2-3-55

I have picked up the loose ends in the investigation of the properties of the standard Kerr-NUT metric

$$ds^2 = 2(du + g\sin\alpha\, d\beta)dx - 2L(du + g\sin\alpha\, d\beta)^2 - M^2 d\Omega^2.$$

with Pratap Bhatt at Rajkot. Referring to pg 94 of the Zeus. The results collected there are:

$$M^2 = \frac{f}{Y}(X^2 + Y^2) \qquad f = f(y) = \frac{(g\sin\alpha)_\alpha}{2\sin\alpha} \qquad g\,d\alpha = dy$$

$$X = C - t. \qquad C_x = -Y_y \qquad C_y = Y_x$$

$$2L = \frac{\Lambda}{3} X^2 - \frac{Y'}{Y} X + 2G + \frac{2FX + 2E}{X^2 + Y^2} \qquad Y' = Y_x$$

$$E = Y^4 \frac{\Lambda}{3} - 2Y^2 G - Y^2 Y_y$$

$$2G = \frac{Y}{f}\left[\frac{g^2}{2}\nabla^2 \log\frac{f}{Y} + f_y - 3f\frac{Y_y}{Y} - 1 + 5fY\frac{\Lambda}{3}\right]$$

$$F_x = -\left(\frac{E}{Y}\right)_y \qquad F_y = \left(\frac{E}{Y}\right)_x.$$

The occurrance of both g and f in G in derivative form makes working with G rather difficult. We carry out a change from g and f to a single variable ϕ. This is done on pp 77, 78 of the Zeus. We reproduce that here.

Put $f = g^2\varphi$

$\therefore$ ~~$\varphi = \frac{g^2}{f} \quad \therefore 2\varphi = 2g\, g\sin\alpha$~~

$$\varphi = \frac{f}{g^2} \qquad 2\varphi = \frac{2f}{g^2} = \frac{2f\sin\alpha}{g\cdot g\sin\alpha} = \frac{(g\sin\alpha)_x}{g\sin\alpha} = \frac{(g\sin\alpha)_y}{g\sin\alpha}$$

Put $g\sin\alpha = l$ $\qquad \therefore \boxed{2\varphi = \frac{l_y}{l}}$

$g\alpha_y = 1$

But $l = g\sin\alpha \quad \therefore \frac{l_y}{l} = \frac{g_y}{g} + \frac{\cos\alpha}{\sin\alpha}\alpha_y$

$$= \frac{g_y}{g} + \frac{\cot\alpha}{g}$$

$$\therefore \frac{g_y}{g} = \frac{l_y}{l} - \frac{\cot\alpha}{g} = 2\varphi - \frac{\cot\alpha}{g}$$

$$\frac{g^2}{2}\nabla^2\log f + f_y = \frac{g^2}{2}\nabla^2\log g^2\varphi + (g^2\varphi)_y$$

$$= \frac{g^2}{2}\nabla^2\log g^2 + \frac{g^2}{2}\nabla^2\log\varphi + 2g g_y\varphi + g^2\varphi_y$$

$$= g^2\nabla^2\log g + \frac{g^2}{2}\nabla^2\log\varphi + g^2\left(\frac{g_y}{g}\cdot 2\varphi\right) + g^2\varphi_y$$

$$= g^2\frac{d}{dy}\left(\frac{g_y}{g}\right) + \frac{g^2}{2}\nabla^2\log\varphi + g^2\left[2\varphi\left(2\varphi - \frac{\cot\alpha}{g}\right)\right] + g^2\varphi_y$$

$$= g^2\frac{d}{dy}\left(2\varphi - \frac{\cot\alpha}{g}\right) + \frac{g^2}{2}\nabla^2\log\varphi + g^2\left(4\varphi^2 - \frac{2\varphi\cot\alpha}{g}\right) + g^2\varphi_y$$

$$= g^2 2\varphi_y - \frac{g^2}{g}(-\operatorname{cosec}^2\alpha)\frac{1}{g} + g^2\frac{\cot\alpha}{g^2}g_y + \frac{g^2}{2}\nabla^2\log\varphi + g^2\left(4\varphi^2 - \frac{2\varphi\cot\alpha}{g}\right)$$

$$+ g^2\varphi_y$$

$$= g^2 3\Phi_y + \operatorname{cosec}^2\alpha + g\cot\alpha\left(2\Phi - \frac{\cot\alpha}{g}\right) + \frac{g^2}{2}\nabla^2\log\Phi + g^2\Phi^2 - 2g\cot\alpha$$

$$= g^2 3\Phi_y + \operatorname{cosec}^2\alpha - \cot^2\alpha + \frac{g^2}{2}\nabla^2\log\Phi + g^2\Phi^2$$

$$= g^2\left[\frac{1}{2}\nabla^2\log\Phi + 3\Phi_y + \Phi^2\right] + 1.$$

$$\therefore 2G = \frac{\gamma}{f}\left[g^2\left(\frac{1}{2}\nabla^2\log\Phi + 3\Phi_y + \Phi^2\right) + 1 - \frac{3f\gamma_y}{\gamma} - 1 + 5f\gamma\frac{\Lambda}{3}\right]$$

$$= \frac{\gamma}{f}\left[\frac{g^2}{f}\left\{\frac{1}{2}\nabla^2\log\Phi + 3\Phi_y + \Phi^2\right\} - \frac{g^2}{2f}\nabla^2\log\gamma - \frac{3\gamma_y}{\gamma} + 5\gamma\frac{\Lambda}{3}\right]$$

$$= \gamma\left[\frac{1}{2\Phi}\nabla^2\log\Phi + \frac{3}{2}\frac{\Phi_y}{\Phi} + \Phi - \frac{1}{2\Phi}\nabla^2\log\gamma - \frac{3\gamma_y}{\gamma} + 5\gamma\frac{\Lambda}{3}\right]$$

$$\therefore 2G = \gamma\left[\frac{1}{2\Phi}\nabla^2\log\frac{\Phi}{\gamma} + 3\left(\frac{\Phi_y}{\Phi} - \frac{\gamma_y}{\gamma}\right) + \Phi + 5\gamma\frac{\Lambda}{3}\right].$$

With this simplification we work out two new cases.

Case I $\quad \gamma = \gamma(y)$. Then γ must be a linear function of y

$\gamma = ay + b \qquad X = -ax - t.$

$\gamma \qquad \Phi = k\gamma = k(ay + b)$. Then $\frac{\Phi}{\gamma} = k \quad \frac{\Phi_y}{\Phi} - \frac{\gamma_y}{\gamma} = 0$

$E = \gamma^4\frac{\Lambda}{3} - 2\gamma^2 G - \gamma^2 a$ but

$2G = \gamma\left[4k\gamma + 5\gamma\frac{\Lambda}{3}\right] = \gamma^2\left[4k + 5\frac{\Lambda}{3}\right]$

$$\therefore E = y^4\frac{\Lambda}{3} - \tfrac{1}{2}y^4\left(4k+\frac{5\Lambda}{3}\right) - y^2a$$

$$= y^4\left(-4k-\frac{5\Lambda}{3}+\frac{\Lambda}{3}\right) - y^2a$$

$$= y^4\left(-4k-\frac{4\Lambda}{3}\right) - y^2a$$

$$\therefore \frac{E}{y} = y^3\left(-4k-\frac{4\Lambda}{3}\right) - ya$$

Since $\frac{E}{y}$ is a function of y only $\frac{E}{y}$ must be linear in y $\therefore k = -\frac{\Lambda}{3}$ and $\frac{E}{y} = -a(ay+b)$

$$= -a^2y - ab$$

$$\therefore F = a^2x + c.$$

Therefore all equations are satisfied and we can write down the only surviving component $R_{(1)(1)}$.

$$R_{11} = \frac{1}{M^2}\left[g^2\left\{\nabla^2 g^G + 2x'\left(-\tfrac{1}{2}\frac{y'}{y}\right)' + 2x_y\left(-\tfrac{1}{2}\frac{y'}{y}\right)_y + \frac{\Lambda}{3}(x'^2+y'^2)\right\}\right.$$

$$\left. + 2fg_x + 2fG_y - \frac{2f}{y}F' - 3\left(\frac{f}{y}\right)'F - \frac{2}{3}fyx'\Lambda - 2f\frac{y'^2}{y} + fy''\right]$$

$$2G = y^2\left(4k+\frac{5\Lambda}{3}\right) = y^2\left(-\frac{4\Lambda}{3}+\frac{5\Lambda}{3}\right) = \frac{\Lambda}{3}y^2$$

$$G = \frac{\Lambda}{6}y^2 \qquad G_y = \frac{\Lambda}{3}ya \qquad G_{yy} = \frac{\Lambda}{3}a^2$$

$$y' = 0 \qquad x' = -a$$

$$\therefore R_{11} = \frac{f}{M^2}\left[\frac{g^2}{f}\left\{\frac{\Lambda}{3}a^2 + \frac{\Lambda}{3}a^2\right\} + 2\frac{\Lambda}{3}ya - \frac{2}{y}a^2 - 2y(-a)\frac{\Lambda}{3}\right\}$$

(8)

But $\frac{g^2}{f} = \frac{1}{\varphi} = \frac{1}{\cancel{4\gamma}} \frac{1}{k\gamma} = \frac{1}{\gamma(-\frac{\Lambda}{3})}$

$$\therefore R_{(11)} = \frac{f}{M^2}\left[\frac{2a^2\frac{\Lambda}{3}}{\gamma(-\frac{\Lambda}{3})} + \frac{2\Lambda}{3}\gamma a - \frac{2}{\gamma}a^2 + \frac{2\Lambda}{3}\gamma a\right]$$

$$= \frac{f}{M^2}\left[-\frac{4a^2}{\gamma} + \frac{4\Lambda}{3}\gamma a\right] = \frac{\gamma}{x^2+y^2}\left[-\frac{4a^2}{\gamma} + \frac{4\Lambda}{3}\gamma a\right]$$

$$\therefore R_{(11)} = \frac{1}{x^2+y^2}\left\{-4a^2 + \frac{4\Lambda}{3}\gamma a^2\right\}$$

And $R_{ik} = -4\pi\sigma\,\xi_i\xi_k$. $\therefore R_{(11)} = -4\pi\sigma(\xi_1)^2$

$R_{(11)}$ —ve $\Longrightarrow$ σ +ve

and so we must have a —ve provided Λ is +ve

Again $2\varphi = \frac{l_y}{l}$ $\therefore \frac{(g\sin\alpha)_y}{g\sin\alpha} = \frac{l_y}{l} = 2k\gamma = -\frac{2\Lambda}{3}(ay+b)$

$\therefore l = e^{-\frac{\Lambda}{3a}(ay+b)^2}$. The metric becomes

$$ds^2 = 2(du + l\,d\beta)\,dx - 2L(du + l\,d\beta)^2 - \frac{f(x^2+y^2)}{\gamma g^2}(g^2 d\alpha^2 + g^2\sin^2\alpha\, d\beta^2)$$

$$\text{"} \quad \text{"} \quad \text{"} \quad - \frac{\varphi(x^2+y^2)}{\gamma}(dy^2 + l^2 d\beta^2)$$

$$- \left(-\frac{\Lambda}{3}\right)(x^2+y^2)(dy^2 + l^2 d\beta^2)$$

$$= 2(du + l\,d\beta)\,dx - 2L(du + l\,d\beta)^2 + \frac{\Lambda}{3}(x^2+y^2)(dy^2 + l^2 d\beta^2)$$

$$2L = \frac{\Lambda}{3}x^2 + y^2\frac{\Lambda}{3} + \frac{2Fx - 2ay^2}{x^2+y^2}.$$

$2Fx - 2ay^2 = 2x(a^2u+c) - 2ay^2$ But $x = -au - t$

$\therefore \frac{x+t}{-a} = u \quad \therefore 2Fx - 2ay^2 = 2x\left[a^2\frac{(x+t)}{-a} + c\right] - 2ay^2$

$= 2x[-ax - at + c] - 2ay^2 = -2a(x^2+y^2) + 2x(c-at)$

$\therefore 2L = \frac{\Lambda}{3}(x^2+y^2) - 2a + \frac{2x(c-at)}{x^2+y^2}$

So this completes the solution. One cannot put $\Lambda = 0$ here because then $\varphi = 0$ $f = 0$ $g = a = \text{const}$.

Case II $\quad y = ay + b \qquad x = -au - t$

$E = 0 \quad \therefore y^4\frac{\Lambda}{3} - 2y^2 G - y^2 a = 0$

$2G = y^2\frac{\Lambda}{3} - a$

$\therefore y\left[\frac{1}{2\varphi}\nabla^2 \log\frac{\varphi}{y} + 3\left(\frac{\varphi_y}{\varphi} - \frac{y_y}{y}\right) + 4\varphi + 5y\frac{\Lambda}{3}\right] = y^2\frac{\Lambda}{3} - a$

$\therefore \frac{1}{2\varphi}\nabla^2 \log\frac{\varphi}{y} + 3y\left(\frac{\varphi_y}{\varphi} - \frac{y_y}{y}\right) + 4\varphi y + 5y^2\frac{\Lambda}{3} - y^2\frac{\Lambda}{3} + a = 0$

But $\varphi = \psi\varphi\,\psi y$

$\therefore \frac{1}{2\psi}\nabla^2 \log\psi + 3y\frac{\psi_y}{\psi} + 4\psi y^2 + \frac{4\Lambda}{3}y^2 + a = 0$

$\psi = \psi(y)$ satisfies this differential equation

and $2L = \frac{\Lambda}{3}x^2 + \frac{\Lambda}{3}y^2 - a + \frac{2Fx}{x^2+y^2}$

Put $a = -1 \quad x = u - t = u \quad y = -y$ (we are putting $u = 0$)

$\therefore 2L = 1 + \frac{2mu}{u^2+y^2} + \frac{\Lambda}{3}(u^2+y^2)$ our earlier solution

We can find $R_{(11)}$. $2G = \frac{\Lambda}{3}y^2 - a$ $\quad 2G_y = \frac{\Lambda}{3}\cdot 2y$

$\therefore G_y = \frac{\Lambda}{3}y \quad G_{yy} = \frac{\Lambda}{3}$

$\therefore R_{(11)} = \frac{f}{M^2}\left[\frac{g^2}{f}\left(\frac{\Lambda}{3}+\frac{\Lambda}{3}\right) + \frac{2\Lambda}{3}y + 2y\frac{\Lambda}{3}\right]$

$= \frac{f}{M^2}\left[\frac{1}{\psi y}\cdot\frac{2\Lambda}{3} + 4y\frac{\Lambda}{3}\right)$

$= X^2\,\frac{1}{X^2+Y^2}\cdot\frac{2\Lambda}{3}\left(\frac{1}{\psi} + -2y^2\right)$ where $\psi(y)$ satisfies the equation $\frac{1}{2\psi}\nabla^2\log\psi - 3y\frac{\psi_y}{\psi} + 4y^2(\psi+\frac{\Lambda}{3}) - 1 = 0.$

Case III We take $Y = au+b \quad Y_y = 0 \quad Y_u = a$

Then $X = ay + \cancel{b} - t.$

~~2G~~ In order to remind us that Y is a function of u only we shall write $Y = U = au+b.$

$E = U^4\frac{\Lambda}{3} - 2U^2G \quad \therefore \quad \frac{E}{Y} = Y^3\frac{\Lambda}{3} - 2YG$

$2G = Y\left[\frac{1}{2\phi}\nabla^2\log\phi + 3\frac{\phi_y}{\phi} + 4\phi - \frac{1}{2\phi}\nabla^2\log U + 5Y\frac{\Lambda}{3}\right]$

$\therefore \frac{E}{Y} = U^3\frac{\Lambda}{3} - U^2\left[\frac{1}{2\phi}\nabla^2\log\phi + 3\frac{\phi_y}{\phi} + 4\phi\right] + 2\frac{U^2}{2\phi}\nabla^2\log U - 5U^3\frac{\Lambda}{3}$

$= -\frac{4\Lambda}{3}U^3 - U^2F(y) + \frac{U^2}{2\phi}\frac{d}{du}\left(\frac{a}{au+b}\right)$ $\qquad F(y) = \frac{1}{2\phi}\nabla^2\log\phi + 3\frac{\phi_y}{\phi} + 4\phi$

$= -\frac{4\Lambda}{3}U^3 - \cancel{\nabla^2}U^2F(y) + \frac{U^2}{2\phi}\frac{(-a^2)}{(au+b)^2}$

$= -\frac{4\Lambda}{3}U^3 - U^2\left\{F(y) + \frac{a^2}{2\phi}\right\}$

We must have $\boxed{\Lambda = 0}$

~~$\frac{E}{Y} = -U^2\left\{F(y) + \frac{a^2}{2\phi}\right\}$~~ $\quad \frac{E}{Y} = -U^2F(y) - \frac{a^2}{2\phi}$

Two cases suggest themselves

Case III a $\quad E=0 \quad \therefore F(y) + \frac{a^2}{2\varphi} = 0$

$$\frac{1}{2\varphi}\nabla^2 \log\varphi + \frac{3\varphi_y}{\varphi} + 4\varphi + \frac{a^2}{2\varphi} = 0$$

Also $E=0 \Longrightarrow \ell = 0$

$\therefore$ We can take $F(y)=0$ and $\frac{1}{2\varphi} = ky$ if the two have a common solution. Then $\frac{\varphi_y}{\varphi}, \frac{E}{\gamma} = -\frac{a^2}{2\varphi} = -a^2 ky$ and we have a valid case. ~~And~~ If $\frac{1}{2\varphi} = ky$ $2\varphi = \frac{1}{ky}$

$\varphi = \frac{1}{2ky}$. With this φ ~~F(y)~~ we find $F(y)$

$\varphi = \frac{1}{2ky} \quad \log\varphi = \log\frac{1}{2k} - \log y \quad \therefore \frac{\varphi_y}{\varphi} = \{\log\varphi\}_y = -\frac{1}{y}$

& $\nabla^2 \log\varphi = \{\log\varphi\}_{yy} = \frac{1}{y^2}$

$$\therefore F(y) = \frac{1}{2\varphi}\nabla^2\log\varphi + \frac{3\varphi_y}{\varphi} + 4\varphi$$

$$= \frac{2ky}{2}\cdot\frac{1}{y^2} + 3\left\{-\frac{1}{y}\right\} + \frac{4}{2ky}$$

$$= \frac{1}{y}\left\{k - 3 + \frac{2}{k}\right\} = \frac{1}{ky}\left\{k^2 - 3k + 2\right\}$$

$$= \frac{1}{ky}(k-2)(k-1)$$

$\therefore F(y) = 0$ if $k=2$ or $k=1$

$\therefore \varphi = \frac{1}{4y}$ or $\varphi = \frac{1}{2y}$ gives $\frac{E}{y} = -4 \cdot 2a^2 y$ or $-a^2 y$

$\therefore y = au + b, \quad x = ay - t \qquad \frac{E}{y} = -2a^2 y$ or $\frac{E}{y} = -a^2 y$

$F = 2a^2 u$ or $F = a^2 u$

$\frac{E}{y} = -2y\lambda \Rightarrow 2\lambda = -\frac{E}{y} \cdot \frac{1}{y} \quad 2\lambda = \frac{2a^2 y}{au+b}$ or $2\lambda = \frac{a^2 y}{au+b}$

$$2L = -\frac{a}{au+b}(ay-t) + \frac{2a^2 y}{au+b} + \frac{2\{2a^2 u(ay-t) + (-2a^2 y)(au+b)\}}{(ay-t)^2 + (au+b)^2}$$

or

$$2L = -\frac{a}{au+b}(ay-t) + \frac{a^2 y}{au+b} + \frac{2[a^2 u(ay-t) - a^2 y\{au+b\}]}{x^2+y^2}$$

$$= \frac{-a^2 y + at + a^2 y}{au+b} + \frac{2}{x^2+y^2}[a^3 uy - a^2 ut - a^2 uy - ba^2 y]$$

$$= \frac{at}{au+b} + \frac{2\{-2a^2\}(ut+by)}{x^2+y^2}$$

P.T.O.

In the case when $Y = Y(y)$ and $\therefore$ $Y = ay + b$ $X = -au - b$ there is still simpler way of solving this case.

We carried out one simplification by introducing ~~$\varphi = g^2$~~ $\varphi = \frac{f}{g^2}$ This $\varphi = \varphi(y)$ ~~is~~ whatever X, Y.

~~The~~ and so the analysis we carried out earlier by using φ remained valid even when $Y = au + b$.

The simplification which we are now introducing is valid only in the case $Y = ay + b$ $X = -au + b$.

$$M^2 = \frac{f}{Y}(X^2 + Y^2)$$

Let $J^2 = \frac{f}{Y}$ Put $J\,d\alpha = d\psi$ (This assumes that J is a function of α only i.e. $Y = Y(y) = ay + b$)

So $J\,d\alpha = d\psi$ and $J \sin\alpha = h$

$$\therefore d\psi = J\,d\alpha = \frac{J\,dy}{g} = \frac{J\sin\alpha}{g\sin\alpha}\,dy = \frac{h}{l}\,dy$$

$h = J\sin\alpha$

$l = g\sin\alpha$

$$\therefore \boxed{\frac{dy}{d\psi} = \frac{l}{h}}$$

Also $f = \frac{(g\sin\alpha)_\alpha}{2\sin\alpha} = \frac{(g\sin\alpha)_\psi J}{2\sin\alpha} = \frac{l_\psi J . J}{2J\sin\alpha} = \frac{l_\psi J^2}{2h} = \frac{l_\psi f}{Y 2h}$

Cancelling f $\boxed{\frac{l_\psi}{2h} = Y}$

Again $G = \frac{Y}{2\Phi}\left[\frac{1}{2}\nabla^2 \log\frac{\Phi}{Y}\right] + \frac{Y}{2}\left[\frac{3\Phi_x}{\Phi} - \frac{3Y_x}{Y} + 4\Phi + \frac{5\Lambda}{3}Y\right]$

But $\Phi = \frac{f}{g^2}$ $\frac{\Phi}{Y} = \frac{f}{g^2 Y} = \frac{F^2}{g^2} = \frac{(F\sin\alpha)^2}{(g\sin\alpha)^2} = \frac{h^2}{l^2}$ $\boxed{\Phi = Y\frac{h^2}{l^2}}$

$\log\frac{\Phi}{Y} = 2\log\frac{h}{l}$ $\therefore \left[\log\left(\frac{\Phi}{Y}\right)\right]_y = \frac{2\left(\frac{h}{l}\right)_y}{\frac{h}{l}} = \frac{2\left(\frac{h}{l}\right)_\Psi \frac{h_\Psi}{l}}{\frac{h}{l}} = 2\left(\frac{h}{l}\right)_\Psi$

$\therefore \left[\log\frac{\Phi}{Y}\right]_y = 2\left(\frac{h}{l}\right)_\Psi$ $\left(\log\frac{\Phi}{Y}\right)_{yy} = 2\left(\frac{h}{l}\right)_{\Psi\Psi}\frac{h_\Psi}{l}$.

We now take up G

$G = \frac{Y}{2\Phi}\left[\frac{1}{2}\left(\log\frac{\Phi}{Y}\right)_{yy}\right] + \frac{Y}{2}\left[3\left(\log\frac{\Phi}{Y}\right)_y + 4\Phi + \frac{5\Lambda}{3}Y\right]$

$= \frac{l^2}{2h^2}\left[\left(\frac{h}{l}\right)_{\Psi\Psi}\frac{h_\Psi}{l}\right] + \frac{Y}{2}\left[6\left(\frac{h}{l}\right)_\Psi + 4Y\frac{h^2}{l^2} + \frac{5\Lambda}{3}Y\right]$

$= \frac{l}{2h}\left(\frac{h}{l}\right)_{\Psi\Psi} + 3Y\left(\frac{h}{l}\right)_\Psi + 2Y^2\frac{h^2}{l^2} + \frac{5\Lambda}{6}Y^2$

Now $\left(\frac{h}{l}\right)_\Psi = \frac{h_\Psi}{l} - \frac{h}{l^2}l_\Psi = \frac{h_\Psi}{l} - \frac{h}{l^2}2hY = \frac{h_\Psi}{l} - 2Y\frac{h^2}{l^2}$

and $\left(\frac{h}{l}\right)_{\Psi\Psi} = \frac{h_{\Psi\Psi}}{l} - \frac{h_\Psi}{l^2}2hY - \frac{h_\Psi}{l^2}2hY - \frac{4hh_\Psi}{l^2}Y - \frac{2h^2}{l^2}a\frac{l}{h} + 4Y\frac{h^2}{l^3}2Yh$

$= \frac{h_{\Psi\Psi}}{l} - \frac{6hh_\Psi}{l^2}Y - 2a\frac{h}{l} + 8Y^2\frac{h^3}{l^3}$.

$\therefore$ G now becomes

$$G_4 = \frac{l}{2h}\left[\frac{h_{44}}{l} - \frac{6hh_4}{l^2}y - 2a\frac{h}{l} + 9y^2\frac{h^3}{l^3}\right] + 3y\left[\frac{h_4}{l} - 2y\frac{h}{l^2}\right] + 2y^2\frac{h^2}{l^2} + \frac{5\Lambda}{6}y^2$$

$$= \frac{h_{44}}{2h} + \frac{yh_4}{l}(-3+3) - a + y^2\frac{h^2}{l^2}(4-6+2) + \frac{5\Lambda}{6}y^2$$

$$\boxed{G_4 = \frac{h_{44}}{2h} - a + \frac{5\Lambda}{6}y^2}$$

Now $E = y^4\frac{\Lambda}{3} - 2y^2G_4 - y^2\gamma_y$ $\quad\therefore E = E(y)$ $\quad\therefore \frac{E}{y} = cy+d$

Push the above value of G_4 here

$$\therefore \frac{E}{y^2} = \frac{\Lambda}{3}y^2 - 2\left(\frac{h_{44}}{2h} - a + \frac{5\Lambda}{6}y^2\right) - a$$

$$= -\frac{h_{44}}{h} + a - \frac{\Lambda}{3}y^2(5-1)$$

$$\therefore \frac{h_{44}}{h} + 4\frac{\Lambda}{3}y^2 - a + \frac{E/y}{y} = 0$$

$$\text{or } \frac{h_{44}}{h} + h_{44} + h\left(\frac{4\Lambda}{3}y^2 - a + \frac{cy+d}{ay+b}\right) = 0$$

$$\text{or } h_{44} + h\left(\frac{4\Lambda}{3}y^2 - a + \frac{\frac{c}{a}(ay+b) + d - \frac{bc}{a}}{ay+b}\right) = 0$$

$$\text{or } h_{44} + h\left(\frac{4\Lambda}{3}y^2 + \frac{c}{a} - a + \frac{d - \frac{bc}{a}}{y}\right) = 0$$

This is the general equation to be solved. The solutions of these equations corresp. to $n=0$ and $d-\frac{bc}{a}=0$ have been found in the Ξιμβεν ~~Equa~~ We try here some singular solutions.

The equations to be solved are:

I $h_{\psi\psi} + h\{AY^2 + B + \frac{C}{Y}\} = 0$, A, B, C are given constants.

$$Y_\psi = \frac{al}{h} \quad \text{or} \quad al = hY_\psi \quad \text{and} \quad l_\psi = 2hY$$

We solve I under the assumption $C=0$ i.e. $d-\frac{bc}{a}=0$

$$\text{or} \quad \frac{E}{Y} = cy + d = cy + \frac{bc}{a} = \frac{c}{a}(ay+b) = \frac{c}{a}Y.$$

So I becomes $h_{\psi\psi} + h(AY^2 + B) = 0$.

Let us consider and equation $h_{\psi\psi} + h(AY^m + B) = 0$ so that both cases $m=2$ and $m=-1$ (i.e. $C=0$, $B=0$ are worked out ~~simpult~~ simultaneously

We work out several singular solutions.

Let us take $h = \cos^n k\psi$. Note that changing k to ik will ~~be~~ lead to ~~hyper~~ $\cosh^n k\psi$. Also since any constant added to ψ does not change the equations for h, l and Y, the same case will include $\sin^n k\psi$ and $\sinh^n k\psi$.

$h = \cos^n k\psi \qquad h_\psi = -kn\cos^{n-1} k\psi \sin k\psi$

$h_{\psi\psi} = k^2 n(n-1)\cos^{n-2} k\psi(1-\cos^2 k\psi) - k^2 n \cos^n k\psi$ ∴ I becomes

$$k^2 n(n-1)\cos^{n-2} k\psi - k^2 n \cos^n k\psi(n-1+1) + \cos^n k\psi\, AY^m + \cos^n k\psi\, B = 0$$

$$\therefore k^2 n(n-1)\sec^2 k\psi - k^2 n^2 + AY^m + B = 0$$

Clearly $B = k^2n^2$ $y^m = -k^2n(n-1)\sec^2 k\psi$

$B = k^2n^2$ i/m

$y = \left[\frac{-k^2n(n-1)}{A}\right]\sec^{\frac{2}{m}} k\psi$

1/m

Now $al = k'_\psi \Longrightarrow$ $al = \cos^n k\psi \frac{2}{m}\left[\frac{-k^2n(n-1)}{A}\right]\sec^{\frac{2}{m}} k\psi \tan k\psi$

$\therefore l = \frac{2k}{am}\left[\frac{-k^2n(n-1)}{A}\right]^{1/m} \cos^{n-\frac{2}{m}-1} k\psi \sin k\psi$

Lastly $l_\psi = 2k'_\psi$ $\Longrightarrow$

$\frac{2k}{am}\left[\frac{-k^2n(n-1)}{A}\right]^{1/m}\left\{(n-\frac{2}{m}-1)(-)\cos^{n-\frac{2}{m}-2}\psi\ (1-\cos^2 k\psi)\,k\right.$

$\left. + k\cos^{n-\frac{2}{m}} k\psi\right\}$

$= 2\cos^n k\psi \left[\frac{-k^2n(n-1)}{A}\right]^{1/m} \cos^{-\frac{2}{m}} k\psi$

$\therefore$ ~~One condition is~~ One condition is

if $n - \frac{2}{m} - 1 = 0$ $\frac{2k}{am} = 2$ $am = k$

$\therefore$ ~~Finally~~ Finally $h = \cos^n k\psi$, & $y = \left[\frac{-k^2n(n-1)}{A}\right]^{1/m}\sec^{\frac{2}{m}} k\psi$

$l = \frac{2k}{am}\left[\frac{-k^2n(n-1)}{A}\right]^{1/m} \sin k\psi$ $B = k^2n^2$

$n = \frac{2}{m} + 1$ $am = k^2$

Putting ~~$m=0$~~ $m=2$, we get $n=2$, $k^2 = 2a$, $B = 4k^2$

$h = \cos^2 k\psi$, $y = k\sqrt{\frac{-2}{A}}\sec k\psi$, $l = 2\sqrt{\frac{-2}{A}}\sin k\psi$

∴ Case I $h_{\psi\psi} + h(Ay^2 + B) = 0$ $al = h_\psi^2$, $l_\psi = 2hy$ has solution

$h = \cos^2 k\psi$ $y = k\sqrt{\frac{-2}{A}} \sec k\psi$ $l = 2\sqrt{\frac{-2}{A}} \sin k\psi$ $B = 4k^2$ $k^2 = 2a$

Replacing k by ik we have another solution

$h = \cosh^2 k\psi$, $y = k\sqrt{\frac{2}{A}} \operatorname{sech} k\psi$, $l = 2\sqrt{\frac{2}{A}} \sinh k\psi$, $B = -4k^2$, $2a = -k^2$.

and replacing A by C

Setting $m = -1$ & $n = -1$, $-a = k^2$, $B = k^2$

$h = \sec k\psi$, $y = \left[\frac{-k^2(-1)(-2)}{C}\right]^{-1} \cos^2 k\psi$, $l = -\frac{2k}{a}\left[\frac{-c}{2k^2}\right] \sin k\psi$

Case II $h_{\psi\psi} + h\left(B + \frac{c}{y}\right) = 0$ $al = h_\psi$ $l_\psi = 2hy$ has the solution

$h = \sec k\psi$, $y = \frac{-c}{2k^2} \cos^2 k\psi$, $l = \frac{c}{ak} \sin k\psi$, $k^2 = -a$, $B = k^2$.

Replacing k by ik, we have another solution

$h = \operatorname{sech} k\psi$, $y = \frac{c}{2k^2} \cosh^2 k\psi$ $l = \frac{c}{ak} \sinh k\psi$ $k^2 = a$. $B = -k^2$.

In Next $G = \frac{h_{\psi\psi}}{2h} - a + \frac{5\Lambda}{6} y^2$

$= -\frac{ha}{2}\left(Ay^2 + B + \frac{C}{y}\right) - a + \frac{5\Lambda}{6} y^2$

$= 4$.

Note.

But $A = \frac{4\Lambda}{3}$, $B = \frac{c}{a} - a$, $C = d - \frac{bc}{a}$. ~~But we shall~~

~~$\therefore G = \frac{4A}{3}$.~~ Now

$\therefore G = \frac{4\Lambda}{3} y^2 + \frac{c}{a} - a + \frac{d - \frac{bc}{a}}{y} - a + \frac{5\Lambda}{6} y^2$

~~However we shall obtain A, B, C~~

$$G_4 = -\frac{1}{2}\frac{4\Lambda}{3}y^2 - \frac{1}{2}\left(\frac{c}{a} - a\right) - \frac{1}{2}\frac{d - \frac{bc}{a}}{y} - a + \frac{5\Lambda}{6}y^2$$

$$= \frac{\Lambda}{6}y^2 - \frac{1}{2}\left(\frac{c}{a} + a\right) - \frac{1}{2}\frac{d - \frac{bc}{a}}{y}$$

Case I $\quad d - \frac{bc}{a} = 0 \quad$ ~~G_4~~ $\quad G_4 = \frac{\Lambda}{6}y^2 - \frac{1}{2}\left(\frac{c}{a} + a\right)$

$$G_y = a\frac{\Lambda}{3}y \qquad G_{yy} = a^2\frac{\Lambda}{3}$$

~~$R_{(11)}$~~ Let us take the sub-case $E = 0$ $F =$ constant

so that $c = 0, d = 0$ $\quad G_4 = \frac{\Lambda}{6}y^2 - \frac{1}{2}a$ We calculate $R_{(11)}$

$R_{(11)}$ 2 is written on p. 4 corresponding to $Y = ay + b$. But we reproduce here the general form of $R_{(11)}$ derived in Lecture 3 and recorded on p. 77 there

$$R_{(11)} = \frac{1}{M^2}\left[g^2\left\{\nabla^2 G + 2X'\left(-\frac{1}{2}\frac{Y'}{Y}\right)' + 2X_y\left(-\frac{1}{2}\frac{Y'}{Y}\right)_y + \frac{\Lambda}{3}(X'^2 + Y'^2)\right\}\right.$$
$$\left. + 2f G_y - \frac{2f}{y}F' - 3\left(\frac{f}{Y}\right)'F - 4fYX'\left(\frac{\Lambda}{6}\right) - 2\frac{f}{y}Y'^2 + fY''\right]$$

In our case $Y' = 0$, $G' = 0$ $X_y = 0$ $F' = 0$

$$\therefore R_{(11)} = \frac{g^2}{M^2}\left\{G_{yy} + \frac{\Lambda}{3}x'^2\right\} + \frac{2f}{M^2}\left\{G_y - 2YX'\frac{\Lambda}{6}\right\}$$

$$= \frac{g^2 y}{f(X^2+Y^2)}\left[a^2\frac{\Lambda}{3} + a^2\frac{\Lambda}{3}\right] + \frac{2Y}{X^2+Y^2}\left\{a\frac{\Lambda}{3}y - 2Y(-a)\frac{\Lambda}{6}\right\}$$

$$= \frac{1}{\psi}\frac{y}{x^2+y^2}\frac{2a^2\Lambda}{3} + \frac{2y^2}{x^2+y^2}\,2a\frac{\Lambda}{3}$$

But $\psi = \frac{yk^2}{l^2}$

$$= \frac{l^2}{yk^2}\cdot\frac{2a^2\Lambda}{3}\cdot\frac{y}{x^2+y^2} + 4a\frac{\Lambda}{3}\frac{y^2}{x^2+y^2}$$

$$= \frac{2a^2\Lambda}{3\nu}\cdot\frac{1}{x^2+y^2}\,\frac{4\left(-\frac{2}{A}\right)\sin^2 k\psi}{\cos^4 k\psi} + \frac{4a\frac{\Lambda}{3}k^2\left(-\frac{2}{A}\right)\sec^2 k\psi}{x^2+y^2}$$

But $A = \frac{4\Lambda}{3}$ But $2a = k^2$

$$= -\frac{4a^2}{x^2+y^2}\tan^2 k\psi\sec^2 k\psi - \frac{2ak^2}{x^2+y^2}\sec^2 k\psi$$

$$= -\frac{k^4}{x^2+y^2}\sec^2 k\psi\left[\tan^2 k\psi + 1\right]$$

$$= -\frac{k^4}{x^2+y^2}\sec^4 k\psi$$

But this solution has a logical inconsistency as pointed out by Prof of Bhatt. You see, $B = 4k^2$ $a = 2k^2$

$\therefore B = 2a$ But $B = \frac{c}{a} - a$ and so if you put $c = 0$ you get $B = -a$ i.e. $2k^2 = -a$ $a > 0$ ψ constant $\therefore k = 0$ $k = \text{const.}$ $l = 0$, $\psi = 0$. You do not get anything

This means $h\psi\psi + h\left(A\psi^2 + B + \frac{C}{\psi}\right) = 0$ is not a convenient form to get particular solutions and again this form is useful in the case $\psi = \psi(y)$ only

10-3-98

But then following the method of simple metric giving here in de Sitter space, we can find the most general solution in this case.

$$Y = Y(y) \quad \therefore Y = ay + b \quad X = -aa - t$$

And $2G = Y\left[\frac{1}{2\varphi}\nabla^2 \log\frac{\varphi}{Y} + 3\left(\frac{\varphi_y}{\varphi} - \frac{Y_y}{Y}\right) + 4\varphi + 5Y\frac{\Lambda}{3}\right]$

$G = G(y)$. $\quad G' = 0$. $\quad \varphi = \frac{f}{g^2}$

Next $\frac{E}{Y} = Y^3\frac{\Lambda}{3} - 2Y^2G - Y^2Y_y$ $\quad \therefore \frac{E}{Y}$ becomes a function of y only and so a linear function

$\therefore \frac{E}{Y} = cy + d \qquad F = -cu + k \quad c, d, k$ constants

The $\therefore cy + d = Y^3\frac{\Lambda}{3} - 2Y^2G - Y^2Y_y$

$\therefore \frac{cy+d}{Y^2} = Y^2\frac{\Lambda}{3} - 2G - a \qquad [Y_y = a]$

$\therefore 2G = Y^2\frac{\Lambda}{3} - aY - \frac{cy+d}{Y^2} = Y^2\frac{\Lambda}{3} - a - \frac{\frac{c}{a}(ay+b) + d - \frac{bc}{a}}{ay+b} \qquad a \neq 0$

$\therefore 2G = Y^2\frac{\Lambda}{3} - a - \frac{c}{a} - \frac{d - \frac{bc}{a}}{Y}$.

$\therefore$ The equation to determine $g = g(\alpha)$ is

$g\,d\alpha = dy \qquad 2f\sin\alpha = (g\sin\alpha)_\alpha \qquad \frac{f}{g^2} = \varphi$ where φ satisfied the diff. eqn $\frac{1}{2\varphi}\left[\log\frac{\varphi}{Y}\right]_{yy} + 3\left[\log\frac{\varphi}{Y}\right]_y + 4\varphi + 5Y\frac{\Lambda}{3} = \frac{2G}{Y}$ where $2G$ is given above

Next $2L = \frac{\Lambda}{3}x^2 - \frac{y'}{y}x + 2G + \frac{2Fx + 2E}{x^2+y^2}$

$= \frac{\Lambda}{3}x^2 + \frac{\Lambda}{3}y^2 - a\mp\frac{c}{a} \mp \frac{d-\frac{bc}{a}}{y} + \frac{2\{-cu+k\}\{-ax-b\} + 2\{cy+b\}\{cy+d\}}{x^2+y^2}$

$= \frac{\Lambda}{3}(x^2+y^2) - a\mp\frac{c}{a} \mp \frac{d-\frac{bc}{a}}{y} + \frac{2(-cu+k)x + 2(cy+d)y}{x^2+y^2}$

The numerator of the last term is

$2\left[\mp\frac{c}{a}(a-au-b) + \frac{cb}{a} + k\right]x + 2\left[\frac{c}{a}(ay+b) - \frac{cb}{a} + d\right]y$

$= 2\frac{c}{a}(x^2+y^2) + 2\left(\frac{cb}{a}+k\right)x + 2\left(d-\frac{bc}{a}\right)y$

$\therefore 2L = \frac{\Lambda}{3}(x^2+y^2) - a\mp\frac{c}{a} \mp \frac{\left(d-\frac{bc}{a}\right)}{y} + \frac{2c}{a} + \frac{2\left(\frac{cb}{a}+k\right)x}{x^2+y^2} + \frac{2\left\{d-\frac{bc}{a}\right\}y}{x^2+y^2}$

$= \frac{\Lambda}{3}(x^2+y^2) \cancel{- a \mp \frac{3c}{a}} - a + \frac{c}{a} + \frac{\left(d-\frac{bc}{a}\right)}{y}\frac{d(x^2-y^2)}{x^2+y^2} + \frac{2cb}{x^2+y^2} + \frac{2\left(\frac{cb}{a}+k\right)x}{x^2+y^2}$

and one can now find σ from $R_{(11)} = -\kappa\sigma$.

Take the general expression for $R_{(11)}$ on p.16 and now put

$R_{(11)} =$

$y' = 0,\ X_y = 0$ ~~$G_y = 0$~~ $G' = 0$

$\therefore R_{(11)} = \frac{g^2}{M^2}\left[G_{yy} + \frac{\Lambda}{3}x'^2\right] + \frac{2t}{M^2}\left[G_y - \frac{F'}{y} - 2yx'\left(\frac{\Lambda}{6}\right)\right]$

$2G = \frac{\Lambda}{6}y^2 - \frac{1}{2}\left(a + \frac{c}{a}\right) - \frac{1}{2}\frac{d-\frac{bc}{a}}{y}$

$\therefore G_y = \frac{\Lambda}{3}ya + \frac{1}{2}\frac{\left(d-\frac{bc}{a}\right)}{y^2}a = \frac{\Lambda}{3}ya + \frac{1}{2}\frac{(ad-bc)}{y^2}$

$G_{yyy} = \frac{\Lambda}{3}a^2 + 2\,\frac{1}{2}(-2)\frac{(ad-bc)}{y^3}\cdot a = \frac{\Lambda}{3}a^2 - \frac{(ad-bc)a}{y^3}$

Also $g^2 = \frac{f}{\varphi}$

$$\therefore R_{(11)} = \frac{2f}{M^2}\left[\frac{1}{2\varphi}\left\{\frac{\Lambda}{3}a^2 - \frac{a(ad-bc)}{Y^3} + \frac{\Lambda}{3}a^2\right\}\right.$$

$$\left. + \frac{\Lambda}{3}Ya + \frac{1}{2}\frac{(ad-bc)}{Y^2} - \frac{(-c)}{Y} - 2Y(-a)\frac{\Lambda}{6}\right]$$

But $M^2 = \frac{f}{Y}(x^2+y^2)$ $\therefore$ $\frac{Y}{x^2+y^2} = \frac{f}{M^2}$

$$\therefore R_{(11)} = \frac{2Y}{x^2+y^2}\left[\frac{2\Lambda}{3}a\left(\frac{a}{2\varphi}+Y\right) + \frac{c}{Y} + \frac{ad-bc}{Y^3}\left(-\frac{a}{2\varphi}+\frac{Y}{2}\right)\right]$$

Physical details of this general solution are being worked out Pratap Bhatt. We go to work out cases where $Y = Y(x,y)$

11-3-88

γ satisfies Laplaces Equation $\gamma_{xx} + \gamma_{yy} = 0$. So we begin with homogeneous functions of x and y which satisfy $\nabla^2 \gamma = 0$. They are

$\gamma =$ const. $\gamma = ax + by$ $\gamma = 2axy + b(y^2 - x^2)$, ~~$\gamma = 3a(3y^2x - y^3)$~~

$\gamma = a(3y^2x - x^3) + b(3x^2y - y^3)$ ~~and so on~~ $\frac{ax}{\sqrt{(x^2+y^2)}} + \frac{by}{\sqrt{(x^2+y^3)}}$ ~~etc~~

and so on. We take some of these.

I $\gamma = ay + bx$ $\gamma_y = -x_x$ $\gamma_x = x_y$ $\therefore x = -ax + by - t$.

$x = -ax + by - t$

$$2G = \gamma\left[\frac{1}{2\varphi}\nabla^2 \log\varphi + \frac{3\varphi_y}{\varphi} + 4\varphi\right] + \gamma\left[\frac{1}{2\varphi}(-\nabla^2 \log\gamma) - 3(\log\gamma)_y + 5\gamma\frac{\Lambda}{3}\right]$$

$$= \text{~~}a\gamma J(\gamma) + \gamma\text{~~}\left[\frac{1}{2\varphi}\right. \qquad (\log\gamma)_y = \frac{a}{\gamma} \qquad (\log\gamma)_x = \frac{b}{\gamma}$$

$$(\log\gamma)_{yy} = -\frac{a^2}{\gamma^2} \qquad (\log\gamma)_{xx} = \frac{-b^2}{\gamma^2} \qquad \therefore \nabla^2(\log\gamma) = -\frac{a^2+b^2}{\gamma^2}$$

$$\therefore 2G = \gamma J(\gamma) + \gamma\left[\frac{1}{2\varphi}\frac{(a^2+b^2)}{\gamma^2} - \frac{3a}{\gamma} + 5\gamma\frac{\Lambda}{3}\right]$$

$$= \text{~~}\gamma\text{~~}\left[\gamma J(\gamma) + \frac{a^2+b^2}{2\varphi}\frac{1}{\gamma} - 3a + \frac{5\Lambda}{3}\gamma^2\right.$$

But $$\frac{E}{\gamma} = \gamma^3\frac{\Lambda}{3} - \gamma a - \gamma\left[\gamma J(\gamma) + \frac{a^2+b^2}{2\varphi}\frac{1}{\gamma} - 3a + \frac{5\Lambda}{3}\gamma^2\right]$$

$$= -\frac{4\Lambda}{3}\gamma^3 + 4a - \gamma a - \gamma^2 J(\gamma) - \frac{(a^2+b^2)}{2\varphi} + 3a\gamma$$

$$= -\frac{4\Lambda}{3}\gamma^3 + 4 - \gamma^2 J(\gamma) + 2a\gamma - \frac{(a^2+b^2)}{2\varphi}$$

We have shown on p. 8 that $\frac{1}{2\varphi} = k\gamma$ makes $J(\gamma) = 0$ if $k = 1$ or $k = 2$. ∴ We choose

$\Lambda = 0$, $k = 1$ or 2 $\frac{1}{2\varphi} = k\gamma$ to get $\frac{E}{\gamma} = 2a(ax + by) - (a^2+b^2)k\gamma$

$= -\left[2a^2 - k(a^2+b^2)\right]\gamma + 2ab\,x$

So that $\frac{E}{Y}$ also satisfies Laplace's equation

$$\frac{E}{Y} = \{(2-k)a^2 - kb^2\}y + 2abu \quad \text{and so}$$

$$F = 2 - \{(2-k)a^2 - kb^2\}u + 2aby \text{ etc.}$$

$$2G = \frac{(a^2+b^2)ky}{Y} - 3a - \frac{ky(a^2+b^2) - 3a(ay+bu)}{Y}$$

$$= \frac{((k-3)a^2 + kb^2)y - 3abu}{ay+bu}$$

$$2L = -\frac{b}{Y}(-au+by) + \frac{((k-3)a^2+kb^2)y - 3abu}{Y}$$

$$+ \frac{2\left[\{kb^2+(k-2)a^2\}u + 2aby\}(-au+by-t) + (-\{(k-2)a^2 - kb^2\}y + 2abu)(ay+bu)\right]}{X^2+Y^2}$$

4-8-94

Papapetrou in Random Walk in G.R. has shown that inward flow of a flowing radiation leads to a naked singularity at the origin which is the centre of spherical symmetry. We wish to consider the same situation not in flat background (∵ in flat back ground, one does not know from where this inflowing radiation comes!) but in the background of Einstein Universe ~~or~~ and then of the Expanding Robertson-Walker Universe.

1 Radiation absorbing Centre in Einstein Universe.

Following Pramana paper and setting the rotation parameter a=0 there, we write Einstein's universe metric as

$$ds^2 = 2dudt - du^2 - M^2(d\alpha^2 + \sin^2\alpha\, d\beta^2)$$

$$M^2 = R^2 \sin^2 \frac{r}{R}.$$

(2) ∴ We take the metric

$$ds^2 = 2\,du\,dt - e^{\lambda} du^2 - R^2 \sin^2 \frac{r}{R}\, d\sigma^2$$

with $u = t + r$ or $r = u - t$.

Turning back to r and t coords with $u = t + r$ we find

$$ds^2 = 2(dt + dr)dt - e^{\lambda}(dt^2 + dr^2 + 2dt\,dr) - M^2 d\sigma^2$$

$$= dt^2(2 - e^{\lambda}) - e^{\lambda} dr^2 + 2\,dt\,dr(1 - e^{\lambda}) - M^2 d\sigma^2$$

if $M = r$ and $e^{\lambda} = 1 + \frac{2m}{r}$ we get

$$ds^2 = dt^2\left(1 - \frac{2m}{r}\right) - \left(1 + \frac{2m}{r}\right) dr^2 + 2\,dt\,dr\left(-\frac{2m}{r}\right) - r^2 d\sigma^2$$

which is the form of radiating star metric used by Papapetrou for infalling "null dust".

∴ For inward flowing radiation in the back-ground of Einstein Universe we take the metric

$$ds^2 = 2\,du\,dt - e^{\lambda} du^2 - M^2(d\alpha^2 + \sin^2\alpha\, d\beta^2) \qquad (1)$$

$M^2 = R^2 \sin^2 \frac{r}{R}$ $\quad r = u = t + r$ or $r = u - t$

$x^1 = u$, $x^2 = \alpha$, $x^3 = \beta$, $x^4 = t$. $\quad \lambda = \lambda(u, t)$

For metric (1)

$$g_{ik} = \begin{pmatrix} -e^{\lambda} & 0 & 0 & 1 \\ 0 & -M^2 & 0 & 0 \\ 0 & 0 & -M^2\sin^2\alpha & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore g = -1\left[1\left(M^4\sin^2\alpha\right)\right] = -M^4\sin^2\alpha \qquad \therefore \sqrt{-g} = M^2\sin\alpha$$

$$g^{11} = 0, \quad g^{12} = 0 \quad g^{13} = 0 \quad g^{14} = \frac{-1\,M^4\sin^2\alpha}{-M^4\sin^2\alpha} = 1$$

$$g^{22} = \frac{1\,M^2\sin^2\alpha}{-M^4\sin^2\alpha} = -\frac{1}{M^2} \quad g^{23} = 0 \quad g^{24} = 0$$

$$g^{33} = \frac{M^2}{-M^4\sin^2\alpha} = -\frac{1}{M^2\sin^2\alpha} \quad g^{34} = 0 \quad g^{44} = \frac{-e^{\lambda}M^4\sin^2\alpha}{-M^4\sin^2\alpha} = e^{\lambda}$$

$$g^{ik} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -\frac{1}{M^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{M^2\sin^2\alpha} & 0 \\ 1 & 0 & 0 & e^{\lambda} \end{pmatrix}$$

Now we work out the 3-index symbols.

$$\left\{ {i \atop jk} \right\} = \frac{1}{2} g^{kn}\left(g_{in,j} + g_{jn,i} - g_{ij,n}\right)$$

(γ) $\left\{ {1 \atop 11} \right\} = \frac{1}{2} g^{1n} (g_{1n,1} + g_{1n,1} - g_{11,n})$

$= \frac{1}{2} g^{14} (g_{14,1} + g_{14,1} - g_{11,4}) = \frac{1}{2} \cdot (e^{\lambda} \dot{\lambda})$

$Z = e^{\lambda}$

$Z' = \frac{\partial Z}{\partial u} \quad \dot{Z} = \partial Z / \partial t$

$\left\{ {1 \atop 11} \right\} = \frac{\dot{Z}}{2} = \frac{e^{\lambda} \dot{\lambda}}{2}.$

$\left\{ {2 \atop 11} \right\} = \frac{1}{2} g^{22} (-g_{11,2}) = 0 \quad \left\{ {3 \atop 11} \right\} = \frac{1}{2} g^{33} (-g_{11,3}) = 0$

$\left\{ {4 \atop 11} \right\} = \frac{1}{2} g^{41} (g_{11,1}) + \frac{1}{2} g^{44} (-g_{11,4})$

$= \frac{1}{2} (-e^{\lambda} \lambda') + \frac{1}{2} e^{\lambda} (e^{\lambda} \dot{\lambda})$

$\left\{ {4 \atop 11} \right\} = \frac{1}{2} (-Z' + Z\dot{Z})$

$\left\{ {1 \atop 12} \right\} = \frac{1}{2} g^{14} (0) = 0, \quad \left\{ {2 \atop 12} \right\} = \frac{1}{2} g^{22} (g_{22,1}) = \frac{M'}{M}$

$\left\{ {2 \atop 12} \right\} = \frac{M'}{M}$

$\left\{ {3 \atop 12} \right\} = \frac{1}{2} g^{33} (0) = 0 \quad \left\{ {4 \atop 12} \right\} = \frac{1}{2} g^{41} (g_{11,2}) + \frac{1}{2} g^{44} (0) = 0$

$\left\{ {1 \atop 13} \right\} = \frac{1}{2} g^{14} (0) = 0 \quad \left\{ {2 \atop 13} \right\} = \frac{g}{2}^{22} (0) = 0$

$\left\{ {3 \atop 13} \right\} = \frac{1}{2} g^{33} (g_{33,1}) = \frac{M'}{M} \quad \left\{ {4 \atop 13} \right\} = \frac{1}{2} g^{41} (g_{11,3}) + \frac{1}{2} g^{44} (0) = 0$

$\left\{ {3 \atop 13} \right\} = \frac{M'}{M}$

$\left\{ {1 \atop 14} \right\} = \frac{1}{2} g^{14} (g_{44,1}) = 0 \quad \left\{ {2 \atop 14} \right\} = \frac{1}{2} g^{22} (0) = 0 \quad \left\{ {3 \atop 14} \right\} = \frac{1}{2} g^{33} (0) = 0$

$\left\{ {4 \atop 14} \right\} = \frac{1}{2} g^{41} (g_{11,4}) + \frac{1}{2} g^{44} (0) = -e^{\lambda} \frac{\lambda'}{2}$

$\left\{ {4 \atop 14} \right\} = -\frac{Z'}{2}$

$\left\{ {1 \atop 22} \right\} = \frac{1}{2} g^{14} (-g_{22,4}) = \frac{1}{2} (2 M \dot{M}) = M \dot{M}$

$\left\{ {1 \atop 22} \right\} = M \dot{M}$

$\left\{ {2 \atop 22} \right\} = \frac{1}{2} g^{22} (0) = 0 \quad \left\{ {3 \atop 22} \right\} = \frac{1}{2} g^{33} (0) = 0$

$\left\{ {4 \atop 22} \right\} = \frac{1}{2} g^{41} (-g_{22,1}) + \frac{1}{2} g^{44} (-g_{22,4}) = M M' + e^{\lambda} M \dot{M}$

$\left\{ {4 \atop 22} \right\} = M M' + Z M \dot{M}$

$\left\{ {1 \atop 23} \right\} = \frac{1}{2} g^{14} (0) = 0 \quad \left\{ {2 \atop 23} \right\} = \frac{1}{2} g^{22} (g_{22,3}) = 0$

$\left\{ {3 \atop 23} \right\} = \frac{1}{2} g^{33} (g_{33,2}) = \cot\theta \quad \left\{ {4 \atop 23} \right\} = \frac{1}{2} g^{41} (0) + \frac{1}{2} g^{44} (0) = 0$

$\left\{ {3 \atop 23} \right\} = \cot\theta$

$\left\{ {}^{1}_{24} \right\} = \frac{1}{2} g^{14}(g_{44,2}) = 0, \quad \left\{ {}^{2}_{24} \right\} = \frac{1}{2} g^{22}(g_{22,4}) = \frac{\dot{M}}{M}$ $\qquad \left\{ {}^{2}_{24} \right\} = \frac{\dot{M}}{M}$

$\left\{ {}^{3}_{24} \right\} = \frac{1}{2} g^{33}(0) = 0 \quad \left\{ {}^{4}_{24} \right\} = \frac{1}{2} g^{41}(0) + \frac{1}{2} g^{44}(0) = 0$

$\left\{ {}^{1}_{33} \right\} = \frac{1}{2} g^{14}(-g_{33,4}) = M\dot{M} \sin^2\theta$ $\qquad \left\{ {}^{1}_{33} \right\} = M\dot{M} \sin^2\theta$

$\left\{ {}^{2}_{33} \right\} = \frac{1}{2} g^{22}(-g_{33,2}) = -\sin\theta\cos\theta$ $\qquad \left\{ {}^{2}_{33} \right\} = -\sin\theta\cos\theta$

$\left\{ {}^{3}_{33} \right\} = \frac{1}{2} g^{33}(g_{33,3}) = 0 \quad \left\{ {}^{4}_{33} \right\} = \frac{1}{2} g^{41}(-g_{33,1}) + \frac{1}{2} g^{44}(-g_{33,4})$ $\qquad \left\{ {}^{4}_{33} \right\} = (M\dot{M} + 2M\dot{M}) \sin^2\theta$

$\left\{ {}^{1}_{34} \right\} = \frac{1}{2} g^{14}(0) = 0 \quad \left\{ {}^{2}_{34} \right\} = \frac{1}{2} g^{22}(0) = 0$

$\left\{ {}^{3}_{34} \right\} = \frac{1}{2} g^{33}(g_{33,4}) = \frac{\dot{M}}{M}$ $\qquad \left\{ {}^{3}_{34} \right\} = \frac{\dot{M}}{M}$

$\left\{ {}^{1}_{44} \right\} = \frac{1}{2} g^{14}(g_{44,4}) = 0 \quad \left\{ {}^{2}_{44} \right\} = \frac{1}{2} g^{22}(-g_{44,2}) = 0$

$\left\{ {}^{3}_{44} \right\} = \frac{1}{2} g^{33}(0) \quad \left\{ {}^{4}_{44} \right\} = \frac{1}{2} g^{41}(0) + \frac{1}{2} g^{44}(0) = 0.$

Now we work out R_{ik}

$$R_{ik} = -\left\{ {}^{n}_{ik} \right\}_{n} + \frac{\partial}{\partial x^i} \left\{ {}^{n}_{kn} \right\} + \left\{ {}^{n}_{iL} \right\} \left\{ {}^{L}_{nk} \right\} - \left\{ {}^{n}_{ik} \right\} \left\{ {}^{L}_{nL} \right\}$$

We begin with R_{44}.

$$R_{44} = -\left\{ {}^{n}_{44} \right\}_{,n} + \frac{\partial}{\partial t} \left\{ {}^{n}_{4n} \right\} + \left\{ {}^{n}_{4L} \right\} \left\{ {}^{L}_{n4} \right\} - \left\{ {}^{n}_{44} \right\} \left\{ {}^{L}_{nL} \right\}$$

$$= 0 + \frac{\partial}{\partial t} \left(2\frac{\dot{M}}{M} \right) + \left\{ {}^{4}_{41} \right\} \left\{ {}^{1}_{44} \right\} + \left\{ {}^{2}_{42} \right\} \left\{ {}^{2}_{24} \right\} + \left\{ {}^{3}_{43} \right\} \left\{ {}^{3}_{34} \right\} - 0$$

$$= \frac{2\ddot{M}}{M} - \frac{2\dot{M}^2}{M^2} + 0 + \frac{\dot{M}^2}{M^2} + \frac{\dot{M}^2}{M^2} = \frac{2\ddot{M}}{M}$$

But $M = R \sin\frac{t}{R}$ $\qquad \dot{M} = R\cos\frac{t}{R}\left(-\frac{1}{R}\right) = -\cos\frac{t}{R}$

$\dot{M} = \sin\frac{t}{R}\left(-\frac{1}{R}\right) \qquad \therefore \frac{\ddot{M}}{M} = -\frac{1}{R^2}$

$$\therefore \boxed{R_{44} = -\frac{2}{R^2}}$$

(9)

$$R_{41} = -\{41^n\}_{,n} + \frac{\partial}{\partial t}\{1n^n\} + \{4n^l\}\{l1^n\} - \{41^n\}\{nl^l\}$$

$$= -\{41^4\}_{,4} + \frac{\partial}{\partial t}\left(\frac{2M'}{M}\right) + \{41^4\}\{41^1\} + \{42^2\}\{21^2\} + \{43^3\}\{31^3\}$$
$$- \{41^4\}\{41^1\}$$

$$= -\left(-\frac{\dot{z}}{2}\right)^{\cdot} + \frac{2\dot{M}'}{M} - \frac{2M'}{M}\frac{\dot{M}}{M} + 0 + \frac{\dot{M}}{M}\frac{M'}{M} + \frac{\dot{M}}{M}\frac{M'}{M} - \left(-\frac{\dot{z}}{2}\right)\frac{2\dot{M}}{M}$$

$$= \frac{\ddot{z}}{2} + \frac{2\dot{M}'}{M} + \frac{\dot{z}\dot{M}}{M}$$

$M = R\sin\frac{z}{R} \quad \frac{\dot{M}}{M} = +\cot\frac{z}{R}\left(-\frac{1}{R}\right)$

$\dot{M} = -\cos\frac{z}{R} \quad \dot{M}' = \sin\frac{z}{R}\left(\frac{1}{R}\right)$

$\therefore \frac{\dot{M}'}{M} = \frac{1}{R^2}$

$$\therefore \boxed{R_{41} = \frac{\ddot{z}}{2} - \frac{\dot{z}}{R}\cot\frac{z}{R} + \frac{2}{R^2}}$$

$$R_{22} = -\{22^n\}_{,n} + \frac{\partial}{\partial\theta}\{2n^n\} + \{2n^l\}\{l2^n\} - \{22^l\}\{l1^n\}$$

$$= -\{21^1\}_{,1} - \{22^4\}_{,4} + \frac{\partial}{\partial\theta}(\cot\theta) + 2\{21^2\}\{21^1\} + 2\{22^4\}\{42^2\} + \{23^3\}\{32^3\}$$
$$- \{21^1\}\{11^1\} - \{22^4\}\{41^1\}$$

$$= -(M\dot{M})' - (MM' + 2M\dot{M})^{\cdot} - \operatorname{cosec}^2\theta + 2\frac{M'}{M}M\dot{M} + 2\frac{\dot{M}}{M}(MM' + 2M\dot{M})$$
$$+ \cot^2\theta - M\dot{M}\left(\frac{2M'}{M}\right) - (MM' + 2M\dot{M})\frac{2\dot{M}}{M}$$

$$= -M\dot{M}' - M'\dot{M} - \dot{M}M' - M\dot{M}' - \dot{z}M\dot{M} - 2(M\ddot{M} + \dot{M}^2) - 1$$
$$+2M'\dot{M} \quad +2M'\dot{M} \quad + \quad +2\cdot 2\dot{M}^2$$
$$-2\dot{M}M' \quad -2\dot{M}M' \quad \quad -2\cdot 2\dot{M}^2$$

$$= -2M\dot{M}' - 2\dot{M}M' - \dot{z}M\dot{M} - 2(M\ddot{M} + \dot{M}^2) - 1$$

$$\therefore \frac{R_{22}}{M^2} = -\frac{2\dot{M}'}{M} - \frac{2\dot{M}}{M}\frac{M'}{M} - \dot{Z}\frac{\dot{M}}{M} - Z\frac{\ddot{M}}{M} - Z\frac{\dot{M}^2}{M^2} - \frac{1}{M^2}$$

$$\frac{\dot{M}'}{M} = \frac{1}{R^2} \quad \frac{\ddot{M}}{M} = -\frac{1}{R^2} \quad \frac{\dot{M}}{M} = -\frac{1}{R}\cot\frac{r}{R} \quad \frac{M'}{M} = +\frac{1}{R}\cot\frac{r}{R}$$

$$\therefore \frac{R_{22}}{M^2} = -\frac{2}{R^2} + \frac{2}{R^2}\cot^2\frac{r}{R} + \frac{\dot{Z}}{R}\cot\frac{r}{R} + \frac{Z}{R^2} - \frac{Z}{R^2}\cot^2\frac{r}{R} - \frac{1}{R^2}\operatorname{cosec}^2\frac{r}{R}$$

$$= -\frac{3}{R^2} + \frac{1}{R^2}\cot^2\frac{r}{R} + \frac{\dot{Z}}{R}\cot\frac{r}{R} + \frac{Z}{R^2}\left(1-\cot^2\frac{r}{R}\right)$$

or rather

$$\boxed{\frac{R_{22}}{M^2} = \frac{\dot{Z}}{R}\cot\frac{r}{R} + \frac{(Z-2)}{R^2}\left(1-\cot^2\frac{r}{R}\right) - \frac{\operatorname{cosec}^2\frac{r}{R}}{R^2}}$$

$$R_{11} = -\left\{{n \atop 11}\right\}_{,n} + \frac{\partial}{\partial u}\left\{{n \atop 1n}\right\} + \left\{{l \atop 1n}\right\}\left\{{n \atop l1}\right\} - \left\{{n \atop 11}\right\}\left\{{l \atop nl}\right\}$$

$$= -\left\{{1 \atop 11}\right\}_{,1} - \left\{{4 \atop 11}\right\}_{,4} + \left(\frac{2M'}{M}\right)' + \left\{{1 \atop 11}\right\}\left\{{1 \atop 11}\right\} + \left\{{4 \atop 11}\right\}\left\{{1 \atop 41}\right\} + \left\{{2 \atop 12}\right\}\left\{{2 \atop 21}\right\}$$

$$+ \left\{{3 \atop 13}\right\}\left\{{3 \atop 31}\right\} + \left\{{4 \atop 14}\right\}\left\{{4 \atop 41}\right\} - \left\{{1 \atop 11}\right\}\frac{2M'}{M} - \left\{{4 \atop 11}\right\}\frac{2\dot{M}}{M}$$

$$= -\left(\frac{\dot{Z}}{2}\right)' - \frac{1}{2}(-Z'+Z\dot{Z})^{\cdot} + \frac{2M''}{M} - \frac{2M'^2}{M^2} + \frac{\dot{Z}^2}{4} + 0 + \frac{M'^2}{M^2} + \frac{M'^2}{M^2}$$

$$+ \frac{\dot{Z}^2}{4} - \frac{\dot{Z}}{2}\frac{2M'}{M} - \left(-\frac{Z'}{2} + \frac{Z\dot{Z}}{2}\right)\frac{2\dot{M}}{M}$$

$$= -\frac{Z'}{2} + \frac{\dot{Z}'}{2} - \frac{1}{2}\dot{Z}^2 - \frac{1}{2}Z\ddot{Z} + \frac{2M''}{M} + \frac{\dot{Z}^2}{4} + \frac{\dot{Z}^2}{4} - \frac{\dot{Z}M'}{M} + Z'\frac{\dot{M}}{M} - Z\dot{Z}\frac{\dot{M}}{M}$$

$$\boxed{R_{11} = -\frac{Z\ddot{Z}}{2} - \frac{2}{R^2} - \frac{(\dot{Z}+Z')}{R}\cot\frac{r}{R} + \frac{Z\dot{Z}}{R}\cot\frac{r}{R}}$$

(1)

$$R_{ik} = -8\pi\left[(\beta+\rho)v_i v_k - \frac{(\rho-\beta)}{2}g_{ik} + \sigma w_i w_k\right]$$

Let $v_i = (v_1, 0, 0, v_4)$ $w_i = (w_1, 0, 0, w_4)$

$g^{ik} v_i v_k = 1 \quad \therefore g^{14} v_1 v_4 + g^{41} v_4 v_1 + g^{44} v_4^2 = 1$

$\therefore 2 v_1 v_4 + 2 (v_4)^2 = 1$

$g^{ik} w_i w_k = 0 \quad \therefore 2 w_1 w_4 + 2 (w_4)^2 = 0 \quad \therefore w_4 = 0$

$g^{ik} v_i w_k = 1 \quad \therefore g^{14} v_1 w_4 + g^{41} v_4 w_1 + g^{44} v_4 w_4 = 1$ But $w_4 = 0$

$v_4 w_1 = 1. \qquad w_1 = 1/v_4$

Thus $v_i = (\alpha, 0, 0, \gamma)$ $w_i = (\frac{1}{\gamma}, 0, 0, 0)$

and $2\alpha\gamma + 2\gamma^2 = 1. \qquad \alpha = (1 - 2\gamma^2)/2\gamma$

We now find β, ρ, γ and σ in terms of $R_{11}, R_{22}, R_{44}, R_{14}$

$$R_{11} = -8\pi\left[(\beta+\rho)(v_1)^2 + e^{\lambda}\frac{(\rho-\beta)}{2} + \sigma(w_1)^2\right] \qquad (11)$$

$$R_{22} = -8\pi\left[-\frac{(\rho-\beta)}{2}(-M^2)\right] \quad \therefore \frac{R_{22}}{M^2} = -8\pi\frac{(\rho-\beta)}{2} \qquad (22)$$

$$R_{44} = -8\pi\left[(\beta+\rho)v_4^2\right] \qquad R_{44} = -8\pi(\beta+\rho)(v_4)^2 \qquad (44)$$

$$R_{41} = -8\pi\left[(\beta+\rho)v_4 v_1 - \frac{(\rho-\beta)}{2}\right] \qquad (41)$$

Take $(44)2 + 2(41)$ and we get

* $2R_{44} + 2R_{41} = -8\pi\left[(\rho+p)-(\rho-p)\right] = -8\pi(2p)$

$\therefore$ ~~$8\pi\phi = -\frac{2}{R}$~~ $\boxed{+8\pi\phi = -\frac{2}{2}R_{44} - R_{41}}$

Next $\frac{2R_{22}}{M^2} = -8\pi\rho + 8\pi\phi$ $\therefore$ $\boxed{8\pi\rho = -\frac{2R_{22}}{M^2} - \frac{2}{2}R_{44} - R_{41}}$

Then $\boxed{(v_4)^2 = y^2 = \frac{R_{44}}{-8\pi(\phi+\rho)}}$

Having found y $x = \frac{1-2y^2}{2y}$ gives x. Thus v_i is known

Again $w_1 = \frac{1}{y}$ gives w_1 and hence finally R_{11} gives σ.

6-8-94

Let us make a small amendment by introducing the cosmological constant Λ.

So on the r.h.s. of R_{ik} add Λg_{ik}

" " " R_{11} " -2Λ

" " " R_{22} " $-M^2\Lambda$

" " " R_{14} " Λ

And R_{44} remains unaltered

$\therefore$ $8\pi\phi$ becomes $-\frac{2}{2}R_{44} - R_{41} + \Lambda$

$8\pi\rho$ becomes $(-2R_{22}/M^2) - 2\Lambda - (2/2)R_{44} - R_{41} + \Lambda$

Thus $8\pi\rho = -\frac{Z}{2}R_{44} - R_{41} + \Lambda$

$$8\pi\rho = -\frac{2R_{22}}{M^2} - \frac{Z}{2}R_{44} - R_{41} - \Lambda$$

Take $8\pi\rho$. Push in the values of R_{44} and R_{41}

$$\therefore 8\pi\rho = -\frac{Z}{2}\left(-\frac{2}{R^2}\right) - \frac{\ddot{Z}}{2} + \frac{\dot{Z}}{R}\cot\frac{r}{R} - \frac{2}{R^2} + \Lambda$$

$$= -\frac{\ddot{Z}}{2} + \frac{\dot{Z}}{R}\cot\frac{r}{R} + \frac{Z}{R^2} - \frac{2}{R^2} + \Lambda$$

For undisturbed Einstein Universe $Z = 1$

$$\text{So } 8\pi\rho_E = -\frac{1}{R^2} + \Lambda$$

One can take $\rho_E = 0$ so that $\Lambda = \frac{1}{R^2}$

However we shall write $\Lambda = 8\pi\rho_E + \frac{1}{R^2}$

$$\therefore 8\pi\rho = -\frac{\ddot{Z}}{2} + \frac{\dot{Z}}{R}\cot\frac{r}{R} + \frac{Z}{R^2} - \frac{2}{R^2} + \frac{1}{R^2} + 8\pi\rho_E.$$

One can recover Pramana $8\pi\rho$ to 0 by setting

$$Z = 1 + \frac{2m(u)}{R}\cot\frac{r}{R}.$$

However, we shall set $\rho_E = 0$ and $\rho = 0$

$$\therefore \quad -\frac{\ddot{z}}{2} + \frac{\dot{z}}{R}\cot\frac{z}{R} + \frac{z-1}{R^2} = 0$$

$$\therefore \quad \ddot{z} - \frac{2\dot{z}}{R}\cot\frac{z}{R} - \frac{2(z-1)}{R^2} = 0 \qquad \text{Let } z-1 = y$$

$$\therefore \quad \ddot{y} - \frac{2\dot{y}}{R}\cot\frac{z}{R} - \frac{2y}{R^2} = 0$$

Put it as $\ddot{y} + P\dot{y} + Qy = 0$. Then $y = v e^{-\frac{1}{2}\int P dt}$ changes the equation to $\ddot{v} + Iv = 0$ where $I = Q - \frac{1}{2}\frac{dP}{dt} - \frac{1}{4}P^2$. We use this B.Sc. method.

$$e^{-\frac{1}{2}\int P dt} = e^{-\frac{1}{2}\int(-\frac{2}{R})\cot\frac{z}{R}\,dt} = e^{\int \frac{1}{R}\frac{\cos(u-t)/R}{\sin(u-t)/R}(-dt)(-1)}$$

$$= e^{-\log \sin\frac{u-t}{R}} = \operatorname{cosec}\frac{z}{R}.$$

$$\therefore \text{ Put } \quad y = v\operatorname{cosec}\frac{z}{R} \qquad \dot{y} = \dot{v}\operatorname{cosec}\frac{z}{R} + v\operatorname{cosec}\frac{z}{R}\cot\frac{z}{R}\left(\frac{1}{R}\right)$$

$$\ddot{y} = \ddot{v}\operatorname{cosec}\frac{z}{R} + \frac{2}{R}\dot{v}\operatorname{cosec}\frac{z}{R}\cot\frac{z}{R} - \frac{v}{R}\left[-\operatorname{cosec}\frac{z}{R}\cot^2\frac{z}{R} - \operatorname{cosec}^3\frac{z}{R}\right](-1)$$

$$\therefore \text{ Our equation becomes}$$

$$\ddot{v}\operatorname{cosec}\frac{z}{R} + \frac{\dot{v}}{R}\operatorname{cosec}\frac{z}{R}\cot\frac{z}{R}(2+2) + \frac{v}{R^2}\operatorname{cosec}\frac{z}{R}\left[\cot^2\frac{z}{R} + \operatorname{cosec}^2\frac{z}{R} - 2\cot^2\frac{z}{R} - 2\right] = 0$$

$$\therefore \quad \ddot{v}\operatorname{cosec}\frac{z}{R} + \frac{v}{R^2}\operatorname{cosec}\frac{z}{R}(-1) = 0$$

$$\therefore \ddot{v} - \frac{v}{R^2} = 0 \quad \therefore v = A(u)\cosh\frac{r}{R} + B(u)\sinh\frac{r}{R}.$$

$$\therefore Y = r\operatorname{cosec}\frac{r}{R} = \frac{A(u)\cosh\frac{r}{R}}{\sin\frac{r}{R}} + \frac{B(u)\sinh\frac{r}{R}}{\sin\frac{r}{R}}$$

$$\therefore Z = e^{\lambda} = 1 + Y = 1 + \frac{A(u)\cosh\frac{r}{R}}{\sin\frac{r}{R}} + \frac{B(u)\sinh\frac{r}{R}}{\sin\frac{r}{R}}.$$

So we are considering a dust distribution through which radiation is flowing ~~out~~ inwards. Let us find the density of the distribution. From equation top at p. 10 we find $8\pi(P-\rho) = -\frac{2R_{22}}{M^2} - 2\Lambda$ But $\Lambda = \frac{1}{R^2}$ and $\rho = 0$

$$\therefore 8\pi P = -\frac{2R_{22}}{M^2} - \frac{2}{R^2} = -2\left[\frac{\dot{Z}}{R}\cot\frac{r}{R} + \frac{Z-2}{R^2}\left(1-\cot^2\frac{r}{R}\right) - \frac{-\operatorname{cosec}^2\frac{r}{R}}{R^2} + \frac{1}{R^2}\right]$$

$$= -2\left[\frac{\dot{Z}}{R}\cot\frac{r}{R} + \frac{(Z-1)}{R^2}\left(1-\cot^2\frac{r}{R}\right) + \frac{1}{R^2}\left\{-1+\cot^2\frac{r}{R} - \operatorname{cosec}^2\frac{r}{R} + 1\right\}\right]$$

$$= -2\left[\frac{\dot{Z}}{R}\cot\frac{r}{R} + \frac{Z-1}{R^2}\left(1-\cot^2\frac{r}{R}\right) - \frac{1}{R^2}\right]$$

Now $Z = 1 + \left(A\cosh\frac{z}{R} + B\sinh\frac{z}{R}\right)\operatorname{cosec}\frac{z}{R}$ $\qquad z = u - t$

$$\therefore \dot{Z} = \left(A\cosh\frac{z}{R} + B\sinh\frac{z}{R}\right)\operatorname{cosec}\frac{z}{R}\cot\frac{z}{R}\left(\frac{1}{R}\right)$$

$$- \left(A\sinh\frac{z}{R} + B\cosh\frac{z}{R}\right)\operatorname{cosec}\frac{z}{R}\left(\frac{1}{R}\right)$$

$$= \frac{1}{R}\operatorname{cosec}\frac{z}{R}\left[\cosh\frac{z}{R}\left(A\cot\frac{z}{R} - B\right) + \sinh\frac{z}{R}\left(B\cot\frac{z}{R} - A\right)\right]$$

$$\therefore \frac{\dot{Z}}{R}\cot\frac{z}{R} = \frac{1}{R^2}\operatorname{cosec}\frac{z}{R}\left[\cosh\frac{z}{R}\left(A\cot^2\frac{z}{R} - B\cot\frac{z}{R}\right)\right.$$

$$\left. + \sinh\frac{z}{R}\left(B\cot^2\frac{z}{R} - A\cot\frac{z}{R}\right)\right]$$

$$\frac{Z-1}{R^2}\left(1-\cot^2\frac{z}{R}\right) = \frac{1}{R^2}\operatorname{cosec}\frac{z}{R}\left(A\cosh\frac{z}{R} + B\sinh\frac{z}{R}\right)\left(1-\cot^2\frac{z}{R}\right)$$

$$\therefore \frac{\dot{Z}}{R}\cot\frac{z}{R} + \frac{Z-1}{R^2}\left(1-\cot^2\frac{z}{R}\right) = \frac{1}{R^2}\operatorname{cosec}\left(\frac{z}{R}\right)\left[\cosh\frac{z}{R}\left(A\cot^2\frac{z}{R}\right.\right.$$

$$\left. - B\cot\frac{z}{R} + A - A\cot^2\frac{z}{R}\right) + \sinh\frac{z}{R}\left(B\cot^2\frac{z}{R} - A\cot\frac{z}{R}\right.$$

$$\left.\left. + B - B\cot^2\frac{z}{R}\right)\right]$$

$$= \frac{1}{R^2\sin\frac{z}{R}}\left[\left(A - B\cot\frac{z}{R}\right)\cosh\frac{z}{R} + \left(B - A\cot\frac{z}{R}\right)\sinh\frac{z}{R}\right]$$

$$\therefore 8\pi p = -2\left[\frac{1}{R^2\sin^2\frac{z}{R}}\left(A - B\cot\frac{z}{R}\right)\cosh\frac{z}{R} + \frac{1}{R^2\sin\frac{z}{R}}\left(B - A\cot\frac{z}{R}\right)\sinh\frac{z}{R}\right.$$

$$\left. - \frac{1}{R^2}\right]$$

$$\therefore 8\pi p = \frac{2\left(B\cot\frac{r}{R} - A\right)\cosh\frac{r}{R}}{R^2 \sin\frac{r}{R}} + \frac{2\left(A\cot\frac{r}{R} - B\right)\sinh\frac{r}{R}}{R^2 \sin\frac{r}{R}} + \frac{2}{R^2}$$

Let us proceed to the limit as $r \to 0$

as $r \to 0$ $\sin\frac{r}{R} \to \frac{r}{R}$ $\sinh\frac{r}{R} \to \frac{r}{R}$ $\left.\begin{matrix}\cos\frac{r}{R} \\ \cosh\frac{r}{R}\end{matrix}\right\} \to 1$

and $p \to p_0$

$$\therefore 8\pi p_0 = \frac{2\left(\frac{BR}{r} - A\right)}{R^2\frac{r}{R}} + \frac{2\left(\frac{AR}{r} - B\right)\frac{r}{R}}{R^2\frac{r}{R}} + \frac{2}{R^2}$$

$$\sim \frac{2B}{r^2} - \frac{2A}{Rr} + \frac{2A}{rR} - \frac{2B}{R^2} + \frac{2}{R^2}$$

$$= \frac{2B}{r^2} + \frac{2}{R^2} - \frac{2B}{R^2}$$

Since $\int 4\pi r^2 p\, dr$ remains finite an essential singularity does not develope at $r=0$.

Again if $B=0$ i.e. $e^{\lambda} = 2 = 1 + A\frac{\cosh\frac{r}{R}}{\sin\frac{r}{R}}$ ..

~~There is no singularity at $r=0$ either in e^{λ} or in p.~~ As a matter of fact taking terms upto } K

$\frac{3}{2}$ in p one can show that A must be -ve to make p +ve near the origin and so we do not get inflowing radiation. We shall work out details of this case later.

We now take the case with $\Lambda = 0$ so we are not worried about Einstein's universe as such. We have a non-static case and hence Λ is not necessary to make pressure $p = 0$

$\therefore \Lambda = 0$

$$8\pi p = -\frac{2\ddot{r} - R_{44}}{2}, \qquad 8\pi p = \frac{-2R_{22}}{M^2} - \frac{2}{2}R_{44} - R_{41}$$

$$-\frac{2}{2}R_{44} - R_{41}$$

Let us put $8\pi p = 0$ $\therefore -\frac{2}{2}\left(-\frac{2}{R^2}\right) - \frac{\ddot{z}}{2} + \frac{\dot{z}}{R}\cot\frac{r}{R} - \frac{2}{R^2} = 0$

$$\therefore -\frac{\ddot{z}}{2} + \frac{\dot{z}}{R}\cot\frac{r}{R} + \frac{2-2}{R^2} = 0$$

$$\therefore \ddot{z} - \frac{2\dot{z}}{R}\cot\frac{r}{R} - \frac{2(z-2)}{R^2} = 0$$ Now put $z - 2 = y$

$$\therefore \ddot{y} - \frac{2\dot{y}}{R}\cot\frac{r}{R} - \frac{2y}{R^2} = 0$$ which is the same as the earlier (y, t) eqn. on p. (92)

$$\therefore Y = \left(A(u)\cosh\frac{z}{R} + B(u)\sinh\frac{z}{R}\right)\operatorname{cosec}\frac{z}{R} \qquad \text{But } Y \text{ is now } Z-z$$

$$\therefore e^{\lambda} = Z = z + \left(A(u)\cosh\frac{z}{R} + B(u)\sinh\frac{z}{R}\right)\operatorname{cosec}\frac{z}{R}$$

This makes $8\pi\rho = 0$. Thus now $8\pi\rho = -\frac{2R_{zz}}{M^2}$

$$\therefore 8\pi\rho = -2\left[\frac{\dot{Z}}{R}\cot\frac{z}{R} + \frac{(Z-z)}{R^2}\left(1-\cot^2\frac{z}{R}\right) - \frac{\operatorname{cosec}^2\frac{z}{R}}{R^2}\right]$$

$$Z = z + \left(A\cosh\frac{z}{R} + B\sinh\frac{z}{R}\right)\operatorname{cosec}\frac{z}{R}$$

$$\therefore \dot{Z} = \left(A\cosh\frac{z}{R} + B\sinh\frac{z}{R}\right)\operatorname{cosec}\frac{z}{R}\cot\frac{z}{R}\left(\frac{1}{R}\right)$$

$$- \left(A\sinh\frac{z}{B} + B\cosh\frac{z}{R}\right)\operatorname{cosec}\frac{z}{R}\left(\frac{1}{R}\right)$$

$$= \frac{1}{R}\operatorname{cosec}\frac{z}{R}\left[\cosh\frac{z}{R}\left(A\cot\frac{z}{R} - B\right) + \sinh\frac{z}{R}\left(B\cot\frac{z}{R} - A\right)\right]$$

$$\therefore \frac{\dot{Z}}{R}\cot\frac{z}{R} = \frac{1}{R^2}\operatorname{cosec}\frac{z}{R}\left[\cosh\frac{z}{R}\left(A\cot^2\frac{z}{R} - B\cot\frac{z}{R}\right) + \sinh\frac{z}{R}\left(B\cot^2\frac{z}{R} - A\cot\frac{z}{R}\right)\right]$$

$$\frac{Z-z}{R^2}\left(1-\cot^2\frac{z}{R}\right) = \frac{1}{R^2}\operatorname{cosec}\frac{z}{R}\left[\cosh\frac{z}{R}\left(A - A\cot^2\frac{z}{R}\right) + \sinh\frac{z}{R}\left(B - B\cot^2\frac{z}{R}\right)\right]$$

$$\therefore \frac{\dot{Z}}{R}\cot\frac{z}{R} + \frac{(Z-z)}{R^2}\left(1-\cot^2\frac{z}{R}\right) = \frac{1}{R^2}\operatorname{cosec}\frac{z}{R}\left[\left(A - B\cot\frac{z}{R}\right)\cosh\frac{z}{R} + \left(B - A\cot\frac{z}{R}\right)\sinh\frac{z}{R}\right]$$

$$\therefore 8\pi\rho = -2\left[\frac{1}{R^2\sin\frac{z}{R}}\left\{\left(A - B\cot\frac{z}{R}\right)\cosh\frac{z}{R} + \left(B - A\cot\frac{z}{R}\right)\sinh\frac{z}{R} - \frac{1}{\sin\frac{z}{R}}\right\}\right]$$

$$8\pi\rho = -\frac{2}{R^2}\left[\frac{1}{\sin\frac{r}{R}}\left(A - B\cot\frac{r}{R}\right)\cosh\frac{r}{R} + \left(B - A\cot\frac{r}{R}\right)\frac{\sinh\frac{r}{R}}{\sin\frac{r}{R}} - \frac{1}{\sin^2\frac{r}{R}}\right]$$

• when r is small $\sin\frac{r}{R} = \frac{r}{R}$ $\cos\frac{r}{R} = 1$

$$\sinh\frac{r}{R} = \frac{r}{R} \quad \cosh\frac{r}{R} = 1$$

$\therefore (8\pi\rho)$ when r is small, r^2 neglected $= -\frac{2}{R^2}\left[\frac{AR}{r} - \frac{BR^2}{r^2} + B - \frac{AR}{r} - \frac{R^2}{r^2}\right]$

$$= \frac{2}{r^2} + \frac{2B}{r^2}\left(1 - \frac{R^2}{R^2}\right),$$

$$= \frac{2}{r^2}(1+B) - \frac{2B}{R^2}.$$

It is obvious that we should study to distinct cases $B = 0$ and $B = -1$.

It is also seen that as far as solution near $r=0$ both the our cases have an identical behaviour. We shall take any one of them and work out details. But before that let us consider initial steps for expanding universe back-ground.

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$ds^2 = 2(du\,dt) - du^2 - M^2 dr^2$ $\quad M = R\sin\frac{r}{R}$

is Einstein Universe and ∴

$ds^2 = T[2du\,dt - du^2 - M^2 dr^2]$ is the R-W. expanding universe

Also $ds^2 = 2\,du\,dt - 2du^2 - M^2 dr^2$ is gives a radiating particle in Einstein Universe. So we expect

$ds^2 = T\left[2du\,dt - 2du^2 - M^2 dr^2\right]$ to give a radiating particle in the expanding Universe

∴ We work with the above metric with

$T = T(t)$ $\quad Z = Z(u,t)$ $\quad M = R\sin\frac{r}{R}$ $\quad dr^2 = d\alpha^2 + \sin^2\alpha\, d\beta^2$

$x^1 = u$, $x^2 = \alpha$, $x^3 = \beta$, $x^4 = t$.

$$\therefore g_{ik} = \begin{pmatrix} -2T & 0 & 0 & T \\ 0 & -TM^2 & 0 & 0 \\ 0 & 0 & -TM^2\sin^2\alpha & 0 \\ T & 0 & 0 & 0 \end{pmatrix}$$

$\therefore g = -T \times T \times T^2 M^4 \sin^2\alpha$

$= -T^4 M^4 \sin^2\alpha$

$\therefore \sqrt{(-g)} = T^2 M^2 \sin\alpha$

$$g^{ik} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{T} \\ 0 & -\frac{1}{TM^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{TM^2\sin^2\alpha} & 0 \\ \frac{1}{T} & 0 & 0 & \frac{2}{T} \end{pmatrix}$$

$$\left\{{1\atop 11}\right\} = \frac{1}{2} g^{14}\left(2g_{14,1} - g_{11,4}\right) = \frac{1}{2T}(2T)\dot{} = \frac{\dot{Z}}{2} + Z\frac{\dot{T}}{2T} = \left\{{1\atop 11}\right\}$$

$$\left\{{2\atop 11}\right\} = \frac{1}{2} g^{22}(0) = 0, \quad \left\{{3\atop 11}\right\} = \frac{1}{2} g^{33}(0) = 0$$

$$\left\{{4\atop 11}\right\} = \frac{1}{2} g^{41}\left(g_{11,1}\right) + \frac{1}{2} g^{44}\left(-g_{11,4}\right) = \frac{1}{2T}(-2T)' + \frac{Z}{T}(2T)\dot{} = -\frac{Z'}{2} + \frac{2\dot{Z}}{2} + \frac{Z\dot{T}}{2T} = \left\{{4\atop 11}\right\}$$

$$\left\{{1\atop 12}\right\} = \frac{1}{2} g^{14}(0) = 0 \qquad \left\{{2\atop 12}\right\} = \frac{1}{2} g^{22}\left(g_{22,1}\right) = \frac{M'}{M}$$

$$\left\{{2\atop 12}\right\} = \frac{M'}{M}$$

$$\left\{{3\atop 12}\right\} = \frac{1}{2} g^{33}(0) = 0 \qquad \left\{{4\atop 12}\right\} = \frac{1}{2} g^{41}(0) + \frac{1}{2} g^{44}(0) = 0$$

$$\left\{{1\atop 13}\right\} = \frac{1}{2} g^{14}(0) \qquad \left\{{2\atop 13}\right\} = \frac{1}{2} g^{22}(0), \quad \left\{{3\atop 13}\right\} = \frac{1}{2} g^{33}\left(g_{33,1}\right) = \frac{M'}{M}$$

$$\left\{{3\atop 13}\right\} = \frac{M'}{M}$$

$$\left\{{4\atop 13}\right\} = \frac{1}{2} g^{41}(0) + \frac{1}{2} g^{44}(0) = 0$$

$$\left\{{1\atop 14}\right\} = \frac{1}{2} g^{14}\left(g_{14,4} + g_{44,1} - g_{14,4}\right) = 0 \qquad \left\{{2\atop 14}\right\} = \frac{1}{2} g^{22}(0) \qquad \left\{{3\atop 14}\right\} = \frac{1}{2} g^{33}(0) = 0$$

$$\left\{{4\atop 14}\right\} = -\frac{1}{2} g^{14}\left(g_{11,4} + g_{14,1} - g_{14,1}\right) + \frac{1}{2} g^{44}\left(g_{14,4} + g_{44,1} - g_{14,4}\right)$$

$$= \frac{1}{2T}(-2T)\dot{} = -\frac{\dot{Z}}{2} - Z\frac{\dot{T}}{2T}$$

$$\left\{{4\atop 14}\right\} = -\frac{\dot{Z}}{2} - Z\frac{\dot{T}}{2T}$$

$$\left\{{1\atop 22}\right\} = \frac{1}{2} g^{14}\left(-g_{22,4}\right) = \frac{1}{2T}\left(TM^2\right)\dot{} = M^2\frac{\dot{T}}{2T} + M\dot{M}$$

$$\left\{{1\atop 22}\right\} = M^2\frac{\dot{T}}{2T} + M\dot{M}$$

$$\left\{{2\atop 22}\right\} = \frac{1}{2} g^{22}(0) = 0 \qquad \left\{{3\atop 22}\right\} = \frac{1}{2} g^{33}(0) = 0 \qquad \left\{{4\atop 22}\right\} = \frac{1}{2} g^{41}\left(-g_{22,1}\right)$$

$$+ \frac{1}{2} g^{44}\left(-g_{22,4}\right) = \frac{1}{2T}\left(T \cdot 2MM'\right) + \frac{Z}{2T}\left(M^2\dot{T} + T2M\dot{M}\right) \therefore \left\{{4\atop 22}\right\} = MM' + ZM^2\frac{\dot{T}}{2T} + ZM\dot{M}$$

$$\left\{{1\atop 23}\right\} = \frac{1}{2} g^{14}(0) = 0 \qquad \left\{{2\atop 23}\right\} = \frac{1}{2} g^{22}(0) = 0 \qquad \left\{{3\atop 23}\right\} = \frac{1}{2} g^{33}\left(g_{33,2}\right) = \cot\alpha$$

$$\left\{{3\atop 23}\right\} = \cot\alpha$$

$$\left\{{4\atop 23}\right\} = \frac{1}{2} g^{41}(0) + \frac{1}{2} g^{44}(0) = 0$$

$$\left\{{1\atop 24}\right\} = \frac{1}{2} g^{14}(0) = 0 \qquad \left\{{2\atop 42}\right\} = \frac{1}{2} g^{22}\left(g_{22,4}\right) = \frac{\dot{T}}{2T} + \frac{\dot{M}}{M}$$

$$\left\{{2\atop 24}\right\} = \frac{\dot{T}}{2T} + \frac{\dot{M}}{M}$$

$$\left\{{3\atop 24}\right\} = \frac{1}{2} g^{33}(0) = 0 \qquad \left\{{4\atop 24}\right\} = \frac{1}{2} g^{41}(0) + \frac{1}{2} g^{44}(0) = 0$$

$$\left\{{1\atop 33}\right\} = \frac{1}{2} g^{14}\left(-g_{33,4}\right) = \frac{1}{2T}\left(TM^2\right)\dot{}\sin^2\alpha$$

$$\left\{{2\atop 33}\right\} = \frac{1}{2} g^{22}\left(-g_{33,2}\right) = -\sin\alpha\cos\alpha$$

$$\therefore \left\{{1\atop 33}\right\} = \left(M^2\frac{\dot{T}}{2T} + M\dot{M}\right)\sin^2\alpha$$

$$\left\{{2\atop 33}\right\} = -\sin\alpha\cos\alpha$$

(20) $\left\{ {3 \atop 33} \right\} = \frac{1}{2} g^{33}(0) = 0 \quad \left\{ {4 \atop 33} \right\} = \frac{1}{2} g^{41}(-g_{33,1}) + \frac{1}{2} g^{44}(-g_{33,4})$

$= \left(\frac{1}{2T}(TM^2)' + \frac{2}{2T}(TM^2)' \right) \sin^2\alpha \quad \therefore \boxed{\left\{ {4 \atop 33} \right\} = \left[MM' + 2MM\dot{} + 2M^2 \frac{\dot{T}}{2T} \right] \sin^2\alpha}$

$\left\{ {1 \atop 34} \right\} = \frac{1}{2} g^{14}(0) = 0 \quad \left\{ {2 \atop 34} \right\} = \frac{1}{2} g^{22}(0) = 0 \quad \left\{ {3 \atop 34} \right\} = \frac{1}{2} g^{33}(g_{33,4}) = \boxed{\frac{\dot{T}}{2T} + \frac{\dot{M}}{M} = \left\{ {3 \atop 34} \right\}}$

$\left\{ {4 \atop 34} \right\} = \frac{1}{2} g^{41}(0) + \frac{1}{2} g^{44}(0) = 0$

$\left\{ {1 \atop 44} \right\} = \frac{1}{2} g^{14}(0) = 0, \quad \left\{ {2 \atop 44} \right\} = \frac{1}{2} g^{22}(0) = 0 \quad \left\{ {3 \atop 44} \right\} = \frac{1}{2} g^{33}(0) = 0$

$\left\{ {4 \atop 44} \right\} \frac{1}{2} g^{14}(g_{41,4} + g_{41,4}) + \frac{1}{2} g^{44}(0) = \frac{\dot{T}}{T} \qquad \left\{ {4 \atop 44} \right\} = \frac{\dot{T}}{T}$

Now we find R_{ik}

$R_{44} = -\left\{ {n \atop 44} \right\}_{,n} + \frac{\partial}{\partial x^4}\left\{ {n \atop 4n} \right\} + \left\{ {l \atop 4n} \right\}\left\{ {n \atop l4} \right\} - \left\{ {n \atop 44} \right\}\left\{ {l \atop nl} \right\}$

$= -\left\{ {4 \atop 44} \right\}_{,4} + \left(\left\{ {2 \atop 42} \right\} + \left\{ {3 \atop 43} \right\} + \left\{ {4 \atop 44} \right\} \right)_{,4} + \left\{ {4 \atop 41} \right\}\cancel{\left\{ {4 \atop 44} \right\}} + \left\{ {2 \atop 42} \right\}\left\{ {2 \atop 24} \right\} + \left\{ {3 \atop 43} \right\}\left\{ {3 \atop 34} \right\}$

$+ \left\{ {4 \atop 44} \right\}\left\{ {4 \atop 44} \right\} - \left\{ {4 \atop 44} \right\}\left(\left\{ {2 \atop 42} \right\} + \left\{ {3 \atop 43} \right\} + \left\{ {4 \atop 44} \right\} \right)$

$= \left(\frac{\dot{T}}{T} + 2\frac{\dot{M}}{M} \right)\dot{} + 2\left(\frac{\dot{T}}{2T} + \frac{\dot{M}}{M} \right)^2 - \frac{\dot{T}}{T} \cdot 2\left(\frac{\dot{T}}{2T} + \frac{\dot{M}}{M} \right)$

$= \frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2} + 2\frac{\ddot{M}}{M} - 2\frac{\dot{M}^2}{M^2} + \frac{\dot{T}^2}{2T^2} + \frac{2\dot{M}^2}{M^2} + 2\frac{\dot{T}}{2T}\frac{\dot{M}}{M} - 2\frac{\dot{T}^2}{2T^2} - 2\frac{\dot{T}}{T}\frac{\dot{M}}{M}$

$= \frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2}\left(-1 + \frac{1}{2} - 1 \right) + 2\frac{\ddot{M}}{M} \quad \therefore \boxed{R_{44} = \frac{\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} - \frac{2}{R^2}}$

$R_{41} = -\left\{ {n \atop 41} \right\}_{,n} + \frac{\partial}{\partial x^4}\left\{ {n \atop 1n} \right\} + \left\{ {l \atop 4n} \right\}\left\{ {n \atop l1} \right\} - \left\{ {n \atop 41} \right\}\left\{ {l \atop nl} \right\}$

$= -\left\{ {4 \atop 41} \right\}_{,4} + \frac{\partial}{\partial t}\left(\frac{2M'}{M} \right) + \cancel{\left\{ {4 \atop 41} \right\}\left\{ {4 \atop 41} \right\}} + \left\{ {2 \atop 42} \right\}\left\{ {2 \atop 21} \right\} + \left\{ {3 \atop 43} \right\}\left\{ {3 \atop 31} \right\} + \left\{ {4 \atop 44} \right\}\left\{ {4 \atop 41} \right\}$

$- \left\{ {4 \atop 41} \right\}\left\{ {2 \atop 42} \right\} + \left\{ {3 \atop 43} \right\} + \left\{ {4 \atop 44} \right\}$

$= 0$

$$= \left(\frac{\dot{z}}{2} + \frac{z\dot{T}}{2T}\right)\cdot + \frac{2\dot{M}'}{M} - \frac{2\dot{M}M'}{M^2} + 2\frac{M'}{M}\left(\frac{\dot{T}}{2T} + \frac{\dot{M}}{M}\right) + \left(\frac{\dot{z}}{2} + \frac{z\dot{T}}{2T}\right)\left(\frac{\dot{T}}{T} + \frac{2\dot{M}}{M}\right)$$

$$= \frac{\ddot{z}}{2} + \frac{\dot{z}\dot{T}}{2T} + \frac{z\ddot{T}}{2T} - \frac{z\dot{T}^2}{2T^2} + \frac{2\dot{M}'}{M} + \frac{2M'}{M}\frac{\dot{T}}{2T} + \frac{\dot{z}}{2}\frac{\dot{T}}{2T} + \frac{z\dot{M}}{M} + \frac{z\dot{T}^2}{2T^2} + \frac{2z\dot{T}}{2T}\frac{\dot{M}}{M}$$

$$R_{41} = \frac{\ddot{z}}{2} + \dot{z}\left(\frac{\dot{T}}{T} + \frac{\dot{M}}{M}\right) + z\left(\frac{\ddot{T}}{2T} + \frac{\dot{T}}{T}\frac{\dot{M}}{M}\right) + \frac{2\dot{M}'}{M} + \frac{M'}{M}\frac{\dot{T}}{T}$$

Put $z = 1 + \frac{3}{T}$ $\therefore \dot{z} = \frac{\dot{3}}{T} - \frac{3}{T}\frac{\dot{T}}{T}$

and $\ddot{z} = \frac{\ddot{3}}{T} - \frac{2\dot{3}}{T}\frac{\dot{T}}{T} - \frac{3}{T}\frac{\ddot{T}}{T} + \frac{2\cdot 3}{T}\frac{\dot{T}^2}{T^2}$ $\therefore R_{41}$ now becomes

$$R_{41} = \frac{\ddot{3}}{2T} - \frac{\dot{3}}{T}\frac{\dot{T}}{T} - \frac{3}{T}\frac{\ddot{T}}{2T} + \frac{3}{T}\frac{\dot{T}^2}{T^2} + \left(\frac{\dot{3}}{T} - \frac{3}{T}\frac{\dot{T}}{T}\right)\left(\frac{\dot{T}}{T} + \frac{\dot{M}}{M}\right)$$
$$+ \left(1 + \frac{3}{T}\right)\left(\frac{\ddot{T}}{2T} + \frac{\dot{T}}{T}\frac{\dot{M}}{M}\right) + \frac{2\dot{M}'}{M} + \frac{M'}{M}\frac{\dot{T}}{T}$$

$$= \frac{\ddot{3}}{2T} + \cancel{\frac{\dot{3}}{T}\left(-\frac{\dot{T}}{T} + \frac{\dot{T}}{T} + \frac{\dot{M}}{M}\right)} + \frac{3}{T}\left(-\frac{\dot{T}}{T} + \frac{\dot{T}}{T} + \frac{\dot{M}}{M}\right)$$
$$+ \frac{3}{T}\left(-\frac{\ddot{T}}{2T} + \frac{\dot{T}^2}{T^2} - \frac{\dot{T}^2}{T^2} - \frac{\dot{T}}{T}\frac{\dot{M}}{M} + \frac{\ddot{T}}{2T} + \frac{\dot{T}}{T}\frac{\dot{M}}{M}\right) + \frac{\ddot{T}}{2T} + \frac{\dot{T}}{T}\frac{\dot{M}}{M} + \frac{2\dot{M}'}{M} + \frac{M'}{M}\frac{\dot{T}}{T}$$

$$= \frac{1}{T}\left[\frac{\ddot{3}}{2} + \dot{3}\frac{\dot{M}}{M}\right] + \frac{2\dot{M}'}{M} + \frac{\ddot{T}}{2T} + \frac{\dot{T}}{T}\left(\frac{\dot{M}}{M} + \frac{M'}{M}\right)$$

But $M = R\sin\frac{z}{R}$ $\therefore \frac{\dot{M}}{M} = \frac{\cos\frac{z}{R}\left(-\frac{1}{R}\right)}{\sin\frac{z}{R}}$ $\frac{M'}{M} = \frac{\cos\frac{z}{R}\left(\frac{1}{R}\right)}{\sin\frac{z}{R}}$

$\therefore \frac{\dot{M}}{M} + \frac{M'}{M} = 0.$ Now $\dot{M} = -\cos\frac{z}{R}$

$$\dot{M}' = \sin\frac{z}{R}\cdot\frac{1}{R} \quad \therefore \frac{\dot{M}'}{M} = \frac{1}{R^2}$$

$$\therefore R_{41} = \frac{1}{T}\left(\frac{\ddot{3}}{2} - \frac{\dot{3}}{R}\cot\frac{z}{R}\right) + \frac{2}{R^2} + \frac{\ddot{T}}{2T}$$

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$$R_{22} = -\left\{ {}^{n}_{22} \right\}_{,n} + \frac{\partial}{\partial x^2}\left\{ {}^{n}_{2n} \right\} + \left\{ {}^{l}_{2n} \right\}\left\{ {}^{n}_{l2} \right\} - \left\{ {}^{n}_{22} \right\}\left\{ {}^{l}_{nl} \right\}$$

$$= -\left\{ {}^{1}_{22} \right\}_{,1} - \left\{ {}^{4}_{22} \right\}_{,4} + \frac{\partial}{\partial x^2}(\cot\alpha) + \left\{ {}^{2}_{21} \right\}\left\{ {}^{1}_{22} \right\} + \left\{ {}^{1}_{22} \right\}\left\{ {}^{2}_{12} \right\} + \left\{ {}^{4}_{22} \right\}\left\{ {}^{2}_{42} \right\}$$

$$+ \left\{ {}^{3}_{23} \right\}\left\{ {}^{3}_{32} \right\} + \left\{ {}^{2}_{24} \right\}\left\{ {}^{4}_{22} \right\} - \left\{ {}^{1}_{22} \right\}\left(\frac{2M'}{M} \right) - \left\{ {}^{4}_{22} \right\}\left(\frac{\dot{I}}{T} + \frac{2\dot{M}}{M} + \frac{\dot{I}}{T} \right)$$

$$= -\left(M\dot{M} + M^2\frac{\dot{I}}{2T} \right)' - \left(MM' + 2M\dot{M} + 2M^2\frac{\dot{I}}{2T} \right)^{\cdot} - \operatorname{cosec}^2\alpha + 2\left(M\dot{M} + \frac{M^2\dot{I}}{2T} \right)\frac{M'}{M}$$

$$+ 2\left(MM' + 2M\dot{M} + 2M^2\frac{\dot{I}}{2T} \right)\left(\frac{\dot{I}}{2T} + \frac{\dot{M}}{M} \right) + \cot^2\alpha$$

$$- \left(M\dot{M} + M^2\frac{\dot{I}}{2T} \right)\frac{2M'}{M} - \left(MM' + 2M\dot{M} + 2M^2\frac{\dot{I}}{2T} \right)\left(\frac{2\dot{I}}{T} + \frac{2\dot{M}}{M} \right)$$

$$= -M\dot{M} - M\dot{M}' - 2MM'\frac{\dot{I}}{2T} - 2M\dot{M} - 2M^2\frac{\dot{I}}{2T} - 2M\dot{M} - 2M^2\frac{\dot{I}}{2T}$$

$$- \dot{M}M' - M\dot{M}' + 2\dot{M}M' + 2MM'\frac{\dot{I}}{2T} + 2M^2\frac{\dot{I}}{T} + 2MM'\frac{\dot{I}}{2T} + 2$$

$$+ 2M^2\frac{\ddot{I}}{2T} + \frac{2M}{2}\frac{\dot{I}^2}{T^2} - 1$$

$$= -M'\dot{M} - 2M\dot{M} - M\dot{M}' - 2MM'\frac{\dot{I}}{2T} - \dot{M}M' - M\dot{M}'$$

$$- 2\dot{M}M - 2M\ddot{M} - 2M^2\frac{\ddot{I}}{2T} - 2M\dot{M}\frac{\dot{I}}{2T} - 2M^2\left(\frac{\ddot{I}}{2T} - \frac{\dot{I}^2}{2T^2} \right)$$

$$- 1 + 2\dot{M}M' + 2MM'\frac{\dot{I}}{2T} + 2MM'\frac{\dot{I}}{2T} + 2M\dot{M}\frac{\dot{I}}{2T} + 2M^2\frac{\dot{I}^2}{4T^2}$$

$$+ 2M'\dot{M} + 2\dot{M}^2 + 2M\dot{M}\frac{\dot{I}}{2T} - 2\dot{M}M' - 2MM'\frac{\dot{I}}{2T}$$

$$= -2M'\dot{M} - 2M\dot{M}' - 2M\dot{M}\frac{\dot{I}}{2T} - \dot{2}M\dot{M} - 2M\ddot{M} - 2\dot{M}^2 - \dot{2}M^2\frac{\dot{I}}{2T}$$

$$-2\,2M\dot{M}\frac{\dot{I}}{2T} - 2M^2\frac{\ddot{T}}{2T} + 2M^2\frac{\dot{T}^2}{2T^2} - 1 - \frac{\dot{T}}{T}\left(M\dot{M} + 2M\dot{M} + 2M^2\frac{\dot{I}}{2T}\right)(2-1)$$

$$= -2M'\dot{M} - 2M\dot{M}' - \dot{2}M\dot{M} - 2M\ddot{M} - 2\dot{M}^2 + \frac{\dot{T}}{2T}\left[-2M\dot{M}' - 2\dot{M}^2 - 2\,2M\dot{M} - 2M\dot{M}' - 2\dot{M}\dot{M}\right]$$

$$-2M^2\frac{\ddot{T}}{2T} + 2M^2\frac{\dot{T}^2}{2T^2}(1-1) - 1$$

$$\therefore \frac{R_{22}}{M^2} = -\frac{2M'}{M}\frac{\dot{M}}{M} - 2\frac{\dot{M}'}{M} - \dot{2}\left(\frac{\dot{M}}{M} + \frac{\dot{I}}{2T}\right) - 2\left(\frac{\ddot{M}}{M} + \frac{\dot{M}^2}{M^2} + 4\frac{\dot{M}}{M}\frac{\dot{I}}{2T} + \frac{\ddot{T}}{2T}\right)$$

$$-\frac{1}{M^2} - 4\frac{M'}{M}\frac{\dot{T}}{2T}$$

Before we find R_{14} let us find ϕ and p

$$R_{ik} = -8\pi\left[(\phi + p)v_i v_k - \frac{(\rho - p)}{2}g_{ik} + \sigma\omega_i\omega_k\right]$$

Let $v_i = (v_1, 0, 0, v_4)$ $\quad \omega_i = (\omega_1, 0, 0, \omega_4)$

$$g^{ik}v_i v_k = 1 \quad \therefore \frac{1}{1}v_1 v_4 + \frac{1}{T}v_4 v_1 + \frac{2}{2T}(v_4)^2 = 1$$

$$\therefore 2v_1 v_4 + 2(v_4)^2 = T$$

$$g^{ik}\omega_i\omega_4 = 0 \quad \therefore 2\omega_1\omega_4 + 2(\omega_4)^2 = 0$$

$$\therefore \omega_4 = 0 \text{ or else } 2\omega_1 + 2\omega_4 = 0.$$

We shall treat both these cases one by one.

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Case A $w_4 = 0$. Case B $2w_1 + 2w_4 = 0$.

Case A $w_4 = 0$ Nesct $g^{iR} v_i w_R = 1$ $\frac{1}{T} v_1 w_4 + \frac{1}{T} v_4 w_1 + \frac{2}{T} v_4 w_4 = 1$

But $w_4 = 0$ $\therefore \frac{1}{T} v_4 w_1 = 1$ $\therefore w_1 = \frac{T}{v_4}$, $w_4 = 0$

$2 v_1 v_4 + 2 (v_4)^2 = T$ } leave only one

v_4 undetermined.

$$R_{11} = -8\pi \left[(p+\rho)(v_1)^2 - \frac{(\rho - p)}{2}(-2T) + \sigma (w_1)^2 \right]$$

$$R_{22} = -8\pi \left[-\frac{(\rho - p)}{2}(-TM^2) \right]$$

$$R_{44} = -8\pi \left[(p+\rho) v_4^2 - \frac{(\rho - p)}{2} 0 + 0 \right]$$

$$R_{41} = -8\pi \left[(p+\rho) v_1 v_4 - \frac{(\rho - p)}{2} \frac{1}{v_4} T \right]$$

Take $2R_{44} + 2R_{41} = -8\pi \left[(p+\rho) \{ 2(v_4)^2 + 2 v_1 v_4 \} - (\rho - p) \frac{1}{v_4} T \right]$

$= -8\pi \left[(p+\rho) T - (\rho - p) T \right] = -8\pi \cdot 2p T$

$\therefore$ $8\pi p = -\frac{1}{2T} \left[2R_{44} + 2R_{41} \right]$

Again $2 \frac{R_{22}}{M^2} \frac{1}{T} = -8\pi \rho + 8\pi p$

$\therefore$ $8\pi \rho = \left[-\frac{2R_{22}}{M^2} - \frac{2}{2} R_{44} - R_{41} \right] \frac{1}{T}$

Next let us take case B.

Case B $2w_1 + 2w_4 = 0$ $g^{ik} v_i w_k = 1$. $\frac{1}{T} v_1 w_4 + \frac{1}{T} v_4 w_1 + \frac{2}{T} v_4 w_4 = 1$

$\therefore v_1 w_4 + v_4 (w_1 + 2w_4) = T$ $\therefore v_1 w_4 + v_4 (-w_1) = T$

$\therefore v_1 w_4 - v_4 w_1 = T$. $\therefore v_1 w_4 - v_4 \left(-\frac{2}{2} w_4\right) = T$ $\therefore w_4 \left(v_1 + \frac{2}{2} v_4\right) = T$

$\therefore w_4 (2v_1 + 2v_4) = 2T$ But $2v_1 v_4 + 2v_4^2 = T$ or

$v_4 (2v_1 + 2v_4) = T$ $\therefore w_4 = 2v_4$. or $-\frac{2}{2} w_1 = 2v_4$

$$w_1 = -2v_4$$

Thus $w_1 = -2v_4$, $w_4 = 2v_4$, $2v_1 v_4 + 2(v_4)^2 = T$

leave only v_4 undetermined.

$$R_{11} = -8\pi \left[(\rho + p) v_1^2 - \frac{(\rho - p)}{2} (-2T) + \sigma (w_1)^2 \right]$$

$$R_{22} = -8\pi \left[-\frac{(\rho - p)}{2} (-TM^2) \right]$$

$$R_{44} = -8\pi \left[(\rho + p)(v_4)^2 + \sigma (w_4)^2 \right]$$

$$R_{41} = -8\pi \left[(\rho + p) v_1 v_4 - \frac{(\rho - p)}{2} T + \sigma w_1 w_4 \right]$$

Again $2R_{44} + 2R_{41} = -8\pi 2\rho T$ and thus

$$8\pi \rho = -\frac{1}{2T} (2R_{44} + 2R_{41})$$

and $$8\pi p = \left[-\frac{2R_{22}}{M^2} - \frac{2}{2} R_{44} - R_{41} \right] \frac{1}{T}$$ so ρ and p are the same in both cases A and B

Let us take the dust case viz $p=0$

$\therefore$ For both cases A and B $\quad p=0 \Longrightarrow$

$$2R_{44} + 2R_{41} = 0$$

$$\therefore 2\left[\frac{\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} - \frac{2}{R^2}\right] + 2\left[\frac{\ddot{z}}{2} + \dot{z}\left(\frac{\dot{T}}{T} + \frac{\dot{M}}{M}\right) + z\left(\frac{\ddot{T}}{2T} + \frac{\dot{T}}{T}\frac{\dot{M}}{M}\right) + \frac{2\dot{M}'}{M} + 2\frac{M'}{M}\frac{\dot{T}}{T}\right] = 0$$

$$\therefore \ddot{z} + 2\dot{z}\left(\frac{\dot{T}}{T} + \frac{\dot{M}}{M}\right) + z\left(\frac{2\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} - \frac{2}{R^2} + 2\frac{\dot{T}}{T}\frac{\dot{M}}{M}\right) + \frac{4\dot{M}'}{M} + 2\frac{M'}{M}\frac{\dot{T}}{T} = 0$$

Put $z = 1 + V$

$$\therefore \ddot{V} + 2\dot{V}\left(\frac{\dot{T}}{T} + \frac{\dot{M}}{M}\right) + V\left(\frac{2\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} + 2\frac{\dot{T}}{T}\frac{\dot{M}}{M} - \frac{2}{R^2}\right)$$

$$+ \frac{2\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} + 2\frac{\dot{T}}{T}\left(\frac{\dot{M}}{M} + \frac{M'}{M}\right) - \frac{2}{R^2} + \frac{4\dot{M}'}{M} = 0$$

But $\dot{M} = -M'$ as $M = M(r)$ and $r' = -\dot{r} = 1$

Also $M = R\sin\frac{r}{R} \quad \frac{M'}{M} = \frac{\cos\frac{r}{R}}{\sin\frac{r}{R}}\frac{1}{R} = \frac{1}{R}\cot\frac{r}{R}$

$$M' = R\cos\frac{r}{R}\left(\frac{1}{R}\right) = \cos\frac{r}{R} \quad \dot{M}' = -\sin\frac{r}{R}\cdot\left(-\frac{1}{R}\right) = \frac{\sin\frac{r}{R}}{R}$$

$$\therefore \frac{\dot{M}'}{M} = \frac{\sin\frac{r}{R}}{R\cdot R\sin\frac{r}{R}} = \frac{1}{R^2}$$

$\therefore$ Eq. $p=0 \Longrightarrow \quad \ddot{V} + 2\dot{V}\left(\frac{\dot{T}}{T} - \frac{1}{R}\cot\frac{r}{R}\right) + V\left(\frac{2\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} - 2\frac{\dot{T}}{T}\frac{1}{R}\cot\frac{r}{R} - \frac{2}{R^2}\right\}$

$$+ \frac{2\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} + \frac{2}{R^2} = 0.$$

Sub case (i) of cases A and B.

Let $R \to \infty$ i.e. consider inward moving radiation in Einstein-de Sitter Universe with zero pressure

Then $p=0 \Rightarrow \ddot{V} + 2\dot{V}\left(\frac{\dot{T}}{T} - \frac{1}{r}\right) + V\left(\frac{2\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} - \frac{2\dot{T}}{T}\frac{1}{r}\right)$

$$+ \frac{2\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} = 0$$

In the undisturbed universe, $p=$ dust E-de Sitter universe corresponds to $V=0$ i.e. $2\ddot{T} - \frac{3}{2}\frac{\dot{T}^2}{T} = 0$

$\therefore \frac{\ddot{T}}{T} - \frac{3}{4}\frac{\dot{T}^2}{T^2} = 0$ if $T = y^4$ this gives $\ddot{y} = 0 \therefore y = at+b$

$$\therefore T = (at+b)^4$$

And then if the disturbed universe also has $p=0$, then

$$\ddot{V} + 2\dot{V}\left(\frac{\dot{T}}{T} - \frac{1}{r}\right) + V\left(-\frac{2\dot{T}}{T}\frac{1}{r}\right) = 0$$

$V = \frac{A(t)}{r}$ is a solution $\therefore V = \frac{A}{r}$ $\dot{V} = \frac{\dot{A}}{r^2}$ $\ddot{V} = \frac{2A}{r^3}$

$$\therefore \frac{2A}{r^3} + \frac{2\dot{A}}{r^2}\left(\frac{\dot{T}}{T} - \frac{1}{r}\right) + \frac{A}{r}\left(-\frac{2\dot{T}}{T}\frac{1}{r}\right) = 0$$

Thus $V = A\frac{1}{r}$ is one solution. We can then find the other solution by putting $V = \frac{\varphi(t)}{r}$

$\dot{V} = \frac{\dot{\varphi}}{r} + \frac{\varphi}{r^2}$. $\ddot{V} = \frac{\ddot{\varphi}}{r} + \frac{2\dot{\varphi}}{r^2} + \frac{2\varphi}{r^3}$

(2c)

$$\therefore \frac{\ddot{\phi}}{r} + \frac{2\dot{\phi}}{r^2} + \frac{2\phi}{r^3} + 2\left(\frac{\dot{\phi}}{r} + \frac{\phi}{r^2}\right)\left(\frac{\dot{T}}{T} - \frac{1}{r}\right) + \frac{\phi}{r}\left(-\frac{2\dot{T}}{T}\frac{1}{r}\right) = 0$$

$$\therefore \frac{\ddot{\phi}}{r} - \frac{2\phi}{r^2}\frac{\dot{T}}{T} + \frac{\dot{\phi}}{r} + \frac{2\phi}{r}\frac{\dot{T}}{T} = 0 \quad \therefore \ddot{\phi} + 2\dot{\phi}\frac{4a}{at+b} = 0$$

$$\therefore \frac{\ddot{\phi}}{\dot{\phi}} + \frac{8a}{at+b} = 0 \qquad \dot{\phi}(at+b)^8 = A(u)$$

$$\dot{\phi} = \frac{A(u)}{(at+b)^8} \qquad \phi = -\frac{A(u)}{7(at+b)^7} + B(u)$$

$$\therefore V = \frac{\phi}{r} = \frac{B(u)}{r} + \frac{\left(-\frac{A}{7}\right)}{r(at+b)^7}$$

$$\text{as } Z \to \infty \quad 1 + \frac{2m(u)}{r} + \frac{A(u)}{r(at+b)^7}$$

⊖ and so we have a very relevant solution.

Let us retain the cosmological constant so that when T = const. our case reduces to Einstein Universe

Then $$R_{11} = -8\pi\left[(\phi+p)v_1^2 - \frac{(\rho-p)}{2}(-2T) + \sigma(\omega_1)^2\right] - 2T\Lambda$$

$$R_{22} = -8\pi\left[-\frac{(\rho-p)}{2}(-TM^2)\right] - TM^2\Lambda$$

$$R_{44} = -8\pi\left[(\phi+p)v_4^2 + \sigma(\omega_4)^2\right]$$

$$R_{41} = -8\pi\left[(\phi+p)v_1v_4 - \frac{(\rho-p)}{2}T + \sigma\omega_1\omega_4\right] + T\Lambda$$

$$\therefore\ 2R_{44} + 2R_{41} = -8\pi\left[(\phi+p)T - (\rho-p)T\right] + 2T\Lambda$$

$$= -8\pi(2pT) + 2T\Lambda$$

$$\therefore\ 8\pi(2pT) = 2T\Lambda - (2R_{44} + 2R_{41})$$

$$\therefore\ 8\pi p = \Lambda - \frac{1}{2T}(2R_{44} + 2R_{41})$$

The eqn. $p=0$ now becomes $\frac{1}{2T}(2R_{44} + 2R_{41}) = \Lambda$

$\therefore\ 2R_{44} + 2R_{41} = 2T\Lambda$. The left hand side of this equation has been simplified on p.26. Using that we find

$$Z = 1 + V$$

$$\ddot{V} + 2\dot{V}\left(\frac{\dot{T}}{T} - \frac{1}{R}\cot\frac{r}{R}\right) + V\left(\frac{2\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} - \frac{2\dot{T}}{T}\frac{1}{R}\cot\frac{r}{R} - \frac{2}{R^2}\right)$$

$$+ \frac{2\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} + \frac{2}{R^2} = 2T\Lambda$$

Setting $V=0$, we find that for the undisturbed dust universe $\frac{2\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} + \frac{2}{R^2} = 2T\Lambda$

Put $T = y^4$ $\frac{\dot{T}}{T} = 4\frac{\dot{y}}{y}$ $\frac{\ddot{T}}{T} = 4\frac{\ddot{y}}{y} - 4\frac{\dot{y}^2}{y^2} + 16\frac{\dot{y}^2}{y^2} = 4\frac{\ddot{y}}{y} + 12\frac{\dot{y}^2}{y^2}$

So our eqn. becomes $4\frac{\ddot{y}}{y} + 12\frac{\dot{y}^2}{y^2} - \frac{3}{4}\times 16\frac{\dot{y}^2}{y^2} + \frac{1}{R^2} = y^4\Lambda$

$\therefore 4\frac{\ddot{y}}{y} + \frac{1}{R^2} = \Lambda y^4$ or $4\ddot{y} + \frac{y}{R^2} - \Lambda y^5 = 0$

Put $\dot{y} = q$ $\ddot{y} = \frac{dq}{dy}q$ $\therefore 4q\frac{dq}{dy} + \frac{y}{R^2} - \Lambda y^5 = 0$

$\therefore 4q\,dq = *\left(\Lambda y^5 - \frac{y}{R^2}\right)dy$. Integrating

$2q^2 = \frac{\Lambda y^6}{6} - \frac{y^2}{2R^2} + \text{const.}$ $\therefore$ $\boxed{q^2 = \frac{\Lambda y^6}{12} - \frac{y^2}{4R^2} + A}$

Our q is $\dot{y}$ and so we shall get (y,t) relation containing elliptic integrals. However a simple solution can be obtained by taking the arbitrary constant A to be zero

So consider $A=0$

$\therefore q^2 = \frac{\Lambda y^6}{12} - \frac{y^2}{4R^2}$. It is clear that we can not now take $\Lambda = 0$.

Setting $\Lambda = \frac{1}{R^2}$ as obtained from $\rho = 0$ in Einstein's Universe

we get $q^2 = \frac{y^6}{12R^2} - \frac{y^2}{4R^2} = \frac{y^2}{4R^2}\left[\frac{y^4}{3} - 1\right]$

$\therefore q = \frac{y}{2R}\sqrt{\left(\frac{y^4}{3} - 1\right)} = \frac{dy}{dt}$

$\therefore \frac{dy}{y\sqrt{\left(\frac{y^4}{3}-1\right)}} = \frac{dt}{2R} \quad \therefore \frac{y^3\,dy}{y^4\sqrt{\left(\frac{y^4}{3}-1\right)}} = \frac{dt}{2R}$

Setting track $y^4 = T \quad 4y^3dy = dT$ we get

$\frac{dT}{4T\sqrt{\left(\frac{T}{3}-1\right)}} = \frac{dt}{2R}$ Set $\frac{T}{3} - 1 = x^2$

$\therefore \frac{3 \times 2x\,dx}{4(x^2+1)3 \cdot x} = \frac{dt}{2R} \quad \therefore \frac{dx}{x^2+1} = \frac{dt}{R}$

$\therefore \tan^{-1} x = \frac{t}{R} + B. \quad \therefore x = \tan\left(\frac{t}{R} + B\right)$

$\therefore \frac{T}{3} - 1 = x^2 = \tan^2\left(\frac{t}{R} + B\right)$

$\therefore \frac{T}{3} = 1 + \tan^2\left(\frac{t}{R} + B\right) = \sec^2\left(\frac{t}{R} + B\right)$

$\therefore \boxed{T = 3\sec^2\left(\frac{t}{R} + B\right)}$ $\quad \therefore \frac{\dot{T}}{T} = \frac{2}{R}\tan\left(\frac{t}{R} + B\right)$ $\quad b = \text{const}$

Also $2\frac{\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} = -\frac{2}{R^2} + 2T\Lambda = -\frac{2}{R^2} + \frac{2T}{R^2}$ we now get our V-equ. (i.e. equ. $\phi = 0$ in the disturbed universe) as

$\ddot{V} + 2\dot{V}\left(\frac{\dot{T}}{T} - \frac{1}{R}\cot\frac{t}{R}\right) + V\left(-\frac{4}{R^2} + \frac{2T}{R^2} - 2\frac{\dot{T}}{T}\frac{1}{R}\cot\frac{t}{R}\right) = 0$

Now let $V = W \operatorname{cosec} \frac{x}{R}$

$\therefore \dot{V} = \dot{W} \operatorname{cosec} \frac{x}{R} + \frac{W}{R} \operatorname{cosec} \frac{x}{R} \cot \frac{x}{R}$.

$\ddot{V} = \ddot{W} \operatorname{cosec} \frac{x}{R} + 2\frac{\dot{W}}{R} \operatorname{cosec} \frac{x}{R} \cot \frac{x}{R} + \frac{W}{R^2} \operatorname{cosec} \frac{x}{R} \cot^2 \frac{x}{R} + \frac{W}{R^2} \operatorname{cosec}^3 \frac{x}{R}$

$\therefore$ Our (V, t) equation changes to

$$\ddot{W} \operatorname{cosec} \frac{x}{R} + 2\frac{\dot{W}}{R} \operatorname{cosec} \frac{x}{R} \cot \frac{x}{R} + \frac{W}{R^2} \operatorname{cosec} \frac{x}{R} \left(\cot^2 \frac{x}{R} + \operatorname{cosec}^2 \frac{x}{R} \right)$$
$$+ \left(2\dot{W} \operatorname{cosec} \frac{x}{R} + 2\frac{W}{R} \operatorname{cosec} \frac{x}{R} \cot \frac{x}{R} \right) \left(\frac{\dot{T}}{T} - \frac{1}{R} \cot \frac{x}{R} \right)$$
$$+ W \operatorname{cosec} \frac{x}{R} \left(-\frac{4}{R^2} + \frac{2T}{R^2} - 2\frac{\dot{T}}{T} \frac{1}{R} \cot \frac{x}{R} \right) = 0$$

Removing the common factor $\operatorname{cosec} \frac{x}{R}$, we find

$$\ddot{W} + 2\dot{W}\left(\frac{\dot{T}}{T}\right) + \frac{W}{R^2} \left[\cot^2 \frac{x}{R} + \operatorname{cosec}^2 \frac{x}{R} - 2\cot^2 \frac{x}{R} - \frac{4}{R} + 2T \right] = 0$$

$\therefore \ddot{W} + 2\dot{W}\frac{\dot{T}}{T} + \frac{W}{R^2}(-3 + 2T) = 0$ Putting the values of $T = 3\sec^2\left(\frac{t}{R} + B\right)$ and $\frac{\dot{T}}{T} = \frac{2}{R}\tan\left(\frac{t}{R} + B\right)$, we get

$$\ddot{W} + 4\frac{\dot{W}}{R} \tan\left(\frac{t}{R} + B\right) + 3\frac{W}{R^2}\left(-1 + 2\sec^2\left(\frac{t}{R} + B\right)\right) = 0$$

Using the result: For eqn $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

$y = v e^{-\frac{1}{2}\int P\,dx}$ changes the eqn. to

$\frac{d^2v}{dx^2} + Iv = 0$ where

$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$.

In our case $P = \frac{4}{R}\tan\alpha$, $Q = \frac{3}{R^2}(-1 + 2\sec^2\alpha)$ $\alpha = \frac{t}{R} + B$

$$\therefore I = \frac{-3 + 6\sec^2\alpha}{R^2} - \frac{1}{2}\cdot\frac{4\sec^2\alpha}{R^2} - \frac{1}{4}\frac{16\tan^2\alpha}{R^2}$$

$$= -\frac{3}{R^2} + \frac{6(1+\tan^2\alpha)}{R^2} - \frac{2(1+\tan^2\alpha)}{R^2} - \frac{4\tan^2\alpha}{R^2}$$

$$= \frac{1}{R^2}(-3+6-2) + \frac{1}{R^2}\tan^2\alpha(6-2-4)$$

$$= \frac{1}{R^2}$$

Next $\frac{1}{2}\int P\,dt = \frac{1}{2}\int \frac{4}{R}\tan\left(\frac{t}{R}+B\right)dt = -\frac{2}{4}\log\cos\left(\frac{t}{R}+B\right)$

$$\therefore e^{-\frac{1}{2}\int P\,dt} = \cos^2\left(\frac{t}{R}+B\right)$$

$$\therefore W = ve^{-\frac{1}{2}\int P\,dt} = v\cos^2\left(\frac{t}{R}+B\right)$$ and we have

$$\frac{d^2v}{dt^2} + \frac{1}{R^2}v = 0 \qquad v = A(u)\sin\left(\frac{t}{R}\right) + B(u)\cos\frac{t}{R}$$

$$\therefore W = \left[A(u)\sin\frac{t}{R} + B(u)\cos\frac{t}{R}\right]\cos^2\left(\frac{t}{R}+b\right)$$

$$\therefore Z = 1 + V = 1 + W\operatorname{cosec}\frac{z}{R}$$

$$1 + \left[A(u)\sin\frac{t}{R} + B(u)\cos\frac{t}{R}\right]\operatorname{cosec}\frac{z}{R}\cos^2\left(\frac{t}{R}+b\right)$$

See page ~~38~~ (35)

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(37) One can work out a solution of the equation

$$\ddot{W} + 2\dot{W}\frac{\dot{T}}{T} - \frac{3W}{R^2} = 0 \qquad \text{i.e. } \beta = 0 \text{ in the case } \Lambda = 0.$$

Here T satisfies $2\frac{\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} + \frac{2}{R^2} = 0$ Setting $T = y^4$ and following steps on p.30 we find $4\frac{\ddot{y}}{y} + \frac{1}{R^2} = 0$

$$\therefore y = a\cos\left(\frac{t}{2R} + b\right) \quad \text{or } y = \text{or } y =$$

$$\therefore T = a^4\cos^4\left(\frac{t}{2R} + b\right) \quad \therefore \frac{\dot{T}}{T} = -\frac{4}{2R}$$

$$\therefore T = a^4\cos^4\left(\frac{t}{2R} + b\right) \quad \therefore \frac{\dot{T}}{T} = \frac{-4}{2R}\tan\left(\frac{t}{2R} + b\right)$$

$$\therefore \ddot{W} - \frac{4}{R}\dot{W}\tan\alpha - \frac{3W}{R^2} = 0 \qquad \alpha = \frac{t}{2R} + b$$

Try $\frac{\dot{W}}{W} = \frac{n}{2R}\cot\alpha + \frac{m}{2R}\tan\alpha$

$$\therefore \frac{\ddot{W}}{W} - \frac{\dot{W}^2}{W^2} = -\frac{n}{4R^2}\operatorname{cosec}^2\alpha + \frac{m}{4R^2}\sec^2\alpha$$

$$\therefore \frac{\ddot{W}}{W} = -\frac{n}{4R^2}(1 + \cot^2\alpha) + \frac{m}{4R^2}(1 + \tan^2\alpha) + \frac{n^2}{4R^2}\cot^2\alpha + \frac{m^2}{4R^2}\tan^2\alpha + \frac{2mn}{4R^2}$$

$\therefore$ Our W-equation gives

$$\cot^2\alpha\left(\frac{n^2}{4R^2} - \frac{n}{4R^2}\right) + \tan^2\alpha\left(\frac{m}{4R^2} + \frac{m^2}{4R^2}\right) - \frac{n}{4R^2} + \frac{m}{4R^2} + \frac{2mn}{4R^2}$$

$$-\frac{4}{R}\tan\alpha\left(\frac{n}{2R}\tan\alpha - \frac{m}{2R}\tan\alpha\right) - \frac{4}{R}\tan\alpha\left(\frac{n}{2R}\cot\alpha + \frac{m}{2R}\tan\alpha\right) - \frac{3}{R^2} = 0$$

$$-\frac{3}{R^2}$$

$\therefore$ ~~$\cot^2\alpha$~~ $\therefore \cot^2\alpha\left(m\frac{n^2-n}{4}\right)+\tan^2\alpha\left(\frac{m^2+m}{4}-2m\right)+\frac{m}{4}-\frac{n}{4}+\frac{mn}{2}-2n-3=0$

~~If we set~~

$\therefore n(n-1)\cot^2\alpha + m(m-7)\tan^2\alpha + m - 9n + 2mn - 12 = 0$

Set $n=1$, $m=7$ then we find that $m-9n+2mn-12 = 7-9+14-12=0$

So with $n=1$, $m=7$ $\frac{\dot W}{W} = \frac{n}{2R}\cot\alpha + \frac{m}{2R}\tan\alpha$ Eq satisfies our equation

$$\therefore \frac{\dot W}{W} = \frac{1}{2R}\cot\alpha + \frac{7}{2R}\tan\alpha$$

$$\therefore \log W = \log \sin\alpha - 7\log\cos\alpha + \log A(w)$$

$$\therefore W = A\frac{\sin\alpha}{\cos^7\alpha}$$

~~To find the general solution~~ Take In order to recover the solution $W = \frac{A(w)}{R}\frac{1}{(at+b)^7}$ of p. 25 we let $\alpha = \frac{t}{2R} + b$ we set $b = \frac{\pi}{2}$

and incorporate a $-$ sign in A to get

$$W = \frac{A\cos\frac{t}{2R}}{\sin^7\left(\frac{t}{2R}\right)}$$

as $R\to\infty$ $\cos\frac{t}{2R}\to 1$ $\sin\frac{t}{2R}\to\frac{t}{2R}$

and $W \to \frac{\bar A}{t^7}$ with the factor $\csc\frac{1}{R}$ we find $V \to \frac{A(w)}{2(t^7)}$

which essentially the 2nd integral on p. 25

However, we shall continue our work with $\alpha = \frac{t}{2R} + b$.

$W = A\frac{\sin\alpha}{\cos^7\alpha}$. ~~Also~~ To find the general solution

assume $W = \Phi(t)\frac{\sin\alpha}{\cos^7\alpha}$.

(39) $\therefore \frac{\dot W}{W} = \frac{\dot\varphi}{\varphi} + \frac{1}{2R}\tan\alpha + \frac{7}{2R}\tan\alpha$ $\quad \frac{\dot W}{W} = \frac{\dot\varphi}{\varphi} + \frac{1}{2R}\cot\alpha + \frac{7}{2R}\tan\alpha$

$$\therefore \frac{\ddot W}{W} = \frac{\ddot\varphi}{\varphi} - \frac{\dot\varphi^2}{\varphi^2} - \frac{\operatorname{cosec}^2\alpha}{4R^2} + \frac{7}{4R^2}\sec^2\alpha\tan\alpha + \frac{\dot\varphi^2}{\varphi^2} + \frac{\dot\varphi}{\varphi}\frac{1}{R}(\cot\alpha + 7\tan\alpha)$$

$$\frac{1}{4R^2}\operatorname{cosec}^2\alpha \quad \frac{\cot^2\alpha}{4R^2} + \frac{49}{4R^2}\tan^2\alpha + \frac{14}{4R^2}$$

$$-\frac{4}{R}\frac{\dot W}{W}\tan\alpha = -\frac{4}{R}\tan\alpha\left(\frac{\dot\varphi}{\varphi} + \frac{1}{2R}\cot\alpha + \frac{7}{2R}\tan\alpha\right)$$

$$-\frac{3}{R^2} = -\frac{3}{R^2}$$

$$\therefore 0 = \frac{\ddot\varphi}{\varphi} + \frac{\dot\varphi}{\varphi}\frac{1}{R}(\cot\alpha + 3\tan\alpha) + \cot^2\alpha\left(-\frac{1}{4R^2} + \frac{1}{4R^2}\right) + \tan^2\alpha\left(\frac{7}{4R^2} + \frac{49}{4R^2} - \frac{28}{2R^2}\right)$$

$$-\frac{1}{4R^2} + \frac{7}{4R^2} + \frac{14}{4R^2} - \frac{2}{R^2} - \frac{3}{R^2}$$

$$\therefore 0 = \frac{\ddot\varphi}{\varphi} + \frac{\dot\varphi}{\varphi}\frac{1}{R}(\cot\alpha + 3\tan\alpha) \quad \therefore \frac{\ddot\varphi}{\dot\varphi} + \frac{1}{2R}\cdot\frac{2\cos\alpha}{\sin\alpha} + \frac{1}{2R}\,6\,\frac{\sin\alpha}{\cos\alpha} = 0$$

$$\therefore \frac{\dot\varphi\sin^2\alpha}{\cos^6\alpha} = k(u)$$

$$\therefore \dot\varphi = k\frac{\cos^6\alpha}{\sin^2\alpha} \cdot k\cos^5\alpha\cot\alpha\operatorname{cosec}\alpha$$

$$\therefore \varphi = k\left[\cos^5\alpha(-\operatorname{cosec}\alpha)2R + \int 2R\operatorname{cosec}\alpha\, 5\cos^4\alpha(-\sin\alpha)\frac{dt}{2R}\right] + L$$

⇒ We work out the coeff. of k. It is

$$-2R\cos^5\alpha\operatorname{cosec}\alpha - 5\int\cos^4\alpha\,dt$$

$$= -2R\cos^5\alpha\operatorname{cosec}\alpha - \frac{5}{4}\int(2\cos^2\alpha)^2\,dt = -2R\cos^5\alpha\operatorname{cosec}\alpha - \frac{5}{4}\int(1+\cos2\alpha)^2\,dt$$

$$= -2R\cos^5\alpha\operatorname{cosec}\alpha - \frac{5}{4}\left[t + 2\sin2\alpha\int(1+2\cos2\alpha+\cos^2 2\alpha)\,dt\right]$$

$$= -2R\cos^5\alpha\operatorname{cosec}\alpha - \frac{5}{4}\left[t + \sin2\alpha\cdot 2R + \frac{1}{2}\int(1+\cos4\alpha)\,dt\right]$$

$$= -2R\cos^5\alpha \operatorname{cosec}\alpha - \frac{5}{4}\left[t + 2R\sin 2\alpha + \frac{1}{2}t + \frac{1}{2}\sin 4\alpha \cdot \frac{1}{4}\cdot 2R\right]$$

$$\therefore \varphi = k\left[-2R\cos^5\alpha\operatorname{cosec}\alpha - \frac{15}{16}t - \frac{15}{8}t - \frac{5}{2}R\sin 2\alpha - \frac{5}{16}R\sin 4\alpha\right] + L$$

∴ The final solution of W-eqn. is

$$W = \varphi \frac{\sin\alpha}{\cos^7\alpha}$$

$$= k\left[-2R\cos^5\alpha \frac{\operatorname{cosec}\alpha \sin\alpha}{\cos^7\alpha} - \frac{15}{8}t\frac{\sin\alpha}{\cos^7\alpha} - \frac{5}{2}R\frac{\sin 2\alpha \sin\alpha}{\cos^7\alpha} - \frac{5}{16}R\frac{\sin 4\alpha \sin\alpha}{\cos^7\alpha}\right]$$

$$+ L\frac{\sin\alpha}{\cos^7\alpha}$$

$$W = k\left[-2R\sec^2\alpha - \frac{15}{8}t\sin\alpha\sec^7\alpha - \frac{5}{R}R\sin^2\alpha\sec^6\alpha - \frac{5}{4}R\sin^2\alpha\sec^6\alpha\cos 2\alpha\right]$$

$$+ L\frac{\sin\alpha}{\cos^7\alpha}.$$

$$Z = 1 + V = 1 + W\operatorname{cosec}\frac{z}{R}.$$

(37) ~~With~~ T = We take up from p. 233.

With $T = 3\sec^2(\alpha)$ $\alpha = \frac{t}{R} + b$ let us find R_{ik}.

$$\dot{T} = \frac{6}{R}\sec^2\alpha\tan\alpha \qquad \ddot{T} = \frac{12}{R^2}\sec^2\alpha\tan^2\alpha + \frac{6}{R^2}\sec^4\alpha$$

$$\therefore \frac{\dot{T}}{T} = \frac{2}{R}\tan\alpha \qquad \frac{\ddot{T}}{T} = \frac{2}{R^2}(2\tan^2\alpha + \sec^2\alpha)$$

$$\therefore R_{44} = \frac{\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} - \frac{2}{R^2} = \frac{4}{R^2}\tan^2\alpha + \frac{2}{R^2} + \frac{2}{R^2}\tan^2\alpha - \frac{3}{2}\frac{4}{R^2}\tan^2\alpha - \frac{2}{R^2}$$

$$= 0$$

But $R_{44} = -8\pi(p+\rho)(v_4)^2$ and $R_{44} = 0 \Longrightarrow p + \rho = 0$

i.e. the back-ground is de Sitter's universe

18-12-94

Before we go over to solving differential equations, let us find R_{ik}, ϕ, P, σ, v_i and w_i for the metric

$$ds^2 = T\left[2\,du\,dt - 2\,dw^2 - M^2 d\Omega^2\right] \qquad T = T(t),\ Z = Z(w,t),\ M = R\sin\frac{z}{R}$$

$$z = w - t.$$

We have worked out g_{ij}, g^{ij}, $\{ij,k\}$ on p. 18, 19, 20.

We have also shown that

$$R_{44} = \frac{\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} - \frac{2}{R^2}$$

and

$$R_{41} = \frac{1}{T}\left[\frac{\ddot{\beta}}{2} - \frac{\dot{\beta}}{R}\cot\frac{z}{R}\right] + \frac{2}{R^2} + \frac{\dot{T}}{2T} \qquad Z = 1 + \frac{\beta}{T}$$

Also $$\frac{R_{22}}{M^2} = -\frac{2M'}{M}\frac{\dot{M}}{M} - \frac{2\dot{M}'}{M} - \dot{Z}\left(\frac{\dot{M}}{M} + \frac{\dot{T}}{2T}\right) - Z\left(\frac{\ddot{M}}{M} + \frac{\dot{M}^2}{M^2} + 4\frac{\dot{M}}{M}\frac{\dot{T}}{2T} + \frac{\ddot{T}}{2T}\right)$$

$$- \frac{1}{M^2} - 4\frac{M'}{M}\frac{\dot{T}}{2T}$$

$$M = R\sin\frac{z}{R} \quad \therefore \quad \frac{\dot{M}}{M} = -\frac{1}{R}\cot\frac{z}{R}, \quad \frac{M'}{M} = \frac{1}{R}\cot\frac{z}{R}$$

$$\dot{M} = -\cos\frac{z}{R}; \quad \ddot{M} = -\sin\frac{z}{R}\frac{1}{R} \qquad \dot{M}' = \sin\frac{z}{R}\frac{1}{R}$$

$$\therefore \frac{\ddot{M}}{M} = -\frac{1}{R^2} \qquad \frac{\dot{M}'}{M} = \frac{1}{R^2}$$

$$\therefore \frac{R_{22}}{M^2} = -2\left(\frac{1}{R}\cot\frac{z}{R}\right)\left(-\frac{1}{R}\cot\frac{z}{R}\right) - 2\left(\frac{1}{R^2}\right) - \frac{1}{R^2}\operatorname{cosec}^2\frac{z}{R} - 4\left(\frac{1}{R}\cot\frac{z}{R}\right)\frac{\dot{T}}{2T}$$

$$-\dot{Z}\left(-\frac{1}{R}\cot\frac{z}{R} + \frac{\dot{T}}{2T}\right) - Z\left(-\frac{1}{R^2} + \frac{1}{R^2}\cot^2\frac{z}{R} + 4\left(-\frac{1}{R}\cot\frac{z}{R}\right)\frac{\dot{T}}{2T} + \frac{\ddot{T}}{2T}\right)$$

$$= +\frac{2}{R^2}\cot^2\frac{z}{R} - \frac{2}{R^2} - \frac{1}{R^2}\left(1+\cot^2\frac{z}{R}\right) - \frac{2}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} - \dot{Z}\left(\frac{1}{2}\frac{\dot{T}}{T} - \frac{1}{R}\cot\frac{z}{R}\right)$$

$$-Z\left(-\frac{1}{R^2} + \frac{1}{R^2}\cot^2\frac{z}{R} - \frac{2}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} + \frac{1}{2}\frac{\ddot{T}}{T}\right)$$

Put $Z = 1 + \frac{z}{T}$. $\dot{Z} = \frac{\dot{z}}{T} - \frac{z}{T^2}\dot{T}$

$$\therefore \frac{R_{22}}{M^2} = \frac{1}{R^2}\cot^2\frac{z}{R} - \frac{3}{R^2} - \frac{2}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} - \left(\frac{\dot{z}}{T} - \frac{z}{T}\frac{\dot{T}}{T}\right)\left(\frac{1}{2}\frac{\dot{T}}{T} - \frac{1}{R}\cot\frac{z}{R}\right)$$

$$-\left(1+\frac{z}{T}\right)\left(-\frac{1}{R^2} + \frac{1}{R^2}\cot^2\frac{z}{R} - \frac{2}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} + \frac{1}{2}\frac{\ddot{T}}{T}\right)$$

$$= \frac{1}{R^2}\cot^2\frac{z}{R} - \frac{3}{R^2} - \frac{2}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} + \frac{1}{R^2} - \frac{1}{R^2}\cot^2\frac{z}{R} + \frac{2}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} - \frac{1}{2}\frac{\ddot{T}}{T}$$

$$-\frac{\dot{z}}{T}\left(\frac{1}{2}\frac{\dot{T}}{T} - \frac{1}{R}\cot\frac{z}{R}\right) + \frac{z}{T}\left(\frac{1}{2}\frac{\dot{T}^2}{T^2} - \frac{\dot{T}}{T}\frac{1}{R}\cot\frac{z}{R} + \frac{1}{R^2} - \frac{1}{R^2}\cot^2\frac{z}{R}\right.$$

$$\left. + \frac{2}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} - \frac{1}{2}\frac{\ddot{T}}{T}\right)$$

$$\frac{R_{22}}{M^2} = \frac{\dot{z}}{T}\left(\frac{1}{R}\cot\frac{z}{R} - \frac{1}{2}\frac{\dot{T}}{T}\right) + \frac{z}{T}\left(-\frac{1}{2}\frac{\ddot{T}}{T} + \frac{1}{2}\frac{\dot{T}^2}{T^2} + \frac{1}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} + \frac{1}{R^2} - \frac{1}{R^2}\cot^2\frac{z}{R}\right)$$

$$-\frac{2}{R^2} - \frac{\ddot{T}}{2T}$$

Next we find R_{11}. ~~from~~

$$R_{11} = -\left\{{}^{n}_{11}\right\}_{,n} + \frac{\partial}{\partial x^1}\left\{{}^{n}_{1n}\right\} + \left\{{}^{l}_{1n}\right\}\left\{{}^{n}_{l1}\right\} - \left\{{}^{n}_{11}\right\}\left\{{}^{l}_{nl}\right\}$$

$$= -\left\{{}^{1}_{11}\right\}_{,1} - \left\{{}^{4}_{11}\right\}_{,4} + \left(\frac{2M'}{M}\right)' + \left\{{}^{1}_{11}\right\}\left\{{}^{1}_{11}\right\} + \left\{{}^{4}_{11}\right\}\left\{{}^{1}_{41}\right\} + \left\{{}^{2}_{12}\right\}\left\{{}^{2}_{21}\right\}$$

$$+ \left\{{}^{3}_{13}\right\}\left\{{}^{3}_{31}\right\} + \left\{{}^{4}_{14}\right\}\left\{{}^{4}_{41}\right\} - \left\{{}^{1}_{11}\right\}\left\{{}^{l}_{1l}\right\} - \left\{{}^{4}_{11}\right\}\left\{{}^{l}_{4l}\right\}$$

$$= -\left(\frac{\dot z}{2} + \frac{z}{2}\frac{\dot T}{T}\right)' - \left(-\frac{z'}{2} + \frac{z\dot z}{2} + \frac{z^2}{2}\frac{\dot T}{T}\right)^{\cdot} + \frac{2M''}{M} - \frac{2M'^2}{M^2} + \left(\frac{\dot z}{2} + \frac{z}{2}\frac{\dot T}{T}\right)^2$$

$$+ \frac{M'^2}{M^2} + \frac{M'^2}{M^2} + \left(-\frac{\dot z}{2} - \frac{z}{2}\frac{\dot T}{T}\right)^2 - \left(\frac{\dot z}{2} + \frac{z}{2}\frac{\dot T}{T}\right)\frac{2M'}{M}$$

$$- \left(-\frac{z'}{2} + \frac{z\dot z}{2} + \frac{z^2}{2}\frac{\dot T}{T}\right)\left(\frac{2\dot T}{T} + 2\frac{\dot M}{M}\right)$$

$$= -\frac{\dot z'}{2} - \frac{z'}{2}\frac{\dot T}{T} + \frac{\dot z'}{2} - \frac{\dot z^2}{2} - \frac{z}{2}\ddot z - z\dot z\frac{\dot T}{T} - \frac{z^2}{2}\left(\frac{\ddot T}{T} - \frac{\dot T^2}{T^2}\right) - \frac{2}{R^2} - \frac{2}{R^2}\cot^2\frac{r}{R}$$

$$+ \frac{\dot z^2}{4} + \frac{z\dot z}{2}\frac{\dot T}{T} + \frac{z^2}{4}\frac{\dot T^2}{T^2} + \frac{2}{R^2}\cot^2\frac{r}{R} - \frac{1}{R}\cot\frac{r}{R}\left(\dot z + z\frac{\dot T}{T}\right)$$

$$+ \frac{z'^2}{4} + \frac{z\dot z}{2}\frac{\dot T}{T} + \frac{z^2}{4}\frac{\dot T^2}{T^2} - \left(\frac{\dot T}{T} - \frac{1}{R}\cot\frac{r}{R}\right)\left(-z' + z\dot z + z^2\frac{\dot T}{T}\right)$$

$$= -\frac{z}{2}\ddot z + \frac{\dot z^2}{2} + z'\left(-\frac{1}{2}\frac{\dot T}{T} + \frac{\dot T}{T} - \frac{1}{R}\cot\frac{r}{R}\right) + z\dot z\left(-\frac{\dot T}{T} + \frac{1}{2}\frac{\dot T}{T} + \frac{1}{2}\frac{\dot T}{T} - \frac{\dot T}{T} + \frac{1}{R}\cot\frac{r}{R}\right)$$

$$+ z^2\left(-\frac{1}{2}\frac{\ddot T}{T} + \frac{1}{2}\frac{\dot T^2}{T^2} + \frac{1}{4}\frac{\dot T^2}{T^2} + \frac{1}{4}\frac{\dot T^2}{T^2} - \frac{\dot T^2}{T^2} + \frac{1}{R}\cot\frac{r}{R}\frac{\dot T}{T}\right) - \frac{2}{R^2}$$

$$- \frac{1}{R}\cot\frac{r}{R}z' - z - z\frac{1}{R}\cot\frac{r}{R}\frac{\dot T}{T}$$

$$= -\frac{z\ddot z}{2} + z\dot z\left(-\frac{\dot T}{T} + \frac{1}{R}\cot\frac{r}{R}\right) + z^2\left(-\frac{1}{2}\frac{\ddot T}{T} + \frac{1}{R}\cot\frac{r}{R}\frac{\dot T}{T}\right) + z'\left(\frac{1}{2}\frac{\dot T}{T} - \frac{1}{R}\cot\frac{r}{R}\right)$$

$$- \dot z\frac{1}{R}\cot\frac{r}{R} - \frac{z}{R}\cot\frac{r}{R}\frac{\dot T}{T} - \frac{2}{R^2}$$

(42) But from p (29)

$$R_{41} = \frac{\ddot{z}}{2} + \dot{z}\left(\frac{\dot{T}}{T} + \frac{\dot{M}}{M}\right) + z\left(\frac{\ddot{T}}{2T} + \frac{\dot{T}}{T}\frac{\dot{M}}{M}\right) + \frac{2\dot{M}'}{M} + \frac{M'}{M}\frac{\dot{T}}{T}$$

$$= \frac{\ddot{z}}{2} + \dot{z}\left(\frac{\dot{T}}{T} - \frac{1}{R}\cot\frac{z}{R}\right) + z\left(\frac{\ddot{T}}{2T} - \frac{1}{R}\cot\frac{z}{R}\frac{\dot{T}}{T}\right) + \frac{2}{R^2} + \frac{1}{R}\cot\frac{z}{R}\frac{\dot{T}}{T}$$

$$\therefore +zR_{41} = +\frac{z\ddot{z}}{2} + z\dot{z}\left(\frac{\dot{T}}{T} - \frac{1}{R}\cot\frac{z}{R}\right) + z^2\left(\frac{\ddot{T}}{2T} - \frac{1}{R}\cot\frac{z}{R}\frac{\dot{T}}{T}\right)$$

$$+\frac{2z}{R^2} + \frac{z}{R}\cot\frac{z}{R}\frac{\dot{T}}{T}$$

But

$$R_{11} = -\frac{z\ddot{z}}{2} + z\dot{z}\left(-\frac{\dot{T}}{T} + \frac{1}{R}\cot\frac{z}{R}\right) + z^2\left(-\frac{\ddot{T}}{2T} + \frac{1}{R}\cot\frac{z}{R}\frac{\dot{T}}{T}\right)$$

$$+z'\left(\frac{1}{2}\frac{\dot{T}}{T}\right) + \dot{z}'\left(\frac{1}{2}\frac{\dot{T}}{T} - \frac{1}{R}\cot\frac{z}{R}\right) - \dot{z}\frac{1}{R}\cot\frac{z}{R} - \frac{z}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} - \frac{2}{R^2}$$

Add up

$$\therefore R_{11} + zR_{41} = \dot{z}'\left(\frac{1}{2}\frac{\dot{T}}{T} - \frac{1}{R}\cot\frac{z}{R}\right) - \dot{z}\frac{1}{R}\cot\frac{z}{R} + \frac{2z}{R^2} - \frac{2}{R^2}$$

On the R.H.S. but $z = 1 + \frac{z}{T}$ $\therefore z' = \frac{z}{T}'$

$$\dot{z} = \frac{\dot{z}}{T} - \frac{z}{T}\frac{\dot{T}}{T}$$

$$\therefore R_{11} + zR_{41} = \frac{z}{T}'\left(\frac{1}{2}\frac{\dot{T}}{T} - \frac{1}{R}\cot\frac{z}{R}\right) - \frac{1}{R}\cot\frac{z}{R}\left(\frac{\dot{z}}{T} - \frac{z}{T}\frac{\dot{T}}{T}\right)$$

$$+2\left(1+\frac{z}{T}\right)\frac{1}{R^2} - \frac{2}{R^2}.$$

$$\therefore \boxed{R_{11} + zR_{41} = \frac{z}{T}'\left(\frac{1}{2}\frac{\dot{T}}{T} - \frac{1}{R}\cot\frac{z}{R}\right) - \frac{1}{R}\cot\frac{z}{R}\frac{\dot{z}}{T} + \frac{z}{T}\left(\frac{2}{R^2} + \frac{1}{R}\cot\frac{z}{R}\frac{\dot{T}}{T}\right)}$$

This settles the geometrical aspect. Let us now take the physical aspect through Einstein's Equations

$$R_{ik} = -8\pi\left[(\phi+p)v_i v_k - \frac{(p-\phi)}{2} g_{ik} + \sigma w_i w_k\right] + \Lambda g_{ik}$$

Let $v_i = (v_1, 0, 0, v_4)$ $w_i = (w_1, 0, 0, w_4)$

$g^{ik} v_i v_k = 1 \quad \therefore \frac{1}{T} v_1 v_4 + \frac{1}{T} v_4 v_1 + \frac{2}{T}(v_4)^2 = 1 \quad \therefore$ $\boxed{2v_1 v_4 + 2(v_4)^2 = T}$

$g^{ik} w_i w_k = 0 \quad \therefore \frac{2}{T} w_1 w_4 + \frac{2}{T}(w_4)^2 = 0 \qquad 2w_1 w_4 + 2(w_4)^2 = 0$

$\therefore$ either $w_4 = 0$ or $2w_1 + 2w_4 = 0$

And

$g^{ik} v_i w_k = 1 \quad \therefore \frac{1}{T} v_1 w_4 + \frac{1}{1} v_4 w_1 + \frac{2}{T} v_4 w_4 = 1$

Case A: $w_4 = 0 \quad \frac{1}{T} v_4 w_1 = 1 \quad \therefore w_1 = \frac{T}{v_4}$

$\therefore$ $\boxed{\text{Case A: } w_4 = 0,\ w_1 = \frac{T}{v_4} \qquad 2v_1 v_4 + 2(v_4)^2 = T}$

Case B: $2w_1 + 2w_4 = 0 \quad \therefore v_1 w_4 + v_4(w_1 + 2w_4) = T$

$\therefore v_1 w_4 - v_4 w_1 = T$

But $w_1 = -\frac{2}{2} w_4 \quad \therefore v_1 w_4 + \frac{2}{2} v_4 w_4 = T$

$\therefore w_4(2v_1 + 2v_4) = 2T$

But $v_4(2v_1 + 2v_4) = T$

On division $\frac{w_4}{v_4} = 2 \qquad w_4 = 2v_4$

$2w_1 = -2w_4 = -2(2v_4)$ or $w_1 = -2v_4$

Case B: $w_1 = -2v_4 \quad w_4 = 2v_4 \quad 2v_1v_4 + 2(v_4)^2 = T$

Take case A $\quad w_4 = 0 \quad w_1 = \frac{T}{v_4} \quad 2v_1v_4 + 2(v_4)^2 = T$

$$R_{11} = -8\pi\left[(\phi+\rho)(v_1)^2 - \frac{(\rho-\phi)}{2}(-2T) + \sigma(w_1)^2\right] + \Lambda(-2T)$$

$$R_{22} = -8\pi\left[-\frac{(\rho-\phi)}{2}(-TM^2)\right] + \Lambda(-TM^2)$$

$$R_{44} = -8\pi\left[(\phi+\rho)(v_4)^2\right]$$

$$R_{41} = -8\pi\left[(\phi+\rho)v_1v_4 - \frac{(\rho-\phi)}{2}T\right] + \Lambda T$$

$$2R_{44} + 2R_{41} = -8\pi\left[(\phi+\rho)(2(v_4)^2 + 2v_1v_4) - (\rho-\phi)T\right] + 2\Lambda T$$

$$= -8\pi\left[(\phi+\rho)T - (\rho-\phi)T\right] + 2\Lambda T$$

$$= -8\pi\left[2\phi T\right] + 2\Lambda T$$

$$\therefore \frac{2}{2}R_{44} + R_{41} = T(-8\pi\phi + \Lambda)$$

$$\therefore 8\pi\phi = \Lambda - \frac{1}{T}\left(\frac{2}{2}R_{44} + R_{41}\right)$$

Next $\quad \frac{R_{22}}{TM^2} = -8\pi\frac{(\rho-\phi)}{2} - \Lambda \quad \therefore \frac{-2R_{22}}{TM^2} = 8\pi(\rho-\phi) + 2\Lambda$

$$-R_2 - \frac{2R_{22}}{TM^2} + 8\pi\phi - 2\Lambda = 8\pi\rho$$

$8\pi\rho = 8\pi\phi - \frac{2R_{22}}{TM^2} - 2\Lambda$ Next $(v_4)^2 = \frac{R_{44}}{-8\pi(\phi+\rho)}$

So we have ϕ, ρ and v_4. $2v_1v_4 + 2(v_4)^2 = T$ will give v_1.

and lastly $R_{11} + 2R_{41} = -8\pi\left[(\phi+\rho)\{(v_1)^2 + 2v_1v_4\} + \sigma(\omega_1)^2\right]$

gives $8\pi\sigma(\omega_1)^2$ But $\omega_1 = \frac{T}{v_4}$ So we have $8\pi\sigma$.

Thus once g_{ik} are known, the ϕ all the physical parameters can be determined.

Next take case B: $2\omega_1 + 2\omega_4 = 0$ $2v_1v_4 + 2(v_4)^2 = T$

$$R_{11} = -8\pi\left[(\phi+\rho)(v_1)^2 - \frac{(\rho-\phi)}{2}(-2T) + \sigma(\omega_1)^2\right] + \Lambda(-2T)$$

$$R_{22} = -8\pi\left[-\frac{(\rho-\phi)}{2}(-TM^2)\right] + \Lambda(-TM^2)$$

$$R_{44} = -8\pi\left[(\phi+\rho)(v_4)^2 + \sigma(\omega_4)^2\right]$$

$$R_{41} = -8\pi\left[(\phi+\rho)v_1v_4 - \frac{(\rho-\phi)}{2}T + \sigma\omega_1\omega_4\right] + \Lambda T$$

$$\therefore 2R_{44} + 2R_{41} = -8\pi\left[(\phi+\rho)T - (\rho-\phi)T\right] + 2\Lambda T$$

So once again $8\pi\phi = \Lambda - \frac{1}{T}\left(\frac{2}{2}R_{44} + R_{41}\right)$

and in the same way $8\pi\rho = 8\pi\phi - \frac{2R_{22}}{TM^2} - 2\Lambda$

Next we know that ~~$4(\omega_1)^2 + 2$~~ $2\omega_1 + 2\omega_4 = 0$.

(45) $\therefore$ Take $2R_{11} + 2R_{41} = -8\pi\left[(p+\rho)(2(v_1)^2 + 2v_1v_4) - \frac{p-\rho}{2}(2ZT - ZT) + \sigma(2(\omega_1)^2 + 2\omega_1\omega_4)\right] + \Lambda(-2ZT + ZT)$

$\therefore 2R_{11} + 2R_{41} = -8\pi\left[(p+\rho)v_1(2v_1 + 2v_4) + \frac{p-\rho}{2}ZT + \sigma\omega_1(2\omega_1 + 2\omega_4)\right] - \Lambda ZT$

$$2R_{11} + 2R_{41} = -8\pi\left[(p+\rho)v_1\frac{T}{v_4} + \frac{(p-\rho)}{2}ZT\right] - \Lambda ZT$$

This gives $\frac{v_1}{v_4}$. $2v_1v_4 + 2(v_4)^2 = T$, then gives v_4

and then $R_{44} = -8\pi\left[(p+\rho)(v_4)^2 + \sigma(\omega_4)^2\right]$

gives $\sigma(\omega_4)^2$.

We now work out particular solutions. The one solution in which we are interested is an inward flowing radiation in the back-ground of expanding universe. We have taken the metric as

$$ds^2 = T\left[2dudt - Zdu^2 - R^2\sin^2\frac{Z}{R}(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

when $Z = 1$, we have the expanding universe.

and when $Z = \frac{1}{2}$, $T = 1$ we have Einstein's universe.

In the standard Einstein-Universe we have

$ds^2 = dt^2 - dr^2 - R^2 \sin^2 \frac{r}{R}\, d\Omega^2$. [Really in more familiar notations we have $ds^2 = dt^2 - R^2 dx^2 - R^2 \sin^2 x\, d\Omega^2$. We put $Rx = r$ and then get our form]. In this form we use comoving coords.

$\therefore V^\mu V_\mu = 1 \quad V^1 = 0,\ V^2 = 0,\ V^3 = 0 \quad V^4 = 1.$

Similarly for Expanding universe metric $ds^2 = T\left[dt^2 - dr^2 - R^2 \sin^2 \frac{r}{R}\, d\Omega^2\right]$ we again have co-moving coordinates i.e. $V^1 = V^2 = V^3 = 0 \quad T(V^4)^2 = 1$

Now we keep r unaltered and replace r by u

$r = u - t$

$\therefore \quad x^4 = x'^4 = t \qquad x^1 = x'^1 - x'^4 \quad$ or $\quad x'^1 = x^1 + x^4$

$x'^4 = x^4$

$\therefore V^\mu$ will change to V'^μ

$$V'^1 = \frac{\partial x'^1}{\partial x^\alpha} V^\alpha = \frac{\partial x'^1}{\partial x^1} V^1 + \frac{\partial x'^1}{\partial x^4} V^4 \qquad \therefore V'^1 = V^4$$

$$V'^4 = \frac{\partial x'^4}{\partial x^\alpha} V^\alpha = \frac{\partial x'^4}{\partial x^4} V^4 \qquad \therefore V'^4 = V^4$$

(7c) For our metric

$$ds^2 = T\left[2du\,dt - Z\,du^2 - R^2 \sin^2\frac{r}{R}\,d\sigma^2\right] \qquad r = u - t$$

$v^1 = V^4 \quad v^4 = V^4$ where V^4 corresponds to usual comoving coordinates

But $v_1 = g_{1\alpha} v^\alpha = g_{11} v^1 + g_{14} v^4 = -ZT V^4 + T V^4 = T V^4 (1-Z)$

$$v_4 = g_{4\alpha} v^\alpha = g_{41} v^1 = T V^4$$

or $v_1 = (1-Z) v_4$. This is the condition for comoving coords in our scheme of working

We take components of R_{ik} as given in case A on p (72)

$$R_{11} = -8\pi\left[(\phi + p)(1-Z)^2 (v_4)^2 + \frac{(\rho - p)}{2} ZT + \sigma (\omega_1)^2\right] - \Lambda Z T$$

$$R_{22} = -8\pi\left[\frac{(\rho - p)}{2} T M^2\right] - \Lambda T M^2$$

$$R_{44} = -8\pi\left[(\phi + p)(v_4)^2\right]$$

$$R_{41} = -8\pi\left[(\phi + p)(1-Z)(v_4)^2 - \frac{(\rho - p)}{2} T\right] + \Lambda T$$

Of course these will give $8\pi p$ and $8\pi \rho$ as before.

$$R_{11} + Z R_{41} = -8\pi\left[(\phi + p)(v_4)^2 (1-Z)(1-Z+Z) + \sigma (\omega_1)^2\right]$$

But $-8\pi(p+\rho)(\omega_4)^2 = R_{44}$

$\therefore R_{11} + 2R_{41} = (1-2)R_{44} - 8\pi\sigma(\omega_1)^2$. This is the equation for σ.

Next $R_{41} + \frac{R_{22}}{M^2} = -8\pi(p+\rho)(\omega_4)^2(1-2) = R_{44}(1-2)$

$\therefore$ The condition of comoving ~~coords~~ coords in standard forms leads to the differential equation for z;

$$R_{41} + \frac{R_{22}}{M^2} - (1-2)R_{44} = 0 .$$

Substituting the forms of these R_{ik}-components from p. 39 onwards we get

$$\frac{1}{T}\left(\frac{\ddot{z}}{2} - \frac{\dot{z}}{R}\cot\frac{z}{R}\right) + \frac{2}{R^2} + \frac{\ddot{T}}{2T} + \frac{\dot{z}}{T}\left(\frac{1}{R}\cot\frac{z}{R} - \frac{1}{2}\frac{\dot{T}}{T}\right) - \frac{2}{R^2} - \frac{\ddot{T}}{2T}$$

$$+ \frac{z}{T}\left(-\frac{1}{2}\frac{\ddot{T}}{T} + \frac{1}{2}\frac{\dot{T}^2}{T^2} + \frac{1}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} + \frac{1}{R^2} - \frac{1}{R^2}\cot^2\frac{z}{R}\right)$$

$$- \left(-\frac{z}{T}\right)\left(\frac{\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} - \frac{2}{R^2}\right) = 0$$

$$\therefore \frac{\ddot{z}}{2} - \frac{1}{2}\frac{\dot{z}}{T}\frac{\dot{T}}{T} + \frac{z}{2}\left(\frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2} + \frac{1}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} - \frac{1}{R^2}\operatorname{cosec}^2\frac{z}{R}\right) = 0$$

This is the equation to determine z and hence $Z = 1 + \frac{z}{T}$.

We ~~shall~~ shall try to solve the equation in 3 stages

Stage (1) $T = 1$. So that the back-ground is Einstein's universe

Stage (2) When $R \to \infty$ so that the back-ground is Einstein-de Sitter universe.

Stage (3) General case $T \neq 1$ $R \to \infty$.

Stage (1) $T = 1$. Our equation becomes

$\frac{\ddot{z}}{2} - \frac{z}{R^2}\operatorname{cosec}^2\frac{z}{R} = 0$. It can be verified that $z = \cot\frac{z}{R}$ is a solution. Just check! $z = \cot\frac{z}{R}$ $\dot{z} = -\operatorname{cosec}^2\frac{z}{R}\left(-\frac{\dot{z}}{R}\right)$

$\dot{z} = \frac{\dot{z}}{R}\operatorname{cosec}^2\frac{z}{R}$ $\therefore \ddot{z} = \frac{\dot{z}}{R}\left(2\operatorname{cosec}\frac{z}{R}\right)\left(-\operatorname{cosec}\frac{z}{R}\cot\frac{z}{R}\right)\left(-\frac{\dot{z}}{R}\right) = \frac{2\dot{z}^2}{R^2}\operatorname{cosec}^2\frac{z}{R}\cot\frac{z}{R}$

$\therefore \frac{\ddot{z}}{2} - \frac{z}{R^2}\operatorname{cosec}^2\frac{z}{R} = \frac{\dot{z}^2}{R^2}\operatorname{cosec}^2\frac{z}{R}\cot\frac{z}{R} - \frac{1}{R^2}\cot\frac{z}{R}\operatorname{cosec}^2\frac{z}{R} = 0$

$\therefore z = \cot\frac{z}{R}$ is a solution

$\therefore$ Put $z = v\cot\frac{z}{R}$ $\dot{z} = \dot{v}\cot\frac{z}{R} + \frac{v}{R}\operatorname{cosec}^2\frac{z}{R}$

$\ddot{z} = \ddot{v}\cot\frac{z}{R} + \frac{2\dot{v}}{R}\operatorname{cosec}^2\frac{z}{R} + \frac{2v}{R^2}\operatorname{cosec}^2\frac{z}{R}\cot\frac{z}{R}$

$\therefore \frac{\ddot{z}}{2} - \frac{z}{R^2}\operatorname{cosec}^2\frac{z}{R} = 0$ gives

$\frac{\ddot{v}}{2}\cot\frac{z}{R} + \frac{\dot{v}}{R}\operatorname{cosec}^2\frac{z}{R} + \frac{v}{R^2}\operatorname{cosec}^2\frac{z}{R}\cot\frac{z}{R} - \frac{v\cot\frac{z}{R}}{R^2}\operatorname{cosec}^2\frac{z}{R} = 0$

$\therefore \frac{\ddot{v}}{2}\cot\frac{z}{R} + \frac{\dot{v}}{R}\operatorname{cosec}^2\frac{z}{R} = 0$ $\therefore \frac{\ddot{v}}{\dot{v}} + \frac{2\operatorname{cosec}^2\frac{z}{R}}{R\cot\frac{z}{R}} = 0$

$\therefore \log\dot{v} + 2\log\cot\frac{z}{R} = A(u)\log A$

$\therefore \dot{v}\cot^2\frac{z}{R} = A$ $A = A(u)$

$\therefore \dot{v} = A\tan^2\frac{r}{R} = A(\sec^2\frac{r}{R} - 1)$. Integrating

$$v = AR\tan\frac{r}{R}(-) - At + B$$

$$\therefore z = v\cot\frac{r}{R} = \cot\frac{r}{R}\left[-AR\tan\frac{r}{R} - At + B\right]$$

Replace A by $\frac{A}{R}$, B by $\frac{B}{R}$

$$\therefore z = -A - A\frac{t}{R}\cot\frac{r}{R} + \frac{B}{R}\cot\frac{r}{R}.$$

$$\therefore Z = 1 + \frac{B}{R}\cot\frac{r}{R} - A\left(1 + \frac{t}{R}\cot\frac{r}{R}\right)$$ [See p. 75.

For this particular solution of stage (1) we find p, ρ, and σ.

$$z = \frac{B}{R}\cot\frac{r}{R} - A - \frac{At}{R}\cot\frac{r}{R} \qquad A = A(u) \quad B = B(u)$$

$$\therefore \dot{z} = +\frac{B}{R^2}\operatorname{cosec}^2\frac{r}{R} - \frac{A}{R}\cot\frac{r}{R} - \frac{At}{R^2}\operatorname{cosec}^2\frac{r}{R}$$

$$z' = -\frac{B}{R^2}\operatorname{cosec}^2\frac{r}{R} \qquad \vee (+)\frac{At}{R^2}\operatorname{cosec}^2\frac{r}{R} \quad + \frac{B'}{R}\cot\frac{r}{R} - A' - \frac{A't}{R}\cot\frac{r}{R}$$

$$\therefore \dot{z} + z' = -\frac{A}{R}\cot\frac{r}{R} + \frac{B'}{R}\cot\frac{r}{R} - A' - \frac{A't}{R}\cot\frac{r}{R}.$$

Now we are ready to find $\frac{R_{22}}{M^2}$ and $R_{11} + 2R_{41}$. From p. 40 we write $\frac{R_{22}}{M^2}$ setting $T=1$ and the above values of z and $\dot{z}$

$$\frac{R_{22}}{M^2} = \frac{1}{R}\cot\frac{r}{R}\left[\frac{B}{R^2}\operatorname{cosec}^2\frac{r}{R} - \frac{A}{R}\cot\frac{r}{R} - \frac{At}{R^2}\operatorname{cosec}^2\frac{r}{R}\right] \cancel{+ \frac{1}{R^2} - \frac{1}{R^2}\cot^2} - \frac{2}{R^2}$$
$$+ \left(\frac{1}{R^2} - \frac{1}{R^2}\cot^2\frac{r}{R}\right)\left(\frac{B}{R}\cot\frac{r}{R} - A - \frac{At}{R}\cot\frac{r}{R}\right)$$

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$$\frac{R_{22}}{M^2} = -\frac{2}{R^2} + \frac{B}{R^3}\cot\frac{z}{R}\left(\operatorname{cosec}^2\frac{z}{R} + 1 - \cot^2\frac{z}{R}\right) - \frac{A}{R^2}\left(\cot^2\frac{z}{R} + 1 - \cot^2\frac{z}{R}\right)$$

$$-\frac{At}{R^3}\left(-\frac{At}{R^3}\cot\frac{z}{R}\left(\operatorname{cosec}^2\frac{z}{R} + 1 - \cot^2\frac{z}{R}\right)\right)$$

$$\frac{R_{22}}{M^2} = -\frac{2}{R^2} + \frac{2B}{R^3}\cot\frac{z}{R} - \frac{A}{R^2} - \frac{2At}{R^3}\cot\frac{z}{R}$$

Now $8\pi\rho = \Lambda - \left(\frac{7}{2}R_{44} + R_{41}\right)$ But $R_{41} + \frac{R_{22}}{M^2} = (1-7)R_{44}$

$$\therefore 8\pi\rho = \Lambda - \left[\frac{7}{2}R_{44} + (1-7)R_{44} - \frac{R_{22}}{M^2}\right]$$

$$= \Lambda - \left[\left(1-\frac{7}{2}\right)R_{44} - \frac{R_{22}}{M^2}\right] = \Lambda - \left(1 - \frac{(1+8)}{2}\right)R_{44} + \frac{R_{22}}{M^2}$$

$$= \Lambda - \left(\frac{1}{2} - \frac{8}{2}\right)\left(-\frac{2}{R^2}\right) + B - \frac{2}{R^2} + \frac{2B}{R^3}\cot\frac{z}{R} - \frac{A}{R^2} - \frac{2At}{R^3}\cot\frac{z}{R}$$

$$= \Lambda + \frac{1}{R^2} - \frac{1}{R^2}\left[\frac{B}{R}\cot\frac{z}{R} - A - \frac{At}{R}\cot\frac{z}{R}\right] - \frac{2}{R^2} + \frac{2B}{R^3}\cot\frac{z}{R} - \frac{A}{R^2} - \frac{2At}{R^3}\cot\frac{z}{R}$$

$$\boxed{8\pi\rho = \Lambda - \frac{1}{R^2} + \frac{B}{R^3}\cot\frac{z}{R} - \frac{At}{R^3}\cot\frac{z}{R}.}$$ ~~This agrees with Prem~~

Next $8\pi p = 8\pi\rho - \frac{2R_{22}}{M^2} - 2\Lambda = \Lambda - \frac{1}{R^2} + \frac{B}{R^3}\cot\frac{z}{R} - \frac{At}{R^3}\cot\frac{z}{R}$

$$-2\Lambda + \frac{4}{R^2} - \frac{4B}{R^3}\cot\frac{z}{R} + \frac{4At}{R^3}\cot\frac{z}{R} + \frac{2A}{R^2}$$

$$\boxed{\therefore 8\pi p = -\Lambda + \frac{3}{R^2} - \frac{3B}{R^3}\cot\frac{z}{R} + \frac{3At}{R^3}\cot\frac{z}{R} + \frac{2A}{R^2}}$$

Next $8\pi\sigma(w_1)^2 = (1-7)R_{44} - (R_{11} + 7R_{41})$

$$= \left(+\frac{1+8}{8}\right)1 - \frac{1}{8}R_{44} - (R_{11} + 7R_{41})$$

$$8\pi\sigma(\omega_1)^2 = -\tfrac{1}{3}\left(-\frac{2}{R^2}\right) - \left[-\frac{1}{R}\cot\frac{z}{R}\left(\tfrac{1}{3}\dot{\gamma}+\dot{\gamma}\right) + \frac{2\dot{\gamma}}{R^2}\right]$$

$$= \frac{2\dot{\gamma}}{R^2} + \frac{1}{R}\cot\frac{z}{R}\left[\frac{B'}{R}\cot\frac{z}{R} - \frac{A't}{R}\cot\frac{z}{R} - A - \frac{A}{R}\cot\frac{z}{R}\right] - \frac{2\dot{\gamma}}{R^2}$$

$$\therefore \boxed{\frac{B'}{R^2}\cot^2\frac{z}{R} - \frac{A't}{R^2}\cot^2\frac{z}{R} - \frac{A'}{R}\cot\frac{z}{R} - \frac{A}{R^2}\cot^2\frac{z}{R} = 8\pi\sigma(\omega_1)^2}$$

It is clear that if σ is to vanish we must have $A=0$ $B=$ const.

Setting $A=0$ $B = 2m(u)$ $\qquad z = 1 + \frac{2m(u)}{R}\cot\frac{z}{R}$

gives a radiating star in the back-ground of Einstein's Universe

$$8\pi\rho = \Lambda - \frac{1}{R^2}\left(1 - \frac{2m}{R}\cot\frac{z}{R}\right)$$

$$8\pi p = \frac{3}{R^2}\left(1 - \frac{2m}{R}\cot\frac{z}{R}\right) - \Lambda$$

} These agree with Pramana results.

$$\text{and } 8\pi\sigma(\omega_1)^2 = \frac{2\,dm/du}{R^2}\cot^2\frac{z}{R}.$$

It will be seen that $\frac{dm}{du}$ is +ve, radiation flows inwards.

We now go over to stage (2)

Stage (2) Take $R\to\infty$ so we Einstein-deSitter Universe in the back-ground.

The equation to be solved is

$$\frac{\ddot{\gamma}}{2} - \frac{1}{2}\dot{\gamma}\frac{\dot{T}}{T} + \gamma\left(\frac{1}{2}\frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2}\right) = 0$$

No. as $R\to\infty$ $\frac{1}{R}\cot\frac{z}{R} \to \frac{1}{z}$

$\frac{1}{R^2}\operatorname{cosec}^2\frac{z}{R} \to \frac{1}{z^2}$. They do not tend to zero

~~Try $\gamma = AT^m + BT^n$~~

$\dot{\gamma} = Am\,T^{m-1}\dot{T} + Bn\,T^{n-1}\dot{T}$ $\quad \ddot{\gamma} = Am(m-1)T^{m-2}\dot{T}^2 + Am\,T^{m-1}\ddot{T}$

$+ Bn(n-1)T^{n-2}\dot{T}^2 + Bn\,T^{n-1}\ddot{T}$

$\therefore Am(m-1)T^m\frac{\dot{T}^2}{T^2} + Am\,T^m\frac{\ddot{T}}{T} - \frac{1}{2}Am\,T^m\frac{\dot{T}^2}{T^2} + \frac{1}{2}AT^m\frac{\ddot{T}}{T} - 2AT^m\frac{\dot{T}^2}{T^2}$

$+ Bn(n-1)T^n\frac{\dot{T}^2}{T^2} + Bn\,T^n\frac{\ddot{T}}{T} - Bn\,T^n\frac{\dot{T}^2}{T^2} + BT^n\frac{\ddot{T}}{T} - 2BT^n\frac{\dot{T}^2}{T^2} = 0$

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$$\frac{\dot{T}^2}{T^2}\left[A\{m(m-1)-m-2\}T^m + B\{n(n-1)-n-2\}T^n + \frac{\ddot{T}}{T}\left[\{(m+1)AT^m + (n+1)BT^n\}\right]\right] = 0$$

$\therefore A(m^2-2m-2) + B(n^2-2n-2) = 0$ and $A(m+1) + B(n+1) = 0$

For consistancy we get

This does not work

$$\frac{A}{B} = -\frac{n^2-2n-2}{m^2-2m-2} = -\frac{n+1}{m+1}$$

$\therefore$ we have

or $[n^2-2(n+1)](m+1) = [m^2-2(m+1)](n+1)$

or $n^2m + n^2 - 2(n+1)(m+1) = m^2n + m^2 - 2(m+1)(n+1)$

or $n^2m + n^2 = m^2n + m^2$ or $n^2m - m^2n + n^2 - m^2 = 0$

or $nm(n-m) + (n-m)(n+m) = 0$

$\therefore (n-m)(nm+n+m) = 0$ But $n = m \Rightarrow z = 0$

$\therefore nm + n + m = 0$ $\therefore m(n+1) + n = 0$ or $m = \frac{-n}{n+1}$

Also $\frac{A}{B} = -\frac{n+1}{m+1} = -\frac{(n+1)}{\frac{-n}{n+1}+1} = \frac{-(n+1)^2}{-n+n+1}$

$\therefore \frac{A}{B} = -(n+1)^2$ $\therefore A = -B(n+1)^2$

$\therefore z = AT^m + BT^n$ gives $z = -B(n+1)^2 T^{\frac{-n}{n+1}} + BT^n$

$= B\left[T^n - (n+1)^2 T^{\frac{-n}{n+1}}\right]$

is a solution of our equation

We pick up Stage (2) Equation to be solved as

$$\ddot{\delta} - \frac{3}{2}\frac{\dot{T}}{T} + \frac{3}{2}\left(\frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2}\right) = 0$$

$$\ddot{\delta} - \frac{3}{2}\frac{\dot{T}}{T} + \frac{3}{2}\left(\frac{\dddot{T}}{\dot{T}} - \frac{\ddot{T}^2}{T^2} + \frac{1}{R}\frac{\dot{T}}{T} - \frac{1}{R^2}\right) = 0$$

or I $\quad \ddot{\delta} - \frac{3}{2}\frac{\dot{T}}{T} + \frac{3}{2}\left(\frac{\ddot{T}}{T} - \frac{2\dot{T}^2}{T^2}\right) = 0$

Let us change from t to τ defined by $T\,dt = d\tau$. Then $\dot{\lambda} = \frac{d\lambda}{d\tau}\cdot\frac{d\tau}{dt} = T\frac{d\lambda}{d\tau} = T\lambda_1$

$$\ddot{\lambda} = \dot{T}\lambda_1 + T^2\lambda_{11} = TT_1\lambda_1 + T^2\lambda_{11}.$$

Thus our equation becomes $TT_1\delta_1 + T^2\delta_{11} - T\delta_1 TT_1$

$$+\frac{3}{2}\delta\left[\frac{TT_1^2}{T} + \frac{T^2T_{11}}{T} - 2\frac{T^2T_1^2}{T^2}\right] = 0$$

$$\therefore T^2\delta_{11} + \frac{3}{2}T^2\left[\frac{T_{11}}{T} - \frac{T_1^2}{T^2}\right] = 0$$

or II $\quad \delta_{11} + \frac{3}{2}\left[\frac{T_{11}}{T} - \frac{T_1^2}{T^2}\right] = 0 \qquad T = T(\tau) \qquad T\,dt = d\tau$

It can be verified that keeping $T = T(\tau)$ undetermined, it is not possible to solve this equation in known functions. We ∴ try to solve I or II for a particular form of $T(t)$ or $T(\tau)$. We choose this function which will make $\rho = 0$ when $z = 1$ or make $\rho = 3p$ when $z = 1$, i.e. we consider inward-flowing radiation in the undisturbed Einstein-DeSitter Universe with either $p = 0$ or $\omega = p = 3p$.

We shall solve I so that we revert to the time coordinate t.

Assume $p=0$ in undisturbed universe with $R\to\infty$ and $\lambda=0$

The last equ. on p. 26 gives undisturbed $p=0$ as

$$2\frac{\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} + \frac{2}{R^2} = 0 \qquad R\to\infty$$

$$\therefore 2\frac{\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} = 0 \quad \text{Put } T = y^n \quad \frac{\dot{T}}{T} = n\frac{\dot{y}}{y},\ \frac{\ddot{T}}{T} = n\frac{\ddot{y}}{y} - n\frac{\dot{y}^2}{y^2} + \frac{n^2\dot{y}^2}{y^2}$$

$$\therefore 2\left[n\frac{\ddot{y}}{y} + (n^2-n)\frac{\dot{y}^2}{y^2}\right] - \frac{3}{2}n^2\frac{\dot{y}^2}{y^2} = 0 \quad \text{Put } 2n^2 - 2n - \frac{3}{2}n^2 = 0$$

$$\propto \frac{1}{2}n^2 - 2n = 0$$

$$\therefore n = 0,\ n = 4$$

$$\therefore T = y^4$$

and the equ. $p=0$ becomes $\ddot{y}=0 \quad \therefore y = At+B$

$$\therefore T = (At+B)^4.$$

$$\frac{\dot{T}}{T} = \frac{4A}{(At+B)} \qquad \frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2} = -\frac{4A^2}{(At+B)^2}.$$

Eq. I now becomes $\ddot{z} - \dot{z}\frac{4A}{(At+B)} + z\left[\frac{-4A^2}{(At+B)^2} + \frac{16A^2}{(At+B)^2} - \frac{2\times 16A^2}{(At+B)^2}\right] = 0$

Put $At + B = y$ $\quad \dot{z} = \frac{dz}{dy} A \quad \ddot{z} = \frac{d^2 z}{dy^2} A^2$

$\therefore$ Our equation becomes

$$\frac{d^2 z}{dy^2} - 4 \frac{dz}{dy} \frac{1}{y} + z \left(\frac{-20}{y^2} \right) = 0$$

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We do not go over to stage 2. But pick up the radiating particle in de Sitter Universe.

For ~~de~~ undisturbed de Sitter Universe $p = 0, \rho = 0$

$\therefore p + \rho = 0 \quad \therefore R_{44} = 0$ ~~Again~~ $\therefore \frac{\ddot{T}}{T} - \frac{3}{2} \frac{\dot{T}^2}{T^2} - \frac{2}{R^2} = 0$

$\therefore T = A \sec^2 \frac{t+b}{R}$. Also $p = 0 \Rightarrow \Lambda - \frac{1}{T} \left(\frac{2}{2} R_{44} + R_{41} \right) = 0$

But $R_{44} = 0 \quad \therefore \Lambda - \frac{1}{T} (R_{41}) = 0$. In the undisturbed universe $R_{41} = \frac{2}{R^2} + \frac{\ddot{T}}{2T} \quad T = A \sec^2 \frac{t+b}{R} \quad \dot{T} = \frac{2A}{R} \sec^2 \frac{t+b}{R} \tan \frac{t+b}{R}$

$$\therefore \ddot{T} = \frac{4A}{R^2} \sec^2 \frac{t+b}{R} \tan^2 \frac{t+b}{R} + \frac{2A}{R^2} \sec^4 \frac{t+b}{R}$$

$$\therefore \Lambda - \frac{1}{T} \left[\frac{2}{R^2} + \frac{2}{R^2} \tan^2 \frac{t+b}{R} + \frac{1}{R^2} \sec^2 \frac{t+b}{R} \right] = 0$$

$$\therefore \Lambda - \frac{1}{A \sec^2 \frac{t+b}{R}} \left[\frac{3}{R^2} \sec^2 \frac{t+b}{R} \right] = 0 \quad \therefore \Lambda = \frac{3}{AR^2} \quad A = \frac{3}{\Lambda R^2}$$

$\therefore$ For de Sitter Universe $T = \frac{3}{\Lambda R^2} \sec^2 \frac{(t+b)}{R}$.

(45)

$\therefore$ For ~~ra~~ inward flowing radiation in de Sitter universe we have to solve the equation

$$\frac{\ddot{z}}{2} - \frac{1}{2}\dot{z}\frac{\dot{T}}{T} + z\left(\frac{1}{2}\frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2} + \frac{1}{R}\cot\frac{r}{R}\frac{\dot{T}}{T} - \frac{1}{R^2}\operatorname{cosec}^2\frac{r}{R}\right) = 0$$

Here $T = \frac{3}{\Lambda R^2}\sec^2\frac{(t+b)}{R}$ $\quad\therefore \frac{\dot{T}}{T} = \frac{2}{R}\tan\frac{t+b}{R}$. $\quad \frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2} = \frac{2}{R^2}\sec^2\frac{t+b}{R}$

$$\therefore \frac{\ddot{z}}{2} - \frac{\dot{z}}{2}\cdot\frac{2}{R}\tan\frac{t+b}{R} + z\left[\frac{1}{2}\cdot\frac{2}{R^2}\sec^2\frac{t+b}{R} - \frac{1}{2}\frac{4}{R^2}\tan^2\frac{t+b}{R} + \frac{2}{R^2}\cot\frac{r}{R}\tan\frac{t+b}{R} - \frac{1}{R^2}\left(1+\cot^2\frac{r}{R}\right)\right] = 0$$

$$\therefore \ddot{z} - \dot{z}\frac{2}{R}\tan\frac{t+b}{R} + 2z\left[\frac{1}{R^2}\tan^2\frac{t+b}{R} - \frac{2}{R^2}\tan^2\frac{t+b}{R} + \frac{2}{R^2}\cot\frac{r}{R}\tan\frac{t+b}{R} - \frac{1}{R^2}\cot^2\frac{r}{R}\right] = 0$$

$$\therefore \ddot{z} - \dot{z}\frac{2}{R}\tan\frac{t+b}{R} + \frac{2z}{R^2}\left[-\tan^2\frac{t+b}{R} + 2\cot\frac{r}{R}\tan\frac{t+b}{R} - \cot^2\frac{r}{R}\right] = 0$$

$$\therefore \ddot{z} - \dot{z}\frac{2}{R}\tan\frac{t+b}{R} - \frac{2z}{R^2}\left(\tan\frac{t+b}{R} - \cot\frac{r}{R}\right)^2 = 0$$

$$\therefore \ddot{z} - \dot{z}\frac{2}{R}\tan\frac{t+b}{R} - \frac{2z}{R^2}\left[\frac{\sin\frac{t+b}{R}}{\cos\frac{t+b}{R}} - \frac{\cos\frac{u-t}{R}}{\sin\frac{u-t}{R}}\right]^2 = 0$$

$$\therefore \cancel{\ddot{z} - \dot{z}\frac{2}{R}\tan\frac{t+b}{R} - \frac{2z}{R^2}\frac{\sin\frac{t+b}{R}\sin\frac{u-t}{R} - \cos\frac{t+b}{R}\sin\frac{u-t}{R}}{\cos\frac{t+b}{R}\sin\frac{u-t}{R}}}$$

$$\therefore \ddot{z} - \dot{z}\frac{2}{R}\tan\frac{t+b}{R} - \frac{2z}{R^2}\left(\frac{\sin\frac{t+b}{R}\sin\frac{u-t}{R} - \cos\frac{t+b}{R}\cos\frac{u-t}{R}}{\cos\frac{t+b}{R}\sin\left(\frac{u-t}{R}\right)}\right)^2 = 0$$

$$\therefore \ddot{z} - \frac{2\dot{z}}{R}\tan\frac{t+b}{R} + \frac{2z}{R^2}\frac{\cos^2\left[\frac{t+b+u-t}{R}\right]}{\cos^2\frac{t+b}{R}\sin^2\frac{z}{R}} = 0$$

$$\therefore \ddot{z} - \frac{2\dot{z}}{R}\tan\frac{t+b}{R} - \frac{2z}{R^2}\cos^2\frac{u+b}{R}\sec^2\frac{t+b}{R}\operatorname{cosec}^2\frac{z}{R} = 0$$

Put $\tan\frac{t+b}{R} = y$ $\quad \dot{z} = z_y \frac{1}{R}\sec^2\frac{t+b}{R}$

$$\ddot{z} = z_{yy}\frac{1}{R^2}\sec^4\frac{t+b}{R} + \frac{2}{R^2}z_y\sec^2\frac{t+b}{R}\tan\frac{t+b}{R}.$$

∴ Our eqn. now becomes

$$\frac{z_{yy}}{R^2}\sec^4\frac{t+b}{R} + \frac{2}{R^2}z_y\sec^2\frac{t+b}{R}\tan\frac{t+b}{R} - \frac{2}{R}\tan\frac{t+b}{R}z_y\frac{1}{R}\sec^2\frac{t+b}{R}$$

$$- \frac{2z}{R^2}\cos^2\frac{(u+b)}{R}\sec^2\frac{t+b}{R}\operatorname{cosec}^2\frac{z}{R} = 0$$

$$\therefore z_{yy} - 2z\frac{\cos^2\frac{(u+b)}{R}\cos^2\frac{t+b}{R}}{\sin^2\frac{t-u}{R}} = 0$$

But $\frac{z}{R} = \frac{u-t}{R} = \frac{u-t-b+b}{R} = \frac{u+b-(t+b)}{R}$

$$\therefore z_{yy} - 2z\frac{\cos^2\frac{u+b}{R}\cos^2\left(\frac{t+b}{R}\right)}{\left[\sin\frac{u+b}{R}\cos\frac{t+b}{R} - \cos\frac{u+b}{R}\sin\frac{t+b}{R}\right]} = 0$$

$$\therefore z_{yy} - 2z\left|\left(\tan\frac{u+b}{R} - \tan\frac{t+b}{R}\right)^2\right. = 0$$

$$z_{yy} - \frac{2z}{\left(\tan\frac{u+b}{R} - y\right)^2} = 0$$

Put $\tan\frac{u+b}{R} - y = x$ $\quad z_y = -z_x$ $\quad z_{yy} = z_{xx}$

$\therefore z_{xx} - \frac{2z}{x^2} = 0$ $\quad x^2 z_{xx} - 2z = 0$ $\quad D \equiv x\frac{d}{dx}$

$$[D(D-1) - 2]z = 0 \qquad [D^2 - D - 2]z = 0$$

$$(D+1)(D-2)z = 0 \qquad \therefore z = \frac{A_1}{x} + A_2 x^2$$

$$\therefore z = \frac{A_1}{\tan\frac{u+b}{R} - y} + A_2\left[\tan\frac{u+b}{R} - y\right)^2$$

$$= \frac{A_1}{\tan\frac{u+b}{R} - \tan\frac{t+b}{R}} + A_2\left[\tan\frac{u+b}{R} - \tan\frac{t+b}{R}\right]^2$$

$$= A_1 \frac{\cos\frac{u+b}{R}\cos\frac{t+b}{R}}{\sin\frac{u+b}{R}\cos\frac{t+b}{R} - \cos\frac{u+b}{R}\sin\frac{t+b}{R}} + A_2\left[\quad\right]^2$$

$$= A_1 \frac{\cos\frac{u+b}{R}\cos\frac{t+b}{R}}{\sin\frac{u-t}{R}} + A_2 \frac{\sin^2\frac{2}{R}}{\cos^2\frac{u+b}{R}\cos^2\frac{t+b}{R}},$$

$\therefore$ Put $A_1 \cos\frac{ct+b}{R} = \frac{2m(u)}{R}$. Similarly put $A_2 \sec^2\frac{ct+b}{R} = \frac{P(u)}{R^2}$

$$\therefore \quad z = \frac{2m(u)}{R\sin\frac{r}{R}}\cos\frac{ct+b}{R} + P(u)\frac{1}{R^2}\sin^2\frac{r}{R}\sec^2\frac{ct+b}{R}$$

$$Z = 1 + \frac{z}{T} = 1 + \frac{\frac{1}{3}\Lambda R^2\cos^2\left(\frac{ct+b}{R}\right)}{3}$$

$$= 1 + \frac{\Lambda R^2}{3}\left[\frac{2m(u)}{R\sin\frac{r}{R}}\cos^3\left(\frac{ct+b}{R}\right) + \frac{P(u)}{R^2}\sin^2\frac{r}{R}\right]$$

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Now we take the case of expanding Universe. Z has to be $1 + \frac{z}{T}V$ ~~because the~~ but because of the conformal factor T V has to be $\frac{z}{T}$. Thus Z will contain t explicitly. Therefore in the orthogonal (r, t) coordinates $T_1^4 \neq 0$ and even when $\sigma = 0$, $T_4^1 \neq 0$ and so $V' \neq 0$. Thus we cannot derive the condition $v_1 = (1-Z)v_4$ of comoving coordinates (as done on p. 89, 90 above).

Let us $\therefore$ assume $\boxed{v_1 = k(1-Z)v_4}$. We know that for expanding universe case $k \neq 1$. Following treatment of p. 90. We have

(52)

$$R_{11} = -8\pi\left[(\rho+p)k^2(1-2)^2(v_4)^2 + \frac{\rho-p}{2}2T + \sigma(\omega_1)^2\right] - \Lambda 2T$$

$$R_{22} = -8\pi\left[\frac{\rho-p}{2}TM^2\right] - \Lambda TM^2$$

$$R_{44} = -8\pi\left[(\rho+p)(v_4)^2\right]$$

$$R_{41} = -8\pi\left[(\rho+p)k(1-2)(v_4)^2 - \left(\frac{\rho-p}{2}\right)T\right] + \Lambda T$$

$$\therefore R_{41} + \frac{R_{22}}{M^2} = -8\pi(\rho+p)k(1-2)(v_4)^2 = k(1-2)R_{44}.$$

$\therefore$ Our condition $v_1 = k(1-2)v_4$ leads to the equation

$$R_{41} + \frac{R_{22}}{M^2} = k(1-2)R_{44}$$

$$\therefore \frac{1}{T}\left[\frac{\ddot{\delta}}{2} - \frac{\dot{\delta}}{R}\cot\frac{z}{R}\right] + \frac{2}{R^2} + \frac{\ddot{T}}{2T} + \frac{\dot{\delta}}{T}\left(\frac{1}{R}\cot\frac{z}{R} - \frac{1}{2}\frac{\dot{T}}{T}\right) + \frac{\delta}{T}\left(-\frac{1}{2}\frac{\ddot{T}}{T} + \frac{1}{2}\frac{\dot{T}^2}{T^2}\right.$$

$$\left. + \frac{1}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} + \frac{1}{R^2} - \frac{1}{R^2}\cot^2\frac{z}{R}\right) - \frac{2}{R^2} - \frac{\ddot{T}}{2T}$$

$$-k\left(-\frac{\delta}{T}\right)\left(\frac{\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} - \frac{2}{R^2}\right) = 0$$

$$\therefore \frac{\ddot{\delta}}{2} + \dot{\delta}\left(-\frac{1}{R}\cot\frac{z}{R} + \frac{1}{R}\cot\frac{z}{R} - \frac{1}{2}\frac{\dot{T}}{T}\right) + \delta\left[\frac{\ddot{T}}{T}\left(-\frac{1}{2}+k\right) + \frac{\dot{T}^2}{T^2}\left(\frac{1}{2} - \frac{3k}{2}\right) + \frac{1}{R^2}(1-2k)\right.$$

$$\left. + \frac{1}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} - \frac{1}{R^2}\cot^2\frac{z}{R}\right] = 0$$

$$\therefore \frac{\ddot{\delta}}{2} - \frac{\dot{\delta}}{2}\frac{\dot{T}}{T} + \delta\left[\frac{2k-1}{2}\frac{\ddot{T}}{T} - \frac{(3k-1)}{2}\frac{\dot{T}^2}{T^2} - \frac{(2k-1)}{R^2} + \frac{1}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} - \frac{1}{R^2}\cot^2\frac{z}{R}\right] = 0$$

$$\therefore \ddot{\zeta} - \dot{\zeta}\frac{\dot{T}}{T} + \zeta\left[(2k-1)\frac{\ddot{T}}{T} - (3k-1)\frac{\dot{T}^2}{T^2} - \frac{2(2k-1)}{R^2} + \frac{2}{R}\cot\frac{\zeta}{R}\frac{\dot{T}}{T} - \frac{1}{R^2}\cot^2\frac{\zeta}{R}\right] = 0$$

Set $\zeta = Y$ ~~$\sqrt{T}\,\mathrm{Dn}\frac{\zeta}{R}$~~ $\qquad \zeta = Y\,T^n \operatorname{cosec}^m\frac{\zeta}{R}$

~~$\therefore \zeta = \sqrt{T}\operatorname{cosec}\frac{\zeta}{R}$~~ $\qquad \therefore \frac{\dot{\zeta}}{\zeta} = \frac{\dot{Y}}{Y} + n\frac{\dot{T}}{T} + \frac{m}{R}\cot\frac{\zeta}{R}$

The above equation transforms to

$$\frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} + n\frac{\ddot{T}}{T} - n\frac{\dot{T}^2}{T^2} + \frac{m}{R^2}\operatorname{cosec}^2\frac{\zeta}{R} + \frac{\dot{Y}^2}{Y^2} + n^2\frac{\dot{T}^2}{T^2} + \frac{m^2}{R^2}\cot^2\frac{\zeta}{R}$$

$$+ \frac{2\dot{Y}}{Y}\left(n\frac{\dot{T}}{T} + \frac{m}{R}\cot\frac{\zeta}{R}\right) + \frac{2nm}{R}\cot\frac{\zeta}{R}\frac{\dot{T}}{T} - \frac{\dot{T}}{T}\left[\frac{\dot{Y}}{Y} + n\frac{\dot{T}}{T} + \frac{m}{R}\cot\frac{\zeta}{R}\right]$$

$$+ (2k-1)\frac{\ddot{T}}{T} - (3k-1)\frac{\dot{T}^2}{T^2} - \frac{2(2k-1)}{R^2} + \frac{2}{R}\cot\frac{\zeta}{R}\frac{\dot{T}}{T} - \frac{2}{R^2}\cot^2\frac{\zeta}{R} = 0$$

$$\therefore \frac{\ddot{Y}}{Y} + \frac{\dot{Y}}{Y}\left(2n\frac{\dot{T}}{T} + \frac{2m}{R}\cot\frac{\zeta}{R} - \frac{\dot{T}}{T}\right) + \frac{\ddot{T}}{T}(n+2k-1) + \frac{\dot{T}^2}{T^2}\left(-n+n^2-n-(3k-1)\right)$$

$$+ \frac{1}{R^2}(m-2(2k-1)) \qquad + \cot^2\frac{\zeta}{R}\left(\frac{m+m^2-2}{R^2}\right)$$

$$+ \frac{1}{R}\cot\frac{\zeta}{R}\frac{\dot{T}}{T}(2nm-m+2) = 0$$

We can choose our parameters n, m and k in a way which will make the above ~~integrals~~ equ. integrable.

We write the equ. as $\ddot{y} + P\dot{y} + Q\zeta = 0$

$$P = (2n-1)\frac{\dot{T}}{T} + \frac{2m}{R}\cot\frac{\zeta}{R} \qquad Q = \frac{\ddot{T}}{T}[(n-1)+2k] + \frac{\dot{T}^2}{T^2}[(n-1)^2 - 3k] + \frac{m+2-4k}{R^2}$$

$$+ \left(\frac{m^2+m-2}{R^2}\right)\cot^2\frac{\zeta}{R} + \left(\frac{2nm-m+2}{R}\right)\cot\frac{\zeta}{R}\frac{\dot{T}}{T}$$

(58) One can now take various cases

(i) $Q=0$ (2) $Q-\frac{1}{2}\frac{dP}{dt}-\frac{1}{4}P^2=0$ (3) $Q-\frac{1}{2}\frac{dP}{dt}-\frac{1}{4}P^2=\frac{\text{constant}}{R^2}$

(4) $Q-\frac{1}{2}\frac{dP}{dt}-\frac{1}{4}P^2=-\frac{2}{R^2}\operatorname{cosec}^2\frac{z}{R}$ etc.

We take Case I $Q=0$ $\therefore n-1+2k=0$ $\quad m+2-4k=0$

$(n-1)^2-3k=0$ $\quad m^2+m-2=0$

$2nm-m+2=0$

First two equations give $4k^2-3k=0$ $k=0$ or $k=\frac{3}{4}$

$k=0$ make $P+P=0$ and so leads to the de Sitter model case which we have already dealt with separately.

$\therefore$ Now take $k=\frac{3}{4}$ $\therefore m+2-4k=0 \Rightarrow m=1$. Then $m^2+m-2=0$ is satisfied and the last eqn. yields $2n-1+2=0$

$n=-\frac{1}{2}$

$\therefore \frac{\dot{z}}{z}=\frac{\dot{Y}}{Y}-\frac{1}{2}\frac{\dot{T}}{T}+\frac{1}{R}\cot\frac{z}{R}$ leads to

$\ddot{Y}+P\dot{Y}=0$ $\quad P=-\frac{2\dot{T}}{T}+\frac{2}{R}\cot\frac{z}{R}$.

One solution is $\dot{Y}=0$ $\quad Y=A(4)$ $\quad z=\frac{A}{\sqrt{T}.\,\text{Am}\frac{z}{R}}$

The other solution is $\frac{\ddot{Y}}{\dot{Y}}+P=0$ or $\frac{\ddot{Y}}{\dot{Y}}-\frac{2\dot{T}}{T}+\frac{2}{R}\cot\frac{z}{R}=0$

$$\therefore \frac{\dot{y}}{T^2 \sin^2 \frac{z}{R}} = B \qquad \dot{y} = BT^2 \sin^2 \frac{z}{R} \qquad y = B\int T^2 \sin^2 \frac{z}{R}\, dt + A(u)$$

$$\therefore z = \frac{A(u)}{\sqrt{T}\sin \frac{z}{R}} + \frac{B(u)}{\sqrt{T}\sin \frac{z}{R}} \int T^2 \sin^2 \frac{z}{R}\, dt$$

$$\therefore Z = 1 + \frac{z}{T} = 1 + \frac{A(u)}{T^{3/2}\sin \frac{z}{R}} + \frac{B(u)}{T^{3/2}\sin \frac{z}{R}} \int T^2 \sin^2 \frac{z}{R}\, dt$$

T is left undetermined.

Case (2) $Q - \frac{1}{2}\frac{dP}{dt} - \frac{1}{4}P^2 = 0$ or $I = 0$ $\quad \therefore \frac{\ddot{T}}{T}[n-1+2k] + \frac{\dot{T}^2}{T^2}[(n-1)^2 - 3k] + \frac{m+2-4k}{R^2}$

$$+\frac{(m^2+m-2)}{R^2}\cot^2\frac{z}{R} + \frac{2nm-m+2}{R}\cot\frac{z}{R}\frac{\dot{T}}{T} - \frac{1}{2}\left[(2n-1)\left(\frac{\ddot{T}}{T} - \frac{\dot{T}^2}{T^2}\right) + \frac{2m}{R^2}\operatorname{cosec}^2\frac{z}{R}\right]$$

$$-\frac{1}{4}\left[(2n-1)^2\frac{\dot{T}^2}{T^2} + \frac{4m^2}{R^2}\cot^2\frac{z}{R} + \frac{4m(2n-1)}{R}\cot\frac{z}{R}\frac{\dot{T}}{T}\right] = 0 = I.$$

$$I = \frac{\ddot{T}}{T}\left[n-1+2k - \frac{(2n-1)}{2}\right] + \frac{\dot{T}^2}{T^2}\left[(n-1)^2 - 3k + \frac{2n-1}{2} - \frac{1}{4}(2n-1)^2\right]$$

$$+ \frac{1}{R^2}\left[m+2-4k - \frac{1}{2}\cdot 2m\right] + \frac{1}{R^2}\cot^2\frac{z}{R}\left[m^2+m-2-m-m^2\right]$$

$$+ \frac{1}{R}\cot\frac{z}{R}\frac{\dot{T}}{T}\left[2nm-m+2-m(2n-1)\right]$$

$$= \frac{\ddot{T}}{T}\left[\frac{-1+4k}{2}\right] + \frac{\dot{T}^2}{T^2}\left[1-\frac{1}{2}-\frac{1}{4}-3k\right] + \frac{1}{R^2}(2-4k) - \frac{2}{R^2}\cot^2\frac{z}{R}$$

$$+ \frac{2}{R}\cot\frac{z}{R}\frac{\dot{T}}{T}$$

The last two terms of coeff of z in the given equation

(55) have reappeared here. So cases (2) and (3) are not possible

But we can take up case (4) $I = -\frac{2}{R^2}\operatorname{cosec}^2\frac{r}{R}$. One has, to take k not as constant but as a function of u, t.

Case (4) $I = -\frac{2}{R^2}\operatorname{cosec}^2\frac{r}{R}$

$$\therefore \frac{\ddot{T}}{T}\frac{(4k-1)}{2} - \frac{\dot{T}^2}{T^2}\frac{(3k-4)}{4} \quad \frac{\ddot{T}}{T}\left(2k-\frac{1}{2}\right) - \frac{\dot{T}^2}{T^2}\left(3k-\frac{1}{4}\right) + \frac{1}{R^2}(2-4k)$$

$$-\frac{2}{R^2}\cot^2\frac{r}{R} + \frac{2}{R}\cot\frac{r}{R}\frac{\dot{T}}{T} = -\frac{2}{R^2}\operatorname{cosec}^2\frac{r}{R}$$

$$\therefore k\left[\frac{2\ddot{T}}{T} - 3\frac{\dot{T}^2}{T^2} - \frac{4}{R^2}\right] - \frac{\ddot{T}}{2T} + \frac{1}{4}\frac{\dot{T}^2}{T^2} + \frac{2}{R^2} + \frac{2}{R}\cot\frac{r}{R}\frac{\dot{T}}{T} = -\frac{2}{R^2}$$

so our k is given by $2k\left[\frac{\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} - \frac{2}{R^2}\right] = \frac{\ddot{T}}{2T} - \frac{\dot{T}^2}{4T^2} - \frac{2}{R}\cot\frac{r}{R}\frac{\dot{T}}{T} - \frac{4}{R^2}$

or $2k R_{44} = \frac{\ddot{T}}{2T} - \frac{\dot{T}^2}{4T^2} - \frac{2}{R}\cot\frac{r}{R}\frac{\dot{T}}{T} - \frac{4}{R^2}$.

But looking back to our original (z, t) equation we can carry this out there itself before going to ~~(X, t)~~ equation

So we now take this case (B)

Case (B) Our original (z, t) equation is

$$\ddot{z} - \dot{z}\frac{\dot{T}}{T} + z\left[(2k-1)\frac{\ddot{T}}{T} - (3k-1)\frac{\dot{T}^2}{T^2} - \frac{4k-2}{R^2} + \frac{2}{R}\cot\frac{r}{R}\frac{\dot{T}}{T} - \frac{2}{R^2}\cot^2\frac{r}{R}\right] = 0$$

Here $P = -\frac{\dot{T}}{T}$ and Q is the coeff. of $\frac{1}{2}$

$$\therefore I = (2k-1)\frac{\ddot{T}}{T} - (3k-1)\frac{\dot{T}^2}{T^2} - \frac{4k-2}{R^2} + \frac{2}{R}\cot\frac{r}{R}\frac{\dot{T}}{T} - \frac{2}{R^2}\cot^2\frac{r}{R}$$
$$- \frac{1}{2}\left(-\frac{\ddot{T}}{T} + \frac{\dot{T}^2}{T^2}\right) - \frac{1}{4}\frac{\dot{T}^2}{T^2} \quad . \quad \text{Put this} = -\frac{2}{R^2}\operatorname{cosec}^2\frac{r}{R}$$

$$\therefore 2k\left(\frac{\ddot{T}}{T} - \frac{3}{2}\frac{\dot{T}^2}{T^2} - \frac{2}{R^2}\right) - \frac{\ddot{T}}{2T} + \frac{\dot{T}^2}{T^2}\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \frac{2}{R^2} + \frac{2}{R}\cot\frac{r}{R}\frac{\dot{T}}{T} - \frac{2}{R^2}\cot^2\frac{r}{R}$$
$$= -\frac{2}{R^2}\operatorname{cosec}^2\frac{r}{R}$$

$$\therefore 2k R_{44} = \frac{\ddot{T}}{2T} - \frac{1}{4}\frac{\dot{T}^2}{T^2} - \frac{2}{R}\cot\frac{r}{R}\frac{\dot{T}}{T} - \frac{4}{R^2}.$$

Choose $k = k(r,t)$ as defined above.

Then $\quad \frac{1}{2} = v\, e^{-\frac{1}{2}\int P\,dt} = v\, e^{-\frac{1}{2}\int -\frac{\dot{T}}{T}\,dt} = v\sqrt{T}$

The equation becomes

$$\ddot{v} - \frac{2v}{R^2}\operatorname{cosec}^2\frac{r}{R} = 0 \qquad \text{which is the eqn. of stage (1) on p. 20.}$$

The solution is

$$v = \frac{B}{R}\cot\frac{r}{R} - A - \frac{At}{R}\cot\frac{r}{R}.$$

$\frac{1}{2} = v\sqrt{T}$

$$Z = 1 + \frac{\frac{1}{2}}{T} = 1 + \frac{v}{\sqrt{T}} = 1 + \frac{B}{R}\frac{\cot\frac{r}{R}}{\sqrt{T}\sin^2\frac{r}{R}} - \frac{\left(A + \frac{At}{R}\cot\frac{r}{R}\right)}{\sqrt{T}}$$

(5c) Case C. If $\alpha = \operatorname{cosec}\frac{z}{R}$ $\dot\alpha = \operatorname{cosec}\frac{z}{R}\cot\frac{z}{R}\,\frac{1}{R}$

$$\ddot\alpha = \frac{1}{R^2}\operatorname{cosec}\frac{z}{R}\left(\cot^2\frac{z}{R} + \operatorname{cosec}^2\frac{z}{R}\right) = \frac{\alpha}{R^2}\left(1 + 2\cot^2\frac{z}{R}\right)$$

$$\therefore \ddot\alpha - \frac{\alpha}{R^2}\left(1 + 2\cot^2\frac{z}{R}\right) = 0 \Rightarrow \alpha = \operatorname{cosec}\frac{z}{R}$$

$\therefore$ One can take $I = -\frac{1}{R^2} - \frac{2}{R^2}\cot^2\frac{z}{R}$

$$\therefore 2k R_{44} - \frac{\ddot T}{2T} + \frac{1}{4}\frac{\dot T^2}{T^2} + \frac{2}{R}\cot\frac{z}{R}\frac{\dot T}{T} + \frac{2}{R^2} - \frac{2}{R^2}\cot^2\frac{z}{R} = -\frac{1}{R^2} - \frac{2}{R}\cot^2\frac{z}{R}$$

or choose $k(z,t)$ in such a way that

$$2k R_{44} = \frac{\ddot T}{2T} - \frac{1}{4}\frac{\dot T^2}{T^2} - \frac{2}{R}\cot\frac{z}{R}\frac{\dot T}{T} - \frac{3}{R^2}$$

Then $\zeta = v\, e^{-\frac{1}{2}\int P\,dt} = v\sqrt{T}$ gives $\ddot v - \frac{1}{R^2}\left(1 + 2\cot^2\frac{z}{R}\right)v = 0$

$$v = A\operatorname{cosec}\frac{z}{R}$$

One can find a more general solution by setting $v = W\operatorname{cosec}\frac{z}{R}$

$$\dot v = \dot W \operatorname{cosec}\frac{z}{R} + \frac{W}{R}\operatorname{cosec}\frac{z}{R}\cot\frac{z}{R}$$

$$\ddot v = \ddot W \operatorname{cosec}\frac{z}{R} + \frac{2\dot W}{R}\operatorname{cosec}\frac{z}{R}\cot\frac{z}{R} + \frac{W}{R^2}\operatorname{cosec}\frac{z}{R}\left(\cot^2\frac{z}{R} + \operatorname{cosec}^2\frac{z}{R}\right)$$

$\therefore$ Our (v,t) equation becomes

$$\ddot{W}\operatorname{cosec}\frac{z}{R} + 2\dot{W}\operatorname{cosec}\frac{z}{R}\cot\frac{z}{R} + \frac{W}{R^2}\operatorname{cosec}\frac{z}{R}\left(1+2\cot^2\frac{z}{R}\right)$$

$$- \frac{W}{R^2}\operatorname{cosec}\frac{z}{R}\left(1+2\cot^2\frac{z}{R}\right) = 0$$

$$\therefore \ddot{W} + 2\frac{\dot{W}}{R}\cot\frac{z}{R} = 0 \qquad \therefore \frac{\ddot{W}}{\dot{W}} + \frac{2}{R}\frac{\cos\frac{z}{R}}{\sin\frac{z}{R}} = 0 \qquad \therefore \frac{\dot{W}}{\sin^2\frac{z}{R}} = \frac{2B(u)}{R}$$

$$\therefore \dot{W} = \frac{2B(u)}{R}\sin^2\frac{z}{R} \qquad \therefore W = B\int\frac{2\sin^2\frac{z}{R}}{R}dt + \frac{A}{R}$$

$$= B\int\frac{1-\cos\frac{2z}{R}}{R}dt + \frac{A}{R} = B\left[\frac{t}{R} + \frac{1}{2}\sin\frac{2z}{R}\right] + \frac{A}{R}$$

$$W = \frac{A}{R} + \frac{B}{R}\left[t + \sin\frac{z}{R}\cos\frac{z}{R}\right]$$

$$\therefore \zeta = v\sqrt{T} = W\sqrt{T}\operatorname{cosec}\frac{z}{R} = \sqrt{T}\left[\frac{A}{R\sin\frac{z}{R}} + \frac{Bt}{R\sin\frac{z}{R}} + \frac{B}{R}\cos\frac{z}{R}\right]$$

$$\therefore \boxed{Z = 1 + \frac{\zeta}{T} = 1 + \frac{(A+Bt)}{R\sqrt{T}\sin\frac{z}{R}} + \frac{B}{R\sqrt{T}}\cos\frac{z}{R}.}$$

(90) Let us collect our results.

I for a central radiation-flow in the back-ground of Einstein Univers

Ia flow throug dust filled universe

$p=0$

$$Z = 1 + \frac{A(u)}{R} \frac{\cosh \frac{z}{R}}{\operatorname{Sin} \frac{z}{R}} .$$ See p. 12

Ib general case $p \neq 0$ $v_1 = (1-2) v_4$.

$$Z = 1 + \frac{A(u)}{R} \cot \frac{z}{R} - B\left(1 + \frac{t}{R} \cot \frac{z}{R}\right)$$ See p. 251

See further p. 75.

II for central flow of radiation in expanding universes

IIa Case $p=0$

IIa1 flow in de Sitter's Universe

$$Z = 1 + \frac{\Lambda R^2}{3} \left[\frac{2m(u)}{R \operatorname{Sin} \frac{z}{R}} \cos^3 \left(\frac{t+b}{R}\right) + \frac{P(u)}{R^2} \operatorname{Sin}^2 \frac{z}{R^2} \right]$$ See p. 61

See also p. 33

IIa2 $\Lambda = 0$, expanding dust universe

$p=0$

$$Z = 1 + \frac{A(u) \cot \frac{t}{R}}{R T^{3/2}}$$

~~$T = \operatorname{Sin}^4 \left(\frac{t}{2R}\right)$~~

$$Z = 1 + \frac{A \cot \frac{\tau}{2R}}{T^{3/2} R \sin \frac{\tau}{R}} \qquad T = \sin^4 \frac{\tau}{2R}$$

See p. 35

II$_{a.3}$ $\rho = 0$ Einstein de Sitter dust universes

$$Z = 1 + \frac{2m(u)}{r} + \frac{A(u)}{r(at+b)^2}$$

See p. 28

II$_b$ $\rho \neq 0$ $\quad v_i = k(1-Z) v_4$

II$_{b_1}$ Assumption $k = \frac{3}{4}$ $\quad Z = 1 + \frac{A(u)}{T^{3/2} \sin \frac{\tau}{R}} + \frac{B(u) \int T^2 \sin^2 \frac{\tau}{R} d\tau}{T^{3/2} \sin \frac{\tau}{R}}$

T undetermined

see p. 65

II$_{b_2}$ $2k R_{44} = \frac{\ddot{T}}{2T} - \frac{1}{4} \frac{\dot{T}^2}{T^2} - \frac{2}{R} \cot \frac{\tau}{R} \frac{\dot{T}}{T} - \frac{4}{R^2}$

$$Z = 1 + \frac{A(u) \cot \frac{\tau}{R}}{R\sqrt{T}} - \frac{B\left(1 + \frac{\tau}{R} \cot \frac{\tau}{R}\right)}{\sqrt{T}}$$

See p. 67

T undetermined

II$_{b_3}$ $2k R_{44} = \frac{\ddot{T}}{2T} - \frac{1}{4} \frac{\dot{T}^2}{T^2} - \frac{2}{R} \cot \frac{\tau}{R} \frac{\dot{T}}{T} - \frac{3}{R^2}$

$$Z = 1 + \frac{A(u)}{\sqrt{T} \cdot R \sin \frac{\tau}{R}} + \frac{B(u)}{RT} \left(\frac{\tau}{\sin \frac{\tau}{R}} + \cos \frac{\tau}{R} \right)$$

see p. 69

T undetermined

26 March 1995

Let us first check whether McVittie's solution for a mass particle in Robertson-Walker universe is in coving coordinates. McVittie's metric is

$$ds^2 = \left[\frac{1-\mu}{1+\mu}\right]^2 dt^2 - \frac{(1+\mu)^4}{\left(1+\frac{1}{4}kr^2\right)^2} R^2(t)\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right]$$

$$\mu = \mu(t, r) = \frac{m}{2rR}\left(1 + k\frac{r^2}{4}\right)^{1/2}.$$

(McVittie M.N.R.A.S. 93, 325, 1933) [Krasinski p. (42)

We shall show that for this metric $T_1^4 \equiv 0$ so ~~the~~ $V^1 = V^2 = V^3 = 0$.

For the metric $ds^2 = e^{\nu} dt^2 - e^{\mu}\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right)$

$$T_1^4 = -e^{\nu}\left[\dot{\mu}' - \dot{\mu}\frac{\nu'}{2}\right]$$

For McVittie's metric $e^{\nu/2} = \frac{1-\mu}{1+\mu}$, $e^{\mu} = \frac{(1+\mu)^4}{\left(1+k\frac{r^2}{4}\right)^2} R^2(t)$

$$\mu = \frac{m}{2rR}\left(1 + k\frac{r^2}{4}\right)^{1/2} \qquad m = \text{const.}$$

$$\frac{\dot{\mu}}{\mu} = -\frac{\dot{R}}{R} \qquad \frac{\mu'}{\mu} = -\frac{1}{r} + \frac{1}{2}\frac{2k\frac{r}{4}}{1+k\frac{r^2}{4}}.$$

$$= \frac{-1 - k\frac{r^2}{4} + k\frac{r^2}{4}}{r\left(1+k\frac{r^2}{4}\right)} = \frac{-1}{r\left(1+k\frac{r^2}{4}\right)}.$$

$$\dot{\mu} = -\frac{\dot{R}}{R}\mu \quad \therefore \dot{\mu}' = -\frac{\dot{R}}{R}\mu'$$

Consider $e^{\mu} = \frac{(1+\mu)^4}{\left(1+k\frac{r^2}{4}\right)^2} R^2(t)$

$\therefore (\mu) = 4\log(1+\mu) - 2\log\left(1+k\frac{r^2}{4}\right) + 2\log R.$

$\therefore (\dot{\mu}) = \frac{4\dot{\mu}}{1+\mu} + \frac{2\dot{R}}{R} \qquad (\mu)' = \frac{4\mu'}{1+\mu} - \frac{4\mu\mu'}{(1+\mu)^2}.$

$e^{\nu/2} = \frac{1-\mu}{1+\mu} \quad \therefore \frac{\nu}{2} = \log(1-\mu) - \log(1+\mu) \quad \therefore \frac{\nu'}{2} = -\frac{\mu'}{1-\mu} - \frac{\mu'}{1+\mu}$

$\therefore \frac{\nu'}{2} = -\frac{\mu'(1+\mu+1-\mu)}{1-\mu^2} = \frac{-2\mu'}{1-\mu^2}$

$\therefore (\dot{\mu})' - (\dot{\mu})\frac{\nu'}{2} = \frac{4\dot{\mu}'}{1+\mu} - \frac{4\mu\mu'}{(1+\mu)^2} - \left(\frac{4\dot{\mu}}{1+\mu} + \frac{2\dot{R}}{R}\right)\left(\frac{-2\mu'}{1-\mu^2}\right)$

$= \frac{4\dot{\mu}'}{1+\mu} - \frac{4\dot{\mu}\mu'}{(1+\mu)^2} + \frac{8\dot{\mu}\mu'}{(1+\mu)(1-\mu^2)} + \frac{4\dot{R}}{R}\frac{\mu'}{1-\mu^2}$

$= \frac{4}{1+\mu}\left(-\frac{\dot{R}}{R}\right)\mu' + \frac{4\dot{\mu}\mu'}{(1+\mu)^2(1-\mu)}(-1+\mu+2) + \frac{4\dot{R}}{R}\frac{\mu'}{1-\mu^2}$

$= -\frac{4\mu'}{1+\mu}\frac{\dot{R}}{R} + \frac{4\dot{\mu}\mu'}{1-\mu^2}$ ~~But $\frac{\dot{R}}{R} = \frac{\dot{\mu}}{\mu}$~~ $+ 4\frac{\dot{R}}{R}\frac{\mu'}{1-\mu^2}$

$= \frac{4\mu'\dot{\mu}}{\mu(1+\mu)} = 4\frac{\dot{R}}{R}\frac{\mu'}{1-\mu^2}(-1+\mu+1) + \frac{4\mu'}{1-\mu^2}\left(-\frac{\dot{R}}{R}\mu\right)$

$= 4\frac{\dot{R}}{R}\frac{\mu\mu'}{1-\mu^2} - 4\frac{\dot{R}}{R}\frac{\mu\mu'}{1-\mu^2} \equiv 0$

$\therefore$ For this metric $T_1^4 = 0 \quad \therefore V^1 = V^2 = V^3 = 0$

This increases our faith in $v_1 = (1-z)v_4$.

So But McVittie's solution also says that $e^{\nu} \neq 1$

(98) Therefore in null coordinates setting a ~~fo~~ of $2a\,du\,dt$ as 1 is also not warranted. This implies that one should begin with

$$ds^2 = T\left[2a\,du\,dt - 2\,du^2 - R^2 \sin^2 \tfrac{z}{R}\, d\Omega^2\right]$$

$$a \neq 1.$$

On p. (42) we have derived a general solution for flowing radiation in Einstein's Universe containing 2 undetermined functions $A(u)$ and $B(u)$ and we have put $A=0$ because $8\pi\sigma(\omega_1)^2$ turns out to be non zero even when A is constant. But one can choose this undetermined function in such a way that this unsatisfactory feature of σ is absent.

Our solution on p. (49) is

$$\xi = -A - A\frac{t}{R}\cot\frac{z}{R} + \frac{B}{R}\cot\frac{z}{R}$$

$$= -A + \frac{1}{R}\cot\frac{z}{R}\,(B - At)$$

Choose $B = 2m(u) + A(u)\cdot u$

$$\therefore\quad \xi = -A + \frac{1}{R}\cot\frac{z}{R}\left(2m(u) + A(u-t)\right)$$

$$\therefore\quad \xi = \frac{2m(u)}{R}\cot\frac{z}{R} - A\left(1 - \frac{z}{R}\cot\frac{z}{R}\right) \qquad A = A(u)$$

Let us now rewrite our ρ, p and σ.

$$8\pi\rho = \Lambda - \frac{1}{R^2} + \frac{B}{R^3}\cot\frac{z}{R} - \frac{At}{R^3}\cot\frac{z}{R} = \Lambda - \frac{1}{R^2} + \frac{1}{R^3}\cot\frac{z}{R}(B - At)$$

$$= \Lambda - \frac{1}{R^2} + \frac{1}{R^3}\cot\frac{z}{R}\left(2m(u) + u\cdot A - tA\right)$$

$$= \Lambda - \frac{1}{R^2} + \frac{2m(u)}{R^3}\cot\frac{z}{R} + \frac{A}{R^3}\,z\cot\frac{z}{R}$$

$$= \Lambda - \frac{1}{R^2}\left(1 - \frac{2m(u)}{R}\cot\frac{z}{R} - \frac{A}{R}\,z\cot\frac{z}{R}\right), \qquad A = A(u)$$

$$8\pi p = -\Lambda + \frac{3}{R^2} - \frac{3B}{R^3}\cot\frac{z}{R} + \frac{3At}{R^3}\cot\frac{z}{R} + \frac{2A}{R^2}$$

(29) $= -\Lambda + \frac{3}{R^2} - \frac{3}{R^3}\cot\frac{\chi}{R}(B - At) + \frac{2A}{R^2}$

$= -\Lambda + \frac{3}{R^2} - \frac{3}{R^3}\cot\frac{\chi}{R}(2m(u) + A\chi) + \frac{2A}{R^2}$

$8\pi P = -\Lambda + \frac{3}{R^2}\left[1 - \frac{2m(u)}{R}\cot\frac{\chi}{R} - \frac{A}{R}\chi\cot\frac{\chi}{R} + \frac{2A}{3}\right]$

and $8\pi\sigma(\omega_i)^2 = \frac{B'}{R^2}\cot^2\frac{\chi}{R} - \frac{A't}{R^2}\cot^2\frac{\chi}{R} - \frac{A'}{R}\cot\frac{\chi}{R} - \frac{A}{R^2}\cot^2\frac{\chi}{R}$

But $B = 2m + u.A \quad B' = 2m' + A + uA'$

$\therefore 8\pi\sigma(\omega_i)^2 = \frac{2m'}{R^2}\cot^2\frac{\chi}{R} + \frac{A}{\chi^2}\cot^2\frac{\chi}{R} + \frac{uA'}{R^2}\cot^2\frac{\chi}{R} - \frac{tA'}{R^2}\cot^2\frac{\chi}{R} - \frac{A'}{R}\cot\frac{\chi}{R} - \frac{A}{R^2}\cot^2\frac{\chi}{R}$

$= \frac{2m'}{R^2}\cot^2\frac{\chi}{R} + \frac{A'}{R^2}\chi\cot^2\frac{\chi}{R} - \frac{A'}{R}\cot\frac{\chi}{R}$

$= \frac{2m'}{R^2}\cot^2\frac{\chi}{R} + \frac{A'}{R}\cot\frac{\chi}{R}\left(\frac{\chi}{R}\cot\frac{\chi}{R} - 1\right)$:

7-4-1995.

There is another method to ~~do~~ deal with the general case.

Let us take the metric as

$$ds^2 = e^{\varphi}\left[2a\,du\,dt - Z\,du^2 - M^2 dr^2\right] \quad \varphi = \varphi(u,t) \quad a = a(u,t),\ Z = Z(u,t)$$

Since Robertson-Walker metric is conformal to flat space let us choose $M^2 = r^2$ then $a = 1$, $Z = 1$ can give Robertson Walker universe. The conformal factor $\varphi = \varphi(u,t)$ has been given ~~in~~ by Infeld in a paper published in 1954 or so in Cadian Journal of Mathematics.

But before we take this case: let us try to immediately generalize I_4 to expanding universe case with $v_1 = (1-Z)v_4$ as also to the case $k = \frac{3}{4}$.

$v_1 = (1-Z)v_4$ leads to $Z = 1 + \frac{\xi}{T}$

$$\ddot{\xi} - \dot{\xi}\frac{\dot{T}}{T} + \xi\left(\frac{\ddot{T}}{T} - \frac{2\dot{T}^2}{T^2} + \frac{2}{R}\cot\frac{r}{R}\frac{\dot{T}}{T} - \frac{2}{R^2}\operatorname{cosec}^2\frac{r}{R}\right) = 0$$

McVittie's soln suggests that $\frac{1}{r}$ i.e. $\frac{1}{R\sin\frac{r}{R}}$ must be replaced by $\frac{1}{VTr}$ i.e. by $\frac{1}{VTR\sin\frac{r}{R}}$

(9c)

So we replace z by Y by the substitution

$$z = \frac{Y}{R\sqrt{T}} \cot \frac{r}{R}$$

$$\cancel{\dot{z} = \frac{\dot{Y}}{R\sqrt{T}} \cot \frac{r}{R} - \frac{1}{2}\frac{Y}{T^{3/2}}\frac{\dot{T}}{R} \cot \frac{r}{R} + \frac{Y}{\sqrt{T}}\frac{1}{R^2}}$$

$$\frac{\dot{z}}{z} = \frac{\dot{Y}}{Y} - \frac{1}{2}\frac{\dot{T}}{T} + \frac{\operatorname{cosec}^2 \frac{r}{R}}{\cot \frac{r}{R}} \cdot \frac{1}{R}$$

$$= \frac{\dot{Y}}{Y} - \frac{\dot{T}}{2T} + \frac{2}{2 \sin \frac{r}{R} \cos \frac{r}{R}} \cdot \frac{1}{R}$$

$$= \frac{\dot{Y}}{Y} - \frac{\dot{T}}{2T} + \frac{2}{R} \operatorname{cosec} \frac{2r}{R}$$

$$\frac{\ddot{z}}{z} = \frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} - \frac{\ddot{T}}{2T} + \frac{\dot{T}^2}{2T^2} + \frac{4}{R^2} \operatorname{cosec} \frac{2r}{R} \cot \frac{2r}{R}$$

$$+ \frac{\dot{Y}^2}{Y^2} + \frac{\dot{T}^2}{4T^2} + \frac{4}{R^2} \operatorname{cosec}^2 \frac{2r}{R}$$

$$+ \frac{2\dot{Y}}{Y}\left(-\frac{\dot{T}}{2T} + \frac{2}{R} \operatorname{cosec} \frac{2r}{R}\right) - \frac{2\dot{T}}{T} \operatorname{cosec} \frac{2r}{R}$$

∴ Our (ξ, τ) eqn changes to (Y, t) eqn

$$\frac{\ddot{Y}}{Y} + \frac{\dot{Y}}{Y}\left(-\frac{\dot{T}}{T} + \frac{2}{R}\operatorname{cosec}\frac{2\xi}{R}\right) - \frac{\ddot{T}}{2T} + \frac{3}{4}\frac{\dot{T}^2}{T^2} - 2\frac{\dot{T}}{T}\operatorname{cosec}\frac{2\xi}{R}$$

$$+ \frac{4}{R^2}\operatorname{cosec}\frac{2\xi}{R}\left(\operatorname{cosec}\frac{2\xi}{R} + \cot\frac{2\xi}{R}\right)$$

$$- \frac{\dot{T}}{T}\left(\frac{\dot{Y}}{Y} - \frac{1}{2}\frac{\dot{T}}{T} + \frac{2}{R}\operatorname{cosec}\frac{2\xi}{R}\right) + \frac{\ddot{T}}{T} - 2\frac{\dot{T}^2}{T^2} + \frac{2}{R}\cot\frac{\xi}{R}\frac{\dot{T}}{T} - \frac{2}{R^2}\operatorname{cosec}^2\frac{\xi}{R}$$

$$= 0$$

$$\therefore \frac{\ddot{Y}}{Y} + \frac{\dot{Y}}{Y}\left(-\frac{2\dot{T}}{T} + \frac{4}{R}\operatorname{cosec}\frac{2\xi}{R}\right) + \frac{\ddot{T}}{2T} - \frac{3}{4}\frac{\dot{T}^2}{T^2} + \frac{2\dot{T}}{T}\left(\cot\frac{\xi}{R}\right.$$

$$\left. + \frac{2\dot{T}}{T}\left(\cot\frac{\xi}{R} - \operatorname{cosec}\frac{2\xi}{R}\right) - \operatorname{cosec}^2\frac{2\xi}{R}\right) + \frac{4}{R^2}\operatorname{cosec}\frac{2\xi}{R}\left(\operatorname{cosec}\frac{2\xi}{R} + \cot\frac{2\xi}{R}\right)$$

$$- \frac{2}{R^2}\operatorname{cosec}^2\frac{\xi}{R} = 0$$

Now $\cot\frac{\xi}{R} - \operatorname{cosec}\frac{2\xi}{R} = \frac{2\cos^2\frac{\xi}{R}}{2\sin\frac{\xi}{R}\cos\frac{\xi}{R}} - \frac{1}{\sin\frac{2\xi}{R}} = \frac{2\cos^2\frac{\xi}{R} - 1}{\sin\frac{2\xi}{R}} = \cot\frac{2\xi}{R}$

Next $\operatorname{cosec}^2\frac{\xi}{R} = \frac{1}{\sin^2\frac{\xi}{R}} = \frac{\cos^2\frac{\xi}{R}}{\sin^2\frac{\xi}{R}\cos^2\frac{\xi}{R}} = \frac{2\left(2\cos^2\frac{\xi}{R}\right)}{\sin^2\frac{2\xi}{R}}$

$$\therefore \operatorname{cosec}^2\frac{k}{R} = 2\operatorname{cosec}^2\frac{2k}{R}\left(1+\cos\frac{2k}{R}\right)$$

$$= \frac{2\operatorname{cosec}^2 2k}{R} = 2\operatorname{cosec}\frac{2k}{R}\left(\operatorname{cosec}\frac{2k}{R} + \cot\frac{2k}{R}\right)$$

$$\therefore 4\operatorname{cosec}\frac{2k}{R}\left(\operatorname{cosec}\frac{2k}{R} + \cot\frac{2k}{R}\right) - \frac{2}{R^2}\operatorname{cosec}^2\frac{k}{R} = 0$$

$(Y.t)$ equation becomes

$$\frac{\ddot{y}}{Y} + \frac{\dot{Y}}{Y}\left(-\frac{2\dot{T}}{T} + \frac{4}{R}\operatorname{cosec}\frac{2k}{R}\right) + \frac{\dot{T}}{2T} - \frac{3}{4}\frac{\dot{T}^2}{T^2} + 2\frac{\dot{T}}{T}\cot\frac{2k}{R}\frac{2k}{R} = 0$$

24-5-95

In a 1957 paper in Proc. Nat. Inst. of Sc. (India) there is a paper by Vaidya and K.B. Shah where an approx. solution of a radiating particle in Einstein-de Sitter universe is given. We propose, in the following, the same approximation method for the general expanding universe $k \neq 0$.

Our first step is to convert our form of mass-particle in Einstein's universe to McVittie's form. Begin with

$$ds^2 = 2du\,dt - \left(1+\frac{2m}{R}\cot x\right)du^2 - R^2\sin^2 x\, d\Omega^2.$$

Put $u = Rx + t$

$$\therefore ds^2 = 2(R\,dx + dt)\,dt - \left(1+\frac{2m}{R}\cot x\right)(R\,dx+dt)^2 - R^2\sin^2 x\, d\Omega^2$$

$$= dt^2\left(1-\frac{2m}{R}\cot x\right) - \left(1+\frac{2m}{R}\cot x\right)R^2dx^2 - 4m\cot x\, R\,dx\,dt - R^2\sin^2 x\, d\Omega^2$$

Put $r = R\sin x$ $\quad \sin x = \frac{r}{R}$ $\quad \cos x = \sqrt{\left(1-\frac{r^2}{R^2}\right)}$

$$\therefore ds^2 = dt^2\left(1-\frac{2m}{R}\frac{\sqrt{\left(1-\frac{r^2}{R^2}\right)}}{\frac{r}{R}}\right) - \left(1+\frac{2m}{R}\frac{\sqrt{\left(1-\frac{r^2}{R^2}\right)}}{\frac{r}{R}}\right)\frac{dr^2}{1-\frac{r^2}{R^2}}$$

$$-4m\frac{\sqrt{\left(1-\frac{r^2}{R^2}\right)}}{\frac{r}{R}}\frac{dr}{\sqrt{\left(1-\frac{r^2}{R^2}\right)}}\,dt - r^2 d\Omega^2$$

$$= dt^2\left(1-\frac{2m\sqrt{\left(1-\frac{r^2}{R^2}\right)}}{r}\right) - \left(1+\frac{2m\sqrt{\left(1-\frac{r^2}{R^2}\right)}}{r}\right)\frac{dr^2}{1-\frac{r^2}{R^2}} - \frac{4m}{r}dr\,dt - r^2 d\Omega^2$$

We now change the present r to isotropic radial coordinate ρ. Put $r = \frac{\rho}{\quad}$

(c2)

Put $r = \dfrac{\rho}{1+\frac{\rho^2}{4R^2}}$ $\quad 1-\dfrac{r^2}{R^2} = 1 - \dfrac{\rho^2/R^2}{\left(1+\frac{\rho^2}{4R^2}\right)^2} = \dfrac{\left(1+\frac{\rho^2}{4R^2}\right)^2 - \frac{\rho^2}{R^2}}{\left(1+\frac{\rho^2}{4R^2}\right)^2}$

$$1-\frac{r^2}{R^2} = \frac{\left(1+\frac{\rho}{R}+\frac{\rho^2}{4R^2}\right)\left(1-\frac{\rho}{R}+\frac{\rho^2}{4R^2}\right)}{\left(1+\frac{\rho^2}{4R^2}\right)^2} = \frac{\left(1+\frac{\rho}{2R}\right)^2\left(1-\frac{\rho}{2R}\right)^2}{\left(1+\frac{\rho^2}{4R^2}\right)^2}$$

$$= \left(1-\frac{\rho^2}{4R^2}\right)^2 \Big/ \left(1+\frac{\rho^2}{4R^2}\right)^2 \quad \text{Also } dr = \frac{d\rho}{1+\frac{\rho^2}{4\rho^2}} - \frac{2\rho^2 d\rho/4R^2}{\left(1+\frac{\rho^2}{4}R^2\right)^2}$$

$$= \frac{d\rho}{1+\quad^2}$$

∴ Our metric becomes

$$ds^2 = dt^2\left[1 - \frac{2m\left(1-\frac{\rho^2}{4R^2}\right)}{\left(1+\frac{\rho^2}{4R^2}\right)\frac{\rho}{1+\frac{\rho^2}{4R^2}}}\right] - \left[1+\frac{2m\left(1-\frac{\rho^2}{4R^2}\right)}{\rho}\right]$$

$$dr = \frac{d\rho}{\left(1+\frac{\rho^2}{4R^2}\right)^2}\left(1+\frac{\rho^2}{4R^2} - \frac{2\cdot\rho^2}{4R^2}\right) = \frac{d\rho\left(1-\frac{\rho^2}{4R^2}\right)}{\left(1+\frac{\rho^2}{4R^2}\right)^2}$$

∴ Our metric becomes

$$ds^2 = dt^2\left[1-\frac{2m}{\rho}(1-\rho^2/4R^2)\right] - \left[1+\frac{2m}{\rho}(1-\rho^2/4R^2\right]\frac{d\rho^2\left(1-\frac{\rho^2}{4R^2}\right)^2}{\left(1+\frac{\rho^2}{4R^2}\right)^4}\frac{\left(1+\frac{\rho^2}{4R^2}\right)^2}{\left(1-\frac{\rho^2}{4R^2}\right)^2}$$

$$-\frac{4m}{\frac{\rho}{1+\frac{\rho^2}{4R^2}}}\frac{d\rho\left(1-\frac{\rho^2}{4R^2}\right)}{\left(1+\frac{\rho^2}{4R^2}\right)^2}dt - \frac{\rho^2}{\left(1+\frac{\rho^2}{4R^2}\right)^2}d\Omega^2$$

$$= dt^2\left[1-\frac{2m}{\rho}(1-\rho^2/4R^2)\right] - \left[1+\frac{2m}{\rho}(1-\rho^2/4R^2)\right]\frac{d\rho^2}{\left(1+\frac{\rho^2}{4R^2}\right)^2}$$

$$-\frac{4m}{\rho}\frac{(1-\rho^2/4R^2)}{1+\rho^2/4R^2}d\rho\, dt - \frac{\rho^2}{\left(1+\frac{\rho^2}{4R^2}\right)^2}d\Omega^2.$$

To remove the cross term $dp\,dt$ replace t by τ

$t = \phi(p,\tau)$ $\quad p = p$ $\quad dt = \phi_p\,dp + \phi_\tau\,d\tau$

$$\therefore ds^2 = \left[1-\frac{2m}{p}\left(1-\frac{p^2}{4R^2}\right)\right]\left[\phi_p\,dp+\phi_\tau\,d\tau\right]^2 - \left[1+\frac{2m}{p}\left(1-\frac{p^2}{4R^2}\right)\right]\frac{dp^2}{\left[1+\frac{p^2}{4R^2}\right]^2}$$

$$-\frac{4m}{p}\frac{(1-p^2/4R^2)}{1+p^2/4R^2}\,dp(\phi_p\,dp+\phi_\tau\,d\tau) - \frac{p^2}{\left(1+\frac{p^2}{4R^2}\right)^2}\,d\Omega^2.$$

Coeff of $dp\,d\tau$ is $2\phi_p\phi_\tau\left[1-\frac{2m}{p}\left(1-\frac{p^2}{4R^2}\right)\right] - \frac{4m}{p}\frac{(1-p^2/4R^2)}{1+p^2/4R^2}\phi_\tau = 0$

$\phi_\tau \neq 0$ $\quad \phi_p = \frac{2m}{p}\frac{(1-p^2/4R^2)}{1+p^2/4R^2}\cdot\frac{1}{1-\frac{2m}{p}\left(1-\frac{p^2}{4R^2}\right)}$.

$\mu = \frac{2m}{p}\left(1-\frac{p^2}{4R^2}\right)$

Coeff of $dp^2 = \phi_p^2(1-\mu) - \frac{(1+\mu)}{\left(1+\frac{p^2}{4R^2}\right)^2} - \frac{2\mu\phi_p}{1+p^2/4R^2}$

But $\phi_p = \frac{\mu}{1+\frac{p^2}{4R^2}}\,\frac{1}{1-\mu}$ $\quad \therefore (1-\mu)\phi_p = \frac{\mu}{1+\frac{p^2}{4R^2}}$

$$\therefore \text{Coeff of } dp^2 = \phi_p\cdot\frac{\mu}{1+\frac{p^2}{4R^2}} - \frac{1+\mu}{\left(1+\frac{p^2}{4R^2}\right)^2} - \frac{2\mu\phi_p}{1+p^2/4R^2}$$

$$= -\frac{\mu}{1+\frac{p^2}{4R^2}}\,\frac{\mu}{1+\frac{p^2}{4R^2}}\,\frac{1}{1-\mu} - \frac{(1+\mu)}{\left(1+\frac{p^2}{4R^2}\right)^2}$$

$$= -\frac{\mu^2+1-\mu^2}{\left(1+\frac{p^2}{4R^2}\right)^2(1-\mu)} = -\frac{1}{\left(1+\frac{p^2}{4R^2}\right)^2(1-\mu)}$$

Again taking $\phi_\tau = 1$ we get

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$$ds^2 = (1-\mu)\,d\tau^2 - \frac{1}{1-\mu}\frac{dp^2}{\left(1+\frac{p^2}{4R^2}\right)^2} - \frac{p^2}{\left(1+\frac{p^2}{4R^2}\right)^2}\,d\Omega^2.$$

We must now transform this to McVittie's form

$$ds^2 = e^{\nu}\,d\tau^2 - \frac{e^{\alpha}}{\left(1+\frac{x^2}{4R^2}\right)^2}\left[dx^2 + x^2\,d\Omega^2\right]$$

by the substitution $\dfrac{x^2 e^{\alpha}}{\left(1+\frac{x^2}{4R^2}\right)^2} = \dfrac{p^2}{\left(1+\frac{p^2}{4R^2}\right)^2}$

$$g'^{11} = +\frac{\partial x'^1}{\partial x^1}\frac{\partial x'^1}{\partial x^1}\,g^{11} \quad \therefore\quad -e^{-\alpha}\left(1+\frac{x^2}{4R^2}\right)^2 = -(1-\mu)\left(1+\frac{p^2}{4R^2}\right)^2 x'^2$$

$$\therefore\quad \frac{x^2}{p^2}\left(1+\frac{p^2}{4R^2}\right)^2 = (1-\mu)\left(1+\frac{p^2}{4R^2}\right)^2 x'^2$$

$$\therefore\quad \frac{x'^2}{x^2} = \frac{1}{p^2(1-\mu)} = \frac{1}{p^2\left[1-\frac{2m}{p}\left(1-\frac{p^2}{4R^2}\right)\right]}$$

$$= \frac{1}{p\left[p-2m+\frac{2m}{4R^2}p^2\right]}$$

$$\therefore\quad \frac{dx'}{x} = \frac{1}{\sqrt{p}\left[\frac{2m}{4R^2}p^2+p-2m\right]^{1/2}}$$

It is ~~no~~ does not seem possible to evaluate

the integral $I = \int \frac{dp}{\sqrt{p}\sqrt{\left(\frac{2m}{4R^2}p^2 + p - 2m\right)}}$, in terms of known functions

Put $p = 2m\sec^2\theta$. $dp = 4m\sec^2\theta\tan\theta\, d\theta$

$$\therefore I = \int \frac{4m\sec^2\theta\tan\theta\, d\theta}{\sqrt{2m}\sec\theta\sqrt{\left[\frac{2m}{4R^2}4m^2\sec^4\theta + 2m\sec^2\theta - 2m\right]}}$$

$$= \int \frac{4m\sec\theta\tan\theta\, d\theta}{2m\sqrt{\left[\frac{4m^2}{R^2}\sec^4\theta + \tan^2\theta\right]}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{\tan\theta\sqrt{\left[1 + \frac{4m^2}{R^2}\frac{\sec^4\theta}{\tan^2\theta}\right]}}$$

$$= 2\int \frac{\sec\theta\, d\theta}{\sqrt{\left(1 + \frac{4m^2}{R^2}\frac{\sec^4\theta}{\tan^2\theta}\right]}}$$

There seems to be a simple way of working out this integral. Begin with $I = \int \frac{dp}{\sqrt{p}\sqrt{\left(\frac{2m}{4R^2}p^2 + p - 2m\right)}}$

Put $p = z^2$ $\therefore I = \int \frac{2z\, dz}{z\sqrt{\left(\frac{m}{2R^2}z^4 + z^2 - 2m\right)}} = \int \frac{2\, dz}{\sqrt{\frac{m}{2}}\frac{1}{R}\left(z^4 + \frac{2R^2}{m}z^2 - 4R^2\right)^{1/2}}$

(24) It does not seem possible to work out this integral in terms of known functions, unless we are prepared to work out an approximate solution. And if we have to follow an approximation procedure, why not follow the Vaidya-Shah procedure of 1957. There R was ~~tant~~ taken infinite. Now we ~~p~~ use the same process but donot take $R \to \infty$ i.e. we do not begin with flate space-time but with Einstein's universe.

So we begin with

$$ds^2 = e^{g(\tau)}\left[d\tau^2 - \frac{1}{\left(1+\frac{\rho^2}{4R^2}\right)^2}\left(d\rho^2 + \rho^2 d\Omega^2\right)\right] \qquad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

or

$$ds^2 = e^{g(\tau)}\left[d\tau^2 - R^2 dx^2 - R^2 \sin^2 x\, d\Omega^2\right] \qquad \text{Put } \cancel{rd}\ R\,dx = dr$$

$$\therefore ds^2 = e^{g(\tau)}\left[d\tau^2 - dr^2 - R^2 \sin^2 \frac{r}{R}\, d\Omega^2\right]$$

Following 1957 paper ~~choose~~ we can choose $r - T = t$ which is ~~equ~~ the same as our earlier choice (in this note book of $r - T = u$. So we continue to consider inflowing radiation. But now we are not to use u as a space-like coordinate. We continue to use r as the coordinate and use u as ~~the~~ time-coordinate. To agree with notations of 1957 ~~paper~~ we do not use u for retarded time, but call it t

$$\therefore r - T = t \qquad \cancel{dr} \qquad r - t = T$$

$$\therefore dr - dt = dT$$

$\therefore$ The metric ~~dr~~ changes to

$$ds^2 = e^{g(r-t)}\left[-R^2\sin^2\tfrac{r}{R}\,d\sigma^2 - dt\,(dr - dt + dr)\right]$$

$$= e^{g(r-t)}\left[-R^2\sin^2\tfrac{r}{R}\,d\sigma^2 + dt^2 - 2dr\,dt\right]$$

So we now consider the metric

$$ds^2 = T\left[-2dr\,dt + (1+V)\,dt^2 - M^2\,d\sigma^2\right] \qquad M = R\sin\tfrac{r}{R} \quad \dot{M} = 0.$$

$r = x^1$, $\theta = x^2$, $\phi = x^3$ $t = x^4$ (Note t is the "advance time" $-t$ is the "retarded time")

$T = T(r-t)$

$$\therefore g_{ik} = \begin{pmatrix} 0 & 0 & 0 & -T \\ 0 & -M^2T & 0 & 0 \\ 0 & 0 & -M^2T\sin^2\theta & 0 \\ -T & 0 & 0 & (1+V)T \end{pmatrix}$$

$$\therefore g = T(-T)M^4T^2\sin^2\theta$$

$$= -T^4M^4\sin^2\theta$$

$$\therefore \sqrt{(-g)} = T^2M^2\sin\theta$$

$$g^{11} = \frac{M^4T^3\sin^2\theta(1+V)}{-M^4T^4\sin^2\theta} = -\frac{(1+V)}{T} \qquad g^{14} = \frac{-(-T)M^4T^2\sin^2\theta}{-T^4M^4\sin^2\theta} = -\frac{1}{T}$$

$$g^{ik} = \begin{pmatrix} -\frac{(1+V)}{T} & 0 & 0 & -\frac{1}{T} \\ 0 & -\frac{1}{M^2T} & 0 & 0 \\ 0 & 0 & \frac{-1}{M^2T\sin^2\theta} & 0 \\ -\frac{1}{T} & 0 & 0 & 0 \end{pmatrix}$$

$T = T(r-t)$

if $r-t = x$

$T' = T_x = T^*$

$\dot{T} = T_x(-1) = -T^*$.

We shall now find the 3-index symbols.

(c-8) $\left\{ {1 \atop 11} \right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{14}\left(g_{41,1} + g_{14,1} - g_{11,4}\right) = -\frac{1}{2T} 2(-T') = \frac{T'}{T} = \frac{T^*}{T}$

$\left\{ {2 \atop 11} \right\} = \frac{1}{2} g^{22}(0) = 0 \quad \left\{ {3 \atop 11} \right\} = \frac{1}{2} g^{33}(0) \quad \left\{ {4 \atop 11} \right\} = g^{41}(0) = 0.$

$\left\{ {1 \atop 11} \right\} = \frac{T^*}{T}$

$\left\{ {1 \atop 12} \right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{14}(0) = 0 \quad \left\{ {2 \atop 12} \right\} = \frac{1}{2} g^{22}(g_{22,1}) = \frac{1}{2}\left(-\frac{1}{TM^2}\right)(-TM^2)_{,1}$

$\therefore \left\{ {2 \atop 12} \right\} = \frac{T'}{2T} + \frac{2M'}{M} = \frac{T^*}{2T} + \frac{M'}{M}$

$\left\{ {3 \atop 12} \right\} = \frac{1}{2} g^{33}(0) = 0 \quad \left\{ {4 \atop 12} \right\} = \frac{1}{2} g^{41}(0) = 0$

$\left\{ {1 \atop 13} \right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{14}(0) = 0. \quad \left\{ {2 \atop 13} \right\} = \frac{1}{2} g^{22}(0) \quad \left\{ {3 \atop 13} \right\} = \frac{1}{2} g^{33}(g_{33,1})$

$\left\{ {2 \atop 12} \right\} = \left\{ {3 \atop 13} \right\} = \frac{T^*}{2T} + \frac{M'}{M}$

$\left\{ {4 \atop 13} \right\} = \frac{1}{2} g^{41}(0) = 0$

$\left\{ {1 \atop 14} \right\} = \frac{1}{2} g^{41}\left(g_{14,1} - g_{14,1}\right) + \frac{1}{2} g^{14}\left(g_{14,4} + g_{44,1} - g_{14,4}\right) = -\frac{1}{2T}\left[(1+V)T' + TV'\right]$

$\left\{ {2 \atop 14} \right\} = \frac{1}{2} g^{22}(0) = 0 \quad \left\{ {3 \atop 14} \right\} = \frac{1}{2} g^{33}(0) = 0 \quad \left\{ {4 \atop 14} \right\} = \frac{1}{2} g^{41}\left(g_{14,1} - g_{14,1}\right) = 0$

$\therefore \left\{ {1 \atop 14} \right\} = -(1+V)\frac{T^*}{2T} + \frac{1}{2}V'$

$\left\{ {1 \atop 22} \right\} = \frac{1}{2} g^{11}(-g_{22,1}) + \frac{1}{2} g^{14}(-g_{22,4})$

$= -\frac{1}{2}\frac{(1+V)}{T}\left[T'M^2 + T2MM'\right] - \frac{1}{2T}\left(\dot{T}M^2\right)$

$= -M^2(1+V)\left(\frac{T'}{2T} + \frac{M'}{M}\right) - M^2\frac{\dot{T}}{2T}$

$\therefore \left\{ {1 \atop 22} \right\} = -M^2(1+V)\left(\frac{T^*}{2T} + \frac{M'}{M}\right) + M^2\frac{T^*}{2T}$

$\left\{ {2 \atop 22} \right\} = \frac{1}{2} g^{22}(0) = 0 \quad \left\{ {3 \atop 22} \right\} = \frac{1}{2} g^{33}(0) = 0 \quad \left\{ {4 \atop 22} \right\} = \frac{1}{2} g^{41}(-g_{22,1}) = -\frac{1}{2T}\left(T'M^2 + T2MM'\right)$

$\therefore \left\{ {4 \atop 22} \right\} = -M^2\left(\frac{T^*}{2T} + \frac{M'}{M}\right)$

$\left\{ {1 \atop 23} \right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{14}(0) = 0 \quad \left\{ {2 \atop 23} \right\} = \frac{1}{2} g^{22}(0) = 0$

$\left\{ {3 \atop 23} \right\} = \frac{1}{2} g^{33}(g_{33,2}) = \cot\theta, \quad \left\{ {4 \atop 23} \right\} = \frac{1}{2} g^{41}(0) = 0$

$\left\{ {3 \atop 23} \right\} = \cot\theta$

$\left\{ {1 \atop 24} \right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{14}(0) = 0 \quad \left\{ {2 \atop 24} \right\} = \frac{1}{2} g^{22}(g_{22,4}) = \frac{\dot{T}}{2T}$

$\left\{ {3 \atop 24} \right\} = \frac{1}{2} g^{33}(0) = 0 \quad \left\{ {4 \atop 24} \right\} = \frac{1}{2} g^{41}(0) = 0$

$\left\{ {2 \atop 24} \right\} = -\frac{T^*}{2T}$

$\left\{ {1 \atop 33} \right\} = \frac{1}{2} g^{11}(-g_{33,1}) + \frac{1}{2} g^{14}(-g_{33,4}) = -\frac{1}{2}\frac{(1+V)}{T}\left(T'M^2 + T2MM'\right)\sin^2\theta - \frac{1}{2T}\left(\dot{T}M^2\sin^2\theta\right)$

$\therefore \left\{ {1 \atop 33} \right\} = -M^2(1+V)\sin^2\theta\left(\frac{T^*}{2T} + \frac{M'}{M}\right) + M^2\sin^2\theta\frac{T^*}{2T}$

$\left\{ {2 \atop 33} \right\} = \frac{1}{2} g^{22}(-g_{33,2}) = -\sin\theta\cos\theta.$

$\left\{ {2 \atop 33} \right\} = -\sin\theta\cos\theta$

$\left\{ {3 \atop 33} \right\} = \frac{1}{2} g^{33}(0) = 0 \quad \left\{ {4 \atop 33} \right\} = \frac{1}{2} g^{41}(-g_{33,1}) = -\frac{1}{2T}\left(TM^2\sin^2\theta\right)'$

$\left\{ {4 \atop 33} \right\} = -M^2\sin^2\theta\left(\frac{T^*}{2T} + \frac{M'}{M}\right)$

$\left\{ {1 \atop 34} \right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{14}(0) = 0 \quad \left\{ {2 \atop 34} \right\} = \frac{1}{2} g^{22}(0) = 0 \cdot \quad \left\{ {3 \atop 34} \right\} = \frac{1}{2} g^{33}(g_{33,4}) = \frac{T^*}{2T} = -\frac{T^*}{2T}$ (CC)

$\left\{ {4 \atop 34} \right\} = \frac{1}{2} g^{41}(0) = 0 .$

$$\boxed{\left\{ {3 \atop 34} \right\} = -\frac{T^*}{2T}}$$

$$\left\{ {1 \atop 44} \right\} = \frac{1}{2} g^{11}(2g_{14,4} - g_{44,1}) + \frac{1}{2} g^{14}(g_{44,4}) = -\frac{(1+V)}{2T}\left[2(-\dot{T}) - ((1+V)T)'\right] - \frac{1}{2T}((1+V)T)^{\cdot}$$

$$= (1+V)\frac{\dot{T}}{T} + (1+V)^2 \frac{T'}{2T} + \frac{(1+V)}{2} V' - (1+V)\frac{\dot{T}}{2T} - \frac{\dot{V}}{2}$$

$$\therefore \boxed{\left\{ {1 \atop 44} \right\} = -(1+V)\frac{T^*}{2T} - \frac{\dot{V}}{2} + (1+V)\frac{V'}{2} + (1+V)^2 \frac{T^*}{2T}}$$

$\left\{ {2 \atop 44} \right\} = \frac{1}{2} g^{22}(0) = 0 \quad \left\{ {3 \atop 44} \right\} = \frac{1}{2} g^{33}(0) = 0 \quad \left\{ {4 \atop 44} \right\} = \frac{1}{2} g^{41}(2g_{14,4} - g_{44,1})$

$$\left\{ {4 \atop 44} \right\} = -\frac{1}{2T}\left[2(-\dot{T}) - V'T - (1+V)T'\right] = \frac{\dot{T}}{T} + \frac{V'}{2} + (1+V)\frac{T'}{2T}$$

$$\therefore \boxed{\left\{ {4 \atop 44} \right\} = \frac{\dot{T}}{T} - \frac{T^*}{T} + \frac{V'}{2} + (1+V)\frac{T^*}{2T}}$$

Now we calculate R_{ik}.

$$R_{ik} = -\left\{ {n \atop ik} \right\}_{,n} + \left\{ {n \atop i,n} \right\}_{,k} + \left\{ {l \atop in} \right\}\left\{ {n \atop lk} \right\} - \left\{ {n \atop ik} \right\}\left\{ {l \atop nl} \right\}.$$

$$+ \left\{ {1 \atop 11} \right\}\left\{ {1 \atop 11} \right\} + \left\{ {2 \atop 12} \right\}\left\{ {2 \atop 21} \right\} + \left\{ {3 \atop 13} \right\}\left\{ {3 \atop 31} \right\}$$

$$\therefore R_{11} = -\left\{ {n \atop 11} \right\}_{,n} + \left\{ {n \atop 1n} \right\}_{,1} + \left\{ {1 \atop 11} \right\}\left\{ {1 \atop 11} \right\} - \left\{ {n \atop 11} \right\}\left\{ {l \atop 1l} \right\}$$

$$+ \left\{ {1 \atop 11} \right\}\left\{ {1 \atop 11} \right\} + \left\{ {2 \atop 12} \right\}^2 + \left\{ {3 \atop 13} \right\}^2 - \left\{ {1 \atop 11} \right\}^2$$

$$= -\left\{ {1 \atop 11} \right\}_{,1} + \left\{ {1 \atop 11} \right\}_{,1} + \left\{ {2 \atop 12} \right\}_{,1} + \left\{ {3 \atop 13} \right\}_{,1} + \left\{ {1 \atop 11} \right\}^2 + \left\{ {2 \atop 12} \right\}^2 + \left\{ {3 \atop 13} \right\}^2 - \left\{ {1 \atop 11} \right\}^2$$

$$- \left\{ {1 \atop 11} \right\}\left\{ {2 \atop 12} \right\} - \left\{ {1 \atop 11} \right\}\left\{ {3 \atop 13} \right\}$$

$$= 2\left(\frac{T^*}{2T} + \frac{M'}{M}\right)' + 2\left(\frac{T^*}{2T} + \frac{M'}{M}\right)^2 - 2\frac{T^*}{T}\left(\frac{T^*}{2T} + \frac{M'}{M}\right)$$

$$= 2\frac{T^{**}}{2T} - 2\frac{T^{*2}}{2T^2} + \frac{2M''}{M} - \frac{2M'^2}{M^2} + \frac{2T^{*2}}{4T^2} + \frac{2M'^2}{M^2} + 4\frac{T^*}{2T}\frac{M'}{M} - \frac{2T^{*2}}{2T^2} - 2\frac{T^*}{T}\frac{M'}{M}$$

$$= \frac{T^{**}}{T} + \frac{T^{*2}}{T^2}\left(-1 + \frac{1}{2} - 1\right) + \frac{2M''}{M} = \frac{T^{**}}{T} - \frac{3}{2}\frac{T^{*2}}{T^2} - \frac{2}{R^2}$$

$$\boxed{R_{11} = \frac{T^{**}}{T} - \frac{3}{2}\frac{T^{*2}}{T^2} - \frac{2}{R^2}}$$

(-8)

$$R_{14} = -\left\{{}^{1}_{14}\right\}_{,1} + \left\{{}^{n}_{14}\right\}\left\{{}^{n}_{1n}\right\}_{,4} + \left\{{}^{l}_{1n}\right\}\left\{{}^{n}_{l4}\right\} - \left\{{}^{n}_{14}\right\}\left\{{}^{l}_{nl}\right\}$$

$$= -\left\{{}^{1}_{14}\right\}_{,1} + \left(2\frac{T^*}{T} + \frac{2M'}{M}\right)_{,4} + \left\{{}^{1}_{11}\right\}\left\{{}^{1}_{14}\right\} + \left\{{}^{2}_{12}\right\}\left\{{}^{2}_{24}\right\} + \left\{{}^{3}_{13}\right\}\left\{{}^{3}_{34}\right\} + \left\{{}^{1}_{14}\right\}\left\{{}^{4}_{14}\right\}$$

$$- \left\{{}^{1}_{14}\right\}\left[\left\{{}^{1}_{11}\right\} + \left\{{}^{2}_{12}\right\} + \left\{{}^{3}_{13}\right\}\right]$$

$$= \left[(1+V)\frac{T^*}{2T} + \frac{V'}{2}\right]' + \left(-\frac{2T^*}{T}\right)^* + 2\left(\frac{T^*}{2T} + \frac{M'}{M}\right)\left(-\frac{T^*}{2T}\right)$$

$$+ \left[(1+V)\frac{T^*}{2T} + \frac{V'}{2}\right]\left(\frac{T^*}{T} + \frac{2M'}{M}\right)$$

$$= V'\frac{T^*}{2T} + (1+V)\left(\frac{T^{**}}{2T} - \frac{1}{2}\left(\frac{T^*}{T}\right)^2\right) + \frac{V''}{2} - \frac{2T^{**}}{T} + 2\left(\frac{T^*}{T}\right)^2 - 2\left(\frac{T^*}{T}\right)^2 \frac{1}{4}$$

$$- \frac{T^*}{2T}\frac{M'}{M} + (1+V)\left(\frac{T^*}{T}\right)^2\frac{1}{2} + (1+V)\frac{T^*}{T}\frac{M'}{M} + V'\frac{T^*}{2T} + V'\frac{M'}{M}$$

$$= \frac{V''}{2} + V'\left(\frac{T^*}{T} + \frac{M'}{M}\right) + V\left[\frac{T^{**}}{2T} - \frac{T^{*2}}{2T^2} + \frac{T^{*2}}{2T^2} + \frac{T^*}{T}\frac{M'}{M}\right)$$

$$+ \frac{T^{**}}{2T} - \frac{T^{*2}}{2T^2} - \frac{2T^{**}}{T} + 2\left(\frac{T^*}{T}\right)^2 - \frac{T^{*2}}{2T^2} - \frac{T^*}{T}\frac{M'}{M} + \frac{T^{*2}}{2T^2}$$

$$+ \frac{T'}{T}\frac{M'}{M}$$

$$= \frac{V''}{2} + V'\left(\frac{T^*}{T} + \frac{M'}{M}\right) + V\left(\frac{T^{**}}{2T} + \frac{T^*}{T}\frac{M'}{M}\right)$$

$$- \frac{3}{2}\frac{T^{**}}{T} + \frac{3}{2}\frac{T^{*2}}{T^2}$$

$$\therefore \boxed{R_{14} = \frac{V''}{2} + V'\left(\frac{T^*}{T} + \frac{M'}{M}\right) + V\left(\frac{T^{**}}{2T} + \frac{T^*}{T}\frac{M'}{M}\right) - \frac{3}{2}\frac{T^{**}}{T} + \frac{3}{2}\frac{T^{*2}}{T^2}}$$

Next we find R_{22}.

$$R_{22} = -\left\{ {n \atop 22} \right\}_{,n} + \left\{ {n \atop 2n} \right\}_{,2} + \left\{ {2n \atop n} \right\}\left\{ {n \atop 12} \right\} - \left\{ {n \atop 22} \right\}\left\{ {l \atop nl} \right\}$$

$$= -\left\{ {1 \atop 22} \right\}_{,1} - \left\{ {4 \atop 22} \right\}_{,4} + (\cot\theta)_{,2} + 2\left\{ {2 \atop 21} \right\}\left\{ {1 \atop 22} \right\} + \cancel{\left\{ {3 \atop 22} \right\}\left\{ {1 \atop 12} \right\}} + 2\left\{ {4 \atop 22} \right\}\left\{ {2 \atop 42} \right\}$$

$$+ \left\{ {3 \atop 23} \right\}\left\{ {3 \atop 32} \right\} + \cancel{\left\{ {2 \atop 24} \right\}\left\{ {4 \atop 22} \right\}} - \left\{ {1 \atop 22} \right\}\left(\left\{ {1 \atop 11} \right\} + 2\left\{ {2 \atop 12} \right\} \right) - \left\{ {4 \atop 22} \right\}\left(\left\{ {1 \atop 41} \right\} + 2\left\{ {2 \atop 42} \right\} + \left\{ {4 \atop 44} \right\} \right)$$

$$= +M^2\left[\frac{2M'}{M}(1+V)\left(\frac{T^*}{2T} + \frac{M'}{M} \right) - \frac{2M'}{M}\frac{T^*}{2T} + V'\left(\frac{T^{**}}{2T} + \frac{M'}{M} \right) + (1+V)\left(\frac{T^{**}}{2T} - \frac{T^{*2}}{2T^2} + \frac{M''}{M} - \frac{M'^2}{M^2} \right) \right.$$

$$\left. - \frac{T^{**}}{2T} + \frac{T^{*2}}{2T^2} \right\} + \left(- \frac{T^{**}}{2T} + \frac{T^{*2}}{2T^2} \right) - \operatorname{cosec}^2\theta + \cot^2\theta$$

$$+M^2\left[(1+V)\left(\frac{T^*}{2T} + \frac{M'}{M} \right) - \frac{T^*}{2T} \right]\frac{T^*}{T} + M^2\left(\frac{T^*}{2T} + \frac{M'}{M} \right)\left[-(1+V)\frac{T^*}{2T} - \frac{1}{2}V' - \frac{T^*}{T} \right.$$

$$\left. + (1+V)\frac{T^*}{2T} + \frac{1}{2}V' \right]$$

$$\therefore \frac{R_{22}+1}{M^2} = V'\left(\frac{T^*}{2T} + \frac{M'}{M} \right) + V\left[\frac{2M'}{M}\left(\frac{T^*}{2T} + \frac{M'}{M} \right) + \frac{T^{**}}{2T} - \frac{T^{*2}}{2T^2} + \frac{M''}{M} - \frac{M'^2}{M^2} \right.$$

$$\left. + \frac{T^{*2}}{2T^2} + \frac{M'}{M}\frac{T^*}{T} \right]$$

$$+\frac{2M'}{M}\frac{T^*}{2T} + \frac{2M'^2}{M^2} + \frac{T^{**}}{2T} - \frac{T^{*2}}{2T^2} + \frac{M''}{M}$$

$$-\frac{2M'}{M}\frac{T^*}{2T} - \frac{M'^2}{M^2} - \frac{T^{**}}{2T} + \frac{T^{*2}}{2T^2}$$

$$+\frac{M'}{M}\frac{T^*}{T} - \frac{T^{**}}{2T} + \frac{T^{*2}}{2T^2}$$

$$-\frac{M'}{M}\frac{T^*}{T} + \frac{T^{*2}}{2T^2}$$

$$- \frac{T^{*2}}{2T^2}$$

$$- \frac{T^{*2}}{2T^2}$$

$$\therefore \frac{R_{22}+1}{M^2} = V'\left(\frac{T^*}{2T}+\frac{M'}{M}\right)+V\left(2\frac{M'}{M}\frac{T^*}{T}+\frac{M''}{M}+\frac{M'^2}{M^2}+\frac{T^{**}}{2T}\right)$$
$$+\frac{M'^2}{M^2}-\frac{T^{**}}{2T}+\frac{M''}{M}$$

To find R_{44} - it is better to write $z = 1+V$. Then the relevent 3 index symbols become

$$\left\{{}^{1}_{41}\right\} = -z\frac{T^*}{2T}-\frac{z'}{2} \qquad \left\{{}^{1}_{44}\right\} = z^2\frac{T^*}{2T}+\frac{zz'}{2}-z\frac{T^*}{2T}-\frac{\dot z}{2}$$

$$\left\{{}^{4}_{44}\right\} = -\frac{T^*}{T}+\frac{z'}{2}+z\frac{T^*}{2T}$$

We now write

$$R_{44} = -\left\{{}^{1}_{44}\right\}_{,1}-\left\{{}^{4}_{44}\right\}_{,4}+\left[\left\{{}^{1}_{41}\right\}\left\{{}^{1}_{14}\right\}+\left\{{}^{2}_{42}\right\}\left\{{}^{2}_{24}\right\}+\left\{{}^{3}_{43}\right\}\left\{{}^{3}_{34}\right\}+\left\{{}^{1}_{44}\right\}\left\{{}^{4}_{14}\right\}\right.$$
$$\left.+\left\{{}^{4}_{44}\right\}\left\{{}^{4}_{44}\right\}-\left\{{}^{1}_{44}\right\}\left\{{}^{n}_{1n}\right\}-\left\{{}^{4}_{44}\right\}\left[\left\{{}^{1}_{41}\right\}+\left\{{}^{2}_{42}\right\}+\left\{{}^{3}_{43}\right\}+\left\{{}^{4}_{44}\right\}\right]\right.$$

$$= -\left[z^2\frac{T^*}{2T}+\frac{zz'}{2}-z\frac{T^*}{2T}-\frac{\dot z}{2}\right]_{,1}+\left[-z\frac{T^*}{2T}-\frac{z'}{2}-\frac{T^*}{T}\right]_{,4}$$

$$+\left(-z\frac{T^*}{2T}-\frac{z'}{2}\right)^2+2\frac{T^{*2}}{4T^2}-\left(z^2\frac{T^*}{2T}+\frac{zz'}{2}-z\frac{T^*}{2T}-\frac{\dot z}{2}\right)\left(-z\frac{T^*}{2T}-\frac{z'}{2}-\frac{T^*}{T}\right)$$

$$-\left(z^2\frac{T^*}{2T}+\frac{zz'}{2}-z\frac{T^*}{2T}-\frac{\dot z}{2}\right)\left(2\frac{T^*}{T}+2\frac{M'}{M}\right)+\left(\frac{T^*}{T}+\frac{z'}{2}+z\frac{T^*}{2T}\right)\left(\right.$$

$$-\left(\frac{T^*}{T}+\frac{z'}{2}+z\frac{T^*}{2T}\right)\left(-z\frac{T^*}{2T}-\frac{z'}{2}-\frac{T^*}{T}\right)$$

$$= -2Z\dot{Z}\frac{T^*}{2T} - Z^2\left(\frac{T^{**}}{2T} - \frac{T^{*2}}{2T^2}\right) - \frac{\dot{Z}^2}{2} - \frac{Z\ddot{Z}}{2} + \dot{Z}\frac{T^*}{2T} + Z\left(\frac{T^{**}}{2T} - \frac{T^{*2}}{2T^2}\right) + \frac{\dot{Z}^2}{2}$$

$$- \dot{Z}\frac{T^*}{2T} + Z\left(\frac{T^{**}}{2T} - \frac{T^{*2}}{2T^2}\right) - \frac{\dot{Z}^2}{2} + \frac{T^{**}}{T} - \frac{T^{*2}}{T^2} + Z^2\frac{T^{*2}}{4T^2} + \frac{\dot{Z}^2}{4} + Z\dot{Z}\frac{T^*}{2T}$$

$$+ \frac{T^{*2}}{2T^2} - Z^2\frac{T^*}{T}\left(\frac{T^*}{T} + \frac{M'}{M}\right) - Z\dot{Z}\left(\frac{T^*}{T} + \frac{M'}{M}\right) + Z\frac{T^*}{T}\left(\frac{T^*}{T} + \frac{M'}{M}\right) + \dot{Z}\left(\frac{T^*}{T} + \frac{M'}{M}\right)$$

$$- Z\frac{T^{*2}}{2T^2} - \frac{\dot{Z}}{2}\frac{T^*}{T} - \frac{T^{*2}}{T^2} + Z\dot{Z}\frac{T^*}{4T} + \frac{\dot{Z}^2}{4} + \frac{\dot{Z}}{2}\frac{T^*}{T} + Z^2\left(\frac{T^{*2}}{4T^2}\right) + Z\dot{Z}\frac{T^*}{4T} + Z\frac{T^{*2}}{2T^2}$$

$$= -\frac{Z\ddot{Z}}{2} + \dot{Z}^2\left(-\frac{1}{2} + \frac{1}{4} + \frac{1}{4}\right) + Z\dot{Z}\left(-\frac{T^*}{T} + \frac{T^*}{2T} - \frac{T^*}{T} - \frac{M'}{M} + \frac{T^*}{4T} + \frac{T^*}{4T}\right)$$

$$+ Z^2\left(-\frac{T^{**}}{2T} + \frac{T^{*2}}{2T^2} + \frac{T^{*2}}{4T^2} - \frac{T^{*2}}{T^2} - \frac{T^*}{T}\frac{M'}{M} + \frac{T^{*2}}{4T^2}\right)$$

$$+ \dot{Z}\left(\frac{T^*}{2T} - \frac{T^*}{2T} + \frac{T^*}{2T}\right) + \dot{Z}\left(-\frac{T^*}{2T} + \frac{T^*}{T} + \frac{M'}{M}\right)$$

$$+ Z\left(\frac{T^{**}}{2T} - \frac{T^{*2}}{2T^2} + \frac{T^{**}}{2T} - \frac{T^{*2}}{2T^2} + \frac{T^{*2}}{T^2} + \frac{T^*}{T}\frac{M'}{M} - \frac{T^{*2}}{2T^2} + \frac{T^{*2}}{2T^2}\right)$$

$$+ \frac{T^{**}}{T}(1) + \frac{T^{*2}}{T^2}\left(-1 + \frac{1}{2} - 1\right)$$

$$R_{44} = -\frac{Z\ddot{Z}}{2} - Z\dot{Z}\left(\frac{T^*}{T} + \frac{M'}{M}\right) + Z^2\left(-\frac{T^{**}}{2T} - \frac{T^*}{T}\frac{M'}{M}\right) + \dot{Z}\frac{T^*}{2T}$$
$$+ \dot{Z}\left(\frac{T^*}{2T} + \frac{M'}{M}\right) + Z\left(\frac{T^{**}}{T} + \frac{T^*}{T}\frac{M'}{M}\right) + \frac{T^{**}}{T} - \frac{3}{2}\frac{T^{*2}}{T^2}$$

Let us work out $R_{44} + 2R_{41}$. We rewrite R_{41} in terms of Z

$V = \frac{Z-1}{Z}$, $V' = Z'$, $V'' = Z''$

(42)

$$R_{41} = \frac{Z''}{2} + Z'\left(\frac{T^*}{T} + \frac{M'}{M}\right) + (Z-1)\left(\frac{T^{**}}{2T} + \frac{T^*}{T}\frac{M'}{M}\right) - \frac{3}{2}\frac{T^{**}}{T} + \frac{3}{2}\frac{T^{*2}}{T^2}$$

$$= \frac{Z''}{2} + Z'\left(\frac{T^*}{T} + \frac{M'}{M}\right) + Z\left(\frac{T^{**}}{2T} + \frac{T^*}{T}\frac{M'}{M}\right) - \frac{2T^{**}}{2T} - \frac{T^*}{T}\frac{M'}{M} - \frac{3}{2}\frac{T^{**}}{T} + \frac{3}{2}\frac{T^{*2}}{T^2}$$

$$\therefore ZR_{41} = \frac{ZZ''}{2} + ZZ'\left(\frac{T^*}{T} + \frac{M'}{M}\right) + Z^2\left(\frac{T^{**}}{2T} + \frac{T^*}{T}\frac{M'}{M}\right) + Z\left(-\frac{2T^{**}}{T} + \frac{3}{2}\frac{T^{*2}}{T^2} - \frac{T^*}{T}\frac{M'}{M}\right)$$

$$\therefore R_{44} + ZR_{41} = Z'\left(\frac{T^*}{T}\right) + \dot{Z}\left(\frac{T^*}{2T} + \frac{M'}{M}\right) + Z\left[\frac{T^{**}}{T} + \frac{T^*}{T}\frac{M'}{M} - \frac{2T^{**}}{T} + \frac{3}{2}\frac{T^{*2}}{T^2} - \frac{T^*}{T}\frac{M'}{M}\right]$$

$Z'\frac{T^*}{2T}$

$$+ \frac{T^{**}}{T} - \frac{3}{2}\frac{T^{*2}}{T^2}$$

$$= Z'\left(\frac{T^*}{2T}\right) + \dot{Z}\left(\frac{T^*}{2T} + \frac{M'}{M}\right) + Z\left(-\frac{T^{**}}{T} + \frac{3}{2}\frac{T^{*2}}{T^2}\right) + \frac{T^{**}}{T} - \frac{3}{2}\frac{T^{*2}}{T^2}$$

$$\boxed{R_{44} + 2R_{41} = V'\left(\frac{T^*}{2T}\right) + \dot{V}\left(\frac{T^*}{2T} + \frac{M'}{M}\right) + V\left(-\frac{T^{**}}{T} + \frac{3}{2}\frac{T^{*2}}{T^2}\right)}$$

We now turn to Physics. Refer to p. 43

$$R_{ik} = -8\pi\left[(\phi+p)v_i v_k - \frac{(p-\phi)}{2}g_{ik} + \sigma w_i w_k\right] + \Lambda g_{ik}$$

Raise the suffix k

$$\therefore R_i{}^k = -8\pi\left[(\phi+p)v_i v^k - \frac{(p-\phi)}{2}g_i{}^k + \sigma w_i w^k\right] + \Lambda g_i{}^k$$

Take $v^k = (0,0,0,v^4)$ $\therefore g_{ik} v^i v^k = 1 \longrightarrow -T v^1 v^4 - T v^4 v^1 + (1+V) T (v^4)^2 = 1$

$\therefore$ ~~ZT~~ $ZT(v^4)^2 = 1$ $\boxed{v^4 = \frac{1}{\sqrt{(ZT)}}, \; v_1 = -\sqrt{\frac{T}{2}}, \; v_4 = \sqrt{(ZT)}}$

$v_1 = g_{1\alpha} v^\alpha = g_{14} v^4 = \frac{-T}{\sqrt{(ZT)}} = -\sqrt{\frac{T}{2}}$. $v_4 = g_{4\alpha} v^\alpha = g_{44} v^4 = \frac{ZT}{\sqrt{ZT}} = \sqrt{(ZT)}$

Next take $\omega^k = (\omega^1, 0, 0, 0)$ $\therefore g_{ik} \omega^i \omega^k = 0$ is satisfied

Normalize ω^k by $g_{ik} \omega^i \omega^k = 1$ $\therefore -T v^1 \omega^4 - T v^4 \omega^1 + ZT v^4 \omega^4 = 1$

$\therefore -T v^4 \omega^1 = 1$ $\therefore \omega^1 = -\frac{1}{T v_4} = -\frac{1}{T} \sqrt{(ZT)} = -\sqrt{\frac{Z}{T}}$.

$\omega_1 = g_{1\alpha} \omega^\alpha = g_{14} \omega^4 = 0$ $\omega_4 = g_{4\alpha} \omega^\alpha = g_{41} \omega^1 = \frac{1}{T}(-\sqrt{\;}) - T\left(-\sqrt{\frac{Z}{T}}\right)$

$\boxed{\therefore \omega^1 = -\sqrt{\frac{Z}{T}}, \quad \omega_4 = \sqrt{ZT}}$

$\therefore R_1^{\,1} = -8\pi\left[-8\sigma\left[-\frac{(p-\phi)}{2} \right] + \Lambda \right.$

$R_2^{\,2} = -8\pi\left[-\frac{(p-\phi)}{2} \right] + \Lambda = R_3^{\,3}$

$R_4^{\,4} = -8\pi\left[(\phi+p) - \frac{(p-\phi)}{2} \right] + \Lambda$

$R_4^{\,1} = -8\pi\left[\sigma \sqrt{ZT} \left(-\sqrt{\frac{Z}{T}}\right) \right]$

$R_1^{\,4} = -8\pi\left[(\phi+p)\left(-\sqrt{\frac{T}{2}}\right) \frac{1}{\sqrt{ZT}} \right]$

$\therefore R_1^{\,1} = R_2^{\,2}$ is the field equ.

$R_4^{\,1} = 8\pi\sigma \cdot 2$ gives σ.

(C-9) Reverting back to geometry, let us find $R_i^{\ k}$.

$$R_1^{\ 1} = R_{1\alpha} g^{1\alpha} = R_{11} g^{11} + R_{14} g^{14} = -\frac{2}{T} R_{11} - \frac{1}{T} R_{14}$$

$$R_2^{\ 2} = R_{2\alpha} g^{2\alpha} = R_{22} g^{22} = -\frac{1}{T} \frac{R_{22}}{M^2} = R_3^{\ 3}$$

$$R_4^{\ 4} = R_{4\alpha} g^{4\alpha} = R_{41} g^{41} = -\frac{1}{T} R_{41}$$

$$R_4^{\ 1} = R_{4\alpha} g^{1\alpha} = R_{41} g^{11} + R_{44} g^{14} = -\frac{2}{T} R_{41} - \frac{1}{T} R_{44}$$

$$R_1^{\ 4} = g_{1\alpha} R_{1\alpha} g^{4\alpha} = R_{11} g^{41} = -\frac{1}{T} R_{11}.$$

Our field equ. is $R_1^{\ 1} = R_2^{\ 2}$

$$\therefore -\frac{2}{T} R_{11} - \frac{1}{T} R_{14} = -\frac{1}{T} \frac{R_{22}}{M^2}$$

$$\therefore \boxed{2R_{11} + R_{14} - \frac{R_{22}}{M^2} = 0}$$

$$8\pi\sigma 2 = R_4^{\ 1} = -\frac{2}{T} R_{41} - \frac{1}{T} R_{44} = -\frac{1}{T}(2R_{41} + R_{44})$$

To find ϕ and ρ we take $R_2^{\ 2} = \frac{8\pi(\rho - \phi)}{2} + \Lambda$

and $R_1^{\ 4} = 8\pi \frac{(\phi + \rho)}{2}$

$$\therefore 2R_2^{\ 2} + 2R_1^{\ 4} = 2 \cdot 8\pi\rho + 2\Lambda$$

$$\therefore 8\pi\rho = -\Lambda + R_2^{\ 2} + \tfrac{1}{2} 2 R_1^{\ 4}$$

$$\therefore \boxed{8\pi\rho = -\Lambda - \frac{1}{T}\frac{R_{22}}{M^2} - \frac{1}{2}\frac{2}{T}R_{11}}$$

Again $2R_1^4 - 2R_2^2 = 8\pi(2\rho) - 2\Lambda \quad \therefore 8\pi\rho = \Lambda + \frac{2}{2}R_1^4 - R_2^2$

$$\therefore \boxed{8\pi\rho = \Lambda - \frac{2}{2T}R_{11} + \frac{1}{T}\frac{R_{22}}{M^2}}$$

We now write in full the field equation $2R_{11} + R_{14} - \frac{R_{22}}{M^2} = 0$

$$\therefore (1+V)\left(\frac{T^{**}}{T} - \frac{3}{2}\frac{T^{*2}}{T^2} - \frac{2}{R^2}\right) + \frac{V''}{2} + V'\left(\frac{T^*}{T} + \frac{M'}{M}\right) + V\left(\frac{T^{**}}{2T} + \frac{T^*}{T}\frac{M'}{M}\right) - \frac{3}{2}\frac{T^{**}}{T} + \frac{3}{2}\frac{T^{*2}}{T^2}$$
$$-\left(-\frac{1}{M^2}\right) - V'\left(\frac{T^*}{2T} + \frac{M'}{M}\right) - V\left(2\frac{M'}{M}\frac{T^*}{T} + \frac{M''}{M} + \frac{M'^2}{M^2} + \frac{T^{**}}{2T}\right) + \frac{M'^2}{M^2} + \frac{T^{**}}{2T} + \frac{M''}{M} = 0$$

$$\therefore \frac{V''}{2} + V'\left(\frac{T^*}{2T}\right) + V\left[\frac{T^{**}}{T}\left(1 + \frac{1}{2} - \frac{1}{2}\right) + \frac{T^{*2}}{T^2}\left(-\frac{3}{2}\right) - \frac{2}{R^2} - \frac{M''}{M} - \frac{M'^2}{M^2} + \frac{T^*}{T}\frac{M'}{M}(1-2)\right]$$
$$+ \frac{T^{**}}{T}\left(1 - \frac{3}{2} + \frac{1}{2}\right) + \frac{T^{*2}}{T^2}\left(-\frac{3}{2} + \frac{3}{2}\right) - \frac{2}{R^2} + \frac{1}{M^2} - \frac{M'^2}{M^2} - \frac{M''}{M} = 0$$

But $M = R\sin\frac{r}{R} \quad \frac{M'}{M} = \frac{1}{R}\cot\frac{r}{R} \quad \frac{M''}{M} = -\frac{1}{R^2}$

$$\therefore \frac{V''}{2} + V'\left(\frac{T^*}{2T}\right) + V\left[\frac{T^{**}}{T} - \frac{3}{2}\frac{T^{*2}}{T^2} - \frac{2}{R^2} + \frac{1}{R^2} - \frac{1}{R^2}\cot^2\frac{r}{R} - \frac{T^*}{T}\frac{M'}{M}\right]$$
$$+ -\frac{2}{R^2} + \frac{1}{R^2} + \frac{1}{R^2}\operatorname{cosec}^2\frac{r}{R} - \frac{1}{R^2}\cot^2\frac{r}{R} + \frac{1}{R^2} = 0$$

$$\therefore \boxed{\frac{V''}{2} + V'\left(\frac{T^*}{2T}\right) + V\left(\frac{T^{**}}{T} - \frac{3}{2}\frac{T^{*2}}{T^2} - \frac{1}{R}\cot\frac{r}{R}\frac{T^*}{T} - \frac{1}{R^2}\operatorname{cosec}^2\frac{r}{R}\right) = 0}$$

This is what Narlikar called the Equation of Isotropy.

(6-6)

$$8\pi\rho = +\Lambda - \frac{2}{2T}R_{11} + \frac{1}{T}\frac{R_{22}}{M^2} = \Lambda - \frac{(1+V)}{2T}\left[\frac{T^{**}}{T} - \frac{3}{2}\frac{T^{*2}}{T^2} - \frac{2}{R^2}\right]$$

$$+\frac{1}{T}\left[-\frac{1}{M^2} + V'\left(\frac{T^*}{2T} + \frac{M'}{M}\right) + V\left(\frac{T^{**}}{2T} + \frac{2M'}{M}\frac{T^*}{T} + \frac{M''}{M} + \frac{M'^2}{M^2}\right) - \frac{T^{**}}{2T} + \frac{M''}{M} + \frac{M'^2}{M^2}\right]$$

$$= \Lambda + \frac{1}{T}\left[(1+V)\left(-\frac{T^{**}}{2T} + \frac{3}{4}\frac{T^{*2}}{T^2} + \frac{1}{R^2}\right) + V'\left(\frac{T^*}{2T} + \frac{M'}{M}\right) + V\left(\frac{T^{**}}{2T} + \frac{2M'}{M}\frac{T^*}{T} + \frac{M''}{M} + \frac{M'^2}{M^2}\right)\right.$$

$$\left. - \frac{T^{**}}{2T} + \frac{M''}{M} + \frac{M'^2}{M^2} - \frac{1}{M^2}\right]$$

$$= \Lambda + \frac{1}{T}\left[V'\left(\frac{T^*}{2T} + \frac{M'}{M}\right) + V\left(-\frac{T^{**}}{2T} + \frac{3}{4}\frac{T^{*2}}{T^2} + \frac{1}{R^2} + \frac{T^{**}}{2T} + \frac{2M'}{M}\frac{T^*}{T} + \frac{M''}{M} + \frac{M'^2}{M^2}\right)\right.$$

$$\left. - \frac{T^{**}}{2T} + \frac{3}{4}\frac{T^{*2}}{T^2} + \frac{1}{R^2} - \frac{T^{**}}{2T} + \frac{M''}{M} + \frac{M'^2}{M^2} - \frac{1}{M^2}\right]$$

$M = R\sin\frac{r}{R} \quad \frac{M'}{M} = \frac{1}{R}\cot\frac{r}{R} \quad \frac{M''}{M} = -\frac{1}{R^2}$

$$\therefore 8\pi\rho = \Lambda + \frac{1}{T}\left[V'\left(\frac{T^*}{2T} + \frac{1}{R}\cot\frac{r}{R}\right) + V\left(\frac{3}{4}\frac{T^{*2}}{T^2} + \frac{2}{R}\cot\frac{r}{R}\frac{T^*}{T} + \frac{1}{R^2} - \frac{1}{R^2} + \frac{1}{R^2}\cot^2\frac{r}{R}\right)\right.$$

$$\left. - \frac{T^{**}}{T} + \frac{3}{4}\frac{T^{*2}}{T^2} + \frac{1}{R^2} - \frac{1}{R^2} + \frac{1}{R^2}\cot^2\frac{r}{R} - \frac{1}{R^2}\operatorname{cosec}^2\frac{r}{R}\right]$$

$$\therefore \boxed{8\pi\rho = \Lambda + \frac{1}{T}\left[V'\left(\frac{T^*}{2T} + \frac{1}{R}\cot\frac{r}{R}\right) + V\left(\frac{3}{4}\frac{T^{*2}}{T^2} + \frac{2}{R}\cot\frac{r}{R}\frac{T^*}{T} + \frac{1}{R^2}\cot^2\frac{r}{R}\right)\right.}$$

$$\boxed{\left. - \frac{T^{**}}{T} + \frac{3}{4}\frac{T^{*2}}{T^2} - \frac{1}{R^2}\right]}$$

Next $8\pi p = -\Lambda - \frac{1}{T}\frac{R_{22}}{M^2} - \frac{2}{2T}R_{11}$

$$= -\Lambda - \frac{1}{T}\left[-\frac{1}{M^2} + V'\left(\frac{T^*}{2T} + \frac{M'}{M}\right) + V\left(\frac{T^{**}}{2T} + \frac{2M'}{M}\frac{T^*}{T} + \frac{M''}{M} + \frac{M'^2}{M^2}\right)\right.$$

$$\left. - \frac{T^{**}}{2T} + \frac{M''}{M} + \frac{M'^2}{M^2} + (1+V)\left(\frac{T^{**}}{2T} - \frac{3}{4}\frac{T^{*2}}{T^2} - \frac{1}{R^2}\right)\right]$$

$$= -\Lambda - \frac{1}{T}\left[V'\left(\frac{T^*}{2T} + \frac{1}{R}\cot\frac{r}{R}\right) + V\left(\frac{T^{**}}{T} - \frac{3}{4}\frac{T^{*2}}{T^2} + \frac{2M'}{M}\frac{T^*}{T} - \frac{1}{R^2} + \frac{1}{R^2}\cot^2\frac{r}{R} - \frac{1}{R^2}\right)\right.$$

$$\left. - \frac{3}{4}\frac{T^{*2}}{T^2} - \frac{1}{R^2} - \frac{1}{R^2} + \frac{1}{R^2}\cot^2\frac{r}{R} - \frac{1}{R^2}\operatorname{cosec}^2\frac{r}{R}\right]$$

$$8\pi P = -\Lambda - \frac{1}{T}\left[V'\left(\frac{T^*}{2T} + \frac{1}{R}\cot\frac{r}{R}\right) + V\left(\frac{T^{**}}{T} - \frac{3}{4}\frac{T^{*2}}{T^2} + \frac{2}{R}\cot\frac{r}{R}\frac{T^*}{T} - \frac{3}{R^2} + \frac{1}{R^2}\operatorname{cosec}^2\frac{r}{R}\right) - \frac{3}{4}\frac{T^{*2}}{T^2} - \frac{3}{R^2}\right]$$

Lastly $8\pi\sigma_2 = -\frac{1}{T}\left[2R_{41} + R_{44}\right] = -\frac{1}{T}\left[V'\left(\frac{T^*}{T}\right) + \dot{V}\left(\frac{T^*}{2T} + \frac{1}{R}\cot\frac{r}{R}\right) + V\left(-\frac{T^{**}}{T} + \frac{3}{2}\frac{T^{*2}}{T^2}\right)\right]$

$$\therefore\ 8\pi\sigma = -\frac{1}{T^2}\left[V'\left(\frac{T^*}{2T}\right) + \dot{V}\left(\frac{T^*}{2T} + \frac{1}{R}\cot\frac{r}{R}\right) + V\left(-\frac{T^{**}}{T} + \frac{3}{2}\frac{T^{*2}}{T^2}\right)\right]$$

We now take the equation of isotropy and carry out its approximate solution. Stage 1. $T=1$ $T^*=0$ $T^{**}=0$. The equation is

$$\frac{V''}{2} - \frac{V}{R^2}\operatorname{cosec}^2\frac{r}{R} = 0 \qquad V_0 = -\frac{2m}{R}\cot\frac{r}{R} \qquad \dot{m}=0 \text{ or } m = m(t)$$

at this stage $8\pi\sigma_0 = -\dfrac{1}{1-\frac{2m}{R}\cot\frac{r}{R}} \cdot \frac{1}{R}\cot\frac{r}{R}\cdot\left(-\frac{2m}{R}\cot\frac{r}{R}\right)$

$$= \frac{\frac{2\dot{m}}{R^2}\cot^2\frac{r}{R}}{1-\frac{2m}{R}\cot\frac{r}{R}}$$

$$8\pi\rho_0 = \Lambda + \frac{2m}{R^2}\operatorname{cosec}^2\frac{r}{R}\frac{1}{R}\cot\frac{r}{R} - \frac{2m}{R}\cot\frac{r}{R}\frac{1}{R^2}\cot^2\frac{r}{R} - \frac{1}{R^2}$$

$$= \Lambda + \frac{2m}{R^3}\cot\frac{r}{R} - \frac{1}{R^2} = \Lambda - \frac{1}{R^2}\left(1 - \frac{2m}{R}\cot\frac{r}{R}\right)$$

$$8\pi P_0 = -\Lambda - \left[\frac{2m}{R^3}\operatorname{cosec}^2\frac{r}{R}\cot\frac{r}{R} + \frac{6m}{R^3}\cot\frac{r}{R} - \frac{2m}{R^3}\cot\frac{r}{R}\operatorname{cosec}^2\frac{r}{R}\right] + \frac{3}{R^2}$$

$$= -\Lambda + \frac{3}{R^2}\left(1 - \frac{2m}{R}\cot\frac{r}{R}\right).$$

At this stage we have radiating particle in Einstein Universe

Stage 2. Retain T^* as a small function and neglect $T^{*\,*}$ and $T^{*\,2}$. The equation of isotropy is

$$V = V_0 + V_1 \frac{T^*}{T}$$

$$\therefore \frac{V_0''}{2} + \frac{V_1''}{2}\frac{T^*}{T} + \frac{T^*}{2T}V_0' - \frac{V_0}{R^2}\operatorname{cosec}^2\frac{z}{R} - V_1\frac{T^*}{T}\frac{1}{R^2}\operatorname{cosec}^2\frac{z}{R} - \frac{V_0}{R}\cot\frac{z}{R}\frac{T^*}{T} = 0$$

$$\therefore \frac{V_0''}{2} - \frac{V_0}{R^2}\operatorname{cosec}^2\frac{z}{R} = 0 \qquad V_0 = -\frac{2m}{R}\cot\frac{z}{R} \quad m' = 0$$

$$\frac{V_1''}{2} - \frac{V_1}{R^2}\operatorname{cosec}^2\frac{z}{R} + \frac{V_0'}{2} - \frac{V_0}{R}\cot\frac{z}{R} = 0$$

$$\therefore \frac{V_1''}{2} - \frac{V_1}{R^2}\operatorname{cosec}^2\frac{z}{R} = \frac{V_0}{R}\cot\frac{z}{R} - \frac{V_0'}{2}$$

$$= -\frac{2m}{R^2}\cot^2\frac{z}{R} - \frac{1}{2}\left(\frac{2m}{R^2}\operatorname{cosec}^2\frac{z}{R}\right)$$

$$= -\frac{2m}{R^2}\left(\operatorname{cosec}^2\frac{z}{R} - 1\right) - \frac{m}{R^2}\operatorname{cosec}^2\frac{z}{R}$$

$$= -\frac{3m}{R^2}\operatorname{cosec}^2\frac{z}{R} + \frac{2m}{R^2}$$

$$\therefore \frac{V_1''}{2} - \frac{V_1}{R^2}\operatorname{cosec}^2\frac{z}{R} + \frac{3m}{R^2}\operatorname{cosec}^2\frac{z}{R} = \frac{2m}{R^2}$$

$$\therefore \frac{V_1''}{2} - \frac{1}{R^2}\operatorname{cosec}^2\frac{z}{R}\left(V_1 - 3m\right) = \frac{2m}{R^2} \qquad \text{But } m' = 0$$

$$\therefore \frac{(V_1 - 3m)''}{2} - \frac{\operatorname{cosec}^2\frac{z}{R}}{R^2}(V_1 - 3m) = \frac{2m}{R^2}$$

$V_1 - 3m$ ~~has~~ $= \frac{A}{R}\cot\frac{z}{R}$ is a term in the complementary function. Put $V_1 - 3m = \frac{v}{R}\cot\frac{z}{R}$ $\therefore (V_1-3m)' = \frac{v'}{R}\cot\frac{z}{R} - \frac{v}{R^2}\operatorname{cosec}^2\frac{z}{R}$

$$(V_1-3m)'' = \frac{v''}{R}\cot\frac{z}{R} - \frac{2v'}{R^2}\operatorname{cosec}^2\frac{z}{R} + \frac{2v}{R^3}\operatorname{cosec}^2\frac{z}{R}\cot\frac{z}{R}.$$

$$\therefore (V_1-3m)'' - \frac{(V_1-3m)}{R^2}\operatorname{cosec}^2\frac{z}{R} = \frac{2m}{R^2} \Rightarrow$$

$$\frac{v''}{2R}\cot\frac{z}{R} - \frac{2v'}{R^2}\operatorname{cosec}^2\frac{z}{R} + \frac{2v}{R^3}\operatorname{cosec}^2\frac{z}{R}\cot\frac{z}{R} - \frac{v}{R^3}\cot\frac{z}{R}\operatorname{cosec}^2\frac{z}{R} = \frac{2m}{R^2}$$

$$\therefore \frac{v''}{2R}\cot\frac{z}{R} - \frac{v'}{R^2}\operatorname{cosec}^2\frac{z}{R} = \frac{2m}{R^2} \qquad \therefore v'' - \frac{2v'}{R}\frac{\operatorname{cosec}^2\frac{z}{R}}{\cot\frac{z}{R}} = \frac{4m}{R}\tan\frac{z}{R}$$

Linear in (v', z). I.F. $= e^{-\int \frac{2}{R}\frac{\operatorname{cosec}^2 z/R}{\cot z/R}\,dz}$

$$= e^{2\log\cot\frac{z}{R}} = \cot^2\frac{z}{R}$$

$$\therefore v'\cot^2\frac{z}{R} = \int\frac{4m}{R}\tan\frac{z}{R}\cot^2\frac{z}{R}\,dz + A(t) = \int\frac{4m}{R}\cot\frac{z}{R}\,dz + A(t)$$

$$= \frac{4m}{R}\log\sin\frac{z}{R} + A(t)$$

$$\therefore v' = 4m\left(\log\sin\frac{z}{R}\right)\tan^2\frac{z}{R} + A(t)\tan^2\frac{z}{R}$$

$$\therefore v = 4m\int\tan^2\frac{z}{R}\log\sin\frac{z}{R}\,dz + A(t)\int\tan^2\frac{z}{R}\,dz + B.$$

$$= 4m\,I + A(t)\int\left(\sec^2\frac{z}{R} - 1\right)dz + B$$

$I = \int\left(\log\sin\frac{z}{R}\right)\tan^2\frac{z}{R}\,dz$ Put $\log\sin\frac{z}{R} = y$ $\sin\frac{z}{R} = e^y$

19 July 1995.

In the above investigation there is the problem of the time coordinate Look at the transformations on ρ. $\Rightarrow \sigma_2$ and σ_3. The time coord. t is $t = \Phi(\rho, T)$ Though $\Phi_t = \pm 1$ Φ_ρ is not $\neq \pm 1$ and so t is not $r+T$ or $r-T$. $\therefore$ $T = T(r, t)$ does not seem to be true. Let us therefore change our attitude and begin with McVittie's solution. Consider the metric

$$ds^2 = -e^{\alpha} dr^2 - r^2 e^{\beta}(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu} dt^2 \qquad \alpha = \alpha(r,t),\ \beta = \beta(r,t),\ \nu = \nu(r,t)$$

McVittie's solution has $e^{\beta} = e^{\alpha} = \dfrac{(1+\mu)^4 R^2(t)}{\left(1 + k\frac{r^2}{4}\right)^2}$ $\qquad e^{\nu} = \dfrac{(1-\mu)^2}{(1+\mu)^2}$

$$\mu = \frac{m}{2rR}\sqrt{\left(1 + k\frac{r^2}{4}\right)}.$$

The $G_i^{\ k} = R_i^{\ k} - \frac{1}{2} g_i^{\ k} R$ for this metric are given in Vaidya and Shah Progress of Theoretical Physics 24, 111–125 (1960)

There we have taken $T_i^{\ k} = (\rho + p) v_i v^k - p g_i^{\ k} + \sigma \omega_i \omega^k$.

$v^1 = v^2 = v^3 = 0$ $\quad e^{\nu}(v^4)^2 = 1$ $\quad \therefore v^4 = e^{-\nu/2}$

$\omega^2 = \omega^3 = 0$ $\quad -e^{\alpha}(\omega^1)^2 + e^{\nu}(\omega^4)^2 = 0$ also $-e^{\alpha} v^1 \omega^1 + e^{\nu} v^4 \omega^4 = 1$

$\therefore e^{\nu/2} \omega^4 = 1$

$\omega^4 = e^{-\nu/2}$

$\therefore -e^{\alpha}(\omega^1)^2 + e^{\nu - \nu} = 0$ $\quad \therefore e^{\alpha}(\omega^1)^2 = 1$ $\quad \omega^1 = e^{-\alpha/2}$ (We take ω^1 +ve

$\therefore v^i = (0, 0, 0, e^{-\nu/2})$

$\omega^i = (e^{-\alpha/2}, 0, 0, e^{-\nu/2})$

Next $G_i^k = -8\pi T_i^k$ gives

$$8\pi\rho = G_2^2, \quad 8\pi p = G_2^2 - G_1^1 - G_4^4 \qquad 8\pi\sigma\, w_1\, w^q = -G_1^4$$

But $w_1 = g_{11} w^1 = -e^{\alpha} \cdot e^{-\alpha/2} = -e^{\alpha/2}$

$$\therefore 8\pi\sigma\,(-e^{\alpha/2})\, e^{-\gamma/2} = -G_1^4 \quad \therefore 8\pi\sigma = e^{(\gamma-\alpha)/2}\, G_1^4$$

And we have the field equation

$$G_2^2 - G_1^1 = -\frac{w^1}{w^4} G_1^4 = -e^{-\alpha/2} e^{\gamma/2} G_1^4 = -e^{(\gamma-\alpha)/2} G_1^4$$

This equation is written out in full as equation (3.5).

In order to solve this equation using a method of successive approximations, we note that

$$\mu = \frac{m\sqrt{2}}{2\pi R} \qquad 2 = 1 + k\frac{R^2}{4}.$$

We expand $e^{\alpha}, e^{\beta}, e^{\gamma}$ in powers of μ such that correct to first power of $\mu \left(= \frac{m\sqrt{2}}{2\pi R}\right)$ we get McVittie's solution

To the first power of μ $e^{\alpha} = \frac{R^2}{2^2}(1+4\mu)$ $e^{\beta} = \frac{R^2}{2^2}(1+4\mu)$ $e^{\gamma} = (1-4\mu)$.

$\therefore$ We assume that $e^{\alpha} = \frac{R^2}{2^2}(1+4\mu+a\mu^2)$

$$e^{\beta} = \frac{R^2}{2^2}(1+4\mu+b\mu^2)$$

$$e^{\gamma} = (1-4\mu+c\mu^2)$$

$\mu = \frac{m\sqrt{2}}{2\pi R}$.

We expect that to the first order of μ our field equation will give $\mu = \frac{m\sqrt{2}}{2\pi R}$.

We have unnecessarily undertaken to use t & t and u as coordinates in place of u and z. This we did believing that our t was the standard flat space-time t so that a conformal factor $T(t)$ will take us from Einstein's Universe to Friedman's Universe but our transformation on p 3 robs us of this belief.

So we begin afresh with p. 2.

Take $ds^2 = 2\,du\,dz + e^{\nu} du^2 - M^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ $\quad M^2 = R^2 \sin^2 \frac{z}{R}$.

$\nu = \nu(u, z)$

$x^1 = z,\ x^2 = \theta,\ x^3 = \phi,\ x^4 = u$

$\therefore$

$$g_{ik} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -M^2 & 0 & 0 \\ 0 & 0 & -M^2\sin^2\theta & 0 \\ 1 & 0 & 0 & e^{\nu} \end{pmatrix}$$

$\therefore g = -1 \times L + M^4 \sin^2\theta$

$= -M^4 \sin^2\theta$

$\sqrt{(-g)} = M^2 \sin\theta$

$\left\{ {a \atop 1a} \right\} = 2\frac{M'}{M}$, $\left\{ {a \atop 2a} \right\} = \cot\theta$, $\left\{ {a \atop 4a} \right\} = 0$

$\therefore \dot{M} = 0$

$g^{11} = \dfrac{e^{\nu} M^4 \sin^2\theta}{-M^4 \sin^2\theta} = -e^{\nu}$, $\quad g^{41} = 1 \quad g^{44} = 0$

$$\therefore\ g^{ik} = \begin{pmatrix} -e^{\nu} & 0 & 0 & 1 \\ 0 & -\frac{1}{M^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{M^2\sin^2\theta} & \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Pof $\left\{{}^{1}_{11}\right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{14}(0) = 0$; $\left\{{}^{2}_{11}\right\} = \frac{1}{2} g^{22}(0) = 0$ $\left\{{}^{3}_{11}\right\} = \frac{1}{2} g^{33}(0) = 0$

$\left\{{}^{4}_{11}\right\} = \frac{1}{2} g^{41}(0) = 0$. $\left\{{}^{1}_{12}\right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{14}(0) = 0$ $\left\{{}^{2}_{12}\right\} = \frac{1}{2} g^{22}(g_{22,1})$

$\left\{{}^{2}_{12}\right\} = -\frac{1}{2}\frac{1}{M^2}(-2MM') = \frac{M'}{M}$ $\left\{{}^{3}_{12}\right\} = \frac{1}{2} g^{33}(0) = 0$ $\left\{{}^{4}_{12}\right\} = \frac{1}{2} g^{41}(0) = 0$

$\left\{{}^{1}_{13}\right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{14}(0) = 0$ $\left\{{}^{2}_{13}\right\} = \frac{1}{2} g^{22}(0) = 0$ $\left\{{}^{3}_{13}\right\} = \frac{1}{2} g^{33}(+g_{33,1})$

$\left\{{}^{3}_{13}\right\} = -\frac{1}{2M^2\sin^2\theta}(-2MM'\sin^2\theta) = \frac{M'}{M}$ $\left\{{}^{1}_{14}\right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{14}(g_{44,1})$

$\left\{{}^{1}_{14}\right\} = \frac{1}{2} e^{\nu}\nu'$ $\left\{{}^{2}_{14}\right\} = \frac{1}{2} g^{22}(0)$, $\left\{{}^{3}_{14}\right\} = \frac{1}{2} g^{33}(0)$, $\left\{{}^{4}_{14}\right\} = \frac{1}{2} g^{41}(0) = 0$

$\left\{{}^{1}_{22}\right\} = \frac{1}{2} g^{11}(-g_{22,1}) = \frac{1}{2}(-e^{\nu})(2MM') = -e^{\nu}MM'$

$\left\{{}^{2}_{22}\right\} = \frac{1}{2} g^{22}(0) = 0$, $\left\{{}^{3}_{22}\right\} = \frac{1}{2} g^{33}(0) = 0$, $\left\{{}^{4}_{22}\right\} = \frac{1}{2} g^{41}(-g_{22,1})$

$\left\{{}^{4}_{22}\right\} = \frac{1}{2}(2MM') = MM'$. $\left\{{}^{1}_{23}\right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{14}(0) = 0$, $\left\{{}^{2}_{23}\right\} = \frac{1}{2} g^{22}(0) = 0$

$\left\{{}^{3}_{23}\right\} = \frac{1}{2} g^{33}(g_{33,2}) = \frac{1}{2}\left(\frac{-1}{M^2\sin^2\theta}\right)(-M^2 2\sin\theta\cos\theta) = \cot\theta$

$\left\{{}^{4}_{23}\right\} = \frac{1}{2} g^{41}(0) = 0$, $\left\{{}^{1}_{24}\right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{14}(0) = 0$, $\left\{{}^{2}_{24}\right\} = \frac{1}{2} g^{22}(g_{22,4}) = 0$

$\left\{{}^{3}_{24}\right\} = \frac{1}{2} g^{33}(0) = 0$, $\left\{{}^{4}_{24}\right\} = \frac{1}{2} g^{41}(0) = 0$ $\left\{{}^{1}_{33}\right\} = \left\{{}^{1}_{22}\right\}\sin^2\theta$

$\left\{{}^{2}_{33}\right\} = \frac{1}{2} g^{22}(-g_{33,2}) = \frac{1}{2}\left(-\frac{1}{M^2}\right)(M^2 \cdot 2\sin\theta\cos\theta) = -\sin\theta\cos\theta$

$\left\{{}^{3}_{33}\right\} = \frac{1}{2} g^{33}(0) = 0$ $\left\{{}^{4}_{33}\right\} = \left\{{}^{4}_{22}\right\}\sin^2\theta$ $\left\{{}^{1}_{34}\right\} = \frac{1}{2} g^{11}(0) + \frac{1}{2} g^{44}(0) = 0$

$\left\{{}^{2}_{34}\right\} = \frac{1}{2} g^{22}(0) = 0$, $\left\{{}^{3}_{34}\right\} = \frac{1}{2} g^{33}(g_{33,4}) = 0$, $\left\{{}^{4}_{34}\right\} = \frac{1}{2} g^{41}(0) = 0$.

$\left\{{}^{1}_{44}\right\} = \frac{1}{2} g^{11}(-g_{44,1}) + \frac{1}{2} g^{14}(g_{44,4}) = \frac{1}{2}(-e^{\nu})(-e^{\nu})' + \frac{1}{2}(e^{\nu})'$

$\left\{{}^{2}_{44}\right\} = \frac{1}{2} g^{22}(0) = 0$, $\left\{{}^{3}_{44}\right\} = \frac{1}{2} g^{33}(0) = 0$ $\left\{{}^{4}_{44}\right\} = \frac{1}{2} g^{41}(-g_{44,1})$

$\left\{{}^{2}_{12}\right\} = \frac{M'}{M}$

$\left\{{}^{3}_{13}\right\} = \frac{M'}{M}$

$\left\{{}^{1}_{14}\right\} = e^{\nu}\frac{\nu'}{2}$

$\left\{{}^{1}_{22}\right\} = -e^{\nu}MM'$

$\left\{{}^{4}_{22}\right\} = MM'$

$\left\{{}^{3}_{23}\right\} = \cot\theta$

$\left\{{}^{1}_{33}\right\} = -e^{\nu}MM'\sin^2\theta$

$\left\{{}^{2}_{33}\right\} = -\sin\theta\cos\theta$

$\left\{{}^{4}_{33}\right\} = MM'\sin^2\theta$

$\left\{{}^{1}_{44}\right\} = e^{2\nu}\frac{\nu'}{2} + e^{\nu}\frac{\nu'}{2}$

$\left\{{}^{4}_{44}\right\} = -e^{\nu}\frac{\nu'}{2}$.

$R_{11} = -\left\{{}^{n}_{11}\right\}_{,n} + \left\{{}^{n}_{1n}\right\}_{,1} + \left\{{}^{a}_{1a}\right\}\left\{{}^{b}_{b1}\right\} - \left\{{}^{n}_{11}\right\}\left\{{}^{b}_{nb}\right\}$

$= 0 + \left(\left\{{}^{2}_{12}\right\} + \left\{{}^{3}_{13}\right\}\right)_{,1} + \left\{{}^{2}_{12}\right\}\left\{{}^{2}_{21}\right\} + \left\{{}^{3}_{13}\right\}\left\{{}^{3}_{31}\right\} + \left\{{}^{1}_{14}\right\}\left\{{}^{4}_{11}\right\} - 0$

$= \left(\frac{2M'}{M}\right)' + 2\frac{M'^2}{M^2} = \frac{2M''}{M} - \frac{2M'^2}{M^2} + \frac{2M'^2}{M^2} = -\frac{2}{R^2}$. $\therefore \boxed{R_{11} = -\frac{2}{R^2}}$

$$R_{22} = -\{^{n}_{22}\}_{,n} + \{^{a}_{2a}\}_{,2} + \{^{b}_{2a}\}\{^{a}_{b2}\} - \{^{n}_{22}\}\{^{a}_{na}\}$$

$$= -\{^{1}_{22}\}_{,1} - \{^{4}_{22}\}_{,4} + (\cot\theta)_{,2} + \{^{2}_{21}\}\{^{1}_{22}\} + \{^{1}_{22}\}\{^{2}_{12}\} + \{^{4}_{22}\}\{^{2}_{42}\} + \{^{3}_{23}\}\{^{3}_{32}\}$$
$$- \{^{1}_{22}\}\left(\{^{2}_{12}\} + \{^{3}_{13}\}\right)$$

$$= -(-e^{\nu} MM')' - (MM')\cdot - \operatorname{cosec}^2\theta + \cot^2\theta = (e^{\nu} MM')' - 1 =$$

$$= e^{\nu}\nu' MM' + e^{\nu} M'^2 + e^{\nu} MM'' - 1 = M^2 e^{\nu}\left[\nu'\frac{M'}{M} + \frac{M'^2}{M^2} + \frac{M''}{M}\right] - 1$$

$$M = R\sin\tfrac{r}{R} \qquad M' = \cos\tfrac{r}{R} \qquad M'' = -\tfrac{1}{R}\sin\tfrac{r}{R}$$

$$\therefore \boxed{R_{22} = M^2 e^{\nu}\left[\frac{1}{R}\cot\frac{r}{R}\,\nu' + \frac{1}{R^2}\cot^2\frac{r}{R} - \frac{1}{R^2}\right] - 1} \qquad R_{33} = R_{22}\sin^2\theta .$$

$$R_{44} = -\{^{n}_{44}\}_{,n} + \{^{n}_{4n}\}_{,4} + \{^{b}_{4a}\}\{^{a}_{b4}\} - \{^{n}_{44}\}\{^{a}_{na}\}$$

$$= -\{^{1}_{44}\}_{,1} - \{^{4}_{44}\}_{,4} + 0 + \{^{1}_{41}\}\{^{4}_{14}\} + \{^{1}_{41}\}\{^{1}_{14}\} + \{^{4}_{44}\}\{^{4}_{44}\} - \{^{1}_{44}\}\{^{a}_{1a}\} - \{^{4}_{44}\}\{^{0}_{44}\}$$

$$= -\left[e^{2\nu}\frac{\nu'}{2}\right]' - \left[e^{\nu}\frac{\dot\nu}{2}\right]' - \left[-e^{\nu}\frac{\nu'}{2}\right]\cdot + \left(e^{\nu}\frac{\nu'}{2}\right)^2 + \left(-e^{\nu}\frac{\nu'}{2}\right) - \left[e^{2\nu}\frac{\nu'}{2} + e^{\nu}\frac{\dot\nu}{2}\right]\left(\frac{2M'}{M}\right) - 0$$

$$= -\left[e^{2\nu}\frac{\nu''}{2} + e^{2\nu}\frac{\nu'^2}{2}\times 2\right] + \frac{e^{2\nu}\nu'^2}{2} - e^{2\nu}\frac{\nu'}{R}\cot\frac{r}{R} - e^{\nu}\frac{\dot\nu}{R}\cot\frac{r}{R}$$

$$= e^{2\nu}\left[-\frac{\nu''}{2} - \nu'^2 + \frac{\nu'^2}{2} - \frac{\nu'}{R}\cot\frac{r}{R}\right] - e^{\nu}\frac{\dot\nu}{R}\cot\frac{r}{R}$$

$$\therefore \boxed{R_{44} = -e^{2\nu}\left[\frac{\nu''}{2} + \frac{\nu'^2}{2} + \frac{\nu'}{R}\cot\frac{r}{R}\right] - e^{\nu}\frac{\dot\nu}{R}\cot\frac{r}{R}}$$

$$R_{41} = -\{^{n}_{41}\}_{,n} + \{^{n}_{4n}\}_{,1} + \{^{b}_{4a}\}\{^{a}_{b1}\} - \{^{n}_{41}\}\{^{b}_{nb}\}$$

$$= -\{^{1}_{41}\}_{,1} + 0 + \{^{1}_{41}\}\{^{1}_{11}\} + \{^{4}_{44}\}\{^{4}_{41}\} - \{^{1}_{41}\}\{^{b}_{1b}\}$$

$$= -\left[e^{\nu}\frac{\nu'}{2}\right]' - e^{\nu}\frac{\nu'}{2}\cdot\frac{2M'}{M} = -e^{\nu}\left[\frac{\nu''}{2} + \frac{\nu'^2}{2} + \frac{\nu'}{R}\cot\frac{r}{R}\right]$$

$$\boxed{R_{41} = -e^{\nu}\left[\frac{\nu''}{2} + \frac{\nu'^2}{2} + \frac{\nu'}{R}\cot\frac{r}{R}\right]}$$

(90c) From §.8 $R_{ik} = -8\pi\left[(\phi+p)v_i v_k - \frac{(p-\phi)}{2} g_{ik} + \sigma \omega_i \omega_k\right] + \Lambda g_{ik}$

But $g_{ik}v^i v^k = 1 \quad g_{ik}\omega^i\omega^k = 0 \quad g_{ik}\omega^i v^k = 1$

Take $v^i = (0, 0, 0, v^4) \therefore e^{\nu}(v^4)^2 = 1. \quad v^4 = e^{-\nu/2}$.

$v_1 = g_{1k}v^k = g_{14}v^4 = v^4 = e^{-\nu/2}. \qquad v_4 = g_{4k}v^k = g_{41}v^1 + g_{44}v^4 = e^{\nu}\cdot e^{-\nu/2} = e^{\nu/2}$

$\therefore \boxed{v_i = (e^{-\nu/2}, 0, 0, e^{\nu/2})}$ $\qquad \omega^i = (\omega^1, 0, 0, 0) \therefore g_{ik}\omega^i\omega^k = 0.$

$g_{ik}\omega^i v^k = 1 \Rightarrow \omega^1 v^4 + \omega^4 v^1 + e^{\nu}\omega^4 v^4 = 1 \quad \therefore \omega^1 v^4 = 1 \quad \therefore \omega^1 = e^{\nu/2}$

$\therefore \omega^i = (e^{\nu/2}, 0, 0, 0)$.

$\omega_1 = g_{1k}\omega^k = g_{14}\omega^4 = 0. \qquad \omega_4 = g_{4k}\omega^k = g_{41}\omega^1 = e^{\nu/2}$

$\therefore \boxed{\omega_i = (0, 0, 0, e^{\nu/2})}$

$\therefore R_{11} = -8\pi\left[(\phi+p)(v_1)^2 - \frac{(p-\phi)}{2} g_{11} + \sigma(\omega_1)^2\right] = -8\pi(\phi+p)e^{-\nu} \quad \therefore \boxed{R_{11} = -8\pi(\phi+p)e^{-\nu}}$

$R_{22} = -8\pi\left[-\frac{(p-\phi)}{2} g_{22}\right] = -8\pi\left[-\frac{(p-\phi)}{2}(-M^2)\right] - \Lambda M^2 \therefore \boxed{R_{22} = -8\pi\frac{M^2}{2}(p-\phi) + \Lambda M^2}$

$R_{44} = -8\pi\left[(\phi+p)e^{\nu} - \frac{(p-\phi)}{2}e^{\nu} + \sigma e^{\nu}\right] = -8\pi e^{\nu}\left[\frac{3\phi+p}{2} + \sigma\right] + \Lambda e^{\nu}$

$\boxed{R_{44} - \Lambda e^{\nu} = -8\pi e^{\nu}\left[\frac{p+3\phi}{2} + \sigma\right]}$

$R_{41} = -8\pi\left[(\phi+p) - \frac{(p-\phi)}{2}\right] + \Lambda \qquad \therefore \quad \boxed{R_{41} - \Lambda = -8\pi\left[\frac{p+3\phi}{2}\right]}$

$\therefore e^{\nu}R_{11} + \frac{2R_{22}}{M^2} = -8\pi[\phi+p+p-\phi) = -8\pi(p)\cdot 2$

$\boxed{\therefore 8\pi p = -e^{\nu}\frac{R_{11}}{2} - \frac{R_{22}}{M^2}}$

$$e^{\nu} R_{11} - \frac{2R_{22}}{M^2} = -8\pi[p+p-p+p) = -8\pi(p)2$$

$$\therefore \boxed{8\pi p = -e^{\nu}\frac{R_{11}}{2} + \frac{R_{22}}{M^2} + \Lambda}$$

Next $R_{41} - e^{-\nu}R_{44} = -8\pi\left[\frac{p+3p}{2} - \frac{p+3p}{2} - \sigma\right]$ $\therefore \boxed{8\pi\sigma = R_{41} - e^{-\nu}R_{44}}$

And finally in $R_{41} = -8\pi\left(\frac{p+3p}{2}\right) - \Lambda$ put in $8\pi p$ and $8\pi\sigma$

$$\therefore 2R_{41} = -\left[-e^{\nu}\frac{R_{11}}{2} - \frac{R_{22}}{M^2}\right] - 3\left[-e^{\nu}\frac{R_{11}}{2} + \frac{R_{22}}{M^2}\right]$$

$$\therefore 2R_{41} = e^{\nu}\frac{R_{11}}{2}(1+3) + \frac{R_{22}}{M^2}(1-3) = 2e^{\nu}R_{11} - \frac{2R_{22}}{M^2}$$

$$\therefore \boxed{R_{41} = e^{\nu}R_{11} - \frac{R_{22}}{M^2}}$$ is the field eqn (for isotropy of pressure)

We solve this eqn. to find e^{ν}.

$$\therefore -e^{\nu}\left[\frac{\nu''}{2} + \frac{\nu'^2}{2} + \frac{\nu'}{R}\cot\frac{x}{R}\right] = e^{\nu}\left(-\frac{2}{R^2}\right) - \frac{1}{M^2}\left[M^2 e^{\nu}\left(\frac{1}{R}\cot\frac{x}{R}\nu' + \frac{1}{R^2}\cot^2\frac{x}{R} - \frac{1}{R^2}\right) - 1\right]$$

$$\therefore e^{\nu}\left(\frac{\nu''}{2} + \frac{\nu'^2}{2} + \frac{\nu'}{R}\cot\frac{x}{R}\right] = \frac{2}{R^2}e^{\nu} + \left[e^{\nu}\left(\frac{1}{R}\cot\frac{x}{R}\nu' + \frac{1}{R^2}\cot^2\frac{x}{R} - \frac{1}{R^2}\right) - \frac{1}{M^2}\right]$$

$$\therefore e^{\nu}\left[\frac{\nu''}{2} + \frac{\nu'^2}{2} + \frac{\nu'}{R}\cot\frac{x}{R} - \frac{2}{R^2} - \frac{\nu'}{R}\cot\frac{x}{R} - \frac{1}{R^2}\cot^2\frac{x}{R} + \frac{1}{R^2}\right] = -\frac{1}{R^2\sin^2\frac{x}{R}}$$

$$\therefore e^{\nu}\left(\frac{\nu''}{2} + \frac{\nu'^2}{2} - \frac{1}{R^2}\operatorname{cosec}^2\frac{x}{R}\right) = -\frac{1}{R^2}\operatorname{cosec}^2\frac{x}{R}$$

$$\therefore e^{\nu}\left(\frac{\nu''}{2} + \frac{\nu'^2}{2}\right) = \frac{1}{R^2}\operatorname{cosec}^2\frac{x}{R}(e^{\nu}-1)$$

$$\therefore \frac{1}{2}g'' = \frac{1}{R^2}\operatorname{cosec}^2\frac{x}{R}\,g$$

Let $e^{\nu} - 1 = g$

$e^{\nu}\nu' = g'$

$e^{\nu}\nu'' + e^{\nu}\nu'^2 = g''$

$\therefore \beta'' - \frac{2}{R^2}\operatorname{cosec}^2\frac{z}{R}\beta = 0$ Compare this with eqn. on p. (220) !

$\beta = \cot\frac{z}{R}$ is a solution $\therefore$ Put $\chi = v\cot\frac{z}{R}$

$\chi' = v'\cot\frac{z}{R} - \frac{v}{R}\operatorname{cosec}^2\frac{z}{R}$. $\chi'' = v''\cot\frac{z}{R} - \frac{2v'}{R}\operatorname{cosec}^2\frac{z}{R} + \frac{2v}{R^2}\operatorname{cosec}^2\frac{z}{R}\cot\frac{z}{R}$

$\therefore v''\cot\frac{z}{R} - \frac{2v'}{R}\operatorname{cosec}^2\frac{z}{R} + \frac{2v}{R^2}\operatorname{cosec}^2\frac{z}{R}\cot\frac{z}{R} - \frac{2}{R^2}\operatorname{cosec}^2\frac{z}{R}v\cot\frac{z}{R} = 0$

$\therefore v''\cot\frac{z}{R} - \frac{2v'}{R}\operatorname{cosec}^2\frac{z}{R} = 0$ $\therefore \frac{v''}{v'} + \frac{2}{R}\frac{\operatorname{cosec}^2\frac{z}{R}}{\cot\frac{z}{R}} = 0$

$\therefore \log v' - 2\log\cot\frac{z}{R} = A\log A$ $\therefore \frac{v'}{\cot^2\frac{z}{R}} = A$

$\therefore v' = A\cot^2\frac{z}{R} = A\left[\operatorname{cosec}^2\frac{z}{R} - 1\right] = A\operatorname{cosec}^2\frac{z}{R} - A$

$\therefore v = -AR\cot\frac{z}{R} - Az + B$

$\therefore \chi = v\cot\frac{z}{R} = B\cot\frac{z}{R} + A\left(z + R\cot\frac{z}{R}\right)\cot\frac{z}{R}$

$\therefore e^{\nu} = 1 - \frac{2m(u)}{R}\cot\frac{z}{R} + B(u)\cot\frac{z}{R}\left(z + R\cot\frac{z}{R}\right)$

This is the general solution. Setting $B(u) = 0$ We ~~have~~ we take

$e^{\nu} = 1 - \frac{2m}{R}\cot\frac{z}{R}$ $m = m(u)$. Significance, if any, of the other integral, we shall take up later.

With $e^{\nu} = 1 - \frac{2m}{R}\cot\frac{z}{R}$, we know that $e^{\nu}\left(\frac{\nu''}{2} + \frac{\nu'^2}{2}\right) = \frac{1}{R^2}\operatorname{cosec}^2\frac{z}{R}(e^{\nu} - 1)$

$= -\frac{2m}{R^3}\operatorname{cosec}^2\frac{z}{R}\cot\frac{z}{R}$

$\therefore R_{41} = -e^{\nu}\left[\frac{\nu''}{2} + \frac{\nu'^2}{2} + \frac{\nu'}{R}\cot\frac{z}{R}\right] = \frac{2m}{R^3}\operatorname{cosec}^2\frac{z}{R}\cot\frac{z}{R} - e^{\nu}\frac{\nu'}{R}\cot\frac{z}{R}$

But $e^{\nu} = 1 - \frac{2m}{R}\cot\frac{\xi}{R}$ $\therefore e^{\nu}\nu' = \frac{2m}{R^2}\operatorname{cosec}^2\frac{\xi}{R}$

$$\therefore R_{41} = \frac{2m}{R^3}\operatorname{cosec}^2\frac{\xi}{R}\cot\frac{\xi}{R} - \frac{2m}{R^3}\operatorname{cosec}^2\frac{\xi}{R}\cot\frac{\xi}{R} = 0$$

$\therefore R_{11} = -\frac{2}{R^2}$. $R_{41} = e^{\nu}R_{11} - \frac{R_{22}}{M^2}$ $\therefore 0 = e^{\nu}\left(-\frac{2}{R^2}\right) - \frac{R_{22}}{M^2}$,

$$\frac{R_{22}}{M^2} = -\frac{2}{R^2}\left(1 - \frac{2m}{R}\cot\frac{\xi}{R}\right)$$

$$R_{44} = \frac{2m}{R^2}\cot^2\frac{\xi}{R}.$$

$R_{41} = 0$

Also $R_{44} = e^{\nu}R_{41} - e^{\nu}\frac{\nu'}{R}\cot\frac{\xi}{R} = -\frac{1}{R}\cot\frac{\xi}{R}\left(-\frac{2m}{R}\cot\frac{\xi}{R}\right)$

$$\therefore 8\pi\rho = -e^{\nu}\frac{R_{11}}{2} + \frac{R_{22}}{M^2} + \Lambda = -\left(1 - \frac{2m}{R}\cot\frac{\xi}{R}\right)\left(-\frac{1}{R^2}\right) - \frac{2}{R^2}\left(1 - \frac{2m}{R}\cot\frac{\xi}{R}\right) + \Lambda$$

$$= \Lambda - \frac{1}{R^2}\left(1 - \frac{2m}{R}\cot\frac{\xi}{R}\right) \qquad \therefore \boxed{8\pi\rho = \Lambda - \frac{1}{R^2} + \frac{2m}{R^3}\cot\frac{\xi}{R}}$$

$$8\pi p = -e^{\nu}\frac{R_{11}}{2} - \frac{R_{22}}{M^2} - \Lambda = -\left(1 - \frac{2m}{R}\cot\frac{\xi}{R}\right)\left(-\frac{1}{R^2}\right) + \frac{2}{R^2}\left(1 - \frac{2m}{R}\cot\frac{\xi}{R}\right) - \Lambda$$

$$= -\Lambda + \frac{3}{R^2}\left(1 - \frac{2m}{R}\cot\frac{\xi}{R}\right) \qquad \therefore \boxed{8\pi p = \frac{3}{R^2} - \Lambda - \frac{6m}{R^3}\cot\frac{\xi}{R}}$$

$$8\pi\sigma = R_{41} - e^{-\nu}R_{44} = 0 - \left(1 - \frac{2m}{R}\cot\frac{\xi}{R}\right)^{-1}\frac{2m}{R^2}\cot^2\frac{\xi}{R}$$

$$\therefore \boxed{8\pi\sigma = \frac{-2\frac{dm}{du}\cdot\frac{\cot^2\frac{\xi}{R}}{R^2}}{1 - \frac{2m}{R}\cot\frac{\xi}{R}}}$$

$ds^2 = 2\,du\,dr + \left(1 - \frac{2m(u)}{R}\cot\frac{\xi}{R}\right)du^2 - R^2\sin^2\frac{\xi}{R}\,d\omega^2$

gives radiating mass particle in Einstein Universe

We now replace u by the time coordinate t

$$u = u(z,t)$$

$$\therefore du = u' dz + \dot{u} dt \quad \left(\text{Now } \dot{z} \equiv \frac{\partial z}{\partial t}\right)$$

$$\therefore ds^2 = 2(u' dz + \dot{u} dt) dz + \left(1 - \frac{2m(u)}{R}\cot\frac{z}{R}\right) dt^2 - (u' dz + \dot{u} dt)^2 - R^2 \sin^2\frac{z}{R} dz^2$$

$$= \dot{u}^2 \left(1 - \frac{2m}{R}\cot\frac{z}{R}\right) dt^2 + dz^2 \left[u'^2 \left(1 - \frac{2m(u)}{R}\cot\frac{z}{R}\right) + 2u'\right]$$

$$+ 2 dz dt \left[\dot{u} + u'\dot{u}\left(1 - \frac{2m(u)}{R}\cot\frac{z}{R}\right)\right] - R^2 \sin^2\frac{z}{R} dz^2.$$

Put the coeff of $dzdt$ as zero

$$\therefore \dot{u}\left[1 + u'\left(1 - \frac{2m(u)}{R}\cot\frac{z}{R}\right)\right] = 0 \quad \text{as } \dot{u} \neq 0$$

$$u'\left(1 - \frac{2m(u)}{R\cot}\cot\frac{z}{R}\right) = -1. \quad \therefore u' \text{ is -ve.}$$

On an onward moving ray du is zero $\therefore u'$ -ve implies $\dot{u}$ +ve

Again the coeff. of dz^2 is $u'^2\left(1 - \frac{2m(u)}{R}\cot\frac{z}{R}\right) + 2u'$

$$= u'\left[u'\left(1 - \frac{2m(u)}{R}\cot\frac{z}{R}\right) + 2\right]$$

$$= u'[-1 + 2] = +u' = -\frac{1}{1 - \frac{2m(u)}{R}\cot\frac{z}{R}}$$

$\therefore$ The metric for a radiating mass-particle in Einstein's Universe is

$$ds^2 = u^2\left(1-\frac{2m(u)}{R}\cot\frac{r}{R}\right)dt^2 - \frac{1}{1-\frac{2m(u)}{R}\cot\frac{r}{R}}dr^2 - R^2\sin^2\frac{r}{R}\,d\Omega^2.$$

with $u'\left(1-\frac{2m(u)}{R}\cot\frac{r}{R}\right) = -1.$

11 Jan. 1996

We take up from p.102. In McVittie's solution when $\dot{R}=0$ we get mass-particle in Einstein Universe. We already have a solution for a static mass particle in Einstein Universe in the previous page (set $m=$ const. there). Then

$$\left.\begin{aligned} 8\pi\rho &= -\frac{1}{R^2} + \frac{2m}{R^3}\cot\frac{r}{R} + \Lambda \\ 8\pi p &= \frac{1}{R^2} - \frac{6m}{R^3}\cot\frac{r}{R} - \Lambda \end{aligned}\right\}$$

On setting $\Lambda=0$ we have $8\pi(\rho+3p)=0$.

But to our surprize — I find that in McVittie's solution when you set $R=$ const, you do not get $8\pi(\rho+3p)=0$!!.

Let us check this.

$$ds^2 = e^{\nu}dt^2 - e^{\alpha}\left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\varphi^2\right]$$

$$e^{\nu} = \left(\frac{1-\mu}{1+\mu}\right)^2, \quad e^{\alpha} = (1+\mu)^4\frac{R^2}{z^2} \qquad R=R(t) \qquad z = 1+\frac{kr^2}{4} \qquad z' = \frac{2kr}{4}$$

$$\mu = \frac{m}{2rR}\sqrt{2}$$

Let us take <u>R=const</u> and work out the ρ and p of the field as affected by the mass particle. So $\dot{\mu}=0 \therefore \dot{\alpha}=0, \dot{\nu}=0$

we are dealing with the static case

$$e^{\alpha} = \frac{(1+\mu)^4 R^2}{z^2} \qquad \alpha = 4\log(1+\mu) + 2\log R - 2\log z$$

$$\therefore \alpha' = \frac{4\mu'}{1+\mu} - \frac{2z'}{z} \qquad \alpha'' = \frac{4\mu''}{1+\mu} - \frac{4\mu'^2}{(1+\mu)^2} - \frac{2z''}{z} + \frac{2z'^2}{z^2}$$

~~$\dot{\alpha} = \frac{4\dot{\mu}}{1+\mu}$~~

$$e^{\nu} = \frac{(1-\mu)^2}{(1+\mu)^2} \qquad \therefore \nu' = \frac{-2\mu'}{1-\mu} + \frac{2\mu'}{1+\mu} = \frac{2\mu'}{1-\mu^2}(-1-\mu+1-\mu)$$

~~$\nu' = \frac{-4\mu\mu'}{1-\mu^2}$~~ $\qquad \nu' = \frac{2(-\mu')}{1-\mu} - \frac{2(\mu')}{1+\mu} \qquad \nu'' = \frac{-2\mu''}{1-\mu} - \frac{2\mu'^2}{(1-\mu)^2} - \frac{2\mu''}{1+\mu} + \frac{2\mu'^2}{(1+\mu)^2}$

$$\therefore \alpha' + \nu' = \frac{4\mu'}{1+\mu} - \frac{2z'}{z} - \frac{2\mu'}{1-\mu} - \frac{2\mu'}{1+\mu} = \frac{2\mu'}{1+\mu} - \frac{2\mu'}{1-\mu} - \frac{2z'}{z}$$

$$= \frac{2\mu'}{1-\mu^2}(1-\mu-1-\mu) - \frac{2z'}{z} = \frac{-4\mu\mu'}{1-\mu^2} - \frac{2z'}{z}.$$

$$G_i^k = -8\pi T_i^k$$

$$\therefore -G_1^1 = 8\pi T_1^1 = -e^{-\alpha}\left[\frac{\alpha'^2}{4} + \frac{\alpha'\nu'}{2} + \frac{\alpha'+\nu'}{r}\right] \qquad \therefore G_1^1 = e^{-\alpha}\left[\frac{\alpha'^2}{4} + \frac{\alpha'\nu'}{2} + \frac{\alpha'+\nu'}{r}\right]$$

$$\therefore G_1^1 = \frac{z^2}{R^2} e^{-\alpha}\left[\left(\frac{4\mu'}{1+\mu} - \frac{2z'}{z}\right)\frac{1}{4} + \left(\frac{4\mu'}{1+\mu} - \frac{2z'}{z}\right)\left(\frac{-2\mu'}{1-\mu} - \frac{2\mu'}{1+\mu}\right)\frac{1}{2} + \frac{1}{r}\left(\frac{-4\mu\mu'}{1-\mu^2} - \frac{2z'}{z}\right)\right]$$

$$= e^{-\alpha}\left[\frac{4\mu'^2}{(1+\mu)^2} - 4\frac{z'}{z}\frac{\mu'}{1+\mu} + \frac{z'^2}{z^2} - \frac{4\mu'^2}{1-\mu^2} - \frac{4\mu'^2}{(1+\mu)^2} + \frac{4z'}{z}\frac{(1+\mu+1-\mu)\mu'}{2(1-\mu^2)}\right.$$

$$\left. - \frac{4\mu\mu'}{1-\mu^2}\frac{1}{r} - \frac{2z'}{zr}\right]$$

$$= e^{-\alpha}\left[\frac{z'^2}{z^2} - \frac{2z'}{z}\frac{1}{r} + \frac{4z'\mu'}{z}\frac{1}{1+\mu}\left(-1 + \frac{1}{1-\mu}\right) - \frac{4\mu'^2}{1-\mu^2} - \frac{2\mu\mu'}{1-\mu^2}\frac{1}{r}\right]$$

$$= e^{-\alpha}\left[\frac{z'}{z^2 r}(z'r - 2z) + 4\frac{z'}{z}\frac{\mu\mu'}{1-\mu^2} - \frac{4\mu'^2}{1-\mu^2} - \frac{4\mu\mu'}{1-\mu^2}\frac{1}{r}\right]$$

$$= e^{-\alpha}\left[\frac{2kr}{4z^2 r}\left(\frac{2k}{4}z^2 - 2 - \frac{2k}{4}r^2\right) - \frac{4\mu'^2}{1-\mu^2} + \frac{4\mu\mu'}{1-\mu^2}\left(\frac{2z'}{z} - \frac{1}{r}\right)\right]$$

$$\therefore G_1^1 = e^{-\alpha}\left[-\frac{k}{2r^2} - \frac{4\mu'^2}{1-\mu^2} + \frac{4\mu\mu'}{1-\mu^2}\left(\frac{z'}{2} - \frac{1}{r}\right)\right].$$

Next $-G_2^2 = 8\pi T_2^2 = -e^{-\alpha}\left[\frac{\alpha''}{2} + \frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\alpha'+\nu'}{2r}\right]$

$$\therefore G_2^2 = e^{-\alpha}\left[\frac{2\mu''}{1+\mu} - \frac{2\mu'^2}{(1+\mu)^2} - \frac{z''}{z} + \frac{z'^2}{z^2} - \frac{\mu''}{1-\mu} - \frac{\mu'^2}{(1-\mu)^2} - \frac{\mu''}{1+\mu} + \frac{\mu'^2}{(1+\mu)^2}\right.$$
$$\left. + \frac{1}{4}\left(\frac{-2\mu'}{1-\mu^2}\cdot 2\right)^2 + \frac{1}{2r}\left(\frac{-4\mu\mu'}{1-\mu^2} - 2\frac{z'}{z}\right)\right]$$

$$= e^{-\alpha}\left[\frac{\mu''}{1+\mu} - \frac{\mu'^2}{(1+\mu)^2} - \frac{\mu''}{1-\mu} - \frac{\mu'^2}{(1-\mu)^2} + \frac{4\mu'^2}{(1-\mu^2)^2} - \frac{z''}{z} + \frac{z'^2}{z^2} - \frac{z'}{zr} - \frac{2\mu\mu'}{r(1-\mu^2)}\right]$$

$$= e^{-\alpha}\left[\frac{\mu''}{1-\mu^2}(1-\mu-1-\mu) + \frac{\mu'^2}{(1-\mu^2)^2}\left(-(1-\mu)^2 - (1+\mu)^2 + 4\right) - \frac{z''}{z} + \frac{z'^2}{z^2} - \frac{z'}{zr} - \frac{2\mu\mu'}{r(1-\mu^2)}\right]$$ But note $z'' = \frac{2k}{4}$, $z' = \frac{2k}{4}r$

$$= e^{-\alpha}\left[-\frac{2\mu\mu''}{1-\mu^2} + \frac{\mu'^2}{(1-\mu^2)^2}\left(-2-2\mu^2+4\right) - \cancel{\frac{2k}{4}} - \frac{2k}{4} + \frac{4\left(\frac{k}{4}\right)^2 r^2}{z^2} - \frac{2\frac{k}{4}r}{rz} - \frac{2\mu\mu'}{r(1-\mu^2)}\right]$$

$$= e^{-\alpha}\left[-\frac{2\mu\mu''}{1-\mu^2} + \frac{2\mu'^2}{1-\mu^2} + \frac{k}{2r}\left(-r + \frac{k}{4}r^2\right) - \frac{2\mu\mu'}{r(1-\mu^2)}\right]$$

$$G_2^2 = e^{-\alpha}\left[-\frac{k}{2r^2} - \frac{2}{1-\mu^2}\left(\mu\mu'' - \mu'^2 + \frac{\mu\mu'}{r}\right)\right]$$ And lastly

$$-G_4^4 = 8\pi T_4^4 = -e^{-\alpha}\left[\alpha'' + \frac{\alpha'^2}{4} + \frac{2\alpha'}{r}\right]$$

$$\therefore G_4^4 = e^{-\alpha}\left[\frac{4\mu''}{1+\mu} - \frac{4\mu'^2}{(1+\mu)^2} - \frac{2z''}{z} + \frac{2z'^2}{z^2} + \frac{1}{4}\left(\frac{4\mu'}{1+\mu} - \frac{2z'}{z}\right)^2 + \frac{2}{r}\left(\frac{4\mu'}{1+\mu} - \frac{2z'}{z}\right)\right]$$

$$= e^{-\alpha}\left[-2\frac{z''}{z} + 2\frac{z'^2}{z^2} + \frac{z'^2}{z^2} - \frac{4}{r}\frac{z'}{z} + \frac{4\mu''}{1+\mu} + \frac{4\mu'^2}{(1+\mu)^2}(-1+1) - \frac{4\mu'}{1+\mu}\frac{z'}{z} + \frac{4\mu'}{1+\mu}\cdot\frac{2}{r}\right]$$

$$= e^{-\alpha}\left[-\frac{2\cdot\frac{2k}{4}}{z} + \frac{3\times 4\left(\frac{k}{4}\right)^2 z^2}{z^2} - \frac{8\frac{k}{4}r}{rz} + \frac{4\mu''}{1+\mu} + \frac{4\mu'}{1+\mu}\left(\frac{2}{r} - \frac{z'}{z}\right)\right].$$

$$= e^{-\alpha}\left[-\frac{3k}{2^2}\left(2 - \frac{k}{4}\right) + \frac{4\mu''}{1+\mu} + \frac{4\mu'}{1+\mu}\left(\frac{2}{2} - \frac{2'}{2}\right)\right]$$

$$G_4^4 = e^{-\alpha}\left[-\frac{3k}{2^2} + \frac{4\mu''}{1+\mu} + \frac{4\mu'}{1+\mu}\left(\frac{2}{2} - \frac{2'}{2}\right)\right].$$

$G_1^4 = 0$. We verify that $G_1^1 - G_2^2 = 0$.

$$G_1^1 - G_2^2 = e^{-\alpha}\left[-\frac{k}{2^2} - \frac{4\mu'^2}{1-\mu^2} + \frac{4\mu\mu'}{1-\mu^2}\left(\frac{2'}{2} - \frac{1}{2}\right) + \frac{k}{2^2} + \frac{2}{(1-\mu^2)}\left(\mu\mu'' - \mu'^2 + \frac{\mu\mu'}{2}\right)\right]$$

$$= \frac{e^{-\alpha}}{1-\mu^2}\left[2\mu\mu'' - 2\mu'^2 + \frac{2\mu\mu'}{2} - 4\mu'^2 + 4\mu\mu'\left(\frac{2'}{2} - \frac{1}{2}\right)\right]$$

$$= \frac{2e^{-\alpha}}{1-\mu^2}\mu\mu'\left(\frac{\mu''}{\mu'} - \frac{3\mu'}{\mu} + \frac{2z'}{2} - \frac{1}{2}\right) \qquad \mu = \frac{m}{2zR}\sqrt{2} \quad \therefore \frac{\mu'}{\mu} = -\frac{1}{2} + \frac{2'}{2z}$$

$$= \frac{-2\left(1 + \frac{kz^2}{4}\right) + z\frac{2kz}{4}}{2z^2} = \frac{-2}{2z^2} = -\frac{1}{z^2}$$

$$\mu' = -\frac{\mu}{z^2} = -\frac{m}{2zR}\frac{\sqrt{2}}{z^2} = -\frac{m}{2Rz^2}\sqrt{2}$$

$$\therefore \frac{\mu''}{\mu'} = -\frac{2}{z} - \frac{1}{2z}z'$$

$$\therefore G_1^1 - G_2^2 = \frac{2e^{-\alpha}\mu\mu'}{1-\mu^2}\left[-\frac{2}{z} - \frac{1}{2}\frac{z'}{2} - 3\left(-\frac{1}{z} + \frac{z'}{2z}\right) + \frac{2z'}{2} - \frac{1}{z}\right]$$

$$= 0. \quad \text{Verified.}$$

Now $g_{11} = G_1^1 = e^{-\alpha}\left[-\frac{k}{2^2} - \frac{4\mu'^2}{1-\mu^2} + \frac{4\mu\mu'}{1-\mu^2}\left(\frac{2'}{2} - \frac{1}{2}\right)\right]$

$$= e^{-\alpha}\left[-\frac{k}{2^2}\right] - \frac{4\mu^2}{1-\mu^2}e^{-\alpha}\left[\frac{\mu'^2}{\mu^2} - \frac{\mu'}{\mu}\left(\frac{2'}{2} - \frac{1}{2}\right)\right]$$

$$= e^{-\alpha}\left[-\frac{k}{2^2}\right] - \frac{4e^{-\alpha}}{1-\mu^2}\mu^2\left[\frac{1}{2^2} - \frac{2'}{2z} + \frac{2'^2}{4z^2} - \left(\frac{2'}{2z} - \frac{1}{2}\right)\left(\frac{2'}{2} - \frac{1}{2}\right)\right]$$

$$= e^{-\alpha}\left[-\frac{k}{2^2}\right] - \frac{4\mu^2}{1-\mu^2}e^{-\alpha}\left[\frac{1}{2^2}(1-1) + \frac{2'}{2z}\left(-1 + \frac{1}{2} + 1\right) + \frac{2'^2}{2^2}\left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$= e^{-\alpha}\left[-\frac{k}{2^2}\right] - \frac{4\mu^2}{1-\mu^2}e^{-\alpha}\left(\frac{2'}{2}\right)\left(\frac{1}{2z} - \frac{1}{4}\frac{2'}{2}\right)$$

(22c) $8\pi\rho = \bar{e}^{\alpha}\left[-\frac{k}{Z^2}\right] - \frac{4\mu^2}{1-\mu^2}\bar{e}^{\alpha}\cdot\frac{2\frac{k}{4}r}{Z}\left(\frac{1}{2r} - \frac{1}{4}\frac{2\frac{k}{4}r}{Z}\right)$

$= \bar{e}^{\alpha}\left[-\frac{k}{Z^2}\right] - \frac{4\mu^2}{1-\mu^2}\bar{e}^{\alpha}\frac{2\frac{k}{4}}{Z}\left(\frac{2Z - 2\frac{k}{4}r^2}{4rZ}\right)$

$= \bar{e}^{\alpha}\left[-\frac{k}{Z^2}\right] - \frac{4\mu^2}{1-\mu^2}\cdot\bar{e}^{\alpha}\frac{\frac{k}{4}}{Z^2}$

$= \bar{e}^{\alpha}\left(-\frac{k}{Z^2}\right)\left(1 + \frac{\mu^2}{1-\mu^2}\right) = \bar{e}^{\alpha}\frac{(-k)}{Z^2}\frac{1}{1-\mu^2}$ $\quad\therefore 8\pi\rho = \frac{\bar{e}^{\alpha}}{1-\mu^2}\left(-\frac{k}{Z^2}\right)$

$8\pi p = -G^4_4 = -\bar{e}^{\alpha}\left[-\frac{3k}{Z^2} + \frac{4\mu''}{1+\mu} + \frac{4\mu'}{1+\mu}\left(\frac{2}{r} - \frac{Z'}{Z}\right)\right]$

$= \bar{e}^{\alpha}\left(\frac{3k}{Z^2}\right) - \bar{e}^{\alpha}\cdot\frac{4}{1+\mu}\mu'\left(\frac{\mu''}{\mu'} + \frac{2}{r} - \frac{Z'}{Z}\right)$

$= \bar{e}^{\alpha}\frac{3k}{Z^2} - \bar{e}^{\alpha}\frac{4\mu}{1+\mu}\frac{\mu'}{\mu}\left(-\frac{2}{r} - \frac{Z'}{2Z} + \frac{2}{r} - \frac{Z'}{Z}\right)$

$= \bar{e}^{\alpha}\frac{3k}{Z^2} - \bar{e}^{\alpha}\cdot\frac{4\mu}{1+\mu}\left(-\frac{1}{r} + \frac{Z'}{2Z}\right)\left(-\frac{3}{2}\frac{2\frac{k}{4}r}{Z}\right)$

$= \bar{e}^{\alpha}\frac{3k}{Z^2} + \bar{e}^{\alpha}\frac{3kr}{2Z}\frac{\mu}{1+\mu}\left(-\frac{1}{r} + \frac{2\frac{k}{4}r}{2Z}\right)$

$= \bar{e}^{\alpha}\frac{3k}{Z^2} + \bar{e}^{\alpha}\frac{3kr}{Z}\frac{\mu}{1+\mu}\frac{\left(-Z + \frac{kr^2}{4}\right)}{8rZ}$

$= \bar{e}^{\alpha}\frac{3k}{Z^2} - \frac{3k}{Z^2}\frac{\mu}{1+\mu} = \bar{e}^{\alpha}\frac{3k}{Z^2}\frac{1+\mu-\mu}{1+\mu}$ $\quad\therefore 8\pi p = \frac{\bar{e}^{\alpha}}{1+\mu}\left(\frac{3k}{Z^2}\right)$

$\therefore 8\pi(\rho + 3p) = \frac{\bar{e}^{\alpha}}{1-\mu^2}\frac{3k}{Z^2}(1-\mu-1)$

$= \frac{-\mu}{1-\mu^2}\bar{e}^{\alpha}\frac{3k}{Z^2}$

$\neq 0.$

14 Jan. 1996

Let us now work out an approximate solution for a radiating mass-particle of small mass m when immersed in an Expanding Universe, based on McVittie's solution. We assume

$$ds^2 = e^{\nu} dt^2 - e^{\alpha}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

We know that in McVittie's solution $e^{\nu} = \left(\frac{1-\mu}{1+\mu}\right)^2$, $e^{\alpha} = \frac{(1+\mu)^4 R^2}{Z^2}$

$\mu = \frac{m}{2rR}\sqrt{Z}$, $R = R(t)$ $Z = 1 + \frac{k}{4}r^2$. We expand e^{ν} and e^{α} in powers of μ. $e^{\alpha} = \frac{R^2}{Z^2}(1 + 4\mu + 6\mu^2 + \dots)$ $e^{\nu} = (1-2\mu+\mu^2)(1+2\mu+\mu^2)^{-1}$

$e^{\nu} = (1-2\mu+\mu^2)(1-(2\mu+\mu^2) + (2\mu+\mu^2)^2 \dots)$

$= 1 - 2\mu - 2\mu + \mu^2 - \mu^2 + 4\mu^2 + 4\mu^2 + \dots) = 1 - 4\mu + 8\mu^2 + \dots)$

Let us choose $e^{\alpha} = \frac{R^2}{Z^2}(1 + 4(\mu + a) + 6(\mu^2 + b))$ $a = a(r,t)$, $b = b(r,t)$

$e^{\nu} = 1 - 4(\mu + c) + 8(\mu^2 + f)$ $c = c(r,t)$, $f = f(r,t)$

a and c are of the order of μ i.e. m
b and f are of the order of μ^2 i.e. m^2.

Let us begin with the case correct upto 1st power of μ. i.e. let us neglect μ^2 and higher powers

$ds^2 = e^{\nu} dt^2 - e^{\alpha}\left[dr^2 + r^2 d\Omega^2\right]$

$e^{\alpha} = \frac{R^2}{Z^2}(1+4(\mu+a))$ $\quad \alpha = 2\log R - 2\log Z + \log(1+4(\mu+a))$

$= 2\log R - 2\log Z + 4(\mu+a)$ neglecting μ^2

$\therefore \alpha' = -\frac{2Z'}{Z} + 4(\mu'+a')$, $\quad \alpha'' = -\frac{2Z''}{Z} + \frac{2Z'^2}{Z^2} + 4(\mu''+a'')$

$\dot{\alpha} = \frac{2\dot{R}}{R} + 4(\dot{\mu}+\dot{a})$, $\quad \ddot{\alpha} = \frac{2\ddot{R}}{R} - \frac{2\dot{R}^2}{R^2} + 4(\ddot{\mu}+\ddot{a})$

$\dot{\alpha}' = 4(\dot{\mu}'+\dot{a}')$

$e^{\nu} = 1-4(\mu+c)$ $\quad \nu = \log(1-4(\mu+c)) = -4(\mu+c)$

$\nu' = -4(\mu'+c')$ etc.

We now write down $-G_i^k = 8\pi T_i^k$ from Tolman's book. p. 252

$-G_1^1 = 8\pi T_1^1 = -e^{-\alpha}\left[\frac{\alpha'^2}{4} + \frac{\alpha'\nu'}{2} + \frac{\alpha'+\nu'}{r}\right] + e^{\nu}\left[\ddot{\alpha} + \frac{3}{4}\dot{\alpha}^2 - \frac{\dot{\alpha}\dot{\nu}}{2}\right]$

$\therefore G_1^1 = \frac{Z^2}{R^2}(1-4(\mu+a))\left[\frac{1}{4}\left(-\frac{2Z'}{Z} + 4(\mu'+a')\right)^2 + \frac{1}{2}\left(-\frac{2Z'}{Z} + 4(\mu'+a')\right)(-4)(\mu'+c')\right.$

$\left. + \frac{1}{2}\left(-\frac{2Z'}{Z} + 4(\mu'+a') - 4(\mu'+c')\right)\right] - (1+4(\mu+c))\left[\frac{2\ddot{R}}{R} - \frac{2\dot{R}^2}{R^2} + 4(\ddot{\mu}+\ddot{a})\right.$

$\left. + \frac{3}{4}\left(\frac{2\dot{R}}{R} + 4(\dot{\mu}+\dot{a})\right)^2 - \frac{1}{2}\left(\frac{2\dot{R}}{R} + 4(\dot{\mu}+\dot{a})\right)(-4)(\dot{\mu}+\dot{c})\right]$

$= -\frac{Z^2}{R^2}(1-4(\mu+a))\left[\frac{Z'^2}{Z^2} - \frac{2}{r}\frac{Z'}{Z} - 4(\mu'+a')\frac{Z'}{Z} + 4\frac{Z'}{Z}(\mu'+c') + \frac{4}{r}(a'-c')\right]$

$- (1+4(\mu+c))\left[\frac{2\ddot{R}}{R} - \frac{2\dot{R}^2}{R^2} + 3\frac{\dot{R}^2}{R^2} + 4(\ddot{\mu}+\ddot{a}) + 12\frac{\dot{R}}{R}(\dot{\mu}+\dot{a}) + 4\frac{\dot{R}}{R}(\dot{\mu}+\dot{c})\right.$

$G_1^1 = \frac{Z^2}{R^2}\left[\left(\frac{Z'^2}{Z^2} - \frac{2}{r}\frac{Z'}{Z}\right)(1-4(\mu+a)) - 4(a'-c')\frac{Z'}{Z} + \frac{4}{r}(a'-c')\right]$

$\left. - \left[\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right)(1+4(\mu+c)) + 4(\ddot{\mu}+\ddot{a}) + 16\dot{\mu}\frac{\dot{R}}{R} + 4\frac{\dot{R}}{R}(3\dot{a}+\dot{c})\right]\right.$

$$-G_2^2 = 8\pi T_2^2 = -e^{-\alpha}\left[\frac{\alpha''}{2}+\frac{\nu''}{2}+\frac{\nu'^2}{4}+\frac{\alpha'+\nu'}{2r}\right]+e^{-\nu}\left[\ddot{\alpha}+\frac{3}{4}\dot{\alpha}^2-\frac{\dot{\alpha}\dot{\nu}}{2}\right]$$

$$\therefore G_2^2 = \frac{Z^2}{R^2}(1-4(\mu+a))\left[\frac{1}{2}\left(-\frac{2Z''}{Z}+\frac{2Z'^2}{Z^2}+4(\mu''+a'')\right)+\frac{1}{2}(-4)(\mu''+c'')+\frac{1}{4}\times 16(\mu'+c')^2\right.$$
$$\left.+\frac{1}{2r}\left(-\frac{2Z'}{Z}+4(\mu'+a')-4(\mu'+c')\right)\right]-(1+4(\mu+c))\left[2\frac{\ddot{R}}{R}-2\frac{\dot{R}^2}{R^2}+4(\ddot{\mu}+\ddot{a})\right.$$
$$\left.+\frac{3}{4}\left(\frac{2\dot{R}}{R}+4(\dot{\mu}+\dot{a})\right)^2-\frac{1}{2}\left(\frac{2\dot{R}}{R}+4(\dot{\mu}+\dot{a})\right)(-4)(\dot{\mu}+\dot{c})\right]$$
$$=\frac{Z^2}{R^2}(1-4(\mu+a))\left[-\frac{Z''}{Z}+\frac{Z'^2}{Z^2}-\frac{Z'}{rZ}+2(\mu''+a'')-2(\mu''+c'')+\frac{2}{r}(a'-c')\right]$$
$$-(1+4(\mu+c))\left[2\frac{\ddot{R}}{R}-2\frac{\dot{R}^2}{R^2}+3\frac{\dot{R}^2}{R^2}+4(\ddot{\mu}+\ddot{a})+12\frac{\dot{R}}{R}(\dot{\mu}+\dot{a})+4\frac{\dot{R}}{R}(\dot{\mu}+\dot{c})\right]$$

$$\boxed{G_2^2 = \frac{Z^2}{R^2}\left[\left(-\frac{Z''}{Z}+\frac{Z'^2}{Z^2}-\frac{Z'}{rZ}\right)(1-4(\mu+a))+2(a''-c'')+\frac{2}{r}(a'-c')\right]}$$
$$\boxed{-\left[\left(2\frac{\ddot{R}}{R}+\frac{\dot{R}^2}{R^2}\right)(1+4(\mu+c))+4(\ddot{\mu}+\ddot{a})+16\frac{\dot{R}}{R}\dot{\mu}+4\frac{\dot{R}}{R}(\cancel{4\dot{\mu}}\,3\dot{a}+\dot{c})\right]}$$

$$-G_4^4 = 8\pi T_4^4 = -e^{-\alpha}\left[\alpha''+\frac{\alpha'^2}{4}+\frac{2\alpha'}{r}\right]+\frac{3}{4}e^{-\nu}\dot{\alpha}^2$$

$$\therefore G_4^4 = \frac{Z^2}{R^2}(1-4(\mu+a))\left[-\frac{2Z''}{Z}+\frac{2Z'^2}{Z^2}+4(\mu''+a'')+\frac{1}{4}\left(-\frac{2Z'}{Z}+4(\mu'+a')\right)^2+\frac{2}{r}\left(-\frac{2Z'}{Z}+4(\mu'+a')\right)\right.$$
$$\left.-\frac{3}{4}(1+4(\mu+c))\left(\frac{2\dot{R}}{R}+4(\dot{\mu}+\dot{a})\right)^2\right.$$
$$=\frac{Z^2}{R^2}(1-4(\mu+a))\left[-\frac{2Z''}{Z}+2\frac{Z'^2}{Z^2}+\frac{Z'^2}{Z^2}-\frac{4Z'}{rZ}+4(\mu''+a'')-4\frac{Z'}{Z}(\mu'+a')+\frac{8}{r}(\mu'+a')\right]$$
$$\left.-\frac{3}{4}(1+4(\mu+c))\left(4\frac{\dot{R}^2}{R^2}+16\frac{\dot{R}}{R}(\dot{\mu}+\dot{a})\right)\right.$$

$$\boxed{G_4^4 = \frac{Z^2}{R^2}\left[\left(-\frac{2Z''}{Z}+3\frac{Z'^2}{Z^2}-4\frac{Z'}{rZ}\right)(1-4(\mu+a))+4(\mu''+a'')-4(\mu'+a')\left(\cancel{\frac{4}{r}}\frac{Z'}{Z}-\frac{2}{r}\right)\right]}$$
$$\boxed{-3\left[\frac{\dot{R}^2}{R^2}(1+4(\mu+c))+4\frac{\dot{R}}{R}(\dot{\mu}+\dot{a})\right]}$$

$$-G_1^4 = \cancel{\cdot}\,8\pi T_1^4 = -e^{-\nu}\left(\dot{\alpha}'-\frac{\dot{\alpha}\nu'}{2}\right) \quad \therefore G_1^4 = (1+4(\mu+c))\left[4(\dot{\mu}'+\dot{a}')-\frac{1}{2}\left(\frac{2\dot{R}}{R}+4(\dot{\mu}+\dot{a})\right)\times(-4)(\mu'+c')\right]$$

$$\therefore G_1^4 = (1+4(\mu+c))\left(4(\dot{\mu}'+\dot{a}')+4\frac{\dot{R}}{R}(\mu'+c')\right)$$

$$\boxed{G_1^4 = 4\left[\dot{\mu}'+\dot{a}'+\frac{\dot{R}}{R}(\mu'+c')\right]}$$

(922) We now take following cases"

Case (1) $k=0$ $R=1$ $G_i^k=0$ Schwarzschild's soln. to 1st order.

(2) $k=0$, $R=1$ $G_i^k=-\sigma w_i w^k$ Radiating soln to 1st order

(3) $k\neq 0$ $R=$ const ~~mas~~ $G_1^1=G_2^2$ mass particle in Einstein Universe to 1st order

(4) $k\neq 0$ $R=$ const $G_1^1-G_2^2=e^{(\nu-\alpha)/2}G_1^4$ radiating particle in Einstein Universe to the 1st order

(5) $k\neq 0$ $R=R(t)$ $G_1^1=G_2^2$ mass particle in Expanding Universe to the 1st order

(6) $k\neq 0$, $R=R(t)$ $G_1^1-G_2^2=e^{(\nu-\alpha)/2}G_1^4$ radiating particle in Expanding Universe to the 1st order

(1) $k=0$, $R=1$ $G_i^k=0$ $\Rightarrow$ $\frac{4}{r}(a'-c')-4(\mu''+a'')=0$

$2(a''-c'')+\frac{2}{r}(a'-c')-4(\mu'+a')=0$

$4(\mu''+a'')+\frac{8}{r}(\mu'+a')=0$

$\mu'+a'+\frac{r}{2}=0$

simplest solution is $\mu=\frac{m}{2r}$ $a=0$, $c=0$. The Schwarzschild's solution.

(2) $k=0$ $R=1$ $G_i^k=-8\pi\sigma w_i w^k$ $w_1 w^1+w_4 w^4=0$ $\therefore G_1^1+G_4^4=0$, $G_2^2=0$

$G_1^1=e^{(\nu-\alpha)/2}G_1^4$

$G_2^2=0 \longrightarrow 2(a''-c'')+\frac{2}{r}(a'-c')-4(\mu''+a'')=0$

i.e. $a''-c''+\frac{1}{r}(a'-c')=2(\mu''+a'')$ ——— I

$G_1^1 + G_4^4 = 0 \Rightarrow \frac{4}{2}(a'-c') - 4(\mu''+a'') + 4(\mu''+a'') + \frac{8}{2}(\mu'+a') = 0$

$$\mu'' + a'' + \frac{2}{2}(\mu'+a') + \frac{1}{2}(a'-c') = \mu''+a'' \quad \text{——— II}$$

$G_1^1 = e^{(\omega-\alpha)/2} G_1^4$ or $G_1^1 - G_2^2 = e^{(\omega-\alpha)/2} G_1^4 \Rightarrow \frac{4}{2}(a'-c') - 2(a''-c'') - \frac{2}{2}(a'-c') = 1 \times 4(\mu'+a')$

$$\therefore -(a''-c'') + \frac{1}{2}(a'-c') = 2(\mu'+a') \quad \text{——— III}$$

Take I + III. and get $\frac{2}{2}(a'-c') = 2(\mu''+a'') + 2(\mu'+a')$ IV

$$\text{or } \frac{a'-c'}{2} = \mu'' + a'' + \mu' + a' \quad \text{IV}$$

Take II − IV and get $\mu'' + a'' + \frac{2}{2}(\mu'+a') = -\mu'-a'$

Let $\mu + a = x$ $\quad x'' + \frac{2}{2}x' + x' = 0$

$$2^2 x'' + 2 2 x' + 2^2 x' = 0$$

$$\therefore (2^2 x')' + (2^2 x') = 0$$

$$\therefore 2^2 x' = f(t-2). \quad x' = \frac{f(t-2)}{2^2}$$

$$\therefore \mu' + a' = \frac{f(t-2)}{2^2}. \qquad \text{Choose } f(t-2) = \lambda(t-2) - \frac{m}{2}$$

$$\therefore \mu' + a' = \frac{\lambda(t-2)}{2^2} - \frac{m}{2 2^2}. \qquad \text{Setting } \mu = \frac{m}{2 2} \quad \mu' = -\frac{m}{2 2^2}$$

$$\therefore a' = \frac{\lambda(t-2)}{2^2} = \frac{\lambda t}{2^2} - \frac{\lambda}{2}$$

$$\therefore a = -\frac{\lambda t}{2} - \lambda \log 2 + \text{co function of } t$$

$$= -\frac{\lambda t}{2} - \lambda \log 2 + \lambda$$

$$= -\frac{\lambda(t-2)}{2} - \lambda \log 2$$

IV will now give c

(227) $\frac{a'-c'}{r} = \mu' + 4\dot{a} + \mu' + a' \quad \therefore \frac{\lambda(t-r)}{r^2}\cdot\frac{1}{r} - \frac{c'}{r} = 0 + \frac{\lambda}{r^2}$

$$\therefore \frac{\lambda t}{r^2} - \frac{\lambda}{r} - \frac{\lambda}{r} = c' \qquad c = -\frac{\lambda(t-r)}{r^2} - 2\lambda \log r$$

$\mu = \frac{m}{2r} \quad a = -\frac{\lambda(t-r)}{r} - \lambda \log r. \qquad c = -\frac{\lambda(t-r)}{r} - 2\lambda \log r$

gives a radiating mass particle in isotropic coords correct to 1st power of m & λ is of the order of m.

Case (3) $k \neq 0 \quad R = \text{const.} \quad G_1^1 = G_2^2 \quad G_1^1 - G_2^2 = 0$

$$\frac{z^2}{R^2}\left[\left(\frac{z'^2}{z^2} - \frac{2}{r}\frac{z'}{z}\right)(1-4(\mu+a)) - 4(a'-c')\left(\frac{z'}{z} - \frac{1}{r}\right)\right.$$

$$\left. -\left(-\frac{z''}{z} + \frac{z'^2}{z^2} - \frac{z'}{rz}\right)(1-4(\mu+a)) - 2(a''-c'') - \frac{2}{r}(a'-c')\right] = 0$$

$z = 1 + \frac{kr^2}{4} \quad z' = 2k\frac{r}{4} \quad z'' = \frac{2k}{4} \quad \therefore z'' = \frac{z'}{r}$

$$\therefore -\frac{z''}{z} + \frac{z'^2}{z^2} - \frac{z'}{rz} = -\frac{2z'}{rz} + \frac{z'^2}{z^2}$$

$$\therefore G_1^1 - G_2^2 = 0 \implies -2(a''-c'') - 2(a'-c')\left(\frac{2z'}{z} - \frac{1}{r}\right) = 0$$

simplest soln is $a = c = 0$ μ undetermined. Take $\mu = \frac{m/2}{2\pi R}$.

1st order McVittie Soln in Einstein Universe

Case (r) $R \neq 0$, R: const $\quad G_1^1 - G_2^2 = \ell^{(\nu-\alpha)/2} G_1^4$

$$\frac{z^2}{R^2}\left[-2(a''-c'') - 2(a'-c')\left(\frac{2z'}{z} - \frac{1}{r}\right)\right] = \left[1 \times 4(\mu'+a')\right]\frac{z}{R}$$

$$\frac{z}{R}\,a''-c''+(a'-c')\left(\frac{2z'}{z}-\frac{1}{z}\right)+\frac{2R}{z}\left(\mu''+a'\right)=0$$

Take $\mu=\frac{m\sqrt{z}}{2zR}$ $\therefore \mu=0$ $\therefore$ Above equ. becomes

$$a''-c''+(a'-c')\left(\frac{2z'}{z}-\frac{1}{z}\right)=-\frac{2R}{z}a'$$

I.F. is $\frac{z^2}{z}$ $\therefore (a'-c')\frac{z^2}{zR}=\int-\frac{2R}{z}a'\frac{z}{R}^2+\text{const.}$

$$(a'-c')\frac{z^3}{zR}=-2R\int\frac{a''}{z}z\,dz+\text{const.}$$

In following case (2) we had $a'=\frac{\lambda(t-z)}{z^2}$

But with $z\neq 1$, $R\neq 1$, radial null lines with μ neglected are not $dt^2-dz^2=0$ but are $dt^2-\frac{z^2}{R^2}dz$

$$dt^2-\frac{R^2}{z^2}dz^2=0 \qquad \therefore t-z \text{ has to be replaced by}$$

$$t-R\int\frac{1}{z}dz$$

So we choose $a'=\frac{\lambda\left(t-R\int\frac{1}{z}dz\right)}{z^2}$ λ a const.

$$\therefore\ a'=\frac{\lambda}{z^2}$$

$$(a'-c')\frac{z^2}{z}=-2R\int\frac{z}{z}\frac{\lambda}{z^2}dz+\Phi(t).$$

$$=-2R\lambda\int\frac{z}{z^3}dz+\Phi(t)$$

$$=-2R\lambda\int\frac{1+kz^2/4}{z^3}dz+\Phi(t)$$

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at Madras

$$\therefore (\dot{a}-\dot{c})\frac{z^2}{r} = -2R\lambda\int\left(\frac{1}{r^3}+\frac{k}{4}\frac{1}{r}\right)dr + \phi(t)$$

$$= -2R\lambda\left[-\frac{1}{2r^2}+\frac{k}{4}\log r\right] + \phi(t)$$

$$= \frac{R\lambda}{r^2} - \frac{R\lambda k}{2}\log r + \phi(t)$$

$$\therefore \dot{a}-\dot{c} = \frac{R\lambda}{rz^2} - \frac{R\lambda k}{2z^2} r\log r + \phi(t)\frac{r}{z^2}.$$

On integrating this we shall get $a-c$ and integrating equ. for $\dot{a}$ we shall get a and hence also c.

We shall first integrate this equation for $\dot{a}$.

$$\dot{a} = \frac{\lambda(t-R\int\frac{1}{2}dr)}{r^2} = \frac{\lambda t}{r^2} - \lambda R\frac{\int\frac{1}{2}dr}{r^2}.$$

$$\therefore a = -\frac{\lambda t}{r} - \lambda R\int\frac{1}{r^2}\int\frac{1}{2}dr\,.dr + B(t)$$

$$= -\frac{\lambda t}{r} - \lambda R\left[-\frac{1}{r}\int\frac{1}{2}dr + \int\frac{1}{r}\frac{1}{2}dr\right] + B(t)$$

$$= -\frac{\lambda t}{r} + \frac{\lambda}{r}\int\frac{R}{2}dr - \lambda R\int\frac{1}{r\left(1+\frac{kr^2}{4}\right)}dr + B(t)$$

$$= -\frac{\lambda(t-R\int\frac{1}{2}dr)}{r} - \lambda R\,I + B(t)$$

$$I = \int\frac{dr}{r\left(1+\frac{kr^2}{4}\right)} = \frac{1}{2}\int\frac{2r\,dr}{r^2\left(1+\frac{kr^2}{4}\right)} = \frac{1}{2}\int 2r\,dr\left[\frac{1}{r^2}-\frac{k/4}{1+k\frac{r}{4}}\right]$$

$$= \frac{1}{2}\log r^2 - \frac{1}{2}\log\left(1+\frac{kr^2}{4}\right) = \frac{1}{2}\log\frac{r^2}{1+\frac{kr^2}{4}}$$

$$\therefore \quad a = -\frac{\lambda(t - R\int \frac{1}{z}\,dy)}{z} - \frac{\lambda R}{2}\log\frac{z^2}{2} + B(t).$$

Next let us integrate a-c eqn.

$$a' - c' = \frac{R\lambda}{2z^2} - \frac{R\lambda k}{2z^2}\,z\log z + \Phi(t)\frac{z}{2z}$$

$$\therefore a - c = R\lambda \int \frac{dz}{z\left(1+k\frac{z^2}{4}\right)^2} - \frac{R\lambda k}{2}\int \frac{z}{\left(1+k\frac{z^2}{4}\right)^2}\log z + \Phi(t)\int \frac{z}{\left(1+k\frac{z^2}{4}\right)^2}dz$$

$$= \frac{R\lambda}{2}\int \frac{2z\,dz}{z^2\left(1+k\frac{z^2}{4}\right)^2} - \frac{R\lambda k}{4}\int \frac{2z\,k/4}{\left(1+k\frac{z^2}{4}\right)^2}\cdot\log z + \Phi(t)\int \frac{z}{\left(1+k\frac{z^2}{4}\right)^2}dz$$

$$= \frac{R\lambda}{2}\int \frac{dx}{x\left(1+\frac{k}{4}x\right)^2} - R\lambda\left[\log z\left(-\frac{1}{1+k\frac{z^2}{4}}\right) + \int \frac{1}{z}\,\frac{1}{1+k\frac{z^2}{4}}dz\right]$$

$$+ \frac{\Phi(t)}{2}\int \frac{dx}{\left(1+\frac{k}{4}x\right)^2}$$

$$= \frac{R\lambda}{2}\int dx\left(\frac{A}{x} + \frac{B}{1+\frac{k}{4}x} + \frac{C}{\left(1+\frac{k}{4}x\right)^2}\right) + \frac{R\lambda\log z}{1+k\frac{z^2}{4}} - R\lambda I$$

$$+ \frac{\Phi(t)}{2}\int \frac{dx}{\left(1+\frac{k}{4}x\right)^2}$$

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22c) $\frac{1}{x(1+\frac{k}{4}x)^2} = \frac{A}{x} + \frac{B}{1+\frac{kx}{4}} + \frac{C}{(1+\frac{k}{4}x)^2}$

$\therefore 1 = A(1+\frac{kx}{4})^2 + Bx(1+\frac{kx}{4}) + Cx$

$\therefore 1 = A.$ $\quad 0 = 2A\frac{k}{4} + B + C$ and $0 = A(\frac{k}{4})^2 + B(\frac{k}{4})$ $\therefore B = -\frac{k}{4}$

$0 = 2\frac{k}{4} - \frac{k}{4} + C$ $\qquad C = -\frac{k}{4}$

$\therefore a - c = \frac{R\lambda}{2}\int dx \left(\frac{1}{x} - \frac{k/4}{1+\frac{k}{4}x} - \frac{k/4}{(1+\frac{k}{4}x)^2} \right) + \frac{R\lambda \log z}{1+R\frac{z^2}{4}} - R\lambda\left[\frac{1}{2}\log\frac{z^2}{2}\right]$

$+ \frac{\varphi(t)}{2}\left(\frac{-1}{k/4}\right)\frac{1}{1+\frac{k}{4}x}$

$= \frac{R\lambda}{2}\left[\log x - \log(1+\frac{kx}{4}) + \frac{1}{(1+\frac{k}{4}x)} \right] + \frac{R\lambda}{2z}\log x - \frac{R\lambda}{2}\log x + \frac{R\lambda}{2}\log z$

$- \frac{2\varphi(t)}{k}\frac{1}{z}$ $\qquad$ choose $\frac{R\lambda}{2} = \frac{2\varphi(t)}{k}$

$\therefore$ ~~$a - c = -R\lambda\log z + \frac{R\lambda}{2}\log z =$~~ $a - c = \frac{R\lambda}{2z}\log x = \frac{R\lambda}{2}\log z$

$\therefore$ ~~$a + \frac{R\lambda}{2}\log z = c$ i.e. $c = -\lambda(t - R\int\frac{1}{2}dz)$~~ $-$

$\therefore a - \frac{R\lambda}{2}\log z = c$ $\quad$ i.e. $c = \frac{-\lambda(t - R\int\frac{1}{2}dz)}{z} - R\lambda\log z + \frac{\lambda R}{2}\log z + B(t) - \frac{\lambda R}{2}\log k$

Put $\mathfrak{B}(t)=0$

$$\therefore c = -\frac{\lambda\left(t - R\int\frac{1}{2}dz\right)}{z} - R\lambda\log z(1+z) - R\lambda\log z\left(1+\frac{1}{2}\right) + \frac{\lambda R}{2}\log z$$

$$a = -\frac{\lambda\left(t - R\int\frac{1}{2}dz\right)}{z} - R\lambda\log z + \frac{\lambda R}{2}\log z$$

If we put $R=1$, $Z=1$ we recover radiating star metric to the first order.

So we had full 4 pages of calculations to work case (r) when $k\neq 0$ $R=$ const. and I found that when case (s) $k\neq 0$ $R=R(t)$ we cannot carry the work in full. We end with an integral of the type $\int\frac{e^{-2\dot{R}z}}{z}$! This only means that our assumption for a' is not proper we take $a' = \frac{\lambda\left(\int\frac{1}{R}dt - \int\frac{1}{2}dz\right)}{z^2}$ from Physics. ~~but one can always~~ We now follow pure mathematics even in choosing our assumption between a and c. Field equations of Physics give us only one equation

$$G_1^1 - G_2^2 = e^{(\nu-\alpha)/2}\cdot G_1^4$$

Using G_i^j from p. 120, 121, we find $\frac{Z^2}{R^2}\left[\left(\frac{Z'^2}{Z^2} - \frac{Z}{R}\frac{Z'}{Z}\right)(1-4(\mu+a)) - 4(a'-c')\frac{Z'}{Z}\right.$

$$\left. + \frac{4}{2}(a'-c') - \left(-\frac{Z''}{Z} + \frac{Z'^2}{Z^2} - \frac{Z'}{RZ}\right)(1-4(\mu+a)) - 2(a''-c'') - \frac{2}{R}(a'-c')\right]$$

$$= (1-2(\mu+c))\frac{Z}{R}(1-2(\mu+a)\left[4\right]\left[\dot{\mu}'+\dot{a}'+\frac{\dot{R}}{R}(\mu'+c')\right]$$

We have simplified l.h.s. on p. 228

$$\therefore \frac{Z^2}{R^2}\left[-2(a''-c'') - 2(a'-c')\left(\frac{2Z'}{Z} - \frac{1}{R}\right)\right] = 4\frac{Z}{R}\left(\dot{\mu}'+\dot{a}'+\frac{\dot{R}}{R}(\mu'+\dot{a})\right)$$

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$$a''-c''+(a'-c')\left(\frac{2Z'}{Z}-\frac{1}{r}\right) = -\frac{2R}{Z}\left[\dot{\mu}'+\dot{a}'+\frac{\dot{R}}{R}(\mu'+c')\right]$$

But $\mu = +\frac{m}{2rR}\sqrt{2}$ $\therefore \frac{\dot{\mu}}{\mu} = -\frac{\dot{R}}{R}$ $\therefore \frac{\dot{\mu}'}{\mu} - \frac{\dot{\mu}\mu'}{\mu^2} = 0$ $\therefore \dot{\mu}' =$

$\therefore \dot{\mu}' - \frac{\dot{\mu}}{\mu}\mu' = 0$ $\therefore \dot{\mu}' + \frac{\dot{R}}{R}\mu' = 0$

$$\therefore a''-c''+(a'-c')\left(\frac{2Z'}{Z}-\frac{1}{r}\right) = -\frac{2R}{Z}\left[a'+\frac{\dot{R}}{R}c'\right]$$

~~We are no~~

$$= -\frac{2R}{Z}\left(a'+\frac{\dot{R}}{R}a - \frac{\dot{R}}{R}(a-c)\right)$$

$$\therefore a''-c''+(a-c')\left(\frac{2Z'}{Z}-\frac{1}{r}-\frac{2\dot{R}}{Z}\right) = -\frac{2R}{Z}\left(a''+\frac{\dot{R}}{R}a'\right) \qquad \text{I}$$

To solve this equation we need an assumption between a and c. Earlier we assumed a relation for a. ~~from~~ But we would rather assume that $a-c$ is the same as was in the pure radiation case. Eqn. at top of p. 124 gives $\frac{a-c}{r} = \frac{\lambda}{r^2}$. So

~~We assume~~ $a-c = \frac{\lambda}{r}$ $\lambda = \frac{\mu}{R}$ μ const. ~~a const.~~ $\therefore$ Eq. I above gives

$$-\frac{\lambda}{r^2}+\frac{\lambda}{r}\left(\frac{2Z'}{Z}-\frac{1}{r}-\frac{2\dot{R}}{Z}\right) = -\frac{2R}{Z}(Ra)^{\cdot}$$

$$\therefore -\frac{2\lambda}{r^2}+\frac{2\lambda}{r}\frac{Z'}{Z}-\frac{2\dot{R}}{Z}\frac{\lambda}{r} = -\frac{2R}{Z}(Ra)^{\cdot}$$

$$\therefore (Ra)'' = \frac{\lambda Z}{Rr^2}-\frac{\lambda}{r}\frac{Z'}{R}+\frac{\dot{R}\lambda}{Rr}$$

Integrate w.r.t. r

$$\therefore (Ra)^{\cdot} = \frac{\lambda}{R}\int\frac{1+k\frac{r^2}{4}}{r^2}dr + \frac{\lambda}{R}\int\frac{2\frac{R}{4}r}{r}dr + \dot{R}\lambda\log r + B(t)$$

We set $B(t) = 0$

~~∴ Ra (Ra)'~~ We assume

$$a' - c' = \frac{\lambda}{R}\cdot\frac{1}{r} \qquad \lambda = \text{constant.} \qquad \text{Eq. I now gives}$$

$$-\frac{\lambda}{R}\frac{1}{r^2} + \frac{\lambda}{R}\left(\frac{1}{r}\right)\left(\frac{2Z'}{Z} - \frac{1}{r} - \frac{2\dot{R}}{Z}\right) = -\frac{2R}{Z}\frac{(Ra')^{\cdot}}{R}$$

$$\therefore (Ra')^{\cdot} = \cancel{\frac{\lambda}{R^2}} \; \frac{Z}{r^2}\frac{\lambda}{R} - \frac{\lambda}{R}\frac{Z'}{r} + \frac{\lambda}{R}\dot{R}\frac{1}{r}. \qquad \text{Integrate w.r.t. } t$$

$$\therefore Ra' = \frac{1+kr^2/4}{r^2}\lambda\int\frac{1}{R}dt - \lambda\cdot\frac{2k}{4}\int\frac{1}{R}dt + \frac{\lambda}{r}\log R + f(r)$$

choose $f(r) = -\frac{\lambda}{r^2}\int\frac{1}{Z}dr$

$$= \frac{\lambda}{r^2}\left(\int\frac{1}{R}dt - \int\frac{1}{Z}dr\right) - \lambda\frac{k}{4}\int\frac{1}{R}dt + \frac{\lambda}{r}\log R$$

$$\therefore a' = \frac{\lambda}{R}\frac{\left(\int\frac{1}{R}dt - \int\frac{1}{Z}dr\right)}{r^2} - \frac{\lambda}{R}\frac{k}{4}\int\frac{1}{R}dt + \frac{\lambda}{R}\frac{\log R}{r}$$

$$\therefore a = \frac{\lambda}{R}\int\frac{1}{R}dt\int\frac{1}{r^2}dr - \frac{\lambda}{R}\int\left(\frac{1}{r^2}\int\frac{1}{Z}dr\right)dr - \frac{\lambda}{R}\frac{kr}{4}\int\frac{1}{R}dt$$

$$+ \frac{\lambda}{R}\log r\log R + \phi(t) \qquad \text{We put } \phi(t)=0$$

$$= \frac{\lambda}{R}\int\frac{1}{R}dt\left(-\frac{1}{r}\right) - \frac{\lambda}{R}\left[-\frac{1}{r}\int\frac{1}{Z}dr + \int\frac{1}{rZ}dr\right] - \frac{\lambda}{R}\frac{kr}{4}\int\frac{1}{R}dt$$

$$+ \frac{\lambda}{R}\log r\log R$$

$$\int\frac{1}{rZ}dr = \int\frac{1}{r\left(1+k\frac{r^2}{4}\right)}$$

$$= \frac{1}{2}\int\frac{2r}{r^2\left(1+k\frac{r^2}{4}\right)} = \frac{1}{2}\int 2r\left(\frac{1}{r^2} - \frac{\frac{k}{4}}{1+\frac{kr^2}{4}}\right)dr$$

$$= \frac{1}{2}\log\frac{r^2}{1+kr^2/4} = \frac{1}{2}\log\frac{r^2}{Z}$$

$$\therefore a = -\frac{\lambda\left(\int \frac{1}{R}\,dt - \int \frac{1}{z}\,dr\right)}{r} - \frac{\lambda}{R}\log\frac{r}{\sqrt{z}} - \frac{\lambda}{R}\frac{kz}{4}\int\frac{1}{R}\,dt + \frac{\lambda}{R}\log r \log R$$

Now $\dot{a}-\dot{c} = \frac{\lambda}{R}\frac{1}{r}$ or $a - c = \frac{\lambda}{R}\log z$ will give c.

We can now work out ϕ, p, σ to this first order.

We started working with the exact approximation, i.e choosing e^{α} and e^{γ} as on p. 226, but we found a very complicated situation with of course $6b$ and $8f$ following our a, c pattern above but complicated by appearance of $(\dot{a}^2)^{\cdot}$, $\ddot{a}a$, $\dot{a}\dot{d}$ and so on. And that set us thinking. We are regarding a, c of order of μ in first approximation. In that approximation let us take a pure radiating particle, on p. 227 we have

$$\mu = \frac{m}{2k} \qquad a = -\frac{\lambda(t-k)}{r} - \lambda\log k. \quad \text{For this } 8\pi\sigma = 4(\dot{\mu}+\dot{a})$$

$$\dot{a} = -\frac{\lambda}{r}$$

$$= \frac{4\lambda}{r^2}$$

$\therefore \lambda$ is the order of Luminosity $(-dm/dt)$ of the star

But our λ is of the order of μ i.e. $\frac{m}{r}$. So our approximation uses the luminosity of a star to be of the same

order as $\frac{m}{r}$. But for the Sun $\frac{m}{r} \sim 10^{-5}$ while luminosity $L_\odot \sim 10^{-20}$

$\therefore$ Luminosity $\sim \left(\frac{m}{r}\right)^4$!!!

Of course the radiating star solution is ~~valid~~ useful and not for normal stars but for highly dense stars with much higher rate of energy generation i.e. for stars for which L need not go as $\left(\frac{m}{r}\right)^4$. But even for such stars $L \sim \left(\frac{m}{r}\right)$ is practically no-use assumption.

We $\therefore$ assume that the luminosity of the order of $\frac{m^2}{r^2}$ i.e. of the order of μ^2. $\therefore$ We begin with

$$ds^2 = e^\nu dt^2 - e^\alpha \left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] \quad \text{with}$$

McVittie's form $e^\nu = \left(\frac{1-\mu}{1+\mu}\right)^2$, $e^\mu = (1+\mu)^4 \frac{R^2}{Z^2}$

$$Z = 1 + k\frac{r^2}{4} \qquad k = 0, +1, -1. \qquad R = R(t)$$

$$\mu = \frac{m\sqrt{r}}{2rR}.$$

and consider $e^\alpha = \frac{R^2}{Z^2}(1 + 4\mu + 6\mu^2 + A) \qquad A = A(r,t)$

$$e^\nu = (1 - 4\mu + 8\mu^2 + C) \qquad C = C(r,t)$$

and A and C are of the order μ^2

and we carry our calculations to correct upto μ^2.

(237) We just note that if regard luminosity of the order of μ^3 we would take

$$e^{\alpha} = \frac{R^2}{z^2}(1+4\mu+6\mu^2+4\mu^3+X) \qquad X = X(z,t) \text{ order of the}$$

$$e^{\nu} = 1-4\mu+8\mu^2-12\mu^3+Y \qquad Y = Y(z,t) \text{ order of } \mu^3.$$

However we work correct upto μ^2.

$$e^{\alpha} = \frac{R^2}{z^2}(1+4\mu+6\mu^2+A) \quad \therefore \alpha = 2\log R - 2\log z + \log(1+(4\mu+6\mu^2+A)$$

But $\log(1+x) = x - \frac{x^2}{2}$ $\therefore \alpha = 2\log R - 2\log z + 4\mu+6\mu^2+A - \frac{1}{2}(4\mu)^2$

$$\alpha = 2\log R - 2\log z + 4\mu - 2\mu^2 + A$$

$$\therefore \alpha' = -\frac{2z'}{z} + 4\mu' - 4\mu\mu' + A' \qquad \dot{\alpha} = 2\frac{\dot{R}}{R} + 4\dot{\mu} - 4\mu\dot{\mu} + \dot{A}$$

$$e^{\nu} = 1-4\mu+8\mu^2+B \quad \therefore \nu = \log\left(1+(-4\mu+8\mu^2+B)\right) = -4\mu+8\mu^2+B-\frac{1}{2}(-4\mu)^2$$

$$= -4\mu+8\mu^2+B-8\mu^2 = -4\mu+B$$

$$\therefore \nu' = -4\mu'+B' \qquad \dot{\nu} = -4\dot{\mu}+\dot{B}$$

Now $-G_1' = -e^{-\alpha}\left(\frac{\alpha'^2}{4} + \frac{\alpha'\nu'}{2} + \frac{\alpha'+\nu'}{z}\right) + e^{-\nu}\left(\ddot{\alpha} + \frac{3}{4}\dot{\alpha}^2 - \frac{\dot{\alpha}\dot{\nu}}{2}\right)$

$$\therefore G_1' = e^{-\alpha}\left[\frac{1}{4}\left(-\frac{2z'}{z}+4\mu'-4\mu\mu'+A'\right)^2 + \frac{1}{2}(-4\mu'+B')\left(-\frac{2z'}{z}+4\mu'\right)\right.$$

$$\left. + \frac{1}{2}\left(-\frac{2z'}{z}+4\mu'-4\mu\mu'+A'-4\mu'+B'\right)\right] - e^{-\nu}(X)$$

$$= e^{-\alpha}\left[\frac{z'^2}{z^2} - \frac{2z'}{z^2} - 4\frac{z'}{z}\mu' + 4\mu'^2 - \frac{z'}{z}(-4\mu\mu'+A') + 4\mu'\frac{z'}{z} - 8\mu'^2 - B'\frac{z'}{z}\right.$$

$$\left. + 4\mu\mu' + \frac{A'+B'}{2}\right] - e^{-\nu}(X)$$

$$G_1^1 = e^{-\alpha}\left[\frac{z'^2}{z^2} - \frac{2z'}{zr} - 4\mu'^2 + \left(\frac{z'}{z} - \frac{1}{r}\right)(4\mu\mu' - A' - B')\right] - e^{-\nu}(X)$$

$$(X) = \ddot{\alpha} + \frac{3}{4}\dot{\alpha}^2 - \frac{\dot{\alpha}\dot{\nu}}{2} = \frac{2\ddot{R}}{R} - \frac{2\dot{R}^2}{R^2} + 4\ddot{\mu} - 4\mu\ddot{\mu} - 4\dot{\mu}^2 + \ddot{A}$$
$$+ \frac{3}{4}\left(\frac{2\dot{R}}{R} + 4\dot{\mu} - 4\mu\dot{\mu} + \dot{A}\right)^2 - \frac{1}{2}(-4\dot{\mu} + \dot{B})\left(\frac{2\dot{R}}{R} + 4\dot{\mu}\right)$$
$$= 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}(-2+3) + 4\ddot{\mu} + 8\dot{\mu}\frac{\dot{R}}{R} + 4\dot{\mu}\frac{\dot{R}}{R} - 4\mu\ddot{\mu} + \dot{\mu}^2(-4+12+8)$$
$$+ 3\frac{\dot{R}}{R}(-4\mu\dot{\mu} + \dot{A}) - \frac{\dot{R}}{R}\dot{B} + \ddot{A}$$

$$\therefore \boxed{G_1^1 = e^{-\alpha}\left[\frac{z'^2}{z^2} - \frac{2z'}{zr} - 4\mu'^2 + \left(\frac{z'}{z} - \frac{1}{r}\right)(4\mu\mu' - A' - B')\right] - e^{-\nu}\left[2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + 4\ddot{\mu} + 16\dot{\mu}\frac{\dot{R}}{R} - 4\mu\ddot{\mu} + 16\dot{\mu}^2 + \ddot{A} + \frac{\dot{R}}{R}(-12\mu\dot{\mu} + 3\dot{A} - \dot{B})\right]}$$

$$G_2^2 = e^{-\alpha}\left[\frac{\alpha''}{2} + \frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\alpha' + \nu'}{2r}\right] - e^{-\nu}\left[\ddot{\alpha} + \frac{3}{4}\dot{\alpha}^2 - \frac{\dot{\alpha}\dot{\nu}}{2}\right]$$
$$= e^{-\alpha}\left[\frac{1}{2}\left(-\frac{2z''}{z} + \frac{2z'^2}{z^2} + 4\mu'' - 4\mu\mu'' - 4\mu'^2 + A'' - 4\dot{\mu}' + B''\right) + \frac{1}{4}(-4\mu' + B')^2\right]$$
$$+ \frac{1}{2r}\left(-\frac{2z'}{z} + 4\mu' - 4\mu\mu' + A' - 4\dot{\mu} + B'\right)\Big] - e^{-\nu}(X)$$
$$= e^{-\alpha}\left[-\frac{z''}{z} + \frac{z'^2}{z^2} - \frac{z'}{zr} + 2\mu'' - 2\mu\mu'' - 2\mu'^2 + 4\mu'^2 - 2\frac{\mu\mu'}{r} + \frac{A'' + B''}{2} + \frac{A' + B'}{2r}\right]$$
$$- e^{-\nu}(X)$$

$$\therefore \boxed{G_2^2 = e^{-\alpha}\left(-\frac{z''}{z} + \frac{z'^2}{z^2} - \frac{z'}{zr} - 2\mu\mu'' + 2\mu'^2 - 2\frac{\mu\mu'}{r} + \frac{A'' + B''}{2} + \frac{A' + B'}{2r}\right] - e^{-\nu}\left[2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + 4\ddot{\mu} + 16\dot{\mu}\frac{\dot{R}}{R} - 4\mu\ddot{\mu} + 16\dot{\mu}^2 + \ddot{A} + \frac{\dot{R}}{R}(-12\mu\dot{\mu} + 3\dot{A} - \dot{B})\right]}$$

$$G_4^4 = e^{-\alpha}\left[\alpha'' + \frac{\alpha'^2}{4} + \frac{2\alpha'}{2r}\right] + \frac{3}{4}e^{-\nu}\dot{\alpha}^2 = e^{-\alpha}\left[-\frac{2z''}{z} + \frac{2z'^2}{z^2} + 4\mu'' - 4\mu\mu'' - 4\mu'^2 + A''\right.$$
$$\left. + \frac{1}{4}\left(-\frac{2z'}{z} + 4\mu' - 4\mu\mu' + A'\right)^2 + \frac{2}{2r}\left(-\frac{2z'}{z} + 4\mu' - 4\mu\mu' + A'\right)\right]$$

$$-\frac{3}{4}e^{-\nu}\left(\frac{2\dot{R}}{R}+4\dot{\mu}-4\mu\dot{\mu}+\dot{A}\right)^2$$

$$G_4^4 = e^{-\alpha}\left[-\frac{2Z''}{Z}+\frac{Z'^2}{Z^2}(2+1)-\frac{4Z'}{rZ}+4\mu''-4\mu'\frac{Z'}{Z}+\frac{8\mu'}{r}-4\mu\mu''-4\mu'^2+4\mu'^2-\frac{8\mu\mu'}{r}\right.$$
$$\left.-\frac{Z'}{Z}(-4\mu\mu'+A')+A''+\frac{2}{r}A'\right]-e^{-\nu}\left[+3\frac{\dot{R}^2}{R^2}+12\frac{\dot{R}}{R}\dot{\mu}+12\dot{\mu}^2+\frac{3}{R}\frac{\dot{R}}{R}(-4\mu\dot{\mu}+\dot{A})\right]$$

G_1

$$G_1^4 = e^{-\nu}\left[\alpha'-\dot{\alpha}\nu'\right] = e^{-\nu}\left[4\dot{\mu}'-4\mu'\dot{\mu}-4\mu\dot{\mu}'+\dot{A}'-\frac{1}{2}(-4\mu'+B')\left(\frac{2\dot{R}}{R}+4\dot{\mu}\right)\right]$$

$$G_1^4 = e^{-\nu}\left[4\dot{\mu}'+4\mu'\frac{\dot{R}}{R}-4\dot{\mu}\mu'+8\dot{\mu}\mu'-4\mu\dot{\mu}'+\dot{A}'-\frac{\dot{R}}{R}B'\right]$$

Take the case $k=0$, $R=1$ $\quad G_i^k = -8\pi\sigma\omega_i\omega^k$ Leading to

$$G_1^1+G_4^4=0,\quad G_2^2=0,\quad G_1^1 = e^{(\nu-\alpha)/2}G_1^4$$

We set $\mu=\frac{m}{2r}$ $\quad \mu'=-\frac{m}{2r^2}$, $\mu''=\frac{m}{r^3}$ $\quad \dot{\mu}=0$ $\quad Z=1$, $R=1$

$$\therefore G_1^1+G_4^4=0 \Longrightarrow e^{-\alpha}\left[-4\left(\frac{-m}{2r^2}\right)^2-\frac{1}{2}\left(4\frac{m}{2r}\left(\frac{-m}{2r^2}\right)\right)+\frac{A'+B'}{2}\cdot+4\frac{m}{r^3}+\frac{8}{2}\left(\frac{-m}{2r^2}\right)\right.$$
$$\left.-4\cdot\frac{m}{2r}\left(\frac{m}{r^3}\right)-\frac{8}{2}\frac{m}{2r}\left(\frac{-m}{2r^2}\right)+A''+\frac{2}{r}A'\right]-e^{-\nu}\left[\ddot{A}\right]=0$$

To the order of m^2 $\quad e^{\alpha}=1$, $e^{-\nu}=1$ $\quad \therefore \frac{m^2}{r^4}(-1+1-2+2)+A''+\frac{3A'+B'}{r}-\ddot{A}=0$

$$G_1^1+G_4^4=0 \Longrightarrow A''+\frac{3A'+B'}{r}=\ddot{A} \quad \text{—— I}$$

$$G_2^2=0 \Longrightarrow e^{-\alpha}\left[-2\frac{m}{2r}\left(\frac{m}{r^3}\right)+2\left(\frac{-m}{2r^2}\right)^2-\frac{2}{2}\frac{m}{2r}\left(\frac{-m}{2r^2}\right)+\frac{A''+B''}{2}+\frac{A'+B'}{2r}\right]-e^{-\nu}\left[\ddot{A}\right]=0$$

$$\therefore \frac{m^2}{r^4}\left(-1+\frac{1}{2}+\frac{1}{2}\right)+\frac{A''+B''}{2}+\frac{A'+B'}{2r}-\ddot{A}=0$$

$$G_2^2=0 \Longrightarrow A''+B''+\frac{1}{2}(A'+B')=2\ddot{A} \quad \text{—— II}$$

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$G_1' = e^{(\nu-\alpha)/2} G_1^4 \Rightarrow -4\left(-\frac{m}{2z^2}\right)^2 - \frac{4}{z}\left(\frac{m}{2z}\right)\left(-\frac{m}{2z^2}\right) + \frac{A'+B'}{2} - \ddot{A} = \ddot{A}'$

$\therefore G_1' = e^{(\nu-\alpha)/2} G_1^4 \Rightarrow \frac{A'+B'}{2} = \ddot{A} + A'' \quad$ — III

Take I − III $\quad A'' + \frac{2A'}{z} = -A'' \quad \therefore z^2A'' + 2zA' + z^2\dot{A}' = 0$

$\therefore (z^2A')' + (z^2A')\dot{} = 0 \quad \therefore z^2A' = \lambda(t-z)$ $\quad \lambda$ of the order of μ^2.

$\therefore A' = \frac{\lambda(t-z)}{z^2} \quad A = -\frac{\lambda t}{z} - \lambda \log z + \text{const}\,\lambda \quad A = -\frac{\lambda(t-z)}{z} - \lambda \log z$

$\therefore \dot{A} = -\frac{\lambda}{z} \quad \ddot{A} = 0, \quad \dot{A}' = \frac{\lambda}{z^2} \quad \therefore$ III now $\Rightarrow \frac{\lambda(t-z)}{z^3} + \frac{B'}{z} = \frac{\lambda}{z^2}$

$\therefore B' = -\frac{\lambda(t-z)}{z^2} + \frac{\lambda}{z^2} = -\frac{\lambda t}{z^2} + \frac{\lambda}{z} + \frac{\lambda}{z} \quad \therefore B = \frac{\lambda t}{z} + 2\lambda \log z - 2\lambda$

$\therefore B = \frac{\lambda(t-z)}{z} + 2\lambda \log z$. It can be verified that all the 3 eqn. I, II and III are satisfied with $A = -\frac{\lambda(t-z)}{z} - \lambda \log z$

$B = \frac{\lambda(t-z)}{z} + 2\lambda \log z$

Let us next take our general case $k \neq 0$, $R = R(t)$

$G_1' - G_2^2 = e^{(\nu-\alpha)/2} G_1^4$

$G_1' - G_2^2 = e^{-\alpha}\left[\frac{z'^2}{z^2} - \frac{2z'}{zz} + \frac{z''}{z} - \frac{z'^2}{z^2} + \frac{z'}{zz} + 2\mu\mu'' - 6\mu'^2 + 2\mu\mu'\left(\frac{2z'}{z} - \frac{1}{z}\right) - \frac{(A''+B'')}{2}\right.$

$\left. - \frac{(A'+B')}{2}\left(\frac{2z'}{z} - \frac{1}{z}\right)\right]$

Now $2\mu\mu'' - 6\mu'^2 + 2\mu\mu'\left(\frac{2z'}{z} - \frac{1}{z}\right) = 2\mu^2\left[\frac{\mu''}{\mu} - 3\frac{\mu'^2}{\mu^2} + \frac{\mu'}{\mu}\left(\frac{2z'}{z} - \frac{1}{z}\right)\right]$

$= 2\mu^2\left[\left(\frac{\mu'}{\mu}\right)' - 2\frac{\mu'^2}{\mu^2} + \frac{\mu'}{\mu}\left(\frac{2z'}{z} - \frac{1}{z}\right)\right] = 2\mu^2\left[\left(-\frac{1}{z} + \frac{z'}{zz}\right)' - 2\left(\frac{z'}{zz} - \frac{1}{z}\right)^2 + \left(\frac{z'}{zz} - \frac{1}{z}\right)\left(\frac{2z'}{z} - \frac{1}{z}\right)\right]$

$= 2\mu^2\left[\frac{1}{z^2}(1-2+1) + \frac{z''}{zz} + \frac{z'^2}{z^2}\left(-\frac{1}{2} - \frac{1}{2} + 1\right) + \frac{z'}{zz}\left(2 - \frac{1}{2} - 2\right)\right] = \frac{2\mu^2}{zz}\left[\frac{2k}{4} - \frac{2k}{4}\right] = 0$

$$\therefore G_1^1 - G_2^2 = e^{-\alpha}\left[-\frac{(A''+B'')}{2} - \frac{(A'+B')}{2}\left(\frac{2Z'}{Z} - \frac{1}{\lambda}\right)\right]$$

$$G_1^4 = e^{-\nu}\left[4\dot{\mu}' + 4\mu'\frac{\dot{R}}{R} + 4\dot{\mu}\mu' - 4\mu\dot{\mu}' + \dot{A}' - \frac{\dot{R}}{R}B'\right]$$

But $\frac{\dot{\mu}}{\mu} = -\frac{\dot{R}}{R}$ $\therefore \frac{\dot{\mu}'}{\mu} - \frac{\dot{\mu}}{\mu}\frac{\mu'}{\mu} = 0$ $\therefore \dot{\mu}' + \frac{\dot{R}}{R}\mu' = 0$

also $\mu^2\left(\frac{\dot{\mu}'}{\mu} - \frac{\dot{\mu}\mu'}{\mu^2}\right) = 0$ or $\mu\dot{\mu}' - \dot{\mu}\mu' = 0$

$$\therefore G_1^4 = e^{-\nu}\left[\dot{A}' - \frac{\dot{R}}{R}B'\right] \quad \therefore G_1^1 - G_2^2 = e^{(\nu-\alpha)/2} G_1^4 \Longrightarrow$$

$$-\frac{(A''+B'')}{2} - \frac{(A'+B')}{2}\left(\frac{2Z'}{Z} - \frac{1}{\lambda}\right) = e^{(\nu+\alpha)/2}\, e^{\alpha}\, e^{(\nu-\alpha)/2}\, e^{-\nu}\left[\dot{A}' - \frac{\dot{R}}{R}B'\right]$$

$$\therefore A'' + B'' + (A'+B')\left(\frac{2Z'}{Z} - \frac{1}{\lambda}\right) = -2\frac{R}{Z}\left[\dot{A}' + \frac{\dot{R}}{R}A' - \frac{\dot{R}}{R}(A'+B')\right]$$

$$\therefore A'' + B'' + (A'+B')\left(\frac{2Z'}{Z} - \frac{1}{\lambda} - \frac{2\dot{R}}{Z}\right) = -\frac{2}{Z}(R\dot{A})^{\cdot}$$

Compare this with Eq I on p. 230. We find that $A = a$ there and $B = -c$ there.

So we write down the general solution as

$$A = -\frac{\lambda}{R}\frac{\left(\int\frac{1}{R}dt - \int\frac{1}{Z}dr\right)}{r} - \frac{\lambda}{R}\log\frac{r}{\sqrt{\lambda}} - \frac{\lambda}{R}\frac{k\lambda}{4}\int\frac{1}{R}dt + \frac{\lambda}{R}\log\lambda\log R$$

and B $A + B = \frac{\lambda}{R}\log r$.

This solution is obtained on the assumption $A' + B' = \frac{\lambda}{R}\cdot\frac{1}{r}$

We now work out ϕ, ρ and σ for this solution

We have $G_i^k = -8\pi T_i^k = -8\pi\left[(\phi+p)v_i v^k - p\, g_i^k + \sigma w_i w^k\right]$

$g_{ik} v^i v^k = 1$, $g_{ik} w^i w^k = 0$ $g_{ik} v^i w^k = 1$.

$v^i = (0,0,0,v^4)$ $\therefore e^{\nu}(v^4)^2 = 1$ $\therefore v^4 = e^{-\nu/2}$.

$w^i = (w^1, 0, 0, w^4)$ $\therefore -e^{\alpha}(w^1)^2 + e^{\nu}(w^4)^2 = 0$ and $-e^{\alpha} v^1 w^1 + e^{\nu} v^4 w^4 = 1$

$\therefore e^{\nu} w^4 \cdot e^{-\nu/2} = 1$ $w^4 = e^{-\nu/2}$ $e^{\alpha}(w^1)^2 = e^{\nu - \nu} = 1$ $\therefore w^1 = e^{-\alpha/2}$ (We take out flow of radiation)

$\therefore v^i = (0,0,0,e^{-\nu/2})$, $w^i = (e^{-\alpha/2}, 0, 0, e^{-\nu/2})$.

$G_1^1 = -8\pi\left[-\phi + \sigma(-e^{\alpha/2})e^{-\alpha/2}\right] = 8\pi(\phi+\sigma)$

$G_2^2 = -8\pi[-\phi] = 8\pi\phi$

$G_4^4 = -8\pi\left[(\phi+p)e^{\nu/2}e^{-\nu/2} - \phi + \sigma e^{\nu/2}e^{-\nu/2}\right] = -8\pi p - 8\pi\sigma$

$G_1^4 = -8\pi\left[\sigma(-e^{\alpha/2})e^{-\nu/2}\right] = 8\pi\sigma e^{(\alpha-\nu)/2}$.

We thus have $G_2^2 = 8\pi\phi$ $G_1^1 - G_2^2 = 8\pi\sigma = e^{(\nu-\alpha)/2} G_1^4$

Also $G_4^4 = -8\pi p - (G_1^1 - G_2^2)$ $\therefore 8\pi p = -G_4^4 + G_2^2 - G_1^1$

We have already solved the equation $G_1^1 - G_2^2 = e^{(\nu-\alpha)/2} G_1^4$

We now find σ, ϕ and p

$$8\pi\sigma = G_1^1 - G_2^2 = e^{-\alpha}\left[-\frac{A''+B''}{2} - \frac{A'+B'}{2}\left(\frac{2Z'}{Z} - \frac{1}{2}\right)\right]$$

$$= -\frac{Z^2}{2R^2}\left[A''+B'' + (A'+B')\left(\frac{2Z'}{Z} - \frac{1}{2}\right)\right]$$

But $A'+B' = \frac{\lambda}{R}\frac{1}{2}$

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$$\therefore 8\pi\sigma = -\frac{z^2}{2R^2}\left[-\frac{\lambda}{R}\frac{1}{r^2} + \frac{\lambda}{R}\frac{1}{r}\left(+\frac{2z'}{z} - \frac{1}{r}\right)\right]$$

$$= -\frac{z^2}{2R^2}\left[-\frac{2\lambda}{R}\frac{1}{r^2} + \frac{2\lambda}{R}\frac{z'}{rz}\right]$$

$$= \frac{z^2}{R^2}\left[\frac{\lambda}{R}\frac{1}{r^2} - \frac{\lambda}{R}\frac{2k\frac{r}{4}}{rz}\right] = \frac{z^2}{R^2}\frac{\lambda}{R}\frac{1}{r^2 z}\left(1 + k\frac{r^2}{4} - 2k\frac{r^2}{4}\right)$$

$$\boxed{8\pi\sigma = \frac{z}{R^2}\frac{\lambda}{R}\frac{1}{r^2}\left(1 - k\frac{r^2}{4}\right)}$$

$$8\pi\rho = G_2^2 = e^{-\alpha}\left[-\frac{z''}{z} + \frac{z'^2}{z^2} - \frac{z'}{zr} - 2\mu\mu'' + 2\mu^2 - \frac{2\mu\mu'}{r} + \frac{A'+B'}{2r} + \frac{A''+B''}{2}\right]$$

$$-e^{-\nu}\left[2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + 4\dot{\mu} + 16\mu\frac{\dot{R}}{R} - 4\mu\dot{\mu} + 16\mu^2 - 12\mu\mu\frac{\dot{R}}{R} + \ddot{A} + \frac{\dot{R}}{R}(3\dot{A} - \dot{B})\right]$$

$$= e^{\alpha}\left[T_0 + T_2\right] - e^{\nu}\left[X_0 + X_1 + X_2\right]$$

$$= \frac{z^2}{r^2}$$

$$T_0 = -\frac{z''}{z} + \frac{z'^2}{z^2} - \frac{z'}{zr} = -\frac{2k\frac{r}{4}}{z} + \frac{z'}{z}\left(\frac{z'}{z} - \frac{1}{r}\right)$$

$$= -\frac{2k\frac{r}{4}}{z} + \frac{2k\frac{r}{4}}{z}\left(\frac{-2k\frac{r}{4}}{z} - \frac{1}{r}\right) = \frac{-2k\frac{r}{4}}{z} + \frac{\left(2k\frac{r}{4}\right)^2}{z^2}$$

$$= \frac{4k\frac{r}{4}}{z^2}\left(-z + k\frac{r^2}{4}\right) = -\frac{k}{z^2}$$

$$\boxed{T_0 = -\frac{k}{z^2}}$$

$$T_2 = -2\mu\mu'' + 2\mu'^2 - \frac{2\mu\mu'}{r} + \frac{A''+B''}{2} + \frac{A'+B'}{2r}$$

But $A'+B' = \frac{\lambda}{R}\frac{1}{r^2}$

$$\therefore (A''+B'') + \frac{A'+B'}{r} = 0$$

Now $\mu = \frac{m}{2rR}\sqrt{z}$ $\therefore \frac{\mu'}{\mu} = \frac{z'}{2z} - \frac{1}{r} = \frac{k\frac{r}{4}}{2} - \frac{1}{r} = \frac{k\frac{r^2}{4} - 2}{rz} = -\frac{1}{rz}$

8-4-96 $\therefore T_2 = -2\mu^2\left[\frac{\mu''}{\mu} - \frac{\mu'^2}{\mu^2} + \frac{1}{r}\frac{\mu'}{\mu}\right] = -2\mu^2\left[\left(-\frac{1}{r^2}\right)' + \frac{1}{r}\left(-\frac{1}{r^2}\right)\right]$

$= -2\mu^2\left[\frac{1}{r^3 z} + \frac{1}{r z^2} - \frac{1}{r^3 z}\right] = -2\mu^2\left[\frac{\frac{2R}{4}\,r}{r z^2}\right] = -\frac{\mu^2 k}{z^2}$

$T_2 = -\frac{\mu^2 k}{z^2}$

$X_0 = \frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}$

$X_1 = 4\ddot{\mu} + 16\dot{\mu}\frac{\dot{R}}{R} = 4\mu\left[\frac{\ddot{\mu}}{\mu} + 4\frac{\dot{\mu}}{\mu}\frac{\dot{R}}{R}\right]$ $\left(\frac{\dot{\mu}}{\mu} = -\frac{\dot{R}}{R}\right)$

$= 4\mu\left[\left(\frac{\ddot{R}}{R}\right)\frac{\ddot{\mu}}{\mu} - \frac{\dot{\mu}^2}{\mu^2} + \frac{\dot{\mu}^2}{\mu^2} + 4\frac{\dot{\mu}}{\mu}\frac{\dot{R}}{R}\right]$

$= 4\mu\left[-\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{\dot{R}^2}{R^2} + 4\left(-\frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right] = 4\mu\left[-\frac{\ddot{R}}{R} - \frac{2\dot{R}^2}{R^2}\right]$

$X_1 = -4\mu\left(\frac{\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2}\right)$

$X_2 = -4\dot{\mu}\ddot{\mu} + 16\dot{\mu}^2 - 12\mu\dot{\mu}\frac{\dot{R}}{R} + (A, B)$.

~~$X_2 = -4\dot{\mu}\ddot{\mu} - 16\dot{\mu}\mu\frac{\dot{R}}{R} + 16\dot{\mu}^2 + 4\mu\dot{\mu}\frac{\dot{R}}{R} + (A, B)$~~ ~~But~~

~~$= \mu(X_1)$~~ $X_2 = -4\mu^2\left[\frac{\ddot{\mu}}{\mu} - 4\frac{\dot{\mu}^2}{\mu^2} + 3\frac{\dot{\mu}}{\mu}\frac{\dot{R}}{R}\right] + (A, B)$

$= -4\mu^2\left[\left(\frac{\dot{\mu}}{\mu}\right)' - 3\frac{\dot{\mu}^2}{\mu^2} + 3\frac{\dot{\mu}}{\mu}\frac{\dot{R}}{R}\right] + (A, B)$

$= -4\mu^2\left[-\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - 3\frac{\dot{R}^2}{R^2} + 3\left(-\frac{\dot{R}}{R}\right)\left(\frac{\dot{R}}{R}\right)\right] + (A, B)$

$= -4\mu^2\left(-\frac{\ddot{R}}{R} - 5\frac{\dot{R}^2}{R^2}\right) + (A, B) = 4\mu^2\left(\frac{\ddot{R}}{R} + 5\frac{\dot{R}^2}{R^2}\right) + (A, B)$

Now $(A, B) = \ddot{A} + \frac{\dot{R}}{R}(3\dot{A} - \dot{B}) = \ddot{A} + \frac{\dot{R}}{R}4\dot{A} - \frac{\dot{R}}{R}(\dot{A} + \dot{B})$

But $A + B = \frac{\lambda}{R}\log r \quad \therefore \dot{A} + \dot{B} = -\frac{\lambda}{R}\log r\,\frac{\dot{R}}{R}$

$\therefore X_2 = 4\mu^2\left(\frac{\ddot{R}}{R} + 5\frac{\dot{R}^2}{R^2}\right) + \frac{\lambda}{R}\frac{\dot{R}^2}{R^2}\log r + \ddot{A} + \frac{\dot{R}}{R}4\dot{A}$

(2.72) Now $RA = -\frac{\lambda}{2}\left(\int\frac{1}{R}dt - \int\frac{1}{2}dr\right) - \lambda\log\frac{r}{\sqrt{2}} - \lambda\frac{kr}{4}\int\frac{1}{R}dt + \lambda\log r\log R$

$$\therefore \dot{A}R + A\dot{R} = -\frac{\lambda}{2}\frac{1}{R} - \lambda\frac{kr}{4}\frac{1}{R} + \lambda\log r\,\frac{\dot{R}}{R} = -\frac{\lambda}{R}\frac{1+k^2r^2/4}{r} + \lambda\log r\frac{\dot{R}}{R}$$

$$\therefore \cancel{\dot{A}R = -\frac{\lambda}{R}\frac{r}{2} + \lambda\log r\frac{\dot{R}}{R} - A\dot{R}} \quad \therefore \dot{A}R + A\dot{R} = -\frac{\lambda}{R}\frac{r}{2} + \lambda\log r\frac{\dot{R}}{R}$$

$$\therefore \ddot{A}R + 2\dot{A}\dot{R} + A\ddot{R} = \frac{\lambda}{R}\frac{r}{2}\frac{\dot{R}}{R} + \lambda\log r\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2}\right)$$

$$\therefore \ddot{A}R + 4\dot{A}\dot{R} - 2\dot{A}\dot{R} + A\ddot{R} = \frac{\lambda}{R}\frac{r}{2}\frac{\dot{R}}{R} + \lambda\log r\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2}\right)$$

$$\therefore \ddot{A}R + 4\dot{A}\dot{R} - 2\frac{\dot{R}}{R}(\dot{A}R + A\dot{R}) + A\ddot{R} + 2A\frac{\dot{R}^2}{R} = \frac{\lambda}{R}\frac{r}{2}\frac{\dot{R}}{R} + \lambda\log r\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2}\right)$$

$$\therefore \ddot{A}R + 4\dot{A}\dot{R} = 2\frac{\dot{R}}{R}\left[-\frac{\lambda}{R}\frac{r}{2} + \lambda\log r\frac{\dot{R}}{R}\right] - A\left(\ddot{R} + 2\frac{\dot{R}^2}{R}\right) + \frac{\lambda}{R}\frac{r}{2}\frac{\dot{R}}{R} + \lambda\log r\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2}\right)$$

$$= -\frac{\lambda}{R}\frac{r}{2}\frac{\dot{R}}{R} + \lambda\log r\left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) - A\left(\ddot{R} + 2\frac{\dot{R}^2}{R}\right)$$

$$\therefore \cancel{\therefore} X_2 = 4\mu^2\left(\frac{\ddot{R}}{R} + 5\frac{\dot{R}^2}{R^2}\right) + \frac{\lambda}{R}\frac{\dot{R}^2}{R^2}\log r + -\frac{\lambda}{R^2}\frac{r}{2}\frac{\dot{R}}{R} + \frac{\lambda}{R}\log r\left(\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) - A\left(\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2}\right)$$

$$\boxed{X_2 = 4\mu^2\left(\frac{\ddot{R}}{R} + 5\frac{\dot{R}^2}{R^2}\right) + \frac{\lambda}{R}\log r\left(\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2}\right) - \frac{\lambda}{R^2}\frac{r}{2}\frac{\dot{R}}{R} - A\left(\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2}\right)}$$

$$8\pi\rho = e^{-a}(T_0 + T_2) - e^{-\nu}(X_0 + X_1 + X_2)$$

$$e^{a} = \frac{R^2}{r^2}(1 + 4\mu + 6\mu^2 + A) \qquad e^{-a} = \frac{r^2}{R^2}\left[1 + (4\mu + 6\mu^2 + A)\right]^{-1}$$

$$e^{-a} = \frac{r^2}{R^2}\left[1 - (4\mu + 6\mu^2 + A) + (4\mu + 6\mu^2 + A)^2 - \cdots\right]$$

$$= \frac{r^2}{R^2}\left[1 - 4\mu - 6\mu^2 - A + 16\mu^2\right] = \frac{r^2}{R^2}\left[1 - 4\mu + 10\mu^2 - A\right]$$

$$e^{\nu} = 1-4\mu+8\mu^2+B \quad \therefore e^{-\nu} = [1+(-4\mu+8\mu^2+B)]^{-1}$$

$$\therefore e^{-\nu} = 1-(-4\mu+8\mu^2+B)+(-4\mu+8\mu^2+B)^2-\dots]$$

$$= 1+4\mu-8\mu^2-B+16\mu^2 = 1+4\mu+8\mu^2-B.$$

$$\therefore 8\pi\rho = e^{\frac{z^2}{R^2}}(1-4\mu+10\mu^2-A)(T_0+T_2) - (1+4\mu+8\mu^2-B)(X_0+X_1+X_2)$$

$$= \frac{z^2}{R^2}\left[T_0 - 4\mu T_0 + (10\mu^2-A)T_0 + T_2\right] - \left[X_0 + 4\mu X_0 + (8\mu^2-B)X_0 + X_1 + 4\mu X_1 + X_2\right]$$

$$= \frac{z^2}{R^2}T_0 - X_0 - 4\mu\left(\frac{z^2}{R^2}T_0 + X_0\right) + X_1 + \frac{z^2}{R^2}T_0(10\mu^2-A) - (8\mu^2-B)X_0 - 4\mu X_1 + \frac{z^2}{R^2}T_2 - X_2$$

$$= 8\pi\rho_0 + 8\pi\rho_1 + 8\pi\rho_2$$

$$8\pi\rho_0 = \frac{z^2}{R^2}T_0 - X_0 = \frac{z^2}{R^2}\left(-\frac{k}{z^2}\right) - \left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) = -\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2}\right).$$

$$8\pi\rho_1 = -4\mu\left(\frac{z^2}{R^2}T_0 + X_0\right) - X_1 = -4\mu\left(\frac{z^2}{R^2}\left(-\frac{k}{z^2}\right) + 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) - (-4\mu)\left(\frac{\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2}\right)$$

$$= -4\mu\left[-\frac{k}{R^2} + 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \frac{\ddot{R}}{R} - \frac{2\dot{R}^2}{R^2}\right) = -4\mu\left(\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2}\right)$$

$$8\pi\rho_2 = \frac{z^2}{R^2}T_0(10\mu^2-A) - (8\mu^2-B)X_0 - 4\mu X_1 + \frac{z^2}{R^2}T_2 - X_2$$

$$= \mu^2\left(10\frac{z^2}{R^2}T_0 - 8X_0\right) - 4\mu X_1 + \frac{z^2}{R^2}T_2 - \frac{z^2}{R^2}T_0 A + BX_0 - X_2$$

$$= \mu^2\left[10\frac{z^2}{R^2}\left(-\frac{k}{z^2}\right) - 8\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right)\right] - 4\mu(-4\mu)\left(\frac{\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2}\right) + \frac{z^2}{R^2}\left(-\frac{\mu^2 k}{z^2}\right)$$

$$- \frac{z^2}{R^2}\left(-\frac{k}{z^2}\right)A + B\left(2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right) - 4\mu^2\left(\frac{\ddot{R}}{R} + \frac{5\dot{R}^2}{R^2}\right) - \frac{\Lambda}{R}\log r\left(\frac{\ddot{R}}{R} + \frac{2\dot{R}^2}{R^2}\right)$$

$$+\frac{\lambda}{R^2}\frac{z}{2}\frac{\dot R}{R}+A\left(\frac{\ddot R}{R}+\frac{2\dot R^2}{R^2}\right)$$

$$8\pi\rho_2=\mu^2\left[\frac{k}{R^2}(-10-1)+\frac{\ddot R}{R}(-16+16-4)+\frac{\dot R^2}{R^2}(-8+32-20)\right]+\frac{\lambda}{R^2}\frac{z}{2}\frac{\dot R}{R}$$

$$+A\left(\frac{k}{R^2}+\frac{\ddot R}{R}+\frac{2\dot R^2}{R^2}\right)+B\left(\frac{2\ddot R}{R}+\frac{\dot R^2}{R^2}\right)-\frac{\lambda}{R}\log z\left(\frac{\ddot R}{R}+\frac{2\dot R^2}{R^2}\right)$$

But $A+B=\frac{\lambda}{R}\log z \qquad B=\frac{\lambda}{R}\log z-A$

$$\therefore 8\pi\rho_2=\mu^2\left[-\frac{11k}{R^2}-4\frac{\ddot R}{R}+4\frac{\dot R^2}{R^2}\right]+\frac{\lambda}{R^2}\frac{z}{2}\frac{\dot R}{R}+A\left(\frac{k}{R^2}+\frac{\ddot R}{R}+\frac{2\dot R^2}{R^2}-\frac{2\ddot R}{R}-\frac{\dot R^2}{R^2}\right]$$

$$+\frac{\lambda}{R}\log z\left(\frac{2\ddot R}{R}+\frac{\dot R^2}{R^2}-\frac{\ddot R}{R}-\frac{2\dot R^2}{R^2}\right)$$

$$=-\mu^2\left[4\frac{\ddot R}{R}-4\frac{\dot R^2}{R^2}+\frac{11k}{R^2}\right]+\frac{\lambda}{R^2}\frac{z}{2}\frac{\dot R}{R}-A\left(\frac{\ddot R}{R}-\frac{\dot R^2}{R^2}-\frac{k}{z^2}\right)$$

$$+\frac{\lambda}{R}\log z\left(\frac{\ddot R}{R}-\frac{\dot R^2}{R^2}\right)$$

$$8\pi\rho=8\pi\rho_0+8\pi\rho_1+8\pi\rho_2$$

We shall fo now find $\&\ 8\pi\rho$

$$8\pi\rho=-G_4^4+G_2^2-G_1^1=-e^{\alpha}\left[-\frac{2z''}{z}+\frac{3z'^2}{z^2}-\frac{4z'}{rz}+4\mu''-4\mu'\frac{z'}{z}+\frac{8\mu'}{r}\right.$$

$$\left.-4\mu\mu''-\frac{8\mu\mu'}{r}+4\mu\mu'\frac{z'}{z}+A''+\frac{2}{r}A'-A'\frac{z'}{z}\right]+e^{-\nu}\left[3\frac{\dot R^2}{R^2}+12\frac{\dot R}{R}\dot\mu+12\dot\mu^2\right.$$

$$\left.+12\mu\dot\mu\frac{\dot R}{R}+3\frac{\dot R}{R}\dot A\right]+e^{\alpha}\left[\frac{A''+B''}{2}\right]+-8\pi\sigma.$$

$$\therefore 8\pi\rho+8\pi\sigma=-G_4^4=-e^{\alpha}\left[T_0+T_1+T_2\right]+e^{\nu}\left(X_0+X_1+X_2\right)$$

$$T_0=-\frac{2z''}{z}+\frac{3z'^2}{z^2}-\frac{4z'}{rz}=-\frac{4\left(\frac{k}{4}\right)}{z}+\frac{3\left(2\frac{k}{4}r\right)^2}{z^2}-\frac{4\left(2\frac{k}{4}r\right)}{rz}$$

$$=\frac{-6\left(\frac{k}{4}\right)}{z}+\frac{12\left(\frac{k}{4}r\right)^2}{z^2}=\frac{6\left(\frac{k}{4}\right)}{z^2}\left(-1-\frac{k}{4}r^2+\right.$$

$$= -\frac{3k}{2} + \frac{3}{4}\frac{k^2r^2}{2^2} = \frac{3k}{2^2}\left(-1 - \frac{kr^2}{4} - \frac{k}{4}r^2\right) = -\frac{3k}{2^2}$$

$$T_0 = -\frac{3k}{2^2}$$

$$T_1 = 4\mu'' - 4\mu'\left(\frac{2'}{2} - \frac{2}{r}\right) = 4\mu\left[\frac{\mu''}{\mu} - 2\frac{\mu'}{\mu}\left(\frac{2'}{22} - \frac{1}{r}\right)\right] = 4\mu\left[\frac{\mu''}{\mu} - \frac{\mu'^2}{\mu^2} + \frac{\mu'^2}{\mu^2} - 2\frac{\mu'}{\mu}\left(\frac{2'}{22} - \frac{1}{r}\right)\right]$$

But we know that $\frac{\mu'}{\mu} = \frac{2'}{22} - \frac{1}{r} = -\frac{1}{r2}$

$$\therefore T_1 = 4\mu\left[\frac{1}{r^2 2} + \frac{1}{r2^2}2' + \frac{1}{r^2 2^2} - 2\left(-\frac{1}{r2}\right)\left(-\frac{1}{r2}\right)\right] = 4\mu\left[\frac{1}{r^2 2} + \frac{2\left(\frac{k}{4}\right)}{2^2} - \frac{1}{r^2 2^2}\right]$$

$$= \frac{4\mu}{r^2 2^2}\left[1 + \frac{kr^2}{4} + 2\frac{k}{4}r^2 - 1\right] = \frac{4\mu \cdot \frac{3kr^2}{4}}{r^2 2^2} = \frac{3\mu k}{2^2}$$

$$T_1 = \frac{3\mu k}{2^2}$$

$$T_2 = -4\mu\mu'' - 8\mu\mu'\frac{1}{r} + 4\mu\mu'\frac{2'}{2} + (A\text{ terms}) = \cancel{-4\mu^2\left[\frac{\mu''}{\mu}\right.}$$

$$= -\mu\left[4\mu'' - \mu'\left(\frac{2'}{2} - \frac{2}{r}\right)\right] + (A\text{ terms}) = \cancel{-4\mu T_1 + A\text{ terms}} - \mu T_1 + (A\text{ terms})$$

$$= -\frac{3\mu^2 k}{2^2} + A'' + \frac{2}{r}A' - A'\frac{2'}{2} = -\frac{3\mu^2 k}{2^2} + A'' + \frac{2A'}{r2}$$

Referring to p. (231) (a is the same as A)

$$A' = \frac{\frac{\lambda}{R}\int\frac{1}{R}dt}{r^2} - \frac{\lambda}{R}\frac{\int\frac{1}{2}dr}{r^2} - \frac{\lambda}{R}\frac{k}{4}\int\frac{1}{R}dt + \frac{\lambda}{R}\frac{\log R}{r}$$

$$\therefore A'' = \frac{-2\frac{\lambda}{R}\int\frac{1}{R}dt}{r^3} + \frac{2\frac{\lambda}{R}\int\frac{1}{2}dr}{r^3} - \frac{\lambda}{R}\frac{1}{2r^2} - \frac{\lambda}{R}\frac{\log R}{r^2}$$

$$\frac{2A'}{r2} = \frac{2\frac{\lambda}{R}\int\frac{1}{R}dt}{r^3 2} - \frac{2\frac{\lambda}{R}\int\frac{1}{2}dr}{r^3 2} - \frac{2\frac{\lambda}{R}\frac{k}{4}\int\frac{1}{R}dt}{r2} + \frac{2\frac{\lambda}{R}\log R}{r^2 2}$$

$$\therefore A'' + \frac{2A'}{r2} = \frac{-2\frac{\lambda}{R}\int\frac{1}{R}dt}{r^3 2}\left(2 - 1 + \frac{k}{4}r^2\right) + \frac{2\frac{\lambda}{R}\int\frac{1}{2}dr}{r^3 2}(2 - 1) - \frac{\lambda}{R}\frac{1}{r^2 2}$$

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$$+\frac{\lambda}{R}\frac{\log R}{\lambda^2 z}(2-z).$$

$$= -\frac{2\lambda}{R}\frac{R\frac{z^2}{4}}{\lambda^3 z}\left(\int\frac{1}{R}dt - \int\frac{1}{2}dr\right) - \frac{\frac{2\lambda}{R}\frac{R\lambda^2}{4}\int\frac{1}{R}dt}{\lambda^3 z} - \frac{\lambda}{R}\frac{1}{\lambda^2 z} + \frac{\lambda}{R}\frac{\log R}{\frac{\lambda^2}{4}z}\left(1-R\frac{z^2}{4}\right)$$

$$= \left(\frac{-\lambda}{R}\frac{R}{2\lambda z}\left(\int\frac{1}{R}dt - \int\frac{1}{2}dr\right) - \frac{\lambda}{R}\frac{R}{2\lambda z}\int\frac{1}{R}dt - \frac{\lambda}{R}\frac{1}{\lambda^2 z} + \frac{\lambda}{R}\frac{\log R}{\lambda^2 z}\left(1-R\frac{z^2}{4}\right)\right)$$

$$\therefore T_2 = -\frac{3\mu^2 k}{z^2} +$$

$$X_0 = 3\frac{\dot{R}^2}{R^2} \qquad X_1 = 12\frac{\dot{R}}{R}\dot{\mu} = 12\mu\left(\frac{\dot{R}}{R}\right)\frac{\dot{\mu}}{\mu} = -12\mu\frac{\dot{R}^2}{R^2} = X_1$$

$$X_2 = 12\dot{\mu}^2 + 12\mu\dot{\mu}\frac{\dot{R}}{R} + 3\frac{\dot{R}}{R}\dot{A} = 3\mu^2\left[4\frac{\dot{\mu}^2}{\mu^2} + 4\frac{\dot{\mu}}{\mu}\left(\frac{\dot{R}}{R}\right)\right] + 3\frac{\dot{R}}{R}\dot{A}$$

$$= 3\mu^2\left[4\left(\frac{\dot{R}^2}{R^2}\right) + 4\left(-\frac{\dot{R}}{R}\right)\left(\frac{\dot{R}}{R}\right)\right] + 3\frac{\dot{R}}{R^2}(R\dot{A}) = 3\mu^2\frac{\dot{R}}{R^2}\left(-\frac{\lambda}{R}\frac{z}{2} + \lambda\log\frac{z}{2} - A\right)$$

$$X_2 = 3\mu^2\frac{\dot{R}}{R^2}\left[-\frac{\lambda}{R}\frac{z}{2} + \lambda\frac{\dot{R}}{R}\log\frac{z}{2} - A\dot{R}\right]$$

$$\therefore 8\pi\rho + 8\pi\sigma = -\frac{z^2}{R^2}\left[1-4\mu+10\mu^2-A\right](T_0+T_1+T_2)$$

$$+ (1+4\mu+8\mu^2-B)(X_0+X_1+X_2)$$

$$= -\frac{z^2}{R^2}\left(T_0 - 4\mu T_0 + T_1 \quad \dot{T}_0 - 4\mu\dot{T}_0 + (10\mu^2-A)\dot{T}_0 + \dot{T}_1 - 4\mu\dot{T}_1 + \dot{T}_2\right)$$

$$+ \dot{X}_0 + 4\mu\dot{X}_0 + (8\mu^2-B)X_0 + \dot{X}_1 + 4\mu X_1 + X_2$$

$$\therefore 8\pi\rho = -\frac{z^2}{R^2}T_0 + X_0 + \frac{z^2}{R^2}(4\mu T_0 + T_1) + 4\mu X_0 + X_1 - \frac{z^2}{R^2}\left\{(10\mu^2-A)T_0 - 4\mu T_1 + T_2\right\}$$

$$+ (8\mu^2-B)X_0 + 4\mu X_1 + X_2 - 8\pi\sigma.$$

$\therefore 8\pi\rho = 8\pi\rho_0 + 8\pi\rho_1 + 8\pi\rho_2 .$

$$8\pi\rho_0 = -\frac{Z^2}{R^2}T_0 + X_0 = -\frac{Z^2}{R^2}\left(-\frac{3k}{Z^2}\right) + 3\frac{\dot{R}^2}{R^2} = \frac{3k}{R^2} + 3\frac{\dot{R}^2}{R^2}$$

$$8\pi\rho_1 = \frac{Z^2}{R^2}(4\mu T_0 - T_1) + 4\mu X_0 + X_1 = \mu\cdot 4\frac{Z^2}{R^2}\left(-\frac{3k}{Z^2}\right) - \frac{Z^2}{R^2}\frac{3\mu k}{Z^2} + 4\mu\frac{3\dot{R}^2}{R^2} - 12\mu\frac{\dot{R}^2}{R^2}$$

$$= \mu\left[-\frac{12k}{R^2} - \frac{3k}{R^2}\right] = -15\mu\frac{k}{R^2}$$

$$8\pi\rho_2 = -\frac{Z^2}{R^2}(10\mu^2 - A)T_0 + \frac{Z^2}{R^2}4\mu T_1 - \frac{Z^2}{R^2}T_2 + (8\mu - B)X_0 + 4\mu X_1 + X_2 - 8\pi\sigma$$

$$= \mu^2\left[-10\frac{Z^2}{R^2}\left(-\frac{3k}{Z^2}\right) + 4\frac{Z^2}{R^2}\cdot\frac{3k}{Z^2} - \frac{Z^2}{R^2}\left(-\frac{3k}{Z^2}\right) + 8\cdot 3\frac{\dot{R}^2}{R} + 4(-12)\frac{\dot{R}^2}{R^2}\right]$$

$$+ \frac{Z^2}{R^2}\left(-\frac{3k}{Z^2}\right)A - \frac{Z^2}{R^2}\left(A'' + \frac{2A'}{rZ}\right) - B\cdot 3\frac{\dot{R}^2}{R^2} + 3\frac{\dot{R}}{R}\dot{A} - 8\pi\sigma$$

$$= \mu^2\left[45\frac{k}{R^2} - 24\frac{\dot{R}^2}{R^2}\right]$$

$\left\{ {}^{1}_{11} \right\} = e^{\lambda} \frac{\dot{\lambda}}{2} = \frac{\dot{z}}{2}$

$\left\{ {}^{2}_{11} \right\} = 0$

$\left\{ {}^{3}_{11} \right\} = 0$

$\left\{ {}^{4}_{11} \right\} = \frac{1}{2} e^{\lambda} (-\lambda' + e^{\lambda} \dot{\lambda})$

$= \frac{1}{2} (-z' + z\dot{z})$

$\left\{ {}^{1}_{12} \right\} = 0$

$\left\{ {}^{2}_{12} \right\} = \frac{M'}{M}$

$\left\{ {}^{3}_{12} \right\} = 0$

$\left\{ {}^{4}_{12} \right\} = 0$

$\left\{ {}^{1}_{13} \right\} = 0$

$\left\{ {}^{2}_{13} \right\} = 0$

$\left\{ {}^{3}_{13} \right\} = \frac{M'}{M}$

$\left\{ {}^{4}_{13} \right\} = 0$

$\left\{ {}^{1}_{14} \right\} = 0$

$\left\{ {}^{2}_{14} \right\} = 0$

$\left\{ {}^{3}_{14} \right\} = 0$

$\left\{ {}^{4}_{14} \right\} = -e^{\lambda} \frac{\dot{\lambda}}{2} = -\frac{\dot{z}}{2}$

$\left\{ {}^{1}_{21} \right\}$

$\left\{ {}^{2}_{21} \right\}$

$\left\{ {}^{3}_{21} \right\} = 0$

$\left\{ {}^{4}_{21} \right\} = 0$

$\left\{ {}^{1}_{22} \right\} = M\dot{M}$

$\left\{ {}^{2}_{22} \right\} = 0$

$\left\{ {}^{3}_{22} \right\} = 0$

$\left\{ {}^{4}_{22} \right\} = M M' + e^{\lambda} M \dot{M}$

$\left\{ {}^{1}_{23} \right\} = 0$

$\left\{ {}^{2}_{23} \right\} = 0$

$\left\{ {}^{3}_{23} \right\} = \cot\theta$

$\left\{ {}^{4}_{23} \right\} = 0$

$\left\{ {}^{1}_{24} \right\} = 0$

$\left\{ {}^{2}_{24} \right\} = \dot{M} + \frac{\dot{M}}{M}$

$\left\{ {}^{3}_{24} \right\} = 0$

$\left\{ {}^{4}_{24} \right\} = 0$

$\left\{ {}^{1}_{31} \right\} = 0$

$\left\{ {}^{2}_{31} \right\} = 0$

$\left\{ {}^{3}_{31} \right\} = \frac{M'}{M}$

$\left\{ {}^{4}_{31} \right\} = 0$

$\left\{ {}^{1}_{32} \right\} = 0$

$\left\{ {}^{2}_{32} \right\} = 0$

$\left\{ {}^{3}_{32} \right\} = \cot\theta$

$\left\{ {}^{4}_{32} \right\} = 0$

$\left\{ {}^{1}_{33} \right\} = M\dot{M} \sin^2\theta$

$\left\{ {}^{2}_{33} \right\} = -\sin\theta\cos\theta$

$\left\{ {}^{3}_{33} \right\} = 0$

$\left\{ {}^{4}_{33} \right\} = (M M' + e^{\lambda} M \dot{M}) \sin^2\theta$

$\left\{ {}^{1}_{34} \right\} = 0$

$\left\{ {}^{2}_{34} \right\} = 0$

$\left\{ {}^{3}_{34} \right\} = \frac{\dot{M}}{M}$

$\left\{ {}^{4}_{34} \right\} = 0$

$\left\{ {}^{1}_{41} \right\} = 0$

$\left\{ {}^{2}_{41} \right\} = 0$

$\left\{ {}^{3}_{41} \right\} = 0$

$\left\{ {}^{4}_{41} \right\} = -e^{\lambda} \frac{\dot{\lambda}}{2} = -\frac{\dot{z}}{2}$

$\left\{ {}^{1}_{42} \right\} = 0$

$\left\{ {}^{2}_{42} \right\} = \frac{M'}{M}$

$\left\{ {}^{3}_{42} \right\} = 0$

$\left\{ {}^{4}_{42} \right\} = 0$

$\left\{ {}^{1}_{43} \right\} = 0$

$\left\{ {}^{2}_{43} \right\} = 0$

$\left\{ {}^{3}_{43} \right\} = \frac{\dot{M}}{M}$

$\left\{ {}^{4}_{43} \right\} = 0$

$\left\{ {}^{1}_{44} \right\} = 0$

$\left\{ {}^{2}_{44} \right\} = 1$

$\left\{ {}^{3}_{44} \right\} = 0$

$\left\{ {}^{4}_{44} \right\} = 0$

$z' = \frac{\partial z}{\partial u} \qquad \dot{z} = \frac{\partial z}{\partial t}$

$$a^2R_{|\ell\ell} - R = a^2R_{|00} + a^2R_{|\alpha}(\Gamma^{\alpha}_{\ell\ell} - \Gamma^{\alpha}_{00}) + a^2h^{*\alpha\beta}R_{|\alpha\beta} - a^2h^{*\alpha\beta}R_{|\sigma}\Gamma^{\sigma}_{\alpha\beta} \qquad (4.15)$$

where

$$\begin{aligned}
\Phi_{00} &= -\gamma_{00|ss} + \gamma_{\ell s|\ell s} \\
\Phi_{0n} &= -\gamma_{0n|ss} + \gamma_{0s|sn} \\
\Phi_{mn} &= -\gamma_{mn|ss} + \gamma_{ms|ns} + \gamma_{ns|ms} - \delta_{mn}\gamma_{\ell s|\ell s}
\end{aligned} \qquad (4.16)$$

$$\begin{aligned}
2\Lambda_{00} &= (L'_{00} + L'_{ss}) \\
2\Lambda_{0n} &= \gamma_{ns|s0} - \gamma_{00|n0} + (2L'_{0n}) \\
2\Lambda_{mn} &= -\gamma_{0m|0n} - \gamma_{0n|0m} + 2\delta_{mn}\gamma_{0s|0s} + \gamma_{mn|00} - \\
&\quad - \delta_{mn}\gamma_{00|00} + (2L'_{mn} - \delta_{mn}L'_{ss} + \delta_{mn}L'_{00})
\end{aligned} \qquad (4.17)$$

$$\begin{aligned}
2\,\textcircled{H}_{00} &= \tfrac{1}{3}a^2R_{|\ell\ell} + \tfrac{1}{3}R + \tfrac{1}{3}a^2R_{|00} - \tfrac{1}{3}a^2(\Gamma^{\alpha}_{00} + \Gamma^{\alpha}_{\ell\ell})R_{|\alpha} + \frac{1}{6}a^2R^2 + \tfrac{1}{3}a^2(\Phi_{00} + 2\Lambda \\
&\quad - \tfrac{1}{3}\gamma_{00}R - \frac{1}{6}a^2\gamma_{00}R^2 \\
2\,\textcircled{H}_{0n} &= \tfrac{2}{3}a^2R_{|0n} - \tfrac{1}{3}\gamma_{0n}R - \tfrac{2}{3}a^2R_{|\alpha}\Gamma^{\alpha}_{0n} - \frac{1}{6}a^2\gamma_{0n}R^2 + \tfrac{1}{3}a^2(\Phi_{0n} + 2\Lambda_{0n})R \\
2\,\textcircled{H}_{mn} &= -\tfrac{1}{3}\delta_{mn}R - \tfrac{1}{3}a^2\delta_{mn}R_{|\ell\ell} + \tfrac{2}{3}a^2R_{|mn} + \tfrac{1}{3}a^2\delta_{mn}R_{|00} - \tfrac{1}{3}\gamma_{mn}R - \\
&\quad - \tfrac{2}{3}a^2(\Gamma^{\alpha}_{mn} - \tfrac{1}{2}\delta_{mn}\Gamma^{\alpha}_{\ell\ell} + \tfrac{1}{2}\delta_{mn}\Gamma^{\alpha}_{00})R_{|\alpha} - \frac{1}{6}a^2\delta_{mn}R^2 + \tfrac{1}{3}a^2(\Phi_{mn} + 2\Lambda_{mn})R \\
&\quad - \frac{1}{6}a^2\gamma_{mn}R^2
\end{aligned} \qquad (4.18)$$

where $L'_{\mu\nu}$ represents the non-linear part of $R_{\mu\nu}$ and the scalar curvature R is explicitly given by

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