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1. Introduction

The condition is asked that a system of differential equations of dynamics of the second order

$$\ddot{q}_i = f_i(q_1, q_2, \dots, q_n, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t) \quad (1)$$

is equivalent to a Lagrangian system

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \quad (2)$$

can be found in the case when the kinetic energy is a homogeneous quadratic in the velocities [1]. Also, the general case is considered in the question of the last multiplier in a system of the second degree [2].

The present article is given for the reduction in the first case is rather cumbersome and I give in this paper a very simple alternative procedure for the same purpose. The new method is well adapted to deduce the conditions of reduction to Lagrangian form when the kinetic energy is of the form $T = T_1 + T_2$. I now consider the relation to the question of the last multiplier, and show the system

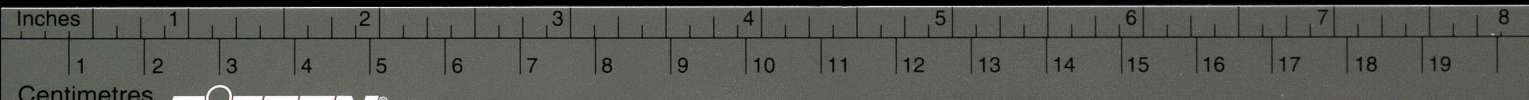
B. S. MADHAVA RAO

ON THE REDUCTION OF DYNAMICAL EQUATIONS TO
THE LAGRANGIAN FORM

Example 1. A homogeneous quadratic

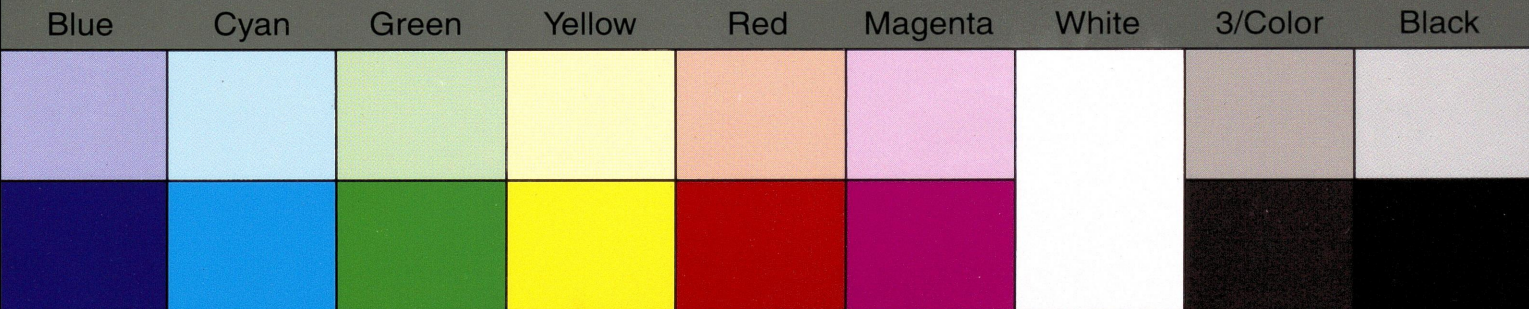
Let the kinetic energy of a system be

$$T = T_1 + T_2 \quad (3)$$



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ON THE REDUCTION OF DYNAMICAL EQUATIONS
TO THE LAGRANGIAN FORM

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§ 1. *Introduction*

The condition in order that a system of differential equations of dynamics of the second order

$$\ddot{q}_k = f_k(\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, q_1, q_2, \dots, q_n, t) \quad (k=1, 2, \dots, n) \quad (1)$$

be equivalent to a Lagrangian system

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} = 0, \quad (2)$$

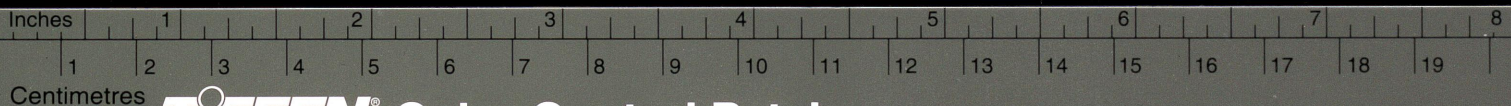
can be found in the case where the Kinetic energy is a homogeneous quadratic in the velocities [1]. Also, the general case can be reduced to the question of the last multiplier for a system with one degree of freedom [1].

The proof usually given for the reduction in the first case is rather cumbersome, and I give in this paper a very simple alternative proof using the tensor notation. This same method is next employed to deduce the conditions of reduction to Lagrangian form when the kinetic energy T is of the form, $T = T_2 + T_1 + T_0$. I next consider the relation to the question of the last multiplier, and show the system with T in the above form can, under certain circumstances, be dealt with under the last multiplier theory even for the case of n degrees of freedom.

§ 2. *Case where T is a homogeneous quadratic*

Let the given system of equations be

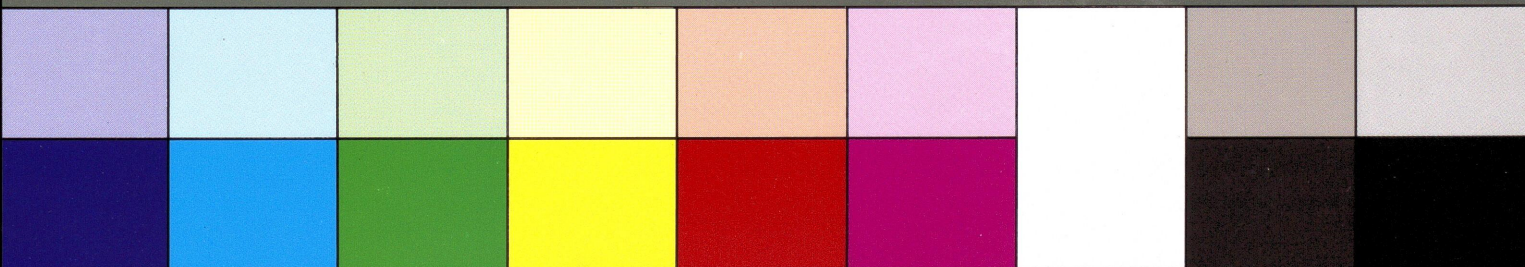
$$\ddot{q}_r = F_r + G_r \quad (3)$$



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where F_r is homogeneous and of the second degree in $(\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n)$, and let these be equivalent to

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = Q_r. \quad (4)$$

Since the value of T is not dependent on (G_1, G_2, \dots, G_n) , we can consider the equivalent problem with $G_i = 0$ and $Q_i = 0$. We can write (3) in the form,

$$\ddot{q}_k = f_k = \frac{1}{2} f_{\mu\nu}^k \dot{q}^\mu \dot{q}^\nu, \quad (5)$$

where the index k is merely one of enumeration. If the kinetic energy T be of the form,

$$T = \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu \quad g_{\mu\nu} = g_{\mu\nu}(q_1, \dots, q_n), \quad (6)$$

it is well known that we can associate with it a manifold M_n with the space element

$$ds^2 = 2T dt^2 = g_{\mu\nu} dq^\mu dq^\nu$$

such that M_n is an n -dimensional Riemann-space [2], and the expression

$$\ddot{q}^\rho + \Gamma_{\mu\nu}^\rho \dot{q}^\mu \dot{q}^\nu = Q^\rho \quad (7)$$

for the contravariant components of the acceleration vector are equivalent to the equations of motion (4). For the case $Q^\rho = 0$, we can write (7) in the form

$$\ddot{q}^k + \Gamma_{\mu\nu}^k \dot{q}^\mu \dot{q}^\nu = 0. \quad (8)$$

Comparing (5) and (8) we have the condition for reduction to Lagrangian form as

$$f_{\mu\nu}^k + 2\Gamma_{\mu\nu}^k = 0 \quad (9)$$

$$i.e., \quad f_{\mu\nu}^k g_{kl} + 2\Gamma_{\mu\nu, l} = 0$$

$$or \quad f_{l\nu}^k g_{k\mu} + 2\Gamma_{l\nu, \mu} = 0,$$

interchanging the indices l and μ .

Adding these equations, and making use of a well-known identity in Christoffel's symbols, we have

$$\frac{1}{2} (f_{\mu\nu}^k g_{kl} + f_{l\nu}^k g_{k\mu}) + \frac{\partial g_{l\mu}}{\partial q_\nu} = 0$$

$$or \quad \frac{1}{2} f_{\mu\nu}^k g_{kl} \dot{q}^\mu \dot{q}^l + \frac{1}{2} f_{l\nu}^k g_{k\mu} \dot{q}^l \dot{q}^\mu + \frac{\partial g_{l\mu}}{\partial q_\nu} \dot{q}^l \dot{q}^\mu = 0.$$

Each of the first

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§ 3.

Let T be

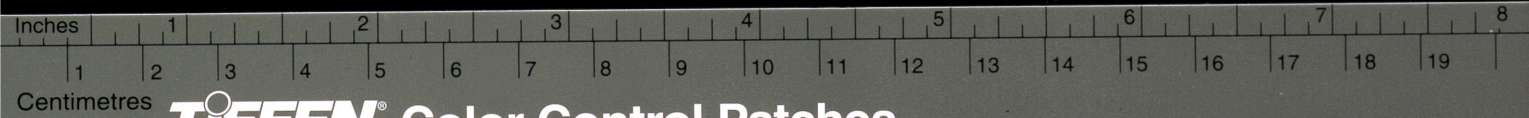
in which $g_{00}, g_{0\rho}$,
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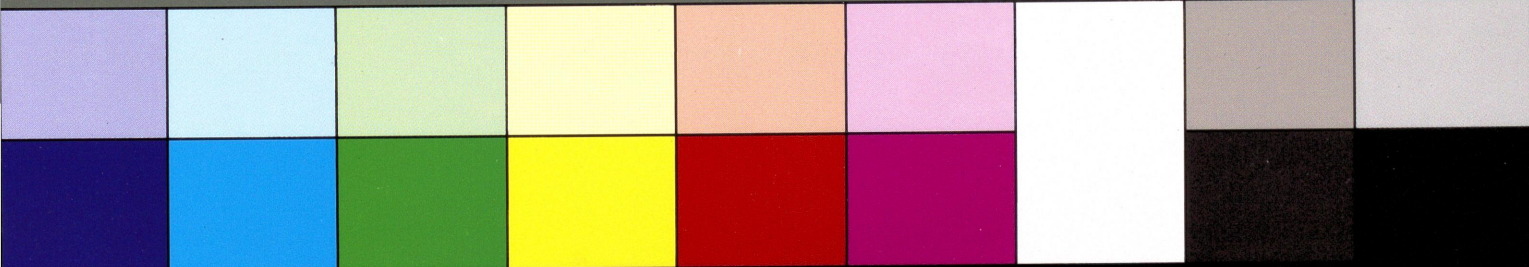
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Each of the first two terms on the left hand side

$$= \frac{1}{2} \frac{\partial f_k}{\partial \dot{q}_v} \frac{\partial T}{\partial \dot{q}_k}, \text{ and the third term} = 2 \frac{\partial T}{\partial \dot{q}_v}.$$

Hence the condition is

$$\frac{1}{2} \frac{\partial f_k}{\partial \dot{q}_v} \frac{\partial T}{\partial \dot{q}_k} + \frac{\partial T}{\partial \dot{q}_v} = 0 \tag{10}$$

which is the same equation as given by Whittaker in [1].

§ 3. Case where T is a General Quadratic

Let T be of the form,

$$T = T_0 + T_1 + T_2 \\ = \frac{1}{2} g_{00} + g_{0\rho} \dot{q}^\rho + \frac{1}{2} g_{\sigma\rho} \dot{q}^\sigma \dot{q}^\rho \tag{11}$$

in which $g_{00}, g_{0\rho}, g_{\sigma\rho}$ are now functions of q_1, \dots, q_n and t . The corresponding Lagrange's equations can be written in the form [3].

$$\ddot{q}^\lambda + \Gamma_{\sigma\rho}^\lambda \dot{q}^\sigma \dot{q}^\rho + 2 \Gamma_{0\rho}^\lambda \dot{q}^\rho + \Gamma_{00}^\lambda = Q^\lambda, \tag{12}$$

where Q^λ are no longer contravariant components of the field vectors since the manifold M_{n+1} of the (q_1, \dots, q_n, t) is not a Riemannian space. The raising and lowering of the indices in (12) is defined with respect to the $g_{\sigma\rho}$ of T_2 . Thus we have

$$\Gamma_{0\rho}^\lambda = g^{\sigma\lambda} \Gamma_{0\rho, \sigma} \tag{13}$$

$$\Gamma_{00}^\lambda = g^{\sigma\lambda} \Gamma_{00, \sigma} \tag{14}$$

$$Q^\lambda = g^{\sigma\lambda} Q_\sigma \tag{15}$$

and,
$$\Gamma_{0\rho, \lambda} = \frac{1}{2} \left(\frac{\partial g_{0\lambda}}{\partial q^\sigma} + \frac{\partial g_{\rho\lambda}}{\partial t} - \frac{\partial g_{0\rho}}{\partial q^\lambda} \right) \tag{16}$$

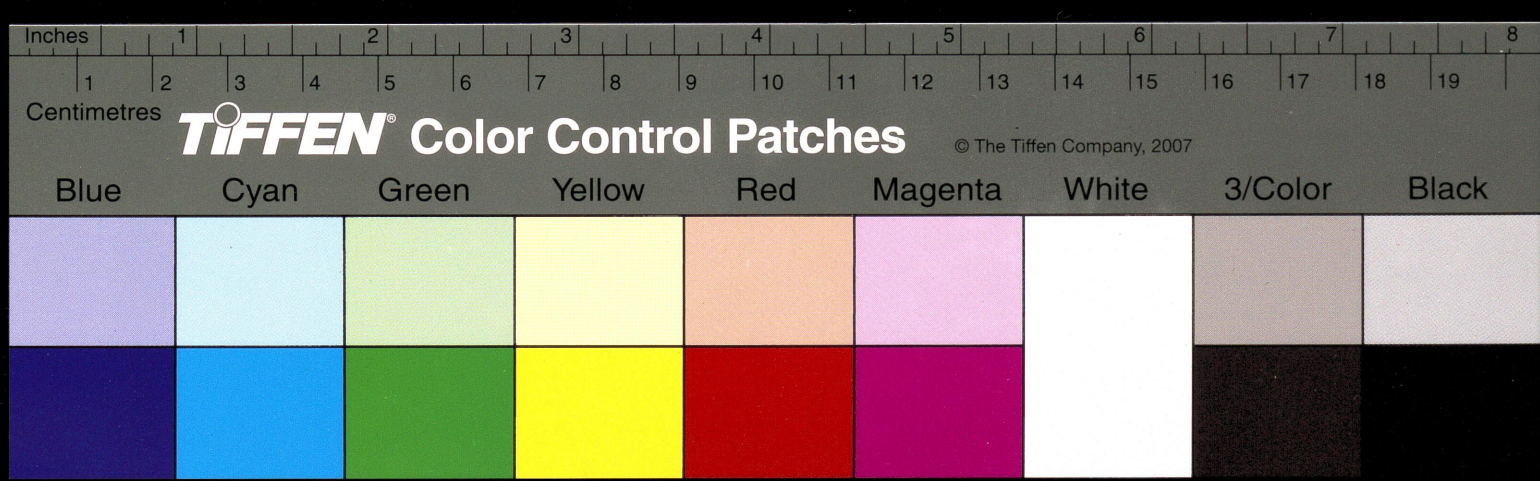
$$\Gamma_{00, \lambda} = \frac{\partial g_{0\lambda}}{\partial t} - \frac{1}{2} \frac{\partial g_{00}}{\partial q^\lambda}. \tag{17}$$

Let the given system of differential equations also be of the form

$$\ddot{q}^\lambda = \frac{1}{2} f_{\sigma\rho}^\lambda \dot{q}^\sigma \dot{q}^\rho + f_{0\rho}^\lambda \dot{q}^\rho + \frac{1}{2} f_{00}^\lambda \tag{18}$$

or
$$\ddot{q}_\lambda = f_\lambda + \phi_\lambda + \psi_\lambda. \tag{19}$$

degree in $(\dot{q}_1,$
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 $(G_1, G_2, \dots, G_n),$
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Comparing (12) and (18) we have the results

$$f_{\sigma\rho}^{\lambda} + 2\Gamma_{\sigma\rho}^{\lambda} = 0 \quad (21)$$

$$f_{0\rho}^{\lambda} + 2\Gamma_{0\rho}^{\lambda} = 0 \quad (22)$$

$$f_{00}^{\lambda} + 2\Gamma_{00}^{\lambda} = Q^{\lambda}. \quad (23)$$

Relation (21) leads just as in §2, to the conditions

$$\frac{1}{2} \frac{\partial f_{\lambda}}{\partial \dot{q}_r} \frac{\partial T_2}{\partial \dot{q}_{\lambda}} + \frac{\partial T_2}{\partial q_r} = 0. \quad (24)$$

Relation (22) can be reduced to

$$f_{0\rho}^{\lambda} g_{\lambda i} + 2\Gamma_{0\rho, i} = 0$$

$$i.e. \quad f_{0i}^{\lambda} g_{\lambda\rho} + 2\Gamma_{0i, \rho} = 0, \text{ interchanging } l \text{ and } \rho.$$

Adding,

$$(f_{0\rho}^{\lambda} g_{\lambda i} + f_{0i}^{\lambda} g_{\lambda\rho}) + 2(\Gamma_{0\rho, i} + \Gamma_{0i, \rho}) = 0.$$

But from (16), the second term is equal to $\partial g_{\rho i} / \partial t$, and hence

$$\frac{1}{2} (f_{0\rho}^{\lambda} g_{\lambda i} + f_{0i}^{\lambda} g_{\lambda\rho}) + (\partial g_{\rho i} / \partial t) = 0.$$

Multiplying by $\dot{q}^{\rho} \dot{q}^i$, each of the terms on the left hand side in the bracket is equal to $\frac{1}{2} \phi_{\lambda} (\partial T_2 / \partial \dot{q}^{\lambda})$, while the second term outside the brackets is $2\partial T_2 / \partial t$. Hence we get the second set of conditions

$$\frac{1}{2} \phi_{\lambda} \frac{\partial T_2}{\partial \dot{q}^{\lambda}} + \frac{\partial T_2}{\partial t} = 0 \quad (25)$$

which are complementary to (24). Finally, (23) can be written in the form

$$g_{\sigma\lambda} f_{00}^{\lambda} + 2g_{\sigma\lambda} \Gamma_{00}^{\lambda} = Q_{\sigma}$$

$$i.e., \quad g_{\sigma\lambda} f_{00}^{\lambda} + 2\Gamma_{00, \sigma} = Q_{\sigma}$$

$$g_{\sigma\lambda} f_{00}^{\lambda} + 2 \frac{\partial g_{0\sigma}}{\partial t} - \frac{\partial g_{00}}{\partial q^{\sigma}} = Q_{\sigma} \quad \text{using (17)}$$

$$i.e., \quad \psi_{\lambda} \frac{\partial^2 T_2}{\partial \dot{q}^{\sigma} \partial \dot{q}^{\lambda}} + \frac{\partial^2 T_1}{\partial \dot{q}^{\sigma} \partial t} - \frac{\partial T_0}{\partial q^{\sigma}} = P_{\sigma} \quad (Q_{\sigma} = 2P_{\sigma})$$

$$i.e., \quad \frac{\partial}{\partial \dot{q}^{\sigma}} \left(\psi_{\lambda} \frac{\partial T_2}{\partial \dot{q}^{\lambda}} + \frac{\partial T_1}{\partial t} \right) = R_{\sigma} \quad (26)$$

$$(R_{\sigma} = P_{\sigma} + \partial T_0 / \partial q^{\sigma})$$

(24)–(26) constitute the Lagrangian form

§ 4.

In the case in the velocities the determination of the last multiplier is that L should satisfy the conditions.

$$\sum_k \left(\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_k} \right)$$

Let us consider the left hand member of (24) by differentiating each of the terms and write down the

After some

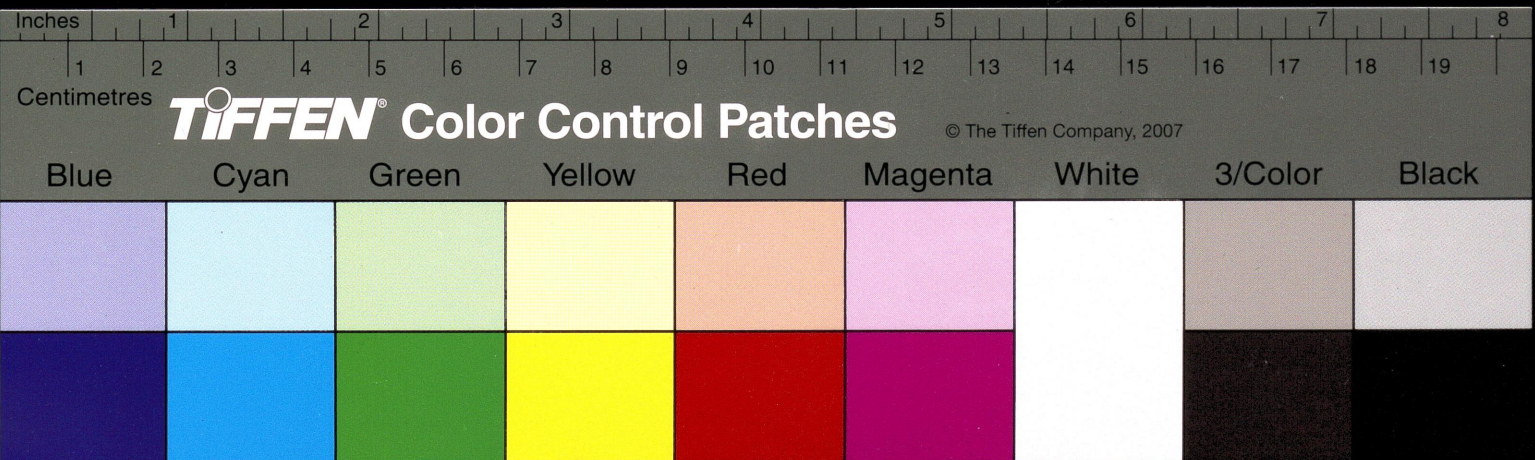
$M_{rs} = \frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_s}$ we obtain the conditions as,

$$\frac{\partial}{\partial \dot{q}_r} \sum_k (M_{sk} f_{rk})$$

$$\frac{\partial}{\partial \dot{q}_r} \sum_k (M_{sk} f_{rk})$$

$$\frac{\partial}{\partial \dot{q}_r} \sum_k (M_{sk} f_{rk})$$

8



(24)–(26) constitute the conditions for the reduction to the Lagrangian form.

§ 4. Connection with the last multiplier

In the case where the Lagrangian is a general quadratic in the velocities we shall now find the conditions under which the determination of L can be reduced to the problem of the last multiplier. The condition that (1) be equivalent to (2) is that L should satisfy the system of partial differential equations.

$$\sum_k \left(\frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_k} f_k + \frac{\partial^2 L}{\partial \dot{q}_r \partial q_k} \dot{q}_k \right) + \frac{\partial^2 L}{\partial \dot{q}_r \partial t} - \frac{\partial L}{\partial q_r} = 0 \quad (27)$$

$(r=1, \dots, n)$

Let us consider the r^{th} and s^{th} equations and denote their left hand members by (A) and (B), respectively. Differentiating each of them partially with respect to \dot{q}_r and \dot{q}_s , we write down the equations,

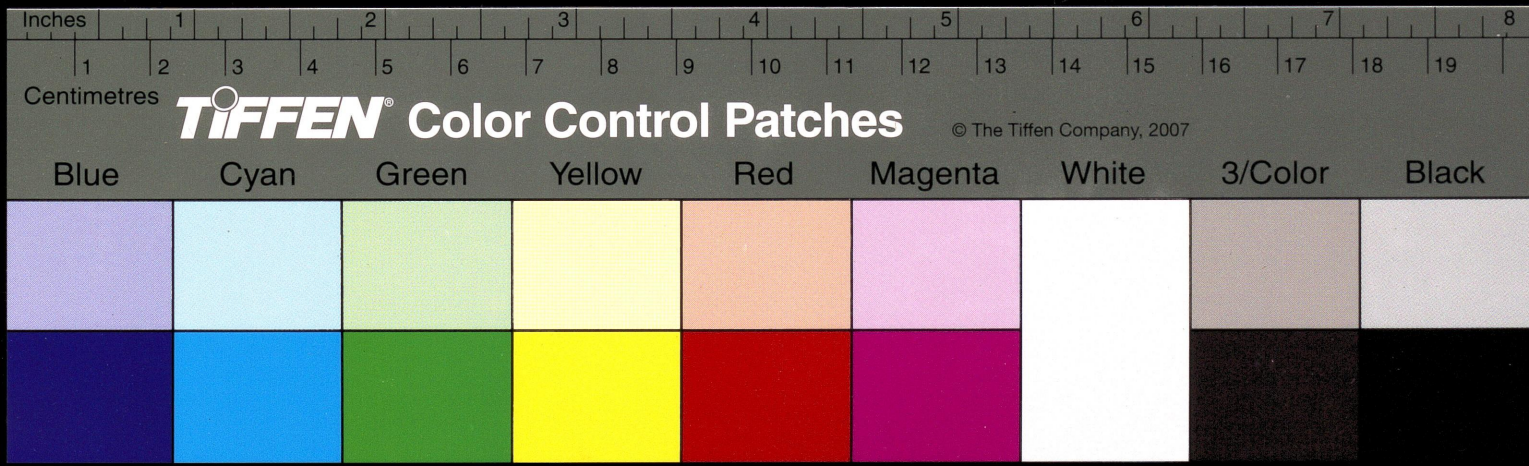
$$\left. \begin{aligned} \frac{\partial(A)}{\partial \dot{q}_r} &= 0 \\ \frac{\partial(A)}{\partial \dot{q}_s} + \frac{\partial(B)}{\partial \dot{q}_r} &= 0 \\ \frac{\partial(B)}{\partial \dot{q}_s} &= 0 \end{aligned} \right\}$$

After some slight simplification, and using the notation $M_{rs} = \frac{\partial^2 L}{\partial \dot{q}_r \partial \dot{q}_s}$ we can write the expanded form of these equations as,

$$\frac{\partial}{\partial \dot{q}_r} \sum (M_{rk} f_k) + \sum \dot{q}_k \frac{\partial M_{rr}}{\partial q_k} + \frac{\partial M_{rr}}{\partial t} = 0 \quad (28)$$

$$\frac{\partial}{\partial \dot{q}_r} \sum (M_{sk} f_k) + \frac{\partial}{\partial \dot{q}_s} \sum (M_{rk} f_k) + 2 \sum q_k \frac{\partial M_{rs}}{\partial q_k} + 2 \frac{\partial M_{rs}}{\partial t} = 0 \quad (29)$$

$$\frac{\partial}{\partial \dot{q}_s} \sum (M_{sk} f_k) + \sum q_k \frac{\partial M_{ss}}{\partial q_k} + \frac{\partial M_{ss}}{\partial t} = 0. \quad (30)$$



If M_{rr} , M_{rs} , M_{ss} be last multipliers of the systems of equations, respectively,

$$dt = \frac{dq_1}{\dot{q}_1} = \dots = \frac{dq_n}{\dot{q}_n} = \frac{d\dot{q}_r}{f_r} \tag{31}$$

$$dt = \frac{dq_1}{\dot{q}_1} = \dots = \frac{dq_n}{\dot{q}_n} = \frac{2d\dot{q}_r}{f_r} = \frac{2d\dot{q}_s}{f_s} \tag{32}$$

$$dt = \frac{dq_1}{\dot{q}_1} = \dots = \frac{dq_n}{\dot{q}_n} = \frac{d\dot{q}_s}{f_s} \tag{33}$$

we have the conditions

$$\frac{\partial M_{rr}}{\partial t} + \sum \frac{\partial}{\partial \dot{q}_k} (M_{rk} \dot{q}_k) + \frac{\partial}{\partial \dot{q}_r} (M_{rr} f_r) = 0 \tag{34}$$

$$2 \frac{\partial M_{rs}}{\partial t} + 2 \sum \frac{\partial}{\partial \dot{q}_k} (M_{rs} \dot{q}_k) + \frac{\partial}{\partial \dot{q}_r} (M_{rs} f_r) + \frac{\partial}{\partial \dot{q}_s} (M_{rs} f_s) = 0 \tag{35}$$

$$\frac{\partial M_{ss}}{\partial t} + \sum \frac{\partial}{\partial \dot{q}_k} (M_{sk} \dot{q}_k) + \frac{\partial}{\partial \dot{q}_s} (M_{ss} f_s) = 0 \tag{36}$$

Equations (29)–(30) will be satisfied in virtue of (34)–(36) if we have

$$\left. \begin{aligned} \frac{\partial}{\partial \dot{q}_r} \sum (M_{rk} f_k) = 0, \quad \frac{\partial}{\partial \dot{q}_s} \sum (M_{sk} f_k) = 0 \\ \text{and} \quad \frac{\partial}{\partial \dot{q}_r} \sum (M_{sk} f_k) + \frac{\partial}{\partial \dot{q}_s} \sum (M_{rk} f_k) = 0 \end{aligned} \right\} (r, s, \neq k)$$

If L be a quadratic in the generalised velocities all the M 's are independent of the \dot{q} 's, and the above conditions are satisfied if

$$\frac{\partial f_r}{\partial \dot{q}_s} = 0 \quad (r \neq s) \tag{37}$$

i.e., f_k is a function only of \dot{q}_k in the velocities.

Thus condition for the possibility of expressing L in terms of last multipliers is

(i) L a quadratic in \dot{q} 's.

(ii) $f_k = f_k(\dot{q}_k, q_1 \dots q_n, t)$.

An alternative condition can also be derived by altering the equation of the last multiplier (32) on replacement of

f_r, f_s , by $2f_r, 2f_s$.
the conditions,

$$\frac{\partial f_r}{\partial \dot{q}_r}$$

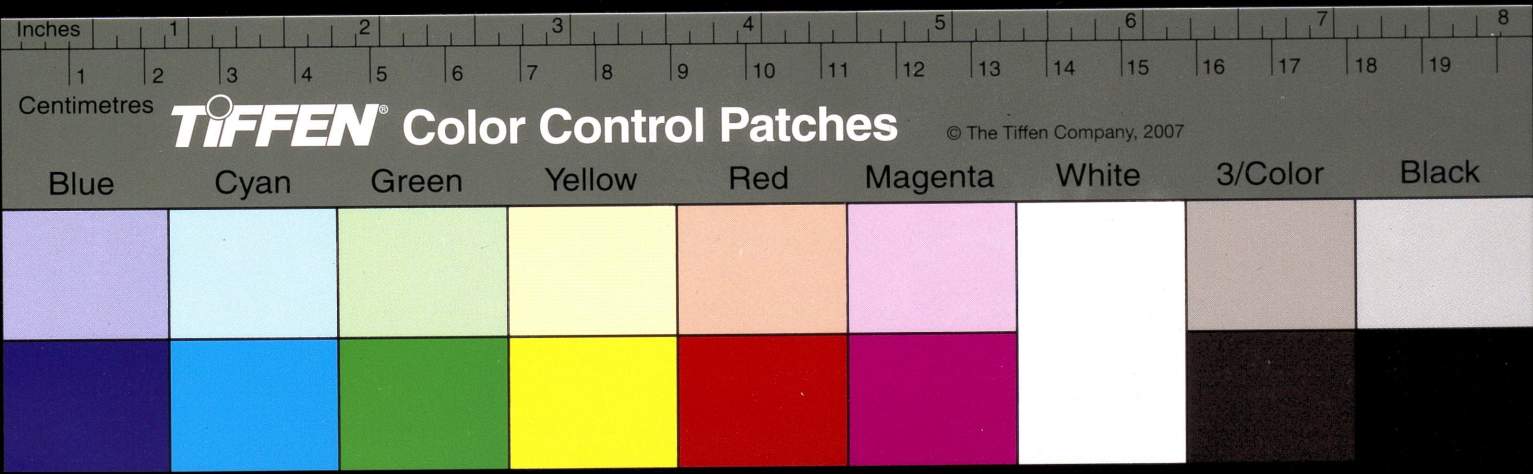
This (37)–(38) conditions of \dot{q}_k 's at a would have no dy the expressions dq and the reduction omitted this trivial fine the requisite la

[1] Whittaker

[2] Prange:

[3] Levi-Civita

[2], p. 556.



systems of equa-

f_r, f_s , by $2f_r, 2f_s$. This would necessitate in addition to (37) the conditions,

$$(31) \quad \frac{\partial f_r}{\partial \dot{q}_r} + \frac{\partial f_s}{\partial \dot{q}_s} = 0 \quad (\text{for all } r, s). \quad (38)$$

$$(32) \quad \frac{2d\dot{q}_s}{f_s}$$

$$(33)$$

$$= 0 \quad (34)$$

$$\frac{\partial}{\partial \dot{q}_s} (M_{rs} f_s) = 0 \quad (35)$$

$$= 0 \quad (36)$$

virtue of (34)–

$\left. \begin{array}{l} \\ \\ \\ \end{array} \right\} (r, s, \neq k)$

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$$(37)$$

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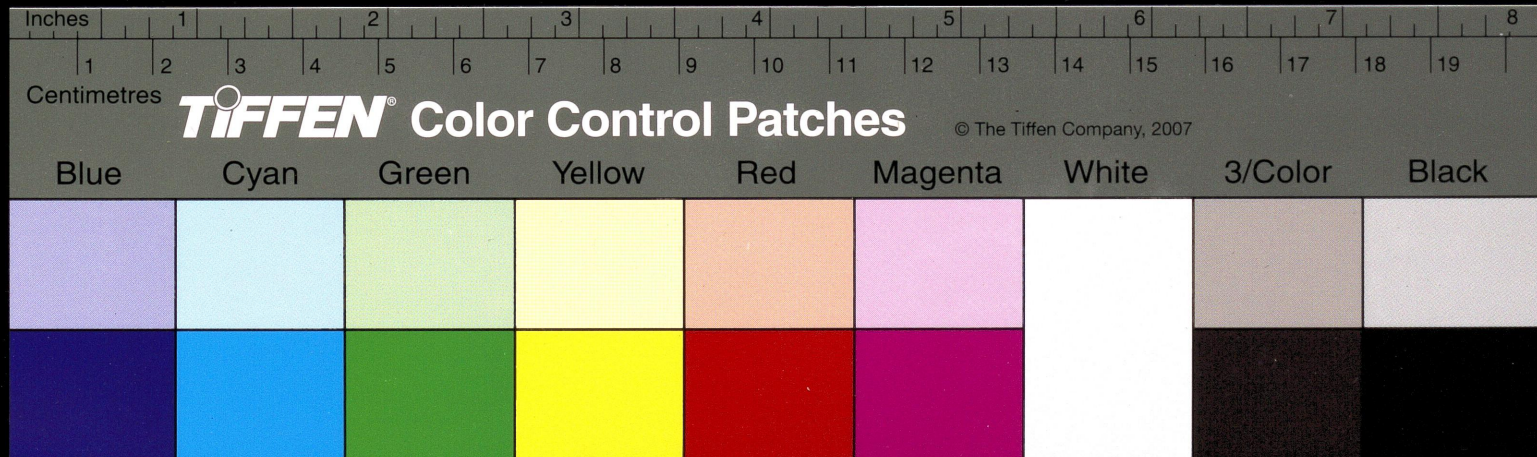
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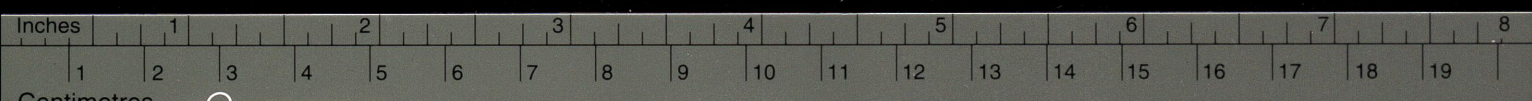
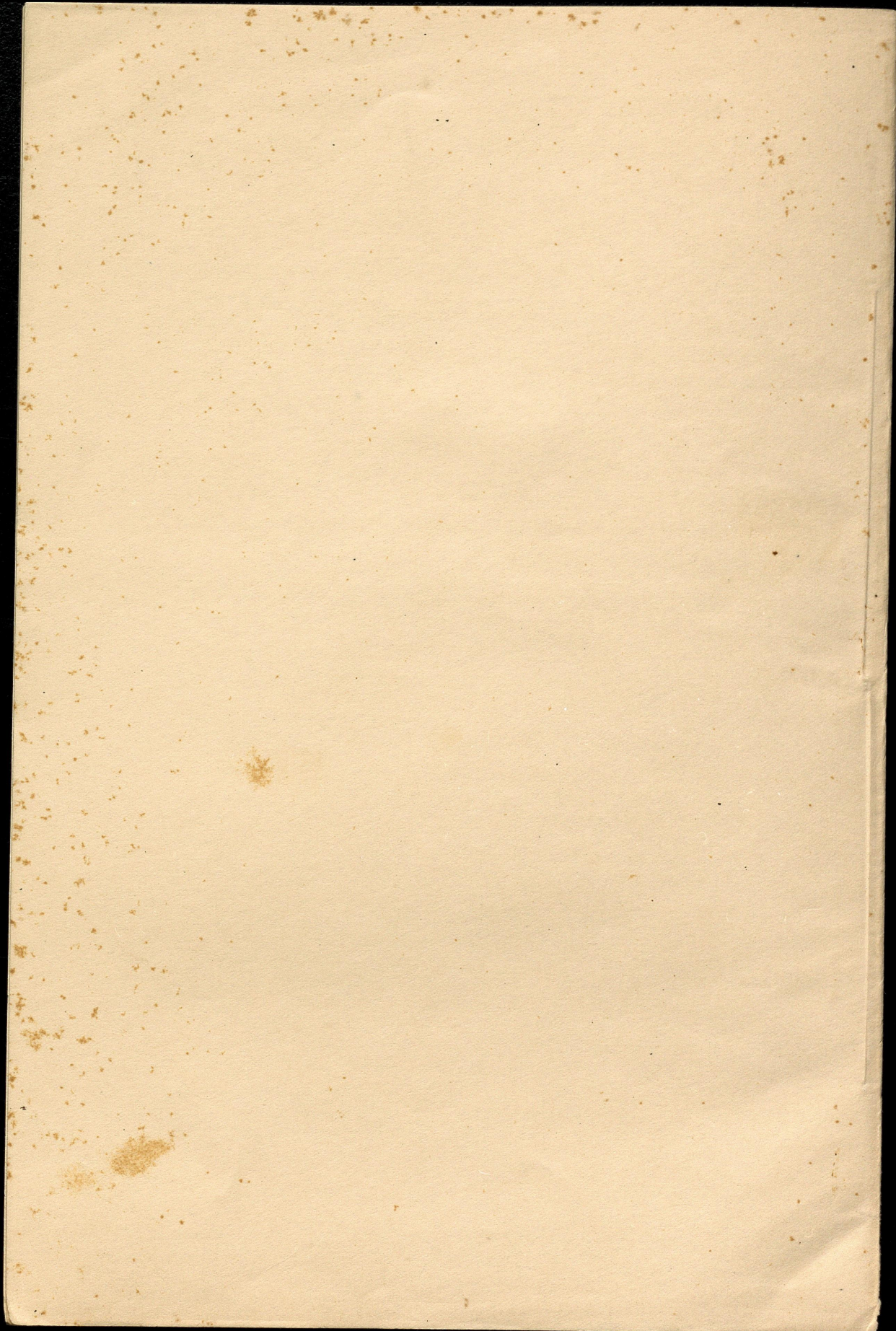
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This (37)–(38) would imply that f_k 's were not functions of \dot{q}_k 's at all. Under these conditions (31)–(33) would have no dynamical significance at all being merely the expressions $dq_1/dt = \dot{q}_1, \dots$; and we should have $T = T_0$ and the reduction $\ddot{q}_k = f_k$ becomes quite trivial. Hence if we omitted this trivial case, equations (34)–(36) appear to define the requisite last multipliers correctly.

References

- [1] *Whittaker*: Analytical Dynamics (1927), p. 284-6.
- [2] *Prauge*: Ency. Math. Wiss IV, 4, p. 550.
- [3] *Levi-Civita*: Turin Atti. 31 (1895), 816. See also [2], p. 556.





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