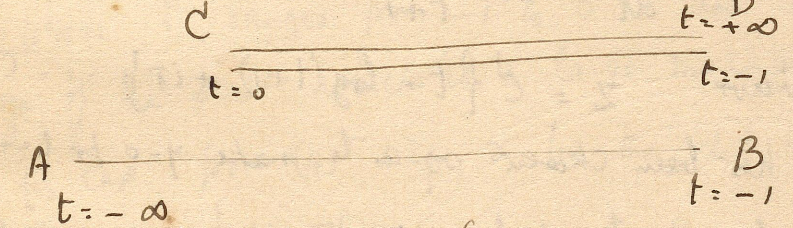
Some Applications of the Schwarz-Christoffel Theorem to Electrostatics

A. Problems connected with infinite plates

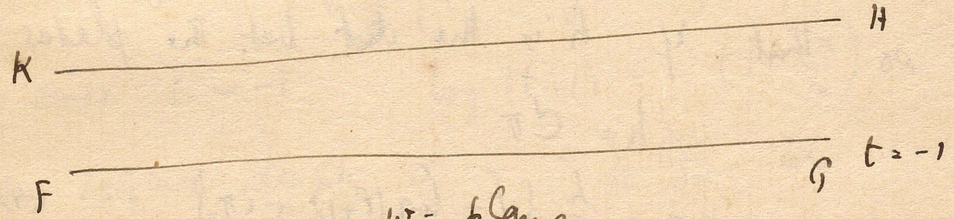
1). Semi-infinite plate with st. edge, at potential V, placed above and parallel to an infinite plate at zero potential

The diagrams in the z and w-planes are as below



($w = \phi + i\psi$
ψ is the potl)

z-plane.



w-plane

The boundary of the z-diagram consists of the infinite st. line AB, the two sides of the st. line CD & an arc of a circle stretching from $x = -\infty$ on AB to $x = +\infty$ on CD. We may assume arbitrarily the values of t corresp. to 3 corners of the diagram, we shall thus assume $t = -\infty$ for $x = -\infty$ on AB, $t = -1$ at $x = +\infty$ on AB, & $t = 0$ at C. The internal angles of the polygon are 0 at B & 2π at C. Hence the transf^n is

$$\frac{dz}{dt} = C \cdot \frac{t}{t+1} \quad \ldots \quad (1)$$

The diagram in the w-plane consists of two par^l st. lines. The internal angle at G ($t=-1$) is 0. Hence the corresp^g transf^n is

$$\frac{dw}{dt} = B \cdot \frac{1}{t+1} \quad \ldots \quad (2)$$

From (1) we have $\quad z = C\left\{t - \log(t+1) + i\pi\right\} \ldots (3)$

where the const. has been chosen so as to make $y=0$ for $t = -\infty$ to -1. When t passes through the value -1, the value of y increases by $C\pi$ so that if h is the dist. bet. the plates

$$h = C\pi$$

& hence $\quad z = \frac{h}{\pi}\left\{t - \log(t+1) + i\pi\right\} \ldots (4)$

From (2) $\quad w = B\left\{\log(t+1) - i\pi\right\} \ldots (5)$

the const. being chosen so as to make $\psi = 0$ from $t = -\infty$ to -1. As t passes through -1, ψ diminishes by $B\pi$. Hence

$$V = -B\pi$$

or $\quad w = \phi + i\psi = -\frac{V}{\pi}\left\{\log(t+1) - i\pi\right\} \ldots (6)$

Eliminating t from (4) & (6) we have

$$z = \frac{h}{V}\left\{w - \frac{V}{\pi}\left(1 + e^{-\frac{w\pi}{V}}\right)\right\} \quad \dots (7)$$

[Compare Jeans, Chap VIII §§ 323, 328

To find the quantity of electricity on a portion of the lower face of the semi-inf. plate, we notice that on this side of the plate t ranges from -1 to 0 & that at a dist. from the edge of the plate which is a large multiple of h, $t \approx -1$. In this case, from (4), if x be the dist from the edge corr^d to t,

$$x = \frac{h}{\pi}\left\{t - \log(1+t)\right\}$$

or since $t \sim -1$, $\log(t+1) = -\left(1 + \frac{\pi x}{h}\right)$

The surface-density $\sigma = -\frac{1}{4\pi}\frac{\partial\psi}{\partial\nu}$. Use $d\nu = \pm dy$, the $+$ or $-$ sign being taken according as the outward-drawn normal is the $+$ or $-$ dir^n of y. i.e. $+$ on the upper surface & $-$ on the lower surface. Thus $\sigma = \mp\frac{1}{4\pi}\frac{\partial\psi}{\partial y} = \mp\frac{1}{4\pi}\frac{\partial\phi}{\partial x}$.

Since $\frac{\partial\psi}{\partial\nu} = \frac{\partial\phi}{\partial s}$, where ds = element of sec^n of conductor,

$$\sigma = -\frac{1}{4\pi}\frac{\partial\psi}{\partial\nu} = -\frac{1}{4\pi}\frac{\partial\phi}{\partial s} = -\frac{1}{4\pi}\frac{d\phi}{dt}\cdot\frac{dt}{ds}.$$

The quantity of electricity on a strip of unit depth

$$= \int\sigma\, ds = -\frac{1}{4\pi}\int\frac{d\phi}{dt}\frac{dt}{ds}\cdot ds = -\frac{1}{4\pi}\left\{\phi(t_2) - \phi(t_1)\right\}$$

where t_1, t_2 correspond to the ends of strip & $t_2 > t_1$ algebraically.

The quantity of electricity on the strip of breadth x is

$$\frac{1}{4\pi}(\varphi_t - \varphi_0)$$

& by (6) this is equal to

$$-\frac{1}{4\pi}\frac{V}{\pi}\log(t+1)$$

$$= \frac{V}{4\pi h}\left\{x + \frac{h}{\pi}\right\}.$$

Thus the quantity of electricity on the lower side of the plate is thus the same as if the density were uniform & equal to that on ~~an~~ inf. plate, the breadth being increased by $\frac{h}{\pi}$.

On the upper side t ranges from 0 to ∞ & if x is a large multiple of h, t is very large. In this case

$$x = \frac{h}{\pi}\left\{t - \log(t+1)\right\}$$

gives

$$t = \frac{\pi x}{h} + \log\left\{1 + \frac{\pi x}{h}\right\} \text{ approx.}$$

& the quantity of electricity on strip of breadth x is $\frac{1}{4\pi}(\varphi_0 - \varphi_t)$ which by (6) equals

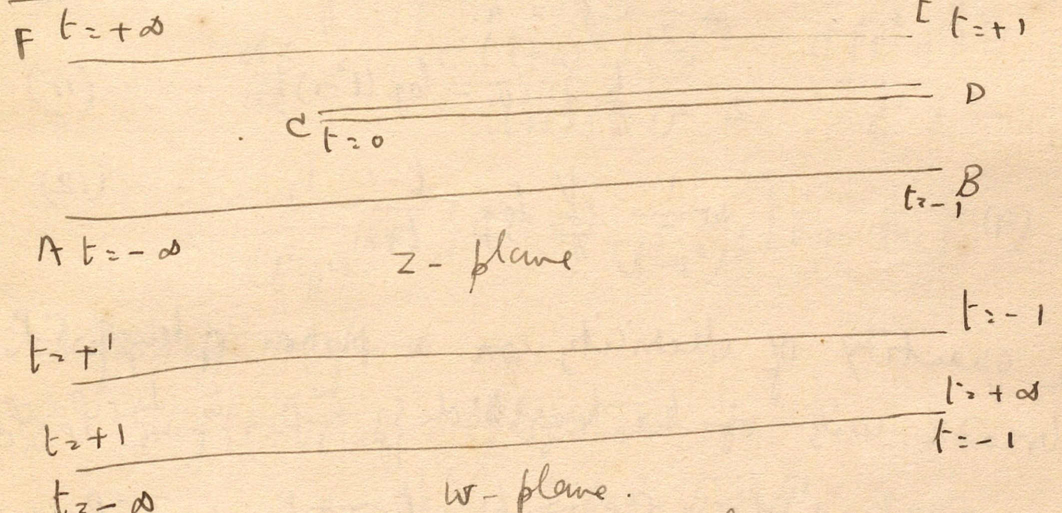
$$\frac{V}{4\pi^2}\log(t+1)$$

$$= \frac{V}{4\pi^2}\log\left\{1 + \frac{\pi x}{h} + \log\left(1 + \frac{\pi x}{h}\right)\right\}$$

The capacity of a breadth x of the upper plate is the ratio of the charge on both surfaces to V i.e.

$$\frac{x}{4\pi h}\left[1+\frac{h}{\pi x}+\frac{h}{\pi x}\log\left\{1+\frac{\pi x}{h}+\log\left(1+\frac{\pi x}{h}\right)\right\}\right]$$

2) <u>Semi-infinite conducting plane (pot'l V) placed midway between two parallel infinite conducting planes (at 0 pot'l).</u>



The ~~boundary~~ diagram in the w-plane consists of one line & 2 sides. The internal angles of the z-polygon are 0 at B, E & 2π at C. Hence

$$\frac{dz}{dt}=\frac{Ct}{(t+1)(t-1)} \quad \ldots \quad (8)$$

The internal angles of the w-polygon are 0 at $t=\pm 1$.

put $t = 1 + Re^{i\theta}$ where R is small, & θ changes from π to 0 as t passes thro' 1. When $t \approx 1$, (13) gives

$$\frac{dz}{dt} = \frac{1}{2} C (1-a^2)^{\frac{1}{2}} \frac{1}{t-1}$$

hence the increase of z as t passes thro' 1 is

$$\frac{1}{2} C (1-a^2)^{\frac{1}{2}} \left[\log R + i\theta \right]_{\pi}^{0}$$

$$= -\frac{i\pi}{2} C (1-a^2)^{\frac{1}{2}}.$$

$$\therefore \quad H - h = -C \frac{\pi}{2} (1-a^2)^{\frac{1}{2}}$$

When t changes from $+\infty$ to $-\infty$, z diminishes by $2iH$.
But for large t, from (13),

$$\frac{dz}{dt} = \frac{C}{t}$$

$$z = C \log t$$

Now $t = Re^{i\theta}$ where R is infinite & θ changes from 0 to π as t changes from $+\infty$ to $-\infty$ $\therefore$ z increases by

$$C \left[\log R + i\theta \right]_{0}^{\pi} = iC\pi$$

$$\therefore \quad H = -C\frac{\pi}{2}.$$

Thus

$$h = H\left\{1 - \sqrt{1-a^2}\right\}.$$

or $\quad a = \sqrt{\dfrac{h(2H-h)}{H^2}}$

The diagram in the w-plane is the same as in 2) above & hence

$$w = \frac{V}{\pi}\log\frac{t-1}{t+1} \quad . \quad . \quad . \quad (15)$$

The quantity of electricity on the portion of the semi-inf. plate between O & a pt P on the upper surface is

$$\frac{1}{4\pi}(\phi_0 - \phi_P)$$

At O $t=0$ & $\therefore \phi_0 = 0$. If $LP = x$ is large compared with H, t at $P \approx 1$. In this case from (14)

$$x = C\log\left\{\frac{1+\sqrt{1-a^2}}{a}\right\} + \tfrac{1}{2}C(1-a^2)^{\frac{1}{2}}\log\frac{a^2}{2(1-a^2)}$$
$$+\tfrac{1}{2}C(1-a^2)^{1/2}\log(t-1)$$

Substituting for C & a, we get

$$-\log(t-1) = \frac{\pi}{H-h}\left\{x + \frac{H}{\pi}\log\frac{2H-h}{h} + \frac{H-h}{\pi}\log\frac{h(2H-h)}{2(H-h)^2}\right\} \cdot (16)$$

From (15) $\quad \phi_P = \frac{V}{\pi}\{\log(t-1) - \log 2\}$

Hence the quantity of electricity on the strip OP is

$$\frac{V}{4\pi(H-h)}\left[x + \frac{H}{\pi}\log\frac{2H-h}{h} + \frac{H-h}{\pi}\log\frac{h(2H-h)}{(H-h)^2}\right]$$

The density of electricity at x on the upper side of middle plate is $-\frac{1}{4\pi}\frac{\partial\phi}{\partial x}$. Now $\frac{\partial\phi}{\partial x} = \frac{d\phi}{dt}\frac{\partial t}{\partial x}$

$$= \frac{2V}{\pi(t+1)(t-1)} \frac{(t+1)(t-1)}{C\,(t^2-a^2)^{1/2}}$$

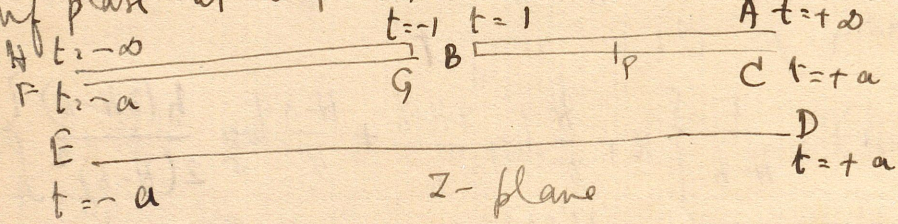
$$= \frac{V}{\pi C} \frac{2}{(t^2-a^2)^{1/2}} = -\frac{V}{H}\frac{1}{\sqrt{t^2-a^2}}.$$

Hence $\sigma = \frac{V}{4\pi H}\frac{1}{\sqrt{t^2-a^2}}$.

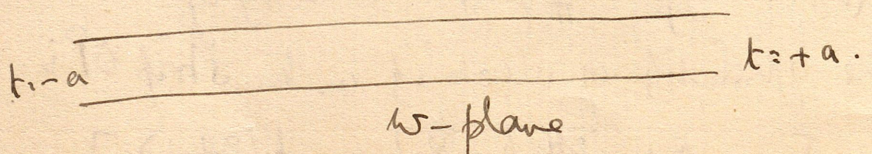
This is infinite at the edges C & E.

4) <u>Two semi-inf. plates separated by a finite interval 2k & placed par^l to an inf. plate at dist h from it</u>

Inf plate at 0 potential & the other two at pot^l V.



$(a<1)$



The internal angles of the z-polygon are 2π at $t=\pm 1$

& zero at $t = \pm a$. Hence

$$\frac{dz}{dt} = C \cdot \frac{t^2-1}{t^2-a^2} \quad \ldots \quad (17)$$

The diagram in the w-plane consists of 2 st. lines par^l to real axis & the pot^l changes by V when t passes thro' $\pm a$. Hence

$$w = \frac{V}{\pi} \log \frac{t+a}{t-a} + iV \quad \ldots \quad (18)$$

Integrating (17) we get

$$z = C\left\{t - \frac{1-a^2}{2a} \log \frac{t-a}{t+a} + \frac{1-a^2}{2a} i\pi\right\} \ldots (19)$$

the const. being determined by $z = 0$ for $t = 0$. (ED = x-axis)

When $t = 1$, $x = k$, $y = h$

$$\therefore \quad k = C\left\{1 - \frac{1-a^2}{2a} \log \frac{1-a}{1+a}\right\}$$

$$h = C \cdot \frac{1-a^2}{2a} \pi.$$

Hence a is given by

$$k = \frac{h}{\pi}\left\{\frac{2a}{1-a^2} + \log \frac{1+a}{1-a}\right\} \ldots (20)$$

The quantity of electricity Q on the lower side between B & P

$$Q = \frac{1}{4\pi}(\varphi_P - \varphi_B) = \frac{V}{4\pi^2}\left\{\log \frac{t_P+a}{t_P-a} - \log \frac{1+a}{1-a}\right\}.$$

362

From (19) & (20)

$$x-k = C\left[t_p-1+\frac{1-a^2}{2a}\left\{\log\frac{t_p-a}{t_p+a}-\log\frac{1-a}{1+a}\right\}\right]$$

$$= C(t_p-1)+\frac{4\pi h}{V}Q$$

$$Q = \frac{V}{4\pi h}\left\{x-k+C(1-t_p)\right\}$$

If $BP \gg h$, $t_p \approx a$, & so

$$Q = \frac{V}{4\pi h}\left\{x-k+C(1-a)\right\} \quad \ldots (21)$$

The quantity of electricity Q_1 on the upper side of the plate from A to B $= \frac{1}{4\pi}(\phi_B-\phi_A)$. $t_A = +\infty$ & $\therefore \phi_A = 0$

$$\therefore \quad Q_1 = \frac{1}{4\pi}\phi_B = -\frac{V}{4\pi^2}\log\frac{1-a}{1+a} \quad \ldots (22).$$

From (20) when k/h is very small, $a \approx \frac{\pi}{4}\frac{k}{h}$, $C \approx \frac{1}{2}k$ neglecting $(k/h)^3$,

$$Q = \frac{V}{4\pi h}\left\{x-\frac{1}{2}k-\frac{\pi}{8}\frac{k^2}{h}\right\}$$

$$Q_1 = \frac{V}{4\pi h}\cdot\frac{k}{2}$$

$$Q+Q_1 = \frac{V}{4\pi h}\left\{x-\frac{\pi}{8}\frac{k^2}{h}\right\}$$

If h/k is very small, $a \approx 1$,

$$1 - a = \frac{h}{\pi k}$$

$$C = k$$

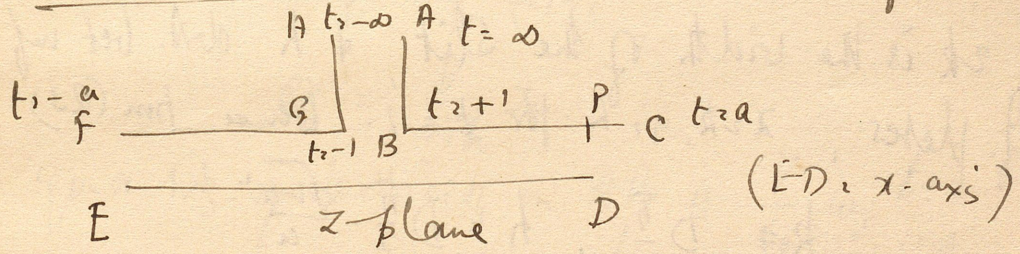
$\therefore$

$$Q = \frac{V}{4\pi h}\left\{x - k + \frac{h}{\pi}\right\}$$

$$Q_1 = \frac{V}{4\pi^2}\log\frac{2\pi k}{h}.$$

$$Q + Q_1 = \frac{V}{4\pi h}\left[x - k + \frac{h}{\pi}\left\{1 + \log\frac{2\pi k}{h}\right\}\right]$$

5) Same as 4) but the upper plates both of inf. thickness



Here
$$\frac{dz}{dt} = C \cdot \frac{(t^2 - 1)^{1/2}}{t^2 - a^2}.$$

For the portion of the plate for which $t < 1$, we write this as

$$\frac{dz}{dt} = iC \cdot \frac{(1 - t^2)^{\frac{1}{2}}}{t^2 - a^2}$$

$$= iC\left[\frac{1 - a^2}{2a}\frac{1}{(1 - t^2)^{1/2}}\left\{\frac{1}{t - a} - \frac{1}{t + a}\right\} - \frac{1}{(1 - t^2)^{1/2}}\right]$$

Integrating & making $Z = 0$ for $t = 0$ we find

$$Z = -iC\left[\frac{\sqrt{1-a^2}}{2a}\log\left\{\frac{(1-at+\sqrt{1-a^2}\sqrt{1-t^2})}{(1+at+\sqrt{1-a^2}\sqrt{1-t^2})}\frac{t+a}{t-a}\right\}+\sin^{-1}t\right]$$

$$+ C\pi\frac{\sqrt{1-a^2}}{2a}.$$

If $D = -iC$, this becomes

$$Z = D\sin^{-1}t + D\frac{\sqrt{1-a^2}}{2a}\log\left\{\frac{(1-at+\sqrt{1-a^2}\sqrt{1-t^2})}{(1+at+\sqrt{1-a^2}\sqrt{1-t^2})}\frac{t+a}{t-a}\right\}$$

$$+ Di\pi\frac{\sqrt{1-a^2}}{2a} \quad \ldots (23)$$

If $2k$ is the width of the slit, & h dist. bet inf & semi-inf. plates, $x = k$, $y = h$ for $t = 1$. Hence from (23)

$$k = D\frac{\pi}{2}, \quad h = D\frac{\pi}{2}\cdot\frac{\sqrt{1-a^2}}{a}$$

or $$a^2 = \frac{k^2}{h^2+k^2}$$

The relation bet w & t is the same as in 4)

$$w = \frac{V}{\pi}\log\frac{t+a}{t-a} + iV$$

The quantity of electricity Q bet A & P is $\frac{1}{4\pi}(\phi_P - \phi_A)$

$\phi_A = 0 \quad \therefore \quad Q = \frac{1}{4\pi}\phi_P$

$$Q = \frac{L}{4\pi} \log \frac{t_p + a}{t_p - a}$$

At P $t_p \approx a$. Hence by (23)

$$\log \frac{t_p + a}{t_p - a} = \frac{\pi}{h}\left\{x - D\sin^{-1} a - \frac{h}{\pi}\log(1-a^2)\right\}$$

$$= \frac{\pi}{h}\left\{x - \frac{2k}{\pi}\sin^{-1}\frac{k}{\sqrt{h^2+k^2}} - \frac{h}{\pi}\log\frac{h^2}{h^2+k^2}\right\}$$

Thus $\boxed{Q} = \frac{V}{4\pi h}\left\{x - \frac{2k}{\pi}\sin^{-1}\frac{k}{\sqrt{h^2+k^2}} - \frac{h}{\pi}\log\frac{h^2}{h^2+k^2}\right\}$

If $k \ll h$, neglecting $(k/h)^2$,

$$Q = \frac{V}{4\pi h} x.$$

The quantity of electricity on AB is $\frac{V}{4\pi^2}\log\frac{1+a}{1-a}$

$$= \frac{V}{4\pi^2}\log\left\{\frac{1+\frac{h}{\sqrt{h^2+k^2}}}{1-\frac{h}{\sqrt{h^2+k^2}}}\right\}$$

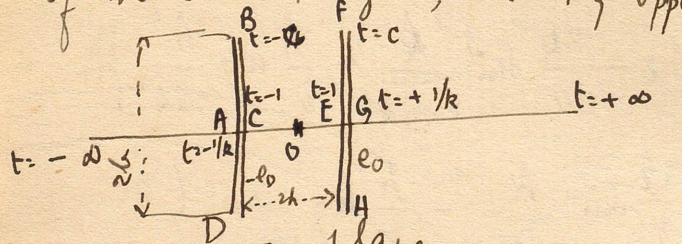
When k/h is small this equals

$$\frac{V}{4\pi h}\cdot\frac{2k}{\pi}.$$

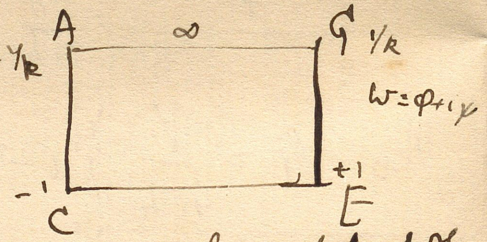
[Cf. Jeans, Chap VIII, Ex

B. Two parallel plates of finite width & related Problems

6). Parallel plate condenser formed by two thin parallel plates with straight parallel edges & of equal breadth, symmetrically situated. The two thin conducting plates cut a plane at right angles to their edges in two straight segments of the same length, directly opposite to each other



z - plane.



w-plane. (φ = pot)

Let the axis of x bisect the two segments at rt $\angle^s$ & let the origin be midway between the segments. In the z-plane the parts of the x-axis as divided by the segments are lines of force. The line of force CE becomes a line ψ = const, starting at the value of ϕ on DB & going to the greater value of ϕ on HF. This is continued by a line ϕ = const rising from the value of ψ at E to its value at G. The height through which it rises is $2\pi e_0$. This is continued by a line ψ = const. along which the potential ϕ diminishes from its value on HF to its value at the infinitely dist. pt. of the z-plane, as $x \to +\infty$. Since the system is a condensing system there is no discontinuity in the value of ψ at ∞; the line of force running from $-\infty$ to A is a continuation of the same line $\psi =$ const & it terminates in the w-plane at the value of ϕ on DB. This is continued by a line $\phi =$ const, falling to the starting point. The figure in the w-plane is therefore a rectangle, its height $= 2\pi e_0$, its length the diff^{ce} of pot^{l} bet the two plates $= 2\Phi$, say.

We associate the pts. O, E, ∞ of the z-plane with the pts. $0, 1, \infty$ in the t-plane. Then to the pts. F, G will correspond some real values of t, $t = c$ & $t = 1/k$. From symmetry the values of t corresp^{g} to A, B, C must be $-1/k$, $-c$, -1.

The transformations are

$$\frac{dz}{dt} = A \cdot \frac{t^2 - c^2}{\left\{\left(t+\frac{1}{k}\right)(t+1)(t-1)\left(t-\frac{1}{k}\right)\right\}^{1/2}} \quad \cdots (24)$$

$$\frac{dw}{dt} = B \cdot \frac{1}{\left\{\left(t+\frac{1}{k}\right)(t+1)(t-1)\left(t-\frac{1}{k}\right)\right\}^{1/2}} \quad \cdots (25)$$

The integrations are performed by putting $t = \operatorname{sn}(\cancel{\zeta}, k)$ and $\cancel{\zeta}$ may be taken to vanish with t. Then w is simply proportional to χ if the arbitrary additive const. in w is suitably adjusted. The values of χ at various pts. of the z-fig are as follows

A	C	E	G	∞
$-K+iK'$	$-K$	K	$K+iK'$	iK'

Since ψ increases by $2\pi e_0$ as χ goes from K to $K+iK'$ we have

$$w = \frac{2\pi e_0}{K'} \chi \quad \ldots \quad (26)$$

& since ϕ increases by $2\phi_1$ as χ goes from $-K$ to K we have

$$2\phi_1 = \frac{4\pi e_0}{K'} K . \quad \ldots \quad (27)$$

Hence the capacity of the condenser per unit length which is $e_0/2\phi_1$ is $K'/4\pi K$.

The value c of t at F is not arbitrary but must be such that z has the same value at G as at E.

Now

$$\frac{dz}{d\chi} = -Ak\left(sn^2\chi - c^2\right)$$

$$= \frac{A}{k}\left(1 - k^2 sn^2\chi - 1 + k^2c^2\right)$$

giving

$$z = \frac{A}{k}\left\{ Z(\chi) + \chi\left(\frac{E}{K} + k^2c^2 - 1\right)\right\} \quad \ldots \quad (28)$$

where $Z(u)$ is Jacobi's Zeta function (Whittaker + Watson, p 518, § 22·731)

Hence $$Z(K+iK') + (K+iK')\left(\frac{E}{K}+k^2c^2-1\right) = Z(K) + K\left(\frac{E}{K}+k^2c^2-1\right)$$

& since $Z(K)=0$, $Z(K+iK') = -\dfrac{i\pi}{2K}$,

$$EK' + E'K - KK' = \frac{\pi}{2}$$

this gives $$k^2c^2 = \frac{E'}{K'}$$

& so $$Z = \frac{A}{k}\left\{Z(\chi) + \frac{\pi\chi}{2KK'}\right\} \quad \ldots (29)$$

Put $c = sn(K+iv_0)$ where $K' > v_0 > 0$.

then $$K'k^2 sn^2(K+iv_0) = E'$$

Since $z=h$ at $\chi = K$, $h = \dfrac{A}{k}\cdot\dfrac{\pi}{2K'}$

& since $z = h+ib$ at $\chi = K+iv_0$

$$\frac{b}{h} = \frac{2K'}{\pi}\left\{-iZ(K+iv_0) + \frac{\pi v_0}{2KK'}\right\} \quad \ldots (30)$$

Since v_0 is known when k is known this eqn. determines k.

37°.

The elementary theory gives for the capacity per unit length the value $b/4\pi h$ when h/b is small. When this is so, K'/K must be of the same order as b/h & so $K \sim \frac{1}{2}\pi$ & k must be very small, $k = O(e^{-K'})$ [Whittaker & Watson, §§ 22.737]

We have
$$Z(x) = \frac{\pi}{K}\sum_1^\infty \frac{\sin(n\pi x/K)}{\sinh(n\pi K'/K)}$$

$$k^2 sn^2 x = 1 - \frac{E}{K} - \frac{\pi^2}{K^2}\sum_1^\infty \frac{n\cos(n\pi x/K)}{\sinh(n\pi K'/K)}$$

[Whittaker & Watson, p520, Ex 5] $\left(q = e^{-\pi K'/K}\right)$

The eqn $sn^2(K+iv_0) = E'/(K'k^2)$ in which K' is large, $E' \approx 1$ & k^2K' very small shows that v_0 must be large compared with K. We may put with suff approxn

$E = K = \frac{\pi}{2}$. Then
$$k^2 sn^2(K+iv_0) = 4\sum_1^\infty (-1)^{n+1} \frac{n\cosh 2nv_0}{\sinh 2nK'}$$
$$= 4\sum_1^\infty (-1)^{n+1}\, n\, e^{-2n(K'-v_0)}$$

approx.

Reducing the series to its 1st term we find that the eqn giving v_0 becomes approx.

$$e^{2(K'-v_0)} = 4K'$$

To the same order of approxn

$$Z(K+iv_0) = -2ie^{-2(K'-v_0)} = -i/(2K')$$

& hence $$\frac{b}{h} = \frac{K'}{K}\left\{1 - \frac{1}{2K'}(1+\log 4K')\right\}$$

or, to the same order

$$\frac{K'}{K} = \frac{b}{h}\left\{1 + \frac{h}{\pi b}\left(1+\log\frac{2\pi b}{h}\right)\right\} \quad \cdots (31)$$

This is the approx. formula given by Bromwich.

7) Plates of equal breadths, asymmetric posn.

If we add to the exprn for $\frac{dz}{dt}$ in (24) a term of the form iC where C is a pos. const., we shall add to Z a term of the form iCt. Then every pt on AF will be displaced in the pos. dirn. of y-axis thro' a dist propl. to the t of the pt & every pt. of DB will be displaced in the opp. dirn. The resulting exprn may be

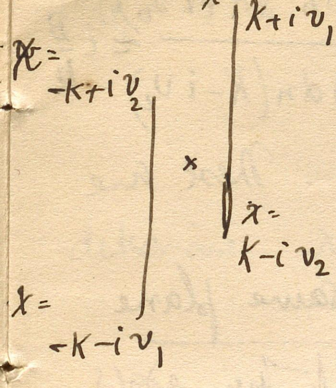
written
$$\frac{dz}{dt} = Ak\left\{\frac{c^2-t^2}{\sqrt{\{(1-t^2)(1-k^2t^2)\}}} + i\frac{B}{k}\right\} \quad \ldots (32)$$
by introducing a new pos. const B in place of C.

If as before $t = \operatorname{sn}\chi$, $\frac{dz}{dt}$ has no zeros within the χ rectangle bounded by $u = \pm K$, $v = \pm K'$ ($\chi = u+iv$). The values of χ at these zeros are of the form

$$K+iv_1, \quad K-iv_2, \quad -K+iv_2, \quad -K-iv_1$$

These values correspond to the ends of the segments. To make z a uniform fn. of t, as before we must have
$$k^2K'c^2 = E'$$
& if $c = \operatorname{sn}(K+iv_0)$ it may be proved that
$$K' > v_1 > v_0 \quad \& \quad v_0 > v_2 > 0$$
The formula for z becomes
$$z = \frac{A}{k}\left[Z(\chi) + \frac{\pi\chi}{2KK'} + iBk\operatorname{sn}\chi\right] \quad \ldots (33)$$
The r.h.s is an odd fn. of χ & thence it follows that the segments are equal & situated symmetrically

w.r.t the origin of z. ~~[illegible]~~ The condenser problem for two par! plates of equal breadths but not directly opposite to each other is solved by combining (33) with $w = 2\pi e_0 x / K'$.



The Capacity per unit length $= K'/4\pi K$ as before.

Let the segments in the z-plane be of length 2b and at a dist 2h apart; & let one project a dist d beyond the other. Then we must have

$$h = \frac{A}{k}\,\frac{\pi}{2K'}$$

$$2ib = \frac{A}{k}\Big[Z(K+iv_1) - Z(K-iv_2) + \frac{i\pi(v_1+v_2)}{2KK'} + iBk\{sn(K+iv_1) - sn(K-iv_2)\}\Big]$$

$$id = \frac{A}{k}\Big[Z(K+iv_1) - Z(-K+iv_2) + \frac{i\pi(v_1-v_2)}{2KK'} + iBk\{sn(K+iv_1) - sn(-K+iv_2)\}\Big]$$

and v_1, v_2, B must be connected by the eqn

$$\frac{sn^2(K+iv_1)-sn^2(K+iv_0)}{cn(K+iv_1)\,dn(K+iv_1)} = \frac{sn^2(K-iv_2)-sn^2(K+iv_0)}{cn(K-iv_2)\,dn(K-iv_2)} = i\frac{B}{k}$$

in which v_0 is known when k is known. These are suff^t eq^{ns} to determine A, B, v_1, v_2, k.

8). Plates of unequal breadths in the same plane

Let the plates cut a plane at rt. L^s to their edges in two segments of unequal lengths in the same plane. Let the ends of the segments in order from left to right be at x_1, x_2, x_3, x_4, let the lengths of the segments x_2-x_1 & x_4-x_3 be b_1 & b_2 and let the length of the gap x_3-x_2 between them be $2a$. Suppose that $b_2 > b_1$. The region in the z-plane bounded internally by the two segments can be transformed into the region in the t-plane bounded internally by two equal segments in line by

$$z = z_0 + \frac{\lambda}{t+\nu} \qquad (34)$$

where $t = -\nu$ corresponds to $z = \infty$. λ, z_0 & ν are real & can be adjusted so that $x_1\ x_2\ x_3\ x_4$ correspond to $1/k, 1, -1, -1/k$ in the t-plane. The eq^{ns}

$$b_1 = x_2 - x_1 = \lambda \frac{1-k}{(\nu+1)(k\nu+1)}, \quad 2a = x_3 - x_2 = \frac{2\lambda}{\nu^2-1}.$$

$$b_2 = x_4 - x_3 = \lambda \cdot \frac{1-k}{(\nu-1)(k\nu-1)}$$

determine λ, ν, k & then

$$Z_0 = x_1 - \frac{\lambda k}{k\nu+1}$$

determines Z_0. k satisfies the eq^n

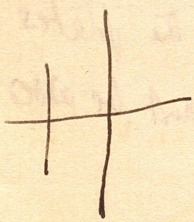
$$k^2 - 2k\left\{1 + \frac{b_1 b_2}{a(2a+b_1+b_2)}\right\} + 1 = 0$$

which has real pos. roots, one of them < 1. This value being taken for k, ν is given by

$$\nu = \frac{b_2 - b_1}{(2a+b_1+b_2)k - 2a}.$$

ν being known, λ is $a(\nu^2-1)$

9) Parallel plates of unequal breadths. In the symmetric position the line bisecting both segments at right $\angle^s$ is made up of lines of force. By



the transf^n

$$\frac{dz}{dz'} = A' \frac{(z'-c_1')(z'-c_2')}{\sqrt{\{(z'-x_1')(z'-x_2')(z'-x_3')(z'-x_4')\}}}$$

we may transform half the fig in the z-plane into the region bounded by the line of two segments in the z'-plane. These segments will be unequal & a further transf.^{n} of the form

$$z' = z_0 + \frac{\lambda}{t+\nu}$$

is reqd. to transform them into equal segments in line. Hence

$$\therefore \frac{dz}{dt} = A \cdot \frac{(t-c_1)(t-c_2)}{(t+\nu)^2\sqrt{\{(1-t^2)(1-k^2t^2)\}}} \qquad (35)$$

secures the transf.^{n} of the z-plane into the t-plane, provided the consts. are such that z is a uniform f.^{n} of t. These condns. for this may be shown to be

$$c_1c_2 = -\frac{\nu^2E'-K'}{k^2\nu^2K'-E'}, \quad (\nu+c_1)(\nu+c_2) = \frac{(\nu^2-1)(k^2\nu^2-1)K'}{k^2\nu^2K'-E'}$$

When these condns. are satisfied

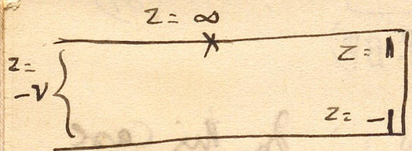
$$z = \text{const} + B\left\{Z(\chi) + \frac{\pi\chi}{2KK'} + \frac{\text{cn}\,\chi\,\text{dn}\,\chi}{\nu+\text{sn}\,\chi}\right\}$$

where B is a real const., & $t = \text{sn}\,\chi$.

For an asymmetric posn., the parallelism of the plates being maintained, a term of the form iC must be added to the above expression for $\frac{dz}{dt}$.

10). Plate + Line charge – Charged line in plane of plate

Let the plate + line meet a transverse plane in a straight segment + a pt. in line with the segment. Let the charged pt. + the ends of the segment be at $z = -\nu, -1, 1$. Let the charges on the line + plate be e_0 + $-e_0$ per unit length.

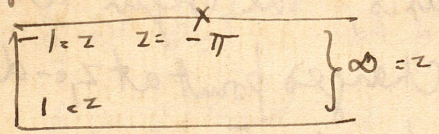


w-plane.

The fig. in the w-plane is a half strip of width $2\pi e_0$. The relation between w and z is of the form

$$\frac{dw}{dz} = \frac{A}{(z+\nu)\sqrt{z^2-1}} \quad --- (36)$$

To integrate it put $z = \cosh\chi$, where $\chi = u + iv$, $u \geq 0$, $v \geq 0$. The upper half of z-plane is transformed into a half-strip of breadth π in the χ plane



Put $\nu = \cosh\beta$, β real + pos.

When $z = -\nu$, $\chi = \beta + i\pi$

Then
$$\frac{dw}{d\chi} = \frac{A}{\cosh\chi + \cosh\beta}$$

$$\therefore \quad w = \frac{A}{\sinh\beta}\log\frac{\cosh\frac{1}{2}(\chi+\beta)}{\cosh\frac{1}{2}(\chi-\beta)} \quad \cdots (37)$$

The value of ϕ at $z = \infty$ is $-2e_0\beta$.

The surface density σ at any pt. of the plate is

$$\sigma = \frac{1}{4\pi}\left|\frac{dw}{dz}\right|_{u=0} = \frac{e_0\sqrt{(\nu^2-1)}}{2\pi(\nu e x)\sqrt{(x^2-1)}} \quad \ldots (38)$$

If O, A, B are the charged pt. & the ends of the segment & P is any pt. of the segment, this may be written

$$\sigma = \frac{e_0}{2\pi}\cdot\frac{\sqrt{(OA\cdot OB)}}{OP\sqrt{(AP\cdot BP)}} \quad \ldots (39)$$

11) Charged line & plate in full view: In this case the pt. in which the line cuts a transverse plane is equidistant from the ends of the segment in which the plate cuts the same plane. The perp. from the pt. to the segment bisects the segment. Take the plane of the pt. & segment as the z_1-plane & the line of symmetry as the x_1-axis, the origin at the mid-pt of segment & the charged point at $z_1 = -d_1$, the length of the segment as $2b$ & the charges on the line & plate as e_0 & $-e$ resp. per unit length.

The upper half of the fig. bounded by the x_1-axis & the segment is transformed into the upper half of the z plane in the previous problem by

$$\frac{dz_1}{dz} = A \cdot \frac{Z-a}{\sqrt{Z^2-1}}$$

where $Z = a$ for $z_1 = ib$. The const. a must be determined by the condn that Z_1 has the same value zero at $Z = 1$ & $Z = -1$. This gives $a = 0$ & hence

$$Z_1 = A\sqrt{Z^2-1} \quad \ldots (40)$$

Since $z_1 = ib$ for $Z = 0$, $A = b$. The present problem will be transformed into the previous problem if $z_1 = -d$ for $Z = -\nu$. The soln is then given by

$$z_1 = b \sinh \chi, \quad w = 2e_0 \log \frac{\cosh \frac{1}{2}(\chi-\beta)}{\cosh \frac{1}{2}(\chi+\beta)}, \; d = \sinh \beta \quad (41)$$

If we put $\cosh \beta = \sec \alpha$, $\sinh \beta = \tan \alpha$

d is $b \tan \alpha$ & $\pi - 2\alpha$ is the angle subtended by the segment at the charged point. The soln holds in the whole of z_1 plane if $u \geq 0$, $2\pi \geq v \geq 0$ where $\chi = u + iv$.

The surface-density σ is given by

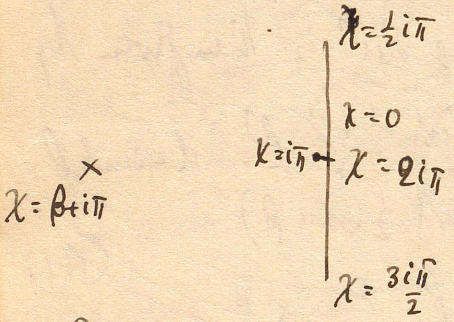
$$\sigma = \frac{1}{4\pi}\left|\frac{dw}{dz_1}\right|_{x=0} = \frac{1}{4\pi}\,\frac{2e_0 \sinh\beta}{|\sin v|(\cosh\beta + \cos v)}\,\frac{1}{b}\left|\frac{\sin v}{\cos v}\right|$$

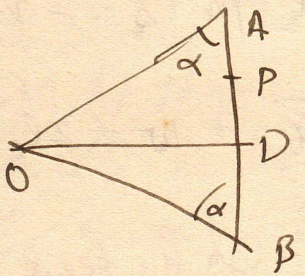
$$= \frac{e_0 \tan\alpha}{2\pi b}\,\frac{\sec\alpha - \cos v}{|\cos v|(\sec^2\alpha - \cos^2 v)}$$

$$= \frac{e_0 \tan\alpha}{2\pi b}\,\frac{\sec\alpha - \cos v}{(\tan^2\alpha + \sin^2 v)|\cos v|}$$

Now $\tan\alpha = d/b$ & on the segment $\sin v = y_1/b$

$$\therefore\ \sigma = \frac{e_0}{2\pi}\,\frac{d}{d^2+y_1^2}\,\frac{\sqrt{(d^2+b^2)} \mp \sqrt{(b^2-y_1^2)}}{\sqrt{(b^2-y_1^2)}}$$





where the upper sign applies to the side remote from the charged pt.

$$\therefore\ \sigma = \frac{e_0}{2\pi}\cdot\frac{OD}{OP^2}\cdot\frac{OA \mp \sqrt{(AP\cdot BP)}}{\sqrt{(AP\cdot BP)}} \qquad (42)$$

12) Free Charge on cylindrical sheet with par[1]. edges: If we transform the 1st fig on the preceding page by a complex inversion w.r.t $z_1 = -d$ the segment becomes an arc of a circle of diameter d & the charged pt. recedes to an infinite distance. We thus have the soln. of the problem of free distribution of charge on a cylindrical sheet bounded by two generators.

Take the plane of the arc to be the plane of a complex variable z_2 & transform the results of the last problem by putting

$$z_2(z_1+d) = d^2$$

Then $$z_2 = d \cdot \frac{\tan\alpha}{\sinh\chi + \tan\alpha} \quad \cdots (43)$$

where $\pi - 2\alpha$ is the angle subtended by the arc at any pt on the circumference & outside the arc.

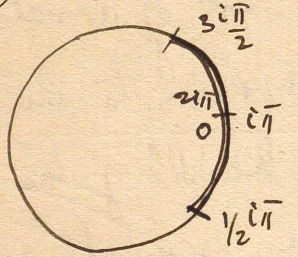
At points on the arc

$$x_2 = d\frac{\tan^2\alpha}{\tan^2\alpha + \sin^2 v}, \quad y_2 = -d \cdot \frac{\tan\alpha \sin v}{\tan^2\alpha + \sin^2 v}$$

so that the eq^n is

$$x_2^2 + y_2^2 = d\, x_2 .$$

The arrangement of the values of χ on the arc is as shown in the fig below: The surface density



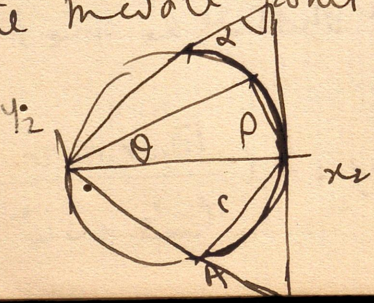
at any point on the cylindrical sheet is

$$\sigma = \frac{1}{4\pi}\left|\frac{dw}{dz_2}\right|_{u=0}$$

$$= \frac{e_0 \tan^2\alpha}{2\pi d}\, \frac{d^2(\sec\alpha - \cos v)}{|z_2^2|_{u=0}(\tan^2\alpha + \sin^2 v)|\cos v|}$$

$$= \frac{e_0}{2\pi d}\, \frac{\sec\alpha - \cos v}{|\cos v|} \quad \ldots \quad (44)$$

In this $\sin v = -(y_2/x_2)\tan\alpha = -\tan\theta\tan\alpha$ where θ = vectorial angle of pt. on the arc measured from the point on the circumference opposite to the middle point of the arc. Let ρ be the dist. of



any pt. on the arc from the mid pt. of arc, c = max. value of ρ.

Then $\rho^2 = d^2\sin^2\theta \qquad c = d\cos\alpha$

$$\sin^2 v = \frac{\rho^2}{d^2-\rho^2}\,\frac{d^2-c^2}{c^2}\,, \quad \cos^2 v = \frac{d^2(c^2-\rho^2)}{c^2(d^2-\rho^2)}\,.$$

$$\therefore \quad \sigma = \frac{e_0}{2\pi d}\left\{\sqrt{\left(\frac{d^2-\rho^2}{c^2-\rho^2}\right)} \mp 1\right\} \quad \ldots \quad (45)$$

where the upper sign applies to the concave side.

13). <u>Plate & line charge - General posn</u>. By a complex inversion of the fig. in the z_2-plane from a point O on the unoccupied arc the circular arc will be transformed into a segment of a st. line & we shall have the soln for this segment under the influence of a charged pt. at O, the segment & pt. being the intersection of a transverse plane by a charged conducting plate (with parl edges) & a charged line parl to its edges. The charges are equal & opp as before. Let the plane of the pt. & segment be the z_3-plane & let $\frac{1}{2}\pi - \gamma$ be the vectorial angle of O measured from the origin of z_2 in the plane of the arc.

Also let D be the pt opp to O on the circumf.

C. Problems solved by given transformations

1). $$x+iy = b\,\mathrm{sn}(\phi+i\psi) \qquad (\mathrm{mod}\ k) \quad \text{-- -- (1)}$$

Let ϕ be the potl. The equipotl $\phi = K$ is given

$$x+iy = b\,\mathrm{sn}(K+i\psi)$$

$$= \frac{b}{\mathrm{dn}(\psi, k')} \quad \text{-- -- (2)}$$

$$\therefore \quad x = \frac{b}{\mathrm{dn}(\psi, k')}, \quad y=0$$

Now $\mathrm{dn}(\psi, k')$ is always >0 & its max value $=1$ for $\psi=0$ or $\psi = 2nK'$, its least value $= k$ for $\psi = (2n+1)K'$

$\therefore$ (1) represents the portion of the real axis bet $x=b$ & $x=b/k$

If we put $\phi = -K$, we have

$$x+iy = b\,\mathrm{sn}(-K+i\psi)$$

$$= -\frac{b}{\mathrm{dn}(\psi, k')}$$

Hence the equipotl $\phi = -K$ consists of the portion of the x-axis bet $x=-b$ & $x=-b/k$.

$\therefore$ The transfn (1) solves the case of two inf. plane strips AB, CD of finite equal breadths $b(1-k)/k$

in one plane with their edges parl. Potl diff $= 2K$.

Quantity of elec. (per unit length) on top of CD

$$= \frac{L}{4\pi}(\psi_C - \psi_D) = \frac{K'}{4\pi}$$

There is an equal quantity of elec. on the lower side

& ∴ total charge on CD $= \dfrac{K'}{2\pi}$.

∴ Capacity per unit length $= \dfrac{1}{4\pi}\dfrac{K'}{K}$. ...(3)

When $k \lll 1$, we have $K \sim \dfrac{\pi}{2}$

$$K' \sim \log(4/k) = \log(4AD/BC)$$

∴ Capacity $\sim \dfrac{1}{2\pi^2}\log(4AD/BC)$. ... (4)

$\sigma = \dfrac{L}{4\pi}\dfrac{\partial\psi}{\partial x}$, since on AB

$$x = -\frac{b}{dn(\psi, k')}$$

$$-\frac{dx}{d\psi} = bk'^2 \frac{sn\psi\, cn\psi}{dn^2\psi}$$

$$= \frac{L}{b}\sqrt{\boxed{(x^2 - b^2)}(b^2 - k^2x^2)}$$

$$= \frac{k}{b}\sqrt{CP.DP.AP.BP}$$

& $\sigma = -\dfrac{b}{4\pi k}\sqrt{CP\cdot DP\cdot AP\cdot BP}$ ... (5)

S 388.

2) : $x + iy = b \log \operatorname{sn}(\phi + i\psi)$ --- (6)

1) When $\phi = K$, $e^{\frac{z}{b}} = \operatorname{sn}(K + i\psi) = \dfrac{1}{\operatorname{dn}(\psi, k')}$

$\therefore \quad y = 0, \ 2\pi b, \ 4\pi b,$ ——

& x varies bet x_1, x_2 where $e^{x_1/b} = 1$, $e^{x_2/b} = \frac{1}{k}$.

When $\phi = -K$, $e^{\frac{z}{b}} = -\dfrac{1}{\operatorname{dn}(\psi, k')}$

$\therefore \quad y = \pi b, \ 3\pi b, \ 5\pi b$ etc

& x is ~~as~~ before. Thus (6) gives the electrical

$\therefore$ distr^n^ for a pile of par^l^ plates of finite width $x_2 - x_1$

dist. bet. consecutive strips being πb, alternate strips

being at the same pot^l^. Pot of one set $= K$, of other

set $= -K$. As before capacity $= \dfrac{L}{4\pi} \dfrac{K'}{K}$.

If ~~d~~ = width of one strip,

$k = e^{-\frac{d}{b}}$,

For $k \ll 1$. $K \sim \pi/2$, $K' \sim \log(4/k) = 2\log 2 + \frac{d}{b}$

$\therefore$ Capacity $= \dfrac{1}{2\pi^2}\left\{2\log 2 + \dfrac{d}{b}\right\}$

3) $\quad x + iy = b\,\mathrm{cn}(\phi + i\psi)$ ——— (7)

Take ψ for pot. When $\psi = 0$

$$\psi = 0, \quad x = b\,\mathrm{cn}\,\phi.$$

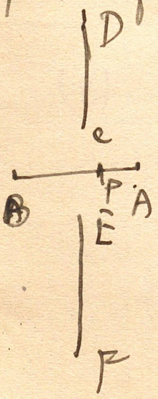
$\therefore$ Equipot $\psi = 0$ is the portion of x-axis bet $x = -b$ & $+b$.

When $\psi = K'$, $\quad x + iy = b\,\mathrm{cn}(\phi + iK')$

$$= -\frac{ib\,\mathrm{dn}\,\phi}{k\,\mathrm{sn}\,\phi}$$

$\therefore x = 0$, & y ranges from $+bk'/k$ to $+\infty$

& from $-bk'/k$ to $-\infty$. Hence equipot $\psi = K'$ is a portn of the y-axis. Now $\phi = 0$ at A & $= 2K$ at B



$\therefore$ quantity of elec on one side of $AB = \dfrac{K}{2\pi}$

& total charge on $AB = \dfrac{K}{\pi}$.

P. D $= K'$ & $\therefore$ capacity $= \dfrac{1}{\pi}\dfrac{K}{K'}$

$\dfrac{k'}{k} = \dfrac{\sqrt{1-k^2}}{k} = \dfrac{EC}{AB}$

If $AB \gg EC$, then $k \approx 1$ we have

$$K = \log(4/k') = \log(4AB/EC)$$

$$K' = \frac{\pi}{2}$$

$$\text{Capacity} = \frac{2}{\pi^2}\log(4AB/EC)$$

$\therefore$ $\cancel{\frac{dx}{d\phi}}$ On AB $x = b\,\mathrm{cn}\,\phi$

$$\therefore \frac{dx}{d\phi} = -b\,\mathrm{sn}\,\phi\,\mathrm{dn}\,\phi$$

$$= -\frac{k}{b}\sqrt{b^2-x^2}\sqrt{\frac{k'^2}{k^2}b^2+x^2}$$

$$= -\frac{k}{b}CP\sqrt{AP\cdot BP}$$

$$\therefore \quad \sigma = -\frac{b}{4\pi k}\frac{1}{CP\sqrt{AP\cdot BP}}$$

4) $$x+iy = b\log\mathrm{cn}(\phi+i\psi) \quad \ldots (8)$$

$\phi =$ pot^l. For $\phi = 0$,

$$e^{\frac{z}{b}} = \mathrm{cn}(i\psi) = \frac{1}{\mathrm{cn}(\psi, k')}$$

$\therefore\quad y = 0,\ \pm\pi b,\ \pm 2\pi b \text{ --}$

while x ranges from 0 to ∞.

For $\phi = K$, $\quad e^{\frac{z}{b}} = \mathrm{cn}(K + i\psi)$

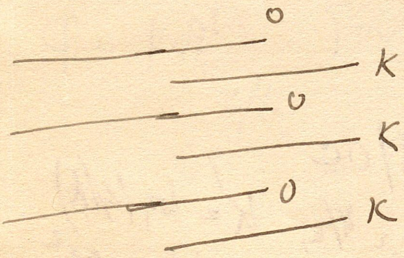
$$= -ik'\,\frac{\mathrm{sn}(\psi, k')}{\mathrm{dn}(\psi, k')}$$

$\therefore\quad y = \pm\tfrac{1}{2}\pi b,\ \pm\tfrac{3}{2}\pi b,\ \pm\tfrac{5}{2}\pi b \text{ --}$

& x ranges from $-\infty$ to x_1 given by

$$e^{x_1/b} = \frac{k'}{k}.$$

Thus (18) gives the distr^n. on a pile of semi-inf. par^l plates placed at equal intervals πb, maintained at pot^l 0 in presence of another similar set at pot^l K. The plates of the 2nd set being midway betw those of the 1st. The 2nd set project a dist x_1 into the 1st set.



5) $z = x + iy = b\,\mathrm{dn}(\phi + i\psi)$ ... (9)

ϕ = pot^l. When $\phi = 0$, $x + iy = b\,\mathrm{dn}\, i\psi$

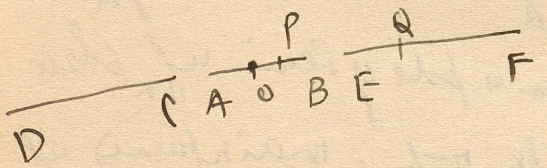
$$= b\,\frac{\mathrm{dn}(\psi, k')}{\mathrm{cn}(\psi, k')}$$

$\therefore y = 0$, x ranges from $+b$ to $+\infty$ & $-b$ to $-\infty$.

[(9) solves the problem of a finite plate placed bet two semi inf plates] When $\phi = K$

$$x + iy = b\,\mathrm{dn}(K + i\psi)$$

$$= bk'\,\frac{\mathrm{cn}(\psi, k')}{\mathrm{dn}(\psi, k')}$$



$\therefore y = 0$ & x ranges between $\pm bk'$.

Quantity of elec. on the 2 sides of AB $= k'/\pi$ & $\therefore$

Capacity $= \frac{1}{\pi}\frac{K'}{K}$.

where $k' = \sqrt{1-k^2} = OA/OC$

When $AC \ll AB$, $k' \sim 1$ & $K = \pi/2$, $K' = \log(4/k)$

$$= \log 4 + \tfrac{1}{2}\log\frac{OC^2}{AC\cdot BC}$$

$\therefore$ Capacity $= \frac{1}{\pi^2}\left\{\log \frac{OC^2}{AC \cdot BC} + 2\log 4\right\}$

At a pt P on AB,

$$\frac{dx}{d\psi} = \frac{1}{\frac{1}{b}\sqrt{(b^2-x^2)(b^2k'^2-x^2)}}$$

$$\therefore \sigma_P = \frac{b}{4\pi}\frac{1}{\sqrt{(b^2-x^2)(b^2k'^2-x^2)}}$$

$$= \frac{b}{4\pi}\frac{1}{\sqrt{AP \cdot BP \cdot CP \cdot EP}}$$

Simly on EF, $\sigma_Q = -\frac{b}{4\pi}\frac{1}{\sqrt{AQ \cdot BQ \cdot CQ \cdot EQ}}$.

6) $z = x+iy = b\log dn(\phi+i\psi)$ --- (10)

ϕ = pot^l. For $\phi = 0$

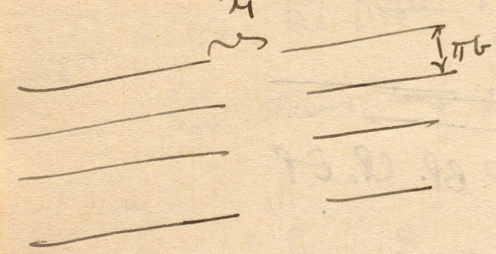
$$e^{z/b} = dn(i\psi) = \frac{dn(\psi, k')}{cn(\psi, k')}$$

$\therefore$ $y = 0, \pm\pi b, \pm 2\pi b, \ldots$ x ranges from 0 to $+\infty$

When $\phi = K$, $e^{z/b} = dn(K+i\psi) = k'\frac{cn(\psi,k')}{dn(\psi,k')}$

$\therefore$ $y = 0, \pm\pi b, \pm 2\pi b, \ldots$ x ranges from $-\infty$ to x_1 where $e^{-x_1/b} = k'$.

$\therefore$ (10) solves problem of two sets of par^l, equidistant semi-inf plates, the 2^nd set being in the same planes as the 1^st set



$$\text{Cap} = \frac{1}{2\pi}\,\frac{K'}{K}.$$

When $x_1 \gg b$, $k' \sim 1$

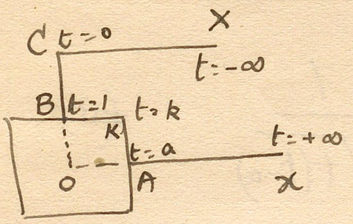
$$K' = \frac{\pi}{2}, \quad K = \log(4/k') = \log 4 + \frac{x_1}{b}$$

$$\therefore \text{ Capacity} = \frac{b}{4(x_1 + b\log 4)}.$$

D. Problems involving Gratings

1) Plane Grating with rectangular bars:

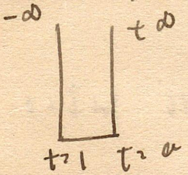
$$A\frac{dz}{dt} = \frac{(t-k)^{1/2}}{(t-a)^{1/2}(t-1)^{1/2}t^{1/2}} \qquad (1)$$



$a>k>1.$

If $k \to a$ or 1 the rectangle flatten into st. lines either all par^l to Ox or all lying along OY.; the bars of the grating approximate to strips of finite breadth & zero thickness. Let $w = \phi + i\psi$, (ϕ = const equipot^{ls}, ψ = const lines of force). Then we have 3 cases:-

a) The bars of the grating carry equal charges: Then AKB is part of an equipotential (say $\phi = 0$) & Ax, BCX are lines of force ($\psi = 0$ & $\psi = \pi/2$). The diagram in the w-plane is as shown. Hence

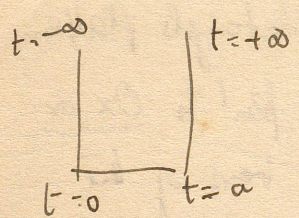


$$B_1\frac{dw}{dt} = \frac{1}{\sqrt{(t-1)(t-a)}} \qquad (\cancel{2})$$

with $w = 0$ when $t = a$ & $w = \frac{1}{2}\pi i$ for $t = 1$

Hence $$t = a\cosh^2 w - \sinh^2 w \qquad (2)$$

b) The uncharged grating placed in a uniform field of force par^l to Ox: - Here $AKBC$ is equipot^l ($\phi = 0$) & Ax, CX are lines of force ($\psi = 0$ & $\psi = \pi/2$). Hence

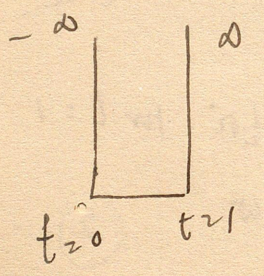


$$B_1 \frac{dw}{dt} = \frac{1}{\sqrt{t(t-a)}} \quad \cancel{(3)}$$

with $w = 0$ for $t = a$ & $w = \frac{1}{2}\pi i$ for $t = 0$.

$$\therefore \quad t = a \cosh^2 w \qquad (3)$$

c) Uncharged grating placed in a uniform field par^l to Oy. - Take ϕ = const as lines of force & ψ = const as equipot^ls. Here $BKAx$ & CX are equipot^ls ($\psi = 0$ & $\psi = \pi/2$) & BC is a line of force ($\phi = 0$)



$$\therefore \quad B_3 \frac{dw}{dt} = \frac{1}{\sqrt{t(t-1)}}$$

with $w = 0$ for $t = 1$, $w = \frac{1}{2}\pi i$ for $t = 0$

Hence $\quad t = \cosh^2 w \quad$ -- (4)