21-9-93

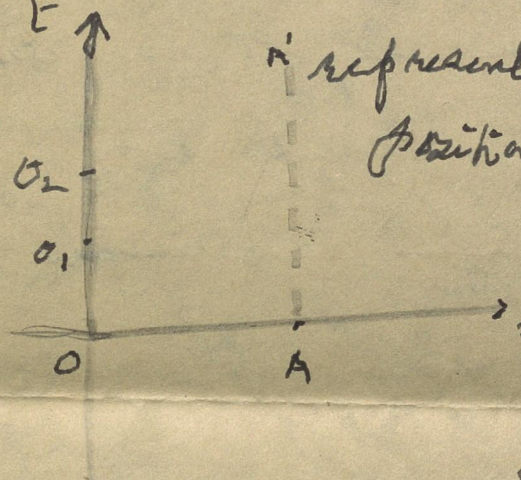
Lecture III

Geometry of Space-time.

We saw that Einstein's ~~prescription of~~ for breaking the logical circle in measuring time-interval between events at different places requires the existence of an agent whose ~~veloc~~ motion is predetermined. ~~And~~ This motion is independent of the ~~subseq~~ kinematics which will be developed subsequently. This forgoes the Newtonian axiom of a true even flowing time, the same for all observers. Einsteinian time will be different for different observers. Different observers O and O′ used different ~~coor~~ space coordinates (x, y, z), (x', y', z') and now they will use different time-reckonings t and t'. So just as we had transformations ~~betwee~~ between (x, y, z) and (x', y', z') in Newtonian scheme we will now have transformations between (x, y, z, t) and (x', y', z', t') in Einsteinian scheme. The problem is how to visualise these transformations geometrically. Since our space of visualisation is 3-dimensional, it is not possible to picture a ~~relatis~~ 4 parametric dependence ~~it~~ in it. In order to get the feel of the fourth parameter (or dimension) t introduced by Einstein's prescription into the geometry let us suppress two space-coordinates y and z i.e. choose observers in such a way that $y = y'$, $z = z'$. i.e. choose the ~~common line of the~~ straight line of the relative velocity of the two observers as the axis of x. Then we are interested in seeing how (x, t) transforms to (x', t') under Einstein's prescription

2. The x-t diagram

Any observer O regards himself at rest. Take his rest position as origin and ~~doto~~ draw the x and t-axis. In the x-t diagram ~~the~~ a point ~~refer~~ represents an event i.e. a position at a given time.



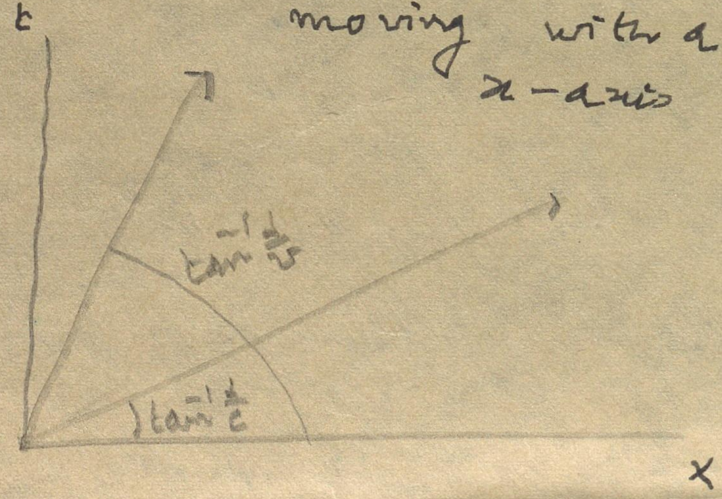
The observer ~~is not~~ at rest ~~not from~~ is not given by O. It is at O when $t=0$, at O_1 when $t=1$, at O_2 when $t=2$. ~~A~~ The observer at rest is represented in this diagram by the t-axis — the world-line of the observer. If there is another particle at A on the x-axis, it is represented by its world-line AA'. and we can state the 1st theorem of space-time geometry.

Theorem: World-lines of bodies at rest in a given reference frame are straight lines ∥ to the t-axis

~~Let us ...~~

Note that in the x-t plane equation of these world lines (of particles at rest) is $x =$ a constant. ($x = 0$ being the t-axis).

Let us next take the case of particles not at rest but moving uniformly along the x-axis. Let us take the ~~agent~~ Einsteinian predetermined Agent moving with uniform velocity c. Its equation of motion is $x = ct$. ~~Since we ...~~ $t = \frac{1}{c}x$, a st. line in the (x, t) plane through the origin with slope $\frac{1}{c}$. Similarly if a train is moving with a velocity v along x-axis

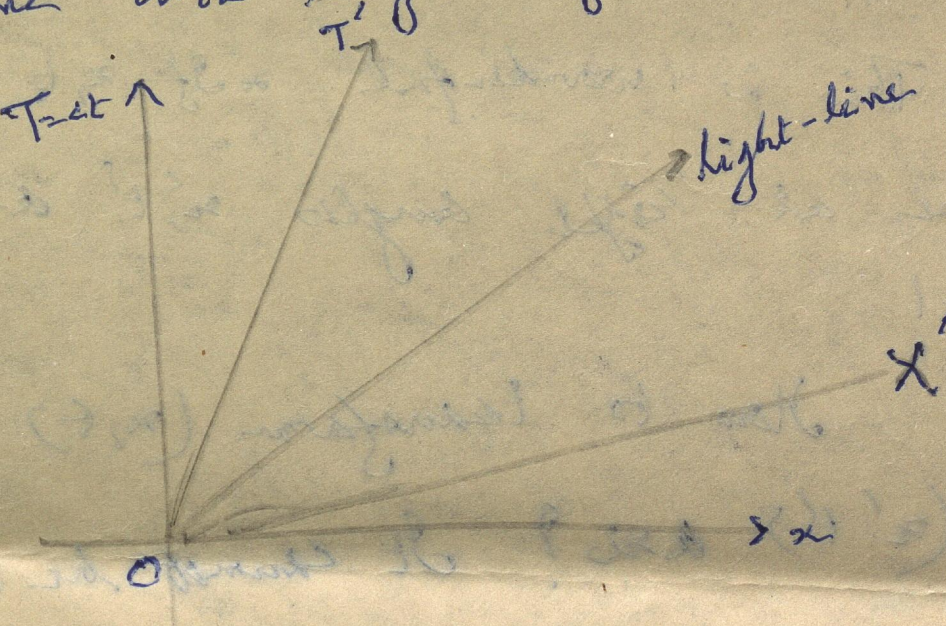


$t = \frac{1}{v}x$ its world line has slope $\tan^{-1}\frac{1}{v}$

Since $v << c$ $\frac{1}{v} >> \frac{1}{c}$ so the world-line of the train is more steep than the world-line of the Agent (light)

3. Lorentz transformations

We shall now redraw the world lines of the last section by making a small change of scale so that figures become convenient to handle. Along the t-axis we change the scale $T = ct$ so that the world-line of light becomes $x = ct$ or $x = T$ the bisector of the angle between x and t-axes and world-line of the particle (velocity v) becomes $x = vt = \frac{v}{c} ct = \frac{v}{c} T$ a line with ~~slope~~ slope $\tan^{-1} \frac{v}{c}$



Now suppose that an observer O' is on this particle i.e. it is moving with uniform velocity v relative to O. Its world line is OT'. But world-line of an observer is its t-axis. $\therefore$ For the observer O' OT' is its t-axis i.e. T'-axis. What is his x axis? Well Einstein's presumption that velocity of light is c for all observers will determine his x'-axis. For O' also $x' = ct' = T'$ i.e. the world-line of light will bisect the angle between ~~of~~ x' and T' axis. This determines his x'-axis as OX'.

This is wonderful. ~~x~~ If x, t axes ~~o~~ were at right angles x', t' axes are not!

How to transform (x, t) axis to $(x' t')$ axis? It cannot be done by simple rotation because in usual rotation of axes angle between the axes is unaltered!

In the usual transformations $x^2+y^2=x'^2+y'^2$

But in our case $x^2-c^2t^2=x'^2-c^2t'^2$ (Hyperbolic!)

So let us put $ct=i\eta$ $\therefore x^2+\eta^2=x'^2+\eta'^2$ and

so we can use our standard rotation

formula
$$x'=x\cos\theta+y\sin\theta$$
$$y'=y\cos\theta-x\sin\theta$$

Hence
$$x'=x\cos\theta+\eta\sin\theta$$
$$\eta'=\eta\cos\theta-x\sin\theta$$

But the t' axis is $x=vt=\frac{v}{c}ct=\frac{v}{c}i\eta$

$\therefore \eta=\frac{c}{vi}x$. But t' axis makes

an angle $90^\circ-\theta$ with x-axis

$$\therefore \frac{c}{vi}=\tan(90-\theta)=\cot\theta$$

$$\therefore \tan\theta=\frac{iv}{c} \quad \sin\theta=\frac{-iv}{\sqrt{c^2-v^2}}=\frac{-iv/c}{\sqrt{1-\frac{v^2}{c^2}}}$$

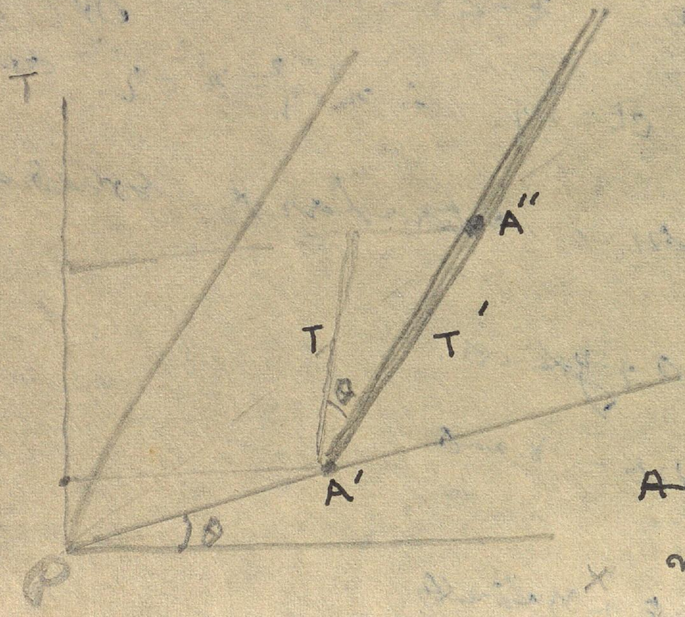
$$\cos\theta=\frac{c}{\sqrt{c^2-v^2}}=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\therefore x'=\frac{x}{\sqrt{1-\frac{v^2}{c^2}}}+\frac{\frac{ct}{i}\,\frac{iv}{c}(-)}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{x-vt}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$$

$$\frac{ct'}{i}=\frac{ct}{i}\frac{1}{\sqrt{(\;)}}-\frac{x\,(-iv/c)}{\sqrt{(\;)}} \quad \therefore t'=\frac{t-vx/c^2}{\sqrt{\left(1-\frac{v^2}{c^2}\right)}}$$

Lecture III

Lengths of moving rods and time by moving clocks

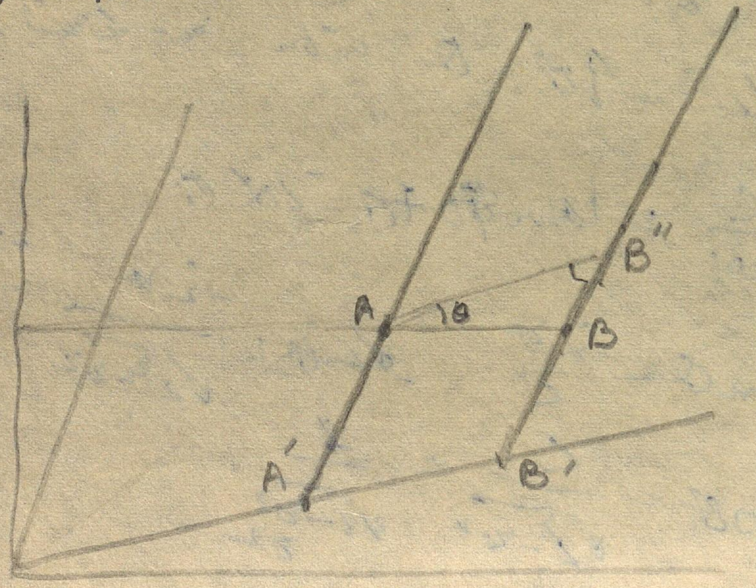


$$T = T' \cos\theta$$

$$= \frac{T'}{\sqrt{1-\frac{v^2}{c^2}}}$$

~~A sta~~ A stationary clock measures 1 second then $T' = 1$. The moving clock measures T seconds.

$T = \cos\theta < 1$. The interval measured by moving clock is smaller. The moving clock goes slow!



AB''B is a rt-led Δ, right angled at B'' $\because$ $\angle\theta$ is imaginary $\therefore$ AB is hypotenuse

$$AB'' = AB\cos\theta$$

$$A'B' = AB\cos\theta$$

$$l' = \frac{l}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$l = l'\sqrt{1-\frac{v^2}{c^2}}$$

If the stationary rod has length l' then the length of the moving rod l is $< l'$. ~~&~~ Length of of moving rod gets shorter!!

Lecture IV

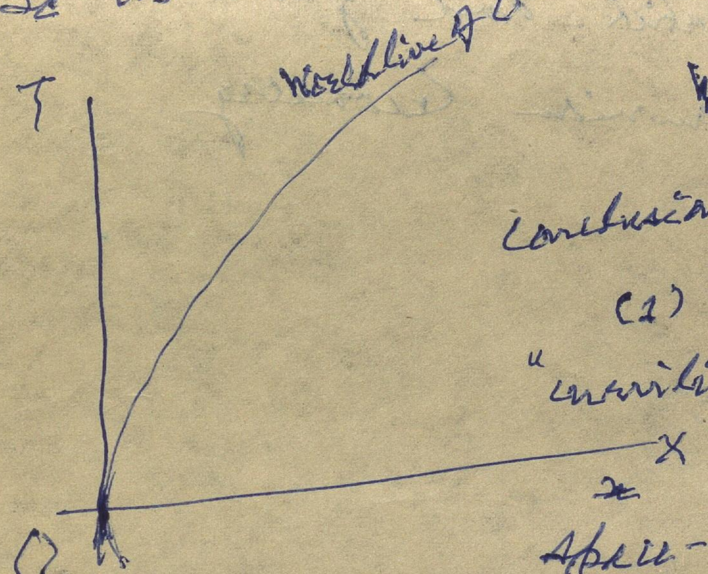
Observers with relative accelerated motion

An observer O finds that O' is moving away from him along his X-axis with uniform acceleration f. Equation of motion of O' is $x = ut + \frac{1}{2} ft^2$

Assume that ~~it~~ initial velocity $u=0$, so that

$$x = \frac{1}{2} ft^2 = \frac{1}{2c^2} f c^2 t^2 = \frac{f}{2c^2} T^2 \quad \text{or} \quad T^2 = \frac{2c^2}{f} x$$

So its world-line of O' is a parabola!



We can draw several conclusions

(1) According to O, O' uses a "curvilinear" coord-system in space-time!

(2) At $x=0$, $t=0$ O' was instantaneously at rest and so momentarily its world line coincided with world line of O i.e. OT.

(3) If instead of starting vel. of O' being zero it were u then the world line of O' will be a parabola with the world line of uniform velocity u as the tangent at O

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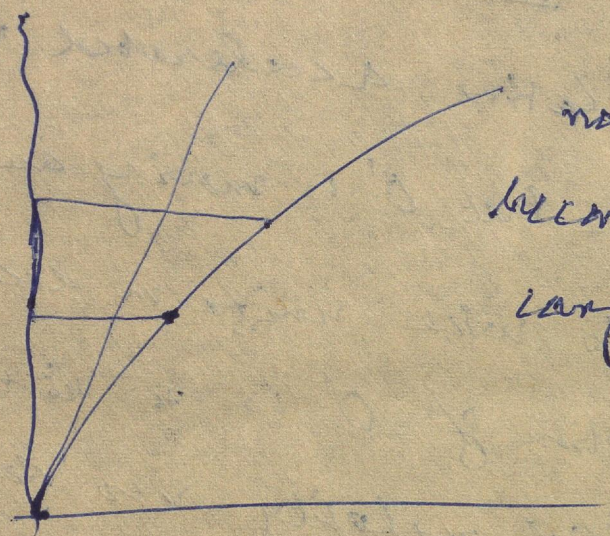
[4] Relative accn. is not uniform the direction becomes geometrically more complex.



World line $x = ut + \frac{1}{2} f t^2$

[5] One can find the relation between time kept by O' and that by O.

Principle of equivalence.

Equality of inertial and gravitational mass.

Riemannian Geometry