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ON THE LIMITS FOR THE ROOTS OF A POLYNOMIAL EQUATION

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ON THE LIMITS FOR THE ROOTS OF A POLYNOMIAL EQUATION

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§ 1. *Introduction*

IT has recently been shown by N. Nicolau[1] that if the polynomial equation

$$f(x) = x^m + px^{m-1} + qx^{m-2} + \cdots = 0 \tag{1}$$

has all its roots real and distinct, then they lie between the numbers

$$l_1, l_2 = \frac{1}{m}\left[-p \mp \{(m-1)^2p^2 - 2m(m-1)q\}^{\frac{1}{2}}\right]. \tag{2}$$

He has proved this result elsewhere for $m = 3$ and generalises it in this paper[2] by the method of induction. It might be mentioned here that this result is really due to Laguerre,[3] whose proof is based on a consideration of the Hessian of a binary quantic. We give here a simple proof of the theorem by a direct method and employ the same to obtain closer limits for the roots in terms of the first four coefficients of the Polynomial.

§ 2. *Proof of Laguerre's Theorem*

Let $\alpha, \beta_1, \beta_2, \cdots, \beta_{m-1}$ be the roots of (1). Then

$$\begin{aligned}\Sigma(\beta_j - \beta_j)^2 &= (m-2)\Sigma\beta_i^2 - 2\Sigma\beta_i\beta_j\\ &= (m-1)\Sigma\beta_i^2 - (\Sigma\beta_i)^2\\ &= (m-1)(S_2 - \alpha^2) - (S_1 - \alpha)^2\\ &> 0.\end{aligned}$$

i.e., $\psi(\alpha) = m\alpha^2 - 2\alpha S_1 - (m-1)S_2 + S_1^2 < 0.$

$\therefore$ α should lie between the roots of the quadratic $\psi(x) = 0$, which gives l_1 and l_2.

Corollary. Let the roots of (1) be now taken as $\alpha, \beta, \gamma_1, \gamma_2, \cdots, \gamma_{m-2}$.

$$\begin{aligned}\Sigma(\gamma_i - \gamma_j)^2 &= (m-3)\Sigma\gamma_i^2 - 2\Sigma\gamma_i\gamma_j\\ &= (m-2)(S_2 - \alpha^2 - \beta^2) - (S_1 - \alpha - \beta)^2\\ &> 0.\end{aligned}$$

i.e., $\phi(\beta) = (m-1)\beta^2 - 2(S_1 - \alpha)\beta + \frac{1}{m-1}\{(m-2)\psi(\alpha) + (S_1-\alpha)^2\} < 0$

$\therefore$ β should lie between the roots of the quadratic $\phi(x) = 0$, *i.e.*, between

$$m_1(\alpha),\ m_2(\alpha) = \frac{1}{m-1}[(S_1-\alpha) \mp \{-(m-2)\psi(\alpha)\}^{\frac{1}{2}}].$$

We now show that $m_1(\alpha) > l_1$ and $m_2(\alpha) < l_2$:

$$m_1(\alpha) > l_1,$$

i.e. $(m\alpha - S_1) + m\{-(m-2)\psi(\alpha)\}^{\frac{1}{2}} < (m-1)\{(m-1)(mS_2 - S_1^2)\}^{\frac{1}{2}}$,

if $2(m\alpha - S_1)\{-(m-2)\psi(\alpha)\}^{\frac{1}{2}} < (m-2)(m\alpha - S_1)^2 - \psi(\alpha)$

on squaring, which is allowed since the second member of the previous step is positive; also, using the equation

$$m\cdot\psi(\alpha) = (m\alpha - S_1)^2 - (m-1)(mS_2 - S_1^2).$$

Or squaring again, we should have

$$0 < [(m-2)(m\alpha - S_1)^2 + \psi(\alpha)]^2$$

which is obviously true. Similarly $m_2(\alpha) < l_2$. Thus, if one root α of the polynomial is known, then $[m_1(\alpha), m_2(\alpha)]$ is the interval for the other roots, and this interval lies wholly within (l_1, l_2).

§ 3. *Extension to four coefficients*

Proceeding as in § 2 above, but taking $\Sigma(\beta_i - \beta_j)^4$ instead of $\Sigma(\beta_i - \beta_j)^2$, we are led to the consideration of the quartic

$\chi(x) = mx^4 - 4S_1x^3 + 6S_2x^2 - 4S_3x - [(m-1)S_4 - 4S_1S_3 + 3S_2^2] = 0;\cdots(3)$

for, $\Sigma(\beta_i - \beta_j)^4 = (m-1)(S_4 - \alpha^4) - 4(S_1-\alpha)(S_3 - \alpha^3) + 3(S_2 - \alpha^2)^2 > 0$. Hence α should lie between the roots of $\chi(x) = 0$. With the usual notation for the invariants of a quartic, $I = -(m-1)\Sigma(\alpha_1 - \alpha_2)^4 < 0$. Therefore, two of the roots of the quartic are imaginary and two are real, and α should lie between these real roots. Also $\Delta = I^3 - 27J^2 < 0$, since only two roots are real. Again $J < 0$, for $H = mS_2 - S_1^2 > 0$, and all the four roots would be imaginary if J and H were both positive.

To solve the quartic let it be written as

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,$$

and let $a(x^2 + 2gx + h)(x^2 + 2g'x + h')$ be the factors, (4)

so that $g + g' = \frac{2b}{a}$; $h + h' + 4gg' = \frac{6c}{a}$; $gh' + g'h = \frac{2d}{a}$ and $hh' = e/a$.

Then $\theta = \frac{c}{a} - gg' = \frac{1}{4}\left(h + h' - \frac{2c}{a}\right)$ satisfies the cubic

$$4a^3\theta^3 - Ia\theta + J = 0.$$

Two roots of this cubic are imaginary since the quartic has only two real roots, and the real value of θ is positive (by Descarte's Rule of Signs). To obtain the real value of θ, write the cubic as

$$z^3 + 2\,H'z + G' = 0$$

where $H' = \frac{-I}{12a^2}$; $G' = \frac{J}{4a^3}$; $\therefore$ $(G'^2 + 4H'^3)^{\frac{1}{2}} = \frac{1}{4a^3}\sqrt{-\Delta/27}$

Hence $\theta = r^{\frac{1}{3}} + s^{\frac{1}{3}}$ where

$$r = \tfrac{1}{2}\,[-G' + (G'^2 + 4\,H'^3)^{\frac{1}{2}}] = \frac{1}{8a^3}\,(-J + \sqrt{-\Delta/27}),$$

$$s = \tfrac{1}{2}\,[-G' - (G'^2 - 4\,H'^3)^{\frac{1}{2}}] = \frac{1}{8a^3}\,(-J - \sqrt{-\Delta/27}).$$

$$\text{So, } \theta = \frac{1}{2a}\,[(-J + \sqrt{-\Delta/27})^{\frac{1}{3}} + (-J - \sqrt{-\Delta/27})^{\frac{1}{3}}].$$

Solving for g, g', h, h' from the equations

$$gg' = \frac{c}{a} - \theta;\; g + g' = \frac{2b}{a}\;;\; h + h' = 4\theta + \frac{2c}{a}\;;\; hh' = e/a$$

and substituting in (4), the quartic is resolved into

$$mx^2 + 2\,(\lambda - S_1)\,x + (2\,m\theta + S_2 + \mu) = 0 \qquad (5)$$

and $$mx^2 - 2\,(\lambda + S_1)\,x + (2\,m\theta + S_2 - \mu) = 0 \qquad (6)$$

where $$\lambda = +\,\{S_1{}^2 - mS_2 + m^2\theta\}^{\frac{1}{2}},$$

and $$\mu = +\,\{(2\,m\theta + S_2)^2 + m\,(\overline{m-1}\,S_4 - 4\,S_1S_3 + 3\,S_2{}^2)\}^{\frac{1}{2}}.$$

These two quadratics differ only in the sign of λ and μ. Also, only one of these quadratics has real roots. First consider the case in which (5) has real roots; then (6) must have imaginary roots (since two roots of the quartic are imaginary) and therefore

$$m\,(2\,m\theta + S_2 - \mu) > (\lambda + S_1)^2$$

and $$m\,(2\,m\theta + S_2 + \mu) < (\lambda - S_1)^2$$

giving $\frac{\mu}{\lambda} < -\frac{2\,S_1}{m}$ on subtraction. (7)

Again since the last term of (5) is positive and its roots are assumed to be real, the roots should be of one sign. But these roots form the range for the roots of the polynomial and so, the roots of the polynomial are of one sign also. This sign cannot be positive on account of (7), whose left-hand side is positive. A necessary condition, therefore, that (5) may have real roots is that all the roots of the given polynomial are negative. And since such a polynomial is easily replaced by one whose roots are all positive, we can take, without loss of generality, the roots of (6) as being real and as fixing the range for the roots of the given polynomial. We have thus the range given by the numbers—

$$L_1, L_2 = \frac{1}{m}\left[(\lambda + S_1) \mp \{(\lambda + S_1)^2 - m(2m\theta + S_2 - \mu)\}^{\frac{1}{2}}\right].$$

To prove that this new range (L_1, L_2) is closer than, and lies wholly within, Laguerre's range. We use Jensen's Theorem* that

$$(\Sigma a_n{}^{K})^{\frac{1}{K}} > (\Sigma a_n{}^{K'})^{\frac{1}{K'}} \quad \text{if } K < K' \text{ and } a_n > 0.$$

Putting $a_n = (\beta_i - \beta_j)^2$, $K = 1$ and $K' = 2$, we get

$$[\Sigma(\beta_i - \beta_j)^2]^2 > \Sigma(\beta_i - \beta_j)^4.$$

$\therefore$ The curve $y = [\psi(x)]^2$ is always above the curve $y = -\chi(x)$. It follows that the roots of $\chi(x) = 0$ lie between those of $\psi(x) = 0$.

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* This method of proof was suggested to us by Prof. K. S. K. Iyengar.

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