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B. S. Madhava Rao

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By B. S. MADHAVA RAO
Department of Mathematics, University of Mysore

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§ 1. *Introduction*

The condition in order that a system of differential equations of dynamics of the second order

$$\ddot{q}_k = f_k(\dot{q}_1, \dot{q}_2, \dots \dot{q}_n, q_1, q_2, \dots q_n, t)(k=1, 2, \dots n) \tag{1}$$

be equivalent to a Lagrangian system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0, \tag{2}$$

can be found in the case where the Kinetic energy is a homogeneous quadratic in the velocities [1]. Also, the general case can be reduced to the question of the last multiplier for a system with one degree of freedom [1].

The proof usually given for the reduction in the first case is rather cumbersome, and I give in this paper a very simple alternative proof using the tensor notation. This same method is next employed to deduce the conditions of reduction to Lagrangian form when the kinetic energy T is of the form, $T = T_2 + T_1 + T_0$. I next consider the relation to the question of the last multiplier, and show the system with T in the above form can, under certain circumstances, be dealt with under the last multiplier theory even for the case of n degrees of freedom.

§ 2. *Case where T is a homogeneous quadratic*

Let the given system of equations be

$$\ddot{q}_r = F_r + G_r \tag{3}$$

where F_r is homogeneous and of the second degree in $(\dot{q}_1, \dot{q}_2, \dots \dot{q}_n)$, and let these be equivalent to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial \dot{q}_r} = Q_r. \tag{4}$$

Since the value of T is not dependent on $(G_1, G_2, \dots G_n)$, we can consider the equivalent problem with $G_i = 0$ and $Q_i = 0$. We can write (3) in the form,

$$\ddot{q}_k = f_k = \tfrac{1}{2} f^k_{\mu\nu}\, \dot{q}^\mu\, \dot{q}^\nu, \tag{5}$$

where the index k is merely one of enumeration. If the kinetic energy T be of the form,

$$T = \tfrac{1}{2} g_{\mu\nu}\, \dot{q}_\mu\, \dot{q}_\nu \qquad g_{\mu\nu} = g_{\mu\nu}(q_1, \dots q_n), \tag{6}$$

it is well known that we can associate with it a manifold M_n with the space element

$$ds^2 = 2T dt^2 = g_{\mu\nu}\, dq^\mu\, dq^\nu$$

such that M_n is an n-dimensional Riemann-space [2], and the expression

$$\ddot{q}^\rho + \Gamma^\rho_{\mu\nu}\, \dot{q}^\mu\, \dot{q}^\nu = Q^\rho \tag{7}$$

for the contravariant components of the acceleration vector are equivalent to the equations of motion (4). For the case $Q^\rho = 0$, we can write (7) in the form

$$\ddot{q}^k + \Gamma^k_{\mu\nu}\, \dot{q}^\mu\, \dot{q}^\nu = 0. \tag{8}$$

Comparing (5) and (8) we have the condition for reduction to Lagrangian form as

$$f^k_{\mu\nu} + 2\Gamma^k_{\mu\nu} = 0 \tag{9}$$

i.e., $$f^k_{\mu\nu}\, g_{kl} + 2\Gamma_{\mu\nu,\, l} = 0$$

or $$f^k_{l\nu}\, g_{k\mu} + 2\Gamma_{l\nu,\, \mu} = 0,$$

interchanging the indices l and μ.

Adding these equations, and making use of a well-known identity in Christoffel's symbols, we have

$$\tfrac{1}{2}\left(f^k_{\mu\nu}\, g_{kl} + f^k_{l\nu}\, g_{k\mu}\right) + \frac{\partial g_{l\mu}}{\partial q_\nu} = 0$$

or $$\tfrac{1}{2} f^k_{\mu\nu}\, g_{kl}\, \dot{q}^\mu \dot{q}^l + \tfrac{1}{2} f^k_{l\nu}\, g_{k\mu}\, \dot{q}^l \dot{q}^\mu + \frac{\partial g_{l\mu}}{\partial q_\mu} \dot{q}^l \dot{q}^\mu = 0.$$

Each of the first two terms on the left hand side

$$=\tfrac{1}{2}\frac{\partial f_k}{\partial \dot q_\nu}\frac{\partial T}{\partial \dot q_k}, \text{ and the third term} = 2\frac{\partial T}{\partial q_\nu}.$$

Hence the condition is

$$\tfrac{1}{2}\frac{\partial f_k}{\partial \dot q_\nu}\frac{\partial T}{\partial \dot q_k}+\frac{\partial T}{\partial \dot q_\nu}=0 \tag{10}$$

which is the same equation as given by Whittaker in [1].

§ 3. *Case where T is a General Quadratic*

Let T be of the form,

$$T=T_0+T_1+T_2$$

$$=\tfrac{1}{2}g_{00}+g_{0\rho}\,\dot q^\rho+\tfrac{1}{2}g_{\sigma\rho}\,\dot q^\sigma \dot q^\rho \tag{11}$$

in which g_{00}, $g_{0\rho}$, $g_{\sigma\rho}$ are now functions of $q_1, \ldots q_n$ and t. The corresponding Lagrange's equations can be written in the form [3].

$$\ddot q^\lambda+\Gamma^\lambda_{\sigma\rho}\,\dot q^\sigma \dot q^\rho+2\,\Gamma^\lambda_{0\rho}\,\dot q^\rho+\Gamma^\lambda_{00}=Q^\lambda, \tag{12}$$

where Q^λ are no longer contravariant components of the field vectors since the manifold M_{n+1} of the $(q_1, \ldots q_n, t)$ is not a Riemannian space. The raising and lowering of the indices in (12) is defined with respect to the $g_{\sigma\rho}$ of T_2. Thus we have

$$\Gamma^\lambda_{0\rho}=g^{\sigma\lambda}\,\Gamma_{0\rho,\,\lambda} \tag{13}$$

$$\Gamma^\lambda_{00}=g^{\sigma\lambda}\,\Gamma_{00,\,\sigma} \tag{14}$$

$$Q^\lambda=g^{\sigma\lambda}\,Q_\sigma \tag{15}$$

and,

$$\Gamma_{0\rho,\,\lambda}=\tfrac{1}{2}\left(\frac{\partial g_{0\lambda}}{\partial q^\sigma}+\frac{\partial g_{\rho\lambda}}{\partial t}-\frac{\partial g_{0\rho}}{\partial q^\lambda}\right) \tag{16}$$

$$\Gamma_{00,\,\lambda}=\frac{\partial g_{0\lambda}}{\partial t}-\tfrac{1}{2}\frac{\partial g_{00}}{\partial q^\lambda}. \tag{17}$$

Let the given system of differential equations also be of the form

$$\ddot q^\lambda=\tfrac{1}{2}f^\lambda_{\sigma\rho}\,\dot q^\sigma\,\dot q^\rho+f^\lambda_{0\rho}\,\dot q^\rho+\tfrac{1}{2}f^\lambda_{00} \tag{18}$$

or

$$\ddot q_\lambda=f_\lambda+\phi_\lambda+\psi_\lambda. \tag{19}$$

Comparing (12) and (18) we have the results

$$f^{\lambda}_{\sigma\rho}+2\Gamma^{\lambda}_{\sigma\rho}=0 \tag{21}$$

$$f^{\lambda}_{0\rho}+2\Gamma^{\lambda}_{0\rho}=0 \tag{22}$$

$$f^{\lambda}_{00}+2\Gamma^{\lambda}_{00}=Q^{\lambda}. \tag{23}$$

Relation (21) leads just as in §2, to the conditions

$$\frac{1}{2}\frac{\partial f_{\lambda}}{\partial \dot{q}_{\nu}}\frac{\partial T_2}{\partial \dot{q}_{\lambda}}+\frac{\partial T_2}{\partial q_{\nu}}=0. \tag{24}$$

Relation (22) can be reduced to

$$f^{\lambda}_{0\rho}\, g_{\lambda l}+2\Gamma_{0\rho,\, l}=0$$

i.e. $f^{\lambda}_{0l}\, g_{\lambda\rho}+2\Gamma_{0l,\,\rho}=0$, interchanging l and ρ.

Adding,

$$(f^{\lambda}_{0\rho}\, g_{\lambda l}+f^{\lambda}_{0l}\, g_{\lambda\rho})+2(\Gamma_{0\rho,\, l}+\Gamma_{0l,\,\rho})=0.$$

But from (16), the second term is equal to $\partial g_{\rho l}/\partial t$, and hence

$$\tfrac{1}{2}(f^{\lambda}_{0\rho}\, g_{\lambda l}+f^{\lambda}_{0l}\, g_{\lambda\rho})+(\partial g_{\rho l}/\partial t)=0.$$

Multiplying by $\dot{q}^{\rho}\,\dot{q}^{l}$, each of the terms on the left hand side in the bracket is equal to $\frac{1}{2}\phi_{\lambda}(\partial T_2/\partial \dot{q}^{\lambda})$, while the second term outside the brackets is $2\partial T_2/\partial t$. Hence we get the second set of conditions

$$\tfrac{1}{2}\phi_{\lambda}\frac{\partial T_2}{\partial \dot{q}^{\lambda}}+\frac{\partial T_2}{\partial t}=0 \tag{25}$$

which are complementary to (24). Finally, (23) can be written in the form

$$g_{\sigma\lambda} f^{\lambda}_{00}+2g_{\sigma\lambda}\,\Gamma^{\lambda}_{00}=Q_{\sigma}$$

i.e., $$g_{\sigma\lambda} f^{\lambda}_{00}+2\Gamma_{00,\,\sigma}=Q_{\sigma}$$

$$g_{\sigma\lambda} f^{\lambda}_{00}+2\frac{\partial g_{0\sigma}}{\partial t}-\frac{\partial g_{00}}{\partial q^{\sigma}}=Q_{\sigma} \quad \text{using (17)}$$

i.e., $$\psi_{\lambda}\frac{\partial^2 T_2}{\partial \dot{q}^{\sigma}\,\partial \dot{q}^{\lambda}}+\frac{\partial^2 T_1}{\partial \dot{q}^{\sigma}\,\partial t}-\frac{\partial T_0}{\partial q^{\sigma}}=P_{\sigma} \quad (Q_{\sigma}=2P_{\sigma})$$

i.e., $$\frac{\partial}{\partial \dot{q}^{\sigma}}\left(\psi_{\lambda}\frac{\partial T_2}{\partial \dot{q}^{\lambda}}+\frac{\partial T_1}{\partial t}\right)=R_{\sigma} \tag{26}$$

$$(R_{\sigma}=P_{\sigma}+\partial T_0/\partial q^{\sigma}]$$

(24)—(26) constitute the conditions for the reduction to the Lagrangian form.

§ 4. *Connection with the last multiplier*

In the case where the Lagrangian is a general quadratic in the velocities we shall now find the conditions under which the determination of L can be reduced to the problem of the last multiplier. The condition that (1) be equivalent to (2) is that L should satisfy the system of partial differential equations.

$$\sum_k \left(\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_k} f_k + \frac{\partial^2 L}{\partial \dot{q}_r \partial q_k} \dot{q}_k \right) + \frac{\partial^2 L}{\partial \dot{q}_r \partial t} - \frac{\partial L}{\partial q_r} = 0 \qquad (27)$$

$$(r = 1, \ldots n)$$

Let us consider the r^{th} and s^{th} equations and denote their left hand members by (A) and (B), respectively. Differentiating each of them partially with respect to $\dot{q}_r$ and $\dot{q}_s$, we write down the equations,

$$\left.\begin{aligned} &\frac{\partial(A)}{\partial \dot{q}_r} = 0 \\ &\frac{\partial(A)}{\partial \dot{q}_s} + \frac{\partial(B)}{\partial \dot{q}_r} = 0 \\ &\frac{\partial(B)}{\partial \dot{q}_s} = 0 \end{aligned}\right\}$$

After some slight simplification, and using the notation $M_{rs} = \dfrac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_s}$ we can write the expanded form of these equations as,

$$\frac{\partial}{\partial \dot{q}_r} \Sigma (M_{rk} f_k) + \Sigma \dot{q}_k \frac{\partial M_{rr}}{\partial q_k} + \frac{\partial M_{rr}}{\partial t} = 0 \qquad (28)$$

$$\frac{\partial}{\partial \dot{q}_r} \Sigma (M_{sk} f_k) + \frac{\partial}{\partial \dot{q}_s} \Sigma (M_{rk} f_k) + 2 \Sigma \dot{q}_k \frac{\partial M_{rs}}{\partial q_k} + 2 \frac{\partial M_{rs}}{\partial t} = 0 \qquad (29)$$

$$\frac{\partial}{\partial \dot{q}_s} \Sigma (M_{sk} f_k) + \Sigma \dot{q}_k \frac{\partial M_{ss}}{\partial q_k} + \frac{\partial M_{ss}}{\partial t} = 0. \qquad (30)$$

If M_{rr}, M_{rs}, M_{ss} be last multipliers of the systems of equations, respectively,

$$dt=\frac{dq_1}{\dot{q}_1}=\ldots=\frac{dq_n}{\dot{q}_n}=\frac{d\dot{q}_r}{f_r} \tag{31}$$

$$dt=\frac{dq_1}{\dot{q}_1}=\ldots=\frac{dq_n}{\dot{q}_n}=\frac{2d\dot{q}_r}{f_r}=\frac{2d\dot{q}_s}{f_s} \tag{32}$$

$$dt=\frac{dq_1}{\dot{q}_1}=\ldots=\frac{dq_n}{\dot{q}_n}=\frac{d\dot{q}_s}{f_s} \tag{33}$$

we have the conditions

$$\frac{\partial M_{rr}}{\partial t}+\Sigma\frac{\partial}{\partial q_k}(M_{rk}\,\dot{q}_k)+\frac{\partial}{\partial \dot{q}_r}(M_{rr}f_r)=0 \tag{34}$$

$$2\frac{\partial M_{rs}}{\partial t}+2\Sigma\frac{\partial}{\partial q_k}(M_{rs}\,\dot{q}_k)+\frac{\partial}{\partial \dot{q}_r}(M_{rs}f_r)+\frac{\partial}{\partial \dot{q}_s}(M_{rs}f_s)=0 \tag{35}$$

$$\frac{\partial M_{ss}}{\partial t}+\Sigma\frac{\partial}{\partial q_k}(M_{ss}\,\dot{q}_k)+\frac{\partial}{\partial \dot{q}_s}(M_{ss}f_s)=0 \tag{36}$$

Equations (29)—(30) will be satisfied in virtue of (34)—(36) if we have

$$\left.\begin{array}{l}\dfrac{\partial}{\partial \dot{q}_r}\Sigma\,(M_{rk}f_k)=0,\quad \dfrac{\partial}{\partial \dot{q}_s}\Sigma\,(M_{sk}f_k)=0\\[2ex] \text{and}\quad \dfrac{\partial}{\partial \dot{q}_r}\Sigma\,(M_{sk}\,f_k)+\dfrac{\partial}{\partial \dot{q}_s}\Sigma\,(M_{rk}f_k)=0\end{array}\right\}(r,\ s,\neq k)$$

If L be a quadratic in the generalised velocities all the M's are independent of the $\dot{q}$'s, and the above conditions are satisfied if

$$\frac{\partial f_r}{\partial \dot{q}_s}=0 \quad (r\neq s) \tag{37}$$

i.e., f_k is a function only of $\dot{q}_k$ in the velocities.

Thus condition for the possibility of expressing L in terms of last multipliers is

(*i*) L a quadratic in $\dot{q}$'s.

(*ii*) $f_k=f_k(\dot{q}_k,\ q_1\ldots q_n,\ t)$.

An alternative condition can also be derived by altering the equation of the last multiplier (32) on replacement of

f_r, f_s, by $2f_r$, $2f_s$. This would necessitate in addition to (37) the conditions,

$$\frac{\partial f_r}{\partial \dot{q}_r}+\frac{\partial f_s}{\partial \dot{q}_s}=0 \quad \text{(for all } r,\, s). \qquad (38)$$

This (37)—(38) would imply that f_k's were not functions of $\dot{q}_k$'s at all. Under these conditions (31)—(33) would have no dynamical significance at all being merely the expressions $dq_1/dt=\dot{q}_1, \ldots$; and we should have $T=T_0$ and the reduction $\ddot{q}_k=f_k$ becomes quite trivial. Hence if we omitted this trivial case, equations (34)—(36) appear to define the requisite last multipliers correctly.

References

[1] *Whittaker:* Analytical Dynamics (1927), p. 284-6.

[2] *Prauge:* Ency. Math. Wiss IV, 4, p. 550.

[3] *Levi-Civita:* Turin Atti. 31 (1895), 816. See also [2], p. 556.