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former Einsteinturm at Potsdam have distinctly shaken my confidence in the numerical agreements in all the three "tests".

The predicted value for the deflection of a ray of light grazing the ~~earth's~~ Sun's limb is seconds of an arc, and the observed value is found to be seconds.* The suspicion that the discord between the two values may be genuine has already begun to arise.

Einstein himself has pointed out two fundamental objections with the present form of the relativity theory.

* Freundlich and Ledermann M.N.R.A.S. 104, 40 (1944). Also Narlikar V.V.

(7). The problem of matter and motion.* In general relativity, ~~motion is given by the~~ the motion in a gravitational field is given by the geodesic hypothesis, whereas the matter is given by the field equations. It is Einstein's contention that ~~the~~ both ~~geodesic hypothesis is superfluous, because~~ motion and matter must ~~both~~ be given by the field-equations. And in a very ~~really~~ ingenius manner he has been able to deduce ~~both~~ matter as well as motion ~~motion and matter~~ from the field equations, ~~in case of a~~ of the gravitational field of a system of n particles.†

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* See. A Einstein and Rosen. Phys. Rev.

† Einstein and others Annals of Maths

5 8. Field and Matter :- The

Newtonian theory of gravitation is based on the concept of matter alone. ~~But~~ General relativity on the other hand makes an attempt to explain gravitation as a field phenomenon. But the theory ~~is not abs~~ does not avoid the concept of matter altogether. The result is that in the present form, the theory considers two fundamental concepts - field and matter.

Einstein and Infeld ~~we~~ write in this connection:

"We have two realities: matter and field. ~~There is no doubt~~ For the moment we accept both the concepts. Can we think of matter and field as two distinct and different realities? Given a small particle of matter, we ~~can~~ could picture in a naive way that there is a definite surface of the particle where it ceases to exist and its gravitational field appears. In our picture the regions in which the laws of field are valid is abruptly separated from the region in which matter is present. But what are the physical criterions distinguishing matter and field? Before we learned about relativity theory we could have tried to answer the question in the following way: matter has mass, whereas field has not. Field represents energy, matter represents mass. But we already know that ~~this~~ such an answer is insufficient in view of the further knowledge gained. ... We cannot in this way distinguish qualitatively between matter and field since the distinction between mass and energy is not a qualitative one."

Thus the theory presents to us a type of internal contradiction: it establishes the identity of the two concepts mass and energy and yet its field equations distinguish the two concepts field and matter.

In spite of all these ~~difficulties~~ objections

to the theory. we must confess
it over comes the two
that ~~there is noth escape~~ we
fundamental difficulties of
~~cannot go back to the older~~
the older Newtonian theory:
~~Newtonian theory. Difficulties~~
the inertial frame and the
of ~~inertial frame and~~
absolute time. Hence it is
~~absolute time are got over by~~
no use going back to
~~general relativity~~
the Newtonian theory. Thus
conscious of the limitations of now go ~~to~~
before ~~us~~
general relativity we ~~set out out~~
the logical working out of
to ~~work out~~ the problem of
radially symmetric material
distributions. ~~logically~~

Chapter II

Line-Elements with Spherical Symmetry

1. Introduction

In this chapter we discuss, what may be called the kinematics of radially symmetric distributions. We deduce the usual simple forms of line-elements with spherical symmetry from their most general forms. We also express these line-elements in quasi-~~Galilean~~ Cartesian forms which are useful for the considerations of the energy of the distribution. We next point but a new and ~~the~~ striking feature of the line-element expressed in isotropic co-ordinates. In fact we actually show that a line-element

$$ds^2 = -e^{\mu}[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2) + e^{\nu} dt^2$$

$$\mu = \mu(r), \quad \nu = \nu(r)$$

can be transformed to a line-element of the ~~same~~ structure but the coeff's e^{μ} and e^{ν} now being functions of both r and t, and derive the integrated equations of transformation. Since the means of exploring a gravitational field are the geodesics and the null-geodesics. We give here a detailed treatment of the geodesics of spherically symmetric line-elements. We next inquire whether there are any positions in the field which are distinguished from others by some singular property, and we conclude the discussion by ~~con~~ a after consi-dering the question of equilibrium and the stability of equilibrium of a particle in a gravitational field with spherical symmetry.

2. Line Elements with Spherical Symmetry

The usual forms of line-elements for describing a radially symmetric gravitational field are

$$ds^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu} dt^2 \qquad (2.1)$$

$$\lambda = \lambda(r, t), \qquad \nu = \nu(r, t)$$

and

$$ds^2 = -e^{\mu}\left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right] + e^{\chi} dt^2 \qquad (2.2)$$

$$\mu = \mu(r, t) \qquad \chi = \chi(r, t)$$

The co-ordinates r and t in (2.2) are of course different from r and t of (2.1).

The most general form of such a line-element is

$$ds^2 = -e^{\alpha} dr^2 - e^{\beta}(r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + e^{\gamma} dt^2 + 2a\,dr\,dt \qquad (2.3)$$

with α, β ~~and~~ γ and a as functions of the present r and t. The problem of deriving (2.1) or (2.2) from (2.3) by a transformation of co-ordinates has been discussed, among others, by Lemaitre and by Tolman.* We notice that Tolman's treatment is not free from fault.

The problem of transformation of co-ordinates is a problem of tensor analysis, and is formulated in this way. Given the line-element

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \tag{2.4}$$

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* A. G. Lemaitre Monthly Notices 91, 490 (1931).
Tolman R. C. Rel. Therm. and Cosmology (1934), 240. The derivation by Tolman is faulty at one place. The equation (94.7) of the reference cited above, does not transform (94.6) to (94.8). The correct derivation is given here on page 17.

and the equations of transformation

$$x'^{\mu} = f_{\mu}(x^1, x^2, x^3, x^4) \qquad (2.5)$$

to find out the transformed line-element

$$ds^2 = g'_{\mu\nu}\, dx'^{\mu} dx'^{\nu}. \qquad (2.6)$$

And this problem is solved by the rules of transformation of the tensor $g_{\mu\nu}$. But for a worker in the field of the ~~th~~ theory of relativity, the problem is different. His problem is — knowing the line-element (2.4) completely and requiring a form (2.6) with some assigned particulars of structure such as isotropy, to arrive at the equations of transformation (2.5) and thus to deduce

(2.6) completely. It is for this reason that he finds it convinient to use different tricks of transformation, rather than the direct use of the rules of tensor analysis. Conscious of this we now transform (2.3) to (2.1) and (2.2).

In

$$ds^2 = -e^{\alpha} dr^2 - r^2 e^{\beta}(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\gamma} dt^2 + 2a\,dr\,dt,$$

which is (2.3),

~~let $e^{\beta} r^2 = \bar{\rho}^2$, t remaining untransformed.~~

~~(2.3) then transforms to~~

$$\cancel{ds^2 = e^{A} d\rho^2 - \bar{\rho}^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{C} dt^2 + 2b\,d\rho\,dt.}$$

~~A, C and b being functions of ρ and t.~~

let $e^{\beta} r^2 = \rho^2$, t remaining untransformed

(2.3) then transforms to

$$ds^2 = -e^{A} dp^2 - p^2(d\theta^2 + \sin^2\theta \, d\phi^2) + e^{C} dt^2 + 2b \, dp \, dt. \quad (2.7)$$ 16

where A, C and b are now functions of p and t.

Next let p remain untransformed and change t to ~~$\mathfrak{T}$~~ T by the substitution

$$dT = \eta\left(b \, dp + e^{C} dt\right) \quad (2.8)$$

η being an integrating factor which makes the right-hand side of (2.8) an exact differential. It may be noted that in principle this η can always be found. (2.8) then transforms (2.7) to the form

$$ds^2 = -e^{\lambda} dp^2 - p^2(d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\nu} dT^2$$

$\lambda = \lambda(p, T)$ $\quad \nu = \nu(p, T)$, which is (2.1) with r and t replaced by ρ and T. So far we agree with Tolman in his treatment.

To deduce (2.2) from (2.3), we first remove the cross term $dr\,dt$ in (2.3).

Starting with

$$ds^2 = -e^{\alpha} dr^2 - r^2 e^{\beta}(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\gamma} dt^2 + 2a\, dr\, dt$$

use the substitutions

$$r = \rho$$

(2.9) and $dT = \eta'(a\, dr + e^{\gamma} dt)$, the former of which is written down to have a uniformity in notations, and in the latter η' is again an integrating factor making the right hand member a perfect differential. (2.3) then transforms to

(2.10) $$ds^2 = -e^{D} d\rho^2 - \rho^2 e^{F}(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{G} dT^2$$

D, F and G being the functions

of present ρ and T. To transform

(2·10) to the form (2·2), ~~viz~~ viz
$\cancel{ds^2 = e^{\mu}}\ ds^2 = -e^{\mu}\left[d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] + e^{\lambda} d\bar{t}^2$,

let $\bar{r} = f(\rho, T)$

and $\bar{t} = \psi(\rho, T)$

The laws of transformation of $g^{\mu\nu}$ give,

$$-e^{-\mu} = -\left(\frac{\partial f}{\partial \rho}\right)^2 e^{-D} + \left(\frac{\partial f}{\partial T}\right)^2 e^{-G} \qquad (2\cdot 11)$$

$$-\frac{e^{-\mu}}{f^2} = -\frac{e^{-F}}{\rho^2} \qquad (2\cdot 12)$$

$$e^{-\lambda} = -\left(\frac{\partial \psi}{\partial \rho}\right)^2 e^{-D} + \left(\frac{\partial \psi}{\partial T}\right)^2 e^{-G} \qquad (2\cdot 13)$$

$$0 = -\left(\frac{\partial f}{\partial \rho}\right)\left(\frac{\partial \psi}{\partial \rho}\right) e^{-D} + \left(\frac{\partial f}{\partial T}\right)\left(\frac{\partial \psi}{\partial T}\right) e^{-G} \qquad (2\cdot 14)$$

(2·11) and (2·12) lead to

$$-\frac{f^2 e^{-F}}{\rho^2} = -\left(\frac{\partial f}{\partial \rho}\right)^2 e^{-D} + \left(\frac{\partial f}{\partial T}\right)^2 e^{-G} \qquad (2\cdot 15)$$

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Equation (2.15) gives the function $f(\rho, T)$. Then (2.14) gives $\psi(\rho, T)$. μ and χ are finally obtained from (2.12) and (2.13). Thus the transformed line-element

$$ds^2 = -e^{\mu}\left[d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] + e^{\chi} d\bar{t}^2$$

(which of the form (2.2)), becomes completely known. By the way, (2.14) suggests that in a radially symmetric field two functions f and ψ ~~can al~~ of the co-ordinates can always be found satisfying the tensor-relation

$$g^{\mu\nu}\left(\frac{\partial f}{\partial x^{\mu}}\right)\left(\frac{\partial \psi}{\partial x^{\nu}}\right) = 0.$$

In the course of our work we would require the line-elements of spherical symmetry, in terms of the so-called quasi-Cartesian co-ordinates. These are deduced by the substitutions

$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \\ t &= t \end{aligned}\right\} \qquad (2.16)$$

in (2.1) or (2.2). The line-element

(2.1) $$ds^2 = -e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu}dt^2$$

then transforms to

$$ds^2 = -\left[(dx)^2 + (dy)^2 + (dz)^2\right] - \frac{e^{\lambda}-1}{r^2}\left[x\,dx + y\,dy + z\,dz\right]^2 + e^{\nu}dt^2, \qquad (2.17)$$

$$r^2 = x^2 + y^2 + z^2.$$

and the line-element (2.2)

$$ds^2 = -e^{\mu}\left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2\right] + e^{\chi}dt^2,$$

21 transforms to

$$ds^2 = -e^{\mu}[dx^2 + dy^2 + dz^2] + e^{\chi} dt^2, \tag{2.18}$$

3. The Isotropic Line-Element

In this section we note certain properties which are peculiar to the line-element,

$$ds^2 = -e^{\mu}[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] + e^{\chi} dt^2 \tag{3.1}$$

$$\mu = \mu(r,t), \qquad \chi = \chi(r,t)$$

We shall show that it is always possible to find equations of transfor-mations which will keep the structure of (3.1) invariant i.e. which will transform an isotropic line-element into another line-element which is again isotropic.

let $\bar{r} = f(r,t)$

$\bar{t} = \psi(r,t)$

transform (3.1) to the line-element

$$ds^2 = -e^{\alpha}\left[d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right] + e^{\beta} d\bar{t}^2. \qquad (3.2)$$

$\alpha = \alpha(\bar{r},\bar{t})$. $\beta = \beta(\bar{r},\bar{t})$.

The laws of transformation of the components of $g^{\mu\nu}$ give

$$-e^{-\alpha} = -f'^2 e^{-\mu} + \dot{f}^2 e^{-\lambda} \qquad (3.3)$$

$$-\frac{e^{-\alpha}}{f^2} = -\frac{e^{-\mu}}{r^2} \qquad (3.4)$$

$$e^{-\beta} = -\psi'^2 e^{-\mu} + \dot{\psi}^2 e^{-\lambda}, \qquad (3.5)$$

and $$0 = -f'\psi' e^{-\mu} + \dot{f}\dot{\psi} e^{-\lambda}. \qquad (3.6)$$

From now on, we make it a convention to denote a differentiation with regard to r by a dash (as in ψ') and that with regard to t by a dot (as in $\dot{f}$), (3.3) and (3.4) together give

$$-e^{-\mu}\frac{f^2}{r^2} = -f'^2 e^{-\mu} + \dot{f}^2 e^{-\chi} \qquad (3.7)$$

(3.7) and (3.6) are, ~~the~~, the two equations to determine non-trivial values of the functions f and ψ. (3.4) and (3.5) then give us e^{α} and e^{β}, thus making the line-element (3.2) completely known.

Two consequences follow immediately from the above property of the line-element (3.1).

Firstly If we try to express the properties of a gravitational field with the help of the isotropic line-element (3.1), the form of the functions e^{μ} and e^{χ} will not be completely determined by the field equations of gravitation. Some arbitrariness will be left in these functions (apart from the usual appearance of arbitrary functions of integrations), owing to the possibility of transformations discussed above. Advantage can be taken of this circumstance to impose further restrictions on the line-element consistent with the gravitational properties of the field to be described. And this is what is actually done in deriving the line-elements of relativistic cosmology. There, in addition to the field-equations the line-element is also made to satisfy certain 'co-ordinate conditions'; also

It may be mentioned here, by way of comparision that the line-element

$$ds^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu} dt^2,$$

is of a unique structure, in as much as non-trivial transformations can not be found which will leave its structure unaltered.

Secondly Every static isotropic line-element can be transformed into a non-static isotropic one. [By a static line-element, we mean one for which $g_{\mu\nu}$ are independant of the time-like co-ordinate]. For if the line-element

(3.1) $$ds^2 = -e^{\mu}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] + e^{\chi} dt^2$$

be a static ~~try~~ one $\left[\mu = \mu(r),\ \chi = \chi(r)\right]$, the equations of transformation (3.7) and (3.6) are immediately integrable leading to

$$\bar{r} = f(r,t) = b\, e^{at}\, e^{\int\sqrt{a^2 e^{\mu-\chi} + \frac{1}{r^2}}\cdot dr} \tag{3.8}$$

and $$\bar{t} = \psi(r,t) = t + \int \frac{a\, e^{\mu-\chi}}{\sqrt{a^2 e^{\mu-\chi} + \frac{1}{r^2}}}\, dr \tag{3.9}$$

a and b being arbitrary

and the transformed line-element is a non-static one

$$ds^2 = -e^{\alpha}\left[d\bar{r}^2 + \bar{r}^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] + e^{\beta} d\bar{t}^2 .$$

$$\alpha = \alpha(\bar{r}, \bar{t}), \qquad \beta = \beta(\bar{r}, \bar{t}).$$

Solitary instances of transformations where a static line-element is transformed into a non-static one are to be found in the works of ~~Friedman~~ Lanczos*, ~~Robertson~~ Lemaitre†, Robertson†† and others. But ~~we don't~~ to our knowledge the general result ~~stated~~ derived above is not explicitly stated anywhere. Robertson has ~~devised~~ used a special name for those non-static line-elements

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* Lanczos Physik. Zeits. 23, 539 (1922).

† Lemaitre J. Math and Phys. (M.I.T) 4, 188 (1925)

†† Robertson Phil. Mag. 5, 835 (1928)

which are obtained by transformations of static line-elements. He calls them stationary line-elements.

In view of the above discussions, whenever one comes across a line-element of the type (3.1)

$$ds^2 = -e^{\mu}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] + e^{\chi}dt^2 \qquad (3.1)$$

$$\mu = \mu(r,t), \quad \chi = \chi(r,t)$$

a question arises whether this line-element is really non-static or merely stationary. It is desirable to have a set of necessary and sufficient conditions to be satisfied by μ and χ as functions of r and t in order that the line-element (3.1) may be stationary. And such conditions can be obtained by starting with the line-element (3.1) and trying to transform it to the form

$$ds^2 = -e^{\lambda} d\bar{r}^2 - \bar{r}^2 (d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\nu} d\bar{t}^2$$

$$\cancel{\lambda(r,t)} \qquad \lambda = \lambda(\bar{r}), \qquad \nu = \nu(\bar{r}).$$

These calculations are included at the end of the chapter. They are not to be included in the thesis

Proceeding in this way, after a series of intricate calculations, the desired conditions can be stated in the following form:

The line-element

$$ds^2 = -e^{\mu}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)\right] + e^{\chi} dt^2,$$

$$\mu = \mu(r,t), \qquad \chi = \chi(r,t),$$

is a stationary line-element, i.e. it can be transformed to a

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static line-element, provided

$1 + r\frac{u'}{2}$ and $u \cdot e^{-X/2}$ can each

be expressed as some function

(~~diff~~ not necessarily the same for each)

of $r e^{u/2}$ and conversely.

4 Geodesics — Principle of Equivalence

The general theory of relativity asserts that the space-time in the neighbourhood of a gravitating system is curved. If we have to make any measurements in such a curved space-time at a point, we take the osculating flat-space time at the point and make the measurements. This procedure is justified by the principle of equivalence, which states that in the space-time with the line-element

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu, \qquad (4.1)$$

it is possible to find co-ordinates at a point such that in the immediate neighbourhood of that point the line-element will be

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2 dt^2. \qquad (4.2)$$

In flat space-time the means of exploration are a test particle and a light pulse.

Their equations of motion in co-variant form are

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\cdot\frac{dx^\beta}{ds} = 0, \qquad (4.3)$$

with the addition of $ds = 0$ for a light pulse. It is the principle of equivalence which suggests that since these equations are valid in the osculating flat space-time at the point (given by 4.21) they may be valid throughout the curved space-time. We accept the suggestion, test its conse-quences and find that they are reasonably in accord with experience.

We thus obtain the geodesics and the null-geodesics as the means of determining the nature of a gravi-tational field* ~~The~~

The above is an observational method of completely determining the nature of gravitational fields. But theoretically too, the field is uniquely determined by the geodesics, because it is known that 'the projective and conformal properties of metric space determine its metric relations uniquely'†

We then go to determine the geodesics and null-geodesics of radially

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* ~~For observational significance~~ See B. Hoffman Rev. of Mod. Phys. 4, 173 (1932)

† Thomas. Diff. Invariants of Generalized Spaces ( ) 21.

symmetric gravitational fields. But a few remarks about the principle of equivalence will not be out of place here, an account of the role it plays in the developement of the general theory. This role has often been mis-interpretted e.g. Milne has quite recently remarked, " But we have been accustomed in Einstein's relativity to the dis-appear-ance of a gravitational field consequent on a change of co-ordinates"*

This is a gross mis-interpretation of the principle of equivalence. The

* Milne E.A. Monthly Notices. 104, 133 (1944)

disappearance of the a gravitational field is characterized by the vanishing of the Reiman tensor $R^{t}_{\mu\nu\sigma}$, and no Gaussian co-ordinate system can be selected in a curved space-time which will make $R^{t}_{\mu\nu\sigma}$ vanish. Even in the osculating flat a spay-time at a point allowed by the principle of equivalence, It is possible for the Christoffel symbols $\Gamma^{\sigma}_{\mu\nu}$ to vanish in the neighbourhood of the point, but not for the components of $R^{t}_{\mu\nu\sigma}$. So there is no a justification behind an assertion that at a point it is possible to remove gravitation by a change of co-ordinates. We can only remove, in a certain sense, the effects of gravitation,* in the immediate neighbourhood of a point by a change to proper co-ordinates at the point, if we measure these effects by the motion of test particles and light rays as given by (4.3) and by the distribution of matter in motion characterized by $(T_\mu^\nu)_{,\nu} = 0$.

An observer can use a co-ordinate system in which a

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* Tolman Rel. Therm. and Cosm (1934). 181.

particular particle has no acceleration, the observer ~~be~~ himself being at the particle. Such a co-ordinate system is on the same footing with any other permissible co-ordinate system of the field. This is the essence of the principle of equivalence.

5. Geodesics of Radially Symmetric Fields.

We choose the line-element of the radial distribution as given by

$$ds^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\nu} dt^2. \tag{5.1}$$

The equations of geodesics (4.3)

written out in full give

$$\frac{d^2r}{ds^2} + \frac{\lambda'}{2}\left(\frac{dr}{ds}\right)^2 + e^{\nu-\lambda}\frac{\nu'}{2}\left(\frac{dt}{ds}\right)^2 + \dot\lambda\left(\frac{dr}{ds}\right)\left(\frac{dt}{dt}\right)$$

$$- r e^{-\lambda}\left[\left(\frac{d\theta}{ds}\right)^2 + \sin^2\theta\left(\frac{d\phi}{ds}\right)^2\right] = 0, \qquad (5.2)$$

$$\frac{d^2\theta}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\theta}{ds} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0, \qquad (5.3)$$

$$\frac{d^2\phi}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{d\phi}{ds} + 2\cot\theta\,\frac{d\theta}{ds}\frac{d\phi}{ds} = 0, \qquad (5.4)$$

$$\frac{d^2t}{ds^2} + \frac{1}{2}e^{\lambda-\nu}\dot\lambda\left(\frac{dr}{ds}\right)^2 + \frac{\dot\nu}{2}\left(\frac{dt}{ds}\right)^2 + \nu'\left(\frac{dr}{ds}\right)\left(\frac{dt}{ds}\right) = 0 \qquad (5.5)$$

We, here, remind ourselves of the convention adopted in equations (3.2) to (3.6), so that $\lambda' \equiv \frac{\partial\lambda}{\partial r}$, $\dot\nu \equiv \frac{\partial\nu}{\partial t}$ and so on.

The five equations (5.1) to (5.5) are not independent, for any one of them can be obtained from the remaining four. They, then, constitute four equations between five variables r, θ, ϕ, t and s. If s and t are eliminated between them, we shall be left with two relations between r, θ and ϕ. Each one of these two relations represents a system of surfaces in space and so the two relations together give a series of curves. These are then the geodesics. If in the above discussion, we take the left hand member of (5.1) as zero, we get null-geodesics.

It is possible to eliminate all, r, t and s at the same time from the two equations (5.3) and (5.4) and then one is left with

$$\frac{d^2 v}{d\varphi^2} + v = 0, \qquad v \equiv \cot\theta. \qquad (5.6)$$

Thus the geodesics of a radial field are plane-curves lying in the plane

$$\cot\theta = A\cos\varphi + B\sin\varphi$$

or with the obvious substitutions (2.16)

$$z = Ax + By,$$

A and B being arbitrary constants.

Owing to this circumstance, the equations of geodesics can be much simplified for suitably chosen initial conditions.

If initially the particle, or the light pulse is in the plane $\theta = \pi/2$, ~~it remains in this plane through~~ with a velocity in this plane, ~~the particle~~ we see from (5·3) that

$\frac{d^2\theta}{ds^2} = 0$ initially and so the particle never leaves the plane $\theta = \pi/2$.

Considering then geodesics in the plane $\theta = \pi/2$ the system of equations (5.2) to (5.5) reduces to

$$\frac{d^2 r}{ds^2} + \frac{\lambda'}{2}\left(\frac{dr}{ds}\right)^2 + e^{\nu-\lambda}\frac{\nu'}{2}\left(\frac{dt}{ds}\right)^2 + \dot{\lambda}\left(\frac{dr}{ds}\right)\left(\frac{dt}{ds}\right) - r e^{-\lambda}\left(\frac{d\phi}{ds}\right)^2 = 0 \tag{5.7}$$

$$r^2 \frac{d\phi}{ds} = h, \quad h \text{ being a constant} \tag{5.8}$$

$$\frac{d^2t}{ds^2} + \frac{1}{2} e^{\lambda-\nu}\dot{\lambda}\left(\frac{dr}{ds}\right)^2 + \frac{\dot{\nu}}{2}\left(\frac{dt}{ds}\right)^2 + \nu'\left(\frac{dr}{ds}\right)\left(\frac{dt}{ds}\right) = 0 \qquad (5.9)$$

$$\text{and} \quad ds^2 = -e^{\lambda}dr^2 - r^2 d\varphi^2 + e^{\nu}dt^2 \qquad (5.10)$$

Eleminating s between (5.7) (5.8) and (5.9) and noting that (5.10) can be made identical with any one of these three, we get *

$$\frac{d^2r}{dt^2} - \left(\frac{dr}{dt}\right)^3 e^{\lambda-\nu}\frac{\dot{\lambda}}{2} + \left(\frac{dr}{dt}\right)^2\left[\frac{\lambda'}{2} - e^{\lambda-\nu}\frac{\dot{\lambda}\nu'}{2}\right.$$

$$\left. 4\frac{h^2}{r(r^2+h^2)}\right] + \frac{dr}{dt}\left[\dot{\lambda} - e^{\lambda-\nu}\frac{\dot{\lambda}\dot{\nu}}{4}\right] + e^{\nu-\lambda}$$

$$\left[\frac{\nu'}{2} - \frac{h^2}{r(r^2+h^2)}\right] = 0 \qquad (5.11)$$

$$\text{and} \quad r^2\left(\frac{d\varphi}{dt}\right)^2(r^2+h^2) = h^2\left(e^{\nu} - e^{\lambda}\left(\frac{dr}{dt}\right)^2\right), \qquad (5.12)$$

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* This treatment of the equations of geodesics for the line-element (5.1) follows the similar treatment for more general line-element in Narlikar V.V. Monthly Notices $\underline{96}$, 263. (1936)

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(5.11) and (5.12) give r and ϕ as functions of t. The elimination of t between them is not possible unless we know the functions

$$\lambda = \lambda(r, t), \qquad \nu = \nu(r, t).$$

We add a few simple results, following from (5.11) and (5.12). If the particle be at rest initially at the point $(r, \pi/2, \phi)$

i.e. $\frac{dr}{dt} = 0$, $\frac{d\phi}{dt} = 0$ initially,

(5.12) gives $h = 0$. Hence $\frac{d\phi}{dt}$ is zero throughout the motion of the particle.

(5.11) gives initially

$$\frac{d^2r}{dt^2} + e^{\nu-\lambda}\frac{\nu'}{2} = 0 \tag{5.13}$$

a $\frac{d^2r}{dt^2} + \frac{\partial\psi}{\partial r} = 0$, to the first order of approximation, ψ being the Newtonian potential expressed in relativistic units. A particle in a radially symmetric field if initially at rest, begins to move in a radial direction. (5·13) suggests that if the particle is to be attracted towards the centre of the spherical distribution, $\ddot{r}$ must always be +ve. If $\dot{r}=0$ at $(r, \pi/2, \phi)$ the particle is momentarily at rest at that point.

6. Geodesics of Static Radially Symmetric Fields — The (Ch. p) Equations.

The equations of geodesics (5.7) to (5.10) can be explicitly written as differential equations between r and ϕ when the line-element is a static one

$$ds^2 = -e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) + e^{\nu} dt^2 \tag{6.1}$$

$$\cancel{\lambda = \lambda(r,t) \qquad \nu = \nu(r,t)}$$

$$\lambda = \lambda(r) \qquad \nu = \nu(r)$$

The geodesics in the plane $\theta = \pi/2$ are given by.

$$\frac{d^2 r}{ds^2} + \frac{\lambda'}{2}\left(\frac{dr}{ds}\right)^2 + e^{\nu-\lambda}\frac{\nu'}{2}\left(\frac{dt}{ds}\right)^2 - r e^{-\lambda}\left(\frac{d\phi}{ds}\right)^2 = 0 \tag{6.2}$$

$$r^2 \frac{d\phi}{ds} = h \tag{6.3}$$

$$ds^2 = -e^{\lambda} dr^2 - r^2 d\phi^2 + e^{\nu} dt^2 . \tag{6.4}$$

t ~~and θ~~ can be eliminated between (6.2) ~~and~~ (6.2), ~~these~~ equations and then one obtains

$$\frac{d^2 r}{ds^2} + \left(\frac{dr}{ds}\right)^2\left(\frac{\lambda' + \nu'}{2}\right) + e^{-\lambda} r \left(\frac{d\phi}{ds}\right)^2\left[r\frac{\nu'}{2} - 1\right] + e^{-\lambda}\frac{\nu'}{2} = 0 \qquad (6.5)$$

& remove ds between (6.5) and (6.3)

one gets

$$\frac{d^2 u}{d\phi^2} - \frac{1}{u^2}\left(\frac{du}{d\phi}\right)^2\left(\frac{\lambda' + \nu'}{2}\right) - e^{-\lambda} u \left(r\frac{\nu'}{2} - 1\right) - e^{-\lambda}\frac{\nu'}{2}\cdot\frac{1}{h^2u^2} = 0 \qquad (6.6)$$

$$u = \frac{1}{r}$$

This then is the differential equation between r and ϕ, giving geodesics in the plane $\theta = \pi/2$.

The (r, β) equations of the geodesics in the plane $\theta = \pi/2$ can be obtained equally easily

Replacing the variable ϕ by the variable β through the relation

$$\frac{p}{\sqrt{r^2-p^2}} = \frac{r\,d\phi}{dr}$$

(6.5) and (6.3) lead to

$$(6.7) \quad -\frac{h^2}{p^3}\frac{dp}{dr} + \frac{h^2}{r^3} + h^2\left(\frac{\lambda'+\nu'}{2}\right)\left(\frac{1}{p^2}-\frac{1}{r^2}\right)$$

$$+ e^{-\lambda}\frac{h^2}{r^3}\left(\frac{r\nu'}{2}-1\right] + e^{-\lambda}\frac{\nu'}{2} = 0$$

(6.7) admits of easy integration, giving

$$(6.8) \quad \frac{e^{\lambda+\nu}}{p^2} = \int e^{\lambda+\nu}\left[\frac{\lambda'+\nu'}{r^2} + \frac{e^{-\lambda}}{r^3}(2-r\nu') - \frac{e^{-\lambda}\nu'}{h^2} - \frac{2}{r^3}\right]dr$$

+ a constant

This is then the explicit (r, p) equation of the orbit in the plane $\theta = \pi/2$. (6.7) can be used with advantage to solve the relativistic analogue of the classical problem: given the orbit to find the law of force. For example if the geodesics are to be circles with centres at the origin $(r = p)$. (6.7) requires that

$$\frac{h^2}{r^3}\left(r\frac{\nu'}{2}-1\right) + \frac{\nu'}{2} = 0. \qquad (6.9)$$

It should be noted here that the condition that $p = f(r)$ be a geodesic of the field gives only one restriction on the two functions λ and ν, the other restriction being supplied by the field equations of relativity, as we shall see in the next chapter.

Chapter III.

Static Distributions of Matter with Radial Symmetry

1. Introduction

We now present a number of material distributions which are static and radially symmetric. The solutions obtained, show a variety of density-distributions with possible astro-physical applications. In the first part of the chapter* we deal with distributions characterized by the line-element

$$ds^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu} dt^2.$$

foot note * The main results of investigations included in this part of the chapter are published in G. K. Patwardhan and P. C. Vaidya ~~Journ~~ J. Univ. Bombay 12, 23–36.

$\lambda = \lambda(r)$, $\nu = \nu(r)$.

Having obtained the equation of isotropy for this line-element we give a number of particular mathematical solutions of this equation. A perfectly general solution for a static spherical distribution is next developed correct to the third order of the density. We also consider some material distributions which do not satisfy the isotropy equation.

In the second part of this chapter * the same problem is treated with the help of the line-element

$$ds^2 = -e^{\mu}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] + e^{\chi} dt^2.$$

$\mu = \mu(r)$, $\chi = \chi(r)$.

Footnote * The principle results of the investigation included in this part of the chapter are published in Narlikar V.V. Patwardhan G.K. and Vaidya P.C. Proc. Nat. I of Science of India 9, 229-236.

But now, not only do we get certain new mathematical solutions of the equation of isotropy, but we are also able to show that the solutions are physically significant.

As the question of boundary conditions is dealt with in detail in a subsequent chapter, we merely touch the topic here to the extent required in our work of this chapter.

2. The Equation of Isotropy.

Let the line-element characterizing the gravitational field of ~~a~~ the material

of Science of India 9, 229-236.

material distribution be taken as

$$ds^2 = -e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu} dt^2, \tag{2.1}$$

$$\lambda = \lambda(r) \text{ and } \nu = \nu(r),$$

and let the material contents of the distribution be given by

$$T_\mu^\nu = (p+\rho) v_\mu v^\nu - p g_\mu^\nu, \tag{2.2}$$

with $v_\mu v^\mu = 1.$

The field and matter are connected through the gravitational equations of Einstein

$$R_\mu^\nu - \frac{1}{2} g_\mu^\nu R = -8\pi T_\mu^\nu \tag{2.3}$$

We have neglected here the cosmo-logical constant Λ. As the material distributions to be considered in this chapter are static, the velocity vector v^μ in (2.2) can be chosen with the components

$$v^1 = v^2 = v^3 = 0, \qquad v^4 = e^{-\nu/2} \tag{2.4}$$

The components of the energy-tensor T_μ^ν, then, are

$$T_1^1 = T_2^2 = T_3^3 = -p, \qquad T_4^4 = \rho \tag{2.5}$$

$$T_\mu^\nu = 0 \quad \text{if } \mu \neq \nu.$$

Making the connection of this tensor with the line-element (2.1) through the field equations (2.3) one obtains.*

$$8\pi T_1^1 = -8\pi p = -e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] + \frac{1}{r^2}. \tag{2.6}$$

$$8\pi T_2^2 = 8\pi T_3^3 = -8\pi p = -e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu'-\lambda'}{2r}\right] \tag{2.7}$$

* Tolman R. C. Rel. Therm. + Cosm (1934)

$$8\pi T_4^4 = 8\pi\rho = e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2} \qquad (2.8)$$

Any one of the above three equations (2.6), (2.7) and (2.8) can be replaced by the equation

$$\frac{dp}{dr} = -(p+\rho)\frac{\nu'}{2} \qquad (2.9)$$

which is a consequence of the relations

$$(T_\mu^\nu)_\nu = 0.$$

Thus (2.6), (2.7), (2.8) and (2.9) constituting among themselves three independent relations, are not sufficient to determine the four quantities λ, ν, p and ρ as functions of r. The necessary fourth relation is supplied by the so-called 'equation of state' of the material distribution considered. This 'equation of state' is usually written as a relation between the pressure p and the density ρ,

$$p = f(\rho) \qquad (2 \cdot 10)$$

Theoretically, therefore, in general relativity the problem of studying a given material distribution (with the given equation of state (2·10)), is equivalent to studying the four simultaneous equations (2·6), (2·7), (2·8) and (2·10). But the solution of these four equations involves such mathematical complications that it has not been possible to solve them with the equation of state however simple, except in the simplest case

$\rho =$ a constant.

To avoid these difficulties the problem is handled in a different way. We do not assume any equation of state but write the following equation which results from (2.6) and (2.7),

$$e^{-\lambda}\left[\frac{\nu''}{2}+\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}-\frac{\lambda'\nu'}{4}\right] = e^{-\lambda}\left[\frac{\nu'}{r}+\frac{1}{r^2}\right]-\frac{1}{r^2} \qquad (2.11)$$

We call this the equation of isotropy since it follows from the equality of the radial and the transverse pressure of the distribution. $(T_1^1 = T_2^2)$ omit

We solve this equation for λ and ν

with the help of an <u>assumed</u> relation between λ and ν. This assumed relation must, of course, be such that the two equations together can be solved in terms of known analytical functions.

By assuming different such relations we get a number of particular solutions.

Having obtained λ and ν, (2.6) and (2.8) will determine the march of the physical variables p and ρ within the distribution. For the solution to have a physical significance, the functions p and ρ, ~~must~~ so obtained, must satisfy the following conditions:

for $0 \leq r < a$ $\quad p \not< 0$.

for $r = a$ $\quad p = 0$.

for $0 \leq r \leq a$ $\quad \rho > 0$.

$r = a$ will then define the boundary of the distribution. At this boundary the line-element must go over continuously to the Schwarzschild's line-element

$$ds^2 = -\left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left(1 - \frac{2M}{r}\right) dt^2 \qquad (212)$$

for $r \geq a$.

~~Here~~ The constant M gives the mass of the material distribution as reckoned by an outside observer

Thus now the whole problem hangs upon the solution of the equation (2.11). Since this equation is obtained from the equality of the radial and the transverse pressure of the distribution ($T_1^1 = T_2^2$) we call it the equation of isotropy.

(2.12)

3. The Isotropic Solution in Successive Approximation

The equation (2.8)

$$8\pi\rho = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$$

can be immediately integrated, leading to

$$e^{-\lambda} = 1 - \frac{2m}{r} + \frac{A}{r}, \qquad m = \int_0^r 4\pi r^2 \rho \, dr, \qquad (3\cdot 1)$$

being

A is a constant of integration. ~~Since~~ If the field inside the fluid-sphere is to reduce to flat space-time when $\rho = 0$, the constant A must be taken as zero. Thus we write (3·1) finally as

$$e^{-\lambda} = 1 - \frac{2m}{r}, \qquad m = \int_0^r 4\pi r^2 \rho \, dr. \qquad (3\cdot 2)$$

The equation of isotropy (2·11)

$$e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\lambda'\nu'}{4} + \frac{\nu' - \lambda'}{2r}\right] = e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2}$$

can now be written in the form

$$\frac{\zeta''}{r^2} - \frac{\zeta'}{r^3} = 2\zeta''x + \zeta'x' + \frac{x'\zeta}{r} \qquad (3.3)$$

where $\zeta \equiv e^{\nu/2}$, $x \equiv \frac{m}{r^3}$.

The density ρ appears through x and x'.

To the first order of ρ we have

$$\zeta = 1 + a_1 + \int_0^r \frac{m}{r^2}\,dr + c_1\frac{r^2}{2}, \qquad (3.4)$$

where a_1 and c_1 are constants of the order ρ. All terms involving ρ^2 and higher powers of ρ have been neglected in deriving (3.4). This means that the latter is actually the solution of

$$\frac{\zeta''}{r^2} - \frac{\zeta'}{r^3} = \frac{x'\zeta}{r} \qquad (3.5)$$

ζ being assumed to be such that its value is unity when ρ vanishes. As determined by the ~~p~~ boundary conditions viz. the continuity of $g_{\mu\nu}$ and the vanishing of p at the surface of the sphere $(r = a)$,

$$a_1 = -\frac{M}{a} - \int_0^a \frac{m}{r^2}\,dr, \qquad c_1 = 0. \tag{3.6}$$

It is well-known that for $r \geq a$, the field is described by (2.12)

$$ds^2 = -\left[1 - \frac{2M}{r}\right]^{-1}(dr)^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \left[1 + \frac{2M}{r}\right]dt^2.$$

$$M = m(a).$$

Carrying the computations to the second order of the density we get

$$\xi = (1+a_1)\left(1 + \int_0^r \frac{m}{r^2}\,dr\right) + a_2 + \frac{1}{2} c_2 r^2$$

$$+ \frac{5}{2} \int_0^r \frac{m^2}{r^3}\,dr - r^2 \int_0^r \frac{m^2}{r^5}\,dr + \int_0^r \frac{m}{r^2} \int_0^r \frac{m}{r^2}\,dr\,dr \qquad (3.7)$$

The constants of integration at this stage are a_2 and c_2 at this stage, and they are determined as before, by ~~un~~ using the boundary conditions:

$$a_2 = -\frac{1}{2}\frac{M^2}{a^2} - a_1 \int_0^a \frac{m}{r^2}\,dr - \frac{5}{2} \int_0^a \frac{m^2}{r^3}\,dr +$$

? $$+ a^2 \int_0^a \frac{m^2}{r^5}\,dr - \int_0^a \frac{m}{r^2}\,dr \int_0^A \frac{m}{r^2}\,dr \overset{-}{+} \frac{1}{2} c_2 a^2;$$

$$c_2 = \frac{M^2}{2a^4} + 2 \int_0^a \frac{m^2}{r^5}\,dr. \qquad (3.8)$$

Even at the second stage the

relativistic effect does not appear prominently in ~~field~~ the field expressions. For example, the expression for pressure obtained at this stage agrees exactly with the classical expression signifying that the relativistic effect on ~~the~~ pressure is nil, so far as these calculations go. Going a stage further we get,

$$
\begin{aligned}
\xi = (1+a_1)\Big[1 &+ \int_0^r \frac{m}{r^2}\,dr + \frac{5}{2}\int_0^r \frac{m^2}{r^3}\,dr - r^2\int_0^r \frac{m^2}{r^5}\,dr \\
&+ \int_0^r \frac{m}{r^2}\int_0^r \frac{m}{r^2}\,dr\,dr\Big] \\
&+ a_2\Big[1 + \int_0^r \frac{m}{r^2}\,dr\Big] + a_3 \\
&+ c_2\Big[\frac{1}{2}r^2 + 2\int_0^r m\,dr - \frac{r^2}{2}\int_0^r \frac{m}{r^2}\,dr\Big] + \frac{1}{2}c_3 r^2 .
\end{aligned}
$$

$$+ \cancel{\int_0^r}$$

$$\zeta = 1 + a_1 + \int_0^r \frac{m}{r^2}\,dr$$

$$+ \frac{5}{2}\int_0^r \frac{m^2}{r^3}\,dr - r^2\int_0^r \frac{m^2}{r^5}\,dr + \int_0^r \frac{m}{r^2}\int_0^r \frac{m}{r^2}\,dr\,dr$$

$$+ a_1\int_0^r \frac{m}{r^2}\,dr + a_2 + c_2\frac{r^2}{2}$$

$$+ \frac{21}{4}\int_0^r \frac{m^3}{r^4}\,dr - \frac{11}{4}r^2\int_0^r \frac{m^3}{r^6}\,dr + \int_0^r \frac{m}{r^2}\int_0^r \frac{m}{r^2}\int_0^r \frac{m}{r^2}\,dr\,dr\,dr$$

$$+ \frac{5}{2}\int_0^r \frac{m}{r^2}\int_0^r \frac{m^2}{r^3}\,dr + \int_0^r r\int_0^r\left(\frac{5m}{r^2} - \frac{3m'}{r}\right)\int_0^r \frac{m^2}{r^5}\,dr\,dr\,dr$$

$$+ \int_0^r r\int_0^r\left(\frac{3mm'}{r^4} - \frac{8m^2}{r^5}\right)\int_0^r \frac{m}{r^2}\,dr\,dr\,dr$$

$$+ a_1\left[\frac{5}{2}\int_0^r \frac{m^2}{r^3}\,dr - r^2\int_0^r \frac{m^2}{r^5}\,dr + \int_0^r \frac{m}{r^2}\int_0^r \frac{m}{r^2}\,dr\,dr\right]$$

$$+ a_2\int_0^r \frac{m}{r^2}\,dr + a_3 +$$

$$+ 4c_2\left[2\int_0^r m\,dr - \frac{r^2}{2}\int_0^r \frac{m}{r^2}\,dr\right] + \frac{1}{2}c_3 r^2. \qquad (3.9)$$

The constants a_3 and c_3 can be determined as before. We have for pressure, the expression

$$8\pi p = \frac{1}{r}\left(1-\frac{2m}{r}\right)\frac{2\zeta'}{\zeta} - \frac{2m}{r^3} \qquad (3.10)$$

To the second order of ρ we find

$$8\pi p = 2c_2 - \frac{m^2}{r^4} - 4\int_0^r \frac{m^2}{r^5}\,dr. \qquad (3.11)$$