$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

$$x = r\sin\alpha\cos\beta - a\sin\alpha\sin\beta$$

$$y = r\sin\alpha\sin\beta + a\sin\alpha\cos\beta$$

$$z = r\cos\alpha$$

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1.$$

$$ds^2 = dt^2 - (dr - a\sin^2\alpha\, d\beta)^2 - (r^2 + a^2\cos^2\alpha)(d\alpha^2 + \sin^2\alpha\, d\beta^2)$$

Replace r by u ~~wrt~~ $r = (u-t)$

$$ds^2 = 2(du + a\sin^2\alpha\, d\beta)dt - (du + a\sin^2\alpha\, d\beta)^2 - (r^2 + a^2\cos^2\alpha)(d\alpha^2 + \sin^2\alpha\, d\beta^2)$$

$$r = u - t$$

$$ds^2 = 2(du + g\sin\alpha\, d\beta)dt - 2L(du + g\sin\alpha\, d\beta)^2 - M^2(d\alpha^2 + \sin^2\alpha\, d\beta^2)$$

$$g = g(\alpha), \quad L = L(u,t,\alpha) \quad M = M(u,t,\alpha) \qquad \text{I}$$

Choose real tetrads

$$\theta^1 = du + g\sin\alpha\, d\beta, \quad \theta^2 = M\,d\alpha, \quad \theta^3 = M\sin\alpha\, d\beta$$

$$\theta^4 = dt - L\theta^1$$

$$ds^2 = 2\theta^1\theta^4 - (\theta^2)^2 - (\theta^3)^2 = g_{(ab)}\,\theta^{(a)}\theta^{(b)}.$$

[Vaidya et al. Gen. Rel. Grav. 16, 355 (1976)] gives $R_{(ab)}$, ~~R~~

$$R_{(ik)} - \tfrac{1}{2} g_{ik} R = -8\pi T_{ik}$$

$$T_{ik} = \mu l_i l_k + (p+\rho)(l_i n_k + l_k n_i) - p g_{ik}$$

$$l_i l^i = 0, \quad n_i n^i = 0 \quad l_i n^i = 1.$$

Tetrad components ~~$l_{(a)} = (0, 0$~~, $l_{(a)} = (1, 0, 0, 0)$, $n_{(a)} = (0, 0, 0, 1)$

Then $R_{(11)} = -8\pi\mu$, $R_{(22)} = R_{(33)} = -8\pi p$, $R_{(14)} = -8\pi \rho$

and $R_{(23)} = R_{(12)} = R_{(13)} = R_{(24)} = R_{(34)} = R_{(44)} = 0.$

For the general metric I, if $R_{(12)} = R_{(13)} = R_{(24)} = R_{(34)} = 0$

then $R_{(44)}$ is a function of t only

$R_{(14)} - h R_{(44)}$ is also a function of t only.

So the metric is likely to represent Cosmological solutions

<u>Field Equations</u> $R_{(23)}$ is identically zero.

$R_{(12)} = R_{(13)} = R_{(24)} = R_{(34)} = R_{(44)} = 0 \Longrightarrow$ a simple solution

$$g = a \sin\alpha, \quad M^2 = (r^2 + a^2\cos^2\alpha)\frac{t^3}{b} \qquad r = u + \frac{b}{t}$$

$$L = A - \frac{2Br}{r^2 + a^2\cos^2\alpha} \qquad A = A(t), \quad B = B(t) \text{ with}$$

$$2B\frac{dB}{dt} + \frac{Ab}{t^2} = \frac{1}{2}.$$

If we make a change of time coordinate from t to $T = -\frac{b}{t}$, the final solution metric is

$$ds^2 = \frac{b}{T^2}\Big[2(du + a\sin^2\alpha\, d\beta)\,dT - (r^2 + a^2\cos^2\alpha)(d\alpha^2 + \sin^2\alpha\, d\beta^2)$$

$$- \left(\lambda - \frac{2mr}{r^2 + a^2\cos^2\alpha}\right)(du + a\sin^2\alpha\, d\beta)^2\Big]$$

$$m = m(T) \qquad \lambda = \lambda(T) \qquad \frac{dm}{dT} = 1 - \lambda + \frac{2m}{T}$$

If $m = 0$, $\lambda = 1$ and our metric represents an expanding Einstein de Sitter Universe.

Particular Solution

Take $m(T) = l\,T^n$ then $\lambda(T) = 1 - l(n-2)\,T^{n-1}$

$$8\pi p = -\frac{3}{b} + \frac{l(n-2)(n-3)(n-4)\,T^{n-1}}{2b}$$

$$8\pi \rho = \frac{3}{b} + \frac{l(n-2)(n-4)}{b}\,T^{n-1} + \frac{l(n-4)\,T^n}{b(r^2 + a^2\cos^2\alpha)}\left[T - r(1-n)\right]$$

$$8\pi \mu = \frac{-l\,(n-1)\,T^{n-2}}{r^2 + a^2\cos^2\alpha}\left[2T + (n-2)\,r\right]$$

$n = 4$ is a simple case $\quad p + \rho = 0$

$$8\pi \mu = \frac{6 l\, T^2 (T + r)}{r^2 + k^2\cos^2\alpha}.$$

Kerr metric $d\sigma^2 = 2(du + b\sin^2\alpha\, d\beta)\,dt$

$$-\left(1+\frac{2mr}{r^2+y^2}\right)(du + b\sin^2\alpha\, d\beta)^2 - (r^2+y^2)(d\alpha^2+\sin^2\alpha\, d\beta^2)$$

$$r = t-u \qquad y = -b\cos\alpha \qquad \text{(K)}$$

If this source moves with a vel. V along z-axis

$$d\sigma^2 = 2\left(du + \frac{b\sin^2\alpha}{(1-V\cos\alpha)^2}d\beta\right)dt - \left(1-V^2+\frac{2mr}{r^2+y^2}\right)\left(du+\frac{b\sin^2\alpha}{(1-V\cos\alpha)^2}d\beta\right)^2$$

$$-\frac{(r^2+y^2)}{(1-V\cos\alpha)^2}(d\alpha^2+\sin^2\alpha\, d\beta^2)$$

$$y = -b\cos\alpha \qquad r = t-(1-V^2)u$$

Lorentz transformations with $b' = \frac{b}{1-V^2}$ $y' = \frac{y}{V(1-V^2)}$,

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$$d\sigma^2 = \frac{b}{T^2}\Big[2(du + a\sin^2\alpha\, d\beta)\,dT - (r^2+a^2\cos^2\alpha)(d\alpha^2+\sin^2\alpha\, d\beta^2)$$

$$-\left\{1-2lT^3\left(1+\frac{rT}{r^2+a^2\cos^2\alpha}\right)\right\}(du+a\sin^2\alpha\, d\beta)^2\Big]$$

satisfies $R_{ik} = \Lambda g_{ik} - 8\pi\mu\, l_i l_k$; $l_i l^i = 0$

with $\Lambda = \frac{3}{b}$.

So it represents expanding de Sitter Universe exhibitting rotation

$$d\sigma^2 = dt^2 - (dx^2 + dy^2 + dz^2) = dt^2 - dr^2 - r^2(d\alpha^2 + \sin^2\alpha\, d\beta^2)$$

Retarded time $u = t - r$. Keep t as it is, replace r by u

$$d\sigma^2 = 2\,du\,dt - du^2 - r^2 \cancel{(\sin}(d\alpha^2 + \sin^2\alpha\, d\beta^2) \qquad (M)$$

$$r = t - u$$

Gravitational field of a mass particle

$$d\sigma^2 = 2\,du\,dt - \left(1 + \frac{2m}{r}\right)du^2 - r^2(d\alpha^2 + \sin^2\alpha\, d\beta^2) \qquad (S)$$

$$r = t - u$$

Gravitational field of a tachion

$$d\sigma^2 = 2\,du\,dt - 2L\,du^2 - M^2(d\alpha^2 + \sin^2\alpha\, d\beta^2)$$

$L = L(r, \alpha)$, $M = M(r, \alpha)$ $\quad r = t - ku$

solve $R_{ik} = 0$ find L, M ~~and~~ r.

$$d\sigma^2 = 2\,du\,dt - \left(1 - V^2 + \frac{2m}{r}\right)du^2 - \frac{r^2}{(1 - V\cos\alpha)^2}(d\alpha^2 + \sin^2\alpha\, d\beta^2)$$

$$r = t - (1 - V^2)u$$

If $V < 1$, Lorentz transformations

$$\beta' = \beta, \quad \cos\alpha' = \frac{\cos\alpha - V}{1 - V\cos\alpha} \quad u' = u\sqrt{(1 - V^2)} \quad t' = \frac{t}{\sqrt{(1 - V^2)}}$$

$$r' = r/\sqrt{(1 - V^2)} \quad m' = \frac{m}{(1 - V^2)^{3/2}}$$