twin primes

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Date .... 195 .

1, 2, 3, 4, 5, 6, 7, 8, 9, 10
11, 12, 13, 14, 15, 16, 17, 18, 19, 20
21, 22, 23, 24, 25, 26, 27, 28, 29, 30
31, 32, 33, 34, 35, 36, 37, 38, 39, 40
41, 42, 43, 44, 45, 46, 47, 48, 49, 50
51, 52, 53, 54, 55, 56, 57, 58, 59, 60
61, 62, 63, 64, 65, 66, 67, 68, 69, 70
71, 72, 73, 74, 75, 76, 77, 78, 79, 80
81, 82, 83, 84, 85, 86, 87, 88, 89, 90
91, 92, 93, 94, 95, 96, 97, 98, 99, 100

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37
41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

25 primes

$4.6 \overline{)\,100}$ 92 80 46 360 334

$\frac{100}{2 \times 2.3026}$

$\frac{100}{\cancel{2 \times 2}} = 25$

$\frac{n}{\log n}$

3 $\frac{4}{\log n}$

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Heaviside function $Y(x)$ which vanishes for values of x not exceeding zero and equal to 1 for positive x $(\geqslant 0)$ – Heaviside calculus – Schenectady Engineers – Operational calculus based on Laplace transformations, rigorous no doubt, but alters the whole question, eliminates functions like $Y(x)$, $\delta(x)$ & has nothing to say about the success of the operational methods.

$Y(x)$ is said to ~~be~~ have the derivative $\delta(x)$ [Dirac] which has the following mathematically impossible properties: it vanishes everywhere except at the origin where its value is so large that

$$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$$

δ and its derivatives have been used with considerable success, – as also relativistic generalisations like $\Delta(x,t)$. δ can be kept and made rigorous by defining it as a measure ie as a set-function in place of an ordinary function. The notion of point-function can be enlarged to include new entities ~~like~~ (distributions & also notion of derivative can be generalised so that in the new system of entities every point function has a rigorously defined derivative. Distributions include all continuous functions, all Lebesgue locally summable functions, and new objects of which a simple example is the Dirac measure function – Distribution is more general than a measure function, thus δ' is not a measure function but a distribution.

Theory of distributions rigorously justifies operational calculus. It can be extended to several variables and provides more complete theory of Fourier series & integrals, convolutions and partial differential equations

Derivative of a point function may be a point function or a Stieltjes measure or a more general distribution. It will be the first if $f(x)$ be a A.C. on every finite closed (a, b) contained in I, and will be the second if & only if $f(x)$ is of bounded variation on every finite closed (a, b) in I – Rigorously $\delta = Y' - \int_c^d f(x)\,dx$ represents in physical problems total charge with $f(x)$ as density. Thus Dirac δ represents concentration of unit charge at a single point, δ' represents a dipole & δ'' etc more complicated multiple layers.

If $f(x) = x^{-\frac{1}{2}}$ for positive x, and $= 0$ for all other x, then f' exists as a distribution although not as a Stieltjes measure.

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Roughly f' corresponds to negative mass continuously distributed on the positive x-axis with an infinite quantity in every neighbourhood of the origin, together with an infinite $+^{ve}$ mass at the origin in such a way that there is a finite total algebraic mass on every finite closed interval.

$$\text{l.f. } F(\phi) = \lim_{\epsilon \to 0} \left[\left(\int_a^{-\epsilon} + \int_\epsilon^b \right) \frac{1}{x} \phi(x)\,dx \right]$$

Corresponds to continuously distributed mass with infinite positive mass along $+^{ve}$ x-axis together with infinite negative mass along the $-^{ve}$ x-axis in such a way that in every neighbourhood of the origin there is a finite total algebraic mass. This distribution is the derivative of the point function $\log|x|$.

$F_1 F_2$ is <u>not</u> defined for arbitrary distributions, but only in certain special cases. Eg. Dirac δ & its $\delta^{(n)}$ multiplied by an $\alpha(x)$ with $\alpha^{(n)}$ [$\alpha(x)$ being a pt. fn with derivatives all point fns]. give

$$\alpha(x)\cdot\delta = \alpha(0)\,\delta$$

$$\alpha(x)\,\delta' = \alpha(0)\,\delta' - \alpha'(0)\,\delta \text{, etc.}$$

order classification of distributions — Hahn-Banach procedure — F. Riesz representation theorem.

Continuity & convergence properties of distributions — If F_m converges to F, then F_m' converges to F' — If $F_t \to F$ as t (parameter) $\to t_0$ ie for each ϕ the c.l.f. $F_t(\phi) \to F(\phi)$ as $t \to t_0$, then $F_t' \to F'$.

Thus from $f(x) = \lim_{t\to\infty} \int_1^t - \frac{\cos wx}{w^2} dw$, differentiating twice in the sense of distribution

$$f'' = \lim_{t\to\infty} \int_1^t \cos wx \cdot dw$$

ie $\int_1^\infty \cos wx \, dw$ is a distribution although not a convergent integral in the usual sense. Now put

$$g_t(x) = \int_0^t 2\cos 2\pi wx \cdot dx = \frac{\sin 2\pi t x}{\pi x}$$

& observing that $g_t(\phi) = \int_{-\infty}^{\infty} \frac{\sin 2\pi t x}{\pi x} \phi(x)\, dx \to \phi(0)$ as $t \to \infty$

(Dirichlet integral of Fourier series).

we see that $g_t \to \delta$. Thus $2\int \cos 2\pi wx \cdot dx = \delta$ a formula indiscriminately used in electricity & in wave mechanics.

Nullity sets & supporting sets to define "local" properties of distributions -

A set x_0 in I is nullity set of F on I if $F_{cd} = 0$ for $c < x_0 < d$.

Complement of x_0 is supporting set (closed) - A distribution has a supporting set consisting of one point only, the origin, if and only if the distribution is a finite linear combination of the Dirac δ & its derivatives

Differential eqns involving distributions - Distn in several variables -

Examples: (i) Generalisation of Dirac δ ie $F(\phi) = \phi(x_0)$ for fixed x_0. It can be identified with n-dimensional Stieltjes measures $d_{x_1} \cdots d_{x_n} Y(x_1 - x_n)$ where $Y(x)$ is the Heaviside function. $\underline{\delta}$ corresponds to unit mass at x_0.

(ii) Let $Y(x) = 1$ for x inside an n-dimensional set J having a hypersurface H or on H and $= 0$ for all other x. Then $\frac{\partial Y}{\partial x_1}$ as a distn corresponds to a surface distribution of mass on H with surface density at a point of H equal to $\cos \theta_1$ ($\angle^{le}$ bet. normal to H & $+^{ve}$ x_1 axis)

(iii) The Laplacian distribution ΔF consists of 3 terms & this expn itself is Green's formula - Similar formulas of Gauss & Stokes can be interpreted

(iv) Fundamental soln of Laplace's eqn: $f(x) = \frac{1}{r^{n-2}}$ $(n \geqslant 3)$ and

$f(x) = \log\left(\frac{1}{r}\right)$ if $n = 2$. $f(x)$ can be considered a distribution. We find that the Laplacian distn is not the identically zero distn, it turns out to be $-N\delta_0$ & $N = (n-2)\, 2\pi^{n/2} / \Gamma(n/2)$

(∅)

convolutions — If δ_x, δ_y are Dirac δ's in x & y spaces, $\delta_x \times \delta_y$ is Dirac δ in (x, y) space. — Potential

$$U^f = \int \frac{f(t)}{|x-t|^{n-2}}\,dt$$

and in distributions $U^T = T * \frac{1}{r^{n-2}}$

giving Poisson's formula $\Delta U^T = -NT$

Fourier coeffts for distns defined by

$$\alpha_k(T) = T \cdot e^{-2\pi i k x}$$

Soln of $\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} = 0$ in the form $V = f(x+y) + g(x-y)$ only if f & g be twice differentiable — General defn of soln has been given by several others & can be made to depend on distributions

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In algebraic topology de Rham's "currents" bear relation to idea of distributions.

Generalised surfaces of L. C. Young etc. are all parents of distn idea

L. Schwartz – Les mathématiques en France pendant et après la guerre – Proc. Canad. Math. Cong. 1949 (1

—

Les équations d'évolution liées au produit de composition – studies certain partial & integro-diff. eqns from the viewpoint of theory of distributions. (1951)

—

Analyse et synthèse harmoniques dans les espaces de distributions (1952)

—

Homomorphismes et applications complètement continues (1953)

—

Sur les multiplicateurs de FL^p (1953). i.e. collection of all distns that are Fourier transforms of elements of L^p on R^n

—

Transforms de Laplace des distributions (1952) – Systematic development of theory of Laplace transformations of distns on the lines of Bochner for fns –

—

~~Theory~~ ~~Theo~~ Théorie des noyaux – Int. Congr. Camb. 1950 – Exposition of mappings between spaces of distributions on euclidean space.

General surface theory of Dirac (Instant Koog) $-\frac{\partial \psi}{\partial \sigma} = H\psi$ with functional differentiation — Tomonaga's method — renormalization ∞'s absorbed in δe & δm — Bare electron & e.m.f. — Dirac's idea

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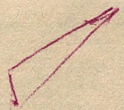
Riez Singularity, Hadamard's criticism — $\delta = \alpha$ on the ring & $\iint \delta\, d\rho\, dz = 1$

Award of Fields medals (1924 instituted) — Committee in 1950 was: H. Bohr, Ahlfors, Borsuk, M. Frechet, Hodge, Kolmogroff, Kosambi & Morse. — Schwartz & Selberg



$$\frac{c}{\sin 45^\circ} = \frac{AP}{\sin(\theta+45^\circ)}$$

$180^\circ - (A-\theta+45^\circ)$

$$AP = \frac{c\sin(\theta+45^\circ)}{\sin 45^\circ} = c(\sin\theta+\cos\theta).$$

$$\frac{b}{\sin 45^\circ} = \frac{AP}{\sin(A-\theta+45^\circ)}$$

$AP = b\{\sin(A-\theta)+\cos(A-\theta)\}$

$$c(\sin\theta+\cos\theta) = b(\sin A\cos\theta - \cos A\sin\theta + \cos A\cos\theta + \sin A\sin\theta)$$

$$c(\sin\theta+\cos\theta) = b\{\sin\theta(\sin A-\cos A) + \cos\theta(\sin A+\cos A)$$

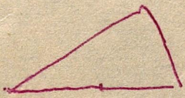
$$c(1+\tan\theta) = b\{(\sin A-\cos A)\tan\theta + (\sin A+\cos A)$$

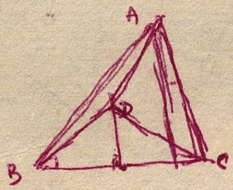
$$c\tan\theta - b(\sin A-\cos A)\tan\theta = b(\sin A+\cos A) - c.$$

$$\tan\theta\{c - b(\sin A - \cos A)\} = b(\sin A+\cos A) - c$$

$$\frac{AD}{c} = \qquad \tan\theta = \frac{b(\sin A+\cos A)-c}{c-b(\sin A-\cos A)} = \frac{b(\sin A+\cos A)-(a\cos B+b\cos A)}{a\cos B+b\cos A-b(\sin A-\cos A)}$$

$BD = a/\sqrt{2}.$





$$AD^2 = AB^2+BD^2-2AB\cdot BD\cos(B-45^\circ)$$

$$= c^2+\frac{a^2}{2}-2c\cdot\frac{a}{\sqrt2}(\cos B\cos 45^\circ+\sin B\sin 45^\circ)$$

$$= c^2+\frac{a^2}{2}-ca(\cos B+\sin B).$$

$$= b^2+\frac{a^2}{2}-ab(\cos C+\sin C)$$

$b^2 = c^2+a^2-2ca\cos B$

$$2AD^2 = 2c^2+a^2-2ca\cos B - 2ca\sin B.$$

$$= \cancel{2c^2+a^2+b^2-c^2-a^2-2ca\sin B}$$

$$\cancel{= c^2+b^2-2ca\sin B}$$

$c^2-b^2-2ca\sin B$

$b^2-c^2-2ba\sin C,$

$\cancel{a\sin A} = c\sin B$

$$\frac{AD}{\sin(A-45^\circ)} = \frac{BD}{\sin\theta}$$

If side of square $= x$

b^2+c^2-

$$x = AP/\sqrt2 = \frac{bc\cos A}{AD\cdot\sqrt2}$$

$2ca\cos B = c^2+a^2-b^2$

$$AP\cdot AD = \frac{bc\cos A}{AD}$$

$(c-b)(c+b) - 4\sqrt{s(s-a)(s-b)(s-c)}$

$(2s-2a)$

$a+b+c = 2s.$

$c+b-2b.$

$2s-2a-2b$

$2s-2(a+b)$

$2s-4(2s-2c)$

$8c-6s,$

$$\frac{b^2+c^2-a^2}{2bc} = \cos A$$

$$2AD^2 = 2c^2+a^2$$

$$= 2c^2+a^2-(c^2+a^2-b^2)-2ca\sin B = b^2+c^2-2bc\sin A.$$

$$= b^2+c^2-2ca\sin B$$

$$= \cancel{b^2+c^2-4A}$$

$$= bc\cos A.$$

$$AD\cdot AP = bc\cos A.$$

$$AP = \frac{bc\cos A}{AD}.$$

$$x = \frac{bc\cos A}{AD\sqrt2} = \frac{bc\cos A}{\sqrt{b^2+c^2-2bc\sin A}}$$

$x = \dfrac{bc\cos A}{\sqrt{b^2+c^2-2bc\sin A}}$. $AL = \dfrac{bc\cos A}{AP} = \sqrt{b^2+c^2-2bc\sin A}$.

$AL^2 = b^2+c^2-2bc\sin A$

$b^2+c^2-2bc\sin A$.

$b^2+c^2-4\sqrt{s(s-a)(s-b)(s-c)}$

$AL^2 = b^2+c^2-2bc\sin A = \dfrac{b^2c^2-4\Delta}{s(s-a)(s-b)(s-c)}$

$\cancel{(b+c)^2}^2$

$b^2\cancel{(b+c)^2}-2bc$. AL

$AP \cdot AD = bc\cos A$

$AL \cdot AR = bc\cos A$

$a^2 = b^2+c^2-2bc\cos A$

$a^2 - AL^2 = 2bc(\cos A - \sin A)$.

$\sin(90^\circ - A) - \sin A$

$2\cos 45^\circ \sin(45^\circ - A)$

$\frac{90^\circ - A + A}{2}$

$\frac{2}{\sqrt{2}}(\sin 45^\circ \cos A - \cos 45^\circ \sin A)$.

$\frac{90^\circ - A - A}{2}$

$\cos A - \sin A$.

$x' = \dfrac{bc\cos A}{AD'}$

$c^2 = a^2 + b^2 - 2ca\cos B$

$b^2 = c^2 + a^2 - 2ca\cos B$

$AP' \cdot AD' = AF \cdot AD = bc\cos A$

$AD'^2 = AB^2 + BD'^2 - 2AB \cdot BD' \cos(B+45^\circ)$

$= c^2 + \frac{a^2}{2} - 2c \cdot \frac{a}{\sqrt{2}}(\cos B \cos 45^\circ - \sin B \sin 45^\circ)$

$ca\sin B = bc\sin A$

$= c^2 + \frac{a^2}{2} - ca(\cos B - \sin B)$.

$AD'^2 = 2c^2 + a^2 - 2ca\cos B + 2ca\sin B$.

$= 2c^2 + a^2 - (c^2+a^2-b^2) + 2bc\sin A$.

$= b^2 + c^2 + 2bc\sin A$

$AP' = \dfrac{bc\cos A \cdot \sqrt{2}}{\sqrt{b^2+c^2+2bc\sin A}}$

$\cancel{AD'} = \dfrac{AD'}{\sqrt{2}\cdot\sqrt{b^2+c^2+2bc\sin A}}$

$x' = \dfrac{AP'}{\sqrt{2}} = \dfrac{bc\cos A}{\sqrt{b^2+c^2+2bc\sin A}}$

To the smaller square ~~the~~ ~~an analogous~~ $\Delta AL AM$.

$AL'^2 = c^2 + BL'^2 + 2c \cdot BL' \sin A$.

$\sin(90^\circ + A)$

$c^2 + b^2 + 2bc\sin A$

$= c^2 + BL'^2 + 2BL' \cdot c\sin A$

$(BL'^2 - b^2) = 2c\sin A(b - BL')$

$BL' + b = -2c\sin A$

$BL' + b = -2c\sin A$.

$BL' = -b - 2c\sin A$.

$= b - 2BF$

$BL' =$

To an analogous ΔLAM

Produce BF backwards

& make $BL' = b$

$AL'^2 = b^2+c^2-2bc\cos(90^\circ+A)$

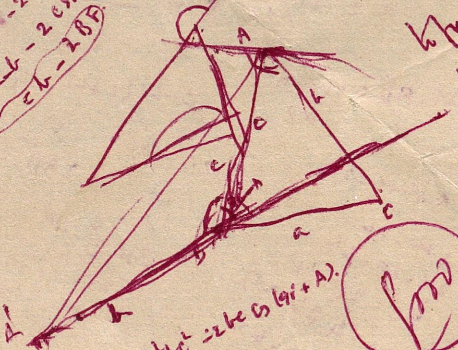
$= b^2+c^2+2bc\sin A$

Say CE backwards

to make $CM' = c$.

$CM'^2 = b^2+c^2+2bc\sin A$

$= AL'^2$



$b^2+c^2-2bc\cos(90^\circ+A)$.

(1) fundamental – ~~real~~ to rational to
real & real to complex

(2)

$$\frac{2+5i}{7-i} = (2+5i)(7+i)/50 = \frac{9}{50} + \frac{37i}{50}.$$

$$= \left(1 - \frac{41}{50}\right) + \left(1 - \frac{13}{50}\right)i = (1+i) - (41+13i)/50$$

$$2+5i = (1+i)(7-i) - (41+13i)(7-i)/50$$

$$= (1+i)(7-i) - (300+50i)/50$$

$$= \quad " \quad - \cancel{(29-2i)/5} - (6+i)$$

$$\frac{7-i}{(29-2i)/5} = \frac{5(7-i)(29+2i)}{(29-2i)(29+2i)} = \frac{(7-i)(29+2i)}{157}$$

$$= \frac{205-15i}{157}.$$

$$7-i = \frac{205-15i}{157} \cdot \frac{29-2i}{5}$$

$$= (41-3i)(29-2i)/157$$

$$\frac{7-i}{2+5i} = \frac{(7-i)(2-5i)}{29} = \frac{9}{29} - \frac{37i}{29}.$$

$$= \cancel{1 - \frac{10}{29}} - \cancel{\frac{(1+?)i}{29}} - \left(1 + \frac{8}{29}\right)i$$

$$\left(1 - \frac{10}{29}\right) - \left(1 + \frac{8}{29}i\right)$$

$$= (1-i) - (10+8i)/29$$

$$7-i = (1-i)(2+5i) + \frac{20-56i}{29}.$$

$$\begin{array}{r} 287 \\ 3 \\ \hline 290 \end{array}$$

$-\cancel{41}+21$

-20

$091 \cdot 41$

29×29

301

58

$\begin{array}{r} 287 \\ 13 \\ \hline 300 \end{array}$

781

4

785

157

203

2

205

29

14

$\cancel{i(7-i)} + 7-i$

$\cancel{-6+i}$

$-6-i$

$1-2i$

$14-5=9$

$$\frac{-20+56i}{29}$$

$$\frac{(2+5i)(10+8i)}{29}$$

$$\frac{(1-t^2)/1+t^2}{1+\frac{(1-t^2)^2}{(1+t^2)^2}} = \frac{(1-t^4)}{2(1+t^4)} = \frac{2\sin\alpha}{1} \qquad t^4 =$$

$$\frac{1}{t^4} = \frac{1+2\sin\alpha}{1-2\sin\alpha}$$

$$t^4 = \frac{1-2\sin\alpha}{1+2\sin\alpha} =$$

$$\cos A = \frac{\sin\delta - x\sin\phi}{\cos\phi\sqrt{1-x^2}} = \frac{\sin\phi(\sin\alpha - x)}{\cos\phi\sqrt{1-x^2}} = \tan\phi \cdot \frac{\frac{2t}{1+t^2} - t}{\sqrt{1-t^2}}$$

$$= \tan\phi \cdot \frac{t\sqrt{1-t^2}}{1+t^2} = \tan\phi \cdot \tan\frac{\alpha}{2} \cdot \sqrt{\cos\alpha} / \cos\frac{\alpha}{2} \cdot \sec^2\alpha/2$$

$$= \tan\phi \cdot \sin\frac{\alpha}{2}\sqrt{\cos\alpha}$$

— o —

$$2+5i = (1+i)(7-i) - (6+i)$$

$$\frac{7-i}{6+i} = \frac{(7-i)(6-i)}{37} = \frac{41}{37} - \frac{13i}{37} = \left(1+\frac{4}{37}\right) - \left(1-\frac{24}{37}\right)i$$

$$= (1-i) + (4+24i)/37$$

$$(7-i) = (1-i)(6+i) + 4(1+6i)(6+i)/37$$

$$= \quad " \quad + 4i$$

$$\frac{6+i}{4i} = \frac{(6+i)(-4i)}{16} = \frac{4-24i}{16} \qquad \frac{4i}{6}+1$$

$$6+i = \frac{4i(4-24i)}{16} = 4i\cdot\frac{1}{4} + 6 = \qquad \frac{4i+6}{6} - 1$$

$$\frac{2i+3}{2} - 1 \qquad \text{HCF} = 1.$$

$$\cancel{\frac{6+i}{6}} \quad \frac{4i}{16} = \frac{2i}{3} \quad \text{HCF}$$

$$2+5i = \frac{2i}{3}(a+bi) = -\frac{2b}{3} + \frac{2ai}{3} \qquad b=-3, \quad a = \frac{15}{2}.$$

$$7-i = \frac{2i}{3}(a+bi) = -\frac{2b}{3} + \frac{2ai}{3} \qquad \left.\begin{array}{l} b = \frac{21}{2} \\ a = -\frac{3}{2} \end{array}\right\}$$

$$\begin{pmatrix} \frac{1}{3} & \frac{16\sqrt{2}}{27} & 0 \\ \frac{2}{3} & -\frac{4\sqrt{2}}{27} & -\frac{1}{\sqrt{2}} \\ \frac{2}{3} & -\frac{4\sqrt{2}}{27} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\frac{2+5i}{7-i} = (2+5i)(7+i)/7^2+1^2 = \frac{9+37i}{50}$$

$$= \frac{9}{50} + \frac{37i}{50}$$

$$= \cancel{\left(1+\frac{1}{5}\right) + \left(1+\frac{18}{50}\right)}$$

$\frac{11}{50} - 1$

$11 - 50$

39

$$= \left(1 - \frac{41}{50}\right) + \left(1 - \frac{13}{50}\right) i$$

$$= (1+i) - (41 + 13i)/50$$

$$2+5i = (1+i)(7-i) - (41+13i)(7-i)/50$$

$$= (1+i)(7-i) - (290 + 50i)/50$$

$$= (1+i)(7-i) - \left(\frac{29}{5} + i\right)$$

273
13

$\cancel{286}$

287
13
290

91
-39
52

$$x^2+y^2 = (x+y)^2 - 2xy = \sigma_1^2 - 2\sigma_2$$
$$x^3+y^3 = (x+y^3)^3 - 3xy(x+y) = \sigma_3 - 3\sigma_2\sigma_1.$$

$\phi\beta^{-1}$ = even $(\phi\beta^{-1})\beta = A_n\beta$ = right coset of A_n

$T^{-1}A_nT = A_n$. normal subgroup

$$(2-9i)N = 85$$

$$(2-9i)(a+bi) = 2a+9b+i(2b-9a)$$

$$2a+9b = 85$$
$$2b-9a = 0$$
$$4b = 18a$$

$$18a+81b = 85$$
$$85b = 85 \quad b = 1$$
$$a = \frac{2}{9}$$

$$(3+11i)\left(\frac{2}{9}+i\right) = \frac{2}{3}+3i+\frac{22i}{9}-11$$

$$(2-9i)(2+9i) = 4+81 = 85$$
$$(3+11i)(3-11i) = 9+121 = 130$$

$$4 = (2+i)(2-i)^2 - (2-i)$$

$$\frac{4}{\cancel{22}\,2-i} = (2+i)$$

MAT-9/1974
13.8.1974

QUANTUM MECHANICS IN FINITE DIMENSIONS

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****

Abstract

We explicitly compute, following the method of Weyl, the commutator $[Q,P]$ of the position operator Q and the momentum operator P of a particle when the dimension of the space on which they act is finite. We show that in the limit of continuous spectrum this reduces to the usual relation of Heisenberg's.

Q. 7(v) – Fundamental existence & uniqueness theorem for space curves

Let $\kappa(s)$, $\tau(s)$ be arbitrary continuous fns on $a \le s \le b$. Then there exists, except for position in space, one & only one space curve (1)

Uniqueness: If κ, κ^*, τ & τ^* be different. At a point s_0 make (t_0^*, n_0^*, b_0^*) (tang., p. n & b. n) coincide with (t, n, b).

Using Frenet eqns

$$\frac{d}{ds}\left(t\cdot t^* + n\cdot n^* + b\cdot b^*\right) = 0$$

$$t\cdot t^* + \cdot \quad + \cdot = \text{Const}$$

but at s_0, $t_0 = t_0^*$, etc. so that $t_0\cdot t_0^* = \cdot \quad = \cdot = 1$. Thus at s_0 & all s

$$t\, t^* + \cdot \quad = 3$$

possible when $t\cdot t^* = 1$, $n\cdot n^* = 1$, $b\cdot b^* = 1$.

Hence for ~~all~~ all s, $t = t^*$, $n = n^*$ & $b = b^*$. Finally since $t = dx/ds = t^* = \frac{dx^*}{ds}$ it follows

that $x(s) = x^*(s) + \text{Const}$, but as s_0, $x(s_0) = x^*(s_0)$

$\therefore x(s) = x^*(s)$ for all s i.e. C & C^* coincide

Existence theorem – given in Appendix I of Schaum's series on Diff. Geom – rather complicated.

Q. 7(a) $\dot{\vec{t}}(s)$ = curvature vector on C at $x(s)$, $\vec{t}$ = tangent vector, $\left[\dot{t}(s) = \frac{dt}{ds}\right]$

$\dot{\vec{b}}(s) = -\tau(s)\,\vec{n}(s)$.

Serret-Frenet eqns:

$$\left.\begin{aligned} \dot{\vec{t}} &= \kappa\vec{n} \\ \dot{\vec{n}} &= -\kappa\vec{t} + \tau\vec{b} \\ \dot{\vec{b}} &= -\tau\vec{n}. \end{aligned}\right\}$$

Q. 8(a). $\vec{x} = \vec{x}(u,v)$

$$\vec{x}(u+du, v+dv) = \vec{x}(u,v) + d\vec{x} + o\left((du^2+dv^2)^{1/2}\right)$$

i.e. $d\vec{x}$ is a 1st order approxn to the vector $\vec{x}(u+du, v+dv) - \vec{x}(u,v)$ from joining the two neighbouring points

1. Introduction

Weyl (1931) has shown that the Schroedinger representation for the momentum operator is a necessary consequence of Heisenberg's commutation relation. He proves this using the ray representations of the Abelian group of rotations. In proving this he uses an ingenious limiting process to go from finite rotations in ray space to a 2-parameter continuous group. More recently, Alladi Ramakrishnan (1972) and his collaborators have studied exhaustively the representation theory of Generalized Clifford Algebra which immediately furnishes the ray representations of the Abelian group of rotations.

In this paper we derive, by limiting to the case of finite dimensions, the explicit expression for the commutator $[Q, P]$ where Q and P are the position and momentum operators respectively. We show that by going to the limit of continuous parametrisation (valid as the dimension goes to infinity) we recover the standard Heisenberg commutation relations.

We believe that this work will open up the possibility of studying quantum mechanics in finite dimensions.

2. Weyl's form of the Heisenberg Relations.

Suppose A and B are two elements of the Abelian group of unitary rotations on a ray space so that

$$AB = \omega BA \qquad (2.1)$$

Conversi

$1 = d\vec{x}\cdot d\vec{x} = (x_u\,du + x_v\,dv)\cdot(x_u\,du + x_v\,dv)$

$= (\vec{x}_u\cdot\vec{x}_u)\,du^2 + 2(\vec{x}_u\cdot\vec{x}_v)\,du\,dv + (\vec{x}_v\cdot\vec{x}_v)\,dv^2$

$= E\,du^2 + 2F\,du\,dv + G\,dv^2$

is the 1st fundamental form

ie $I(du, dv) = E\,du^2 + 2F\,du\,dv + G\,dv^2$

If $dx = x_u\,du + x_v\,dv$ & $\delta x = x_u\,\delta u + x_v\,\delta v$ are two vectors $\parallel^l$ to tangent plane at $\vec{x}$,

If α = angle between dx & δx, then

$$\cos\alpha = \frac{dx\cdot\delta x}{|dx||\delta x|} = \frac{E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v}{[E\,du^2 + 2F\,du\,dv + G\,dv^2]^{1/2}\,[E\,\delta u^2 + 2F\,\delta u\,\delta v + G\,\delta v^2]^{1/2}}$$

In particular, if β = angle between the u & v – parameter curves at $\vec{x}$ ie angle between $\vec{x}_u$ & $\vec{x}_v$, then $\cos\beta = \dfrac{\vec{x}_u\cdot\vec{x}_v}{|x_u||x_v|} = \dfrac{F}{\sqrt{EG}} = \dfrac{F}{\sqrt{EG}}$.

(a) Tangent directions $d\vec{x}$ & $\delta\vec{x}$ are $\perp^r$ if & only if

$$E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v = 0$$

(b) u & v – curves are $\perp^r$ if & only if $F = 0$.

2.5 (a) Scalar field $\phi = \phi(x,y,z)$ for each point of a region R.
scalar field ϕ defined in R

Vector field – If to each (x,y,z) of R there corresponds $\vec{V}(x,y,z)$. ie Vector field $\vec{V}$ defined in R

Gradient $\nabla\phi = \left(\vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial y} + \vec{k}\dfrac{\partial}{\partial z}\right)\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \cdot$

Component of $\nabla\phi$ in direction of unit vector $\vec{a}$ is given by $\nabla\phi\cdot\vec{a}$ = directional derivative
ie rate of change of $\phi(x,y,z)$ at (x,y,z) in the direction $\vec{a}$

(b) $\nabla(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} + (\vec{A}\cdot\nabla)\vec{B} + \vec{B}\times(\nabla\times\vec{A}) + \vec{A}\times(\nabla\times\vec{B})$.

where ω is a primitive n^{th} root of unity. By iteration we get

$$A^k B^l = \omega^{kl} B^l A^k \tag{2.2}$$

from which it follows that A^n commutes with B and B^n commutes with A and if the representation is irreducible, it follows from Schur's lemma that

$$A^n = I \ , \ B^n = I . \tag{2.3}$$

We take the following representations for A and B (Weyl, 1931):

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{n-1} \end{bmatrix} \tag{2.4}$$

The interesting properties of the algebra satisfied by operators like A and B which is a generalization of the usual Clifford algebra have been systematically studied by Alladi Ramakrishnan (1972) and collaborators. If one identifies

$$A = e^{i\xi P} \ , \ B = e^{i\eta Q} \tag{2.5}$$

where ξ, η are arbitrary real parameters, then it follows that eqn. (2.1) is the Weyl form of the Heisenberg commutation relation $[Q, P] = i$, if we allow power series expansion of

6 (a)

$$\iint_S (\nabla \times \vec{A}) \cdot \vec{n} \, dS$$ (3)

$$= \oint_C \vec{A} \cdot d\vec{r}$$ [Stokes theorem]

(b) Area of a simple closed curve is given by $\frac{1}{2}\oint(x\,dy - y\,dx)$ (*)

For ellipse area $= \frac{1}{2}\oint_C (x\,dy - y\,dx) = \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(b\cos\theta)\,d\theta - (b\sin\theta)(-a\sin\theta)\,d\theta$

$[x = a\cos\theta, y = a\sin\theta]$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta) = \pi ab$$

(*) This is got by using Greens theorem in the plane viz

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

by putting $M = -y$, $N = x$, then

$$\oint_C (x\,dy - y\,dx) = \iint_R \left\{\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right\} dx\,dy = 2\iint dx\,dy = 2A$$

So the Q 6 (c) is really use of Greens theorem and Stokes' theorem

But Greens theorem in the plane is a particular case of Stokes' theorem since the former can be written as

$$\oint_A A \cdot d\vec{r} = \iint_R (\nabla \times \vec{A}) \cdot \vec{k} \, dR.$$

Diff. eqn

$y'' + 4y = 0$ has roots $y_1 = \sin 2x$, $y_2 = \cos 2x$.

$\therefore$ Wronskian $W(\sin 2x, \cos 2x) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x, & -2\sin 2x \end{vmatrix} = -2 \neq 0$

ie they are independent.

operator exponentials (which is justified if A is bounded but not otherwise) (P.Carter, 1966). Weyl takes the limit $n \to \infty$ such that $\xi\eta n = 2\pi$ to show that

$$P = \frac{1}{i}\frac{\partial}{\partial q} \tag{2.6}$$

3. Case of Finite Dimensions

We now solve eqn.(2.5) for P and Q by taking logarithms. We solve the problem that, given

$$e^{i\xi P} = A, \quad e^{i\eta Q} = B, \tag{3.1}$$

where ξ and η are arbitrary real parameters and

$$AB = \omega BA, \quad \omega^n = 1 \tag{3.2}$$

to compute the commutator $[Q, P]$ and show that

$$[Q, P] = i \tag{3.3}$$

in the continuous limit.

We take

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{n-1} \end{bmatrix} \tag{3.4}$$

Coddington's Diff Eq^ns (4)

p. 92 Ex 1(c) use annihilator method to find particular soln

Q.3(b). $y'' - 4y = 3e^{2x} + 4e^{-x}$ (~~[illegible]~~)

Ans. $\psi(2x) = \frac{3x}{4} e^{2x} - \frac{4}{3} e^{-x}$.

[For function e^{ax}, characteristic polynomial of an annihilator is $r-a$]

p. 79. Q.3(a) of Q. paper.

Three solns are $\phi_1(x) = 1$, $\phi_2(x) = e^{2x}$, $\phi_3(x) = e^{-2x}$

Wronskian: $W(\phi_1, \phi_2, \phi_3) = 16 \neq 0$.

Required $\phi(x)$ satisfying given initial cond^s is

$$\phi(x) = \frac{1}{2}\sinh 2x. = \tfrac{1}{4}(e^{2x} - e^{-2x})$$

Q.1(a) of paper is Q 2.3, Chap. 4, sec 7, p. 1,5 of Coddington

Soln satisfying $\phi(2) = 2\phi(1)$ is given by $\phi(x) = \frac{1}{x} - \frac{6}{7x^2}$

[For $y' + a(x)y = b(x)$ — (1)

Let A be a fn such that $A' = a$. Then the fn ψ given by

$$\psi(x) = e^{-A x}\int_{x_0}^{x} e^{A(t)} b t \, dt \quad \text{on interval } I.$$

where x_0 is in I. is a soln of the eq^n (1)

$\phi_1(x) = e^{-A(x}$ is a soln of the homogeneous eq

$y' + a(x)y = 0$.

If c any constant. $\phi = \psi + c\phi_1$ is a soln of (1) & every soln of (1) has this form.

Ex. solve $y' + (\cos x)y = \sin x \cos x$

Here $a(x) = \cos x$, $b(x) = \sin x \cos x$ & a choice for A is $A(x) = \sin x$

Thus if ϕ is any soln $(e^{\sin x}\phi)' = e^{\sin x}\sin x \cos x$

& integration gives $e^{\sin x}\phi(x) = (\sin x - 1)e^{\sin x} + c$

or $\phi(x) = (\sin x - 1) + ce^{-\sin x}$

The diagonal form of A is given by B, for

$$S^{-1} A S = B \tag{3.5}$$

where S is the sylvester matrix

$$S = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ \vdots & & & & \\ 1 & \omega^{n-1} & \omega^{n-2} & \cdots & \omega \end{bmatrix}, \quad S^{-1} = S^{+} . \tag{3.6}$$

we then have, taking logarithms

$$i \xi P = \log A = S (\log B) S^{-1} \tag{3.7}$$

$$i \eta Q = \log B . \tag{3.8}$$

where $\log B$ is given by

$$\log B = (\log \omega) \begin{bmatrix} 0 & & & & \\ & 1 & & 0 & \\ & & 2 & & \\ & 0 & & \ddots & \\ & & & & n-1 \end{bmatrix} \tag{3.9}$$

Since A and B are diagonalizable and non-singular it follows (Gantmacher, 1959) that log A and log B exist. Elementwise, labelling the rows and columns from 0 to n-1,

$$B_{rs} = \omega^{r} \delta_{rs} \tag{3.10}$$

$$S_{rs} = \frac{1}{\sqrt{n}} \omega^{rs} , \quad (S^{-1})_{rs} = \frac{1}{\sqrt{n}} \omega^{-rs} , \tag{3.11}$$

$$(\log B)_{rs} = (\log \omega) r \, \delta_{rs} \tag{3.12}$$

Q.1(b) of paper is Ex 1(b) of Coddington, p.59. with this difference that $y'(0)$ is given $= 10$ in the Q paper while Coddington has $y'(0) = 0$

[Answer to Coddington's problem

$\therefore \varphi(x)$ is found thus.

Coddington Ex 1(c) is

$$y'' + (3i-1)\,y' - 3iy = 0, \quad y(0) = 2, \; y'(0) = 0$$

Ans is $\varphi(x) = \left(\frac{3i+9}{5}\right) e^{x} + \left(\frac{1-3i}{5}\right) e^{-3ix}$]

Q.4(b).

If a fn f defined for (x,y) in a set S. we say f satisfies a Lipschitz cond on S if there exists a constant K such that

$$|f(x,y_1) - f(x,y_2)| \leqq K\,|y_1 - y_2|$$

for all $(x,y_1), (x,y_2)$ in S. K is Lipschitz const.

Ex. 1(a) Coddington Chap. 5 sec 5, p.209 is Q.4(b). Ans to 1(b) gives $K = 2$.

[worked Ex on p.208 of Coddington

An Ex of a fn satisfying a Lipschitz cond is

$$f(x,y) = xy^2 \quad \text{on } R: |x| \leqq 1, \; |y| \leqq 1$$

Here $\left|\frac{\partial f}{\partial y}(x,y)\right| = |2xy| \leqq 2.$

This same method should be applicable to $f = 4x^2 + y^2$ in Q.4(b)

already $\because \left|\frac{\partial f}{\partial y}\right| = |2y| \leqq 2.$

We now compute the commutator

$$\left[i\eta Q, i\xi P\right] = \left[\log B, S(\log B)S^{-1}\right]$$

We have

$$\begin{aligned}
\left[i\eta Q, i\xi P\right]_{rs} &= \left\{(\log B)\, S(\log B)S^{-1}\right\}_{rs} \\
&\quad - \left\{S(\log B)S^{-1}(\log B)\right\}_{rs} \\
&= \sum_{t,u,v=0}^{n-1} (\log B)_{rt} S_{tu} (\log B)_{uv} (S^{-1})_{vs} \\
&\quad - S_{rt}(\log B)_{tu}(S^{-1})_{uv}(\log B)_{vs} \\
&= \sum_{t,u,v} \frac{(\log \omega)^2}{n}\Big[r\delta_{rt}\omega^{tu} u\delta_{uv}\omega^{-vs} \\
&\quad - \omega^{rt} t\delta_{tu}\omega^{-uv} v\delta_{vs}\Big] \\
&= \frac{(\log \omega)^2}{n} \sum_{t,u,v} \omega^{u(r-s)}(ur-us)
\end{aligned}$$

Therefore

$$\left[i\eta Q, i\xi P\right]_{rs} = \frac{(\log \omega)^2}{n}(r-s)\sum_{u=0}^{n-1} u\,\omega^{u(r-s)} \qquad (3.13)$$

Q.1(b). $y'' + (4i+1)y' + y = 0.$

$$y'' + (3i-1)y' - 3iy = 0.$$

$$\lambda^2 + (3i-1)\lambda - 3i = 0.$$

$$\lambda = \frac{-(3i-1) \pm \sqrt{(3i-1)^2 + 12i}}{2}$$

$$= \frac{-(3i-1) \pm \sqrt{9i^2 - 6i + 1 + 12i}}{2} = \frac{-(3i-1) \pm \sqrt{6i-8}}{2}.$$

$$= \frac{-(3i-1)}{2} \pm \frac{1}{2}\sqrt{6i-8}.$$

$$y = e^{-\frac{(3i-1)}{2}x}\left\{Ae^{\frac{x}{2}\sqrt{6i-8}} + Be^{-\frac{x}{2}\sqrt{6i-8}}\right\}.$$

$\sqrt{6i-8} = x + iy$

$6i - 8 = x^2 - y^2 + 2ixy$

$x^2 - y^2 = -8$

$2xy = 3.$

~~A+B = 2~~

$$y = Ae^{\frac{x}{2}\underbrace{\{\sqrt{6i-8} - (3i-1)\}}_{2\lambda_1}} + Be^{\frac{x}{2}\underbrace{\{-\sqrt{6i-8} - (3i-1)\}}_{2\lambda_2}}$$

$$B = -\frac{A\lambda_1}{\lambda_2}$$

$$y' = A\lambda_1 e^{\lambda_1 x} + B\lambda_2 e^{\lambda_2 x}.$$

$$\left.\begin{array}{l} A + B = 2 \\ A\lambda_1 + B\lambda_2 = 0 \end{array}\right\} \quad A - \frac{A\lambda_1}{\lambda_2} = 2 \quad A = \frac{2\lambda_2}{\lambda_2 - \lambda_1}$$

$$A(\lambda_2 - \lambda_1) = 2\lambda_2 \quad B = \frac{2\lambda_1}{\lambda_1 - \lambda_2}$$

$$\lambda_2 - \lambda_1 = \frac{1}{2}\left\{-\sqrt{6i-8} - \cancel{(3i-1)} - \sqrt{6i-8} + \cancel{(3i-1)}\right\} = -\sqrt{6i-8}.$$

$$A = \frac{-\sqrt{6i-8} - (3i-1)}{-\sqrt{6i-8}}, \quad B = \frac{\sqrt{6i-8} - (3i-1)}{\sqrt{6i-8}}.$$

$$A = 1 + \frac{3i-1}{\sqrt{6i-8}}, \quad B = 1 - \frac{3i-1}{\sqrt{6i-8}}.$$

$$y = \frac{1}{2}e^{2x} - \frac{4}{3}e^{-x}$$

$$y' = e^{2x} + \frac{4}{3}e^{-x}$$

$$y'' = 2e^{2x} - \frac{4}{3}e^{-x}.$$

$$y'' - 4y = \cancel{2e^{2x}} - \frac{4}{3}e^{-x} - \cancel{2e^{2x}} + \frac{16}{3}e^{-x} = 4e^{-x}$$

If $\omega^{r-s} = x = 1$

$$\left[i\eta Q, i\xi P \right]_{rs} = \frac{(\log\omega)^2 (r-s)}{n} \frac{n(n-1)}{2} \qquad (3.14)$$

If $\omega^{r-s} = x \neq 1$, then, since $x^n = 1$, we have

$$\sum_{u=0}^{n-1} u x^u = \frac{n}{x-1} \qquad (3.15)$$

and hence

$$\left[i\eta Q, i\xi P \right]_{rs} = \frac{(\log\omega)^2 (r-s)}{n} \left[\frac{n}{\omega^{r-s}-1} \right] \qquad (3.16)$$

Thus we have,

$$\left[Q, P \right]_{rs} = \frac{(s-r)(\log\omega)^2}{n\xi\eta} \frac{n(n-1)}{2}$$

when $\omega^{r-s} = 1$

and

$$= \frac{(s-r)(\log\omega)^2}{n\xi\eta} \frac{n}{\omega^{r-s}-1},$$

when $\omega^{r-s} \neq 1$.

We notice that since n is finite we could choose $\xi = \eta = 1$, and that $[Q,P]$ is strictly off-diagonal and hence is trace free.

Q. 2 — L.T. method

(a) $F(s) = \int_0^\infty f(t) e^{-st} \, dt$

$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} \, ds$

suff. cond. (i) $f(t) = 0$ for $-$ve t

(ii) $|f(t)| \leq M e^{\sigma t}$

for some σ & M & for all t

(b)

$\ddot{x} + 3\dot{x} + 2x = t$

$s^2 X(s) - s + 4 + 3\{sX(s) - 1\} + 2X(s) = \frac{1}{s^2}$

$X(s)\{s^2 + 3s + 2\} = s - 4 + 3 + \frac{1}{s^2} = s - 1 + \frac{1}{s^2} = \frac{s^3 - s^2 + 1}{s^2}$

$X(s) = \frac{s^3 - s^2 + 1}{s^2(s^2 + 3s + 2)} = \frac{s^3 - s^2 + 1}{s^2(s+1)(s+2)}$

$= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$

$As(s+1)(s+2) + B(s+1)(s+2) + Cs^2(s+2) + Ds^2(s+1)$

$\equiv s^3 - s^2 + 1$

$s^3(A + C + D) = s^3$

$s^2(3A + B + 2C + D) = -s^2$

$s(2A + 2B) = 0$

$2B = 1$

$A + C + D = 1$

$3A + B + 2C + D = -1$

$2A + 2B = 0$

$2B = 1$

$-1 + \frac{3}{2} - \frac{1}{2}$

$B = \frac{1}{2}, A = -\frac{1}{2}$

$C + D = 3/2$

$2C + D = 0$

$C = -3/2$, $D = 3$

$\frac{1}{2}(-e^{-t} + 4e^{-2t}) + \frac{3}{2}(1 + e^{-t} - 2e^{-2t})$

$X(s) =$

$+ (t - e^{-t} + e^{-2t})$

$x(t) = -\frac{3}{4}u(t) + \frac{t}{2} + e^{-t} - \frac{1}{4}e^{-2t}$

$u(t) = 0$

$\mathcal{L}[\dot{x}(t)] = sX(s) - x(0)$

$\mathcal{L}[\ddot{x}(t)] = s^2 X(s) - s x(0) - \dot{x}(0)$

$\ddot{x}(0) = \dot{x}(0) = 0$

$\mathcal{L}[\dot{x}] = sX(s)$

$\mathcal{L}[\ddot{x}] = s^2 X(s)$

$\dot{x}(0) = 0$

$\dot{x}(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$

$\dot{x} = \frac{1}{2}\{e^{-t} + 2e^{-2t}\}$

$x(t) = e^{-t} - e^{-2t}, x(0) = 0$

$\ddot{x} = \frac{1}{2}\{-e^{-t} - 4e^{-2t}\}$

$x(0) = $

$= -\frac{3}{4} + 1 - \frac{1}{4} = 0$

$\dot{x} = \frac{1}{2}\{1 + e^{-t} - 2e^{-2t}\}$

$\dot{x}(0) = 0$

$\ddot{x} = \frac{1}{2}(-e^{-t} + 4e^{-2t})$

$s^2 X(s) + 3sX(s) + 2X(s) = \frac{1}{s^2}$

$X(s) = \frac{1}{s^2(s+1)(s+2)} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$

$x = -\frac{3}{4}u(t) + \frac{t}{2} + e^{-t} - \frac{1}{4}e^{-2t}$

$As(s+1)(s+2) + B(s+1)(s+2) + Cs^2(s+2) + Ds^2(s+1) \equiv 1$

$A + C + D = 0$

$3A + B + 2C + D = 0$

$2A = -\frac{3}{2}$

$D = -\frac{3}{4}$

$2A + 3B = 0$

$2B = 1$

$A = 0, B = \frac{1}{2}$

$C + D = 3/4$

$2C + D = \frac{9}{4} - \frac{1}{2} = 7/4$

$B = \frac{1}{2}$

$A = -\frac{3}{4}$

$C = 1$

$D = -\frac{1}{4}$

$C + D = 0$

$2C + D = -\frac{1}{2}$

$D = 1/2$

$x(t) = \frac{1}{2}\{t - e^{-t} + e^{-2t}\}$

$x(0) = 0$

$2c_2e^{2x} - 2c_3e^{-2x}$

$4c_2e^{2x} + 4c_3e^{-2x}$

$\sinh 2x = \frac{e^{2x} - e^{-2x}}{2}$

$\phi = c_1 + c_2e^{2x} + c_3e^{-2x}$

$\phi(0) = 0, \quad c_1 + c_2 + c_3 = 0$

$\phi'(0) = 0 \quad 2(c_2 - c_3) = 1$

$\phi''(0) = 0 \quad c_2 + c_3 = 0$

$\left.\begin{array}{l} c_2 = 1/4 \\ c_3 = -1/4 \end{array}\right\} \begin{array}{l} c_1 = 0. \\ 7 \quad y = \frac{1}{4}(e^{2x} - e^{-2x}) \\ = \frac{1}{2} \sinh 2x \end{array}$

We now prove that the commutation relation given by eqn. (3.13) does indeed yield the Heisenberg commutation relation in the limit as $n \to \infty$. We begin with eqn. (3.13) relabelling the rows and columns from $-\frac{n-1}{2}$ to $\frac{n-1}{2}$ and replace the sum by an integral, that is, we let the matrix index take continuous values. Thus the sum

$$-\frac{(\log \omega)^2}{n \zeta \eta} \sum_{u=0}^{n-1} u\, \omega^{u(r-s)} (r-s)$$

reduces in the limit as $n \to \infty$ to the integral (Dirac, 1927 and Heisenberg 1931)

$$-\frac{(\log \omega)^2}{n \zeta \eta} (r-s) \int_{-\infty}^{\infty} u\, e^{\frac{2\pi i}{n} u(r-s)}\, du$$

$$= \frac{4\pi^2}{n \eta \zeta} (r-s) \int_{-\infty}^{\infty} \frac{u}{n} e^{\frac{2\pi i}{n} u(r-s)}\, d\left(\frac{u}{n}\right)$$

$$= -i(r-s) \frac{d}{d(r-s)} \int e^{2\pi i (r-s)\frac{u}{n}}\, d\left(\frac{u}{n}\right)$$

$$= -i(r-s)\, \delta'(r-s) = i\, \delta(r-s)$$

where we have used $n \zeta \eta = 2\pi$ as $n \to \infty$. This Completes the proof. It should be remembered that in the limit as n approaches infinity continuously, we are taking only the principal value of log ω as this gives the correspondence to the Heisenberg commutation relation.

4. Conclusions

We have calculated the commutator $[Q,P]$ when the space on which the operators act is finite. We elevate the commutator for finite n to what we call 'Finite Quantum Mechanics'. It turns out that the operator is strictly off-diagonal for finite n. This implies no uncertainty and no zero-point energy if these concepts have any meaning for finite n. Of course, in the limiting case as n approaches infinity continuously, the commutator becomes strictly diagonal and reduces to a multiple of the Dirac delta functions, thus restoring the Heisenberg commutation relations.

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The authors are grateful to Professor Alladi Ramakrishnan for his interest in this work and for useful discussions. They also thank Sri R.Jagannathan for some useful remarks.

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1, 14, 15, 4
8, 11, 10, 5
12, 7, 6, 9
13, 2, 3, 16

(1,8), (2,7)
(3,6), (4,5)

1	4	4	1
2	3	3	2
4	1	1	4
3	2	2	3

+

0	4	12	8
12	8	0	4
12	8	0	4
0	4	12	8

1	4	1	4
2	3	2	3
4	1	4	1
3	2	3	2

+

0	4	12	8
12	8	0	4
0	4	12	8
12	8	0	4

=

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

1, 2, 7, 4.

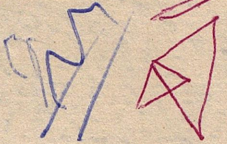
1, 4, 15, 4
8, 11, 10, 5
12, 7, 6, 9
6, 9, 3, 16

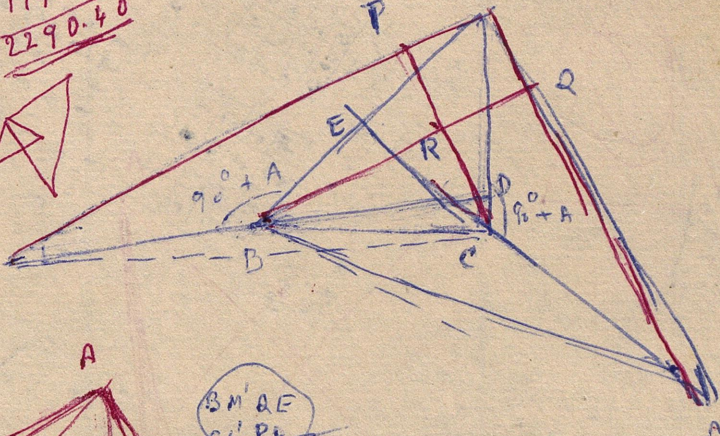
650.
650
1300

1800.
1300
500.00
1790.40
2290.40

$LM^2 = 2(b^2c^2 - 2bc\sin A)$

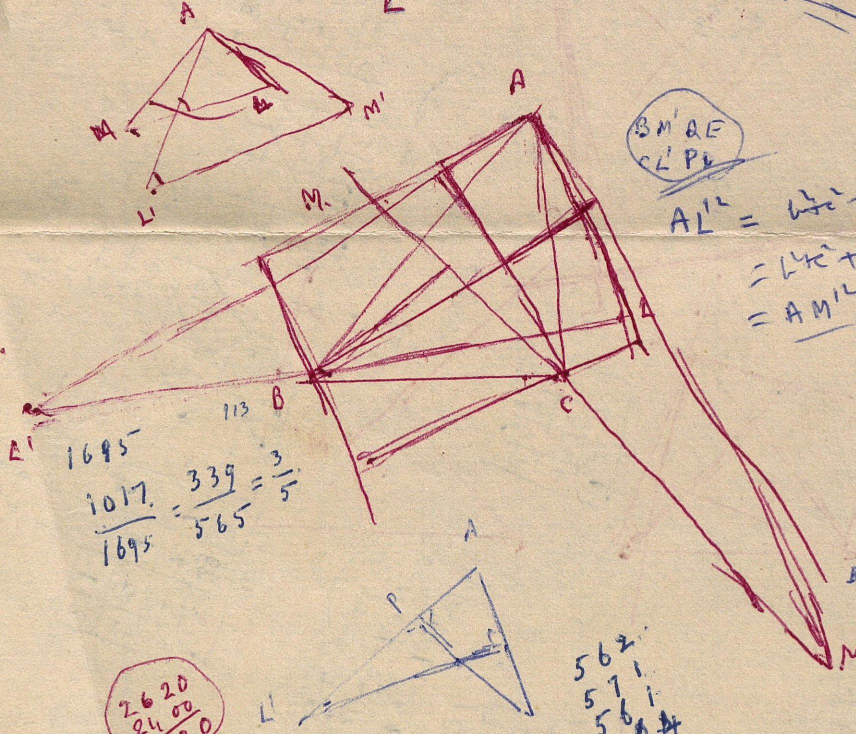
596.60
596.60
596.60

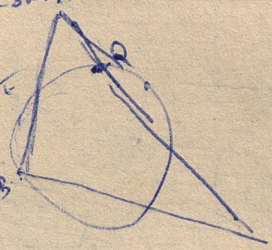




$BM'AE$
$CL'PL$

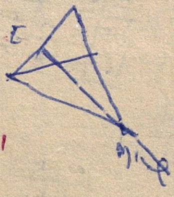
$AL'^2 = b^2c^2 - 2bc\cos(90^\circ + A)$
$= b^2c^2 + 2bc\sin A$
$= AM'^2$





1695

$\frac{1017}{1695} = \frac{339}{565} = \frac{3}{5}$

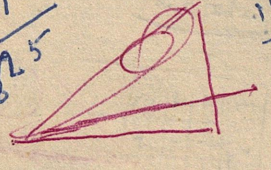


2620
2400
220
170

AL.L'7 =

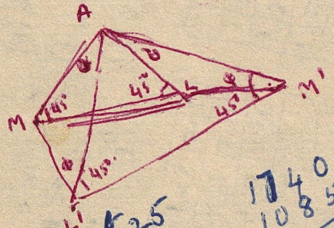
562
571
561
564
567
2825

$\frac{1695}{2825} = \frac{339}{565} = \frac{3}{5}$



1695
2951

113)2957(2
226
697



610
475

625
580
535

1740
1085
2825
1695

343
339

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8, 7, 5, 2, 1, 3, 4, 6
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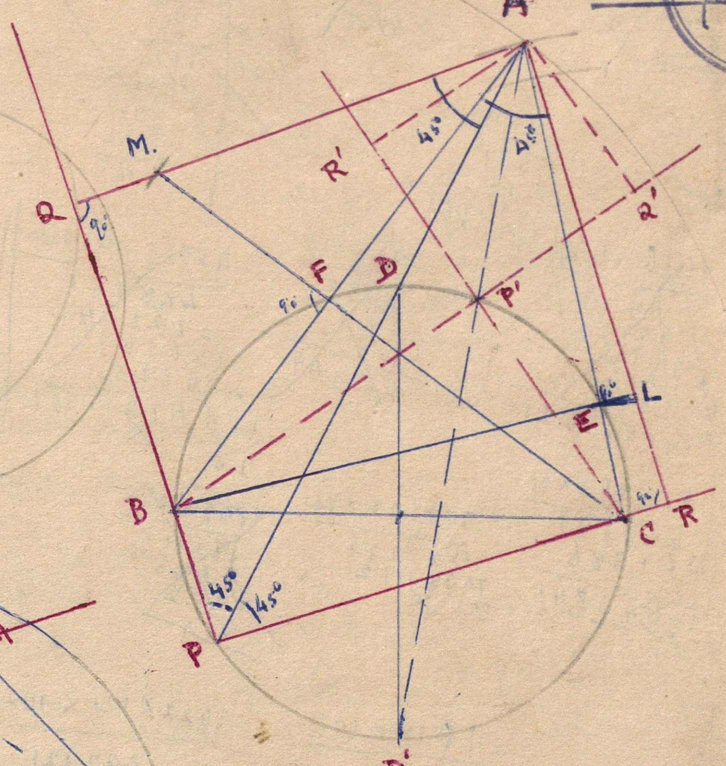
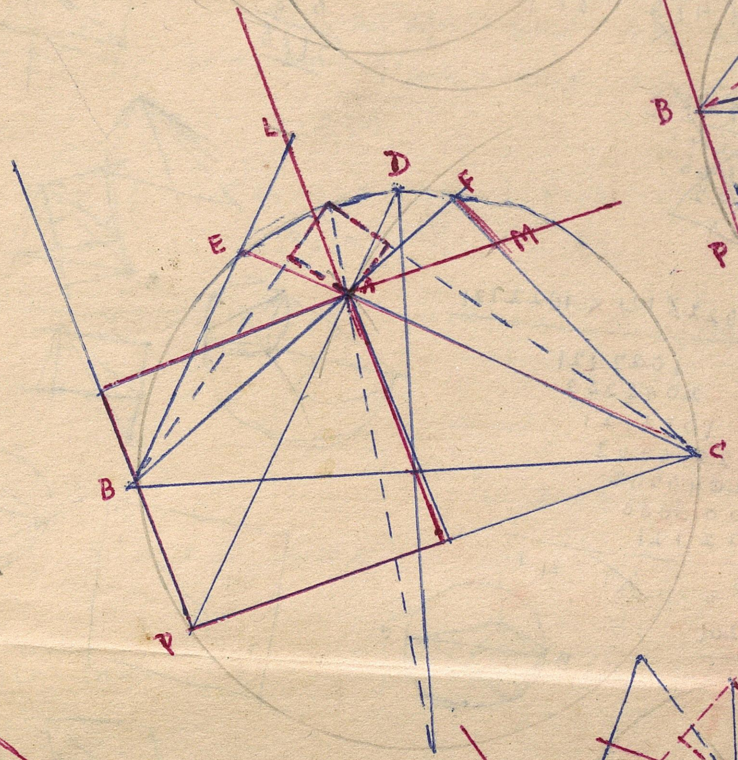
8, 7, 5, 2





Q. Δ^s ABL & CAM

$\hat{ABL} = \hat{ACM}$

$\hat{BAL} = 90^\circ - \hat{FAM} = \hat{AMC}$

Δ^s are similar

$\hat{BLA} = \hat{CAM}$

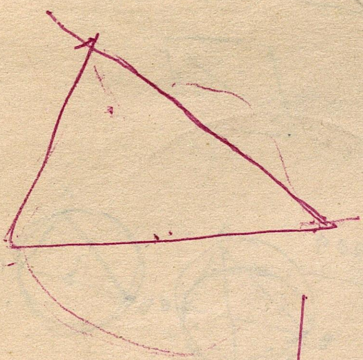
$AL \cdot AR = AE \cdot AC = bc \cos A$

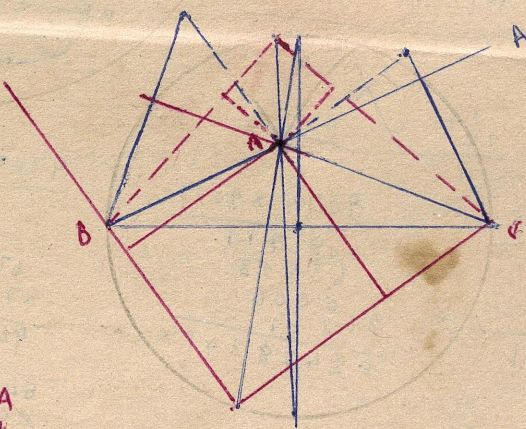
$AM \cdot AQ = AF \cdot AB = bc \cos A$

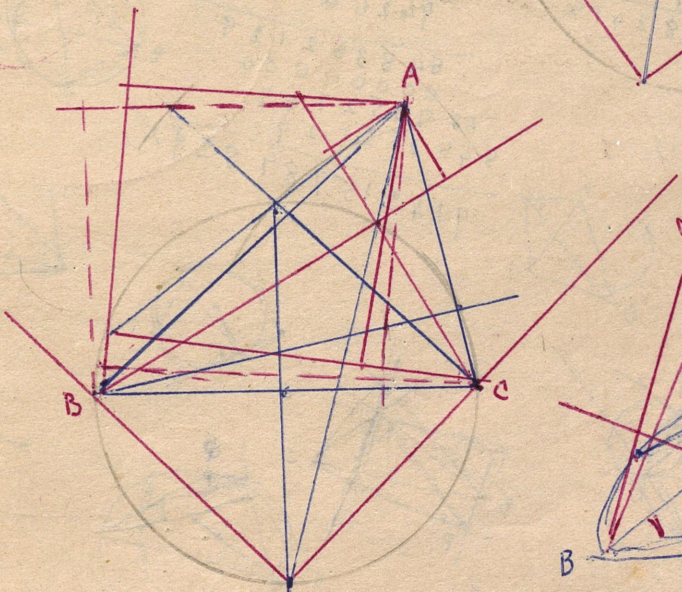
$AQ = AR \therefore AL = AM$

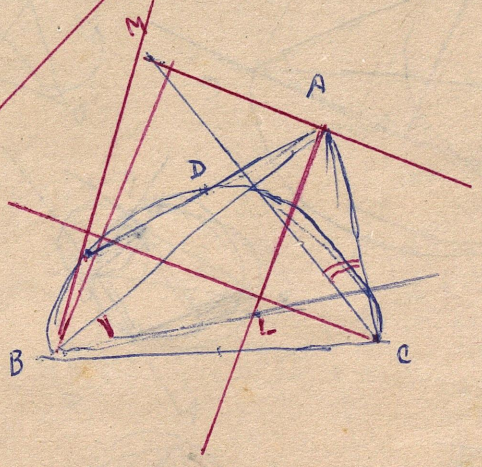
$\therefore \Delta^s$ ~~are~~ ABL & ACM are $\equiv$

$\therefore BL = AC$
CM = AB
Q.E.D.









Δ^s ABL & CAM.

$\hat{ABL} = \hat{ACM}$

AB = CM.

BL = AC

Δ^s are $\equiv$

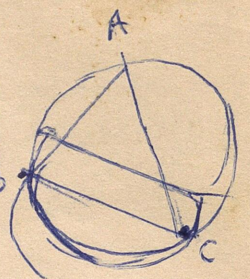
AL = AM.

94249 x 94249
888241
3 14 9 9 6
1 3 8 4 9 8
3 7 4 9 9 6
8 4 8 2 4 1

233 x 233
699
699
466
54289

3 | 3367439839 | 3066
36
60
3674
3636
3337
3672
165

211 x 211
211
211
422
44521 x 44521
44521
89042
222605
178084
1780
198 x 41



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1071
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119
14161

14161 x 14161
14161
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14161
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213 x 213
639
213
426
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61
20

62 x 62
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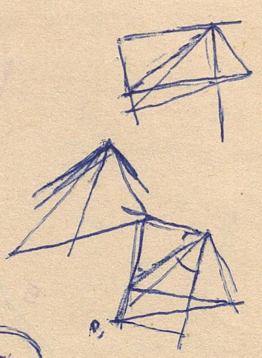
678
3166



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226981
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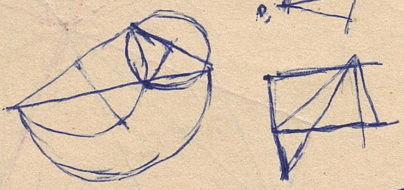
7396
14792
1849



1011 x 1011
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1011
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1011
1022121

1022121 x 1022121
1022121
2044242
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2044242
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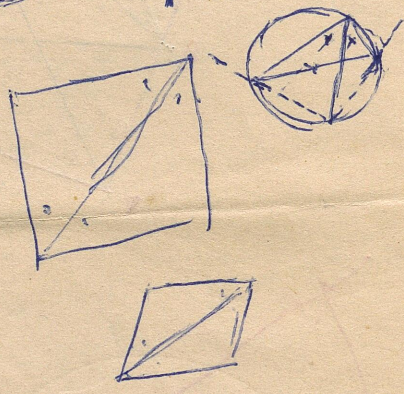
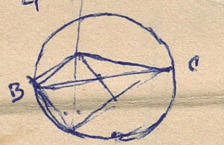
0000000
1022121
10 x 41





1101 x 1101
1101
0000
1101
1101
1212201

1212201 x 1212201
201
24
12
14
01

938313739 x 979
8444823851
6568196173
8444823851
91
61

9079 x 9079
81711
63553
0000
81711
82428241 x

9709 x 9709
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0000
67963
87381
94264681 x 9709
848382129
000000000
659852767
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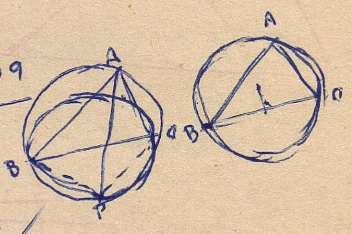
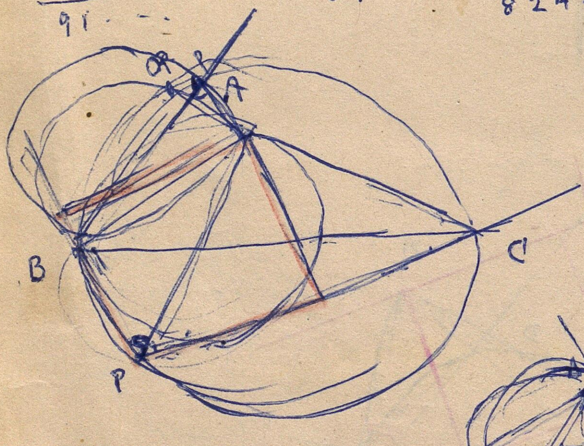
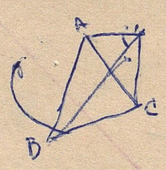
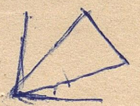
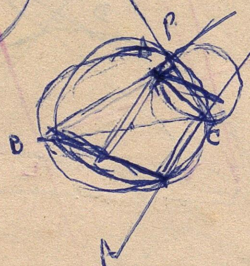
914215787829 x

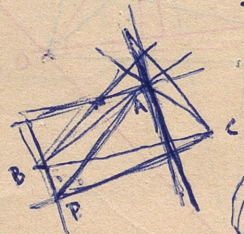


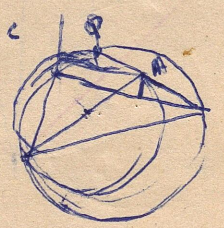


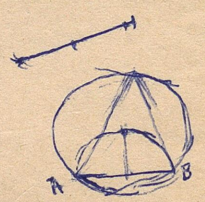


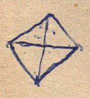


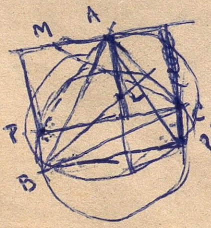


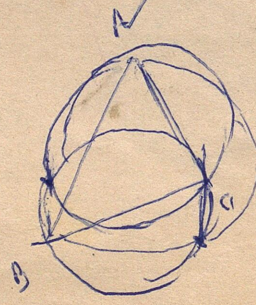


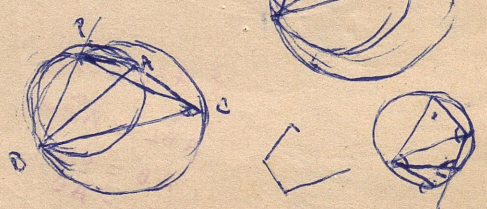


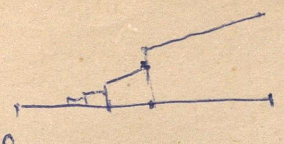






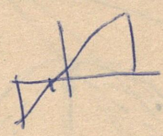


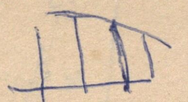


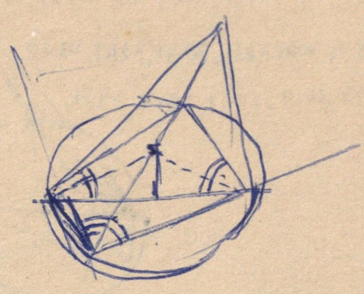


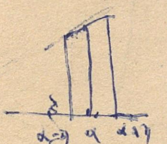
$y = \sin\frac{1}{x}$, $x \gtrless 0$, $y = 0$ for $x = 0$

$f(x)$ $y = \sin\frac{1}{x}$ $x \gtrless 0$,
$f(x) = 0$ $x = 0$







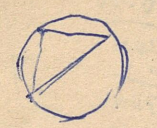


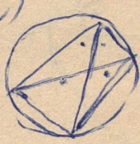
Cauchy's defn – Exx. discontinuous at one pt
Weierstrass defn. – Lebesgue & at every point
Finitely discontinuous (finite or infinite) – Saltus

$\sin^{-1}/x$: $y = E(x)$

$E(n-0) = n-1$, $E(n) = n$, $E(n+0) = n$.

1st & 2nd kinds





$\int f(x)\delta(x-a) = f(a)$

$\int f'(x)\delta(x-a) = f'(a)$

$+ \int f(x)\delta'(x-a) = \cancel{f'(a)}\ 0$

$\int f(x)\delta'(x-a) = -f'(a)$

$y = c$ for all rational x
$y = d$ " irrational x } Totally discontinuous

$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$.

$\delta(x) = 0$, $x \neq 0$

$\delta(x) = \delta(-x)$

$x\delta(x) = 0$

$\delta'(x) = -\delta'(-x)$.

$\delta'(x) = \frac{-\delta(x)}{x}$

$\delta'(x) =$

$\delta'(x) = -\delta'(-x)$

$\delta'(-x) = -\delta'(x)$

$x\delta'(x) = -\delta(x)$

$\int_{-\infty}^{\infty} f(x)\delta(x-b) = f(b)$

$\int_{-\infty}^{\infty} \delta(a-x)\,dx\,\delta(b-x) = \delta(a-b)$

$x = -y$.

$\int \delta(-y) - dy = 1$.

$\int_{-\infty}^{\infty} \delta(-y)\,dy = 1$

$\delta'(-x) = \cancel{\delta'(x)} -\delta'(x)$.

$x\delta'(x) = -\delta(x)$

$\int_{-\infty}^{\infty} f(x)\,\delta'(x-a)\,dx = -f'(a)$.

$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$.

$\int \cancel{\delta(-x)\,dx} = 1$

$\delta(x) = \delta(-x)$.

$\delta'(x) = -\delta'(-x)$

$x\delta'(x) = -\delta\alpha$

$\cancel{x\delta(x)}$ $x\delta(x) = 0$.

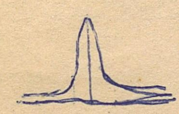
$\int_{-\infty}^{\infty} \delta'(a-x)\,dx\,\cancel{\delta(x-b)}\ \delta(x-b) = \delta'(a-b)$

$\int$

$\int_{-\infty}^{\infty} \cancel{x\delta(x)\,dx} = \frac{x^2}{2}\delta(x) - \int \frac{x^2}{2}\delta'(x)\,dx$

$\delta(x) = 0$.

$x\delta(x) = 0$



$\delta(x)$.

$\delta(-x) = \delta(x)$

$x\delta(x) = 0$

$\int_{-\infty}^{\infty} f(x)\,\delta x \cdot dx = 1$.

$\int_{-\infty}^{\infty} f(x)\,\delta(x-a) = f(a)$.

$\int_{-\infty}^{\infty} \delta(a-x)\,dx\,\delta(x-b) = \delta(a-b)$

$\int_{-\infty}^{\infty} \delta(y-x)\,dx\,\delta(x-b) = \delta(y-b)$

$\int_{-\infty}^{\infty} \delta'(y-x)\,dx\,\delta(x-b) = \delta'(y-b)$.

$\int \delta'(a-x)\,dx\,\delta(x-b) = \delta'(a-b)$

Defn $\int_{-\infty}^{\infty} \delta(x)\cdot dx = 1$, $\delta(x) = 0$, $x \neq 0$.

$\delta(-x) = \delta(x)$

$x\delta(x) = 0$.

$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$

$\int_{-\infty}^{\infty} \delta(a-x)\,dx\,\delta(x-b) = \delta(a-b)$

$\int \delta(x-b)\,dx \cdot \delta(x-a) = \delta(a-b)$.

$\delta'(x) = -\delta'(-x)$.

$x\delta'(x) = -\delta(x)$

$\int_{-\infty}^{\infty} f(x)\cdot\delta'(x-a)\cdot dx = -f'(a)$.

$\int \delta'(a-x)\,dx\,\delta(x-b) = \delta'(a-b)$.

$\int \delta'(x-a)\cdot\delta(x-b)\cdot dx = \delta'(b-a)$

$\int \delta'(a-x)\,\delta(x-b)\,dx = \cancel{\delta}\ \delta'(a-b)$.

$\int f(x)\,\delta(x-y)\,dx = f(y)$.

$\int f(x)\cdot\delta'(x-y)\,dx = f'(y)$.

$\int f(x)\,\delta'(x-a)\,dx = -f'(a)$

264×264
1056
1584
528
69696

2636×2636
15816
7908
15816
5272
6948496

836

$\cancel{409 \times 409}$
3681
000
1636
$\cancel{0}$
164481

307×307
2149
000
921
94249

113, 117, 1123, 1129, 1153, 1171, ~~[illegible]~~,
131, 137, 1319, 1361, 1367, 1373,
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193, 197, 1913, 1931, 1973, 1979, 1997
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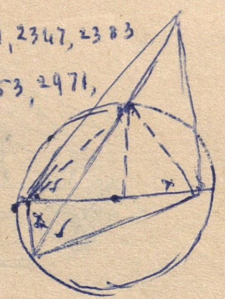
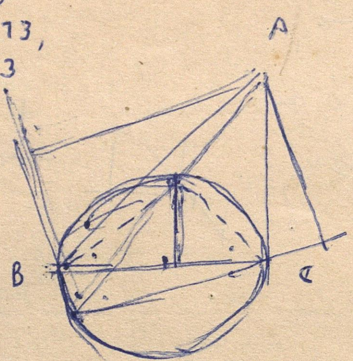

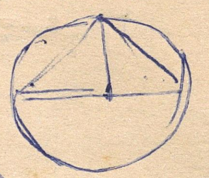

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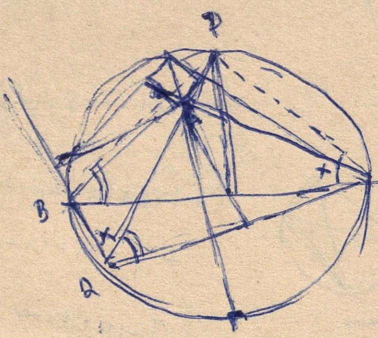

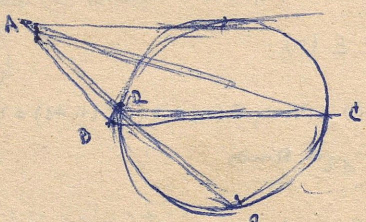

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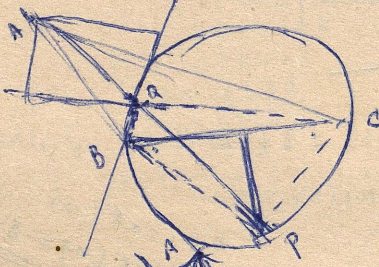

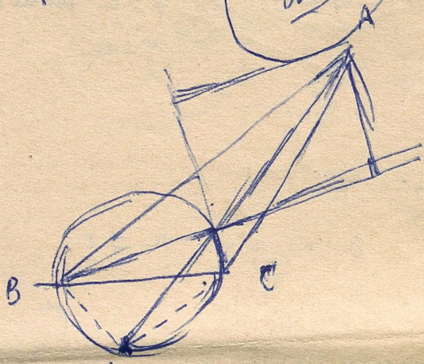

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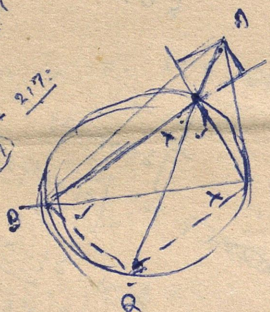
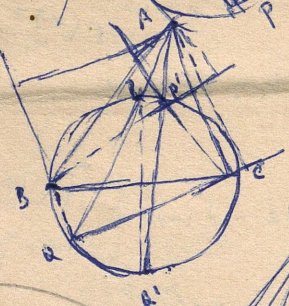
217

217

2967
20.3

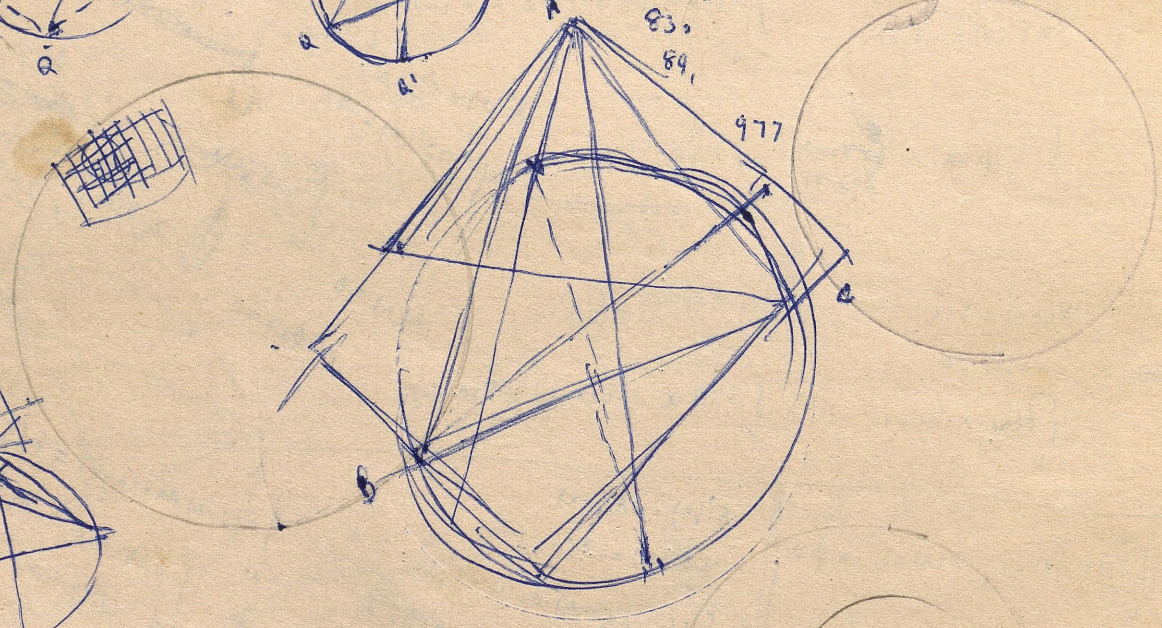


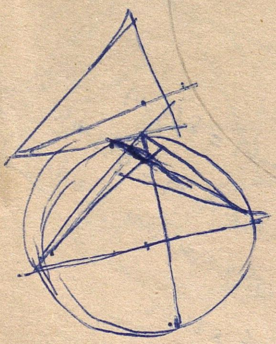
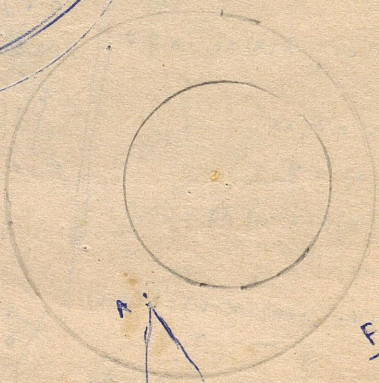


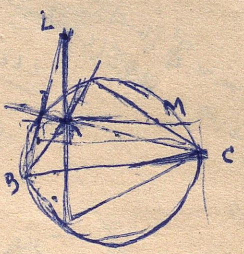
733,
797,
83,
89,

977

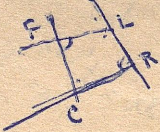
173.
137

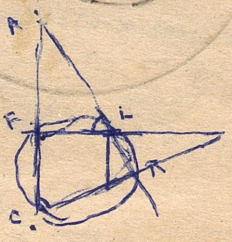




[313] 317.





$\frac{FL}{CR} = \frac{AF}{AR} = \frac{AL}{AC}$

$AL \cdot AR = AF \cdot AC = c\cos A \cdot b$

$AM \cdot AQ = AE \cdot AB = b\cos A \cdot c$

$AL \cdot \cancel{AR} = AM \cdot \cancel{AR}$

$\underline{AL = AM}$

(x, 4, 2)
640854 = ~~599994~~ → (6, 4, 1)

(a) { 750843 → (7, 5, 1) → (785430)
{ 170829 → (1, 7, 1) → (198720)
(b) ~~279018~~ → (2, 8, -1) →

only these two with least but one digit at beginning & remaining in descending order.

we have found another like this 640854 = 685440 − 044586 not in Sunders tables since digits are repeated here. In Sunder's tables, there are just two with same integers as in (a), but more than 20 with same integers as in type (b).

640854
599994
50860
~~39960~~
20900

9990
699993
449950
249943
900
750843

170829
99999
70830
69930
900

279018
199998
79020
79920
900

785430
034587
750843

198720
027891
170829

~~298710~~
~~017892~~
0818

19/6/78

631764
599994
31770
29970
1800

9990
a − f = 6
b − e = 3
c − d = 2

766431
134667
631764

598203
302895
295308

845163
361548
483615

518463
364815
153648

9990

295308
199998
95310
89910
5400

a − f = 2
b − e = 9
c − d = 6.

598203

483615
499995
-16380
19980
3600

a − f = 5
b − e = -2
c − d = 4

845163
638451

638451
154836
483615

900

499995
19980
480015
3600
483615

153648
199998
-46350
49950
3600

a − f = 2
b − e = -5
c − d = 4

518463
635184

518463
364815
153648

635184
481536
153648
3600
380016
2700
490716

383616

~~399996~~
~~59940~~
~~467136~~

4
6
8

420876
399996
20880
19980
900

a − f = 4
b − e = 2
c − d = 1

827604
406728
420876

860832

4 0
8 6
~~8 0~~
72

a − f	6	3	3						
b − e	3	6	6						
c − d	2	2	1						
	✓	×	×						

380816

380016
1800

~~379976~~
~~900~~
380876

9990
59940

8100

399996
59940
1800
2700
460836
461736
~~468036~~

4
6
9

299997
59940
3600
~~360937~~
362637
363537

~~299997~~
~~39960~~
~~339953~~

462636

299997
39960
339957
900
340857

~~299997~~
~~19980~~
~~319977~~

3
2
320977

399996
19980
419976
900
420876

4
2
1

399996
19980
419976
1800
421776

4
2
3

419976
2700
422676

419976
3600
423576

~~399996~~

499995
19980
519975
1800
520875
521775

5
2
2

499995 5, 3, 4

419976
4500
424476

419976
5400
425376

419976
6300
426276

~~419976~~
~~7200~~
~~427176~~

399975
3600
~~403575~~

72
51
75

599975
2700
602675

773152
251377

499995
9990
509985
900
510885

5
1

499995
19980
519975
900
520875

5
2
1

519975
1800
521775

5
2
2

519975
2700
522675

5
2
3

~~599975~~
~~900~~
~~600875~~

9990

499995
29970
529965
900
530865

5
3

529965
1800
531765

519975
3600
523575

5
3
4

5, 3, 2

~~519975~~
~~4500~~
~~524475~~

519975
4500
524475

5
3
5

5
5
2

~~549945~~
~~1800~~
~~551745~~

~~519975~~
~~5400~~
~~525375~~

499995
39960
539955
900
540855

5
4
1

539955
1800
541755

72
52
55

~~539955~~
~~2700~~
~~542655~~

5
4
2

5
4
3

499995
49950
549945
900
550845

5
5
1

599994
9990
609984
900
610884

599994
19980
619974
900
620874

619974
1800
621774

619974
2700
622674

619974
3600
623574

619974
4500
624474

619974
5400
625374

619974
6300
626274

199998
9990
209988

599994
19980
580014
900
580914

580014
1800
581814

580014
2700
582714

599994
29970
629964
~~3600~~
~~633564~~

630864

~~629964~~
~~2700~~
~~632664~~

629964
1800
631764

599994
29970
570024
1800
571824
900
572724
900

573624

599994
29970
629964
~~3600~~
~~630864~~
~~633564~~

631764

129969
99999
29970

99999
79920
179919
900
180819
900
181719

129969
299961

179919
2700
182619

299961
169992
129969

~~749961~~
~~169947~~
~~800014~~

3870
0783
3087

769941
149967
619974

919908
809919
109989

199980
089991

109989 (1,1,0)

91908

309987
~~299997~~
~~09990~~
9990

~~99987~~
199998
0,3,1,0
389970

589923
329985
259938

199998
9900
209898
900
210798
900
211698

(1)	99999	± 9990	± 900
2	199998	± 19980	± 1800
(3)	299997	± 29970	± 2700
(4)	399996	± 39960	± 3600
(5)	499995	± 49950	± 4500
(6)	599994	± 59940	± 5400
(7)	699993	± 69930	± 6300
(8)	799992	± 79920	± 7200
(9)	899991	± 89910	± 8100

899991
9990
909980
900
910880

641754
599994
41760
39960
1800

631764
651744
599994
51750
49950
1800

661734
599994
61740
59940
1800

671724
599994
71730
69930
1800

621774
599994
21780
19980
1800

681714
599994
81720
79920
1800

611784
599994
11790

601794
599994
1800

691704
599994
91710
89910
1800

796401
104697
691704

109989 = 919908 − 809919 ✓ = 199980 − 089991 ✓

~~299961~~ 129969 = 299961 − 169992 ✓

309987 = 389970 − 079983 ✓

259938 = 589923 − 329985 ✓

619974 = 769941 − 149967 ✓

691764 = 766431 − 134667 ✓ (R)

~~691704 = 796401 − 104697~~

190809 = 991008 − 800199 ✓ = 199800 − 008991

~~190361~~

132669 = 296361 − 163692 = 269631 − 136962 ✓✓

330867 = 638703 − 307836 = 368730 − 037863 ✓

296361
269631

132669
99999
32670
29970
2700

296361
163692
132669

269631
136962
132669

110889
99999
10890
9990
900

130819
99999
30820
79920
900

330867
299997
30870
29970
900

100899
99999
10890

190809
99999
90810
89910
900

~~180819~~
900
~~179919~~
~~180819~~
179919

638703
307836
330867

368730
037863
330867

991008
800199
190809

320877
299997
20880
19980
900

199800
008991
190809

891027
569214
~~4936~~
591624

1089 90

235368
199998
35370
29970
5400

2 53
3 63
6 82

2 53 8

598293
392895
205398

~~5683~~ 568233
332865
235368

These above results suggest that if middle two digits be unchanged & the others arranged in descending order, we should get a Kaprekar constant of order 6.

723645

753642 − 246357 = 407285

857240 − 042758 = 814482

884421 − 124488 = 759933

759933 − 339975 = 419958

859941 − 149958 = 709983

879930 − 039978 = 839952

859932 − 239958 = 619974 ✓

769941 − 149967 = 619974

7 steps

394662

964632 − 236469 = 728163

768132 − 231867 = 536265

656253 − 352656 = 303597

973530 − 035379 = 938151

958131 − 131859 = 826272

876222 − 222678 = 653544

653544 − 445356 = 408188

888140 − 041888 = 846252

856242 − 242658 = 612584

862541 − 145268 = 717273

777231 − 132777 = 644454

654444 − 444456 = 209988

889920 − 029988 = 859932

859932 − 239958 = 619974 ✓

12 steps

Try the M-const

723645 → 573623 − 326375 = 247248

487224 − 422784 = 064440

694400 − 004496 = ~~299908~~ 689904

~~889902~~ − ~~209988~~ = ~~679914~~

689904 − 409968 = 279936

679923 − 329976 = 349947

479934 − 439974 = 039960

369900 − 009963 = 359937

579933 − 339975 = 239958

589923 − 329985 = 259938 ✓

9 steps.

7699941 − 1499967 = ✓6199974

5899923 − 3299985 = 2599938 ✓

For 7 digits also.

76999941 − 14999967 = 61999974

8 digits

These extensions of the K-const for 5, 6, 7, 8 & 9 ~~digits~~ etc all appear therefore trivial

Are there ways of obtaining other non-trivial constants?

Consider for e.g. taking case (R) leave 2 & 5th unchanged.

8352 8542

723645

~~18~~ 726543 − 345627 = 380916

986310 − 013689 = 972621

976221 − 122679 = 853542

855442 − 244558 = 610884

6104 8640

~~64~~ 8

816480 − 084618 = 731762

737261 − 162737 = 674424

674424 − 424476 = 249948

949842 − 248949 = 700893

7083

807390 − 093708 = 713682

7362

79 7172

716382 − 283617 = 432765

735462 − 264537 = 470925

975420

796491 − 194697 = 601794

736461 − 164637 = 24

954 − 459 = 495

990 − 099

710496

796410

496710

0, 2, 4, 6, 7, 8

827604

723645

2, 3, 4, 5, 6, 7

736524

367

763 − 367 = 396

693 − 396 = 297

792 − 297 = 495

594 − 495 = 099

891 − 198 = 693

673 − 376 = 297

972 − 279 = 693

936 − 639

954

495

873 − 736 = 063

630 − 036 = 594

945 − 549 = 394

943 − 349 = 594

961038
635184

1,3,4,5,6,8
0,1,3,6,8,9

70254, 0,2,4,5,7
89604 0,4,6,8,9
60273, 0,2,3,6,7

abcde - edcba
9999(a-e)+990(b-d)

$10^9 - 10$

~~10000~~
~~1~~
~~9999~~

100000

69723 | 70254 | 89604
96732 | 54270 | 49680
| | 69480

60273 | ~~29780~~
60732
32760
73260

76941
34780
79380
38970
58923

29961
16992
$\boxed{12969}$

$10^4 a + 10^3 b + 10^2 c + 10 d + e$
$-(10^4 e + 10^3 d + 10^2 c + 10 b + a$
$10^4(a-e) + 10^3(b-d) + 10(d-b) + (e-a)$
$9999\underbrace{(a-e)}_{x} + 990\underbrace{(b-d)}_{y}$ //

2,3,6,7,9
0,2,4,5,7
0,4,6,8,9
0,2,3,6,7

8532
2358
6174

23456

~~74250~~
~~69003~~
~~05247~~

~~65342~~
~~24356~~
~~40986~~

~~78460~~
64352
25346
99006

93060
06039
87021

82170
07128
75042

74250
05247
69003

93060

73260
67023

2,3,6,7,9

697~~23~~
96732.

2,3,4,5,6

46523
32564
13959

1,3,5,9,9
59913
31995
27918

~~79812~~
~~21897~~
~~56925~~

~~69525~~
~~52596~~
~~16929~~

~~69912~~
~~21996~~
~~47916~~

~~69714~~
~~21796~~
~~27918~~

1,5,5,8,9

0,0,1,7,9 0,1,8,9,9 0,3,7,8,9

36927

~~79812~~
~~21897~~
~~57915~~

~~79518~~
~~51597~~
~~27918~~

79812
21897
57915

59715
51795
07910

19700
00791
18909

89901
10998
78903

1980
3870
2961

04689

89604

79803
30897
48906

69804
40896
28908

89802
20898
68904

69804
40896
28908

3,5,7,6,9
3,5,6,7,9

~~69735~~
~~53796~~
~~15936~~

~~59613~~
~~31695~~
~~27918~~

~~89712~~
~~21798~~
~~67914~~

~~79614~~
~~41697~~
~~37917~~

~~79713~~
~~31797~~
~~47916~~

~~79614~~

2,3,5,8,9

48906

79635
53697
25938

89523
32598
56925

69525
52596
16929

99612
21699
77913

~~79613~~
~~31697~~
~~47916~~

~~79614~~
~~41697~~
~~37917~~

~~79713~~
~~31797~~
~~47916~~

89604

1,2,3,4,5

45312
21354
23958

89523
32598
56925

69525

79713
31797
47916

79614
41697
37917

79317
71397
07910

79100
00197
78903

70254
45207
25047

2,5,6,6,9

59625
52695
06926

69620
02696
66924

89703
30798

69624
42696
26928

69822
22869
46953

69534
43596
25928

59822
22895
36927

79623
32697
46926

(2)

0,2,6,6,9

89802
20898
68904

69804
40896
28908

1,2,3,4,5

35412
21453
13959

99513
31599
67914

69714
41796
27918

79812
21897
57915

59715
51795
07920

29700
00792
28908

1,3,5,9,9

0,3,4,5,6

1,2,3,4,5

0,3,4,5,6

51243
34215
17028

~~80272~~
~~27208~~
~~53064~~

~~60354~~
~~45306~~
~~15048~~

0,1,4,5,8

~~80154~~
~~45108~~
~~35046~~

~~60354~~
~~45306~~
~~15048~~

0,3,5,6,7

80154

0,2,2,7,8
0,1,2,7,8

80172
27108
53064

60354
45306
15048

80154
45108
35046

5,6,7,8,9

95687
78659
17028

70365
56307
14058

0,3,4,5,6

0,4,4,5,5

0,0,4,5,9

1,2,5,6,8

0,1,4,5,8

1,4,7,8,9

91487
78419
13068

80163
36108
44055

~~504~~ 50454
45405
05049

90054
45009
45045

1,2,3,4,5

87491
19478
68013

63180
08136
55044

54450
05445
49000

90090
09009
81081

63180
08136
55044

~~81265~~
~~56218~~
~~25047~~

~~98354~~
~~45389~~
~~52965~~

~~93365~~
~~56329~~
~~36036~~

~~70265~~
~~56207~~

0,4,4,5,5.

~~40455~~
55440
04455
50985

85590
09558
76032

63270
07236
56034

54260
06245
48015

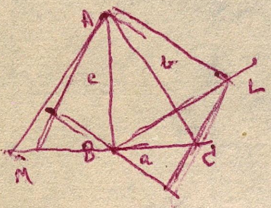
54180
08145
46035

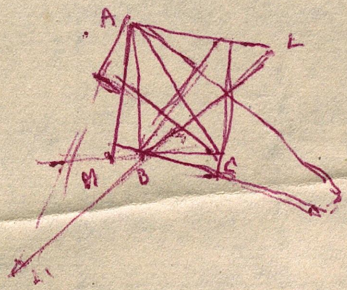
Special cases

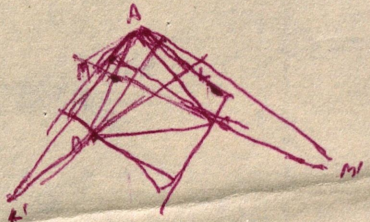
(1) $A = 90°$, $x = x' = 0$, $AL^2 = b^2 + c^2 - 2bc$ or $AL = b - c$, $AL' = b + c$. (no constn possible) X

(2) Δ^{le} isosceles with $b = c$, $AL^2 = 2b^2 - 2b^2 \sin A = 2b^2(1 - \sin A)$ ✓

(3) $B = 90°$, $AL^2 = \checkmark$, and $C' = 90°$ ✓





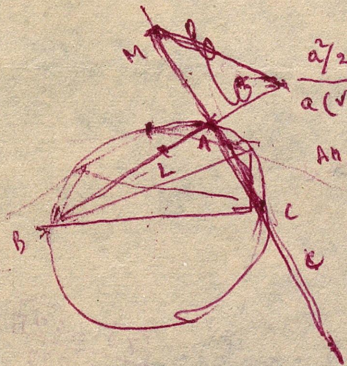


(4) Δ^{le} equilateral

$$AL^2 = a^2 + a^2 - 2a^2 \sin 60° = 2a^2 - a^2\sqrt{3} = a^2(2 - \sqrt{3})$$
$$= a^2(\sqrt{3} - \sqrt{3})^2/2$$

$$(1 - \sqrt{3})^2 = 1 + 3 - 2\sqrt{3} = 2(2 - \sqrt{3})$$

$$AL = \frac{a(\sqrt{3} - 1)}{\sqrt{2}}$$



$$\frac{a^2/2}{a(\sqrt{3}-1)/\sqrt{2}} = \frac{a}{\sqrt{2}(\sqrt{3}-1)} = \frac{a(\sqrt{3}+1)}{\sqrt{2} \cdot 2}$$

$AM = c^2 t$.

650
650

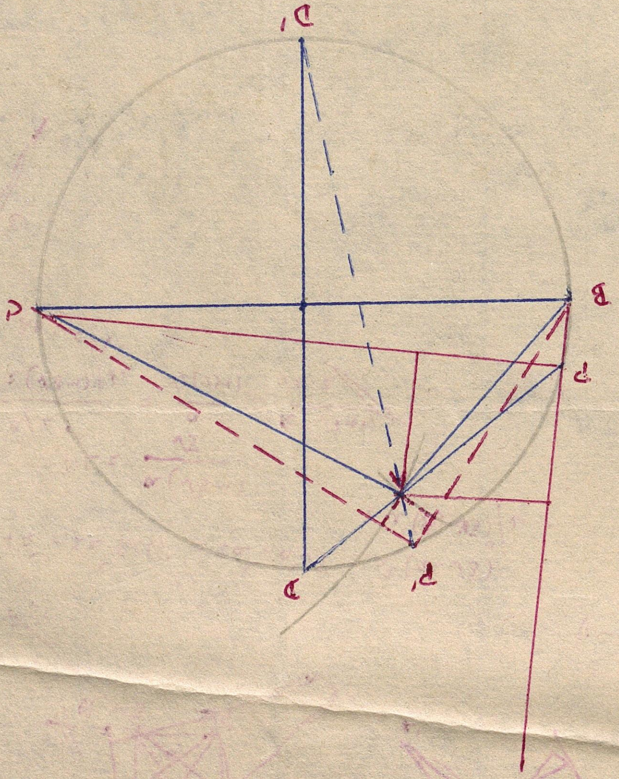
1805.22
1300.00
505.22

1682.62
505.22
1177.40

1177.40
606.40
571.00

588.70.

1682.62
606.40
2289.02
650.00
$\boxed{1639.02}$



$b^2+c^2-2bc\sin A.$

$AD = \frac{1}{2}\sqrt{b^2+c^2+2bc\cos A}$

$AL' = \sqrt{b^2+c^2+2bc\sin A}$

$AL = \sqrt{b^2+c^2-2bc\sin A}.$

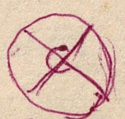
$4AD^2+AL'^2 =$

$AL'^2 - AL^2 = 4bc\sin A,$

$\frac{x^2}{x'^2} = \frac{AL'^2}{AL^2} = \frac{b^2+c^2+2bc\sin A}{b^2+c^2-2bc\sin A}$

$\frac{x^2+x'^2}{x^2-x'^2} = \frac{b^2+c^2}{2bc\sin A}$

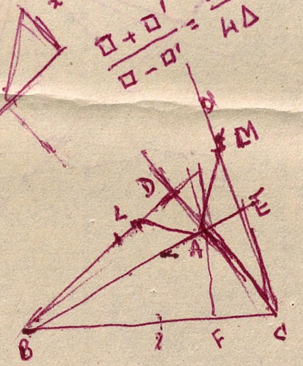
$\frac{\Box+\Box'}{\Box-\Box'} = \frac{b^2+c^2}{4\Delta}$



$\sin(180^\circ+A) = -\sin A.$

$180^\circ - A + 90^\circ$

$\cos(270^\circ - A) = \cos 270^\circ \cos A + \sin 270^\circ \sin A$



$AL^2 = c^2+b^2-2cb\cos(A-90^\circ)$

$= b^2+c^2-2bc\sin A$

$= b^2+c^2-2bc\cos(270^\circ-A).$

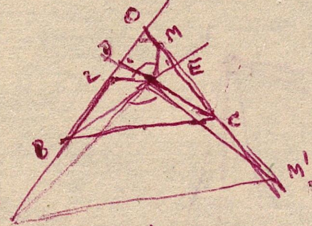
$AL'^2 = b^2+c^2-2bc\sin A$

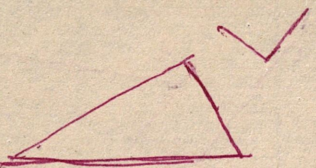
$= b^2+c^2+2bc\sin A$

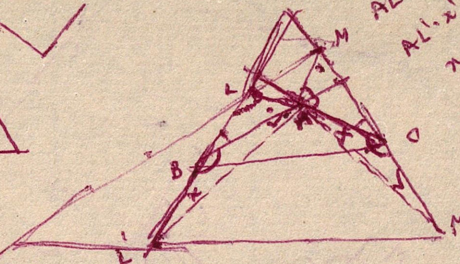
$AL.x = bc\cos A.$

$AL'.x' = bc\cos A$

x & x' are sides of squares







$OL' = OL + 2b$

$OM' = OM + 2c$

$\cancel{AL'+AM}$

$=$

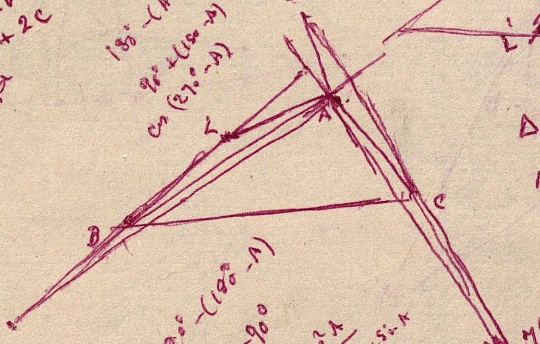
$180^\circ - (A-90^\circ)$

$90^\circ + (180^\circ - A)$

$\cos(270^\circ - A)$

$\Delta BAL \equiv \Delta CMA$

$\Delta BAL' \equiv \Delta CM'A$



$90^\circ - (180^\circ - A)$

$A - 90^\circ$

$x'^2 = \frac{b^2c^2\cos^2 A}{b^2+c^2+2bc\sin A}$

$x^2 = \frac{b^2c^2\cos^2 A}{b^2+c^2-2bc\sin A}$

$x^2+x'^2 = \frac{b^2c^2\cos^2 A\,(b^2+c^2)}{(b^2+c^2)^2-4b^2c^2\sin^2 A}$

$= \frac{2b^2c^2(b^2+c^2)\cos^2 A}{(b^2-c^2)^2+4b^2c^2\cos^2 A}$

$\frac{1}{x'^2} = \frac{b^2+c^2-2bc\sin A}{b^2c^2\cos^2 A}$

$\frac{1}{x^2}+\frac{1}{x'^2} = \frac{2(b^2+c^2)}{4\Delta}$

$AL.x = bc\cos A$

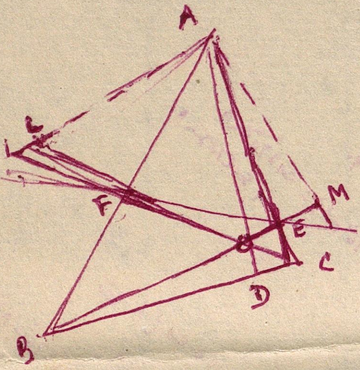
$AL'.x' = bc\cos A.$

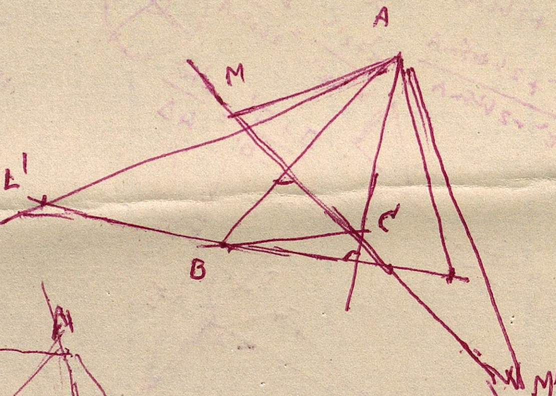
$\frac{x'}{x} = \frac{AL}{AL'}.$

$\frac{x'^2}{x^2} = \frac{AL^2}{AL'^2}$

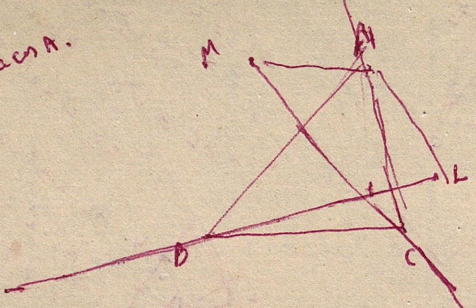
$\cos A = \frac{b^2+c^2-a^2}{2bc}$

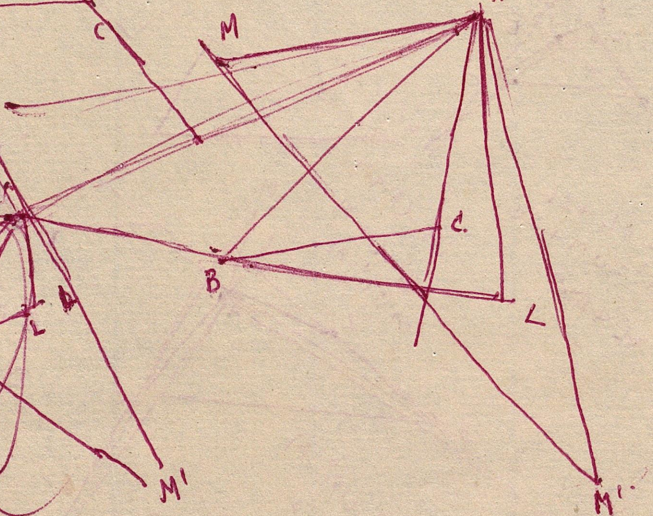
$\frac{2b^2c^2(b^2+c^2)(b^2+c^2-a^2)^2/4b^2c^2}{(b^2-c^2)^2+4b^2c^2(b^2+c^2-a^2)^2/4b^2c^2}$

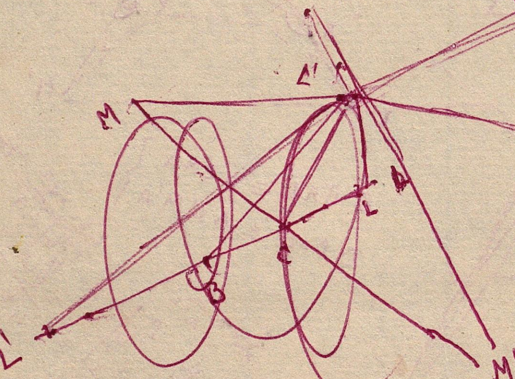




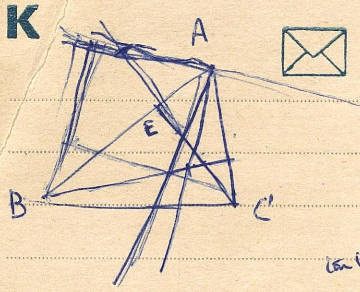
$EF = a\cos A.$







K



$\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$

$\cot(A-B) = \dfrac{1+\tan A\tan B}{\tan A - \tan B} = \dfrac{1+\cot A\cot B}{\cot B - \cot A}$

$\phi = 90^\circ - (A-\theta)$

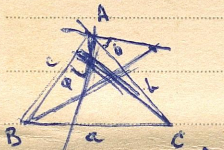
$\tan\phi = \cot(A-\theta) = \dfrac{1+\tan A\tan\theta}{\tan A - \tan\theta} = \dfrac{\cos A + \sin A\tan\theta}{\sin A - \cos A\tan\theta}$

$\dfrac{c\cos^2 A + \sin A(c\sin A - b)}{c\sin A\cos A - \cos A(c\sin A - b)} = \dfrac{c - b\sin A}{b\cos A}$ ✓

Constr with $\triangle^s$ BD & CE give the same square.//

$\dfrac{b - c\sin A}{c\cos A} = \tan\theta$

$\tan\phi = \dfrac{b\sin A - c}{b\cos A}$.

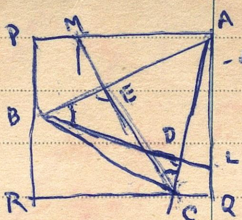


$\theta - \phi = (90^\circ - A)$ $\quad \phi = \theta - (90^\circ - A)$ $\quad \phi = \theta - (90^\circ - A)$ $\quad \phi = \phi + (90^\circ - A)$

$\phi + 90 - \theta = A$ $\quad \phi - \theta = (90^\circ - A)$ $\quad \theta = \phi - (90^\circ - A)$.

$c\sin A < b$.

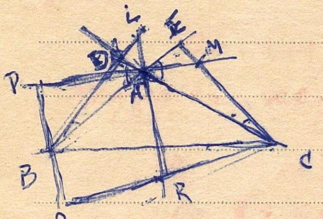
$b\sin A > c$



$\int_{-\infty}^{\infty} \delta(-x)\cdot dx = 1$.

$\int_{-\infty}^{\infty} \delta(-x)\cdot dx = 1$

$\delta(-x) = \delta(x)$



$\delta(x) = 0,\ x \neq 0$.

$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$.

$\delta(x) = \delta(-x)$.

$x\delta(x) = 0$

$\int f(x)\,\delta(x-a)\,dx = f(a)$//

$\int \delta(a-x)\,dx\,\delta(x-b) = \delta(a-b)$

$\triangle^s$ ABL & ACM.

BL = AC

CM = AB

$\triangle^s$ ABE & ACD

$\hat{E} = \hat{D} = 90^\circ$

BAE = DAC

ABE = ACM.

483

~~BAL = MAC~~

from $\triangle^s$ BDA & CEA

BAD = CAE

$\therefore$ ABD = ACE

$\triangle^s$ BAL & CAM.

ABL = ACM.

BAL = CMA

BL = AC. $\therefore$ CM = AB

$\triangle^s$ are $\equiv$.

$\triangle^s$ BDA, CEA

$\hat{ABD} = \hat{ACE}$

BAL & CMA

$\hat{BAL} = \hat{CMA}$

$\hat{ABL} = \hat{ACM}$

BL = AC

CM = AB

500 to 499

$(5,-1,-7) = 483705$

$(6,-9,-3)$ 507384.

$\begin{array}{r} 599994 \\ 507384 \\ \hline 92610 \\ 89990 \\ \hline 2700 \end{array}$

600 / 90 / 510

89990

$[10-x, -(10-4)] \quad -(10-3)]$

$(5,1,-3)$ 507285 582705

$(5,9,-8)$ $10-x, 10-y, -(11-3)$

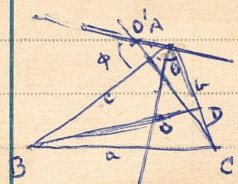
$\begin{array}{r} 499995 \\ 82710 \\ 89910 \\ \hline 7200 \end{array}$

O.K.

L

$\tan\phi = \dfrac{c - b\sin A}{b\cos A}$ | $\dfrac{b\sin A - c}{b\cos A}$

$\tan\theta = \dfrac{b - c\sin A}{c\cos A}$



$\phi = 90^\circ - (A-\theta)$

$\tan\phi = \cot(A-\theta)$

$= \dfrac{1+\tan A\tan\theta}{\tan A - \tan\theta}$.

$\tan\theta = \dfrac{c\sin A - b}{c\cos A}$

$\tan\phi = \dfrac{c - b\sin A}{b\cos A}$

~~$1 + \tan A\,(b - c\sin A)/b\cos A$~~

~~$\tan A - (b - c\sin A)/b\cos A$~~

~~$b\cos A + (b - c\sin A)\tan A$~~

~~$b\cos A\tan A - (b - c\sin A)$~~

~~$b\cos A + (b - c\sin A)\sin A$~~

~~$b\sin A - (b - c\sin A)$~~ =

If $c > b$. ~~$c > c\sin A$~~

If $b > c$, $b > c\sin A$

$c\sin A < b$. $\quad b > c,\ B > C$

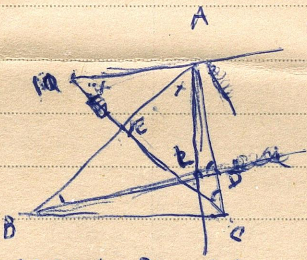
~~$c\sin A > b$~~ If $b\sin A > c \to b > c$

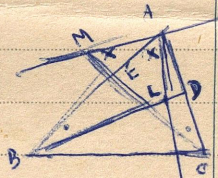
$\int f(x)\,\delta(x-b)\,dx = f(b)$

$\int \delta'(a-x)\,dx\,\delta(x-b) = \delta'(a-b)$

-ve

By parts formula





~~BD = AC~~ ~~BL = AC~~

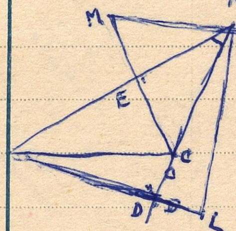
$\triangle^s$ BAD & CAE.

$f(x) + f(-x) = 2f(x)$

$x\cos x + \cos(-x) = 2\cos x$

$\cos(-x)$. BL = AC.

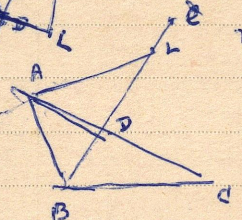
$x\cos(x)$.

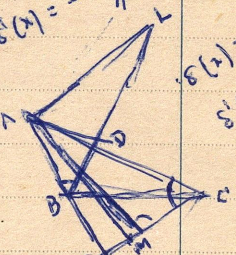


$x\delta(x) = 0$

$x\delta'(x) + \delta(x) = 0$.

$x\delta'(x) = -\delta x$.//





$\delta(x) = \delta(-x)$

$\delta'(x) = -\delta'(-x)$.

$\delta'(-x) = -\delta'(x)$

$-x\delta(-x) = x\delta(x)$.

$x\delta(x) = x\delta(x) + x\delta(x)$

$\int_{-\infty}^{\infty} f(x)\,\delta(x-y)\,dx = f(y)$

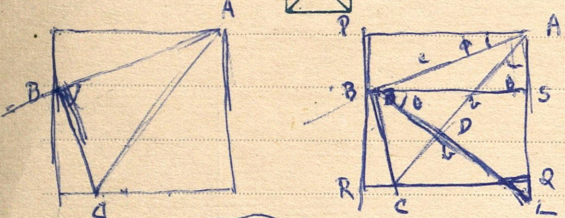
$\int_{-\infty}^{\infty} f(x)\,\delta'(x-y)\,dx = f'(y)$

$\int f(x)\,\delta'(x-y) = -f'(y)$

M

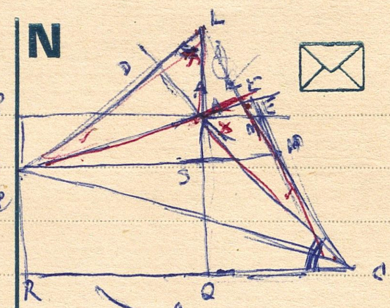
N

m 441

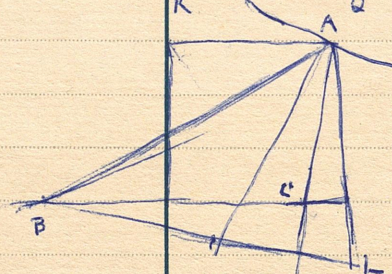


$AQ = b \cos\theta$.
~~$AP = c \cos\phi$~~.
$AP = b \sin(90-\theta)$
$= b \cos\theta$

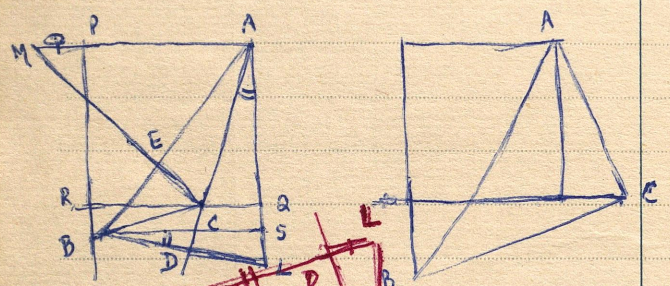
$\Delta BSL \equiv \Delta CRA$

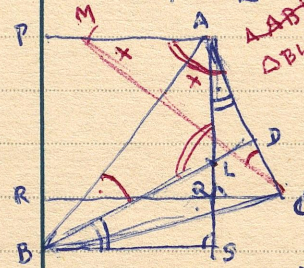


$BS = AQ = AP$
$AP = AQ$

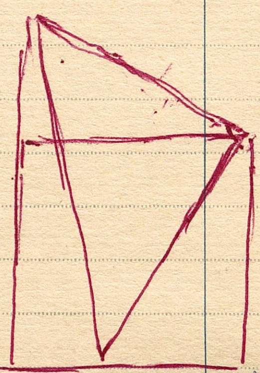


Cantor fails when $\angle A = 90°$

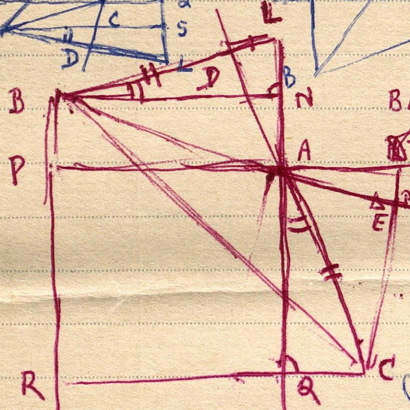




~~$\Delta ABL \equiv \Delta$~~
$\Delta BLA \equiv \Delta CAM$



(1, −1, 1)



BL = AC.

$\Delta BLN \equiv \Delta ACQ$.

BN = AQ = AP

CM = AB.

(5, −5, 10)

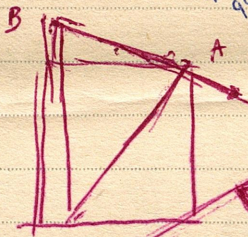
499995
49950
450045
9000
459045.

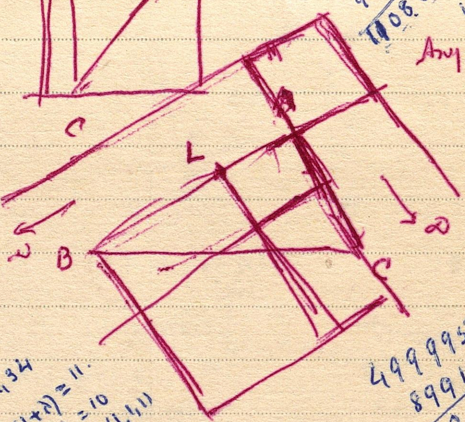
99999
9990
900
110889

999990
99900
9000
1108890

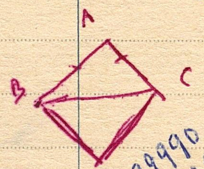
1108890 = ~~10 × 110889~~.
10 × 0110889

99999.
9990.
90009
900
090909.

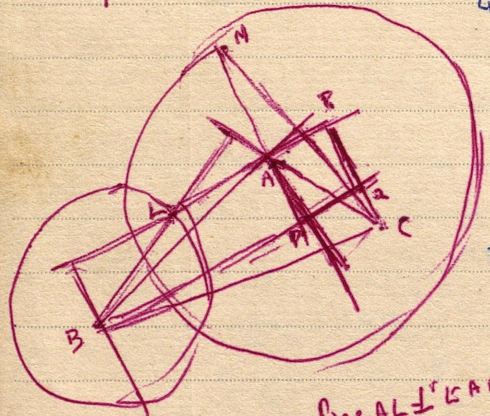




Any square on AB & AC



999990
99900
900090
9000
909090



399996
49950
350046
9000
359046

399996
29970
4500
434466

(4, 3, 5)
(6, 6, 5)

66434

$x(1+d) = 11$.
$d = 10$
$x = (1, 1, 1)$
$11 - x = \ldots$
$11 - y = 14$
$11 - z = \ldots$

499995
89910
410085
9000
419085

(4, −4)

Prove AL ⊥ʳ AM

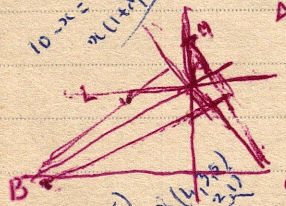
Δ^s ~~ABL~~ & M.
Δ^s ALB & MAC are ≡
Δ^s AQC & BLN "

640953

699993
640953
59040
59940
900

(4, −5, 10)

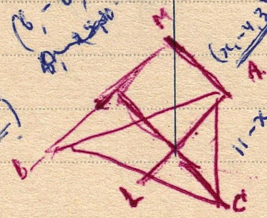
$10 - x = x(1+d) = 10$



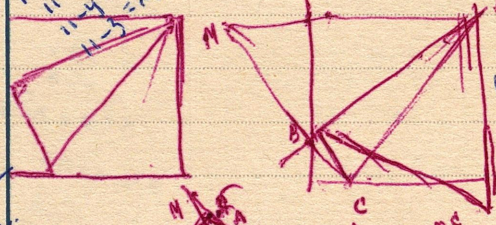
(4, 3, 5)
(6, 6, 3) = (4, ...)
$10 - x = 2x$

6999 (6, −6, 1)

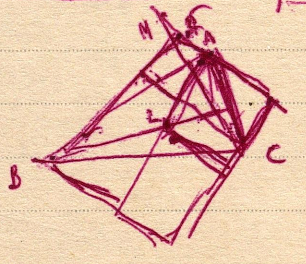
(4, −5, 10)
(7, −6, 1)



(x, −4, 3)
$11 - x, -(11-y), 11 - z$

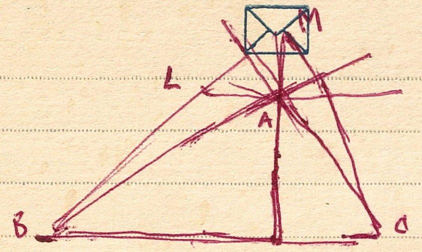


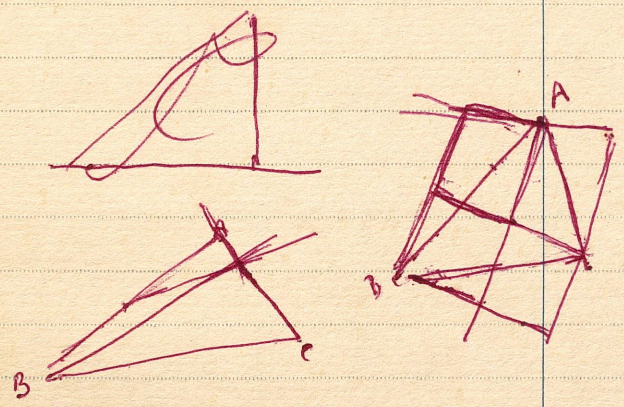
(4, −6, 1)
(7, −5, 10)
$(11 - x, \ldots 11 - 1)$

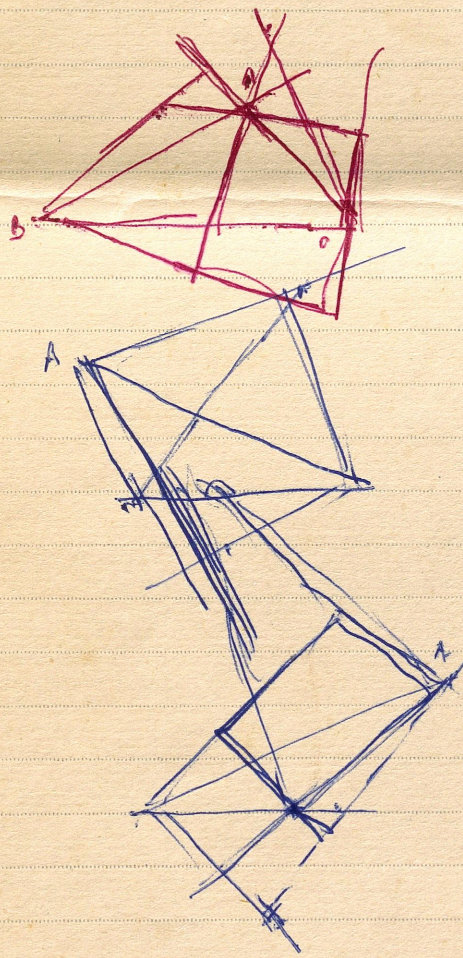


$B\hat{A}L = AMC$
$MAB + AMQ = 90°$
$B\hat{A}L + M\hat{A}B = 90$
∴ AL ⊥ AM

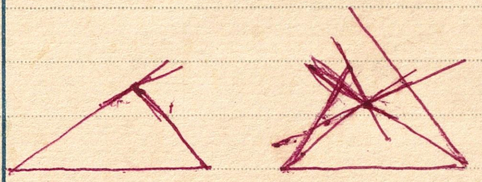
O

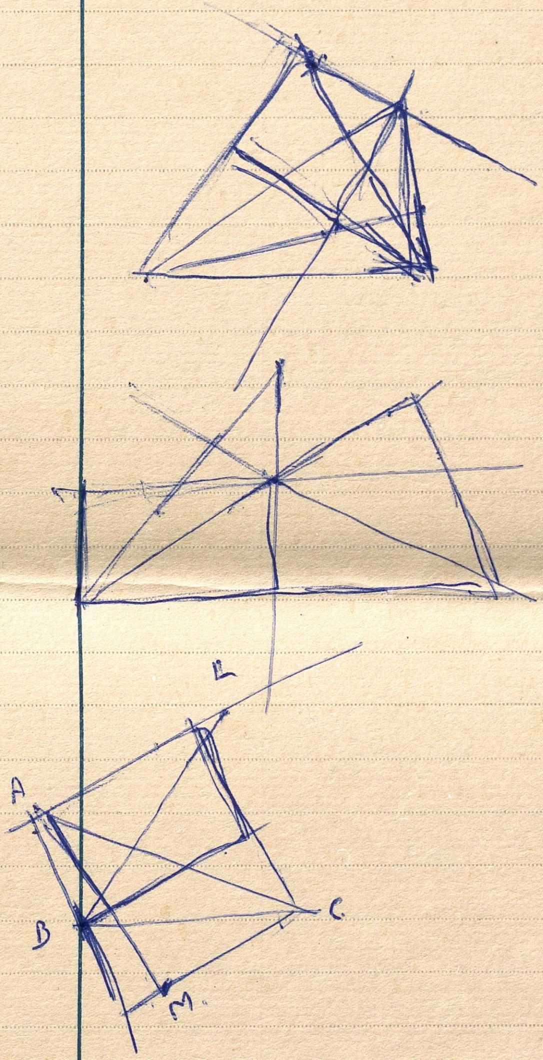




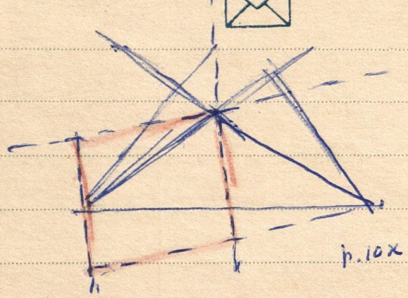


P





Q

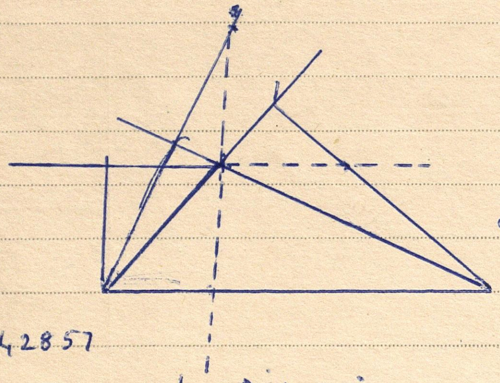


$p.x = q(10x - a_1 10^3 + a_1)$

$p.10x = q(x - a_1 10^3$

$x(10q - p) = q.a_1(10^3 - 1)$.

$p(a_1 \cdot 10^3 + a_2 \cdot 10^2 + a_3 \cdot 10)$
$= q(a_2 \cdot 10^2 + a_3 \cdot 10 + a_1)$



142857

428571

$\frac{1}{7} = 0.\dot{1}4285\dot{7}$

$\frac{3}{7} = 0.\dot{4}2857\dot{1}$

$3 \cdot \dot{4}2857\dot{1}$

142857142857   142857.

428571428571   714285.

$\frac{5}{7} = 714285$

758241.

$\frac{2}{7} = 285714$

$\frac{6}{7} = 857142$

$\frac{1}{11} = \cdot 09$

857142 = 285714 × 3

$\frac{1}{7} = 142857$; $\frac{2}{7} = 285714$, $\frac{3}{7} = 428571$.

$\frac{4}{7} = 571428$, $\frac{5}{7} = 714285$ 2857140

$\frac{6}{7} = 857142$.

$\frac{6}{7} = 857142$
142857

$\frac{1}{17} = \cdot 0529411$

$\frac{1}{13} = \cdot 077$

758241   758241   142857   615384

(6, 1, 6). (2, 5, 3).

6 1 5
4 8 3

171428
$\boxed{571428}$

428571
175824
252747

171428
857142

(2·10³+3·10+1)p
$(3 \cdot 10^2 + 8 \cdot 10 + 2)q$

$\frac{10^2 p - q}{10q - p}$

$10^2(2p - 3q)$
$+10(3p - 8q)$
$+ (p - 2q)$.

p 4 : 3

231 × b/a
312
231 · p = 312 q

$10^2 + 2$.
$4 \cdot 10^2 + 8$
$3 \cdot 10^2 + 10^2 + 10$

10 + 2

12
21

$\frac{48}{3} = 16$

4·10+8
3·10+4+8
3+4

37 + 73 = 100

100 + 001 = 101.

45 + 54 = 9

32 + 23 = 55

79 + 97 = 176

176 + 671 = 847

847 + 748 = 1595

1595 + 5951 = 7546

7546
6457
14003

$\frac{x}{a} = \frac{10^n \mu}{10\mu - 1}$

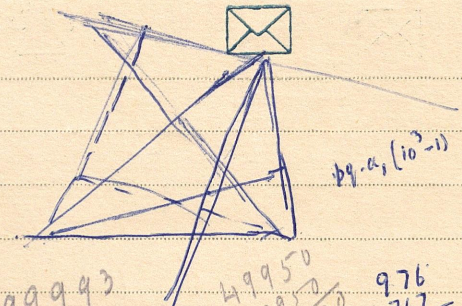
$= \frac{10^n p - q}{10q - p}$

$\frac{10^n - \mu}{10\mu - 1}$

$= \frac{10^5 - 3}{30 - 1}$

$\frac{3 \cdot 10^n - 1}{7}$

R



$pq \cdot a_1(10^3 - 1)$

9990

699993
659043
40950

49950
40950
9000

976
767
1743
3471
5214
4125
9339

169
961
1130
0311
1441

$10pq(a_1 \cdot 10^2$
$+ pq \cdot a_2 10 + pq \cdot a_3 \cdot 10$
$+ pq \cdot a_3 10 + a_1 pq$.

$= a_2 pq \cdot 10^2 + a_3 pq \cdot 10 + a_1 pq$.

976
679
97
79
18

59940
54440

$p(a_1 \cdot 10^2 + a_2 \cdot 10 + a_3)$
$= q(a_2 \cdot 10^2 + a_3 \cdot 10 + a_1)$

$p.x = q(a_2 \cdot 10^2 + a_3 \cdot 10 + a_1 \cdot 10^3 + a_1)$
$= q(10x - a_1 \cdot 10^3 + a_1)$

14003
30041
44044

= 31034

1034482758620689655172413 × 3

$px = q(10x - a_1(10^3 - 1))$.

$x(10q - p) = q.a_1(10^3 - 1)$

49

$\mu = q/p$.

100000
99997 (3444

29) 99997 (344
87
129
116
139
116
239
232

237
999999
142857

$\frac{10^5 - 3}{10 - 3}$

77
2999997
428571

3000000

$a_1 a_2 a_3$ $a_2 a_3 a_1$

$a_1 \cdot 10^3 + a_2 \cdot 10^2 + a_3$

$a_2 \cdot 10^3 + a_3 \cdot 10^2 + a_1$

$a_1(10^3 - 1) + a_2(10^2 - 10^3) + a_3(1 - 10^3)$.

$(a_2 - a_3)(10^3 - 1) = 10 a_2 (10 - 1)$

123
321.
198.

$a_1 a_2 a_3 a_4$ $a_1 a_2 a_3$ $a_2 a_3 a_1$

$a_2 a_3 a_4 a_1$. $a_1 a_2 a_3 a_4 a_5 a_6$

142857
428571.

$a_2 a_3 a_4 a_5 a_6 a_1$

$10(a_1 \cdot 10^6 + a_2 \cdot 10^5 + \ldots + a_5 \cdot 10 + a_6)$

$a_2 \cdot 10^6 + \ldots + a_5 \cdot 10^2 + a_6 \cdot 10 + a_1$

$10a_1 \cdot 10^7 + a_2 \cdot 10^6 + \ldots + a_5 \cdot 10^2 + a_6 \cdot 10 +$

$a_2 \cdot 10^6 + \ldots + a_6 \cdot 10 + a_1$

(7, 5, 9)

999999
9990

699993
659043
40950

49950
9000

599

429
561

450
540   4, 6, 2

(5, 4, 1)

461
558
340

(4, 6, 1)

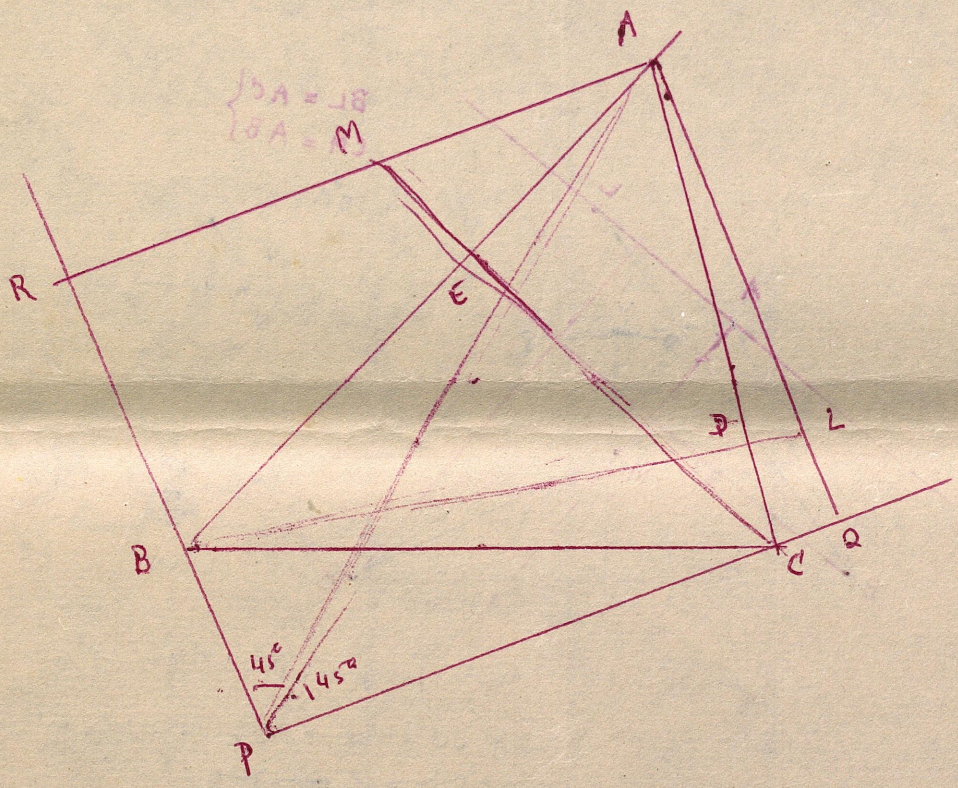
340
659

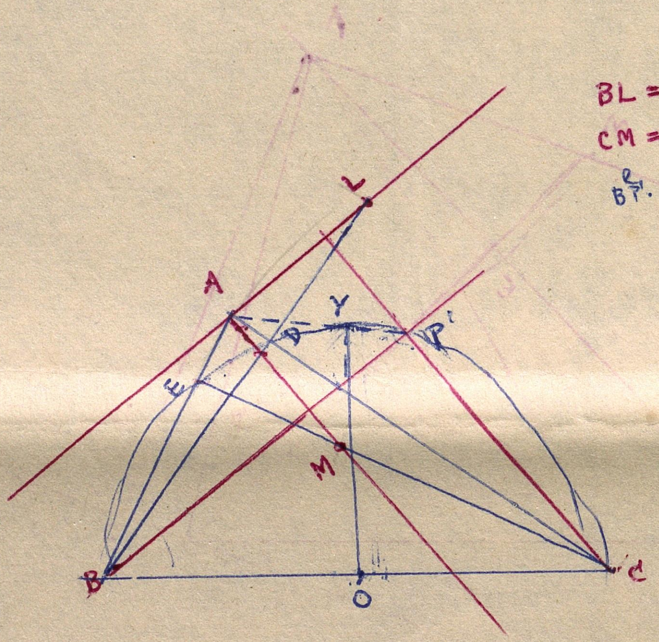
659043

(9, 5, 1)

956

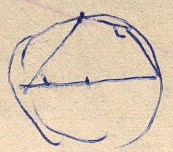
(6, 5, 9)


A
M
R
E
D
L
B
C
Q
P
45°
45°



BL = AC
CM = AB

BP.



If semi-$\odot^{le}$ on AB as diameter passing through D, E & P' meets AP' at Y.

$Y\hat{B}C = 180° - YP'C = 180° - 135° = 45°$

∴ ~~Y is the ex of Y is~~

OY is $\perp^r$ to BC & OY = OB = OC

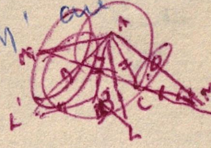
& this is the original Constⁿ

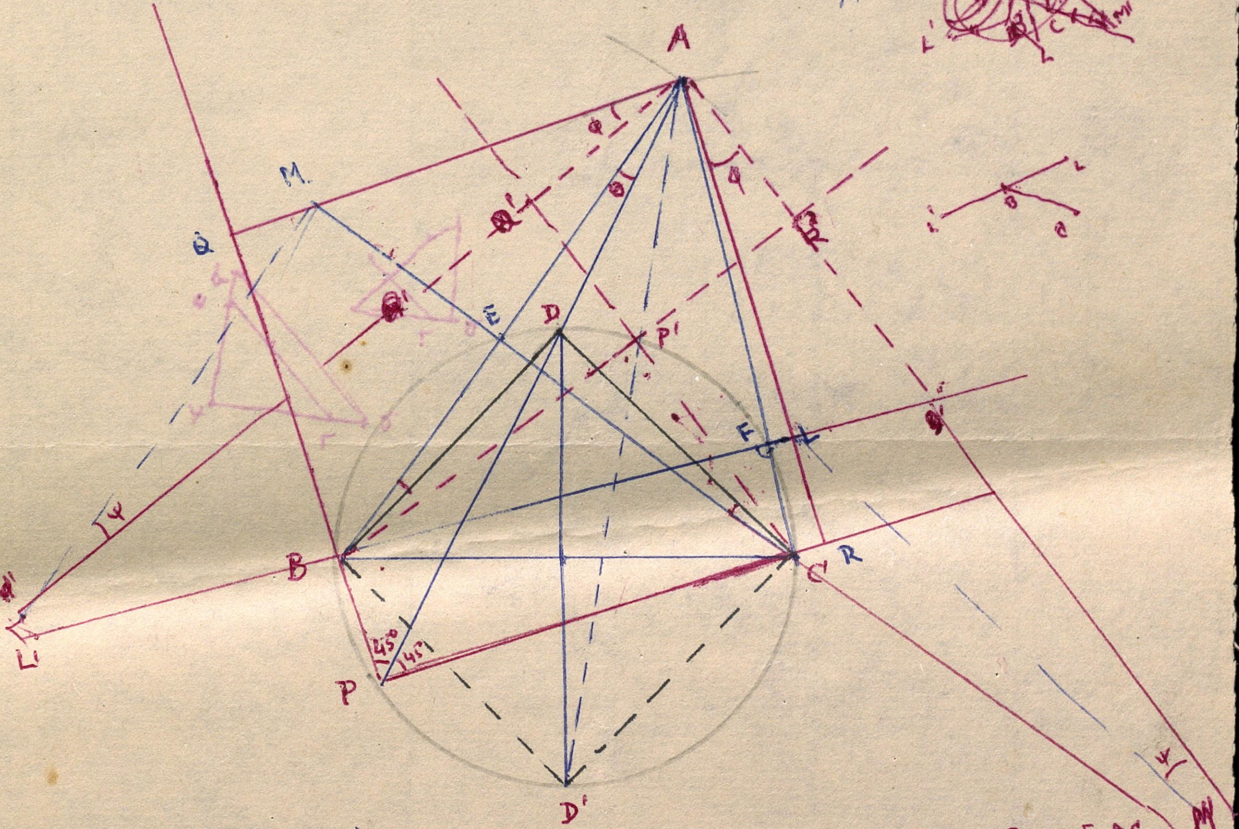
$$\frac{ML'}{\sin\phi} = \frac{\sqrt{b^2+c^2+2bc\sin A}}{\sin(\phi+\theta)} = \frac{\sqrt{b^2+c^2-2bc\sin A}}{\sin\phi}$$

$$ML^2 = b^2+c^2+2bc\sin A + b^2+c^2-2bc\sin A$$
$$-2\sqrt{\ }\sqrt{\ }\cdot\cos\phi$$
$$=2\left(b^2+c^2\right)-2\sqrt{\ }\ \cancel{b^4+c^4+4b^2c^2}$$

$(b^2+c^2)^2-4b^2c^2\sin^2 A$
$b^4+c^4 \mp 4b^2c^2\sin^2 A$
$+2b^2c^2$.

AML' ≅ E.
ALM'





Δ's AFL & ARC are similar

$\therefore \frac{AL}{AF} = \frac{AC}{AR}$ i.e. $AL\cdot AR = AF\cdot AC = c\cos A\cdot b = bc\cos A$

Similarly Δ's AEM & AQB are similar

$\frac{AM}{AE} = \frac{AB}{AQ}$ i.e. $AM\cdot AQ = AE\cdot AB = b\cos A\cdot c = bc\cos A$.

i.e. $AL\cdot AR = AM\cdot AQ$ but $AQ = AR$ $\therefore AL = AM$

Of Δ^s BAL & CAM, these are similar $\because \hat{ADL} = \hat{ACM}$ & $\hat{MAC} = 90^\circ - CAL = \hat{ALB}$, but $AL = AM$

$\therefore$ Δ^s are ≡ i.e. $BL = AC$ and $CM = AB$. Hence the proof.

Alternately FLRC is a cyclic quadrilateral $\therefore AL\cdot AR = AF\cdot AC$

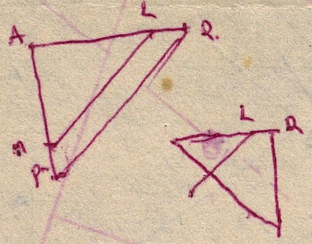
Similarly BEMQ is cyclic $\therefore AM\cdot AQ = AE\cdot AB$

Again BCFE is cyclic $\therefore AF\cdot AC = AE\cdot AB$

$\therefore AL\cdot AR = AM\cdot AQ$

but $AQ = AR \therefore AL = AM$.

etc


A
L
Q
M
P
L
Q

G

(77743)

13 77701

87
69
189

77792+5

77725 + 18

457

77791 − 48 75 1785

17791 17746 + 61

16948

142857 285714

~~724518~~

758214

~~428571~~

428571 428571 857142

$A + 2\theta = 90°$

$\theta = \frac{1}{2}(90° - A)$

148857 758241

8712

428571

857142

$a\cos\theta + c\cos$

$a\cos\theta - c\cos(B-\theta)$

$= a\sin\theta - b\cos(C-\theta)$

$b\cos\theta = c\cos\phi.$

$b\sin\theta + a\sin(\phi - \theta)$

$= a\cos(\phi - \theta)$

$+ c\sin\phi.$

$b\sin\theta + a\sin\phi\cos\theta$

$- a\cos\phi\sin\theta$

$= a\cos\phi\cos\theta + a\sin\phi\sin\theta$

$+ c\sin\phi$

$180° - (A + C + \theta)$

$B - \theta.$

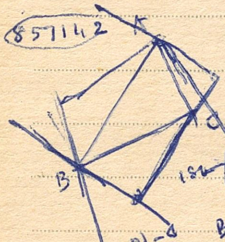
$180° - (A + B + \theta)$

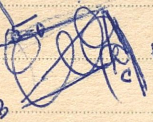
$C - \theta$

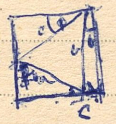
$B + C - 2\theta = 90°$

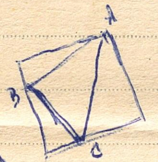
$180° - A - 2\theta = 90°$

$A + 2\theta = 90°$



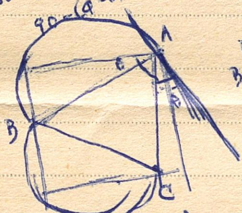


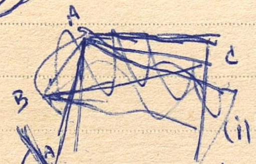




$180 - (90 - \theta) - \phi$

$180 - (90 - (\phi - \theta))$





(i) A acute.

$c\sin\theta = b\cos\phi$ — (i)

$c\sin\theta + a\sin(B - \theta)$

$= b\sin\phi + a\cos\theta(B - \theta).$

$180° - \overline{90 - \theta + B}$

$90° - (B - \theta)$

$c\sin\theta + a(\sin B\cos\theta - \cos B\sin\theta)$

$= b\sin\phi + a(\cos B\cos\theta + \sin B\sin\theta)$

$c\sin\theta + a(\sin B \cdot \cos\theta - \cos B\sin\theta)$

$= b\sin\phi + a(\cos B\cos\theta + \sin B\sin\theta)$

$\frac{a}{b} = \frac{\sin A}{\sin B}$

$c\sin\theta + b\sin A\cos\theta - a\cos B\sin\theta$

$= b\sin\phi + a\cos B\cos\theta + b\sin A\sin\theta$

$90 - \phi + C$

$90° - (C - \phi)$

$= B - \theta.$

$B - \theta + C - \phi = 90°$

$B - \theta - (\phi + \theta) = 90°$

$(B + C) - (\phi + \theta) = 90°$

$(180° - A) - (\phi + \theta) = 90°$

$90° - A = \phi + \theta$

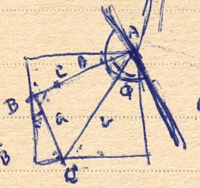
$90° - A - \theta$

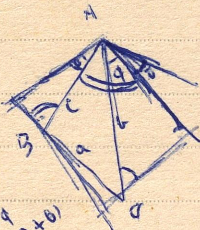
$a = 90° - A$

$\phi + A = 90°$

$\cos\theta = b\cos\phi$

$= b\sin(A + \theta)$

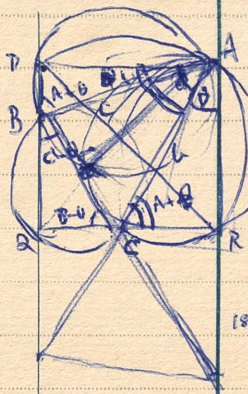




H

$\angle ... = \phi.$

$c\cos\theta = b\cos\theta.$



$A + \theta = 90° - A$

$A + B = 90° - \theta$

$90° - C = 90° - \theta$

$90° - \theta$

$= \phi$

$180° - (B + 90° - \phi)$

$90° - (B - \phi)$

$180° - (A + C + \phi)$ ~~$90 - A + B$~~

$180° - (A + C) - \phi.$ ~~$180° - (90° + B - \phi)$~~

$B - \phi$ $90° - B$

$A + B + B.$

$180° - (A + B + \theta)$

$C - \theta$

$180° - (90° - \phi + B)$ $A + \phi = 90° - \theta$

$90° - (B - \phi)$ $\phi = 90° - A + \theta$

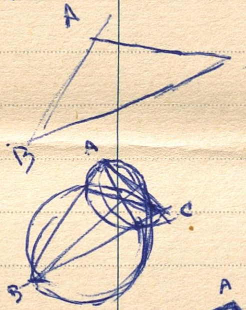
$B - (90° B - 90° + A + \theta$ $90° - B + A$

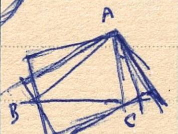
$180° - C - 90° + \theta$ ~~$90 - \theta$~~

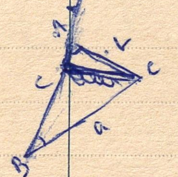
$90° - (C - \theta).$

$c(\sin A\cos\theta + \cos A\sin\theta) = b\cos\theta.$

$\cos\theta(b - c\sin A) = c\cos A\sin\theta.$







$b\cos A/2$

~~$c\cos A/2 - b\cos A/2 = b\cos A/2$~~

~~$c\cos\theta - b\sin\theta = b\sin\theta$~~

~~$c\cos\theta = 2b\sin\theta.$~~

~~$c\sin\theta$~~

$a\sin(B + \theta) - b\sin(A - \theta)$

$= c\cos\theta - b\cos(A - \theta)$

$a(\sin B\cos\theta + a\cos B\sin\theta - b\sin A\cos\theta + b\cos A\sin\theta$

$= c\cos\theta - b\cos A\cos\theta - b\sin A\sin\theta$

$\sin\theta(a\cos B + b\cos A + b\sin A).$

$= \cos\theta(c - b\cos A).$

$\sin\theta(a\cos B + b\cos A - a\sin B + b\sin A).$

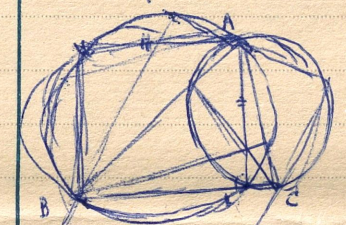
$= \cos\theta(c - b\cos A).$

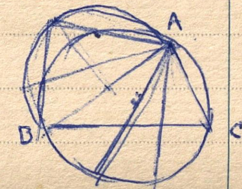
$\sin\theta(c + b\sin A) = \cos\theta(c - b\cos A).$

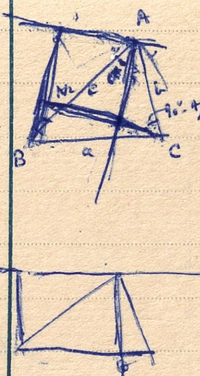
$\tan\theta = \frac{c - b\cos A}{c + b\sin A} = \frac{a\cos B}{c + b\sin A}$

$\frac{\sin A\cos B}{\sin C + \sin A\sin B}.$

$\frac{c}{\sin C} = \frac{a}{\sin A}$







$l = c\cos\phi = b\cos\theta.$

$l = c\sin\phi + a\sin(B - \phi)$

$l = b\sin\theta +$

$l = c\sin(A + \theta) = b\cos\theta$

$l = c\cos(A + \theta) + a\cos(C - \theta)$

$l = a\sin(C - \theta) + b\sin\theta$

$l = c\sin B = b\cos(B - A)$

$c\sin B = b\cos B\cos A + b\sin B\sin A$

E
F
G
H

I

$c\cos\phi = b\sin(A+\phi)$.

$c\cos\phi = b(\sin A \cos\phi + \cos A \sin\phi)$

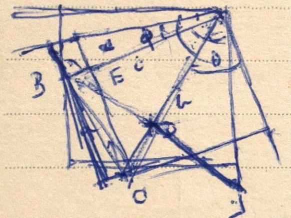
$b\cos A \cdot \sin\phi = \cos\phi(c - b\sin A)$

$b\cos\theta = c\sin(A+\theta)$

$b\cos\theta = c(\sin A\cos\theta + \cos A\sin\theta)$

$c\sin\theta\cos A = \cos\theta(b - c\sin A)$

$\tan\theta = \dfrac{b - c\sin A}{c\cos A}$



$\dfrac{a\cos B}{c + b\sin A}$

$\dfrac{AC - BD}{AD} \cdot \dfrac{AB - CE}{AE}$

$b = c\sin A$

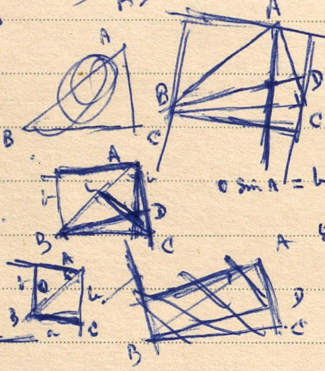
$\sin A = b/c$

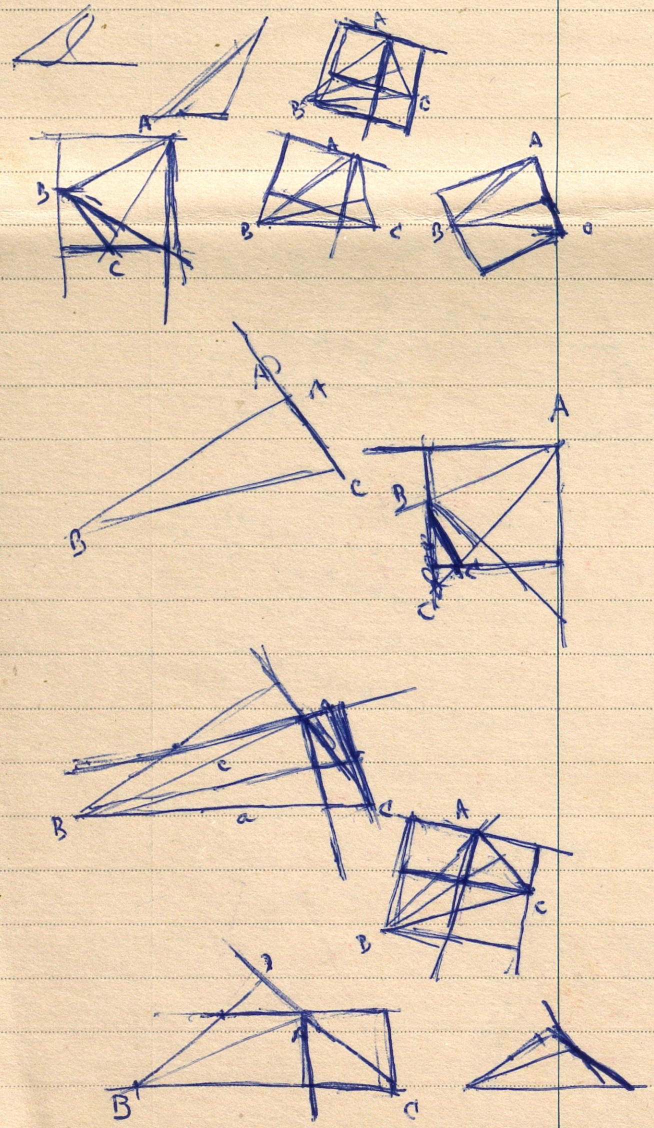
$\frac{1}{2} : \frac{0}{\ } : \boxed{\frac{3}{2}} : \frac{2}{\ }$

$e \quad p \quad q$

$c\sin A = b/c$

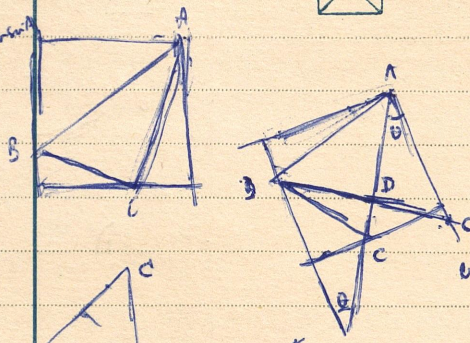
$\sin A = \dfrac{BD}{c}$





J

$\tan\phi = \dfrac{c - b\sin A}{b\cos A}$

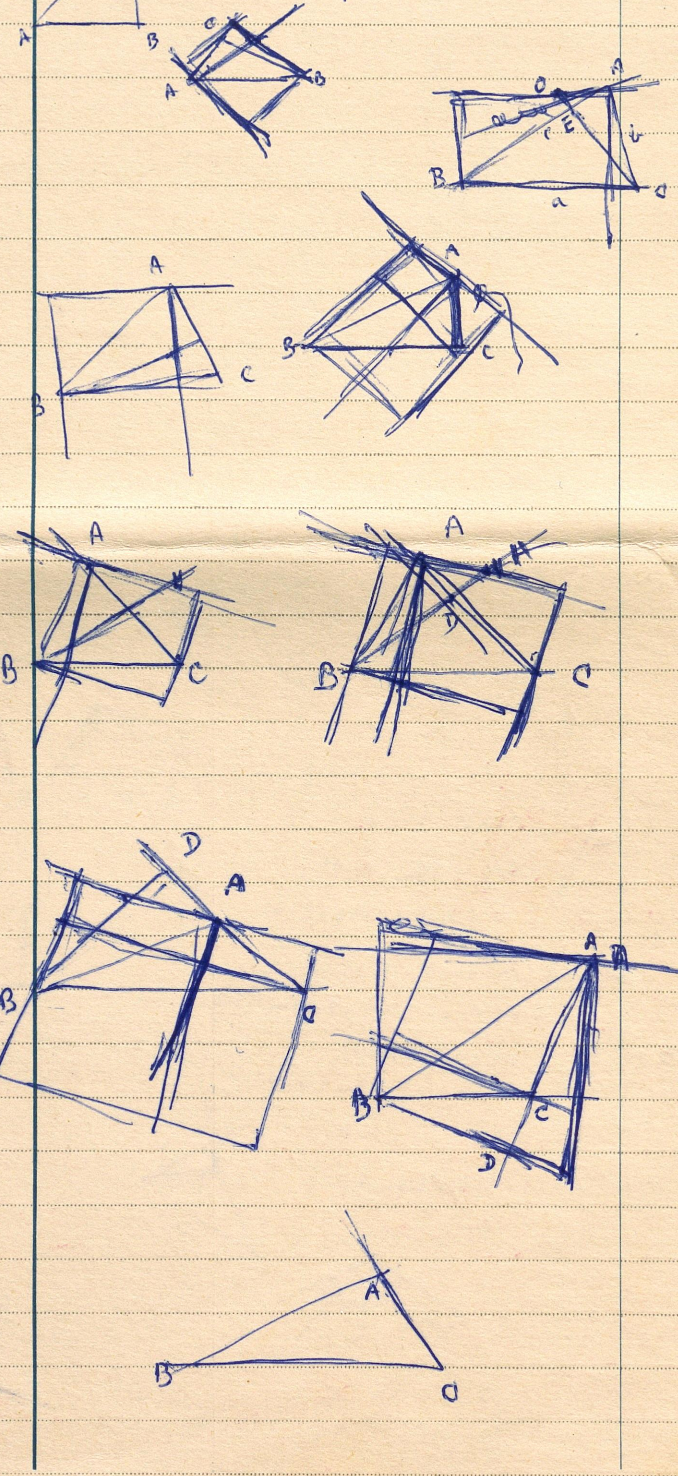


$\tan\theta = \dfrac{OD}{AD}$

$= \dfrac{b - BD}{AD}$

$b\cos\theta = c\sin(A+\theta)$.

$\tan\theta = \dfrac{b - c\sin A}{c\cos A} =$



1, 3
5, 7
11, 13
17, 19
29, 31
41, 43
59, 61
71, 73
101, 103
107, 109
137, 139
149, 151
7 ← ~~161~~, ~~163~~
179, 181
191, 193
197, 199
13 ← ~~221~~, 223
227, 229
239, 241
257, 7 ← ~~259~~

269, 271
281, 283
~~299~~ → 13, 301
311, 313
347, 349
~~371~~ → 7, 373
~~377~~ → 13, 379
401, 403
419, 421
431, 433
~~437~~ → 19, 439
461, 463
467, ~~469~~ → 7
479, ~~481~~ → 13
491, ~~493~~ → 17
509, ~~511~~ → 7
521, 523
557, ~~559~~ → 13
569, 571
587, ~~589~~ → 19
599, 601
~~611~~ → 13, 613

617, 619
~~629~~ → 17, 631
641, 643
647, ~~649~~ → 11
677, ~~679~~ → 7
~~689~~ → 13, 691
701, ~~703~~ → 19
~~707~~ → 7, 709
~~731~~ → 17, 733
761, ~~763~~ → 7
~~767~~ → 13, 769
797, ~~799~~ → 17
809, 811
821, 823
827, 829
839, 841
~~851~~ → 23, 853
857, 859
881, 883
887, ~~889~~ → 7

899, ~~901~~ → 17
929, ~~931~~ → 7
941, ~~943~~ → 23
947, ~~949~~ → 13
~~959~~ → 7, 961
971, ~~973~~ → 7
~~989~~ → 23, 991

1001 × 1001
1001
0000
0000
1001
1002001

10001 × 101
10001
00000
10001
1010101

9091 / 19 = 479

1000001
= 16

Prime pairs between 1 and 1000		= 36
"	1 and 100	= 9
"	100 & 200	= 7
"	200 & 300	= 4
"	300 & 400	= 2
"	400 & 500	= ~~3~~ 4
"	500 & 600	= 3
"	600 & 700	= 2
"	700 & 800	= 0
"	800 & 900	= 6
"	900 & 1000	= 0
		36

909091

101 − p
1001 = 7 × 11 × 13
10001 = 73 × 137
100001 = 11 × 19 × 479
1000001 = 101 × 9901
10000001 = 11 × 909091
100000001 = 17 × 5882353
1000000001 = 11 × x
10000000001 =

67) 1000001 (1492
67
330
268
620
603
170
134
361

73) 1000001 (1369
73
270
219
510
438
720
657
631

79) 1000001 (1265
79
210
158
520
474
460
395
651

71) 1000001 (1408
71
290
284
600
568

83) 1000001 (1226
81
190
166
240
186
540
498
421

89) 1000001 (1123
89
110
89
210
178
320
267
531

97) 1000001 (1030
97
300
291
901

101) 1000001 (9901
909
910
909
101

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