie } (1 - \alpha^2)c^2 = -\alpha^2 v^2 \end{array}$$

$\alpha^2(c^2 - v^2) = c^2$ or $\alpha^2 = 1/(1 - v^2/c^2)$. The +ve sign to be taken for when $v = 0$, $t' = \gamma x + \delta t$
$= \gamma x + \alpha t$
gives $t' = t$.

$\therefore\ x' = \dfrac{x - vt}{\sqrt{1 - v^2/c^2}}$, $y' = y$; $z' = z$; $t' = \dfrac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$.

may be used. Thus Maxwell's eqns, constitutive eqns & the force eqn form the foundation of the whole of electrodynamics — Vector & scalar potentials: $\vec{A}$ & ϕ introduced by $\vec{B} = \text{curl}\,\vec{A}$ and $\vec{E} = -\text{grad}\,\phi - \frac{\partial \vec{A}}{\partial t}$. These automatically satisfy the two Maxwell eqns div $\vec{B} = 0$ & curl $\vec{E} = -\frac{\partial B}{\partial t}$. They also satisfy the other two eqns div $\vec{D} = \rho$ and curl $\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$, (if we choose $\vec{A}$ & ϕ subject to div $\vec{A} + \epsilon\mu \frac{\partial \phi}{\partial t} = 0$) if $\vec{A}$ and ϕ satisfy the eqns $\nabla^2 \vec{A} - \epsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$, and $\nabla^2 \phi - \epsilon\mu \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon}$ — Electromagnetic waves & energy flow: Eqns $\nabla^2 \vec{E} - \epsilon\mu \frac{\partial^2 \vec{E}}{\partial t^2} = 0$, & $\nabla^2 \vec{H} - \epsilon\mu \frac{\partial^2 \vec{H}}{\partial t^2} = 0$ for a non-conducting medium & also $\rho = 0$, $\vec{J} = 0$. / wave eqn $\nabla^2 f - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = 0$ gives transverse waves with velocity v. Here $v = 1/\sqrt{\epsilon\mu}$; Putting $\epsilon/\epsilon_0 = K_e$ and $\mu/\mu_0 = K_m$, we get experimental result $c = 1/\sqrt{\epsilon_0 \mu_0}$ and $v = \frac{c}{n}$ where $n = \sqrt{K_e K_m}$. Also $\vec{E}$ & $\vec{H}$ are ^{r} to dirn of flow, & mutually $\perp^{\text{r}}$, and $\vec{S} = \vec{E} \times \vec{H}$, the Poynting vector is along the dirn of flow & gives the flux of energy i.e. amount of energy crossing unit area $\perp^{\text{r}}$ to the vector per unit time.

16th — 19/8/64 — Soln of wave eqn $\nabla^2 \phi - \epsilon\mu \frac{\partial^2 \phi}{\partial t^2} = 0$ as on p. 4 & derivation of $v = 1/\sqrt{\epsilon\mu}$ — Poynting's vector S — Special relativity — Galilean frames & Galilean transformations — These transformations leave laws of mechanics invariant — Michelson-Morley expt. leading to postulate of constancy of velocity of light in all Galilean frames — Galilean transformations contradict this postulate and hence also electrodynamics i.e. these trns do not leave electromagnetic laws invariant — Einstein's special relativity based on two postulates (i) postulate of relativity viz. all physical laws must remain invariant for Galilean frames, and (ii) invariance of c in all Galilean frames — / Derivation of L.T

17th Lecture — 20/8/64 — Matrix of Poynting's vector — conditions for a L.T (from Kompaneyets) (i) invariance of form when inverted with $-v$ for v, (ii) finite values to finite values, (iii) Trn an identity for $v = 0$, (iv) c invariant in both systems — (i) & (ii) can be replaced by making (ii) a condition of linearity and these lead to the L.T * / Alternative method of finding L.T depending on invariance of $c^2dt^2 - dx^2 - dy^2 - dz^2$. Putting $c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 = k(v)(c^2dt^2 - dx^2 - dy^2 - dz^2)$ since eqns must be linear. This leads to $c'dt'^2 - \ldots = k(-v)(c^2dt^2 - \ldots)$ i.e. $k(v)k(-v) = 1$ & $c^2dt^2 - dx^2 - \ldots$ being $+^{\text{ve}}$ definite & $k(v) = k(-v)$, we have $k = 1$. Putting $u = ict$, the L.T's are characterised by $dx^2 + dy^2 + dz^2 + du^2 = dx'^2 + dy'^2 + dz'^2 + du'^2$. For special L.T $dx^2 + du^2 = dx'^2 + du'^2$ (Neat as in Lichnerowicz, pp. 126-127) — If $\beta = 0$, L.T's $\to$ G.T's

18th lecture — 26/8/64 — Remaining portions of previous lecture — Derivation of $x = (x' + vt')/\sqrt{1-\beta^2}$, and $t = \left(t' + vx'/c^2\right)/\sqrt{1-\beta^2}$ is merely making the trn $v \to -v$ — Fitzgerald contraction and time dilatation and the two limericks from Gamow (1, 2, 3, ---∞, pp 101 & 105) — Minkowski space time, $ds^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta$ — dynamics in special relativity — law of addition of velocities $(v + w)/\left(1 + \frac{vw}{c^2}\right)$ — ~~Energy~~ $E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}$, $\vec{P} = \frac{m_0 \vec{v}}{\sqrt{1 - v^2/c^2}}$ If $v = 0$, $\vec{P} = 0$, but $\vec{E} \neq 0$, $E_0 = m_0 c^2$, $E = Mc^2$.

19th lecture — 27/8/64 — Remaining portions of previous lecture — Relativistic electrodynamics — Invariance of Maxwell's eqns under L.T's — Portions as in 17th lecture of last year / up to writing of a charged mass point in an external field. / done upto $\frac{\partial F^{ik}}{\partial x_k} = 0$.

20th Lecture – Remaining portions of last lecture.

(1) $\frac{\partial F^{ik}}{\partial x_k} = 0$. Using $F^{23} = F_{23}$; $F^{14} = -F_{14}$, these eqns give $\operatorname{div}\vec{E} = 0$, $\operatorname{curl}\vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

For the general case, introducing $s^i = \left(\frac{4\pi}{c}\vec{j}, 4\pi\rho\right)$, $\frac{\partial F^{ik}}{\partial x_k} = s^i$ give $\operatorname{div}\vec{E} = 4\pi\rho$, $\operatorname{curl}\vec{H} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j}$.

(2) $\frac{\partial^2 F_{ik}}{\partial x_i \partial x_k} = 0$, which follows from antisymmetry of F_{ik} leads to $\frac{\partial s^i}{\partial x_i} = 0$ which $\rightarrow \frac{\partial \rho}{\partial t} + \operatorname{div}\vec{j} = 0$ (continuity eqn).

(3) Scalar & ~~pot~~ vector potentials $(A_x, A_y, A_z, -\phi) = A_i$ – The eqn $F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k}$ give $\vec{H} = \operatorname{curl}\vec{A}$

and $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \operatorname{grad}\phi$ – Condn $\frac{\partial A^i}{\partial x_i} = 0$ gives the usual condn imposed $\frac{1}{c}\frac{\partial \phi}{\partial t} + \operatorname{div}\vec{A} = 0$.

(4) Transformation of $\vec{E}$ & $\vec{H}$ under the special L.T using $T'^{ik} = \frac{\partial x_i'}{\partial x_l}\cdot\frac{\partial x_k'}{\partial x_m} T^{lm}$ getting

$H_x = H_x'$, $E_x = E_x'$, $E_y' = \frac{1}{r}\left(E_y - \frac{vH_z}{c}\right)$, $E_z' = \frac{1}{r}\left(E_z + \frac{vH_y}{c}\right)$, $H_y' = \frac{1}{r}\left(H_y + \frac{vE_z}{c}\right)$

$H_z' = \frac{1}{r}\left(H_z - \frac{vE_y}{c}\right)$.

Verification of $\vec{E}^2 - \vec{H}^2 = \vec{E}'^2 - \vec{H}'^2$ & $\vec{E}\cdot\vec{H} = \vec{E}'\cdot\vec{H}'$ – Also following from

$\vec{E}^2 - \vec{H}^2 = -\frac{1}{2}F^{ik}F_{ik}$, $\vec{E}\cdot\vec{H} = \frac{1}{4}\overset{*}{F}{}^{ik}F_{ik}$

(5) Motion of charged mass point in an external field – Charge $u^i = \frac{dx^i}{d\tau}$, $c\,d\tau = ds$. – $w^i = \frac{du^i}{d\tau}$

Force vector $B_i = -\frac{e}{c}u^k F_{ik}$ & eqns of motion given by $m_0 w^i = B^i$

21st Lecture – Remaining portions of the 20th Lecture up to ~~Ø~~ –

22nd Lecture – 2/9/64 – Remaining portions of 20th lecture after and return of Tutorial Exercises.

23rd Lecture – 3/9/64 – General type of lecture – 92 chemical elements or 92 different kinds of atoms (H to U), plutonium formation of matter by combinations in different proportions (hexamethylbenzene, triphenylpirilium chloride, methyl isopropyl cyclohexane – Relative atomic wts, Bragg's X-ray photography eg. hexa-benzene – Dissecting the atom (i) J.J. Thomson – electrons ($\vec{E}$ & $\vec{H}$ fields) & positive charges uniformly distributed; finding e/m (ii) Millikan finding e & m separately, $e = 4.8\times10^{-10}$ esu, $m = 9.1\times10^{-28}$ gm, (iii) Rutherford's α-particle scattering, nucleus & electrons going round; increase of one electron in ~~system of~~ chemical sequence (iv) periodicity of Mendeleev – noble gases (He, Ne, Ar, Kr, Xe, Rn) (2, 10, 18, 36, 54, 86) valency (Na + Cl) (Na_{11} + Cl_{17}) – Eka & dvi notation Eka-Silicon = Ge (Si_{14} & Ge_{32}).

24th Lecture – 4/9/64 – Bohr working with Rutherford on atomic model showed that electrons would spiral & fall into the nucleus within one hundred millionth of a second – Necessary to modify classical mechanics and classical electromagnetism (radiation theory) (1913) – Jean's ultraviolet catastrophe – (1900) Planck's quantum theory, Light quanta – Photoelectric effect, Compton effect and Raman effect – Bohr atom – Pauli Principle, Periodic law & valency explained.

or (1899 Christmas)

25th Lecture – 8/9/64 – Remaining portions of 24th lecture – 3 postulates of Bohr – applied to the H-atom, explanation of the Balmer Series, Paschen & Lyman series – Sommerfeld's elliptical orbits – Pauli principle – Periodic law and valency – de Broglie waves, $\lambda = h/mv$ ($\underline{n}$ for n^{th} orbit); diffraction of electron beam – wave-particle duality (electrons as waves & particles & photon as particle in photoelectric effect)

$(\vec{p}, E/c)$ * use $E^2 = c^2\vec{p}^2 + m_0^2c^4$

P; wave mechanics (Schrödinger's eqn) – Heisenberg's uncertainty principle $\Delta x \times \Delta v = \frac{h}{m}$ ($h \propto 7\times10^{-27}$, $m_e \propto 10^{-27}$)

– Dirac's positrons as holes

26th Lecture – 10/9/64 – Heisenberg's matrix mechanics, $pq - qp = ih$ – Dirac's theory of antiparticles, positron, by combining quantum mechanics & special relativity*; positrons, anti-protons, neutrons & anti-neutrons, formation in pairs & annihilation, Anderson's observation of cosmic rays – Natural radioactivity due to nuclear transformations, α, β, γ rays; $_{92}U^{238} \rightarrow {}_{90}Th^{234} + {}_2He^4 \rightarrow \ldots \rightarrow {}_{82}Pb^{206} + ?$ – Artificial nuclear transformations; $_7N^{14} + {}_2He^4 = {}_8O^{17} + {}_1H'$ (by α-particles); $_3Li^7 + {}_1H' = 2\cdot{}_2He^4$ and $_5B^{11} + {}_1H' = 3\cdot{}_2He^4$ (by Cockcroft using protons of 10^6 ev, as also by Rutherford); van de Graaff generator (10^7 ev); Lawrence's cyclotron (40 M.ev); all elements up to heaviest ones – Structure of atomic nucleus $_8O^{16}$ = 8 protons + 8 neutrons; $_{92}U^{238}$ = 92 protons + 146 neutrons; nuclear fluid; 1 cm. wire supports 10^{12} tons, surface tension forces – Large scale nuclear reactions; fission by neutron bombardment $_{92}U^{238} + {}_0n^1 = {}_{56}Ba^{145} + {}_{36}Kr^{94}$; U^{238} & U^{235}, slowing down of neutrons by Fermi Manhattan project (1939 last open article).

27th Lecture – 11/9/64 – (1) Mention of van de Graaff generator & Lawrence's cyclotron (40 M.e.v) – nuclear fluid, surface tension (1 cm. wire with rupturing 10^{12} tons); large scale nuclear reactions, fission by neutron bombardment · $_{92}U^{238} + {}_0n^1 = {}_{56}Ba^{145} + {}_{36}Kr^{94}$; U^{238} & U^{235}, slowing down of neutrons by fermi, Manhattan project (1939 last open article by Bohr & Wheeler in Phy. Rev.)

(2) neutrino (energy balance from β-particle emission; $_4Be^7 \rightarrow {}_3Li^7 + e^+ + \nu$ or $_4Be^7 + (e^-) = {}_3Li^7 + \nu'$; $n \rightarrow p^+ + e^- + \nu$; $p + \nu = n + e^+$)

(3) Mesons (Yukawa's theory of exchange forces, π & μ mesons, $\pi^\pm \rightarrow \mu^\pm + \nu$, $\mu^\pm \rightarrow e^\pm + 2\nu$, π^0, $\pi^0 \rightarrow \gamma + \gamma'$, π-mesons are Yukawa particles) – (4) K-mesons, hyperons $\Lambda^0, \Sigma^\pm, \underset{(\chi_i)}{\Xi^\pm}$ ($K^\pm \rightarrow \pi^\pm + \pi^0 + \pi^0 + -$, $K^0 \rightarrow \pi^0 + \pi^0$ or $\pi^+ + \pi^-$, $\chi_i^\pm \rightarrow \Lambda^0 + \pi^\pm$, $\Sigma^\pm \rightarrow n + \pi^\pm$ $\Lambda^0 \rightarrow p + \pi^-$ or $n + \pi^+ + \pi^-$, 2600, 2300, 2200, 1840, 960, 270, 200, 1, 0) – (5) Spin and Statistics in quantisation – CPT theorem of Yang & Lee – Strangeness.

28th Lecture – 15/9/64 – Chap I of Schiff – $E = h\nu$, $\lambda = \frac{h}{p}$ valid both for particles & radiation – for atomic orbits $J = n\hbar$, $E_1 - E_2 = h\nu$ – uncertainty principle & principle of complementarity; for canonical variables $\delta x \cdot \delta p_x \geq \hbar$, $\delta\phi \cdot \delta J \geq \hbar$, $\delta t \cdot \delta E \geq \hbar$ – correspondence principle – wave formalism of associating a wave function $\psi(\vec{r}, t)$ with a particle – Schrödinger's assumptions of $E = i\hbar\frac{\partial}{\partial t}$, $\vec{p} = -i\hbar$ grad & derivation of wave eqns for free particle & particle in a force-field from the classical equation – case of force derivable from a potential energy $F = -\text{grad}\, V(\vec{r}, t)$ – Eqns $E = \frac{p^2}{2m}$ & $E = \frac{p^2}{2m} + V$ – Schrödinger eqns $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$ & $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$ – Probability interpretation of ψ viz $P = \psi\bar{\psi} = |\psi|^2$, where P is prob. $P\,d\tau$ = probability of finding particle in vol. $d\tau$ at time t – normalisation $\int |\psi(r,t)|^2 d\tau = 1$ (integral convergent)

* Discrete and continuous eigen values - Consider a free particle in one dimension & energy eigenvalues

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} = Eu \quad \text{or} \quad \frac{d^2u}{dx^2} = -\frac{2mE}{\hbar^2}u$$

Case (i) u satisfies some boundary conds say $u=0$ for $x=0$, $x=L$ (confined in a box)

general soln is $u = A\sin\alpha x + B\cos\alpha x$, $\alpha = \sqrt{\frac{2mE}{\hbar^2}}$

& cond's $u=0$ for $x=0$, $x=L$ give $u=0$: $A\sin\alpha L = 0$, ie $\alpha L = n\pi$ $(n=0,1,2,\ldots$

ie $\alpha = \frac{n\pi}{L}$ or $\frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{L^2}$ or $E = \frac{n^2\hbar^2\pi^2}{2mL^2}$, $(n=0,1,\ldots$

ie discrete energy eigenvalues

(ii) no boundary conds, any α is permissible & we have continuous range of eigenvalues

Probability interpretation should be independent of t → $\frac{\partial}{\partial t}\int_V P\,d\tau = -\int_A S_n\,dA = 0$ — Probability current density.

28th Lecture – 18/9/64 — Expectation values – Ehrenfest's theorem – as example of correspondence principle.

29th Lecture – 2nd part of Ehrenfest's theorem viz $\frac{d}{dt}\langle p_x\rangle = \langle -\partial V/\partial x\rangle$ — Schrödinger's eqn with V independent of t; separation of the wave eqn putting $\psi = f(t)\,u(\vec{r})$ getting $i\hbar\frac{1}{f}\frac{df}{dt} = E = \frac{1}{u}\left\{-\frac{\hbar^2}{2m}\nabla^2 u + Vu\right\}$; Soln for f given by $f = Ce^{-iEt/\hbar}$ & assuming u normalised ie $\int \bar{u}u\,d\tau = 1$, we have from $\int \bar{\psi}\psi\,d\tau = 1$, the result $1 = C^2\int \bar{u}u\,d\tau$ ie $C = 1$ & a particular soln is thus $\psi = u(\vec{r})\,e^{-iEt/\hbar}$ — for this eqn $\frac{\partial\psi}{\partial t} = ue^{-iEt/\hbar}\cdot\left(-\frac{iE}{\hbar}\right)$ or $i\hbar\frac{\partial\psi}{\partial t} = E\psi$, energy eigenfunctions & energy eigenvalues & eigenvalue eqn; ψ a stationary state since $|\psi|^2 = |u|^2$ = const. in time; u also an energy eigenfunction of the operator $\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)$ with the energy eigenvalue E, and this result is true more generally (non-separable case) as seen from Schrödinger's eqn — Assuming that the only possible results of precise measurements of the total energy are the eigenvalues of E leads to consideration of general character of energy eigenvalues of $-\frac{\hbar^2}{2m}\nabla^2 u + Vu = Eu$; this depends on $V(\vec{r})$ and boundary conditions imposed on $V(\vec{r})$. When $V(r)$ is finite, eqn can be solved if u and grad u are finite, continuous and single valued at every point of space, which also makes $P(u)$ finite & continuous everywhere. Hence these conditions are assumed to be true in general independent of nature of $V(r)$ — Case of $V(r)$ being ∞, or there being an ∞-discontinuity of V across a surface continuous surface, finite one or are side & ∞ on the other; reduction to one dimensional case by taking tangent plane at any point, this point as origin & normal as x-axis.

30th Lecture – 23/9/64 – Final workings of the above case of ∞ potential barrier – Ex x · 5, 4, 8 / 2, p. 40 of Schiff worked out.

31st Lecture – 24/9/64 – Ex. 2, p. 40 worked out, for general case including V also –

32nd Lecture – 25/9/64 – Above example worked again correctly* / Chap III – General eigenvalues functions and eigenvalues – The three physical postulates (uncertainty relations assumed); (i) Ω, $\Omega u_\omega = \omega u_\omega$, (ii) ω only physically possible, (iii) Any ψ is a linear function of the u_ω, $\psi = \sum c_\omega u_\omega$ and probability for number of measurements resulting in ω is $\propto |c_\omega|^2$ — Energy operator considered more in greater detail; orthonormality of energy eigenfunctions & reality of energy eigenvalues.

33rd Lecture – 29/9/64 – Beginning of Chap III with the above topics – up to reality of energy eigenvalues.

34th Lecture – 30/9/64 – Reality of energy eigen-values – completeness & closure property of orthonormal functions – Probability for & expectation value

(1) Reality :- from $\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)u_E = Eu_E$, we have by multiplying both sides by $\bar{u}_E$ & integrating over L^3,

$$-\frac{\hbar^2}{2m}\int \bar{u}_E\nabla^2 u_E\,d\tau + \int \bar{u}_E V u_E\,d\tau = E\int \bar{u}_E u_E\,d\tau = E \quad \cdots(1)$$

Note that this is of the form $\frac{1}{2m}\langle p^2\rangle + \langle V\rangle = E$

The second integral on the L.H.S of (1) is +ve, and the integrand of the first integral on the L.H.S

* Before this, introduce Dirac's delta function $\delta(x)$ defined by $\delta(x) = 0$ if $x \neq 0$ & $\int \delta(x)\,dx = 1$ if $x = 0$ is included in the ranges of integration or alternatively by $\int f(x)\,\delta(x)\,.\,dx = f(0)$, $x = 0$ being included in the region of integration, or again as $\lim_{a \to 0}$ of $y = \frac{1}{\sqrt{\pi a}}\, e^{-x^2/a^2}$ — generalisation to $\delta(x - x')$ i.e. $\delta(x - x') = 0$ if $x \neq x'$ & $\int \delta(x - x')\,dx = 1$ if $x = x'$ is included in the region

or $\int f(x)\,\delta(x - x')\,dx = f(x')$ if x' be in the region — $\delta(\vec{r} - \vec{r}') = \delta(x - x')\,\delta(y - y')\,\delta(z - z')$.

Proof of $\delta(x) = \delta(-x)$: $\int_{-\infty}^{\infty} f(x)\,\delta(-x)\,.\,dx = \int_{\infty}^{-\infty} f(-x)\,\delta(x)\,(-dx) = \int_{-\infty}^{\infty} f(-x)\,\delta(x)\,dx = f(0)$,

leading to $\delta(-x) = \delta(x)$.

$\therefore \int f(x')\,\delta(x - x')\,dx' = f(x)$. or $\int f(x')\,\delta(x' - x)\,dx' = f(x)$.

or $\int f(r')\,\delta(r - r')\,d\tau' = f(r)$.

can be written as $= \operatorname{div}(\bar{u}_E \operatorname{grad} u_E) - \operatorname{grad}\bar{u}_E \cdot \operatorname{grad} u_E$, using the relation $\operatorname{div}(f\cdot\vec{A}) = f\cdot\operatorname{div}\vec{A} + \operatorname{grad} f\cdot\vec{A}$.

Hence first integral of the d.H.S $= \int \operatorname{grad}\bar{u}_E \cdot \operatorname{grad} u_E \cdot d\tau - \int \operatorname{div}(\bar{u}_E \operatorname{grad} u_E)\, d\tau$, omitting $\hbar^2/2m$

the first integral is $+^{ve}$ & the second by Green's theorem $= \int (\bar{u}_E \operatorname{grad} u_E)_n \, d\sigma = 0$

Hence E is $+^{ve}$ & the note mentioned shows $\langle p^2\rangle$ is $+^{ve}$.

(2) Completeness is the third assumption $\psi = \sum A_E u_E$, and to determine A_E

(3) Closure: $\int \bar{u}_{E'} \psi \cdot d\tau = \sum_E \int \bar{u}_{E'} A_E u_E \cdot d\tau = \sum_E A_E \delta_{EE'} = A_{E'}$ (assuming Σ & $\int$ can be interchanged)

i.e. $A_E = \int \bar{u}_E \psi \, d\tau = \int \bar{u}_E(r')\psi(r')\, d\tau'$.

(3) Closure*: using above value of A_E, $\psi(r) = \sum_E' \int \bar{u}_E(r')\psi(r')u_E(r)\, d\tau'$

$$= \int \psi(r') \sum_E \bar{u}_E(r') u_E(r) \cdot d\tau'$$

which leads to $\sum_E \bar{u}_E(r') u_E(r) = \delta(r-r')$

i.e. $\Sigma \bar{u}_E(r') u_E(r) = 0 \quad \forall\ r \neq r'$

& $\int \Sigma \bar{u}_E(r') u_E(r)\, d\tau' = 1$ if volume of integration includes $r = r'$.

(4) Probability function - with this postulate can be associated at probability for $P(E) \propto |A_E|^2$ & :: by factor is 1, for

we have $\sum_E P(E) = 1$, ~~$\sum_E \bar{A}_E A_E = \sum_E \int u_E(r)\psi(r)dr \int \bar{u}_E(r')\psi(r')\cdot d\tau'$~~

~~$\bar{\psi}(r) = \sum \bar{A}_E \bar{u}_E$~~

$\therefore A_E = \int \bar{u}_E \psi \cdot d\tau$

leads to $\bar{A}_E = \int u_E \bar{\psi}\, d\tau$

and $\sum_E A_E \bar{A}_E = \sum_E \int \bar{u}_E(r)\psi(r)dr \int u_E(r')\bar{\psi}(r')dr'$

$= \iint \psi(r)\bar{\psi}(r') \sum_E \bar{u}_E(r) u_E(r')\, d\tau\, d\tau'$

~~$= \iint \psi(r)\psi(r')\delta(r-r')\, d\tau\, d\tau' = \int \psi(r)\bar{\psi}(r)$~~

$= \iint \psi(r)\bar{\psi}(r')\delta(r-r')\, d\tau\, d\tau' = \int \psi(r)\bar{\psi}(r)\, d\tau = \int |\psi(r)|^2 d\tau = 1$

35th lecture - 1/10/64 - Probability function - Derivation of expectation value (or average value) using $P(E)$, and derivation of the old formula for expectation value. -

Formula for expectation value:- $\langle E\rangle = \sum_E E P(E) \Big/ \sum_E P(E) = \sum_E E P(E) = \sum_E E \int \bar{u}_E(r)\psi(r)dr \int u_E(r')\bar{\psi}(r')dr'$

Now, using $\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\bar{u}_E = E\bar{u}_E$, the first integral can be written as

$$\int \psi(r)\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\bar{u}_E \cdot d\tau$$

using $\operatorname{div}(f\cdot\vec{A}) = f\cdot\operatorname{div}A + \operatorname{grad} f\cdot\vec{A}\cdot$, $\psi(r)\nabla^2\bar{u}_E = \operatorname{div}(\psi \operatorname{grad} u_E) - \operatorname{grad}\psi\cdot\operatorname{grad}\bar{u}_E$

& on integration $\int \operatorname{div}(\ )\, d\tau = \int (\ )_n d\sigma = 0$.

Again using div same result with $f = \bar{u}_E$ & $A = \operatorname{grad}\psi$, ~~div~~ $\operatorname{grad}\bar{u}_E \cdot \operatorname{grad}\psi = \operatorname{div}(\bar{u}_E \operatorname{grad}\psi)$

$- \bar{u}_E \nabla^2\psi = -\bar{u}_E \nabla^2\psi$

$\therefore \langle E\rangle = \sum_E \int \bar{u}_E(r)\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi(r)dr \int u_E(r')\bar{\psi}(r')dr'$

$= \iint \bar{\psi}(r')\left\{\left[-\frac{\hbar^2}{2m} + V(r)\right]\psi(r)\right\}\sum_E \bar{u}_E(r)u_E(r')\cdot dr\, dr'$

$= \int \bar{\psi}(r)\left[-\frac{\hbar^2}{2m} + V(r)\right]\psi(r)\, d\tau$, i.e. original defn of $\langle E\rangle$.

36th Lecture – 12/11/64 – Met the class and handed over the terminal examination answer books.

37th Lecture – 13/11/64 – Met the class, took attendance, mentioned linear harmonic oscillator & left the class.

38th Lecture – 17/11/64 – Discrete and continuous energy level eigen-values as on reverse of p. 43 – Two important cases of discrete eigen-values d.H.O and the H-atom – Schrödinger eqn for d.H.O (with $F = -Kx$ $V = \frac{1}{2}Kx^2$) is $-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + \frac{1}{2}Kx^2u = Eu$ – Transformation to non-dimensional form with $\xi = \alpha x$ where $\alpha \sim L^{-1}$ possible with $\alpha^4 = \frac{mK}{\hbar^2}$; $\lambda = \frac{2E}{\hbar}\left(\frac{m}{K}\right)^{1/2} = \frac{2E}{\hbar\omega_c}$ $\left(\omega_c = \sqrt{K/m}\right)$ leading to the eqn

$$\frac{d^2u}{d\xi^2} + (\lambda - \xi^2)u = 0 \quad \text{– (2)}$$

For large ξ, a possible soln of $\frac{d^2u}{d\xi^2} - \xi^2 u = 0$ is $u = \xi^n e^{\pm\frac{1}{2}\xi^2}$ since highest order terms in $\frac{d^2u}{d\xi^2}$ & $\xi^2 u$ are both ξ^{n+2} & this suggests a general soln $H(\xi)\,e^{\pm\frac{1}{2}\xi^2}$ – Boundary condn $u \to 0$ as $\xi \to \infty$ suggests the factor $e^{-\frac{1}{2}\xi^2}$ and also $H(\xi) = H_n(\xi)$ = a finite polynomial

$$\text{ie } u = H_n(\xi)e^{-\frac{1}{2}\xi^2} \quad \text{– (3)}$$

Using (3) in (2) gives recurrence relation $H_n'' - 2\xi H_n' + (\lambda - 1)H_n = 0$ – (4)

Consider Hermite polynomial $H_n(\xi)$ defined by $e^{-t^2+2t\xi} = \sum_{n=0}^{\infty}\frac{H_n(\xi)}{n!}t^n$ – (5) $= S(t,\xi)$ – (5)

Equating coeffts of t^n in $\frac{\partial S}{\partial \xi}$ and its equivalent Σ leads to

$$H_n' = 2nH_{n-1} \quad \text{...(5)}$$

Similarly with $\frac{\partial S}{\partial t}$ leads to $H_{n+1} = 2\xi H_n - 2nH_{n-1}$ – (6)

From (5) & (6), $H_{n+1} = 2\xi H_n - H_n'$; differentiating this and using (5) for H_{n+1}' leads to

$$H_n'' - 2\xi H_n' + 2nH_n = 0 \quad \text{– (7)}$$

Comparing (4) and (7), $\lambda - 1 = 2n$ or $\lambda = 2n+1$ ie $E_n = (n+\frac{1}{2})\hbar\omega_c$ Φ – (8) giving the discrete energy levels – eigen values

39th Lecture – 18/11/64 – Second defn of Hermite polynomials viz. $H_n(\xi) = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}(e^{-\xi^2})$ derived from the above defn (5)

Values of the H_0, H_1, H_2, H_3, H_4 [Using $H_0 = 1$, finding H_1 from second defn & the rest from eqn (6)]

Eigen-functions ie Derivation of 2nd defn from the first – Derivation of orthogonality relations – Expn for the eigen functions – non-degenerate eigen functions of energy – Polynomial method as an alternative direct method of showing that if $u(\xi) = H(\xi)e^{-\frac{1}{2}\xi^2}$, then $H(\xi) = H_n(\xi)$ & also a method of verifying the form of the polynomial $H_n(\xi)$ – Notion of parity for general one-dim Schrödinger equation with $V(x) = V(-x)$ & for L.H.O, parity is even or odd according as n is even or odd –

40th Lecture – 19/11/64 – Degeneracy of eigenvalues given – ie if two or more linearly independent eigenfunctions correspond to the same eigen-value; non-degeneracy in case of L.H.O (orthonormal set of eigenfns) – Notion of parity for general one-dimnl Schrödinger eqn with $V(x) = V(-x)$ & for L.H.O, parity is even or odd according as n is even or odd – Expectation values: $\langle x\rangle = 0$ & values of $\int u_n(x)\cdot x\cdot u_{n+1}(x)\,dx$, and $\int u_n(x)\cdot x\,u_{n-1}(x)\cdot dx$ – $\langle p\rangle = 0$, $\langle V\rangle_n = \frac{1}{2}E_n = \langle K.E\rangle$; comparison with average values in classical case – Uncertainty relation $\Delta x\cdot\Delta p = (n+\frac{1}{2})\hbar$.

41st lecture – 20/11/64 – $\langle p \rangle = 0$ – $\langle V_n \rangle = \frac{1}{2} E_n = \langle K.E \rangle$ – Verification with average values in the classical case – Uncertainty relⁿ viz $\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar$ resulting in the equality sign for the s.H.O (Proof not done last year)

Here $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \langle x^2 \rangle$, $(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = \langle p^2 \rangle$

$\langle x^2 \rangle = (2n+1)/2\alpha^2$

To calculate $\langle p^2 \rangle$, we have to evaluate $-\hbar^2 \int_{-\infty}^{\infty} u_n \frac{d^2 u_n}{dx^2} \cdot dx$, which

$$= -\hbar^2 C_n^2 \alpha \int H_n e^{-\xi^2/2} \frac{d^2}{d\xi^2}\left(H_n e^{-\xi^2/2}\right) \cdot d\xi \qquad \left[\frac{d^2}{dx^2} = \alpha^2 \frac{d^2}{d\xi^2}\right.$$

$$= -\hbar^2 C_n^2 \alpha \int H_n e^{-\xi^2/2} \left\{ H_n'' - 2\xi H_n' + (\xi^2 - 1) H_n \right\} e^{-\xi^2/2} \cdot d\xi$$

$$= -\hbar^2 C_n^2 \alpha \int H_n e^{-\xi^2/2} \left\{ 4n(n-1) H_{n-2} - 4n\xi H_{n-1} + (\xi^2 - 1) H_n \right\} e^{-\xi^2/2} \cdot d\xi$$

[using $H_n' = 2n H_{n-1}$

$$= -\hbar^2 C_n^2 \alpha \int \left\{ -4n\xi H_n H_{n-1} e^{-\xi^2} \cdot d\xi + \xi^2 H_n^2 e^{-\xi^2} d\xi - H_n^2 e^{-\xi^2} \right\} d\xi$$

since $\int H_n H_{n-2} e^{-\xi^2} d\xi = 0$.

$$= \hbar^2 \alpha^2 \cdot \frac{C_n^2}{\alpha} \int \left\{ 4n \cdot H_n \xi H_{n-1} e^{-\xi^2} d\xi - 2 H_n \xi^2 H_n e^{-\xi^2} d\xi + H_n^2 e^{-\xi^2} \right\} d\xi$$

$$= \hbar^2 \alpha^2 \cdot \frac{C_n^2}{\alpha} \left[4n \cdot \sqrt{\pi} \cdot 2^{n-1} \cdot n! - \frac{\sqrt{\pi}}{2} \cdot 2^n \, n! (2n+1) + \right.$$

$$= -\hbar^2 C_n^2 \alpha \int H_n e^{-\xi^2/2} \left\{ -2n H_n + (\xi^2 - 1) H_n \right\} e^{-\xi^2/2} \cdot d\xi, \quad \text{using } H_n'' - 2\xi H_n' + 2n H_n = 0$$

$$= -\hbar^2 C_n^2 \alpha \int \left\{ \xi^2 H_n^2 e^{-\xi^2} - (2n+1) H_n^2 e^{-\xi^2} \right\} d\xi$$

$$= -\hbar^2 C_n^2 \alpha \left\{ \frac{\sqrt{\pi}}{2} \cdot 2^n \cdot n! (2n+1) - (2n+1) \sqrt{\pi} \cdot 2^n \cdot n! \right\} = \hbar^2 C_n^2 \alpha \cdot \frac{\sqrt{\pi}}{2} \cdot 2^n \cdot n! (2n+1).$$

$$= \hbar^2 \alpha^2 \left(n + \tfrac{1}{2}\right).$$

$\langle x^2 \rangle = \frac{2n+1}{2\alpha^2} = (n+1/2)/\alpha^2 \quad \therefore \langle x^2 \rangle \cdot \langle p^2 \rangle = \hbar^2 (n+\tfrac{1}{2})^2$

$\therefore \Delta x \cdot \Delta p = (n + \frac{1}{2})\hbar$; in consonance with $\Delta x \cdot \Delta p \geq \frac{1}{2}\hbar$

$\therefore \Delta x \cdot \Delta p = \frac{1}{2}\hbar$ for the case $n = 0$ ie for the state $u_0(x) = C_0 H_0(\xi) e^{-\frac{1}{2}\xi^2} = \frac{\alpha^{1/2}}{\pi^{1/4}} e^{-\frac{1}{2}\alpha^2 x^2}$.

42nd lecture – 23/11/64 – Remaining portions of above lecture viz $\Delta x \cdot \Delta p = (n+\frac{1}{2})\hbar \;\&\; = \frac{1}{2}\hbar$ for state $n = 0$ –

Ex. 3, Schiff. p. 90 – Evaluation of $\int_{-\infty}^{\infty} u_n(x) x^2 u_m(x) dx$

$$\text{this} = \frac{C_n C_m}{\alpha^3} \int H_n(\xi) \xi^2 H_m(\xi) e^{-\xi^2} \cdot d\xi$$

Using $e^{2st} \int e^{-\{\xi - (s+t)^2\}} \xi^2 d\xi = \sum\sum \frac{s^m}{m!} \cdot \frac{t^n}{n!} \int H_m(\xi) \xi^2 H_n(\xi) e^{-\xi^2} \cdot d\xi$

ie $e^{2st}\left\{\frac{\sqrt{\pi}}{2} + (s+t)^2 \sqrt{\pi}\right\} =$ "

ie $\frac{\sqrt{\pi}}{2} \sum \frac{(2st)^n}{n!} + \sqrt{\pi} \cdot \sum \frac{s^{n+2} t^n + 2 s^{n+1} t^{n+1} + s^n t^{n+2}}{n!}$

we find equating coeffts of $s^{n+2} t^n$ on both sides, $\int H_n(\xi) \xi^2 H_{n+2}(\xi) e^{-\xi^2} d\xi = \sqrt{\pi} \cdot 2^n \cdot (n+2)!$

" $s^{n-2} t^n$ " $\int H_n(\xi) \xi^2 H_{n-2}(\xi) e^{-\xi^2} d\xi = \sqrt{\pi} \cdot 2^{n-2} \cdot n!$

" $s^n t^n$ " $\int H_n(\xi) \xi^2 H_n(\xi) e^{-\xi^2} d\xi = \sqrt{\pi} \cdot 2^n \cdot n! \left(\frac{2n+1}{2}\right)$

* For the Φ eqn $\frac{d^2\Phi}{d\phi^2} + \nu\Phi = 0$, the condn that ϕ is unaltered if $\phi \to \phi + 2s\pi$ where s is an integer leads to ν = square of an integer, for

in $\Phi = Ae^{i\sqrt{\nu}\phi} + Be^{-i\sqrt{\nu}\phi}$

Changing ϕ to $\phi + 2s\pi$ requires and having Φ unaltered requires $e^{2\pi i\sqrt{\nu}s} \cdot e^{-2\pi i\sqrt{\nu}s} = 1$

and this is true if $\sqrt{\nu} = m^2$.

ie $\Phi = Ae^{im\phi} + Be^{-im\phi}$ is the general soln for $\nu \neq 0$

and $\Phi = \Phi = A + B\phi$ " for $\nu = 0$.

for normalisation purposes we can take $\Phi = Ae^{im\phi}$ (where m is $+^{ve}$ or $-^{ve}$ integer)

& $m = 0$ means $\nu = 0$ & the soln is $\Phi = A$ & normalisation gives $A = \frac{1}{\sqrt{2\pi}}$.

ie general normalised soln is $\frac{1}{\sqrt{2\pi}} e^{im\phi}$ ($m +^{ve}$ or $-^{ve}$).

The normalising factor for $e^{im\phi}$ can also be found by putting $A^2 \int_0^{2\pi} e^{im\phi} \cdot e^{-im\phi} \cdot d\phi = 1$ ie $A = \frac{1}{\sqrt{2\pi}}$

since $|\psi|^2 = \psi\bar{\psi} = e^{im\phi} \cdot e^{-im\phi}$.

Hence $\int u_n(x)\, x^2\, u_{n+2}(x)\, dx = \frac{1}{2\alpha^2}\sqrt{(n+1)(n+2)}$

& $\int u_n(x)\, x^2\, u_{n-2}(x)\, dx = \frac{1}{2\alpha^2}\sqrt{n(n-1)}$

$\int u_n(x)\cdot x^2 \cdot u_n(x)\, dx = \frac{1}{2\alpha^2}(2n+1)$.

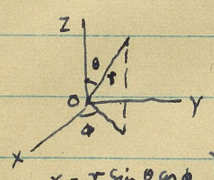
ie $\int u_n(x)\, x^2 u_m(x)\, dx = \frac{1}{2\alpha^2}\sqrt{(n+1)(n+2)}$ if $m = n+2$

$= \frac{1}{2\alpha^2}\sqrt{n(n-1)}$ if $m = n-2$

$= \frac{1}{2\alpha^2}(2n+1)$ if $m = n$

$= 0$, otherwise

- The Hydrogen atom problem is a case where $V(r)$ is a fn of r only & ie $V(r) = -Ze^2/r$ $(Z=1)$ (nucleus+Ze, electron - e) - Schrödinger time independent eqn is $\left\{-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right\}u(r) = E\,u(r)$, and in spherical coordinates, this can be written as



$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]u + V(r)u = Eu$$

with $u = u(r,\theta,\phi)$

$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

Writing $u = R(r)\,Y(\theta,\phi)$ on separating radial and angular coords. and $Y = \Theta(\theta)\,\Phi(\phi)$ on separating angular coordinates

the wave eqn can be solved.

43rd Lecture – 25/11/63 – Starting of H-atom problem ie $V(r)$ – Separation into radial & angular parts – further separation of the angular eqn* – Derivation of $\Phi = \frac{1}{\sqrt{2\pi}}e^{im\phi}$ – Θ-eqn put as

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0 \quad \text{---- (1)}$$

(1) obtained from $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \lambda y = 0$ ---- (2)

by differentiating it m times, putting $u = d^m y/dx^m$, and $v = (1-x^2)^{\frac{1}{2}m}u$

44th Lecture (26/11/64) – Eqn in u is $(1-x^2)\frac{d^2u}{dx^2} - 2x(m+1)\frac{du}{dx} + \{\lambda - m(m+1)\}u = 0$ ---- (3)

Putting $v = (1-x^2)^{\frac{1}{2}m}u$, or $u = (1-x^2)^{-\frac{1}{2}m}v$

ie $\frac{du}{dx} = (1-x^2)^{-\frac{1}{2}m}\frac{dv}{dx} + mx(1-x^2)^{-\frac{1}{2}m-1}\cdot v$;

and, $\frac{d^2u}{dx^2} = (1-x^2)^{-\frac{1}{2}m}\frac{d^2v}{dx^2} + 2mx(1-x^2)^{-\frac{1}{2}m-1}\frac{dv}{dx} + m(1-x^2)^{-\frac{1}{2}m-2}\{1+(m+1)x^2\}v$;

Substituting these values for u, $\frac{du}{dx}$ & $\frac{d^2u}{dx^2}$ in (3), we have

Coeff^t of $\frac{d^2v}{dx^2} = (1-x^2)^{-\frac{1}{2}m+1}$

coefficient of $\frac{dv}{dx} = 2mx(1-x^2)^{-\frac{1}{2}m} - 2x(m+1)(1-x^2)^{-\frac{1}{2}m} = -2x(1-x^2)^{-\frac{1}{2}m}$

coefficient of $u = m(1-x^2)^{-\frac{1}{2}m-1}\{1+(m+1)x^2\} - 2x(m+1)\cdot mx(1-x^2)^{-\frac{1}{2}m-1} + \{\lambda - m(m+1)\}(1-x^2)^{-\frac{1}{2}m}$

$= (1-x^2)^{-\frac{1}{2}m-1}\left[m + m(m+1)x^2 - 2m(m+1)x^2 + \lambda(1-x^2) - m(m+1)(1-x^2)\right]$

$= (1-x^2)^{-\frac{1}{2}m-1}\left[m + m(m+1)x^2 - 2m(m+1)x^2 - m(m+1) + m(m+1)x^2 + \lambda(1-x^2)\right]$

$= (1-x^2)^{-\frac{1}{2}m-1}\{\lambda(1-x^2) - m^2\}$

$e^{-2\pi i\sqrt{\lambda}s} = 1$

$A = \frac{1}{\sqrt{2\pi}}$

ie eqn in v is

$$(1-x^2)^{-\frac{1}{2}m+1}\frac{d^2v}{dx^2} - 2x(1-x^2)^{-\frac{1}{2}m}\frac{dv}{dx} + (1-x^2)^{-\frac{1}{2}m-1}\{\lambda(1-x^2)-m^2\}v = 0$$

Dividing throughout by $(1-x^2)^{-\frac{1}{2}m}$, this reduces to

$$(1-x^2)\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + \left(\lambda - \frac{m^2}{1-x^2}\right)v = 0 \quad \ldots \quad \square$$

ie same as eqn (1) with v in place of y.

– Consideration of (2) & (1); both have singular points $x = \pm 1$, and if a value of λ can be found such that soln of (2) is bounded for $|x| = 1$, the same is true of (1) – Soln in series $y = \sum_{n=0}^{\infty} c_n x^n$ of (2) gives recurrence relation $c_{l+2} = \frac{n(n+1)-\lambda}{(n+1)(n+2)} c_l$. Writing $y = F(x^2) + x\,G(x^2)$, where $F(x^2) = \sum_{l=0}^{\infty} c_{2l} x^{2l}$ & $G(x^2) = \sum_{l=0}^{\infty} c_{2l+1} x^{2l}$

Since $\lim_{n\to\infty} \frac{c_n}{c_{n+2}} = 1$, both $F(x^2)$ & $G(x^2)$ are convergent for $|x| < 1$. If $\lambda = n(n+1)$, either F or G becomes a polynomial & if $\lambda \neq n(n+1)$, it can be shown that both $F(x^2)$ & $G(x^2)$ tend to ∞ for $|x| = 1$, so that physically acceptable solns are polynomials ie the case $\lambda = n(n+1)$ and $y = F(x^2)$ or $x\,G(x^2)$ according as l is even or odd & solns are the Legendre polynomials satisfying the diffl. eqn

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0.$$

– Properties of Legendre's polynomials $P_n(x)$

(i) $(1-2xh+h^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} h^n P_n(x)$

(ii) $P_n = \frac{(2n)!}{2^n.(n!)^2}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} - \ldots\right\}$

(iii) Rodrigues formula, $P_n(x) = \frac{1}{2^n.n!}\frac{d^n}{dx^n}(x^2-1)^n$.

(iv) Recurrence formulae: (a) $(1-x^2)P_n' = -nxP_n + nP_{n-1}$

(b) $(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}$

(v) orthonormality: $\int_{-1}^{+1} P_m(x)P_n(x)\,dx = 0$ if $m \neq n$

$= \frac{2}{2n+1}$ if $m = n$

– Properties of associated Legendre polynomials viz $P_n^m = (1-x^2)^{\frac{1}{2}m}\frac{d^m}{dx^m}P_n(x) \quad (m \leq n)$

(i) Parity $= n-m$, since $(1-x^2)^{\frac{1}{2}m}$ is of even parity whatever m

(ii) orthonormality: $\int_{-1}^{+1} P_n^l(x)P_m^l(x)\,dx = 0$ if $m \neq n$

$= \frac{2}{2n+1}\cdot\frac{(n+l)!}{(n-l)!}$ if $m = n$

45th Lecture – 27/11/64 – Remaining portions of last lecture – Re. query of a student about $\lim_{n\to\infty}\frac{c_n}{c_{n+2}} = 1$; here the test is really about $\lim_{n\to\infty}\frac{c_{n+2}}{c_n}x^2$, which $= x^2$ & hence convergence of $F(x^2)$ & $G(x^2)$ follows for $|x| < 1$.

– General solution of the (θ,ϕ) eqn ie equation satisfied by Y viz

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + l(l+1)Y = 0 \quad (n \to l)$$

with $Y = \Theta.\Phi$ is denoted by $Y_{lm}(\theta,\phi)$ and called a spherical harmonic. Since the equation satisfied by Θ contains m^2, its soln is generalised by allowing +ve as well as −ve values for m such that $|m| \leq l$, and $\Theta = (1-x^2)^{\frac{1}{2}|m|}\frac{d^{|m|}}{dx^{|m|}}P_l(x) = P_l^m(x)$. Normalising factor for $P_l^m = \left\{\frac{2l+1}{2}\cdot\frac{(l-|m|)!}{(l+|m|)!}\right\}^{1/2}$

$$\left.\begin{aligned}
&-\sin\theta\cdot\frac{\partial\theta}{\partial x}=-\frac{z}{r^2}\cdot\frac{\partial r}{\partial x}=-\frac{zx}{r^3};\ \frac{\partial\theta}{\partial x}=+\frac{zx}{r^3\sin\theta}=\frac{r^2\cos\theta\sin\theta\cos\phi}{r^3\sin\theta}=\frac{\cos\theta\cos\phi}{r}\\
&-\sin\theta\cdot\frac{\partial\theta}{\partial y}=-\frac{z}{r^2}\cdot\frac{\partial r}{\partial y}=-\frac{zy}{r^3};\ \frac{\partial\theta}{\partial y}=\frac{zy}{r^3\sin\theta}=\frac{r^2\cos\theta\sin\theta\sin\phi}{r^3\sin\theta}=\frac{\cos\theta\sin\phi}{r}\\
&-\sin\theta\cdot\frac{\partial\theta}{\partial z}=\frac{r-z\,\partial r/\partial z}{r^2}=\frac{r^2-z^2}{r^3};\ \frac{\partial\theta}{\partial z}=-\frac{x^2+y^2}{r^3\sin\theta}=-\frac{\sin\theta}{r}
\end{aligned}\right\}$$

$$\left.\begin{aligned}
&\sec^2\phi\cdot\frac{\partial\phi}{\partial x}=-\frac{y}{x^2};\ \frac{\partial\phi}{\partial x}=-\frac{y\cos^2\phi}{x^2}=-\frac{\sin\phi}{r\sin\theta}\\
&\sec^2\phi\cdot\frac{\partial\phi}{\partial y}=\frac{1}{x^2};\ \frac{\partial\phi}{\partial y}=\frac{\cos^2\phi}{x^2}=\frac{\cos\phi}{r^2\sin^2\theta}\\
&\sec^2\phi\cdot\frac{\partial\phi}{\partial z}=0\quad;\ \frac{\partial\phi}{\partial z}=0
\end{aligned}\right\}$$

(*) $x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta$

$r^2=x^2+y^2+z^2;\ \cos\theta=\frac{z}{r},\ \tan\phi=\frac{y}{x}.$

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial}{\partial\theta}\cdot\frac{\partial\theta}{\partial x}+\frac{\partial}{\partial\phi}\cdot\frac{\partial\phi}{\partial x}=\frac{\partial}{\partial r}\cdot\frac{x}{r}\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}.$$

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial}{\partial\theta}\cdot\frac{\partial\theta}{\partial y}+\frac{\partial}{\partial\phi}\cdot\frac{\partial\phi}{\partial y}=\frac{y}{r}\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi}{r}\cdot\frac{\partial}{\partial\theta}+\frac{\cos\phi}{r^2\sin\theta}\cdot\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial r}\cdot\frac{\partial r}{\partial z}+\frac{\partial}{\partial\theta}\cdot\frac{\partial\theta}{\partial z}+\frac{\partial}{\partial\phi}\cdot\frac{\partial\phi}{\partial z}=\frac{z}{r}\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}.$$

$$M_x=-ih\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)=-ih\left\{y\left(\frac{z}{r}\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}\right)-z\left(\frac{y}{r}\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi}{r}\cdot\frac{\partial}{\partial\theta}+\frac{\cos\phi}{r^2\sin\theta}\cdot\frac{\partial}{\partial\phi}\right)\right\}$$

$$=-ih\left\{-\frac{y}{r}\sin\theta\frac{\partial}{\partial\theta}-\frac{z}{r}\cos\theta\sin\phi\cdot\frac{\partial}{\partial\theta}-\frac{z\cos\phi}{r^2\sin^2\theta}\cdot\frac{\partial}{\partial\phi}\right\}$$

$$=ih\left(\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)$$

$$M_y=-ih\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)=-ih\left\{z\left(\frac{x}{r}\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi}{r}\cdot\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}\right)-x\left(\frac{z}{r}\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\cdot\frac{\partial}{\partial\theta}\right)\right\}$$

$$=-ih\left\{\cos\phi\frac{\partial}{\partial\theta}-\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right\}=ih\left(-\cos\phi\frac{\partial}{\partial\theta}+\cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)$$

$$M_z=-ih\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)=-ih\left\{x\left(\frac{y}{r}\frac{\partial}{\partial r}+\frac{\cos\theta\sin\phi}{r}\cdot\frac{\partial}{\partial\theta}+\frac{\cos\phi}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}\right)\right.$$
$$\left.-y\left(\frac{x}{r}\cdot\frac{\partial}{\partial r}+\frac{\cos\theta\cos\phi}{r}\cdot\frac{\partial}{\partial\theta}-\frac{\sin\phi}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}\right)\right\}$$

$=-ih\left\{\partial/\partial\phi\right\}$, terms in $\partial/\partial r$ & $\partial/\partial\theta$ cancelling.

$M_x^2+M_y^2+M_z^2$ to be derived as on p. 23.

Hence the spherical harmonic $Y_{lm}(\theta,\phi) = N_{lm} P_l^m(\cos\theta) e^{im\phi}$, where the normalising factor $N_{lm} = \left\{\frac{2l+1}{4\pi}\cdot\frac{(l-|m|)!}{(l+|m|)!}\right\}^{1/2}$.

→ orthogonality relations for spherical harmonics is given by $\quad (Y_{00}, Y_{10}, Y_{11}, Y_{1,-1})$

$$\int_0^{\pi}\int_0^{2\pi} \bar{Y}_{lm} Y_{l'm'} \sin\theta\, d\theta\, d\phi = \int_{-1}^{+1} \bar{Y}_{lm} Y_{l'm'}\, dx\, d\phi = 0, \text{ unless } l=l',\ m=m'$$

for the angular part $= \int_0^{2\pi} e^{(-im+im')\phi}\, d\phi = 0$ if $m \neq m'$ whatever l

and the part vanishes only when $l \neq l'$ and $|m| = |m'|$, since $m \neq m'$ is taken care of by integration on ϕ and $l \neq l'$ follows from the orthonormality of P_l^m. When $l = l'$ & $m = m'$, we get $N_{lm} \to \infty$; also the eigen value l is $(2l+1)$-fold degenerate since for given l, m can take all values from $-l$ to $+l$

– Parity for the general case $\vec{r} \to -\vec{r}$ means $r \to r$, $\theta \to \pi - \theta$ from $z = r\cos\theta$ (since $z \to -z$), and $\phi \to \phi + \pi$ as seen from $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ (since $x \to -x$, $y \to -y$). This means Φ has parity $|m|$ & Θ has parity $l - |m|$

$R(r)$ is unchanged as seen & hence Y_{lm} & u has the parity l.

– The equation for $R(r)$ can be transformed by putting $R(r) = \chi(r)/r$: eqn for $R(r)$ is $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left\{\frac{2m}{\hbar^2}(E-V) - \frac{\lambda}{r^2}\right\}R = 0$. $\quad [\lambda = l(l+1)$

Putting $R = \chi/r$, $\frac{dR}{dr} = \frac{1}{r}\frac{d\chi}{dr} - \frac{1}{r^2}\chi$

& eqn becomes $\frac{1}{r^2}\frac{d}{dr}(\chi r - \chi) + \{\quad\}\frac{\chi}{r} = 0$ i.e. $\frac{1}{r^2}\left(r\frac{d^2\chi}{dr^2}\right) + \{\quad\}\frac{\chi}{r} = 0$

i.e. $\frac{d^2\chi}{dr^2} + \{\quad\}\chi = 0$

i.e. $-\frac{\hbar^2}{2m}\frac{d^2\chi}{dr^2} + \left[V(r) + \frac{l(l+1)\hbar^2}{2mr^2}\right]\chi = E\chi$

i.e. the one-dimensional motion of a particle of mass m with potential energy $V + \frac{l(l+1)\hbar^2}{2mr^2}$.

45th Lecture – 1/12/64 – Remaining portion of last lecture – Cf. with eqn $-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + Vu = Eu$, where $V = V(x)$ i.e. motion along x-direction; here V & hence $-\partial V/\partial x$ = rate of change of linear momentum. In the case where $V = V(r)$ i.e. motion in spherical coords, in addition to L.M also an A.M is possible & this additional term can be shown equal to the potential energy associated with the A.M – analogy of the classical case – in quant. mech. A.M is an operator obtained from $\vec{M} = \vec{r} \times \vec{p}$ as

$M_x = yp_z - zp_y = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$, etc – Expression of M_x, M_y, M_z in terms of θ & ϕ (*) – Proof of $\vec{M}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$ & $M_z Y_{lm} = m\hbar Y_{lm}$ – l & m as azimuthal or orbital, and magnetic quantum numbers – Simultaneous specification of M_x, M_y, M_z in classical theory, but only simultaneous specification of $\vec{M}^2$ & M_z in quantum theory as an expression of the uncertainty principle –

– The Hydrogen atom with $V(r) = -e^2/r$.

46th Lecture – 2/12/64 – Remaining parts of previous lecture

Schrödinger eqn (general) for two particles – Simplification in case of relative motion with

$x = x_1 - x_2, \ldots$ & $MX = m_1x_1 + m_2x_2$, (X, Y, Z of C.G), $\ldots$ – Separation of time factor & further separation of the space-eqn with $u = u(x_1, \ldots)\,U(X, \ldots)$ into two eqns for u & U & their interpretation viz eqn in u is the problem of the H-atom in view of m_1/m_2 being small & this being the same eqn as can be transformed to polar coords, the (θ, ϕ) part is the same as in that case & we have only to consider the r-part with $V = -Ze^2/r$

47th Lecture – 3/12/64 – Portions left over from previous lecture ie whole of it – The equation for R with $V = -Ze^2/r$ ($E<0$) – Reduction to dimensionless form by putting $\rho = \alpha r$, where $\alpha^2 = \frac{8\mu|E|}{\hbar^2}$ so that $\alpha = [L^{-1}]$ & ρ is dimensionless – Also putting $\lambda = \frac{2\mu Z e^2}{\alpha\hbar^2}$, redn of the eqn to $\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right) + \left\{\frac{\lambda}{\rho} - \frac{1}{4} - \frac{l(l+1)}{\rho^2}\right\}R = 0$ – Possible soln with $R = \rho^n e^{-\frac{1}{2}\rho}$ for large ρ, since highest order term in $\frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho^2\frac{dR}{d\rho}\right)$ cancels with $-\frac{1}{4}R = -\frac{1}{4}\rho^n e^{-\frac{1}{2}\rho}$ & hence assumption of general soln as $R = F(\rho)e^{-\frac{1}{2}\rho}$ [$F(\rho)$ being a polynomial of order n in ρ] – Eqn $F'' + \left(\frac{2}{\rho}-1\right)F' + \left[\frac{\lambda-1}{\rho} - \frac{l(l+1)}{\rho^2}\right]F = 0$

48th Lecture – 8/12/64. – Left over portions of previous lecture (?) – Using the polynomial method to solve this eqn, if we assume $F(\rho) = a_0 + a_1\rho + a_2\rho^2 + \cdots$, the eqn cannot be satisfied (in view of $F(\rho)$ being finite for $\rho = 0$) since there are terms in $1/\rho$, $1/\rho^2$ on L.H.S. Hence assume $F(\rho) = \rho^s(a_0 + a_1\rho + \cdots) \equiv \rho^s L(\rho)$ – Then eqn for $L(\rho)$ is

$$\rho^2 L'' + \rho\{2(s+1) - \rho\}L' + \{\rho(\lambda - s - 1) + s(s+1) - l(l+1)\}L = 0 \quad \text{—(1)}$$

If this is to be satisfied for $\rho = 0$, $s(s+1) - l(l+1) = 0$ since L, L' & L'' are all finite for $\rho = 0$. ie $s = l$ or $-(l+1)$ & $F(\rho)$ being finite for $\rho = 0$ requires $s = l$ & eqn becomes

$$\rho L'' + \{2(l+1) - \rho\}L' + (\lambda - l - 1)L = 0. \quad \text{—(2)}$$

49th Lecture – 9/12/64 – Completion of the equation in L & remaining parts of previous lecture – Eqn (1) above should hold for $\rho = 0$ & since in L we can take generally $a_0 \neq 0$, $s(s+1) - l(l+1) = 0$ – Substituting the power series for L in (2) & equating all powers of ρ to zero, we get by equation power $\rho^{n'}$

$$a_{n'+1}(n'+1)(n'+2l+2) = a_{n'}(n'+l+1-\lambda)$$

ie $\frac{a_{n'+1}}{a_{n'}} = \frac{n'+l+1-\lambda}{(n'+1)(n'+2l+2)}$ & ~~limit of this as $n' \to \infty$, this limit goes to ∞ as $\frac{1}{n'}$~~ Asymptotic behaviour is that of $1/n'$ which is different for $R = \rho^s e^{-\frac{1}{2}\rho}L(\rho)$ & hence

ie ~~and $\lim_{n'\to\infty} \frac{a_{n'+1}}{a_{n'}} = 1$ ie the radius of convergence is 1 ie convergence only if $\rho < 1$~~ Since ~~we require convergence for ρ,~~ the expn for $L(\rho)$ cannot be an infinite series & must be a polynomial ie it must terminate & condn for this is $\lambda = n' + l + 1$ (~~n' an integer~~) = an integer $= n$, say – n' is radial quantum number & n total quantum number & this value of λ gives discrete energy eigen values viz ~~E_n~~ $|E_n|$ are given by

$$E = \frac{\alpha^2\hbar^2}{8\mu} \text{ \& from } \lambda = \frac{2\mu Z e^2}{\alpha\hbar^2},\ \alpha = \frac{2\mu Z e^2}{\lambda\hbar^2} \quad \text{ie } E = \frac{4\mu^2 Z^2 e^4 \hbar^2}{8\mu\lambda^2\hbar^4} = \frac{\mu Z^2 e^4}{2\lambda^2\hbar^2}$$

and with $\lambda = n$ & noting $E < 0$, we can write eigen-values of E as

$$E_n = -\frac{\mu Z^2 e^4}{2n^2\hbar^2}$$ – Balmer series formula $\nu_n^m = \frac{1}{\hbar}(E_n - E_m) = -T\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$ Bohr's work –

– To find eigenfunctions introduce the Laguerre polynomials

50th Lecture 10/12/64 – Laguerre polynomials from Hilbert – Courant – Starting from Definition $L_n(x) = e^n \frac{d^n}{dx^n}(x^n e^{-x})$, deduction of the generating function and the differential equation

51st Lecture – 11/12/64 – Asymptotic behaviour given in Pauling & Wilson, p. 123 ie for large n', $\frac{a_{n'+1}}{a_{n'}}$ behaves as $\frac{1}{n'}$ ie represented by e^ρ & hence $R(\rho) = e^{-\rho/2}\rho^l L(\rho)$ behaves like $e^{+\rho/2}$ which is contrary to assumed behaviour of $R(\rho)$ for $\rho \to \infty$ – Derivation of ~~associated~~ diffl eqn of associated Laguerre polynomial (P&W, p. 131) or (S, p. 84) & comparison with the diffl. eqn for $L(\rho)$ in the H-atom give $L^{2l+1}_{n+l}(\rho)$ as the of order $(n+l) - (2l+1) = n - l - 1 = n'$ – Normalising factor

* Surfaces ξ = const are given by $r - z = k$ i.e. $r = z + k$ or $x^2+y^2+z^2 = z^2+k^2+2kz$ i.e. $x^2+y^2 = k^2+2kz$

i.e. $x^2+y^2 = 4 \cdot \frac{k}{2}\left(z + k/2\right)$ i.e. with vertex at $(0, 0, -k/2)$ or since latus rectum $= 4 \cdot k/2$, focus is at the origin

Similarly for surfaces η = const. - That ξ = const, η = const, ϕ = const form mutually orthogonal system, it needs only to prove that ξ = const & η = const. form an orthogonal system follows from the result that confocal parabolas ~~intersect~~ with same axes intersect orthogonally.

(†) See p. 30.

for the associated Laguerre ~~polynomials~~ functions (P&W, p. 451) or rather of $R = e^{-\frac{1}{2}\rho}\rho^{l} L^{2l+1}_{n+l}(\rho)$

52nd Lecture – 15/12/64. Class not taken today.

52nd Lecture – 16/12/64 – Degeneracy of energy levels – Degeneracy and different coordinate systems – H-atom problem in parabolic coordinates* – wave eqn in parabolic coordinates – $\nabla^2 u$ in parabolic coordinates

53rd Lecture – 17/12/64 – continuation of wave eqn in parabolic coordinates; could obtain only expn for ds^2 in terms of (ξ, η, ϕ) in the last lecture – Derivation of the equations [with $\rho = \alpha\xi$, $\alpha^2 = \frac{2\mu E}{\hbar^2}$]

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{df}{d\rho}\right) - \left(\frac{m^2}{4\rho^2} + \frac{1}{4} - \frac{\lambda_1}{\rho}\right) f = 0 \quad \text{—(1)} \qquad \left[\alpha\lambda_1 = \frac{\mu Z e^2}{\hbar^2} - \nu\right.$$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dg}{d\rho}\right) - \left(\frac{m^2}{4\rho^2} + \frac{1}{4} - \frac{\lambda_2}{\rho}\right) g = 0. \quad \text{—(2)} \qquad \left[\alpha\lambda_2 = \nu\right.$$

54th Lecture – 18/12/64 – Continuation of same topic – Substitution of $f = e^{-\frac{1}{2}\rho}\rho^{\frac{1}{2}m} L(\rho)$ in (1) leading to the equation in $L(\rho)$ viz

$$\rho L''(\rho) + (m+1-\rho)L'(\rho) + (\lambda_1 - \tfrac{1}{2}m - \tfrac{1}{2})L(\rho) = 0$$

& similarly for (2)

$$\rho L''(\rho) + (m+1-\rho)L'(\rho) + (\lambda_2 - \tfrac{1}{2}m - \tfrac{1}{2})L(\rho) = 0.$$

– comparison with eqn for $L^p_q(\rho)$ viz $\rho L^{p\,\prime\prime}_q + (p+1-\rho)L^{p\,\prime}_q + (q-p)L^p_q = 0$, giving $p = m$, $q - p = \lambda_1 - \frac{1}{2}(m+1)$

ie $q = \lambda_1 + m - \frac{1}{2}(m+1) = n_1 + m$ (requiring $n_1 = \lambda_1 - \frac{1}{2}(m+1)$ to be an integer)

& similarly $q = \lambda_2 + m - \frac{1}{2}(m+1) = n_2 + m$ (" $n_2 = \lambda_2 - \frac{1}{2}(m+1)$ ")

$\lambda_1 + \lambda_2 = n_1 + n_2 + (m+1) =$ an integer defined by n & called principal quantum number

n_1, n_2 are called parabolic quantum numbers (Here there are apparently four quantum numbers, but n_1 & n_2 are not independent, being connected by the relation $n_1 + n_2 + m + 1 = n$ – $\alpha(\lambda_1 + \lambda_2) = \frac{\mu Z e^2}{\hbar^2}$ ie $\alpha n = \frac{\mu Z e^2}{\hbar^2}$

& using $\alpha^2 = \frac{2\mu E}{\hbar^2}$; we have $\frac{2\mu E}{\hbar^2} = \frac{\mu^2 Z^2 e^4}{\hbar^4}$ or $E_n = -\frac{\mu Z^2 e^4}{2\hbar^2 n^2}$ as in the polar eqn case.

– Degeneracy of energy levels – Expn for $u_{n_1 n_2 m} = e^{-\frac{1}{2}\alpha(\xi+\eta)}(\xi\eta)^{\frac{1}{2}m} L^m_{n_1+m}(\alpha\xi)\, L^m_{n_2+m}(\alpha\eta)\, e^{im\phi}$.

$$\alpha = \frac{\mu Z e^2}{\hbar^2(n_1 + n_2 + m + 1)}.$$

– Comparison of ground states ($n_1 = n_2 = m = 0$, ~~and~~ and $n = 1$, $l = m = 0$) in the two cases –

55th Lecture – 22/12/64 – Matrices up to secular equation –

56th Lecture – 23/12/64 – Secular eqn, degeneracy – Matrices of ∞ rank – 3 Theorems (Converse of real eigenvalues theorem also) –

57th Lecture – 15/1/65 – Matrices in Quantum mechanics – Matrix transforming ~~one set of~~ a complete orthonormal set of fns into another is unitary – Schrodinger's eqn in the form $H u_k(\vec{r}) = E_k u_k(\vec{r})$ – Proof of $(SHS^*)_{kl} = \int \bar{u}_k H u_l \cdot d\tau$.

58th Lecture – 19/1/65 – (A) The three results on p. 30 of the notes – (i) $\int \bar{u}_k H u_l\, d\tau = \int E_k \bar{u}_k u_l \cdot d\tau = E_k \delta_{kl}$ or $E_k \delta(k-l)$ ie in the v-repn H is diagonal with eigenvalues E_k or solving Schrödinger eqn is equivalent to finding a repn in which H is diagonalised & eigenvalues of H are energy eigenvalues & $u_k = \bar{S}_{kn} v_n$ determines u_k in terms of v_n through the S diagonalising H. (ii) E_k's are real & hence eigenvalues of H are real ie H is Hermitian in ~~any~~ this representation & from the property (+) that hermiticity is maintained during a unitary transformation, H is hermitian in any representation, (iii) Any dynamical variable hermitian

(f) To find matrices for M_z, $\vec{M}^2$ and M_x & M_y for different values of j, note that

(i) for M_z the eigen values are $m\hbar$ ($\underline{m}$ varying from j to $-j$) (only diagonal)

(ii) for $\vec{M}^2$ " $j(j+1)\hbar^2$ (diagonal, same for all $\underline{m}$ - only $(\vec{M}^2)_{mj,mj}$

(iii) To find these for M_x & M_y, first find L and L^*.

(a) $j = \frac{1}{2}$, all elements of L except $L_{m+1,m}$ are zero ie the only non-zero element

is $L_{1/2,-1/2} = \left[\left\{\frac{1}{2} - (-\frac{1}{2})\right\}\left\{\frac{1}{2} + (-\frac{1}{2}) + 1\right\}\right]^{1/2}\hbar = \hbar$

Hence $L = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $L^* = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$$M_x = \frac{1}{2}(L^* + L) = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad M_y = \frac{1}{2}i(L^* - L) = \frac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$M_z = \frac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{M}^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(b) $j = 1$, $\underline{m}$ has values $1, 0, -1$

$$M_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad \vec{M}^2 = 2\hbar^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$L_{1,0}$ and $L_{0,-1}$ are the only non-vanishing elements in the matrix

$$L = \begin{pmatrix} L_{11} & L_{1,0} & L_{1,-1} \\ L_{01} & L_{0,0} & L_{0,-1} \\ L_{-1,1} & L_{-1,0} & L_{-1,-1} \end{pmatrix}$$

and $L_{1,0} = \{(1-0)(1+0+1)\}^{1/2}\hbar = \sqrt{2}\hbar$, $L_{0,-1} = \{(1+1)(1-1+1)\}^{1/2}\hbar = \sqrt{2}\hbar$

ie $L = \hbar\begin{pmatrix} 0, & \sqrt{2}, & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$ & $L^* = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$

$$\therefore M_x = \cancel{\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \end{pmatrix}} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\& \; M_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0, & -i, & 0 \\ i, & 0, & -i \\ 0, & i, & 0 \end{pmatrix}$$

Similarly for (c) $j = 3/2$, and (d) $j = 2$, mention Madhavarao Algebra

29th lecture - 2/3/63 - Soln of biharmonic equation by means of complex variables viz $2U = \bar{z}\varphi(z) + z\bar{\varphi}(\bar{z}) + \chi(z) + \bar{\chi}(\bar{z})$

- formulas for stresses and displacements in terms of complex variables

because of trn of a diagonal matrix with real diagonal elements → Hermitian matrix &

(B) Eqns $u_k(\vec{r})$ considered a matrix U_{rk} - U_{rk} unitary, using closure & normal properties.

(C) The identity $\int \bar{g}(r)[\Omega f(r)]\,d\tau = \int \overline{[\Omega^* g(r)]} f(r)\,d\tau$, where $\Omega f(r) = \int \Omega(r,r') f(r')\,d\tau'$ [$\Omega(r,r')$ an operator]

(D) Eqns of motion in matrix form.

59th Lecture – 20/1/65 – Eqns of motion in matrix form (Heisenberg's eqns) & classical analogues with P.B – Angular momentum operator in matrix mechanics.

60th Lecture – 21/1/65 – A M operator – Earlier results $\vec{M}^2 Y_{lm}(\theta,\phi) = l(l+1)\hbar^2 Y_{lm}(\theta,\phi)$ & $M_z Y_{lm}(\theta,\phi) = m\hbar Y_{lm}(\theta,\phi)$ – $\vec{M} = \vec{r}\times\vec{p}$ for a particle – Dern of $[M_y, M_z] = i\hbar M_x$, etc & taken as more general results – Two of ~~Not more than one~~ of M_x, M_y, M_z cannot be diagonalised simultaneously since no two commute with each other – Proof of $\vec{M}^2$ commuting with all M_x, M_y, M_z – Simultaneous diagonalisation $\vec{M}^2$ & M_z – Introduction $L = M_x + iM_y$ ($M_x, M_y, M_z, \vec{M}^2$ all hermitian, in consonance with commutation relns $[M_y, M_z] = i\hbar M_x$) – $[\vec{M}^2, L) = 0$ – $[M_z, L] = \hbar L$, $[L, L^*] = 2\hbar M_z$, $\vec{M}^2 = M_z^2 + \frac{1}{2}(LL^* + L^*L)$ – In diagonal repn of M_z, m is not necessarily an integer) & same m can be used in repn of L as seen from $[L, L^*] = 2\hbar M_z$ – repn of L in this diagonal repn from $[M_z, L] = \hbar L$ leading to $L_{m+1,m} = \lambda_m \hbar$ (λ_m may be complex) & $[L, L^*] = 2\hbar M_z$ leading to a difference eqn in $|\lambda_m|^2$ with soln $|\lambda_m|^2 = C - m(m+1)$ & $|\lambda_m|^2 \geq 0$ limiting values of m to be from m_1 to m_2+1 ($m_1 > m_2$) ie $m_1, m_1-1, \ldots m_2+1$

61st Lecture – 22/1/65 – Remaining portions of previous lecture – Introduction of $j = m_1$, $m_2+1 = -j$ leading to eigenvalues of M_z ranging from j to $-j$ by unit steps ie j = integer or half-odd integer – $|\lambda_m|^2 = j(j+1) - m(m+1) = (j-m)(j+m+1)$ – Eigenvalues of $\vec{M}^2$ from $M^2 = M_z^2 + \frac{1}{2}(LL^* + L^*L)$ $= M_z^2 + \frac{1}{2}\{|\lambda_{m-1}|^2 + |\lambda_m|^2 = j(j+1)\hbar^2$ – Matrices for M_z, $\vec{M}^2$ & M_x & M_y for $j = 0, \frac{1}{2}, 1, \frac{3}{2}$. (†)

62nd Lecture – 2/2/65 – Remaining portion of previous lecture going up to $j = \frac{3}{2}$ – Did not take the class.

" – 3/2/65 – Orbital and spin angular momentum; for former case integral $\hbar$ only as seen by Sph. harmonic relations (orbital & magnetic quantum number) – $\vec{M}^2$ does not commute with $\vec{r}$ and $\vec{p}$ in the orbital case for eg. $[\vec{M}^2, x] \neq 0$ – ~~[illegible]~~

63rd Lecture – whole of portions proposed for 3/2/65 ~~Addition of A M~~ presented as follows:–
(4/2/65)

(i) The representations based on $\vec{M} = \vec{r}\times\vec{p}$ correspond to only integral eigen-values as seen from $\vec{M}^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm}$ & $M_z Y_{lm} = m\hbar Y_{lm}$, since l & m are integers as follow from definition of Sph. harmonics. Thus orbital and magnetic quantum numbers are integers.

(ii) Representations based on $[M_x, M_y] = i\hbar M_z$, etc are more general and include those based on $\vec{M} = \vec{r}\times\vec{p}$ (viz. integral repns) & contain besides the half-integral repns which are not contained in $\vec{M} = \vec{r}\times\vec{p}$.

(iii) This more general nature is also brought out by the fact that $\vec{M}^2$ being a diagonal matrix with ~~unity~~ all the diagonal elements unity commutes with matrices representing $\vec{r}$ & $\vec{p}$, but this is

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25th Lecture – 7/2/63 – Ex. 1, p. 78 worked – Strain energy density function W – Clapeyron's formula – Castigliano's formula – Expn of W as a +ve definite quad. form in the e_{ij} in the isotropic case – Expn for W as quad. fn in τ_{ij} (not correctly done!, by not using $e_4 = 2e_{23}$)

26th Lecture – 9/2/63 – (*) Proof of +ve definite form of quadratic in τ_{ij} – W in terms of ϑ_1 and ϑ_2 – W in terms of Θ_1 and Θ_2 – ~~Uniqueness problem~~ Clapeyron's theorem – St. Venant's Reciprocity and uniqueness of solns (Superposition of solutions used in proving uniqueness) – St. Venant's Principle (relating to the first boundary value problem)

27th Lecture – 14/2/63 – Did not take the Class – went to Univ. Office to see Repr & then to the Library.

28th 27th Lecture – 16/2/63 – Uniqueness a consequence of +ve definite character of W, which again is a consequence of the linear theory of elasticity. In non-linear theory uniqueness may not exist – Questions of existence of solns more complicated than in Potential Theory & use is made of Integral equations – Exact solns of problems presents formidable mathematical difficulties & hence approximations are used eg. St. Venant's Principle (Explained with the illustration of blows on the leg & wrists of leg giving equal pain influencing the head equally!) –

Two-dimensional elastostatic problems (Chap. II, 249) & then return to Chap IV – Cases of plane strain & plane stress; – Case of plane strain in long cylinders – described by fns $\tau_{\alpha\beta}(x_1, x_2)$ & $u_\alpha(x_1, x_2)$ [$\tau_{33} \neq 0$, but $= \sigma(\tau_{11} + \tau_{22}) = \sigma\Theta_1$]* – Second boundary value problem: Navier's eqn $\mu \nabla^2 u_\alpha + (\lambda+\mu)\frac{\partial \vartheta}{\partial x_\alpha} + F_\alpha = 0$ on a plane section R $x_3 = \text{const}$ & u_α on the boundary curve C of section R – First boundary value problem: Compatibility eqns are to be satisfied; these eqns reduce to only one viz $\nabla^2\Theta_1 = -\frac{1}{1-\sigma} F_{\alpha,\alpha}$ ie finding solns of $\tau_{\alpha\beta,\beta} = -F_\alpha$ subject to this compatibility eqn & given boundary cond^n $\tau_{\alpha\beta}\nu_\beta = T_\alpha$ – Uniqueness of solns in both cases – Physical explng $\tau_{33} \neq 0$ & St. Venant's Principle for long cylinders.

28th Lecture – 21/2/63 – Remaining parts of previous lecture – Case of plane stress $\perp^r$ to x_1x_2-plane

" – 28/2/63 – Missed the lecture on 21/2/63 being unwell – Case of plane stress; difference from case of plane strain viz $\tau_{\alpha\beta}$ & u_α may be depend on x_3 & not only 2-dimensional – Filon's idea of taking mean values for a plate of small thickness $2h$, to make problems identical – Plane elastostatic problems – Reduction to case of $F_\alpha = 0$ Knowing a set of particular integrals $\tau^{(0)}_{\alpha\beta}$ & putting $\tau_{\alpha\beta} = \tau^{(0)}_{\alpha\beta} + \tau^{(1)}_{\alpha\beta} \to \tau_{\alpha\beta,\beta} = \tau^{(0)}_{\alpha\beta,\beta} + \tau^{(1)}_{\alpha\beta,\beta}$ i.e. $-F_\alpha = -F_\alpha + \tau^{(1)}_{\alpha\beta,\beta}$ or $\tau^{(1)}_{\alpha\beta,\beta} = 0$ – Airy's Stress function – Fundamental biharmonic boundary value problem – Fundamental solution of biharmonic equation

not true of the matrices $\vec{M}^2$ obtained from $\vec{M} = \vec{r} \times \vec{p}$, ~~do not~~ c. For example $[\vec{M}^2, x]$ can be shown, using $[x, p_x] = [y, p_y] = [z, p_z] = i\hbar$ & all other commutators between $\vec{r}$ & $\vec{p}$ zero, equal to $2i\hbar\{(y^2+z^2)p_x - xyp_y - zxp_z\} \neq 0$. Similarly $[\vec{M}^2, p_x] = 2i\hbar\{(p_y^2+p_z^2)x - yp_xp_y - zp_zp_x\} \neq 0$.

(iv) Intrinsic A.M & orbital A.M, former called spin A.M – Electrons, protons, pontrons, antiprotons, neutrons & μ-mesons have all spin $1/2$, π-mesons have $j = 0$ & there are other types of mesons with $j = 1$, but none-observed with $j > 1$.

64th Lecture (5/2/65)

Addition of angular momenta: $\vec{M} = \vec{M}_1 + \vec{M}_2$, with $[\vec{M}_1, \vec{M}_2] = 0$

(a) M_1 & M_2 may be A.M of two separate particles or orbital & spin A.M of same particle [$j = 1$ for former & $j = 1/2$ for latter]

(b) $\vec{M}$ obeys same spin commutation relations as $\vec{M}_1$ & $\vec{M}_2$.

(c) Two types of repns (i) $\vec{M}_1^2, \vec{M}_2^2, M_{1z}, M_{2z}$ diagonal, specified by j_1, j_2, m_1, m_2

(ii) $\vec{M}_1^2, \vec{M}_2^2, \vec{M}^2, M_z$ diagonal, specified by j_1, j_2, j, m.

~~Case C(i):~~ – ~~This In this representation, the $\vec{M}$ & $\vec{M}^2$ matrices have $(2j_1+1)(2j_2+1)$ rows or columns, as seen for~~

Force eqn $\tau_{ij,j} = -F_i \rightarrow F_3 = 0$ & F_1, F_2 are fns of (x_1, x_2) only.
& to find τ_{xp} & u_x we have $\tau_{xp} = \lambda\delta_{xp} + \mu(u_{x,p} + u_{p,x})$ and $\tau_{xp,p} = -F_x$ $(3+2\text{eqns})$.

~~example by the M_z-matrix whose diagonal elements are obtained by adding each of the diagonal $(2j_1+1)$ diagonal elements of M_{1z} to each of the $(2j_2+1)$ diagonal elements of M_{2z} – Particular example of $j_1 = \frac{1}{2}, j_2 = 1$: Here the diagonal elements of M_{1z} and M_{2z} are $(\frac{1}{2}, -\frac{1}{2})$ and $(1, 0, -1)$ and so M_z has diagonal elements $(\frac{1}{2}+1, \frac{1}{2}+0, \frac{1}{2}-1, -\frac{1}{2}+1, -\frac{1}{2}+0, -\frac{1}{2}-1)$ ie $(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2})$ ie matrix with 6 rows or columns.~~

~~For M for given j_1 and j_2~~

(c) Here six observables are available for consideration regarding simultaneous diagonalisation, viz $\vec{M}^2, M_z, \vec{M}_1^2, M_{1z}, \vec{M}_2^2, M_{2z}$. We can see that $\vec{M}^2$ does not commute with either M_{1z} or M_{2z}, but all other pairs of the six quantities (ie $6C_2 - 2 = 13$) ie 13 other pairs commute with each other. This fact leads to the two representations C(i) & C(ii), where in C(i) all four of $\vec{M}_1^2, M_{1z}, \vec{M}_2^2, M_{2z}$ commute with each other, but two of them not with $\vec{M}^2$, thus giving rise to $\vec{M}_1$ and $\vec{M}_2$ being dynamically independent, with $\vec{M}_1^2, M_{1z}$ and $\vec{M}_2^2, M_{2z}$ simultaneously diagonalised, and in C(ii) $\vec{M}_1$ & $\vec{M}_2$ are considered

(*) obtain correct expn for W in terms of τ_{ij} and verify $\frac{\partial W}{\partial \tau_{ij}} = e_{ij}$

as components of $\vec{M}$ [in (i) $\vec{M}$ is considered resultant of $\vec{M}_1$ and $\vec{M}_2$], with $\vec{M}_1^2, \vec{M}_2^2, \vec{M}^2, M_z$ simultaneously diagonised (all the six pairs mutually commuting). We will now consider C(i) & C(ii) separately.

Next adding the second and third equations of (A)

$$\tau_{11,22} + \tau_{11,33} + \tau_{22,11} + \tau_{33,11} - 2\tau_{31,31} - 2\tau_{12,12} = \frac{\sigma}{1+\sigma}\left(\nabla^2\Theta + \Theta_{,11}\right)$$

ie $$\nabla^2\tau_{11} + \Theta_{,11} - 2\left(\tau_{11,11} + \tau_{12,12} + \tau_{31,31}\right) = \frac{\sigma}{1+\sigma}\left(\nabla^2\Theta + \Theta_{,11}\right)$$

From the first eqn of (C) it is easily seen that the third term on the left hand side above is equal to $2F_{1,1}$. Hence

$$\nabla^2\tau_{11} + \frac{1}{1+\sigma}\Theta_{,11} + 2F_{1,1} = \frac{\sigma}{1+\sigma}\nabla^2\Theta = -\frac{\sigma}{1-\sigma}\,\mathrm{div}\,F \text{ , using (E)}$$

ie $$\nabla^2\tau_{xx} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x^2} = -\frac{\sigma}{1-\sigma}\,\mathrm{div}\,F - 2\frac{\partial F_x}{\partial x}$$

which is the first of the B-M eqns. The second & third eqns follow similarly by cyclic symmetry.

Note :- The proof given above is far simpler than that given in Sokolnikoff.

23rd Lecture – 31/1/63 – (*) Writing of the B-M eqns in the form

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} = -\frac{\sigma}{1-\sigma}\delta_{ij}\,\mathrm{div}\,\vec{F} - \left(F_{i,j} + F_{j,i}\right) \quad \text{---} \quad (F)$$

If F be conservative ie $F = \nabla\phi = \mathrm{grad}\,\phi$, in the R.H.S of (F), $\mathrm{div}\,F = \nabla^2\phi$

and $$\left(F_{i,j} + F_{j,i}\right) = 2\phi_{,ij}.$$

and (F) $\rightarrow$ $$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} = -\frac{\sigma}{1-\sigma}\nabla^2\phi - 2\phi_{,ij} \quad \text{---} \quad (G)$$

Particular cases (i) F is constant. ie ϕ is linear, and we have Beltrami's eqn.

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} = 0 \quad \text{---} \quad (H)$$

From (E) we have in this case, since $\mathrm{div}\,F = 0$, $\nabla^2\Theta = 0$ ie $\Theta = \tau_{ii}$ is harmonic

From $\Theta = (3\lambda + 2\mu)\vartheta$, it follows that ϑ is also harmonic $\nabla^2\vartheta = 0$

From Beltrami's eqn we further have $\nabla^2\nabla^2\tau_{ij} = \frac{1}{1+\sigma}(\nabla^2\Theta)_{,ij} = 0$

ie τ_{ij} & hence e_{ij} are biharmonic functions.

(ii) ϕ is harmonic ie $\nabla^2\phi = 0$ ie $\mathrm{div}\,F = \mathrm{div}\,\mathrm{grad}\,\phi = \nabla^2\phi = 0$.

Hence $\nabla^2\Theta = 0$ & also $\nabla^2\vartheta = 0$ (†)

24th Lecture – 2/2/63 – Remaining parts of the Beltrami – Michell compatibility equations viz. the first three, the last three having been done on 31/1/63 – the portions noted above for the lecture on 31/1/63 –

Ex 1, p. 78 (**). Here the B.M. eqns are not satisfied, for $\Theta = c(1+\sigma)(x^2+y^2)$ we $\Theta_{,xx} = 2c(1+\sigma)$

Also $$\nabla^2\tau_{xx} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left\{cy^2 + c(x^2 - y^2)\right\} = 2c\sigma + 2c(1-\sigma) = 2c$$

$$\therefore \quad \nabla^2\tau_{xx} + \frac{1}{1+\sigma}\Theta_{,xx} = 4c \neq 0$$

(*) equm eqns with body forces are satisfied provided $\tau_{ij,j} + \beta_i = 0$ & $\tau_{ij,j} = 0$ i.e.

$$\left.\begin{aligned} \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 &\rightarrow 2c\theta_x - 2c\theta_x = 0 \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 &\rightarrow -2c\theta_y + 2c\theta_y = 0 \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = 0 &\rightarrow 0 = 0. \end{aligned}\right\}$$

Thus is not mentioned in Sokolnikoff, I wonder why?

(†) In Fig. case τ_{ij} and ε_{ij} are biharmonic functions, for the B.M. eqns become

$$\text{i.e.,} \quad \nabla^2 \tau_{ij} + \frac{1}{1+\sigma} \Theta_{,ij} = -2\phi_{,ij} \qquad (\because \nabla^2 \phi = 0.)$$

$$\therefore \quad \nabla^4 \tau_{ij} + \frac{1}{1+\sigma} (\nabla^2 \Theta)_{,ij} = -2(\nabla^2 \phi)_{,ij} \quad \text{i.e.} \quad \nabla^4 \tau_{ij} = 0 \text{ since } \nabla^4 \varepsilon_{ij} = 0$$

C(i):- Let the wave functions associated with $\vec{M}_1$ & $\vec{M}_2$ be denoted by ψ_{j_1, m_1} & ψ_{j_2, m_2}, where for fixed j_1 & j_2, m_1 & m_2 respectively take $(2j_1+1)$ & $(2j_2+1)$ values from $j_1, \ldots$ to $-j_1$ and $j_2, \ldots$ to $-j_2$. We can write

$$M_{1z} \psi_{j_1, m_1} = m_1 \hbar \psi_{j_1, m_1}, \text{ and } M_{2z} \psi_{j_2, m_2} = m_2 \hbar \psi_{j_2, m_2} \quad (\text{say } \psi_1 \text{ & } \psi_2 \text{ for brevity})$$

If $\phi_{j_1, k_1; j_2, k_2}$ be equal to the product of ψ_{j_1, m_1} & ψ_{j_2, m_2}, then (say $\phi = \psi_1 \cdot \psi_2$), then

$$(M_{1z} + M_{2z})\phi = (M_{1z} + M_{2z})\psi_1 \psi_2 = m_1 \hbar \psi_1 \psi_2 + m_2 \hbar \psi_1 \psi_2 = (m_1 + m_2)\hbar \psi_1 \psi_2$$

i.e. if $M_z \phi = (m_1 + m_2)\hbar \phi$, and so the eigen values of M_z are the sums of all possible sums of m_1 and m_2 i.e. M_z has a diagonal representation with $(2j+1)(2j+1)$ rows or columns. As a particular example consider $\vec{M}_1$ to be an orbital A.M corresponding to $j_1 = 1$, and $\vec{M}_2$ a spin A.M, corresponding to $j_2 = \frac{1}{2}$, then m_1 has possible values $1, 0, -1$ & m_2 has $\frac{1}{2}, -\frac{1}{2}$ & all possible combinations of sums of m_1 & m_2 give

$\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$ i.e. $M_z = \begin{pmatrix} 3/2 & & & & & \\ & 1/2 & & & & \\ & & -1/2 & & & \\ & & & 1/2 & & \\ & & & & -1/2 & \\ & & & & & -3/2 \end{pmatrix}$ i.e. a 6×6 matrix.

finding of $\vec{M}^2$ in this case, which is not diagonal is more complicated, and is not considered here.

C(ii):- Here also $m = m_1 + m_2$, and largest value of $m = j_1 + j_2$ & since m varies from j to $-j$, largest value of $j = j_1 + j_2$, the next smaller value of m is $j_1 + j_2 - 1$ & this corresponds to $m_1 = j_1 - 1$ & $m_2 = j$ i.e. the next smaller value of j is also $j_1 + j_2 - 1$. Hence going to smaller & smaller values, the least value of $j = j_1 - j_2$ ($j_1 > j_2$)

(*) Both relating to the problem handed in 31/1/63 — on the slow bend, both the problems left over from last lecture viz. Problems 1 and 2, and the discussion of the Boltzmann-Maxwell equation. Today I could do only half.

give $\tau_{ij,kl} + \tau_{kl,ij} - \tau_{ik,jl} - \tau_{jl,ik} = \frac{\sigma}{1+\sigma}\left(\delta_{ij}\Theta_{,kl} + \delta_{kl}\Theta_{,ij} - \delta_{ik}\Theta_{,jl} - \delta_{jl}\Theta_{,ik}\right)$ --(6)

As in the case of $e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$, these also reduce to six equations corresponding to the cases

$$\left.\begin{array}{l} K=l=2,\ i=j=3 \\ K=l=3,\ i=j=1 \\ K=l=1,\ i=j=2 \end{array}\right\} \text{(A)}_0 \quad \text{and} \quad \left.\begin{array}{l} K=l=1,\ i=2,\ j=3 \\ K=l=2,\ i=3,\ j=1 \\ K=l=3,\ i=1,\ j=2 \end{array}\right\} \text{(B)}_0$$

Writing (6) in full for these cases:

$$\left.\begin{aligned} \tau_{22,33} + \tau_{33,22} - 2\tau_{23,23} &= \frac{\sigma}{1+\sigma}\left(\Theta_{,22} + \Theta_{,33}\right) \\ \tau_{33,11} + \tau_{11,33} - 2\tau_{31,31} &= \frac{\sigma}{1+\sigma}\left(\Theta_{,33} + \Theta_{,11}\right) \\ \tau_{11,22} + \tau_{22,11} - 2\tau_{12,12} &= \frac{\sigma}{1+\sigma}\left(\Theta_{,11} + \Theta_{,22}\right) \end{aligned}\right\} \text{(A)}$$

$$\left.\begin{aligned} \tau_{23,11} + \tau_{11,23} - \tau_{12,31} - \tau_{31,12} &= \frac{\sigma}{1+\sigma}\Theta_{,23} \\ \tau_{31,22} + \tau_{22,31} - \tau_{23,12} - \tau_{12,23} &= \frac{\sigma}{1+\sigma}\Theta_{,31} \\ \tau_{12,33} + \tau_{33,12} - \tau_{31,23} - \tau_{23,31} &= \frac{\sigma}{1+\sigma}\Theta_{,12} \end{aligned}\right\} \text{(B)}$$

We also have the equations of equilibrium $\tau_{ij,j} = -F_i$

$$\text{ie} \quad \left.\begin{aligned} \tau_{11,1} + \tau_{12,2} + \tau_{13,3} &= -F_1 \\ \tau_{21,1} + \tau_{22,2} + \tau_{23,3} &= -F_2 \\ \tau_{31,1} + \tau_{32,2} + \tau_{33,3} &= -F_3 \end{aligned}\right\} \quad \text{(C)}$$

From (C) $-(F_{2,3} + F_{3,2}) = \tau_{21,13} + \tau_{22,23} + \tau_{23,33} + \tau_{31,12} + \tau_{32,22} + \tau_{33,32}$

$= \tau_{12,31} + \tau_{22,23} + \tau_{23,33} + \tau_{31,12} + \tau_{23,22} + \tau_{33,23}$

$= \nabla^2\tau_{23} + \Theta_{,23} - \tau_{23,11} - \tau_{11,23} + \tau_{12,31} + \tau_{31,12}$

$= \nabla^2\tau_{23} + \Theta_{,23} - \frac{\sigma}{1+\sigma}\Theta_{,23}$ using the first eqn of (B)

$= \nabla^2\tau_{23} + \frac{1}{1+\sigma}\Theta_{,23}$

ie $\nabla^2\tau_{yz} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial y\partial z} = -\left(\frac{\partial F_y}{\partial z} + \frac{\partial F_z}{\partial y}\right)$ the fourth of the B-M-eqns //

Similarly the 5th & 6th are obtained by cyclic symmetry.

Again from (C) we have $-\text{div } F = -(F_{1,1} + F_{2,2} + F_{3,3})$

$= \tau_{11,11} + \tau_{22,22} + \tau_{33,33} + 2\tau_{23,23} + 2\tau_{31,31} + 2\tau_{12,12}$ -- (D)

Adding the three equations in (A), we have

$$\tau_{11,22} + \tau_{11,33} + \tau_{22,33} + \tau_{22,11} + \tau_{33,11} + \tau_{33,22} - 2\tau_{23,23} - 2\tau_{31,31} - 2\tau_{12,12} = \frac{2\sigma}{1+\sigma}\nabla^2\Theta$$

or $\nabla^2\Theta - (\tau_{11,11} + \tau_{22,22} + \tau_{33,33} + 2\tau_{23,23} + 2\tau_{31,31} + 2\tau_{12,12}) = \frac{2\sigma}{1+\sigma}\nabla^2\Theta$

which leads to $\nabla^2\Theta = -\frac{1+\sigma}{1-\sigma}\text{div } F$ -- (E)

since j cannot be negative. For each j, there are $2j+1$ states. Hence the order of the 2nd repⁿ is

no. of rows or columns $= \sum_{j_1-j_2}^{j_1+j_2}(2j+1)$, no. of terms $= (j_1+j_2)-(j_1-j_2)+1 = 2j_2+1$

$$\therefore \sum_{j_1-j_2}^{j_1+j_2} 2j + (2j_2+1) = \{2(j_1-j_2)+2(j_1-j_2+1)+\cdots \text{ to } 2j_2+1 \text{ terms}\} + (2j_2+1)$$

$$= 2(j_1-j_2)(2j_2+1) + 2(1+2+\cdots \text{ to } 2j_2 \text{ term}) + (2j_2+1)$$

$$= 2(j_1-j_2)(2j_2+1) + 2\cdot\tfrac{1}{2}\cdot 2j_2(2j_2+1) + (2j_2+1)$$

$$= (2j_2+1)\{2(j_1-j_2)+2j_2+1\} = (2j_2+1)(2j_1+1).$$ first as in C(i).

For the particular case of the combination of orbital A.M., $j_1=1$ and spin A.M., $j_2=\frac{1}{2}$, C(ii) is equivalent to expressing $\Phi_{3/2,3/2}, \Phi_{3/2,\frac{1}{2}}, \Phi_{3/2,-\frac{1}{2}}, \Phi_{3/2,-3/2}, \Phi_{\frac{1}{2},\frac{1}{2}}, \Phi_{\frac{1}{2},-\frac{1}{2}}$ * as linear combinations of the six products of $\psi_{11}, \psi_{10}, \psi_{1,-1}, \psi_{1/2}, \psi_{-1/2}$ two each. — Use in theory of atomic spectra [* $j_1+j_2=\frac{3}{2}$, $j_1+j_2-1=\frac{1}{2}=j_1-j_2$ Hence $\Phi_{3/2}$ & $\Phi_{1/2}$ with comp. m's]

65th Lecture — 9/2/65 — Remaining portion of previous lecture introducing the ~~notation~~ table from Feenberg & Pake, p. 23 viz

m_1	m_2	$m = m_1+m_2$	j
j_1	j_2	j_1+j_2	j_1+j_2
j_1	j_2-1	j_1+j_2-1	j_1+j_2-1
j_1-1	j_2	"	"
j_1	j_2-2	j_1+j_2-2	j_1+j_2-2
j_1-1	j_2-1	"	"
j_1-2	j_2	"	"
j_1	$-j_2$	j_1-j_2	j_1-j_2
j_1-1	$-j_2+1$	"	"
j_1-2	$-j_2+2$	"	"
⋮	⋮	"	"
⋮	⋮	"	"
j_1-2j_2	j_2	"	"

(in the several blocks the ~~same~~ j_2 has values $j_2, j_2-1, j_2-2, \ldots, -j_2$)

which is also in accord with the remark on p. 58 of Condon & Shortley viz "The least value of j_1+j_2-n is obtained when $j_\delta - n = -j_\delta$ where j_δ is the smaller of j_1 and j_2, and this gives $n = 2j_\delta$, and the least value of $j = j_1-j_2$. — $\sum_{j_1-j_2}^{j_1+j_2}(2j+1) = (2j_1+1)(2j_2+1)$ — System of identical particles, symmetry & antisymmetry; formation of these from unsymmetrised functions, exchange degeneracy when ~~H~~ H does not involve the spin; for $n=2$, $u(1,2)$ & $u(2,1)$ can be formed from the two functions $u(1,2) \pm u(2,1)$, but $n=3$, from the two sym. & antisym fns we

$u(1,2,3) + u(2,3,1) + u(3,1,2) \pm u(2,1,3) + u(3,2,1) + u(1,3,2)$

cannot be formed all the six $u(1,2,3)$, etc, but only two linear combinations, ~~the~~ remaining ~~four~~ linear combinations are not of any physical significance —

Pauli's Exclusion Principle [$H_0 = H_0'(1) + H_0'(2) + \cdots + H_0'(n)$, $H_0 = H_0(1,2\ldots n)$

$u(1,2,\ldots n) = v_1(1)\,v_2(2)\cdots v_n(n)$, $E = E_1+E_2+\cdots+E_n$

$H_0'(1)\,v_i(1) = E_i\,v_i(1)$, etc.

$$u_A = \begin{vmatrix} v_1(1), & v_2(1), & \cdots & v_n(1) \\ v_1(2), & v_2(2), & \cdots & v_n(2) \\ v_1(n), & v_2(n), & \cdots & v_n(n) \end{vmatrix}$$

66th Lecture — 12/2/65 — Pauli's Exclusion Principle — Connection with statistical mechanics, Bose-Einstein and Fermi-Dirac statistics (Bosons & fermions). (π-mesons, photons) [electrons, etc. ~~p~~ neutrinos]

Statistics — Connection between spin & statistics ~~[Considering only the spin A.M. of a number of~~ ~~identical particles, of a number as aggregate of fundamental particles~~ [Total spin of a system of identical particles can be an integer or half-odd integer, and experimental evidence shows that wave fns for these cases must be symmetric or antisymmetric] — Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$.

$\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&-i\\i&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ & spin eigen functions $v(\frac{1}{2}) = \begin{pmatrix}1\\0\end{pmatrix}$, $v(-\frac{1}{2}) = \begin{pmatrix}0\\1\end{pmatrix}$.

(V) $K = \lambda + \frac{2}{3}\mu = \frac{\lambda(1+\sigma)}{3\sigma} = \frac{2\mu(1+\sigma)}{3(1-2\sigma)} = \frac{\mu E}{3(3\mu - E)} = \frac{E}{3(1-2\sigma)}$

$K = \lambda + \frac{2}{3}\mu$ and $K = \frac{E}{3(1-2\sigma)}$ are standard relations

Using exprs for σ and E in terms of λ and μ,

$$\frac{\lambda(1+\sigma)}{3\sigma} = \frac{\lambda\left\{1 + \lambda/2(\lambda+\mu)\right\}}{3\lambda/2(\lambda+\mu)} = \frac{\lambda(3\lambda+2\mu)}{3\lambda} = \lambda + \frac{2}{3}\mu = K$$

$$\frac{2\mu(1+\sigma)}{3(1-2\sigma)} = \frac{2\mu\left\{1 + \lambda/2(\lambda+\mu)\right\}}{3\left\{1 - 2\lambda/2(\lambda+\mu)\right\}} = \frac{2\mu(3\lambda+2\mu)}{3\cdot 2\mu} = \lambda + \frac{2}{3}\mu = K.$$

$$\frac{\mu E}{3(3\mu - E)} = \frac{\mu^2(3\lambda+2\mu)/(\lambda+\mu)}{3\left\{3\mu - \mu(3\lambda+2\mu)/(\lambda+\mu)\right\}} = \frac{\mu^2(3\lambda+2\mu)}{3\cdot 3\mu^2} = \lambda + \frac{2}{3}\mu = K.$$

— Eqm eqns for isotropic elastic solid are (1) & (2) subject to (3)

(1) $\tau_{ij,j} = -F_i$

(2) $\tau_{ij} = \lambda\delta_{ij}\vartheta + 2\mu e_{ij}$

(3) (4) $\tau_{ij}\nu_j = \bar{T}_i$ — Boundary conds

(4) $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

(5) $e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0$ — compatibility eqns

(1) & (2) constitute 9 eqns for the nine quantities τ_{ij}, u_i to be found subject to (3) ie given F_i & $\bar{T}_i$ — Statement of the two problems — Problem 1: To find τ_{ij}, u_i in interior given F_i and $\bar{T}_i$ — Problem 2: To find τ_{ij}, u_i in interior given F_i and u_i on surface.

Problem 2: leads to Navier's eqn $\mu\nabla^2 u_i + (\lambda+\mu)\frac{\partial\vartheta}{\partial x_i} + F_i = 0$

to be solved subject to boundary conds $u_i = f_i(x_1, x_2, x_3)$

$u_i \to e_{ij} \to \tau_{ij}$ — Here no need to use compatibility conds since we go to e_{ij} from u_i

Problem 1: Here from (3) τ_{ij} is given on boundary from $\bar{T}_i$, and we have to find τ_{ij} in interior & also u_i. To find latter we have to find e_{ij} from the relation reciprocal to (2) ie

$$e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\tau_{kk} \quad \ldots (2')$$

& then find u_i from (4) & hence the compatibility conds are to be used.

What is done is to express the compatibility conds (5) in terms of τ_{ij} using (2') and also F_i using (1) also, and then take the resulting compatibility eqns and (1) as the reqd eqns for equilibrium. The resulting compatibility eqns are called the Beltrami-Michell compatibility equations.

Derivation of the Beltrami-Michell compatibility conditions: Using (2') these conditions in e_{ij}

– Relativistic wave eqns of – Klein Gordan eqn / $* \left(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - M\right)\psi(x_\nu) = 0$. $\left[\frac{1}{M} = \frac{\hbar c}{mc^2}\frac{\hbar}{mc} = \text{compton wave length}\right]$

67th Lecture – 15/2/64 (Special) – Klein-Gordan eqn* and its inadequacy, because the wave eqn was only one component, while for electron of spin, we have two component wave eqns $v_1 = \begin{pmatrix}1\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\end{pmatrix}$ & with spin matrices $\left[\sigma_x\sigma_y = -\sigma_y\sigma_y = i\sigma_z \quad \sigma_x^2 = \cdots = 1\right]$ $\sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ [ie M_x, M_y, M_z divided by $\frac{1}{2}\hbar$] – Dirac eqn $\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + M\right)\psi(x_\nu) = 0$ with $\psi(x_\nu)$ a 1-column matrix with four elements & γ_μ being 4×4 matrices (Spinors) – Derivation of Klein-Gordon eqn from the Dirac by operating with $\gamma_\lambda \frac{\partial}{\partial x_\lambda} - M$ requires $\gamma_\lambda\gamma_\mu + \gamma_\mu\gamma_\lambda = 2\delta_{\lambda\mu}$, (since rel^n $E^2 = c^2p^2 + m^2c^4$ should hold) as can be shown by taking the Klein Gordan eqn as $\left(\delta_{\lambda\mu}\frac{\partial^2}{\partial x_\lambda \partial x_\mu} - M^2\right)\psi = 0$ – 16 basic quantities in matform & using $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ ($\gamma_5^2 = 1$, $\gamma_\lambda\gamma_5 + \gamma_5\gamma_\lambda = 0$), 1, γ_λ, $i\gamma_\lambda\gamma_\mu$, $i\gamma_5\gamma_\lambda$, γ_5 (in latter four square of each basic quantity $= 1$) – Requirement of hermiticity of the γ_λ – Defn by aid of Pauli matrix: $\gamma_K = \begin{pmatrix}0 & -i\sigma_K\\ i\sigma_K & 0\end{pmatrix}$ $(K = 1,2,3)$, $\gamma_4 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ & explicit expressions – Verification of $\gamma_5^2 = \begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\gamma_\lambda\gamma_\mu + \gamma_\mu\gamma_\lambda = \frac{1}{2}\delta_{\lambda\mu}$ by using above expressions – Schiff notation $\alpha_K = i\gamma_4\gamma_K$, $\beta = \gamma_4$ & explicit expn for them – $\hat{\sigma}_K = \begin{pmatrix}\sigma_K & 0\\ 0 & \sigma_K\end{pmatrix} = i\gamma_4\gamma_5\gamma_K = -i\gamma_{lm}$ (in cyclic order)

68th Lecture – 16/2/65 – (1) $\hat{\sigma}_K$ – (2) particle & current densities, (3) spin of electron [Putting Dirac eqn as $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ with $H = ic\gamma_4\gamma_K p_K + mc^2\psi\gamma_4$

69th Lecture –

69th Lecture – 17/2/65 – Writing $\vec{J} = \vec{M} + \frac{1}{2}\hbar\vec{\sigma}$ or $J_K = M_K + \frac{1}{2}\hbar\hat{\sigma}_K$, where $\vec{M} = (\vec{r}\times\vec{p})$ is the orbital angular momentum and showing that $[M_K, H] \neq 0$ for eg $[M_{K0}, H] = -\hbar c(\vec{r}\times\vec{p})$, and $[\vec{\sigma}_K, H] \neq 0$ by showing that $[\vec{\sigma}, H] = 2c\gamma_4(\vec{r}\times\vec{p})$ & hence $[J_K, H] = 0$ ie J_K is a constant of the motion ie total A.M = orbital A.M + Spin A.M & latter $= \frac{1}{2}\hbar$ in magnitude ie electron has spin $\frac{1}{2}$ – Magnetic moment of the electron. – Motion of magnetic moment due to orbital motion of electron & electron interacting with a magnetic field, the normal Zeeman effect & associated with $\vec{M}$ (orbital A.M) magnetic moment taken as $-\frac{e}{2mc}\vec{M}$ explains Zeeman effect – Anomalous Zeeman effect necessitates notion of spin vector of A.M $\vec{S}$ and associated with it a magnetic moment $-\frac{e}{mc}\vec{S}$ (note difference in not having factor $\frac{1}{2}$, as in case of $\vec{M}$ – Derivation of magnetic moment of electron from Dirac eqn //

70th Lecture – 18/2/65 – up to derivation of $\left\{\gamma_\mu\left(\frac{\partial}{\partial x_\mu} + \frac{ie}{\hbar c}A_\mu\right) + M\right\}\psi = 0$ – Showing magnetic moment $= -\frac{1}{2}\hbar\vec{\sigma}\cdot\frac{e}{mc}$ –

71st Lecture – 20/2/65 – Proof of existence of magnetic moment for the electron.

72nd Lecture – 22/2/65 – Rel. invariance of Dirac's equation – To prove that $|a_{\mu\nu}|^2 = 1$, it is perhaps simplest to prove as follows:- $|a_{\mu\nu}|^2 = |a_{\mu\nu}||a_{\mu\nu}| = |a_{\mu\nu}\cdot a_{\mu\nu}| = 1$ [using $|A||B| = |A\times B|$ and $a_{\mu\alpha}a_{\mu\beta} = \delta_{\alpha\beta}$] instead of the complicated method of Mandl used last year – That $a_{44} > 0$ excludes time-reversal is seen by taking the transformations with $x_K' = x_K$ and the inverse transformation $x_K = x_K'$ giving $x_4' = c_{44}x_4$ & $x_4 = c_{44}x'$ ie $t = t'$ ie $c_{44} = 1$, $t = t'$ ($c_{44} > 0$) ie $c_{44}^2 = 1$ & $c_{44} > 0$ gives $t = t'$ or time inversion or non-inversion only corresponds to $(a_{\mu\nu}) = \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&\mp 1\end{pmatrix}$ – Similarly space inversion without time inversion ie $x_K = -x_K'$ or $x_K' = -x_K$ corresponds to $a_{\mu\nu} = \begin{pmatrix}-1&&&\\&-1&&\\&&-1&\\&&&+1\end{pmatrix}$

22nd Lecture — 12/1/63.

Ex. 4. p. 71 remaining parts:-

(ii) $\mu = \frac{\lambda(1-2\sigma)}{2\sigma} = \frac{3}{2}(K-\lambda) = \frac{E}{2(1+\sigma)} = \frac{3K(1-2\sigma)}{2(1+\sigma)} = \frac{3KE}{9K-E}$

$\mu = \frac{E}{2(1+\sigma)}$ is the standard relation – using $\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}$, $\frac{\lambda(1-2\sigma)}{2\sigma} = \frac{E}{2(1+\sigma)} = \mu$

$\frac{3}{2}(K-\lambda) = \frac{3}{2}\left(\lambda + \frac{2}{3}\mu - \lambda\right) = \mu$

$\frac{3k(1-2\sigma)}{2(1+\sigma)} = \frac{E}{2(1+\sigma)} = \mu$, using $k = \frac{E}{3(1-2\sigma)}$

$\frac{3KE}{9K-E} = \frac{E^2/(1-2\sigma)}{3E/(1-2\sigma) - E} = \frac{E}{3-(1-2\sigma)} = \frac{E}{2(1+\sigma)} = \mu.$

(iii) $\sigma = \frac{\lambda}{2(\lambda+\mu)} = \frac{\lambda}{3K-\lambda} = \frac{E}{2\mu} - 1 = \frac{3K-2\mu}{2(3K+\mu)} = \frac{3K-E}{6K}$

$\sigma = \frac{\lambda}{2(\lambda+\mu)}$ is the standard relation – Also $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is standard

$\frac{\lambda}{3k-\lambda} = \frac{\lambda}{(3\lambda+2\mu)-\lambda} = \frac{\lambda}{2(\lambda+\mu)} = \sigma$

$\frac{E}{2\mu} - 1 = \frac{3\lambda+2\mu}{2(\lambda+\mu)} - 1 = \frac{\lambda}{2(\lambda+\mu)} = \sigma$

$\frac{3K-2\mu}{2(3K+\mu)} = \frac{(3\lambda+2\mu)-2\mu}{2(3\lambda+2\mu+\mu)} = \frac{3\lambda}{6(\lambda+\mu)} = \frac{\lambda}{2(\lambda+\mu)} = \sigma$

$\frac{3K-E}{6K} = \frac{(3\lambda+2\mu) - \mu(3\lambda+2\mu)/(\lambda+\mu)}{2(3\lambda+2\mu)} = \frac{(\lambda+\mu)-\mu}{2(\lambda+\mu)} = \frac{\lambda}{2(\lambda+\mu)} = \sigma$

(iv) $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{\lambda(1+\sigma)(1-2\sigma)}{\sigma} = \frac{9K(K-\lambda)}{3K-\lambda} = 2\mu(1+\sigma) = \frac{9K\mu}{3K+\mu} = 3K(1-2\sigma)$

$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is a standard relation

Also $\mu = \frac{E}{2(1+\sigma)}$ a standard relation gives $E = 2\mu(1+\sigma)$

& $K = \frac{E}{3(1-2\sigma)}$ " " $E = 3K(1-2\sigma)$.

$E = \frac{\lambda(1+\sigma)(1-2\sigma)}{\sigma}$ also follows from a standard reln, $\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}$

$\frac{9K(K-\lambda)}{3K-\lambda} = \frac{3(3\lambda+2\mu)\left(\lambda+\frac{2}{3}\mu-\lambda\right)}{(3\lambda+2\mu)-\lambda} = \frac{2\mu(3\lambda+2\mu)}{2(\lambda+\mu)} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = E.$

$\frac{9K\mu}{3K+\mu} = \frac{3(3\lambda+2\mu)\mu}{3\lambda+2\mu+\mu} = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = E.$

– Proper L·T's forming a group – Proof of a_{k4}, a_{4k}, a_{44} being real & a_{k4}, a_{4k} being imaginary –

73rd Lecture – (1) Proper L·T's forming a group – (2) Correct derivation of $S^{-1}\gamma_\lambda S = a_{\lambda\mu}\gamma_\mu$ (A) (Correcting the mistake in the last step yesterday) – Derivation of $\gamma_4 S^* \gamma_4 = S^{-1}$ (B) – (3) Proof of existence of an S for every L·T. satisfying (A) and (B) – (4) Covariants of the five types constructed out of the ψ, $\bar\psi$ and the γ_μ.

(23/2/65)

(1) [To prove the identity exists for L·T's, consider $x_\mu' = \delta_{\mu\nu} x_\nu$; this an L·T since $\delta_{\mu\alpha}\delta_{\nu\beta} = \delta_{\alpha\beta}$ as can be easily verified. That inverse exists has been proved – Re. product of two L·T's being an L·T. see last years lecture notes. Also proof that c_{kl}, c_{44} are real & c_{4k}, c_{k4} are imaginary follows from $c_{\mu\nu} = b_{\mu\lambda}a_{\lambda\nu}$ & also $|c_{\mu\nu}| = |b_{\mu\lambda}||a_{\lambda\nu}| = 1$]

(2) [(A) $\gamma_\mu \frac{\partial\psi}{\partial x_\mu} + M\psi = 0$, write $\psi = S^{-1}\psi'$ and $\frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x_\lambda'}\cdot\frac{\partial x_\lambda'}{\partial x_\mu} = \frac{\partial}{\partial x_\lambda'} a_{\lambda\mu}$, getting

$\gamma_\mu \frac{\partial}{\partial x_\lambda'} a_{\lambda\mu} S^{-1}\psi' + M S^{-1}\psi' = 0$ ie $a_{\lambda\mu}\gamma_\mu S^{-1}\frac{\partial\psi'}{\partial x_\lambda'} + M S^{-1}\psi' = 0$ & multiplying on the left by S, $a_{\lambda\mu} S\gamma_\mu S^{-1}\frac{\partial\psi'}{\partial x_\lambda'} + M\psi' = 0$. Hence comparing with $\gamma_\lambda \frac{\partial\psi'}{\partial x_\lambda'} + M\psi' = 0$, we get

$a_{\lambda\mu} S\gamma_\mu S^{-1} = \gamma_\lambda$ and again multiplying on the left by S^{-1} and on the right by S, we get $a_{\lambda\mu}\gamma_\mu = S^{-1}\gamma_\lambda S$, which is (A).

[(B), ~~this condition that the continuity eqn $\frac{\partial s_\mu}{\partial x_\mu} = 0$ should remain invariant requires~~

~~$\frac{\partial s_\mu}{\partial x_\mu} \to \frac{\partial s_\mu'}{\partial x_\mu'}$. Putting $\frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x_\lambda'} a_{\lambda\mu}$ and $s_\mu = i\bar\psi\gamma_\mu\psi$ $(\bar\psi = \psi^*\gamma_4)$~~

~~ie i and $\psi = S^{-1}\psi'$, then $\psi^* = \psi'^* S^*$ ie $\bar\psi' = \psi^* S^* \gamma_4 \psi'^* =$~~

Using the condition that s_μ should be a four-vector in both systems requires that s_μ should transform like x_μ ie $s_\mu' = a_{\mu\nu} s_\nu$ (†) & using definition of $s_\mu = i\bar\psi\gamma_\mu\psi$ (where $\bar\psi = \psi^*\gamma_4$), we should have $\bar\psi'\gamma_\mu\psi' = a_{\mu\nu}\bar\psi\gamma_\nu\psi$, and using $\gamma_\mu = a_{\mu\nu} S\gamma_\nu S^{-1}$ this becomes $\bar\psi' a_{\mu\nu}\bar\psi' S\gamma_\nu S^{-1}\psi' = a_{\mu\nu}\bar\psi\gamma_\nu\psi$ or $a_{\mu\nu}\bar\psi' S\gamma_\nu\psi = a_\mu\bar\psi\gamma_\nu\psi$

ie $\bar\psi' S = \bar\psi$ or $\bar\psi' = \bar\psi S^{-1}$.

(†) Alternatively if $\frac{\partial s_\lambda'}{\partial x_\lambda'} = \frac{\partial s_\mu}{\partial x_\mu}$ ie $\frac{\partial s_\lambda'}{\partial x_\lambda'} = \frac{\partial s_\mu}{\partial x_\lambda'} a_{\lambda\mu}$ ie $s_\lambda' = a_{\lambda\mu} s_\mu$ or $s_\mu' = a_{\mu\nu} s_\nu$

Using $\bar\psi' = \psi'^*\gamma_4$ and $\psi'^* = (S\psi)^* = \psi^* S^*$ ie $\bar\psi' = \psi^* S^*\gamma_4 = \bar\psi\gamma_4 S^*\gamma_4$ ($\because \bar\psi = \psi^*\gamma_4 \Rightarrow \psi^* = \bar\psi\gamma_4$ & $\gamma_4^2 = 1$)

hence $\gamma_4 S^*\gamma_4 = S^{-1}$ which is (B)]

(3) [To be supplied from Pauli's article in Handbuch d. Physik] ✗

(4) [(i) $I = \bar\psi\psi$; $\bar\psi'\psi' = \bar\psi S^{-1} S\psi = \bar\psi\psi$ ie I is an invariant

(ii) $s_\mu = i\bar\psi\gamma_\mu\psi$; $s_\mu' = i\bar\psi'\gamma_\mu\psi' = i\bar\psi S^{-1}\gamma_\mu S\psi = i\bar\psi a_{\mu\nu}\gamma_\nu\psi = a_{\mu\nu}\, i\bar\psi\gamma_\nu\psi = a_{\mu\nu} s_\mu$

ie s_μ is a covariant four-vector.

(iii) $M_{\mu\nu} = i\bar\psi\gamma_\mu\gamma_\nu\psi$ $(\mu \neq \nu)$, and $= 0$ $(\mu = \nu)$

$M_{\mu\nu}' = i\bar\psi'\gamma_\mu\gamma_\nu\psi' = i\bar\psi S^{-1}\gamma_\mu\gamma_\nu S\psi = i\bar\psi S^{-1}\gamma_\mu S\cdot S^{-1}\gamma_\nu S\psi = i\bar\psi a_{\mu\rho}\gamma_\rho a_{\nu\sigma}\gamma_\sigma\psi$

$= a_{\mu\rho}a_{\nu\sigma}\cdot i\bar\psi\gamma_\rho\gamma_\sigma\psi = a_{\mu\rho}a_{\nu\sigma}M_{\rho\sigma}$

ie $M_{\mu\nu}$ is a second rank tensor. obviously $M_{\mu\nu} = -M_{\nu\mu}$ if $\mu \neq \nu$

ie $M_{\mu\nu}$ is an antisymmetric tensor of second rank (with components $M_{\mu\mu} = 0$) ie it has six components and called a six-vector ($M_{12}, M_{23}, M_{31}, M_{14}, M_{24}, M_{34}$)

(iv) $K_\mu = i\bar\psi\gamma_5\gamma_\mu\psi$ is a pseudo-vector ie vector like angular-momentum. for such

15th Lecture – 29/9/62 – Hooke's law – Elastic limit and proportional limit – yield point – $T = f(e)$ for elastic & plastic materials – Generalised Hooke's law $\tau_{ij} = c_{ijkl}\, e_{kl}$ with symmetry of c_{ijkl} and i, j and k, l – $\tau_i = c_{ij}\, e_j$ $(i, j = 1, \ldots 6)$ & $e_i = c_{ij}\, \tau_j$ – Reduction from 36 to 21 constants when the strain-energy density fn $W = \frac{1}{2} c_{ij}\, e_i\, e_j$, such that $\tau_i = \frac{\partial W}{\partial e_i}$, exists

16th Lecture – 4/10/62 – Elastic and geometric symmetry – (i) Elastic symmetry w.r.t. a plane, (ii) w.r.t. three mutually ⊥r planes – Elastic symmetry determined from invariance of c_{ij} ie $\tau_i' = c_{ij}\, e_j'$ and $\tau_i = c_{ij}\, e_j$ when transformation of coordinates is made – ~~Matrix for the case of homogeneous isotropic solid~~

17th Lecture – 6/10/62 – Matrix c_{ij} for the case of homogeneous isotropic solid –

18th Lecture (After vacation) – 17/11/62 – Did no work.

19th Lecture – 22/11/62 – Worked out Q.7. given in the terminal examination about maximum shearing stress without using Mohr's diagram & adopting the second method in my last year's notes viz $S^2 = \nu_2^2 \nu_3^2 (\tau_2 - \tau_3)^2 + \nu_3^2 \nu_1^2 (\tau_3 - \tau_1)^2 + \nu_1^2 \nu_2^2 (\tau_1 - \tau_2)^2$ – Also worked out ~~in fact~~ the case $\tau_2 = \tau_3$.

26th Lecture – 6/12/62 – Homogeneous isotropic media – Derivation of $\tau_{ij} = \lambda \delta_{ij} \vartheta + 2\mu e_{ij}$ – $\Theta = (3\lambda + 2\mu)\vartheta$

$$e_{ij} = -\frac{\lambda \delta_{ij}}{2\mu(3\lambda + 2\mu)} \Theta + \frac{1}{2\mu} \tau_{ij}$$

(λ & μ called Lame's consts.) – Principal axes of strain (ie $e_{23} = e_{31} = e_{12} = 0$) coincide with pr. axes of stress – Case of simple tension with $\tau_{11} = T$, all the $\tau_{ij} = 0$, introduction of σ and E – Case of simple shear; $\mu = T/\alpha_{23}$ – From $\mu = E/2(1+\sigma)$ $\mu > 0$ since $E, \sigma > 0$ //.

21st Lecture (After Christmas Vacation) – 10/1/63 – Case of hydrostatic pressure of uniform intensity p over surface of the body ie $\tau_{ij} = -p\delta_{ij}$. Both $\bar{T}_i = \tau_{ij} \nu_j$ and $\tau_{ij,j} = 0$ for surface & interior eqns are satisfied – $\Theta = -3p$, and from $e_{ij} = -\frac{\lambda \delta_{ij}}{2\mu(3\lambda+2\mu)} \Theta + \frac{\tau_{ij}}{2\mu}$ we get

$$e_{11} = e_{22} = e_{33} = -\frac{p}{3\lambda + 2\mu}, \quad e_{23} = e_{31} = e_{12} = 0.$$

satisfying compatibility eqns – $\vartheta = e_{11} + e_{22} + e_{33} = -p/(\lambda + \frac{2}{3}\mu)$ or $\vartheta = -p/R$ or $k = -p/\vartheta$ with $R = \lambda + \frac{2}{3}\mu$ and R = compressive stress / cub. comp^n = modulus of compression > 0. – Derivation of $R = E/3(1-2\sigma)$ & since $R > 0$, $\sigma < 1/2$ & from $\lambda = E\sigma/(1+\sigma)(1-2\sigma)$ it follows that $\lambda > 0$ – Comp^n $e_{ij} = \frac{1+\sigma}{E} \tau_{ij} - \frac{\sigma}{E} \delta_{ij} \Theta$,

p. 71. Ex. 2 – From above eqn $\vartheta = \frac{1+\sigma}{E}\Theta - \frac{3\sigma}{E}\Theta = \frac{1-2\sigma}{E}\Theta = \frac{\Theta}{3R}$ ie $\Theta = 3K\vartheta$

Ex. 4 – Did the first part viz. eqn for λ.

vectors are defined by the transformation (under L·T) by $K_\mu' = |a_{\alpha\beta}|\, a_{\mu\nu} K_\nu$. Here

$$K_\mu' = i\bar\psi' \gamma_5 \gamma_\mu \psi' = i \bar\psi S^{-1} \gamma_5 \gamma_\mu S \psi$$

7/4th Lecture – (3) of last lecture, and the remaining four components of (4) of last lecture.

(24/2/65) (3) Since the properties L·T's form a group, any element can of the group can be considered to be formed out of infinitesimal elements, it is sufficient to prove the existence of S satisfying Ⓐ & Ⓑ for infinitesimal L·T's. Let such a one be specified by $a_{\mu\nu} = \delta_{\mu\nu} + \epsilon_{\mu\nu}$ (squares of ϵ neglected);

then $a_{\mu\alpha} a_{\mu\beta} = \delta_{\alpha\beta}$ requires $(\delta_{\mu\alpha} + \epsilon_{\mu\alpha})(\delta_{\mu\beta} + \epsilon_{\mu\beta}) = \delta_{\mu\alpha}\delta_{\mu\beta} + \delta_{\mu\alpha}\epsilon_{\mu\beta} + \delta_{\mu\beta}\epsilon_{\mu\alpha} = \delta_{\alpha\beta}$

i.e. since $\delta_{\mu\alpha}\delta_{\mu\beta} = \delta_{\alpha\beta}$, we should have $\delta_{\mu\alpha}\epsilon_{\mu\beta} + \delta_{\mu\beta}\epsilon_{\mu\alpha} = 0$ or $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$

This is what characterises an infinitesimal L·T: $x_\mu' = x_\mu + \epsilon_{\mu\nu} x_\nu$ ($\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$).

Let $S = I + \frac{1}{2}\epsilon_{\mu\nu} T^{\mu\nu}$, where $T^{\mu\nu} = -T_{\mu\nu}$; I = unit matrix, and $S^{-1} = I - \frac{1}{2}\epsilon_{\mu\nu} T^{\mu\nu}$

If S should satisfy Ⓐ i.e. $S^{-1}\gamma_\lambda S = a_{\lambda\mu}\gamma_\mu$, we should have

$$(I - \tfrac{1}{2}\epsilon_{\mu\nu} T^{\mu\nu})\gamma_\lambda (I + \tfrac{1}{2}\epsilon_{\mu\nu} T^{\mu\nu}) = a_{\lambda\mu}\gamma_\mu = (\delta_{\lambda\mu} + \epsilon_{\lambda\mu})\gamma_\mu = \gamma_\lambda + \epsilon_{\lambda\mu}\gamma_\mu$$

$$\gamma_\lambda + \tfrac{1}{2}\gamma_\lambda \epsilon_{\mu\nu} T^{\mu\nu} - \tfrac{1}{2}\epsilon_{\mu\nu} T^{\mu\nu}\gamma_\lambda = \gamma_\lambda + \epsilon_{\lambda\mu}\gamma_\mu$$

i.e. $\frac{1}{2}(\gamma_\lambda \epsilon_{\mu\nu} T^{\mu\nu} - \epsilon_{\mu\nu} T^{\mu\nu}\gamma_\lambda) = \epsilon_{\lambda\mu}\gamma_\mu = \frac{1}{2}\epsilon_{\mu\nu}(\delta_{\mu\lambda}\gamma_\nu - \delta_{\nu\lambda}\gamma_\mu)$, as can be easily verified

i.e. this is satisfied for any $\epsilon_{\mu\nu}$ if

$$\gamma_\lambda T^{\mu\nu} - T^{\mu\nu}\gamma_\lambda = \delta_{\mu\lambda}\gamma_\nu - \delta_{\nu\lambda}\gamma_\mu \quad \text{---(1)}$$

This eqn enables determination of the $T^{\mu\nu}$ in terms of the γ-matrices & hence determination of S, thus proving the existence of S satisfying Ⓐ. A solution of (1) of the $T^{\mu\nu}$ skew-symmetric in μ,ν is given ($\mu \neq \nu$)

by $T^{\mu\nu} = \frac{1}{2}\gamma_\mu\gamma_\nu$ ------ (2)

as can be shown by considering cases (i) $\lambda \neq \mu$ or ν, (ii) $\lambda = \mu \neq \nu$; (iii) $\lambda = \nu \neq \mu$.

We have next to show that (1) satisfies Ⓑ also, and it is enough to show that (2) satisfies Ⓑ

Ⓑ is $\gamma_4 S^* \gamma_4 = S^{-1}$.

$S^* = 1 + \frac{1}{2} T^{*\mu\nu}\epsilon^*_{\mu\nu}$; $S^{-1} = 1 - \frac{1}{2}\epsilon_{\mu\nu} T_{\mu\nu}$

and (B) becomes $\gamma_4 (1 + \frac{1}{2} T^{*\mu\nu}\epsilon^*_{\mu\nu})\gamma_4 = 1 - \frac{1}{2}\epsilon_{\mu\nu} T_{\mu\nu}$

i.e. using $\gamma_4^2 = 1$, $\gamma_4 T^*_{\mu\nu}\epsilon^*_{\mu\nu}\gamma_4 = -\epsilon_{\mu\nu} T_{\mu\nu}$

and $T^*_{\mu\nu} = \frac{1}{2}\gamma_\nu^*\gamma_\mu^* = \frac{1}{2}\gamma_\nu\gamma_\mu = -\frac{1}{2}\gamma_\mu\gamma_\nu$. Hence Ⓑ becomes

$$\gamma_4\gamma_\mu\gamma_\nu\epsilon^*_{\mu\nu}\gamma_4 = -\epsilon_{\mu\nu}\gamma_\mu\gamma_\nu \text{ or } \gamma_\mu\gamma_\nu\epsilon^*_{\mu\nu} = -\epsilon_{\mu\nu}\gamma_4\gamma_\mu\gamma_\nu\gamma_4$$

or $\epsilon^*_{\mu\nu}\gamma_\mu\gamma_\nu = +\epsilon_{\mu\nu}\gamma_4\gamma_\mu\gamma_\nu\gamma_4$ — (B)

$\epsilon_{\mu\nu}$ is not treated as a matrix but as a coefficient $\therefore \epsilon^*_{\mu\nu} = \epsilon^c_{\mu\nu}$ ⊕ → or $\epsilon^c_{\mu\nu}\gamma_\mu\gamma_\nu = +\epsilon_{\mu\nu}\gamma_4\gamma_\mu\gamma_\nu\gamma_4$ — (B′) (where ϵ^c is complex conjugate of ϵ)

From the nature of the definition ϵ_{K4}, ϵ_{4K} are imaginary & ϵ_{44} & ϵ_{KL} are real, and we can verify (B′) for these

several cases (i) $\epsilon^c_{K4}\gamma_K\gamma_4 = +\epsilon_{K4}\gamma_4\gamma_K\gamma_4\gamma_4 = -\epsilon_{K4}\gamma_K\gamma_4$

i.e. L.H.S $= -\epsilon^*_{K4}\gamma_K\gamma_4 = \epsilon_{K4}\gamma_K\gamma_4 = \epsilon_{K4}\gamma_4\gamma_K = $ R.H.S

(ii) is similarly proved

⊕ I thought on the point in the class today.

Michell - Kolosov relation in the general case, and its modification for this particular transformation (as on p. 6 of last year's notes) — Summary of results of stress under six headings — Introduction to strain; Rigid & elastic bodies — affine transformations & their properties; infinitesimal affine transformations, and their superposition.

9th Lecture — 25/8/62 — General displacement of vector: $\delta A_i = \alpha_{ij} A_j$ including both rigid body displacements and elastic deformations — Division into symmetric & antisymmetric part of α_{ij} (e_{ij} & ω_{ij}) with e_{ij} ($\overset{sym}{e_{ij}}$) corresponding to elastic deformations — Geometrical interpretation of the components e_{ii} and e_{ij} of the strain tensor ($\delta A_i = e_{ij} A_i$) with e_{11} and e_{23} as particular cases.

10th Lecture — 30/8/62 — Strain quadric — Change of axes & tensor character of e_{ij} — $\delta\vec{A}$ along the normal at P to the strain quadric — Principal axes of strain given by $\| e_{ij} - e\delta_{ij} \| = 0$; roots of this being real — Invariants $\vartheta_1, \vartheta_2, \vartheta_3$ in terms of e_1, e_2, e_3 and e_{ij} — Interpretation of $\vartheta_1 = \vartheta = e_1 + e_2 + e_3$ as the cubical dilatation.

11th Lecture — 1/9/62 — Case of general infinitesimal deformation viz $\delta A_i = u_{i,j} A_j$ ($u_{i,j} = \partial u_i / \partial x_j$) (ie unlike the affine case $u_{i,j}$ varies from point to point, while the α_{ij} is constant ~~throughout~~ throughout; but in the neighbourhood of a point, the trn is affine) — Introduction ω_{ij} & e_{ij} by $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, $\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i})$ — Dilation $\vartheta = u_{i,i}$ varies from point to point — Examples of strain: Four cases: (a) Uniform dilatation, (b) Simple extension in axes ($\zeta_1, \zeta_2, \zeta_3$) — (c) Shearing strain equivalent to equal extension & contraction of two orthogonal linear elements; (d) Plane strain — Strain quadrics in the four cases.

12th Lecture — 6/9/62 — Examples on p. 24 of Sokolnikoff — Deductions of St. Venant's compatibility conditions as necessary conditions following the method of Ex 6, p. 33 of Sokolnikoff and showing partly that the 81 conditions really reduce to only 6.

13th Lecture — 15/9/62 — Revision of the reduction of the 81 conditions to only 6:

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \quad \text{--- (A)}$$

for the case $ij = 11$, reduces to an identity except when $kl = 23$, & similarly for $ij = 22$, $kl = 31$, $ij = 33$, $kl = 12$

Again for $ij = 23$, " $kl = 23$ & $ij = 31$, $kl = 31$, $ij = 12$, $kl = 12$.

thus giving only 6 conditions written out in the usual form.

14th Lecture — 20/9/62 — Suffic (A) also proved to be sufficient conditions as in Sokolnikoff, p. 27, only with this difference* that U_{jl} was taken $= e_{jl} - x_k (e_{lj,K} - e_{kl,j})$ without introducing the x_k' which appears unnecessary — Mentioned that the ~~displacements~~ displacements given by $u_j = u_j^0 + (x_K - x_K^0)\omega_{jk}^0 + \int_{P_0}^{P} U_{jl}\, dx_l$ can be shown to satisfy the differential equations $u_{i,j} + u_{j,i} = 2e_{ij}$ — Postponed this to the next lecture.

* Also in the integration by parts $(x_k \omega_{jk} - x_k^0 \omega_{jk}^0)$ terms were taken $= (x_k - x_k^0)\,\omega_{jk}^0$ since $x_k - x_k^0$ is an infinitesimal displacement.

(iii) E_{44} real & is equal to zero, hence both L.H.S & R.H.S of (B') = 0

(iv) E_{kl} real (we need not consider case of $k = l$, for then $\epsilon_{kl} = 0$ & just as in (iii) (B') holds.

For $k \neq l$, (B') becomes $\epsilon^c_{kl}\, \gamma_k \gamma_l = + \epsilon_{kl}\, \gamma_4 \gamma_k \gamma_l \gamma_4$

L.H.S $= \epsilon_{kl}\, \gamma_k \gamma_l = + \epsilon_{kl}\, \gamma_k \gamma_l$; R.H.S $= + \epsilon_{kl}\, \gamma_4^2 \gamma_k \gamma_l = + \epsilon_{kl}\, \gamma_k \gamma_l$ & hence (B') holds.

(4) (iv) $K_\mu = i \bar{\psi} \gamma_5 \gamma_\mu \psi$ – Here $\gamma_5 \gamma_\mu$ has only 4 values, for eg $\gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_\mu$ ($\mu = 1, 2, 3, 4$) ie $-\gamma_2 \gamma_3 \gamma_4$, $-\gamma_3 \gamma_1 \gamma_4$, $-\gamma_1 \gamma_2 \gamma_4$ and $\gamma_1 \gamma_2 \gamma_3$ & K_μ is called a pseudo-vector or a vector like A.M for which the law of transformation under L.T's is given by $K_\mu' = |a_{\alpha\beta}|\, a_{\mu\nu} K_\mu$ ie $|\ |$ can be ± 1

$$K_\mu' = i \bar{\psi}' \gamma_5 \gamma_\mu \psi' = i \bar{\psi} S^{-1} \gamma_5 \gamma_\mu S \bar{\psi}$$

$$= i \bar{\psi} (S^{-1}\gamma_1 S)(S^{-1}\gamma_2 S)(S^{-1}\gamma_3 S)(S^{-1}\gamma_4 S) S^{-1} \gamma_\mu S \psi$$

$$= a_{1\lambda} a_{2\mu} a_{3\nu} a_{4\sigma}\, i \bar{\psi} \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\sigma S^{-1} \gamma_\mu S \psi$$

$$= a_{1\lambda} a_{2\mu} a_{3\nu} a_{4\sigma}\, \epsilon_{\lambda\mu\nu\sigma} \cdot i \bar{\psi} \gamma_1 \gamma_2 \gamma_3 \gamma_4 S^{-1} \gamma_\mu S \psi$$

$$= |a_{\mu\nu}| \cdot i \bar{\psi} \gamma_5 S^{-1} \gamma_\mu S \psi$$

$$= |a_{\mu\nu}|\, a_{\mu\nu} \cdot i \bar{\psi} \gamma_5 \gamma_\nu \psi$$

$$= |a_{\mu\nu}| \cdot a_{\mu\nu} K_\nu .$$

(v) $N = i \bar{\psi} \gamma_5 \psi$ – Pseudo-scalar ie $N' = |a_{\mu\nu}|\, N$. This is ~~easily~~ proved as in Mandl, p. 221

Lecture 1 – 19/7/62 – Defns of elastic body, stress and strain – To deal with stress first – Body forces and surface forces at any point of the body – Notations (1, 2, 3) & (x, y, z), and mixture of both – Body force: $R_i = \int_\tau F_i \, d\tau$ – Surface force: $\vec{T}_i$ (not along $\vec{\nu}$ as in fluid mechanics); resolved parts along $\vec{\nu}$ and $\perp^r$ to $\vec{\nu}$ ie tension ~~and~~ or compression, and shear – Used of Π^d picture to obtain $T_i^j = \tau_{ji}$ (stress tensor); tension or compression – Resolution of body forces and surfaces using Π^d picture to obtain $\vec{T}_i = \tau_{ji} \nu_j$ – Derivation of equilibrium for the whole body, $\tau_{ji,j} = -F_i$.

Lecture 2 – 21/7/62 – Putting sum of moments equal to zero leading to $\tau_{ji} = \tau_{ij}$ – Reciprocity theorem $\vec{T} \cdot \vec{\nu}' = \vec{T}' \cdot \vec{\nu}$ – Proof tensor character of τ_{ij} – Invariance of τ_{ii} – Exx. 1, 2, 3 on p. 44 solved (Solns on p. 1 of 1961–62 Notes).

Lecture 3 – 28/7/62 – Ex. 4. p. 44 solved (Soln in above) – Stress quadric of Cauchy – Tensor rule for τ_{ij} from invariance of $\tau_{ij} x_i x_j$ for change of axes – Normal at point of quadric $\perp^r$ to T and hence construction of T given N – Principal axes and directions of stress; reality of roots and orthogonality of directions – Invariants $\Theta_1, \Theta_2, \Theta_3$ in terms of τ_1, τ_2, τ_3 only.

Lecture 4 – 2/8/62 – $\Theta_1, \Theta_2, \Theta_3$ in terms of τ_{ij} by expanding $|\tau_{ij} - \delta_{ij}\tau| = 0$, and comparing coefficients of τ^2, τ, and constant term in $-\tau^3 + \Theta_1\tau^2 - \Theta_2\tau + \Theta_3 = 0$ – Transformation to principal axes using $\tau_{ij} = l_{i\alpha} l_{j\alpha} \tau_\alpha$ – Several cases of stress-quadric (i) Ellipsoid (using $N = K^2/A^2$) gives tensile stress with $+K^2$, and compressive stress with $-K^2$, (ii) $\tau_1 > 0, \tau_2 > 0, \tau_3 < 0$, $+K^2$ gives hyperboloid of one sheet, and $-K^2$ of two sheets, and if normal to the element meets the first stress is tensile, and if the second stress is compressive; (iii) $\tau_1 < 0, \tau_2 < 0, \tau_3 > 0$, same as (ii) but only different regions, (iv) Sphere – Stress purely normal whatever the orientation of surface element – Theorem on maximum shearing stress.

5th Lecture – 4/8/62 – Discussion of the Mohr Circle diagram.

6th Lecture (9/8/62) – Problem on p. 53 worked out (Soln. on p. 5 of last year's Lecture Notes) – Discussion of the maximum normal and shear stresses without the use of the Mohr diagram (as in reverse of p. 3 of last year's Lecture notes)

7th Lecture – 16/8/62 – Special examples of stress: (a) Purely normal stress, (b) Simple tension or compression, (c) Shearing stress

8th Lecture – 18/8/62 – The other special case (d) of plane stress – Finding directions of principal axes, maximum of τ'_{12}, and to find $\tau_{11}, \tau_{12}, \tau_{22}$ for any set of axes given τ_1, τ_2 and α – The

Lecture Notes on Elasticity during 1962–63

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(Following Sokolnikoff).

Syllabus

(1) Analysis of Stress and Strain.

(2) Hooke's Law.

(3) Strain-energy function.

(4) Equations of elasticity and uniqueness of solution.

(5) Problems of plane stress and strain.

(6) Extension, torsion and flexure of homogeneous beams.

(7) St. Venant's Problem

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