with best compliments and respects to Shri V.K. Krishna Menon
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SECTION OF MATHEMATICS

President : B. S. MADHAVA RAO

Presidential Address

MODERN ALGEBRA AND THEORY OF ELEMENTARY PARTICLES

1. Introduction

The ever increasing use of the deepest and the most abstract parts of mathematics has been a remarkable feature of modern physical theory. Mathematics of the present day abounds with abstract notions like abstract sets, abstract spaces, abstract algebra, abstract analysis, and so forth, and equally so in the highest departments of modern physics like relativity and quantum theory, it appears that the abstract point of view is likely to yield the most fruitful results. It is also true that the more varied and more subtle contributions from mathematics are in proportion to the wider range of physical fact acquired as a result of more and more accurate experimental research in fields involving a progressive increase in the complexity of experimental facts.

This increase in the application of abstract mathematical notions can be traced to a large extent to the change brought about by quantum mechanics in the meaning to be attached to physical quantities. This basic importance of mathematics arises mainly on account of the fundamental notion of complementarity which, when pressed to its logical conclusion, implies that natural science is not nature itself, but a part of the relation between nature and man, and is, therefore, dependent on man. Thus the understanding of the symmetry laws of nature is nothing but the attainment of the transparent clarity of a mathematics which governs these possible laws. In the old classical physics, a physical quantity was considered as the exact equivalent of the mathematical function assigned to the observable. Mathematics now abounds with concepts which are not mere functions, and yet allow the assignment of numbers under certain conditions, *e. g.* matrices, differential operators, groups, integral operators, and tensors. What quantum mechanics has done is to emphasise that the definition of physical quantities as such operators should be taken, not in a symbolic, but exact sense. With this interpretation, an operator may yield not one but a large set of numbers which can, however, be consistently used with the aid of a statistical theory. In the language of mathematics, we might say that the number concept has been replaced by the more general concept of aggregates.

The widening of the concepts of theoretical physics is bound to be of significance to the mathematician also. The history of mathematics is a clear witness to the fact that stimulating questions arising in the application of mathematics to other fields spur on the progress of mathematics itself. In some concepts like analysis, one can even say that mathematics owes more of its advance in these branches to the physicist than to any other agent. We thus recall how the problem of heat conduction, and wave motion led to the development of the function concept, and the introduction of orthogonal series, which are basic elements of present day analysis. Dirichlet's problem in potential theory had a profound influence on the calculus of variations, and led to the theory of integral equations. In the masterly hands of Hilbert, this became a theory of orthogonal transformations, and reduction of quadratic forms, and created the atmosphere which stimulated basic discoveries on function spaces including the abstract Hilbert space. It is a remarkable, but a common feature in physical theory, that while these developments excited by physics were taking place in mathematics, the physicist himself had little or no interest in them, until the advent of quantum mechanics compelled him to look at them more seriouly. It is common knowledge, nowadays, that the analytical problems of quantum mechanics can be thought of in terms of linear transformations in an abstract Hilbert space.

2. Algebra and physics

On the other hand, the relationship between algebra and physics has been a rather loose one in the earlier years. It is no doubt true that the theory of groups has always played an important part in theoretical physics, as for example, in the domains of molecular physics, crystal physics, and chemical physics. Also the analysis of space and time has involved group-theoretic considerations, and classical dynamics has employed group methods through transtormation theory, and many recent studies have freely used topological notions. But it is only in the last two decades that the more profound portions of the theory of groups, and some other parts of modern algebra have played notably enhanced roles in relativity and quantum theory.

In this case it is perhaps not true, as in the case of analysis, that these branches of group theory and algebra were inspired by the physicist. The researches of Noether, Weatherburn, Dickson, Artin and others on abstract algebras, of Frobenus, Schur, and Weyl on the theory of group representations, and of Cartan on continuous groups were carried on independently as a result of the fusion of several fundamental ideas in mathematics itself. These have now found applications in quantum mechanics, and this relationship is attractive from many points of view.

It is, however, true that in recent years, these and many other branches of mathematics have made advances in some directions as a consequence of this relationship to physics. These advances have resulted sometimes as a consequence of the methods of rigour employed in pure mathematics. In the work of many theoretical physicists one notices sometimes that heavy emphasis is laid on non-invariant (*i.e.* to say possessing no physical significance) properties of the matrices under consideration, and at the same time, their invariant properties are insufficiently explained. Very often, the colourless pseudonym "matrix" hides a multitude of spin tensors of widely varying nature, the properties of which remain almost entirely in the dark. Even more undesirable is the fact that relations among spinor quantities are often written in non-invariant form, a circumstance which naturally does not permit the disclosure of their physical meaning. As examples arising out of such rigorous treatments, we might mention the special types of non-associative algebras including the quantum mechanical algebra, theory of rings of operations, simply reducible groups and their representations, the new methods of abstract lattice theory resulting from the discussions relating to the logic of quantum mechanics, the general theory of spinors, the theory of gauge groups, and above all, the theory of semi-simple Lie groups of which Lorentz groups are particular types, and the more general theory of topological groups, and their different representations.

3. Concept of elementary particles

The subject of elementary particles of nature holds the centre of interest in physics today, and is a striking example of knowledge gained by an intimate fusion of experimental and theoretical methods, specially the relativistic and quantum theories. While the very concept of elementary particles is quantum mechanical in origin, it is certainly true that relativity has been crucial for further developments in quantum theory. Thus, starting from the notion of a photon, the relationship $E = cp$(including $E = mc^2$) between the energy and momentum of a particle is relativistic. Another example is the de Broglie relation $p = h/\lambda$ which followed from Planck's relation $E = h\nu$ on the basis of Lorentz invariance. A third example is the relativistic quantum mechanics inaugurated by Dirac, which led to the concept of anti-particles and to the production and annihilation of pairs, by turning the apparent paradox of negative energy states into the most remarkable success of the theory. In addition to being crucial for the development of quantum theory, relativity has provided some general results which give a deeper insight into the structure of quantum mechanics as illustrated, for example, in the connection between Lorentz invariance and the conservation laws of quantum field theory. A brilliant example of this connection is Pauli's analysis of the relation between spin and statistics, which is one of the most important applications of the theory of representations of the Lorentz group.

With the growth of experimental techniques in cosmic ray work, nuclear reactions, and specially with the completion in recent years of the cosmotron and bevatron, a host of new elementary particles have been discovered, and it is the fashion nowadays to classify them into the old, and new or strange particles. Another classification is to group them under the categories of well-established particles, well-established but not well determined ones, and the less well-established ones.

The oldest known of the old particles are the *electron* and the *proton* with charges $-e$ and $+e$, masses m_e and $m_p \sim 2000 m_e$ and the same spin 1/2, associated with the constant angular momentum $h/2$. The proton-electron constitution of nucleii was rendered untenable as a consequence of Dirac's theory of the electron, and Pauli's relationship between spin and statistics *viz.*, that particles with integral spin obey the Einstein-Bose statistics, while those with half-integral spin obey the Fermi-Dirac statistics, and this led to the discovery of the *neutron* having nearly the same mass as the proton, no charge, and spin 1/2. Next, to explain the phenomenon of β-decay consistently with the proton-neutron constitution of nucleii, the *neutrino* with mass very small compared to m_e, if not exactly zero, spin 1/2, and zero charge, was postulated by Pauli, and the process of β-decay was described by

$$\text{neutron} \rightarrow \text{proton} + \text{electron} + \text{neutrino},$$

in consonance with the conservation of spin and charge. The next old elementary particle to be discovered was the *positron* with the same mass and spin as the electron, but charge $+e$, and this discovery and the demonstration of the processes of annihilation and pair-creation in cosmic ray showers provided a remarkable confirmation of Dirac's relativistic wave equation. The neutron and position also led to a radical revision of the concept of elementary particles *viz.*, that they cannot be considered permanent and immutable, but take part in transformations among themselves. The next elementary particle was the *meson* postulated by Yukawa to explain the exchange nature of nuclear forces changing a proton into a neutron, and a neutron into a proton every time these particles interact, the intermediaries in these interactions being the mesons with charges $\pm e$, integral spin, and mass about 200 to 300 m_e. The discovery of particles in cosmic radiation with masses of this order must be regarded as a confirmation of the general idea of the Yukawa theory. The β-decay is fitted into this theory by assuming that the original process takes place really in two steps, as for *e.g.*,

$$\text{neutron} \rightarrow \text{proton} + -\text{ve meson, and } -\text{ve meson} \rightarrow \text{electron} + \text{neutrino},$$

the latter corresponding to a spontaneous decay of the meson, thus introducing a new feature into the concept of elementary particles. This meson may be considered as a bridge between the old and new particles, and the neutrino postulated in the β-decay, and the spontaneous decay of the meson has always been a mysterious particle, and can well be considered as an old as well as a new particle.

Coming now to the new particles it is best to group them on the basis of their rest masses. On this classification, the old particles *viz.*, the photon r^0, the neutrino γ^0, the electron e^-, and the positron e^+ could all fall in the first group of particles with masses less than or equal to m_e. The second group, termed the group of *L-mensons*, has for its most important member the π-meson which is responsible for the main features of nuclear forces, with mass about $273m_e$ and existing in the forms π^+, π^- and π^0, all having spin zero. This second group of L-mesons is characterised by particles having masses between m_e and m_π, and contains another member, the μ-meson well known in cosmic radiation at sea level, of mass about $207m_e$, and existing as a well-established particle in the forms μ^+ and μ^- both of spin 1/2. The third group consists of the *K-mesons*, the new particles discovered in cosmic radiation with masses intermediate between m_π $\pm$ and m_{proton}. During the last few years, at least eight different K-mesons have been discovered with different decay modes, but the well-established ones among them are the τ^+ and τ^- mesons with masses about $966m_e$, and both of spin 0, and the θ^0-meson of about the same mass as the τ-mesons, and a spin which is integral but $\neq 1$. The fourth group would consist of the old and well-established particles, the proton p^+ with mass $m_p \sim 1836m_e$, and spin 1/2, and the neutron n^0 with mass $\sim 1839\ m_e$, and also spin 1/2, both these particles being designated as *nucleons*. The fifth group consists of the so-called *hyperons* (or Y-particles) mainly resulting from controlled bevatron experiments, with masses lying between m_p and $m_{deuteron}$, and among the well-established hypersons can be listed four, viz., the Λ^0 with mass $\sim 2181m_e$, the $\Sigma^{\pm}$ both of mass $\sim 2327m_e$, and the Ξ^- of mass $\sim 2586m_e$, all the four hypersons having half-integral spin.

Mention may also be made of the particles with properties *not so well determined*, viz., the five types of K-mesons characterised by their decay products, and made up of the so-called $\theta^{\pm}$, $\tau^{1\pm}$, and $k^{\pm}$ mesons. In the same class would fall the *anti-proton* p^- (recently produced at Berkeley) with mass same as m_p to within $\pm 5\%$, spin 1/2, and providing another brilliant confirmation of Dirac's theory. If this theory be generally applicable, every particle should have a corresponding anti-particle, and there should also be phenomena of creation and anihilation of pairs. Even a neutral particle as, for instance, the neutron which has a magnetic moment representable by a closed current must have its counterpart obtained by inverting the current, thus giving rise to an *anti-neutron*. The same is true of the neutrino also if it be associated with a magnetic moment, however, small, and even if the magnetic moment were exactly zero, the anti-neutrino could still be defined as a hole in the negative energy state of the neutrino wave equation, and this is the more usual definition of an anti-particle. The notion of anti-particles also raises the interesting question about the distinctness or otherwise of the particle and antiparticle. Some recent conjectures have been made that $\bar{\pi}^0 \equiv \pi^0$, while the $\bar{\nu}^0$ and ν^0 are different particles (the bars denoting anti-particles.).

The discovery of the new particles has emphasised the fundamental importance of the knowledge of interactions between them, and their decay modes and this emphasis has to some extent changed the weight of the old theoretical arguments, and the lines of experimental research. Also, the abundance and complexity of these interactions have stimulated tremendous activity among experimenters, and caused a great flow of interesting speculations from theoretical physicists. While a correct theoretical understanding is obviously a matter for the future, mention may be made of recent attempts of a phenomenological nature undertaken to elucidate the symmetry laws or invariance properties that hold in such interactions. Regarding the nature of the interactions, it has been found that they can be classified into three categories, the strong, the electromagnetic, and the weak ones, the first type having intensities of around 1 to 10^{-1}, the electromagnetic interaction having 10^{-2}, and the third type having intensities ranging from 10^{-12} to 10^{-14}. Designating the *K*-mesons and the hypersons as *strange particles*, examples of such weak interactions are provided by the decay of strange particles (for e.g. $\tau^+ \rightarrow \pi^+ + \pi^+ + \pi^{-1}$; $\theta^0 \rightarrow \pi^+ + \pi^-$; $\Lambda^0 \rightarrow p^+ + \pi^-$; $\Sigma^- \rightarrow n^0 + \pi^-$, and $\Xi^- \rightarrow \Lambda^0 + \pi^-$) the β-decay, the μ-meson decay viz., $\mu^{\pm} \rightarrow e^{\pm} + \nu^0 + \nu^0$, the π-meson decay viz., $\pi^{\pm} \rightarrow \mu^{\pm} + \nu^0$, and the μ-meson-nucleon interaction. The wide gap separating the weak interactions from the other two, and the fact that while the production of the strange particles falls into the first class, their decay falls into the third, are so striking as to suggest that there should be a fundamental reason for these discrepancies.

Coming now to the phenomenological invariance properties, one notices first of all that the well-established law of the conservation of the number of nucleons necessitates the assignment of a "heavy particle quantum number" N to all elementary particles. A consistent assignment is possible by putting $N=0$ for the first two groups of the mass classification, $N=\pm 1$ for nucleons and anti-nucleons respectively , $N=1$ for the hyperons, and $N=0$ for the K-meson class. Also the charge is conserved for all interactions. There has also been noticed another conservation law if one limits oneself to the strong and electromagnetic interactions only viz., the conservation of "*strangeness*" S which is satisfied by assigning $S=0$ to the nucleons and π-mesons, $S=1$ to K^+ and K^0, $S=-1$ to K^-, $\bar{K}^0$, Λ^0, Σ^+, Σ^0, and Σ^{-1}, and $S=-2$ to Ξ^-. The weak interactions violate this strangeness conservation. Finally taking the strong interactions only into consideration, it is found that the third component τ_3 of the isotopic spin quantum number (given by the Pauli 2×2 spin matrices) is invariantly connected with N and S by the relation

$$\tau_3 = Q - \frac{N}{2} - \frac{S}{2} \qquad \dots \quad (1)$$

where Q is the total charge expressed in multiples of $|e|$.

In the above, we have not mentioned invariance under spacetime transformations, but of recent interest, specially in the case of weak interactions is the concept of *parity* which is $\pm$, according as the wave function associated with the particle does not or does change its sign under space reflections alone. From the experimental information so far available, the well-established L-mesons have all a negative parity, both the $\tau^{\pm}$ are of negative parity and θ^0 is perhaps of positive parity. The question of invariance of parity in weak interactions has very recently been tested by suitable experiments suggested by Lee and Yang, and the important discovery of parity-nonconservation in such interactions has been made. This naturally raises questions of invariance under time reversal, and charge conjugation, and combinations of these operations. We will return to these questions in the last section.

It appears probable that these new points of view do not demand a radical revision of quantum concepts, but a careful reconsideration of the algebraic nature of the transformations under the full Lorentz group, its several sub-groups, and also under charge conjugation. It would, therefore, not be inappropriate if we considered in the next few sections the results already known about the relativistic quantum theory of elementary particles.

4. Relativistic quantum field theory

This theory consists essentially of two stages in its application, the c-number theory with the wave functions and field equations for the particles satisfying the postulates of special relativity, and the q-number theory making a transition to the particle picture obtained by using the quantum conditions expressing the non-commutation of field functions at different points of space-time. Such a passage from the c-number to the q-number theory, a transition from a one-particle to a many-particle picture is called double quantisation.

The essential steps in the c-number theory are the setting up of a Lagrangian which is invariant for transformations of the proper Lorentz group *i.e.* the continuous Lorentz group L_4 in which no reflections are included, the derivation of the field equations from a variational principle, the setting up of the energy-momentum, and angular momentum tensors, and leastly the setting up of a current four-vector by assuming the invariance of the Lagangian against gauge transformations, and dividing the field quantities into $U(x)$, their conjugates $U^*(x)$, and the real quantities $V(x)$. Since the field equations are derived from a Lagrangian, the field quantities themselves should transform according to irreducible representations of the group L_4. Such quantities are, as is well known, the spinors, and the field quantity can be written as $U(j, k)$ characterised by two indices j and k corresponding to spinors with $2j$ undotted and $2k$ dotted indices, and symmetric in them separately. Using the Clebsch-Gordan rule for the reduction of product representations, and the situation in the case of the subgroup of space rotations, one defines the spin of the particle as $j+k$. For the case of $2j+2k =$ even, the spinors reduce to the ordinary world tensors, but not for the case $2j+2k =$ odd. Working with these field quantities, we find the general results that for particles of half-integral spin, the total energy is not necessarily positive and for integral spin the charge density is not necessarily positive. Going to the q-number theory, and using the fact that the expressions for the non-commutation of the field quantities at different points should themselves satisfy invariance relations, these expressions can be written in terms of the bracket forms

$$[U(x), U^*(x)]^{\pm} = D(x, x') \quad \dots \quad (2)$$

the + or − being taken according as the particles satisfy the Fermi-Dirac (F.D) statistics, or the Einstein-Bose (E.B.) statistics, where, further, the transformation properties of the U's under L_4 also require that the D's should transform in a certain way. By merely considering the algebraic nature of the transformations of the D's under L_4, with the further requirements that D is the function of the invariant distance between x and x', that $D = 0$ if they be separated by a space-like distance, one is led to the general results that for integral spin, quantisation according to F.D. statistics is not possible, and that for half-integral spin there is no algebraic contradiction in either statistics being satisfied, but the removal of the negative energy difficulty is not possible if one uses the E.B. statistics. These results are purely negative, but the actual carrying through of the quantisation shows that for half-integral, and integral spin particles a stisfactory theory can be obtained using respectively the F.D. and E. B. statistics.

5. The particle aspect

The general theory sketched in the previous article enables the setting up of a wave equation satisfied by the spinors, the equations being of the second order, and in order that the theory may represent particles of a single spin, it is necessary to assume, besides the symmetry of the spinor in the dotted and un-dotted indices, that the spinors $p^{\dot{\nu}\rho}\ a_{\rho\delta..}^{\ddot{\mu}..}$, and $p_{\dot{\nu}\delta}\ a^{\nu\ddot{\mu}..}_{\rho..}$ where $p^{\dot{\nu}\rho}$ is the gradient spinor) should also be symmetrical. This makes it possible to go from a second order wave equation to a system of two first order equations of the type

$$\left.\begin{aligned} p^{\dot{\nu}\rho}\ a_{\rho\delta..}^{\ddot{\mu}..} &= \chi b_{\delta...}^{\dot{\nu}\ \ddot{\mu}...} \\ p_{\dot{\nu}\rho}\ b_{\delta...}^{\dot{\nu}\ \ddot{\mu}..} &= \chi a_{\rho\delta..}^{\ddot{\mu}..} \end{aligned}\right\}\quad (h\chi = \text{rest mass of the particle}) \qquad \ldots (3)$$

where the spinor b satisfies the same conditions of symmetry as a. The cases where a has $2k$ undotted, and $2k-1$ dotted indices, or $2k$ dotted and $2k-1$ undotted indices according as the spin is half-integral or integral respectively, are specially simple, and denoting the spinors in this case by $a^{(0)}$ and $b^{(0)}$, they have the property of going over into each other by reflections of space-time. In the integral case $a^{(0)} = b^{(0)}$, and in the half-integral case $a^{(0)} \rightleftarrows b^{(0)}$ under reflections, so that together they form a system invariant under the Lorentz group consisting of L_4, and the reflections. The first order wave equations in them are then said to be of the Dirac particle type. Using the general result of semi-simple groups that, if the matrix commuting with all the matrices of an infinitesimal product respresentation is brought to the diagonal form, the product representation is simultaneously split up into its irreducible components, it can be shown that the above type of equations can be reduced to a type involving only one spinor index viz.,

$$\left.\begin{aligned} p^{\dot{\nu}\rho}\ \psi_{\rho}^{A} &= \chi\psi^{\dot{\nu}} \\ p_{\dot{\nu}\rho}\ \psi^{\dot{\nu}B} &= \chi\psi_{\rho}^{A} \end{aligned}\right\} \qquad \ldots (4)$$

where A and B indicate that ψ involves magnitudes like $A^{\dot{r}}_{s}$ and $B^{\dot{i}}_{j}$, and the equations are to be treated as matrix equations. Finally the two equations can be combined into one single equation, the famous Dirac equation representing the particle aspect of elementary particles viz,

$$\partial_\mu \beta_\mu \psi + \chi\psi = 0, \qquad \left(\partial_\mu = \frac{\partial}{\partial \chi_\mu}\ ;\ \mu = 1, ..\ 4\right),$$

where for the case of spin 1/2, the Dirac matrices β_μ satisfy the commutation relations

$$\tfrac{1}{2}(\beta_\mu \beta_\nu + \beta_\nu \beta_\mu) = \delta_{\mu\nu} \qquad \ldots \quad (5)$$

In virtue of these relations, the system of the 16 members 1, β_μ, $\beta_\mu\beta_\nu$, $\beta_\lambda\beta_\mu\beta_\nu$, and $\beta_1\beta_2\beta_3\beta_4$ form a hypercomplex system (the Dirac algebra). As is well-known the theorems relating to the representations of a finite group can be extended to group rings and hence also to a system of hypercomplex numbers or an algebra, satisfying certain conditions. Applying the theorems which hold in the case of a semi-simple algebra of which the Dirac algebra is a particular case, it is very simple to prove that this algebra has only one irreducible representation of order 4, showing that the Dirac equation is unique but for equivalence.

The question arises whether a particle aspect of the theory can be set up for particles of higher spins also. As is well-known, this is possible for the cases of spins 0 and 1. The second order wave equations in these cases with the field quantities being a scalar and four-vector respectively, can be put in the Dirac form with the β-matrices being 5-rowed and 10-rowed respectively, the commutation relations for both cases being combined in one single form :

$$\beta_\lambda\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\beta_\lambda = \delta_{\lambda\mu}\beta_\nu + \delta_{\mu\nu}\beta_\lambda. \qquad \ldots \quad (6)$$

The β-algebra in this case can be shown to have rank 126, resulting in three representations of orders 1, 5, and 10 satisfying $1^2+5^2+10^2 = 126$.

I have considered some time back the question of extending this to the cases of higher spin. As a preliminary to this, it is necessary to derive commutation relations satisfied by the β_μ's and I have shown that we can derive such relations by making some general assumptions :

Let the infinitesimal transformation

$$x'_\mu = x_\mu + \Sigma\, \epsilon_{\mu\nu} x_\nu, (\epsilon_{\mu\nu} = -\epsilon_{\nu\mu},\ \text{numerical}) \qquad \ldots \quad (7)$$

correspond to the transformation $\psi' = A\psi$ of the wave equation with

$$A = 1 + \tfrac{1}{2}\Sigma\ \Sigma\ \epsilon_{\mu\nu}\, s_{\mu\nu}, (s_{\mu\nu} = -s_{\nu\mu},\ \text{matrices}) \qquad \ldots \quad (8)$$

where the $s_{\mu\nu}$ are the spin matrices. The general assumptions are that

(i) the wave equation is invariant against L_4. This leads to

$$\beta_\lambda s_{\mu\nu} - s_{\mu\nu}\beta_\mu = \delta_{\lambda\mu}\beta_\nu - \delta_{\lambda\nu}\beta_\mu \qquad \ldots \quad (9)$$

(ii) $$s_{\mu\nu} = K(\beta_\mu\beta_\nu - \beta_\nu\beta_\mu), \qquad (K, a \text{ constant}) \qquad (10)$$

(iii) each component of $s_{\mu\nu}$ satisfies the algebraic equation whose roots are the $(2f+1)$ eigen values of the spin operator ($f =$ spin).

These assumptions enable us to construct generalised algebras related to particles of arbitrary spin based on the commutation rules. These rules have been obtained, in particular, for the cases $f = 3/2$ and 2, and the algebra in the former case has been studied in detail, and it is shown that this algebra is the direct product of the Dirac algebra, and an associated ξ-algebra (a result true for general half-integral spin). The ξ-algebra is shown to have just three representations of orders 1, 4, and 5 such that the rank of the algebra is equal to 42.

Explicit matrix representations for the non-trivial cases of orders 4 and 5 have also been obtained.

6. Particles of rest mass zero, and the gauge group

The statistical interpretation of the wave functions appearing in the general field theory of section 4 requires that the Lagrangiam be invariant under the gauge transformations of the first kind satisfied by the field quantities viz.,

$$U(x) \rightarrow U(x)e^{id}\ ; \qquad U^*(x) \rightarrow U^*(x)e^{-id} \qquad \ldots \quad (11)$$

with α, an arbitratry constant in the case where no external fields are present. As already mentioned in section 4, the invariance of the Lagrangian under gauge transformations also leads to the possibility of setting up a current vector since such transformations express the non-measurability of the phase of the complex wave function of a charged particle. In the case of a particle of rest mass zero one has the physically significant result that the particle has to have only two independent and really different plane waves for a specified wave number and frequency. For this purpose, it is found necessary to add to (11) another kind of gauge transformation said to be of the second kind. Thus for example in the case of fields of arbitrary integral spin describing particles of zero rest mass, the field quantity would be a tensor $A_{\lambda\mu\cdots\nu}$, and the gauge transformation of the second kind is defined by

$$A'_{\lambda\mu..\nu} = A_{\lambda\mu..\nu} + N_{\lambda\mu..\nu} \qquad \ldots \quad (12)$$

where
$$\left.\begin{aligned} N_{\mu..\nu\rho} &= \frac{\partial c_{\mu..\nu\rho}}{\partial x_\lambda} + \frac{\partial c_{..\nu\rho}}{\partial x_\mu} + \ldots + \frac{\partial c_{\mu..\nu}}{\partial x_\rho} ; \\ \Box C_{\mu..\nu} = 0 \; ; \; & C_{\mu\mu..\nu} = 0 \; ; \; \frac{\partial C_{\mu..\nu}}{\partial x_\mu} = 0 \end{aligned}\right\} \qquad \ldots \quad (12,a)$$

Similar gauge transformations of the second kind can be set up for particles of half-integral spin also, with the aid of spinors. In both cases we have to consider the transformations of the second kind in addition to (11) where α is now to be taken as an arbitrary function of space-time, and we are then led to only two independent components if we consider states which go into each other under such gauge transformations as equivalent.

7. Neutral particles and charge invariance

The case of neutral particles corresponds to real fields with $U = U^*$. The original form with a complex U is equivalent to two real fields $V = V^*$, and $W = W^*$ with

$$U = \frac{1}{\sqrt{2}}(V + iW) \; ; \quad U^* = \frac{1}{\sqrt{2}}(V - iW) \qquad \ldots \quad (13)$$

the numerical factors being introduced for the sake of convenience of quantisation. A theory of neutrical particles can be obtained from (13) by striking out W and such a theory is called an "*abbreviated*" one. To find out whether this method of splitting into two real fields is possible in the case of a particle of spin 1/2, let us consider first the Dirac equation of the electron, and define the α, β matrices by $\alpha_k = i\gamma_4\gamma_k (k = 1, 2, 3)$ and $\beta = \gamma_4$. We can now introduce the Pauli C-matrix cefined by

$$\beta^* = -C\beta C^{-1} \; ; \quad \alpha_k^* = C\alpha_k C^{-1} \qquad \ldots \quad (14)$$

and show that C exists, that $C^* C$ commutes with all the γ_μ, and is hence a constant and that C is symmetric, and hence can be so chosen that $C^*C = 1$. The C-matrix can be used to set up a Lorentz-invariant ordering between the solutions of Dirac's equation with positive and negative frequency viz,

$$u_-^* = Cu_+ \; ; \quad u_+ = C^{-1}u_-^* \qquad \ldots \quad (15)$$

such solutions being called *charge-conjugate* solutions. This terminology can be justified by considering the effects of an external electro-magnetic field, and it can be shown that if u_+ satisfies the wave equation with charge $+e$, then u_- satisfies it with charge $-e$.

These considerations can be generalised to a spinor field u, and the decomposition into the "real" fields v and w could be done according to

$$u = \frac{1}{\sqrt{2}}(v + iw) \; ; \quad u^* = \frac{C}{\sqrt{2}}(v - iw) \qquad \ldots \quad (16)$$

where v and w fulfil the Lorentz-invariant reality conditions

$$v^* = Cv\ ; \qquad w^* = Cw \qquad \dots \quad (17)$$

The whole of the q-number theory can be worked out with the pair u, w in place of u, and the transition to the charge conjugate state is realised by $v \to v$, and $w \to w$, and for this substitution, the current vector changes its sign properly in the q-number theory quantised according to the exclusion principle. To make a transition to a neutral particle of spin 1/2, one makes an "*abbreviation*" of the theory due to Majorana by striking out w and its bracket relations or, what is the same thing, by identifying the charge conjugate states. Making this abbreviation in the q-number theory, it can be shown that the current vector , as also the magnetic moment, vanish identically. It is interesting to notice that the question so far unsettled as to whether the appropriate theory for the neutrino is the abbreviated or unabbreviated one appears to have been decided in favour of the latter alternative as a consequence of the recently established result about the non-conservation of parity in weak interactions mentioned in section 3, *i.e.* the neutrino state, and the anti-neutrino state (defined as a hole in the negative energy state) cannot be the same, and the Majorana theory for such a neutrino is, therefore, not possible. This also makes it possible to build up a theory of the neutrino in terms of two-component spinors instead of four-component spinors.

The notion of charge-conjugate states introduced above has an algebraic significance in that it can be extended to higher spins also. This can be achieved by working with field quantities called *undors* which form a generalisation of the Dirac spinor u_ρ, which is characterised by being a pair of spinors transforming one into the other by a space-time reflection, and are quantities $\psi_{\rho_1\rho_2\dots\rho_n}$ transformaing like products of Dirac spinors, the Dirac spinor u itself being considered as an under of rank one. The Majorana theory of neutral particles identifying charge-conjugate states would, therefore, deal in this case with self charge-conjugated four-spinors, or what are called *neutrettors* of rank one. Similarly, field functions for the case of a particle of spin 1 can be taken as the symmetric under $\psi_{\rho_1\rho_2}$ of rank two, and this can be associated with a charge-conjugated undor according to

$$\psi^{\pounds}_{\rho_1\rho_2} = \pounds^{(1)}\pounds^{(2)}(\psi_{\rho_1\rho_2})^* \qquad \dots \quad (18)$$

where $\pounds$ is identical with the Pauli matrix C^*, and $\psi^{\pounds}$ is called the charge adjoint of ψ. A Majorana abbreviation would now give a *neutrettor of rank two*, which would be appropriate as a field function for neutral particles of spin 1.

To go to higher spins, we first generalise the Pauli matrix operator by putting

$$\pounds = \prod_{\mu=1}^{n} \pounds^{(4)} \qquad \ldots \quad (18,a)$$

and operate with it on the conjugate complex under $\psi^*_{\rho_1 \rho_2 \ldots \rho_n}$, and derive the transformed under

$$\psi^{\pounds}_{=} = L\psi^* \qquad \ldots \quad (18,b)$$

with the property that in the field equations satisfied by the $\psi^{\pounds}$, the constants of the dimensions of a charge, e, for instance, are changed into their opposites according to $e^L = -e$. Further it can be shown that the transformation

$$\psi = \psi^L \text{ together with } e \rightarrow e^L \qquad \ldots \quad (A)$$

where ψ^L is given by $\psi^L = \psi^{\pounds}$ ... (B)

not only ensures the invariance of the fieldequations, but also the invariance of all physically significant quantities *provided that one works with a q-number theory i.e.* only taking account of the commutation rules holding between the undor components. The invariance of physical quantities under (A)—(B) is called *charge invariance.* Pauli and Belinfante have deduced the striking result that the connection between spin and statistics derived in Section 4 follows as a consequence of this postulate of charge invariance under (A)—(B), showing indirectly the partly algebraic nature of this notion.

8. Parity, charge conjugation, and time reversal

The discovery of the host of new and strange particles, and their interactions described in Section 3 including, but still mysterious particle, the neutrino, and the proper understanding of the several conservation laws governing the interactions have necessitated a reconsideration of the algebra of elementary particles presented in sections 4–7 since the emphasis in the earlier years has been on free particles, while today the investigation of interactions between the several kinds of particles is in the centre of interest. Also questions recently raised of invariance separately under space reflections, time reversal, and space-time reflections which all constitute elements of the full Lorentz group, just like the continuous L_4, are bound to be of great algebraic significance. Although the replacement of a function by its complex conjugate is not a linear operator, we have seen in section 7 how the concept of charge invariance wherein such replacement occurs, can also be considered partly algebraic in nature. Pauli (Bohr Commemoration Volume, 1955) has recently reexamined some of these questions, and his work, coming as it did, before the discovery of non-conservation of parity in weak interactions, discusses the invariance under space-time reflections, without going into space reflections alone or time reversal alone separately. Two of his general conclusions, taking interactions properly into consideration, are: (i) for the case of integral spin, the assumption made in equation (2) that $D = 0$ if the points are separated by a space-like distance, so as to exclude the possibility of quantisation with anti-commutators, now appears superfluous, and (ii) for half integral spin, the validity of the exclusion principle, and the consequent Dirac theory of holes still holds.

He introduces the three operations viz., (a) *particle-antiparticle conjugation* (which includes charge conjugation), denoted by AC connecting every spinor field with its own complex conjugate field, also in the case of several fields, (b) simultaneous transformation of every particle into its anti-particle coupled with the reflection of space-time co-ordinates, which is called a *strong reflection*, n and denoted by SR, and (c) reflection of space-time alone without the change of particle into anti-particle which is called the *weak reflection*, and denoted by WR, and indicates the obvious result that each one of the transformations SR, AC and WR is a product of the other two. Further by considering the simplest cases of particles of spin 0, 1/2 and 1, their interactions, he has derived the following interesting results about WR, and SR :

(a) The WR transformation holds whether the normal connection between spin and statistics holds or not.

(b) The transformation law of a quantity with respect to L_4 does not determine uniquely its behaviour for WR; and the invariance with respect to WR imposes further restrictions upon the Lagrangian density of the interaction, besides the invariance for L_4.

(c) The SR is uniquely determined as a consequence of L_4, and the spin-statistics connection. The remarkable result (c) that the SR holds from more general postulates than the WR or AC is referred to as the *Pauli-Luders theorem* and throws light on some problems that have arisen in connection with the interactions between elementary particles. As SR is the product of WR and AC, it follows immediately from the above theorem that the results (a) and (b) are also true for AC with the same additional restrictions imposed on the interaction Lagrangian density, and that the transformation of a certain kind of spinor or tensor for WR uniquely determines its transformation for AC.

Although in the above considerations of Pauli, the conservation properties of space reflection, and time reversal have not been separately taken into account, it can be shown, as has been done by Lee, Oehme, and Yang (Phys. Rev., **106**, p. 340, 1957) that we can derive from (c) above, some general results regarding conservations under space reflections or parity (P), pure time reversal (T), and charge conjugation (C). Noting that SR is a combination of P, C, and T, these authors have considered the transformation properties (in the Schroedinger representation) of wave functions of the doubly quantised spin fields (both integral and half-integral) under the operations P, C and T with these transformations involving phase factors η_p, η_c, η_t of absolute values equal to unity. Working with a local Hermitian operator H invariant under L_4, they have shown that the Pauli-Luders theorem is equivalent to the statement that there always exists a choice of η_p, η_c, η_t such that (a) H commutes with the product of the operators P, C, and T taken in any order, and (b) if this choice of phase does not make H commute with P, for example, then no other choice does, and the theory is not invariant under P, and the same holds for C and T also. Of course, (b) includes the possibility also that the choice of phases made under (a) may make P commute with H. The statement (a) and (b) constitute the *CPT-theorem*, which is thus a simple consequence of the Pauli-Luders theorem. It follows from the CPT theorem that if one of the three operators P, C, and T is not conserved, at least one other must also not be conserved. Thus there are five possibilities of conservation or non-conservation of P, C, T as indicated in the table below:

No.	*Non-conserved operators*	*conserved operators*
1	...	P, C, T
2	C, T	P, CT, TC
3	P, T	C, PT, TP
4	C, P	T, CP, PC
5	P, C, T	PCT and permutations

... (19)

The CPT theorem taken along with the recently established experimental result of non-conservation of P in weak interactions, raises many interesting questions about conservation laws relating to strong and weak interactions in which the old, the new and the strange particles take part, and whether on the basis of such laws one could explain the reason for the existence of these two types of interactions separated by a wide gap. Explanation is also needed for the fact that while the number of nucleons and total charge are conserved for all interactions, the strangeness is conserved onlyin strong interactions. Since strangeness is not conserved in weak interactions, the question arises as to whether there is any relation between this non-conservation, and other types of non-conservation indicated in table (19). This table also shows that the non-conservation of P in a weak interaction does not definitely decide whether C or T, or both are not conserved in the same interaction, since there are three rows in the table including non-conservation of P. Also while energy and momentum are conserved in weak interactions, the question of angular momentum being conserved is still an open one. The explanation of the invariance relation (1), and of the relationship between several K-particles, in particular, whether some of them are identical, is not yet clear. The characteristic feature of strange particles in that their production falls into the class of strong interactions, while their decay falls into the class of weak interactions, requires an explanation. It is impossible to go here into the theoretical questions involved in the numerous problems raised above, and so we will deal with a few of them only.

Considering first, the neutrino involved in the weak interaction of β-decay, the experiments of Dr. Wu and collaborators at Columbia University on β- decay of Cobalt 60 have shown that parity is not conserved in the interaction. As already mentioned in Section 7, this makes it possible to develop a two component theory of the neutrino, and it is easy to see that in such a theory, the spin of a neutrino ν^0 (defined as a particle in the positive energy state) is always parallel to its momentum, while the spin of an anti-neutrino ν^0 (defined as a hole in the $-$ve energy state) is always anti-parallel to its momentum (that of υ^0). Thus the spin and velocity of ν^0 represent the spiral motion of a right-handed screw, while the spin and velocity of $\bar{\nu}^0$ represent the spiral motion of a left-handed screw. The theory also shown that there is no invariance under C, and hence, as indicated in table (19) there may be invariance under CP, and also under T, or T also may be violated with invariance under CPT. By making a deeper analysis of the theory, Lee and Yang (Phys. Rev., **105**, p.1674, 1957) have shown that the correct decay process for the μ^--meson is given by

$$\mu^- \rightarrow e^- + \nu^0 + \bar{\nu}^0. \qquad \ldots \quad (20)$$

The next question is about the status of the CPT theorem for strong interac- tions. Experimental evidence indicates that P is conserved in such actions, and it is usual to assume that C and T are also conserved, so that the first row of table (19) represents strong interactions. Making this assumptiontion, and considering a Hamiltonian $H = H_{strong} + H_{weak}$, with the former invariant under C, P, and T, and both terms invariant under L_4, one can derive some interesting results throwing light on the inter-relationships between non-conservation under C, P, and T.

The third question we will consider is the relationship between the strange K-mesons, τ^+ and θ^+, or the so-called $\tau-\theta$ puzzle. These have the decay processes

$$\left.\begin{aligned} &\tau^+\rightarrow\pi^++\pi^++\pi^- \\ \text{and}\qquad &\theta^+\rightarrow\pi^++\pi^0 \end{aligned}\right\} \quad \ldots \quad (21)$$

and experimental data indicate that the two particles have closely identical masses and life times. On the other hand there is evidence of the non-identity of spin-parity properties of the two particles. Both the above interactions are weak since strangeness is not conserved in them, and non-conservation of P therefore makes it possible to think of τ^+ and θ^+ as one one particle which has a definite parity on production, but which can decay into various partities. A more interesting approach to the mass degenaracy of τ^+ and θ^+ would be to assume, in analogy with the mass degeneracies of electron-positron, and neutron-proton, that an invariance law is responsible for this mass degeneracy. Assuming τ^+ and θ^+ to have the same spin but opposite parity, we can denote this invariance law as "*parity conjugation*", and denote it by C_p. Thus, C_p would commute with the part of the Hamiltonian including strong interactions (H_s) *i.e.*

$$C_pH_s-H_sC_p=0 \quad \ldots \quad (22)$$

The other part H_{weak} does not commute with C_p, producing the small mass difference between τ^+ and θ^+. Consider now the strong interaction

$$\pi^++n^0\rightarrow\Lambda^0+\theta^+ \quad \ldots \quad (23)$$

Equation (1) implies that there exists a parity-conjugated reaction of equal strength corresponding to (23) viz ;

$$\pi^++n^0\rightarrow\Lambda'^0+\tau^+ \quad \ldots \quad (24)$$

where Λ'^0 is the parity-conjugated hyperon of Λ^0. Extending the above reasoning to Σ, one concludes that there are two types of Σ with opposite parity. In general, it can be shown that all particles of odd strangeness S given by (1) must be *parity doublets*". For systems of even strangeness, the operation therefore leaves the parity invariant, and we have

$$C_pP-(-1)\ PC_p=0 \quad \ldots(25)$$

Mention may also be made finally of a result due to Luders and Zumino (Phys. Rev. **106**,, No. 2, p.385, 1957) that the masses, and (for unstable particles) also lifetimes of particles and anti-particles are equal as a consequence of *CPT*.

9. Algebraic significance of thf c,p,t

P, *T* and *PT*, and L_4 all constitute elements of the full Lorentz group, and the operation *C* is also algebraic in nature in that it connects every spinor field with its own complex conjugate. Further from the Pauli-Luders theorem that *PCT* is uniquely determined by L_4, and the spin-statistics connection, and the fact that the latter too has a partly algebraic significance, we derive the algebraic nature of *PCT*.

For a complete understanding of the several invariance laws, we shall have to first of all derive the irreducible representations of the full Lorentz group consisting of all the leements (L_4, PL_4, TL_4, PTL_4) and having the sub-groups (L_4), (L_4, PTL_4), (L_4, PL_4) and (L_4, TL_4). These representations are known in the literature, and a convenient and systematic list, also showing the behaviour of the base vectors of the several representations under complex conjugation has been recently given (Heine, Phys. Rev. 107, No.2, 1957).

I have recently attempted to derive the algebraic significance of some of the symmetry laws relating to elementary particles on the basis of these representations, and found that it is useful to borrow notions from the theory of semi-simple Lie algebras, complex Lie groups, and the theory of topological groups.