NOTE BO

(14)

Lekhak

NO.300 Rs.1.70

k is m... from paper supp... concessio...

NAME

SCHOO...

SUE

$xx' + yy' + zz'$

In spite of all the can.

Vol 21

OK

OK

anything happened?

Dickson II – Chapter XII – Pell equation; ax^2+bx+c made a square

p. 360 – Lagrange's direct method to $a+bt^2=u^2$ in integers.

(13) — $A = p^2 - Bq^2$, where p & q are relatively prime.

(i) Case $A^2 > B$. [$B>0$. Lagrange tells $B<0$ or $B>0$ & $A^2>B$]

Choose integers p_1 & q_1 such that $pq_1 - qp_1 = \pm 1$ & multiply (13) by $A_1 = p_1^2 - Bq_1^2$

Thus $AA_1 = \alpha^2 - B$, where $\alpha = pp_1 - Bqq_1$

p_1, q
p_1, q_1

[$AA_1 = (p^2 - Bq^2)(p_1^2 - Bq_1^2) = (pp_1 - Bqq_1)^2 - B(pq_1 - qp_1)^2$ by Brahmagupta's identity, ie $AA_1 = \alpha^2 - B$]

If $\alpha^2 - B$ is divisible by A for no value of $\alpha < |A|/2$, (13) is not solvable

If such an α exists, the problem reduces to the solution of

(14) $\quad A_1 = p_1^2 - Bq_1^2 \quad |A_1| < |A|$

If solutions of (14) are ~~found~~ found, we deduce, using $pp_1 - Bqq_1 = \alpha$, $pq_1 - qp_1 = \pm 1$, the solutions

$$p = \frac{\alpha p_1 \pm Bq_1}{A_1}, \quad q = \frac{\alpha q_1 \mp p_1}{A_1} \text{ of (13)}$$

Again if $A_1^2 > B$, we proceed as before & see that (14) reduces to

$$A_2 = p_2^2 - Bq_2^2, \quad \alpha_1 < \tfrac{1}{2}|A_1|, \quad |A_2| < |A_1|$$

If at ~~some any~~ no stage the case $A_i^2 < B$ arises, we finally reach a term $\alpha_n = e$, where $e < \sqrt{B}$ and $A_n A_{n+1} = e^2 - B$. Thus $A_n = \pm E$, $A_{n+1} = \mp D$, where D and E are +ve & $DE = B - e^2$. Moreover

$$\mp D = \rho^2 - B\sigma^2, \quad \pm E = r^2 - Bs^2$$

the solution of one of which implies ~~the other~~ that of the other.

Since $DE < B$, one of the equations is of the next type.

The proposed type is $\pm E = r^2 - Bs^2$, where $E < \sqrt{B}$, $B > 0$.

We first seek an integer ϵ, $\sqrt{B} > \epsilon > \sqrt{B} - E$, such that $B - \epsilon^2$ is divisible by E: if no such ϵ exists the equation is impossible. In the contrary case, take a particle equation and determine E_i, ϵ_i, λ_i by means of the equations

$$EE_1 = B - \epsilon^2, \quad E_1E_2 = B - \epsilon_1^2, \quad E_2E_3 = B - \epsilon_2^2, \ldots$$

$$\epsilon_1 = \lambda_1 E_1 - \epsilon, \quad \epsilon_2 = \lambda_2 E_2 - \epsilon_1, \quad \epsilon_3 = \lambda_3 E_3 - \epsilon_2, \ldots$$

$$\frac{\sqrt{B} + \epsilon}{E_1} > \lambda_1 > \frac{\sqrt{B} + \epsilon}{E_1} - 1, \quad \frac{\sqrt{B} + \epsilon_1}{E_2} > \lambda_2 > \frac{\sqrt{B} + \epsilon_1}{E_2} - 1, \ldots$$

where the effect of the inequalities is to ensure that the λ's shall be $+$ve integers making $0 < \epsilon_i < \sqrt{B}$. It is proved at length that if the proposed equation is solvable, we shall finally reach a least positive integer μ such that the term E_μ is identical with E and such that $E_{\mu+1} = E_1$, where $E_{\mu+\nu} = E_\nu$, and also that $E_m = \pm 1$ for a certain m, $0 \leqq m \leqq \mu$. Then ϵ_{m-1} is the greatest integer β for which $\beta < \sqrt{B}$. For brevity set

$$f_j = \frac{(\epsilon + \sqrt{B})(\epsilon_1 + \sqrt{B})(\epsilon_2 + \sqrt{B}) \cdots (\epsilon_{j-1} + \sqrt{B})}{E_1 E_2 \cdots E_{j-1}}$$

$$f_m = R + S\sqrt{B}, \quad f_\mu = X + Y\sqrt{B}$$

Since $E_m = \pm 1$, $f_m \bar{f}_m$ gives $R^2 - BS^2 = \pm E$, and the general solution is given by

$$r + s\sqrt{B} = (R + S\sqrt{B})(X + Y\sqrt{B})^n.$$

By actually multiplying together the factors in f_m, it is shown that

$$R = \beta l_{m-1} + l_{m-2}, \quad S = l$$

where the l's are derived from the relations

$$l = 1, l_1 = \lambda_1 l, l_2 = \lambda_2 l_1 + l, l_3 = \lambda_3 l_2 + l_1, l_4 = \lambda_4 l_3 + l_2, \ldots$$

The notation is at fault if $m = 0$ when we have $R = 1$, $S = 0$, and if $m = 1$, when we have $R \equiv G \equiv \beta$, $S = 1$.

Lagrange made applications to various numerical equations (13). for Pell's eqns, we have $E = 1$, whence $\beta = G$, $m = 0$, $R = 1$, $S = 0$.

$$X = \beta l_{\mu-1} + l_{\mu-2}; \quad Y = l_{\mu-1}, \quad r + s\sqrt{B} = (X + Y\sqrt{B})^n,$$

where n is a positive integer such that $n\mu$ is even or odd according as $r^2 - Bs^2 = +1$ or -1. For the former n is arbitrary if μ is even, but n must be even if μ is odd. Hence if B is any positive integer not a square, $r^2 - Bs^2 = 1$ has +ve integral solutions.

p. 361 - Lagrange derives this preceding formulas for the solution of

$$\pm E = r^2 - Bs^2$$

by a method first applies to eqns of any degree & n (See Lagrange's Ch. XXIII). His method for $t^2 - Au^2 = A$ where A is +ve and not a square is as follows. First Consider the solutions u prime to A. Then we can determine integers θ and y such that $t = \theta u - Ay$, $\theta < \frac{1}{2}A$. For this value of t, the initial equation becomes, after division by A,

$$E, u^2 - 2\theta uy + Ay^2 = 1$$

where $(\theta^2 - \Delta)/A = E_1$ an integer. Employ in turn each value of θ for which $\theta^2 \equiv \Delta \pmod{A}$ and solve the new equation by developing into a continued fraction either root of the corresponding quadratic,

$$E_1 - 2\theta Y + AY^2 = 0$$

Second, for solutions with $u = ru'$, $A = r^2 A'$, where $t = rt'$ with u', A' relatively prime, we have only to treat $t'^2 - \Delta u'^2 = A'$ as before.

The same method applies to $Bt^2 + Ctu + Du^2 = A$, where $C^2 > 4BD$. By the same substitution we now get $E_1 u^2 - Quy + ABy^2 = 1$, ~~where E~~, where $E_1 = (B\theta^2 + C\theta + D)/A$, $Q = 2B\theta + C$.

He noted that a conjecture made by Euler is false since $101 = x^2 - 79y^2$ has no integral solns although $-4 \cdot 4 \cdot 79 + 38^2 - 79 = 101$.

~~Euler stated~~ He applied the method of his former paper to deduce the solution $u = 34$, $t = 123$ of ~~α^2~~ $101 = t^2 - 13q^2$, chosen probably in view of his correspondence next mentioned.

Euler stated that he found trouble in applying Lagrange's method of solving eqn (13) to the case $101 = p^2 - 13q^2$. By that method we seek an integer $\alpha < 101/2$ such that $\alpha^2 - 13$ is divisible by 101. This is true for $\alpha = 35$. Then (14) becomes $A_1 = 12 = p_1^2 - 13q_1^2$. Since 12 is divisible by the square 4, set the quotient 3 equal to $t^2 - 13u^2$.

Then $t = 4$, $u = 1$, whence $p_1 = 8$, $q_1 = 2$. By Lagrange's method

$$p = \frac{\alpha p_1 \pm B q_1}{A_1} = \frac{35 \cdot 8 \pm 13 \cdot 2}{12} \; ; \; q = \frac{\alpha q_1 \mp p_1}{A_1} = \frac{35 \cdot 2 \mp 8}{12}$$

As these are not integers, one should conclude that the problem is impossible. However $p = 123$, $q = 34$ are solutions, which fact led Euler to believe that Lagrange's method is not sufficient. He notes that this solution $(123, 34)$ is given by

$$p = 123 = (35 \cdot 47 - 13 \cdot 13)/12, \quad q = 34 = (35 \cdot 13 - 47)/12$$

But what reason leads us to suppose that $p_1 = 47$, $q_1 = 13$?

To test whether $A = p^2 \pm Bq^2$ is possible or not, Euler gave for the case A a prime the following rule, of which he had no proof: Subtract from A any multiple of $4B$; if $A - 4nB$ is of the form at^2, where a is a prime or unity, and if $a = p^2 \pm Bq^2$ is solvable, then the proposed eqⁿ is solvable. Thus $101 = p^2 - 13q^2$ is solvable since $101 - 4 \cdot 13 = 7^2$ and $1 = p^2 - 13q^2$ is solvable.

Lagrange's reply has not been preserved, but it convinced Euler of the correctness of Lagrange's treatment of $101 = p^2 - 13q^2$, though, being then blind, Euler confesses he did not follow the real meaning of all the deductions, nor the significance of all the letters introduced.

Euler notes that $ax^2 - 4 = z^2$ implies that $ax^2 + 1 = y^2$ holds

for $x = \frac{1}{2}p^2(q^2-1)$, $y = \frac{1}{2}q(q^2-3)$, $p = r3$, $q = s^2+2$.

Thus if $a = 61$, we may take $r = 5$, $s = 39$ and deduce the large numbers x, y in his table

p. 363. Euler treated $ax^2+1 = y^2$ — Euler notes certain forms which are never equal to squares, as $3x^2+2$, $3t^2+(3n+2)u^2$, $5t^2+(5n\pm2)u^2$ — Euler gave a method of solving $ax^2+bx+c = y^2$ given $af^2+bf+c = g^2$ i.e. find new solns. Subtract & factor each member; thus we may set

$$p(x-f) = q(y-g),\quad q(ax+af+b) = p(y+g)$$

Multiply the first by p & the second by q and subtract. Hence

$$x = ng - mf - \frac{b(m+1)}{2a},\quad y = mg - naf - \tfrac{1}{2}bn$$

$$m = \frac{(aq^2+p^2)}{aq^2-p^2};\quad n = \frac{2pq}{aq^2-p^2}$$

To obtain integral solutions take $p^2 = aq^2+1$ and change the sign of g. Thus

$$x = 2gpq + f(aq^2+p^2) + bq^2,\quad y = g(aq^2+p^2) + 2afpq + bpq$$

$$\text{if } p^2 - aq^2 = 1$$

The method for $ax^2+c = y^2$ is similar but simpler, giving

$$x = gq + bf,\quad y = pg + aqf$$ (Cf. Euler's use of Brahmegupta's rule).

p. 365 – Legendre proved that $x^2 - ay^2 = -1$ has integral solutions if a is a prime $4n+1$ – if a is a prime $8n+3$, $ax^2 - y^2 = 2$ is solvable – if a is a prime $8n-1$, $y^2 - ax^2 = 2$ is solvable – Legendre also proved that one of the

$$Mx^2 - Ny^2 = \pm 1$$

is solvable if M and N are primes of the form $4n+3$. Given a positive integer A not a square it is always possible to decompose it into two factors M, N such that one of $Mx^2 - Ny^2 = \pm 1$, $Mx^2 - Ny^2 = \pm 2$ is solvable when the signs are suitably chosen – When $x^2 - Ay^2 = -1$ is solvable, A is a sum of two squares.

p. 367 – P. Paoli notes that if $t = h$, $u = k$ give one set of rational solutions of $At^2 + B = u^2$, all are given by

$$t = (hr^2 - 2kr + Ah)/(r^2 - A), \quad u = k + r(t - h)$$

F. Pezzi shows that $x^2 - Ay^2 = 1$ is solvable in an infinitude of ways, likewise $x^2 - Ay^2 = -1$ if and only if the period length is odd.

p. 369 – P.N.C. Egen gave the 121 values of $A < 1000$ for which $x^2 - Ay^2 = -1$ is solvable.

J.L. Wesell that $x^2 - Ay^2 = \pm C$ is solvable if there occurs in the C.F for $\sqrt{A}$, a complete quotient of denominator C.

J. Bayles found the values of n for which

$$25.\frac{1^4+2^4+\cdots+n^4}{1^2+2^2+\cdots+n^2} = 15n^2+15n-5 = \square$$

Let $n = m+1$, then $15m^2+45m+25 = (mr/s \pm 5)^2$ of

$m = 5s(9s \mp 2r)/D$, where $D = r^2-15s^2$. As by Euler $D=1$

if $(s,r) = (1,4), (8,31), (63,244), (496,1924)$ whence

$n = 6, 86, 401, 5361, \ldots$

p. 370 For $p^2-Aq^2=1$, Dirichlet proved that when $A = 2a$, (a prime)

$2t^2-au^2 = +1$ is solvable for $a = 8n+7$, $2t^2-au^2=-1$

for $a = 8n+3$, and $t^2-2au^2=-1$ for $a = 8n+5$

C. Gill noted $(1364557)^2-369(71036)^2 = 25$ and

that is the least soln of $t^2-940751u^2=1$, u has 55

digits, and t has 58 digits

p. 376 - C. Hermite proved that if D is $+^{ve}$, $x^2-Dy^2=1$ has an

infinitude of integral solutions, and all are given by

$$x+y\sqrt{D} = (a+b\sqrt{D})^i \quad (i = 0, \pm1, \pm2, \ldots$$

where a and b are solutions such that $a+b\sqrt{D}$ is a minimum.

p. 380 - Evans & Martin found the least solution

of $rx^2+1 = \square$ where $r = 940751$ (vide Gill above) and

noted that $rx^2+38 = \square$ has no integral solution

Mrst. Blanc notes that if $x=h, y=k$ be one solution of $2x^2-1=y^2$, then $x=hu+kv$, $y=2hv+ku$ gives a second solution provided $u^2-2v^2=1$, as for $u=3, v=2$

p. 381 – If (p_0, q_0) be the fundamental set of solutions of $p^2-Nq^2=\pm 1$ has been found, ~~D.S. Hart stated that~~ so that a set in addition to $(1,0)$ exists, D. S. Hart stated that the simplest method to find successively all further sets of solutions is to use the relations

$p=2p_0r+r'$, $q=2p_0s \mp s'$, where r, s are the last found values of p, q and r', s' the next preceding values

p. 385 – M. Veall noted that $x^2-Ay^2=n^2$ has the solution $x=Au^2+t^2$, $y=2tu$ if t, u give a solution of $t^2-Au^2=N$.

p. 386 – H. van Aubel proved the statement of Brocard that

$$x_{m+1}=2px_m-x_{m-1}, \quad y_{m+1}=2py_m-y_{m-1}$$

gives the relation between three successive solutions of $x^2-Ay^2=1$ where p, q are the least solutions

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Let us try to solve $x^2-69y^2=25$ using Lagrange's method.

$A/2 = 69/2$ $(p^2-Bq^2)=A$. Here $A^2>B$.

(8') Euler's & Lagrange ~~solns~~ re-solns.

$p^2 - Bq^2 = A$ $\qquad$ $p^2 - 13q^2 = 101.$

$A^2 > B]$ $\qquad$ $\alpha^2 - 13$ to be divisible by 101 for $\alpha \leq 50$

$\alpha^2 = 1225,\ \alpha^2 - 13 = 1212 = 101 \times 12$

2500 / 13 / 2287 $\qquad$ 2401 / 13 / 2388

1010. $\quad \alpha = 35 \qquad A_1 = 12.$

2196 2103 $\qquad$ 1111

2012 1024 / 13 / 1011 $\qquad$ 1212 / 2222

$p_1^2 - 13q_1^2 = 12. \qquad q_1 = 1, p_1 = 5$

2287 / 2500 / 13 / 7 $\qquad$ 1225 / 13 / 1212

$\alpha_1^2 - 13$ to be divisible by 12. & $\alpha_1 < 6$

$\alpha_1 = 5.$ $\qquad$ $5^2 - 13 = 12$

$p_2^2 - 13q_2^2 = 1.$ $\qquad$ $[E = 1$ $\quad EE_1 = 69 - 64 = 5$

$\sqrt{69} > \epsilon > \sqrt{69} - 1$ ie $\epsilon = 8,\ R = 1,\ S = 0,\ \beta = \epsilon = 8.$ $\qquad E_1 = 5.$

$\lambda_1 = 16,\ \epsilon_1 = 16 \cdot 1 - 8 = 8.$ $\quad$ ~~$EE_1 = 69 - 64 = 5,\ E_1 = 5.$~~

$\lambda_1 = 3.$ $\qquad$ ~~$\lambda_2 =$~~ $\quad \mu = 2$ $\qquad E_1 E_2 = 5,\ E_2 = 1$

$\lambda_2 = 16$ $\qquad$ 16 3

$l = 1,\ l_1 = 16,\ l_2 = 16$

$X = 8 \cdot 16 + 1 = 49 \quad Y = 16$

$r + s\sqrt{69} = (\cancel{129} + 16\sqrt{69})^n,\ \cancel{r = 385},\ s = 1.$

~~$f_2 = (8 + \sqrt{B})t$~~

129×129 / 1161 / 258 / 129 / 16641

256×89

$B = 101$ & not 69 you fool!

$p_2^2 - 13q_2^2 = 1.$ or $r^2 - 13s^2 = 1$

$E = 1]$ $\quad \sqrt{101} > \epsilon > \sqrt{101} - E \qquad \epsilon = 10 \qquad EE_1 = 101 - 100 = 1$

$\therefore E_1 = 1.$

$\frac{\sqrt{101}+10}{1} < \lambda_1 < \frac{\sqrt{101}+10}{1} - 1 \qquad \lambda_1 = 20.$

$\epsilon_1 = 20 \cdot 1 - 10 = 10, \quad E_1 E_2 = 101 - 100 = 1 \quad \text{a } E_2 = 1$

$p_2^2 - 13q_2^2 = 1, \quad f_1 = (10 + \sqrt{101})/1$

$\cancel{\sqrt{101} = (10 + \sqrt{101})^n}$

$r + s\sqrt{101} = (10 \pm \sqrt{101})^n. \quad r = 10, s = 1. \text{ with } n = 1.$

$10^2 - 101 \cdot 1^2 = -1. \qquad 61^2 - 101 \cdot 5^2$

$p_1 = \frac{5 \cdot 10 \pm 101}{1} = 61, \quad q_1 = \frac{5 \mp 10}{1} = 5.$

175
210
2275
505
1770 2780/12

$\frac{35 \cdot 65 \pm 101 \cdot 5}{12}$

$\sqrt{13}$

0	3	1	2	1	3	3
1	4	3	3	4	1	4
3	1	1	1	1	6	1

$3 + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1}$

18
5
25 · 13
6

324 − 18, 5

325 + 13 · 25, 2 · 18 · 5
326

(649, 180)

4

$p_1 = \frac{35 \cdot 649 \pm 101 \cdot 180}{1}, \quad q_1 = \frac{35 \cdot 180 \mp 649}{1}$

3245
1947
12715

18180
3245

2209

$\cancel{p_1^2 - 101 q_1^2 = 12} \quad p_1^2 - 13q^2 = 101. \quad A = 101$

$p_1^2 - 13q_1^2 = 12 \qquad AA_1 = \alpha^2 - B$

109×101

$\alpha = 35 \qquad \alpha_1 < 6 \qquad 101 \cdot A_1 = 1212, \; A_1 = 12$

$\alpha^2 - B$ to be divisible by 101 $\qquad \alpha_1 = 5 \qquad p_2 - 13q_2^2 = \qquad A_1 A_2 = \alpha^2 - B = 12$

$\Delta\alpha < |A/2| \qquad \alpha^2 - B \div 12 \qquad A_1 = 12, \; A_2 = 1$

$\cancel{\alpha^2 - B}\, \alpha^2 - 13 \qquad p_2^2 - 13q_2^2 = 1$

10) $a^2 - 69$ is divisible by 25 for $a = 12 < A/2$.

$$AA_1 = a^2 - 69 = 75$$

$$A_1 = 3.$$

144
69
75

$$p_1^2 - 69q_1^2 = 3. \quad \text{Here } A_1^2 < B.$$

296 There is no obvious solving this. 4th is of the type

$$\epsilon = r^2 - 69s^2 \text{ i.e. } 3 = r^2 - 69s^2$$

$\sqrt{69} = 8.3\ldots$] Choose ϵ such that $8.3 > \epsilon > 8.3 - 3$ i.e. $\epsilon = 6, 7, 8$

$\because$ of $B - \epsilon^2$ is divisible by 3 i.e. $3q - \epsilon^2$ is divisible by 3 if $\epsilon = 6$.

i.e. $\epsilon = 6$. $\therefore EE_1 = B - \epsilon^2$ i.e. $3E_1 = 69 - 36 = 33$ i.e. $E_1 = 11$.

$$\frac{\sqrt{69} + 6}{11} > \lambda_1 > \frac{\sqrt{69} + 6}{11} - 6 \text{ i.e. } \frac{14.3}{11} > \lambda > \frac{3.3}{11}$$

14.3

i.e. $\lambda_1 = 1$ and $E_1 = 1 \cdot 11 - 6 = 5$. $E_1E_2 = 69 - 25 = 44$

$\therefore E_2 = 4$.

$$\frac{\sqrt{69} + 5}{4} > \lambda_2 > \frac{\sqrt{69} + 5}{4} - 1 \text{ i.e. } \frac{13.3}{4} > \lambda_2 > \frac{9.3}{4}.$$

i.e. $\lambda_2 = 3$, $E_2 = 3 \cdot 4 - 5 = 7$, $E_2E_3 = 69 - 49 = 20$

$\therefore E_3 = 5$.

Next $\frac{\sqrt{69} + 7}{5} > \lambda_3 = \frac{\sqrt{69} + 7}{5} - 1$ i.e. $\frac{15.3}{5} > \lambda_3 > \frac{10.3}{5}$

$\lambda_3 = 3$, $E_3 = 3 \cdot 5 - 7 = 8$. $\because E_3E_4 = 69 - 64$,

$\therefore E_4 = 1$.

$$\frac{\sqrt{69} + 8}{1} > \lambda_4 > \frac{\sqrt{69} + 8}{1} - 1 \text{ i.e. } \lambda_4$$

$\lambda_4 = 16$.

$E_4 = 9\cdot 1 - 8 = 1$. $E_4 E_5 = 69 - 1 = 69$,

$\therefore E_5 = 68$. $\sqrt{69}+1$ $E_4 = 16\cdot 1 - 8 = 8$.

$E_4 E_5 = 69 - 64 = 5$, $E_5 = 5$.

$8\cdot 3 - 8$ $\lambda_5 = 16$. $E_5 = 16\cdot 5 - 8 = 72$

$E_5 E_6 = 69 - 72^2$. (Something wrong)

[We need to go to higher values of m until $E_m = 1$ for $m = 4$. $\therefore E_3 = 8$, the greatest integer $< \sqrt{69}$.]

$E_4 = 16\cdot 1 - 8 = 8$, $E_4 E_5 = 69 - 64 = 5$, $E_5 = 5$.

$\frac{\sqrt{69}+8}{5} > \lambda_5 > \frac{\sqrt{69}+8}{5} - 1$, $\lambda_5 = 3$.

$E_5 = 3\cdot 5 - 8 = 7$, $E_5 E_6 = 69 - 49 = 20$ $\therefore E_6 = 4$.

$\frac{\sqrt{69}+7}{4} > \lambda_6 > \frac{\sqrt{69}+7}{4} - 1$. i.e. $\lambda_6 = 3$

$E_6 = 3\cdot 4 - 7 = 5$. $E_6 E_7 = 69 - 25 = 44$, $E_7 = 11$

$\frac{\sqrt{69}+5}{11} > \lambda_7 > 0$. $\lambda_7 = 1$

$E_7 = 1\cdot 11 - 5 = 6$ $E_7 E_8 = 69 - 36 = 33$ $\therefore E_8 = 3$ //

i.e. $\mu = 8$, $m = 4$.

$f_m = (6+\sqrt{69})(5+\sqrt{69})(7+\sqrt{69})(8+\sqrt{69}) / 11\cdot 4\cdot 5$.

$f_\mu = (6+\sqrt{69})(5+\sqrt{69})(7+\sqrt{69})(8+\sqrt{69})(8+\sqrt{69})(7+\sqrt{69})$
$(5+\sqrt{69})(6+\sqrt{69}) / 11\cdot 4\cdot 5\cdot 1\cdot 5\cdot 6\cdot 11$

$f_m = \frac{(30+11\sqrt{69}+69)(56+15\sqrt{69}+69)}{11\cdot 4\cdot 5} = \frac{(99+11\sqrt{69})(125+25\sqrt{69})}{11\cdot 4\cdot 5}$

(12) $= \frac{(9+\sqrt{69})(25+3\sqrt{69})}{4} = \frac{225+207+52\sqrt{69}}{4} = \frac{432+52\sqrt{69}}{4}$

$= 108 + 13\sqrt{69}$.

8 $f_{11} = (108+13\sqrt{69})^2 = 108^2 + 169\cdot 69 + 26\cdot 108\sqrt{69}$.

$= 23325 + 2808\sqrt{69}$

108 × 108 = 864, 000, 108 = 11664

169 × 69 = 1521, 1014 = 11661

11664 − 11661 = 3

108 × 26 = 648, 216 = 2808

$\pm 3\sqrt{69} = (108+13\sqrt{69})(23325+2808\sqrt{69})^n$

ie (108, 13) is a soln of $x^2 - 69y^2 = 3$.

$x = \frac{12\cdot 108 \pm 69\cdot 13}{3}$, $y = \frac{12\cdot 13 \pm 108}{3}$

$= \frac{2193 \text{ or } 399}{3}$, $y = \frac{264 \text{ or } 48}{3}$

ie (731, 88), (133, 16) are solns

731 × 731 = 731, 2193, 117 = 534361

88 × 88 = 544, 408 = 4624

4624 × 69 = 41616, 27744 = 319056

7744 × 69 = 19936, 116464 = 534336

534361 − 534336 = 25

To Lagrange's method gives (133, 16) although the C.F. method does not. Also (731, 88) answers

Continue from p(9') $p_2^2 - 13q_2^2 = 1$ Here $A < \sqrt{B}$ E (13)

3 $\cancel{k^2 - 13q^2 = 1}$. $r^2 - 13s^2 = E = 1$. 6·6 2·6 4

$\epsilon = 3$, $EE_1 = 13 - 9 = 4$, $E = 1$, $E_1 = 4$.

$\lambda_1 = 1$, $\epsilon_1 = 4 - 3 = 1$. $E_1E_2 = 13 - 1 = 12$, $E_2 = 3$.

$\lambda_2 = 1$, $\epsilon_2 = 3 - 1 = 2$, $E_2E_3 = 13 - 4 = 9$, $E_3 = 3$

$\lambda_3 = 1$, $\epsilon_3 = 3 - 2 = 1$, $E_3E_4 = 13 - 1 = 12$, $E_4 = 4$.

$\lambda_4 = 1$, $\epsilon_4 = 4 - 1 = 3$, $E_4E_5 = 13 - 9 = 4$, $E_5 = 1$.

$\mu = 5$.

$f_\mu = X + Y\sqrt{B}$

$$f_5 = \frac{(3+\sqrt{13})(1+\sqrt{13})(2+\sqrt{13})(1+\sqrt{13})(3+\sqrt{13})}{4 \cdot 3 \cdot 3 \cdot 4}$$

178
68
289

$= \cancel{(17 + 4\sqrt{13})^2 / 36} = \cancel{289 + 16 \cdot 169 + 136\sqrt{13}}$

$= (16 + 4\sqrt{13})(2+\sqrt{13})(16 + 4\sqrt{13})^{36} / 36$ 256·13

4

289
208
497

$= \cancel{(289 + 16 \cdot 13 + 136\sqrt{13})(2+\sqrt{13})/36}$

$= \cancel{(497 + 136\sqrt{13})(2+\sqrt{13})/36}$

$= \cancel{(994 + 136 \cdot 13 + 767\sqrt{13})/36}$

$= (16 + 4\sqrt{13})^2 (2+\sqrt{13}) / 144$

497
272
767

928
3328
4456

$= (256 + 208 + 128\sqrt{13})(2+\sqrt{13}) / 144$

928
128
1056

$= (464 + 128\sqrt{13})(2+\sqrt{13}) / 144$

58
104
162

14

$= (29 + 8\sqrt{13})(2+\sqrt{13}) / 9$

$= (162 + 45\sqrt{13}) / 9 = 18 + 5\sqrt{13}$.

(14) 13.49 $r + s\sqrt{13} = (18 + 5\sqrt{13})^n$ 24

11 $\alpha_1 = 5$

637
625
12

$r = 18,\ s = 5$

$p_1 = \dfrac{\alpha_1 p_2 \pm 13 q_2}{A_2}$; $q_2 = \dfrac{\alpha_1 q_2 \mp p_2}{A_2}$

$= 90 \pm 65$; $q_1 = 25 \mp 18$

25, 18 3 $= (155, 7), (25, 43)$ $p_1 = (25, 7)$

175
70
875
619
256

$p = \cancel{35.135 \pm}$ $p = \dfrac{35.25 \pm 13.43}{12}$

$\dfrac{35.25 \pm 13.7}{101}$ Really a problem

875
61
960
84

26/5/76 — At Inn — Sc. American

Notes from Math. Games

Vol. 231. No. 2, Aug. 74. p. 103

(1) If a star number multiplied by 3 and 2 added, the result is a number that can be expressed as the sum of two consecutive squares & also as the sum of three consecutive squares.

Square stars are nos. of the form $6n(n-1)+1 = m^2$, where m & n are +ve integers. If the L.H.S. which defines a star is multiplied by 3 & 2 is added, the result is $18n(n-1)+3+2$ $= 18n^2 - 18n + 5 = (3n-1)^2 + (3n-2)^2$. If the right side which

defines a square is multiplied by 3 & 2 is added, the result is $3m^2+2$. This equals $(m-1)^2+m^2+(m+1)^2$

(1) Vol. 231, No 3, Sept 74 — nil

" No. 4, Oct. 74, p. 125

(2) The sum of the first n consecutive stars is square only when n is 1 or 169 first established by John Harris in 1942 on the basis of results reported by L. J. Mordell, ~~p. 279~~ p. 271 of Diophantine equations

(3) It is not true that all square stars begin with 1. The 12th square star begins with 9 & thereafter the numbers can start with any positive digit

Ibid No 5, Nov 74 — Nil

Vol. 229 — No. 1 — July 73 — nil

" No. 2 — Aug. 73 — nil

" Sept 73 — nil

" — Oct 73 — Interesting geometrical illustrations proving formula Σn, Σn^2 & Σn^3 etc.

" — Nov. 73 — Nil

" — Dec. 73 — On Waring's problem (Lagrange's 4-sq. theorem also included)

Vol. 228 — No. 1 — Jan 73 — Nil

" No 2 — Feb 73 — Nil

" No. 3 — March 73, p. 110 — Historical article about logs & Napier

" No. 4 — April 73 — p. 106 — Negative any numbers.

" ~~No 5~~ —

(16) Vol-228 - No-5, May 73

p. 105 - $2^{2^{2^2}} = 2^{2^4} = 2^{16} = 2^{65,536}$. How large is $2^{65,536}$?

Geoffrey Hoffman of W. Germany sent me a computer reprint of this number. It starts 20035... & has 19,729 digits. Adding another 2 to the ladder gives a number that will never be calculated because the answer, as Hoffman put it, would require the age of the Universe in computer time & the space of the Universe to hold the computer printout. Even a ladder as short as three 9's is $9^{387,420,489}$ a number of more than 360 million digits

Skewes paper of 1933 - If x is the number of primes less than $\underline{x}$ and $li(x)$ is the logarithmic integral function, then $\pi(x) - li(x)$ is positive for some $\underline{x}$ less than

$$10^{10^{10^{34}}}$$

an integer said to be the largest known to play a role in a non-trivial theorem.

Polypowers - Not calculated upwards & their relation to those calculated downwards

[for eg: The tripower 9^{9^9} calculated upwards = 81^9 & has only 77 digits]

x^{x^x} where x may be real or complex numbers

Vol 228 No. 6 – June 73, p. 110 – Remarks on polypowers of last month (17)

Vol. 227 No. 1 – ~~Jan~~ July 72 – nil

" No. 2. ~~Feb~~ Aug 72 – p. 106 – Binary Code – the curious properties of the Gray Code. –

" No. 3 – ~~March~~ Sept 72 – nil

" No. 4 – Oct 72. p. 110 – Why the long arm of coincidence is usually not as long as it seems.

Since the beginning of history unusual coincidences have strengthened belief in the influence on life of occult forces

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Star sq. no: ~~$3n^2+2$~~ $3l^2+2 = m^2$

$m^2 - 3l^2 = 2.$ $m^2 - 3l^2 = 1, (m=2, l=1).$

$6n(n+1) = l.$

$36: n=6.$

27/5/76 Vol. 226 – No. 4, April 1972

p. 102 – A problem of Dudeney

Nine digits (0 excluded) are arranged in two groups. On the left a 3-digit number is to be multiplied by a 2-digit number. On the right both numbers have 2 digits. In each case the product must be the same

158	79
23	46
3634	3634

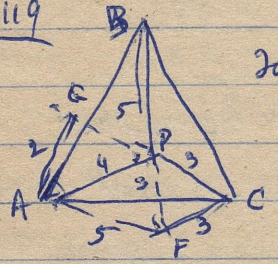
(20)

$a^2+b^2=c^2$
$a^2+d^2=e^2$
$b^2+d^2=f^2$
$b^2+e^2=g^2$

reviver radar
rotator
deified.
saippuakauppias – turnish
word for a soap dealer

The problem has not been shown to be impossible, nor has it been solved

p. 119



To find the side of an equilateral $\triangle$le ABC which contains a point P that is 3, 4, 5 units from the vertices A, B, C.

Trigg's soln – Construct PCF equilateral, draw AE $\perp^r$ to CP produced.

$$\angle PCB = 60^\circ - \angle PCA = 60^\circ - \angle ACF$$

$\therefore \triangle PCB \equiv \triangle FCA$. $\therefore AF = BP = 5$

$\because$ APF is a right angle since PCF being equilateral $PF = 3$, $PA = 4$

$\triangle AF = 5$, $\therefore \angle APE = 180^\circ - 60^\circ - 90^\circ = 30^\circ$. Hence

$AE = 2$ & $EP = 2\sqrt{3}$.

$$\therefore AC = \sqrt{2^2 + (3+2\sqrt{3})^2} = \sqrt{25+12\sqrt{3}}$$

Ibid – No. 2, Aug, 1970

2961,	328	58923	6547
2538	282	69723	7747
6174	686	96732	10748
		76941	8549

$2961 - 1692 = 1269$
$5823 - 3285 = 2538$
$7641 - 1467 = 6174$

1, 2, 3, 4

6742

6724	8442
4276	2448
2448	5994

2431	1980	1980
1342	0891	
1089	1089	

9954	5553	9981	8820
4599	3555	1899	0288
5355	1998	8082	8532

5823	5823	76421	67412	65934
3285			21476	43956
2538			45936	21978

78912	56925	36900	57933	58924
21987	52965	00963	33975	42985
56925	03960	35937	24958	15939

59913	78922	55935	71900	78930	74934
31995	22987	53955	00917	03987	43947
27928	55935	01970	70983	74943	30987

78930
03987
74943

$\frac{\overline{aba}}{\overline{bab}} = \frac{aa}{bb}$ $(100a + 10x + b)(10b + a)$

$= (100b + 10x + a)(10a + b)$

$1000ab + 100a^2 + 100xb + 10xa + ab + 10b^2$

$= 1000ab + 100ab + 100xa + 10xa + 10b^2 + ab$

$101b^2 = 101a^2$. $\|a = b$

7/6/76 Sc. American - Vol. 222

No. 1 — Nil

No. 2 — To find the smallest number that, when it is multiplied by dd, 17d, where d is any digit, gives a product that consists entirely of d's

Such a number obviously must produce a string of 1's when multiplied by 17, to see if we reach a point where there is no remainder. Such a point therefore we divide 1111 — by 17 to see if we reach a point where there is no remainder. Such a point is first reached by the quotient

6 69, 359, 477, 124, 183, the answer to the problem. Since 17 into this number is 1, 111, 111, 111, 111, 111, a multiple of 17 × 2 = 34 will produce a row of 2's and so on for the remaining digits

Because the infinite repeating decimal .111... equals 1/9, it can be shown that each integral magic number is the repeating portion of the decimal form of a reciprocal of uneven multiples of 9 that are not multiples of 5. In this case the magic number is the repetend of the decimal of 1/153. An example of non-integral magic number is 1.375. Multiplied by 8 d, the product consists of entirely of d's provided that the zeros at the right of the decimal point of the product are disregarded

No. 3 — p. 120 121 - Cyclic numbers & their properties

A cyclic ~~is a~~ number is a number of n digits with an unusual property: when multiplied by any number from 1 through n, the product contains the same n digits of the original number in the same cyclic order.

The number 142,857 is the smallest cyclic number — Points out this is 1/7 repeated — Generalisation of any 1/p where p is prime — Ref. to Ogilvy's Tomorrow's Math — other properties — Dickson's Theory of nos. ~~Mudo~~ Midy's theorem — Rademacher & Toeplitz — To be read in detail later

No. 4, p. ~~108~~ 112 — Some more remarks on cyclic numbers

No. 5 — Nil

No. 6 — p. 132 — Elegant triangle theorems not found in Euclid — Ref to W. R. Ransom's "One Hundred Mathematical Curiosities (J. Weston Welch, 1955)

p. 136 — Ref. to ingenious short cuts by which one can search for cyclic nos. sent ~~my~~ by many readers — Postponed to a future date.

Vol. 2~~23~~ 221

No. 1 — Nil

No. 2 — p. 121 — Remarks that 121 is a square in any system with a base greater than 2 (2 does not appear in base 2) since 11 × 11

in any system with base > 2 has a product 121

No. 3 – Special issue on "Oceans"

p. 239 – Geometric constructions compass & ruler & using compass only.

p. 240 – Refers to Bonaparte as an enthusiastic amateur mathematician; friendship of Napoleon & Monge – Napoleon once declared "Monge loved me as one loves a mistress"

p. 242 – Napoleon's problem – Soln by Dudeney in his book Modern Puzzles (1926) – Also in Trigg's Mathematical Quickies (McGraw Hill, 1967), Problem 248. – Also ref. to Mascheroni's problems

No. 4 – p. 126 – Dr Matrix & lunar flight of Apollo 11

No. 5 – p. 144 – The solution to SPIRO × 7 = AGNEW is
14,076 × 7 = 98,532

No. 6 – p. 125 – Forming magic squares with dominos is another old recreation – Ref. to the 360 page book on board games by Podroze W. Krainie Gier (journey into the land of games) published this year (1969) in Warsaw.

28/10/76 – Lasers by O. S. Heavens – Chap. 1 – Intn

A device which produces an intense highly directional beam of light of a very pure colour. It possesses the property of coherence indicating a regularity of the waves in the beam. Direction only in the fine pencil like beam – Even a 1-watt laser would appear many thousand times more intense than the 100-watt ordinary lamp – Most powerful pulsed laser ~~beams~~ flashes will vaporise any known material & have, for eg; been used ~~in~~ to drill holes in diamond – Pure colour; laser light more monochromatic than that of any line source, and this provides one with a powerful tool for precision measurements over long distances – Coherence; waves of different atoms are thoroughly mixed up giving an "incoherent" beam of light; the operation of the laser imposes a strict discipline on the behaviour of the atoms, the resulting light wave has a beautiful regularity rising & falling in perfect sequence as long as the laser is working; use of laser waves ~~in~~ instead of radio waves has truly enormous potential

Chap 2 – Nature of light – Independent of source; in laser the source is all important

Chap 3 – Sources of light – Thermal source gives continuous spectrum of radiation – Flourescent materials like ruby

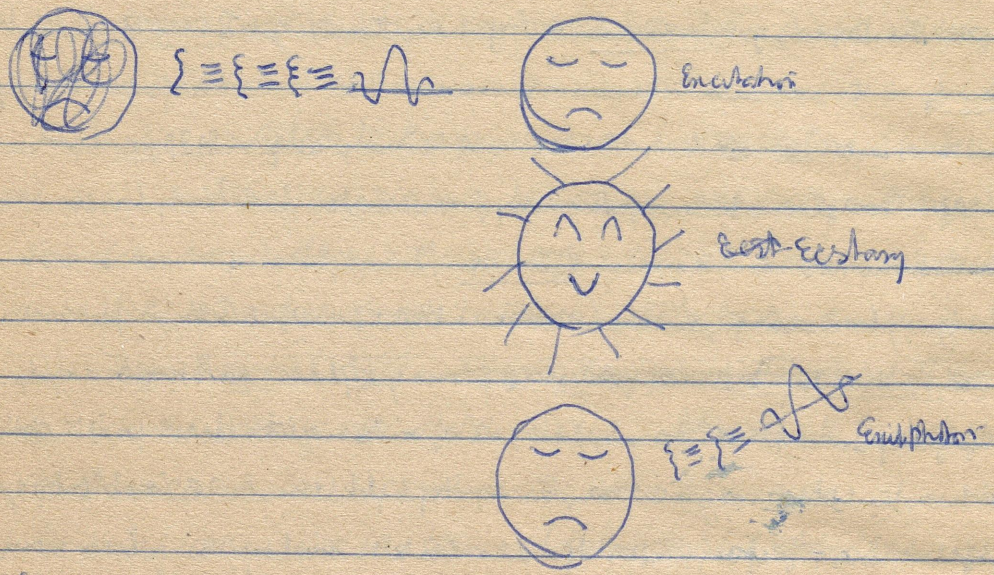
Chap. 4 – How things behave when exposed to light – In some cases

Reflection + transmission + absorption = 100%

In other cases, Reflection + transmission + absorption + scattering = 100% (for eg. scattering in frosted glass) – For laser materials scattering should be minimum.

Special effects in crystals —

Chap 5 – Is light really waves?



(Absorption & re emission of radiation by an atom) —

Chap 6 – Einstein & stimulated emission – It is possible, in right conditions, that radiation will pass in intensity as it passes through matter

Chap 15 – Last Chapter – Impact of lasers in the future

As Space travel propulsion device, no use – of real use in initiating fusion real reaction – of great use in communication – Biological & medical development

From Book on Lasers Vol 4 – ~~Ref~~ New Books Section.

Laser device today is multidisciplinary, and contributes to and draws ~~from~~ upon such fields as solid state, molecular & atomic physics, spectroscopy, acoustics, electronics, plasma physics, semiconductor technology, organic & inorganic chemistry, crystallography, thin film technology, fluid dynamics, aerodynamics, and combustion physics

Math Jems from

Scientific American – Jan 1976:

Although some of the greatest mathematicians have done work on magic squares, and even though such work leads into theories of groups, lattices, Latin squares, determinants, partitions, matrices, congruence arithmetic & other non-trivial areas of mathematics, the most enthusiastic square constructors have been amateurs. The famous Franklin 16×16 square has been called by him the "most magically magic of any magic square ever made by any magician" The literature on magic squares is very vast

3×3 square: There are exactly eight ways in which 15 can be partitioned into a triplet of distinct positive integers.

9+5+1	Central number belongs to two orthogonals & two
9+4+2	diagonals, ie it must be repeated 4 times in the
8+6+1	partitions & hence is 5. 9 belongs to only two triplets
8+5+2	& hence cannot be in a corner cell ie it conforms to a

8+4+3
7+6+2
7+5+3
6+5+4

side cell & it can be any side cell. 2 & 4 appear thrice and can only go in corner cells

Hence neglecting reflections & rotations the square is uniquely determined. Leads of 9 be put above 5 & 4 & 2 in top corners to fig (a)

4	9	2
3	5	7
8	1	6

(a)

(to Shu magic square of Chinese)

of order 4, there are 880 magic squares (excluding rotations & reflections) given by Dudeney Ency. Brit. 14th Edn

How many of order 5? A recent major break through made in the study of magic squares - Candy arrived in 1938 to a total (private publication) at a total of 13,288,952. In 1973 the counting was completed by a computer program developed by Richard Schroeppel, a mathematician and computer programmer at Information International. The program, using a standard back-tracking procedure, consists of about 3500 "words" & took about 100 hours of running time on a PDP-10. A final report written by Michael Beeler was released last October (1975). Not counting rotations and reflections, the report gives 275,305,224 magic squares of order 5. Schroeppel prefers to divide this number by 4 & gives the total as 68,826,306. The reason is that in addition to the eight variants obtained by rotations & reflections there are four other variants generated by the following two transformations:

(i) Exchange the left and right border columns & then exchange the top and

border bottom border rows;

(ii) Exchange rows 1 and 2, and rows 4 and 5. Then exchange columns 1 and 2 & columns 4 & 5

~~Thus 8 × 4~~ When combined with 2 reflections & 4 rotations, the above two give $2 \times 4 \times 2 \times 2 = 32$ forms that can be called isomorphic. With this definition of isomorphism the number reduces to 68,826,306.

The number can be lowered even more by considering the complementary magic square ie one obtained ~~obtained~~ by taking complements of each number in the square. When the centre of an order-5 square is 13, the complement is isomorphic with the original [check this]* If it is not 13, a different square results. If we broaden the concept of isomorphism to include complements, the count drops to about 3.5 million [how many correctly?]

The task of classifying order-5 squares in meaningful ways is staggering. Dudeney once wrote that certain ways of dividing magic squares into types seemed to him as useful as dividing people into those who take snuff and those who do not. Nevertheless, certain divisions yield unexpected results. The numbers of order-5 squares with centres that have numbers 1 to 13 are as follows:-

1 – 4,001,448

2 – 1,366,179

[* This appears true only if the square with centre 13 is associated - for eg. a bordered square may have 13 in centre & not be associated]

3 – 1,914,984
4 – 1,958,837
5 – 2,431,806
6 – 2,600,879
3,076,881 } 7 – 3,016,881
30, 8 – 3,112,161
9 – 3,472,540
10 – 3,344,034
11 – 3,933,818
12 – 3,784,618
13 – 4,769,936

Note – Increase from 1 to 9
Decrease from 9 to 10
Increase from 10 to 11
Decrease from 11 to 12 –
Increase from 12 to 13

rather surprising – In counts of squares with centres 14 to 25, same anamolies occur, since every square with a centre that is not 13 has its complement. Thus there are as many squares with centre 1 as there are with centre 25 & simly for other numbers except 13. Readers interested in details of Schroeppel's program may write to him at 835 Ashland Avenue, Santa Monica, Calif. 90405.

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

(b)

Square (b) is both associative and pandiagonal that is more powerfully magic than any other. To say it is Nasik means that if we ~~tile~~ tile the plane with this square, we can outline a 5x5 square anywhere on this infinite pattern and it will be magic (although not necessarily associative), to be associative 13 must be in the centre as in (b).

The lo-shu is associative but not Nasik. An order-4 square

may be pandiagonal or associative but not both. The order-5 square is the smallest one that can have both properties. Excluding rotations & reflections, 3600 order-5 squares are Nasik & if we also exclude cyclic permutations of rows and columns [what are these?], 144 are pandiagonal. In other words there are 144 infinite patterns of the type (obtained by tiling the plane) each containing 25 pandiagonal order-5 squares [144×25 = 3600]. Of the 144, just 16 contain a square that is also associative. All this, by the way, was known before Schroeppel's ~~work~~ program.

Of the 16 ~~pa~~ ~~Nasik squares~~ associative Nasik squares of order 5, four have 1 in the first cell, four have 1 in the third cell, four have 1 in the seventh and four have 1 in the eighth.

The Moslems were interested in pandiagonal squares having 1 in the centre (of course not associative) since they thought the central 1 was symbolic of the unity of Allah! [what of the Hindus?]

Sc. Amer. Sept 76 has an interesting article on Agriculture in India.

22/11/76

Tantra Art by Ajit Mukherjee (upstairs)

Plate 32 has some kind of a 5×5 magic sq — top right.

— however not very clear — see other squares with letters inscribed instead of numbers.

New Books Section

(1) Mechanics for Technology by Charles D. Bruch — John Wiley & Sons, 1976

No calculus used — Quite elementary; but interesting

(2) Problems & Theorems in Analysis — Polya — Szego — Vol II — Springer-Verlag 1976

Part IV — Comp. fns of one variable

Part V — Location of zeros

Part VI — Polynomials & trigonometric polynomials

Part VII — Determinants & quadratic forms

Part VIII — Number Theory

Part IX — Geometric problems

Part VIII — Problem 30, p. 117

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & & & 0 \\ 1 & 1 & 1 & 0 & \cdots & & & 0 \\ - & - & - & - & - & - & & - \\ 1 & 1 & 1 & 1 & \cdots & & & 1 \end{vmatrix}^2 = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & 3 & \cdots & 3 \\ 1 & 2 & 3 & 4 & \cdots & 4 \\ - & - & - & - & & \\ 1 & 2 & 3 & 4 & \cdots & n+1 \end{vmatrix}$$

Resembles the $11^2 = 121$

$111^2 = 12321$

— — —

p. 118 – Defns the arithmetical functions

$\varphi(n)$ (the Euler function), the no. of numbers less than n that are relatively prime to n

~~$\sigma(n)$~~ $\tau(n) = \sum_{t/n} 1$, the number of divisors of n.

$\sigma(n) = \sum_{t/n} t$, the sum of the divisors of n

$\sigma_\alpha(n) = \sum_{t/n} t^\alpha$, the sum of the αth powers of the divisors of n : $\sigma_1(n) = \sigma(n)$, $\sigma_0(n) = \tau(n)$

$\nu(n)$ = the no. of distinct prime factors of n.

$\mu(n)$ (the Möbius function); $\mu(1) = 1$, $\mu(n) = 0$ if n is divisible by a square (a part from 1), and $\mu(n) = (-1)^{\nu(n)}$ in all other cases.

$\lambda(n)$ (the Liouville function), $\lambda(1) = 1$, $\lambda(n) = (-1)^k$ where k is the number of prime factors of n (counting multiple factors according to multiplicity).

$\Lambda(n)$ (the Mangoldt fn), $\Lambda(n) = \log p$ if $n = p^m$ is a power of a prime number, and otherwise $\Lambda(n) = 0$

– additive & multiplicative number theory

Prob. 587 : If $n = 2^t p_1^{k_1} p_2^{k_2} \cdots p_\mu^{k_\mu} q_1^{l_1} q_2^{l_2} \cdots q_\nu^{l_\nu}$

where $2, p_1, \dots p_\mu, q_1, \dots q_\nu$ are distinct primes,

$p_1 \equiv p_2 \equiv \cdots \equiv p_\mu \equiv 1$, $q_1 \equiv q_2 \equiv \cdots \equiv q_\nu \equiv 3 \pmod 4$

set $\delta(n) = (k_1+1)(k_2+1)\cdots(k_\mu+1)$ if $l_1, l_2, \dots l_\nu$ are all even, and

$\delta(n) = 0$ otherwise

Show then $\delta(n) = \sum_{t'/n} (-1)^{(t'-1)/2}$, $\sum$ over the odd divisors t' of n

Gauss proved that $4\delta(n) =$ no. of lattice points on the circle

$x^2 + y^2 = n$.

(3) Variational methods in theoretical mechanics by J. T. Oden & ~~J. N. Reddy~~ and J. N. Reddy (Springer-Verlag, 1976)

– May take out parts relevant to Analytical Dynamics if & when such a book is written.

(4) Elements of Engineering Mechanics by Peter W. Likins – McGraw Hill, 1973

– may be useful – queer notations

(5) The theory of numbers by – Edited by S. Iyanaga – North-Holland, 1975

Chaps. 1 – Cohomology of groups

2 – Valuation theory

3 – Adele rings and Idele groups

4 – Main theorems of class field theory

5 – Proofs of the main theorems

Appendix 1 – Ideal theory

" 2 – History of class field theory

– The book gives systematic exposition of class field theory which is cohomology theory

(6) Methods of mathematics by Richard R. Goldberg – 2nd Edn

– Last chapter on Fourier series

7) Psychophysics – Method & Theory by George A. Gescheider, 1976

– Lawrence Erlbaum Associates, Publishers

– talking neural quantum theory – best chapter on ~~key~~ psychophysical law viz the discovery of a simple equation

which describes how the intensity of stimuli and our impressions of them are related. – Stevens power law $\psi = k\phi^{a}$; where ψ is sensation magnitude, ϕ the stimulus intensity, k an arbitrary constant & a depends on the sensory modality & stimulus conditions

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Encyclopaedia Americana – 1969.

Magic squares – Article by Aaron Bakst – Quite elementary.

Magic squares were known to men from the early days of civilisation. Originally magic squares were considered to have been endowed with supernatural powers and properties. Magic squares were used as amulets & talismans to ward off illness and evil spirits, and to provide all sorts of protection from disease & misfortune.

The new book of knowledge – Children's Encyclopaedia

p. Under Magic – 21

The digit problem – Here is a mystifying mind reading trick – Ask two persons to assist you. Instruct them that when you turn your back and say "Ready" they must both hold up from 1 to 5 fingers and then call aloud the total of the two hands.

When this has been done, you immediately announce the number of fingers each of them is showing. This may be repeated any number of times and it

becomes more baffling each time you do it.

The secret is simple and ingenious. One of the two persons who participates is your secret assistant. The first time it is worked, your assistant must always hold up two fingers. When the total is called, you simply subtract ~~two~~ 2 and announce the no. of fingers each of them has extended.

You now offer to repeat the trick. This time your assistant knows that he must hold up the same number of fingers that the other person held out previously, and so on.

— Looks very trivial, but the secret agent must be there

Q.N. George W. Kaplan

Author: The fine art of magic

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(1) The Ascent of Man by J. Bronowski — B B C Publication

(2) The philosophical impact of Contemporary Physics by Milič Čapek — Van Nostrand Reinhold Company.

(3)* The Theory of Graphs by Claude Berge — Methuen & Co Ltd, London.

(4) Frequency Response — Edited by Rufus Oldenburger — Macmillan Co, New York

(5) The Atomic Nucleus — M. Korsunsky — Foreign Languages Publishing House Moscow.

(6) Allgemeine Formen und Invariantentheorie I Bd by W. Franz Meyer — G. J. Göschen'sche Verlagshandlung, Leipzig

Theory of graphs & its applns available in Lib)

{ (†) cybernetics or control and communication etc in the Library
Also cybernetics God & Golem Inc) }

371

(7) Radio Today by Radunskaya and Zhabotinsky – Foreign Languages Publishing House, Moscow.

(8) (†) Cybernetics by Norbert Weiner – Hermann & Cie, Paris

(9) Treatise on Set Topology by R. Vaidyanathaswamy – Ind. Math. Society Pub^n – Private by the author

(10) Higher Transcendental Functions, Vol II – California Inst of Technology, by A. Erdelyi – Mc Graw Hill

(11) Space and Time by Emile Borel – Dover Publications Inc, New York

(12) Monte Carlo Method by Cashwell & Everett – Pergamon Press

(13) Waveform Analysis by R. G. Manley – Chapman & Hall Ltd.

(14) Introduction to Measure and Integration by Munroe – Addison Wesley.

(15) Tensors in Electrical Machine Theory by W. J. Gibbs – Chapman & Hall

(16) Lehrbuch der Theoretischen Physik Bd I – Walter Weizel – Springer Verlag

(17) Theory & Problems of Diff & Int ~~Calculus by~~ Calculus by Frank Ayres – Schaum Publishing Company, N.Y.

(18) Elements of Tensor Calculus by Lichnerowicz – Methuen & Co (Monographs)

(19) Wave Mechanics by V. V. Narlikar – Patna University Press

(20) Ramanujan Memorial Number Vol 1 – Muthialpet High School, Madras I

(21) " Vol 2 – "

(22) Quantum Mechanics by Condon & Morse — Mc Graw Hill.

(23) Quantum Theory by David Bohm — Asia Publishing House

(24) Electricity by C. A. Coulson — Oliver and Boyd.

(25) Physics; Foundations & Frontiers by Gamow & Cleveland — Prentice Hall (New Delhi)

(29) Theory of atomic nucleus & nuclear energy sources by Gamow & Critchfield — Oxford, Clarendon Press.

(30) Survey of Modern Algebra by Birkhoff & MacLane — Macmillan Co, New York

(31) Die Atomkerne by von Weizsäcker — Akad. Verlag M.B.H., Leipzig

(32) James Gregory Tercentenary Memorial Volume — G. Bell & Sons, London.

(33) Arithmetical Essays by Pedro A. Piza — Imprenta Soltero, Puerto Rico

(34) Measure, Lebesgue Integrals and Hilbert Space by Kolmogorov & Fomin — Academic Press, N.Y.

(35) ~~Quantenmechanik der~~ Quantenmechanik der materie und strahlung ~~Akad~~ Teil I — Akad. Verlag, Leipzig

(36) Electromagnetism by J. C. Slater — Mc Graw Hill [Slater & Frank in Library]

(37) Wave Mechanics by H. T. Flint — Methuen & Co.

(38) Typical means by Chandrasekharan & Minakshisundaram — Oxford Univ. Press

(39) Theoretical Physics by A. S. Kompaneyets — Foreign Languages Publishing House, Moscow

(40) Oberlikon Pocket Book — Zurich

(41) Quantum Mechanics by Schiff — Mc Graw Hill

(42) An introduction to Vector Analysis by F. Max Stein — Harper & Row.

(43) The challenge of world poverty by Gunnar Myrdal — Penguin International

(44) Transcendental & algebraic numbers by A.O. Gelfond — Dover Publications, N.Y.

(45) Complex variables & applications by R.V. Churchill — Mc Graw Hill

(46) Recent developments in general Relativity — Pergamon Press

(47) Algebra by N.L. Jerrar — Oxford, Clarendon Press

(48) Moderne Algebra Vols 1 & 2 (German) — van d. Waerden — Frederick Ungar Pub. Co, N.Y.

(49) Theory of Elasticity by Filonenko-Borodich — Foreign Languages Publishing Co, Moscow.

(50) Mathematical Theory of Electricity & Magnetism by Sir James Jeans (Paperback) — C.U.P

(51) Concepts from Tensor Analysis & Differential Geometry by T.Y. Thomas — Academic Press

(52) Banach Algebras by Charles E. Rickart — D. Van Nostrand Co Ltd.

(53) Theoretical Astrophysics by S. Rosseland — Oxford, Clarendon Press

(54) Operational Calculus by Jan Mikusinski — Pergamon

(55) Tables for solving the Laplace equation inside an ellipse by A.I. Vzorova — Infosearch Ltd, London

(56) Water Waves by J.J. Stoker — Interscience Publishers, N.Y & London

(57) Essay in Physics by Viscount Samuel with a letter from Dr Albert Einstein — Basil Blackwell, Oxford.

(58) Text book of Statistics by V. Seetharaman – N. Nadanasundaram Bros, Madurai

(59) First Course in Mathematical Statistics by C.E. Weatherburn (Paper Back) – C.U.P

(60) Waves by C.A. Coulson – Methuen Monographs

(61) ~~Unif~~ Unified system concept of nature by St. Th. Bornemisza – Vantage Press, N.Y.

(62) Theory of groups & applns to physical problems – Bhagavantam & Venkataraynd – Andhra Univ, Waltair

(63) The constitution of matter by Max Born – Methuen & Co.

(64) Electricity & Magnetism by Ramsey – C.U.P.

(65) Transient response from frequency response – Infosearch Ltd, London.

(66) Graphical Solutions by C.O. Mackey – John Wiley

29/11/76

Nuovo Cimento: 33 A, No. 1, Europhysics journal.

p. 641. on the violation of an inherent symmetry of spin-2 particles by K. L. Narayana & S. Navare.

Deptt of Physics, Shivaji University, Kolhapur – 416 004, India

Summary – The modified Pauli-Fierz Lagrangian function is given to yield the ratio of masses of spin zero & spin-2 particles as $\sqrt{3}$. The violation of an inherent symmetry principle between the f-mesons and the graviton has been used to ~~deduce~~ define Cosmod transformations

and to estimate rest mass of gravitational particles.

Ref: B. S. Madhava Rao, Proc. Ind. Acad. Sci, 15, 139 (1942)

Also references to Pauli, Fierz, Kemmer & several papers of Sivaram & others

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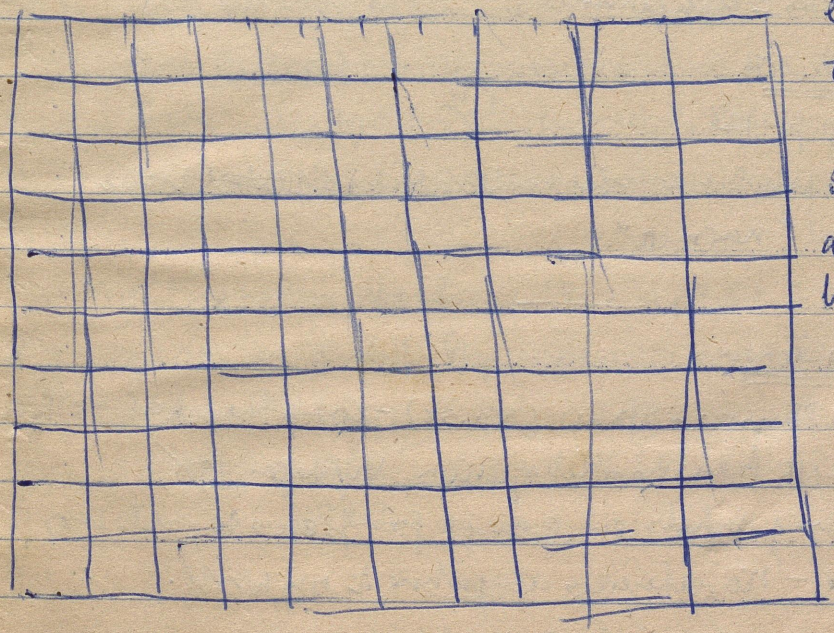
Sanskrit numerals

१, २, ३, ४, ५, ६, ७, ८, ९

Ajit Mukherjee's Tantra, Plate 32 – there are two types of magic squares (?)



but impossible to decipher the numbers there. In the 10 x 10 there appear to be in the borders Sanskrit letters!

3/12/76 New Books Section

(1) General Theory of Relativity by P.A.M. Dirac (Florida State Univ) - John Wiley - 1975

Appears a short simple clearly explained beautiful book written by a master - Based on a course of lectures in Florida State Univ - p. 52 Blackholes $r = 2m$ etc.

(2) The foundations of quantum physics by C. Piron (Univ. of Geneva), 1976 - Benjamin Inc.

Text book for graduate students - Abstract book

Chaps 1-5 - Classical case - quantum case - Realizations - Measurement and probabilities - Applications

(3) Sc. American - July, 1976

Martin Gams, p. 126.

Gardner has some pertinent & useful remarks about use of ~~total~~ pocket calculators & about "new math".

15/12/76 New Books Section

(1) Lecture notes in Physics - Springer (No. 58) - Group theoretical methods in Physics - 4th International Colloquium, Nijmegen, 1975

Large number of articles on gauge fields etc all my recent work - No reference anywhere to my work.

p. 617 - What is so "special" about relativity by Jean-Marc Levy-Leblond, Physique Theorique, Universite Paris VII

appears a popular article (among under "Educational" section).

Author's postal address:

Laboratoire de Physique Théorique
Université Paris VII (tower 33-43 - 1er étage)
2, Place Jussieu
75221 - Paris CEDEX 05

(2) Graph Theory with Applications by J. A. Bondy, U.S.R. Murty, (Macmillan 1976) - Deptt of Combinatorics and Optimization, University of Waterloo, Ontario, Canada. - Dedicated To our parents'

(3) Precalculus Mathematics by Harvey Braverman - Alg, Trig & Analyt. Geom. - Deptt of Maths, N.Y. City Community College of the City Univ. of N.Y. - Williams & Wilkins Company, Baltimore - Elementary Books.

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found two books dtd date 1965 & 1967 viz

(a) Progress in Elem. particle Physics & Cosmic rays. - North-Holland Publication Publishing Co.
(a) Vol VIII - 1965
(b) Vol IX - 1967

Sc. Amer. May 1976 - p. 123 on Math. Games, Gardner states

(44)

(?)

Magic square buffs will welcome an excellent book just published by Dover: New recreations with magic squares by William H. Benson and Oswald Jacoby. The book is filled with original material that appears in print for the first time. All 880 magic squares of order 4 are listed in an Appendix, and the book gives the fantastic tri-magic square of order 32 constructed by Captain Benson in 1949. This is a square that retains its magic not only when every number is squared but also when every number is cubed (originally magic? perhaps yes). The simplest known square of that type was of order 64 before Captain Benson found a way to halve the order.

(28/12/70) Lecture on partial diffns by Courant Inst man at App. Math. Dept

$u_y + u u_x = 0$.

D.V.P. $u(x,0) = h(x)$

Γ ~~x=0~~ $x=0,\ y=0,\ z=h(s)$

$\frac{dx}{dt} = y,\ \frac{dy}{dt} = 1,\ \frac{dz}{dt} = 0$.

~~y = h(s)~~

$x = s + h(s)t,\ y = t,\ z = h(s)$.

28/12/76 Questions for Another paper on Complex Analysis from Ch. VI of Cartan's Book.

(a) Fundamental Theorem of conformal representation viz

Any open set D of the complex plane $\mathbb{C}$ which is simply connected and different from $\mathbb{C}$ is isomorphic to the open disc $|z|<1$.

(b) Prop. 4.1, p. 192

Let f and g be two holomorphic mappings of a complex manifold X into a complex manifold X'; the set U of points of X in a neighbourhood f and g coincide is both open & closed

(c) Definition of ramification index of a holomorphic mapping.

Prop. 6.1 (p. 194) - Any simple holomorphic mapping of a complex manifold X on a complex manifold Y is an isomorphism

(uses fact that ramification index $=1$)

(d) Def of Y a complex manifold - Defn of a Riemann surface over Y - Defn of unramified Riemann surface = Defn of covering space of Y

(e) Ex. 4, p. 207.

Let $w = f(z)$ be a simple holomorphic function in the unit disc $|z|<1$ and let Γ be the image of the circle $|z| = r$, $0<r<1$, under f. Show that the radius curvature ρ of Γ at a point $f(a)$, with $|a| = r$ is given by

$$\frac{1}{\rho} = \frac{\mathrm{Re}\{af''(a)/f'(a)\}+1}{|af'(a)|}.$$

(f) p. 198 – An example of how a many-valued function can be "made "single-valued" by the introduction of a suitable Riemann surface by considering the many valued fn $y = (1-x^3)^{1/3}$.

Ask this ~~post~~ as part of a question

Sequence of primes from 1952 to 1972 – 21 in number.

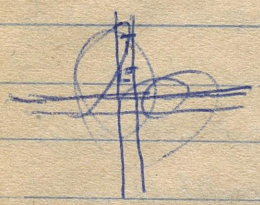
At C.C.B – 6/1/77

(1) Ency. Brit – 1973 – Vol. 14, pp. 573–575.

Article on Magic Squares – Article by S.V.R.C (?) [Same as what I have set in Bk No. 13]

A square matrix often divided into cells filled with numbers or letters in particular arrangements that were once thought to have special properties which led men to consider them as magical. Originally used as religious symbols they later became protective charms or tools for divination, and finally, when the original meanings were lost, people considered them as mere curiosities or puzzles – except for some western mathematicians who continue to study them as problems in number theory.

Fig 1 – Sator square – 1st Century A.D. found in ruins of Pompeii – used in 19th century in



S	A	T	O	B
A	R	E	P	O
T	E	N	E	T
O	P	E	R	A
R	O	T	A	S

Europe & U.S.A for fancied protection against fire, theft, sickness and other disasters – Jews & Arabs often using sacred words or holy names –

Natural magic square (ie 1 to n^2) arranged in order of ~~[illegible]~~ increase ie simply listed in those rows or those columns, not magical but a first step – magical const. denoted by C – Standard magic square (ie normal) with nos to 1 to n^2. – Order of magic square n – 3rd order from 1 to 9 or 5 to 13 or from (2,4,6), (5,7,9), (8,10,12)

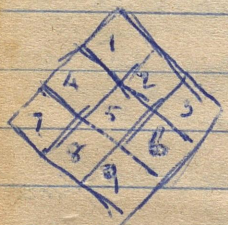


Fig 2 (A)

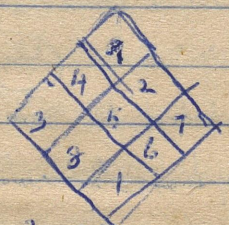


Fig 2 (B)

4	9	2
3	5	7
8	1	6

Fig 2 (C)

China – pp. 573 – 574.
Islamic Work pp. 574 – 574 B
India – 574 B – 575
Western Work – 575.

(A) natural square of order 3 slanted to make a diamond.
(B) Reversing centre row and column
(C) Collapsing diamond B to make a square

} Chinese method.

– Lot of references to Chinese work specially the Lo-shu.

To be read later – Vide Note Book. 13.

* and associated

(2) Encyclopaedia Americana Vol 18 (M to Mexico)

Magic squares – pp. 117 – 121 – Article by A. B. Nelson

Says "Known in India & China before the Christian era" – Knowledge introduced into Europe by Moschopulus who flourished at Constantinople early in 15th Century

(P)

1013	251	449	911	881
839	1301	941	113	311
41	173	701	1229	1361
1091	1289	461	101	563
521	491	953	1151	389

Fig P p. 121

(P) is Schaldham's example – All primes & it is also Nasik* – In addition the sum of any two numbers equidistant from the centre and opposite each other is 1,402 or twice the central number

Appeared in New York Sun. (be associated)

– Some remarks of Mc Mahon

Ref to 4th Hugel's book "Die Magischen Quadrate"

To be read more carefully later on

3505

(3) – (5) Three more Encyclopaedia – To be looked into later

(8/1/77) At Institute

The Sciences - published by the N.Y. Academy of Sciences (1) - Vol. 16, No. 5, Sept-Oct, 1976

p. 17 - The shape of change by James Gorman - A very interesting article on René Thom's (French topologist) Catastrophe Theory.

(2) Ibid, Vol. 16, No. 4, July - August, 76

p. 6 - The J/ψ puzzle by Jeremy Bernstein (The new particle (quark) described by Marshak at the lecture at the C.C. B. Physics Hall) - Very interesting article - mentions "~~ashrom~~ ashrams" at Harvard - Ref. to Glashow & Weinberg's articles in Sc. Amer. Oct 75 & July 74 respectively - Talks of infrared slavery etc.

Institute - 13/1/77

New Books Section (1) Quantum Mechanics, Determinism, Causality & particles by M. Flato, Z. Marić, A. Milojević, D. Sternheimer, J. P. Vigier - Editors. Reidel Pub. co - De Broglie thesis Centenary,

p. 227 - Article by Marie Antoinette Tonnelat contains a lot of references to work on wave eqns for particles of arbitrary spin - Send her my reprints - Address: Université de la Sorbonne, Faculté Pierre et Marie Curie

(7) Springer's Ergebnisse der Math. und ihrer Grenzgebiete - A series of Modern surveys in Maths, No. 91 1976 Riemann surfaces & generalized theta functions by Robert C. Gunning - Princeton, New Jersey.

(3) Elem. Diff. Eqns with applns - William R. Derrick & Stanley J. Grossman
- Addison Wesley, 1976, pp 1-533 - Contains Laplace transforms

4) Elem. Diff Eqns with linear algebra - Rees L. Finney & Donald R. Ostberg - Addison Wesley, 1976, pp. 1-478
- Contains Laplace transforms.

(5) Integral eqns, a reference text - Book by 6 Russian (?) - Monographs & text books on pure & applied mathematics by Noordhoff International Publishing, 1975

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6) Proc. Roy. Soc. A - Vol. 347, No. 1658, 13/1/1976.
Bakerian lecture, 1975 on Global geometry by M.F. Atiyah.
Mathematical Institute, Oxford. pp. 291-299
- Quite a beautiful Number article
- Also found a number of articles by S. Chandrasekhar on the Kerr equation for holes & blackholes & allied topics

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(7) Proc. Indiana Acad. Sci - Vol. 83, 1973 - Published at Indianapolis, Indiana. 1974
Indiana State Library, Indianapolis, Indiana 46 204.

(8) Science.
Vol. 192, No. 4239 - 7/5/1976 -

10/9/76
14/5/76
25/5/76

10/9/76 – p. 990 – ~~Latest~~ new particle caps the evidence for Charm by N.D.M

14/5/76 – p. 645 – DNA sequencing – A new era in molecular biology.

28/5/76 – p. 875 – The moon: not so different from the earth after all –

New evidence to show that they had a common origin – ~~not~~ the moon not originating elsewhere in the solar system & being captured by the earth – Clayton's evidence (Science, 21/5/76, p. 772) indicates that the earth, the moon and certain highly evolved meteorites were all made from a similar batch of material, isotopically different from most meteorites and presumably, from original material in the parts of the solar system where these meteorites were formed. As a result Clayton proposes that the earth & the moon were both formed from the parent bodies of the highly evolved meteorites, which, he views, as sizable planetesimals 10 Km or more in diameter – large enough to have undergone internal melting. Accumulation of these planetesimals – rather than direct condensation of the earth and moon from the solar nebula or capture of a complete moon – is thus what he envisages for the origin for both the earth & the moon

2/7/76, p. 17 – Chirality in Chemistry by V. Prelog (who got Nobel prize jointly with Cornforth) ~~extracts from~~ this article is his Nobel prize lecture. – An object is chiral if it cannot be brought into congruence with its mirror image by translation & rotation – Very interesting article even from a mathematical point of view – Ref to Lord Kelvin, Möbius & Pasteur (the

who first suggested, around 1850, that molecules can be chiral). — Also lots of interesting figures — discussion of 2-dimn chirality with a scalar triangle — Does not mention group-theoretic considerations.

17/9/76 — p. 1117 — Solar neutrinos: Proposal for a new test by M. S. Freedman et al. — No reference to work done by TIFR at KGF.

27/8/76, p. 718 — letter on "TM: Meditation or Sleep. — No of letters — Qn being "Are the beneficial effects reported for TM due to the sleep that occurs during the meditation period or to some other feature of the process?" more research necessary — letters from (i) Maharishi International University, Fairfield, Iowa 52556, (ii) Deptt of Psychology, Univ. of Washington, Seattle 98195. — pp. 759 — 812 all reports about Viking I mission to Mars.

20/8/76, p. 656 — Chinese earthquakes: The Maoist approach to seismology — quite an interesting article pointing out how in China amateurs take great interest in the subject & thus make it a "people's science" ~~and are~~ Amateurs in thousands are not watching for signs of earthquakes & do not depend on one observation in a laboratory — Plate tectonics theory (Chinese & Western also) viz that the India plate is pushing against the Asia plate under the Tibetan Plateau to explain the high seismicity of ~~the~~ Yunan, which is on the eastern ridge of the plateau next to the Burmese border — Soviet union does not favour the tectonic theory — of course

* by Herbert F. York – Freeman, San Frisco, 1976, §6.95.

this theory may not be perfect – Ref to the most devastating disaster on record for eg the great Shensi earthquake of 1556 which killed 820,000 people!)

p. 668 – Review of the book "The Advisers: Oppenheimer, Teller and the Superbomb"* – A reappraisal of the H-bomb decision, review by F. J. Dyson (of Inst. of Adv. Study, Princeton) who calls this an important book – very interesting review.

21/1/77 – B. Univ. Library

(1) Ency. Americana (Contd from p. 48) – Article on magic sqs by A. B. Nelson explanation of Narayana Pandithas method for case of $n = 5$ (Calls it La Hire's method).

(a) Put 3 in the top left hand corner & the remaining 1, 2, 4, 5 in any order in the first row. Repeat the integers in the downward diagonals & also repeat the in in the corresponding broken diagonals (to form square A)

(b) From A form A' by making the first column of A the taken from the bottom the first row of A', & simly the other cols.

~~(c) From~~ (c) From A' form B by replacing 1, 2, 3, 4, 5 resp'ly by 0, 5, 10, 15, 20.

(d) From A & B form C by addition of numbers in corresponding places.

[Looks rather far fetched]

La Loubère's method for odd squares is given next for the case $n = 7$

Remark about a kind of symmetry

From a magic square whether odd or even, a number of other magic squares can be formed by the mere interchange of the row & column which intersect in a diagonal with the row and column which intersect in some other cell in the same diagonal. In this way from each magic square of the 5th order, 48 other magic squares can be formed [For one cell in a diagonal there are 4 other cells for replacement] — this appears a new remark? [nothing new - same as interchange of two rows followed by interchange of mth & n^{th} cols.]

Formation of singly even order square ($n = 6$ is taken) (modification of Baltzer's method).

(i) Fill the cells of the two diagonals with the numbers 1, 2, 3, 4, 5, 6 beginning on the left hand side

(ii) Fill each of the remaining cells of the first column with the same number as that already in two of them (ie 1) or with the complementary number ie 1 or 6 provided that there are the same number of these numbers in the column

(iii) Cells horizontally pairs with those in the first column [two cells in a row equidistant from the ends are known a horizontal pair] & two cells in a column equidistant from the ends are known a vertical pair] are filled with the complementary numbers. [In a series of natural numbers any two equidistant from the ends are said to be complementary]

1
6
6
1
6
1

(iv) The remaining cells in the ~~first~~ second & third columns are filled in an analogous way to that in which the cells in the first column were filled, and then the cells horizontally paired with them are filled with the complementary number.

Note. - In the case of singly even magic square it will be necessary in constructing the square to take care that in step (iii) that in every row at least one cell which is not in a diagonal shall have its vertically paired cell filled with the same number as itself.

Thus for example on has square E

(E)

1	5	4	3	2	6
6	2	4	3	5	1
6	5	3	4	2	1
1	5	3	4	2	6
6	2	3	4	5	1
1	2	4	3	5	6

(F)

0	30	30	0	30	0
24	6	24	24	6	6
18	18	12	12	12	18
12	12	18	18	18	12
6	24	6	6	24	24
30	0	0	30	0	30

To construct F: Rewrite square E in which the left hand column of E (beginning with its top number) becomes the top row of a new square, the second column of E becomes the second row of the new square and so on. Then instead of each of the numbers 1, 2, 3, 4, 5, 6 substitute the corresponding number from the series 0, 6, 12, 18, 24 and 30. The result is square F.

Next to construct G :- Add the corresponding cells in (E) & (F) to get the number in the corresp. cell of (G) -

[Looks rather complicated, but interesting nevertheless]

(G)

1	35	34	3	32	6
30	8	28	27	11	7
24	23	15	16	14	19
13	17	21	22	20	18
12	26	9	10	29	25
31	2	4	33	5	36

Formation of double even magic squares -

Imagine the square to be divided into squarelets of four cells each ; and conceive these squarelets to be of two kinds, alternating with each other. Place 1 in the left-hand upper corner cell, and proceed horizontally to the right counting a number to each cell, but filling successively the squarelets of one kind only. When the end of one row is reached turn to the left-hand cell of the next row and again advance filling cells of one kind as before. and so on. For the 4-square the result of this operation is seen in (H).

1			4
	6	7	
	10	11	
13			16

(H)

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

(I)

Next begin with the right-hand lower corner cell, considering 1 as falling on it but not writing the number, and proceed regularly to the left row after row, filling the empty cells with the numbers belonging to them but scrupulously omitting numbers in the cells already filled. The result of the two operations is the magic square (I).

This is the most perfect magic square of the fourth order. Not only the rows, cols & diags sum up to 34, but there are 38 other ways in which sets of four numbers may be selected whose sum is 34, making 48 ways in all [not at all clear]. By the interchange of rows & cols, other squares may be formed, but none so perfect as this

[Cf. this with the Khajuraho square. Note that (I) is not Nasik]

~~Bo~~ Next ~~briefly~~ mentions constructing composite magic squares for eg $n = 81$ – Also mentions bordering method.

Ref. to Theodor Hugel in "Die Magischen Quadrate" has calculated that the paper required to contain all the magic squares of the 13th order would cover the whole surface of the earth about 348 times [mere statement]

A quotation from Mc Mahon's remarks in Proc. Roy. Inst. of Great Britain 1892 stating that ~~magic squares~~ "What was at first merely a practice of magicians & talisman makers has now for a long time become the serious study of mathematicians . . . . This has in fact proved to be the case, for from a certain point of view the subject has been found to be intimately connected with great departments of science ~~like~~ such as the Infinitesimal Calculus, the Calculus of Operations and the Theory of Groups"

[Absurd remarks!]

(57) 105 £ 10/

[* This is not a magic square]

curiosities in magic construction* may be mentioned the following.

30	21	6	15	28	19
7	16	29	20	5	14
22	31	8	35	18	27
9	36	17	26	13	4
32	23	2	11	34	25
1	10	33	24	3	12

(O)

The square in diagram (O) is filled with the natural numbers in the path of a knight returning to its starting ~~point~~ cell, and possesses the property that the difference of any two numbers equidistant from and on opposite sides of the centre is 18. It is due to Euler the famous mathematician [to be looked into more carefully]

Ref. to Andrews' book about a magic cross filled with 145 numbers

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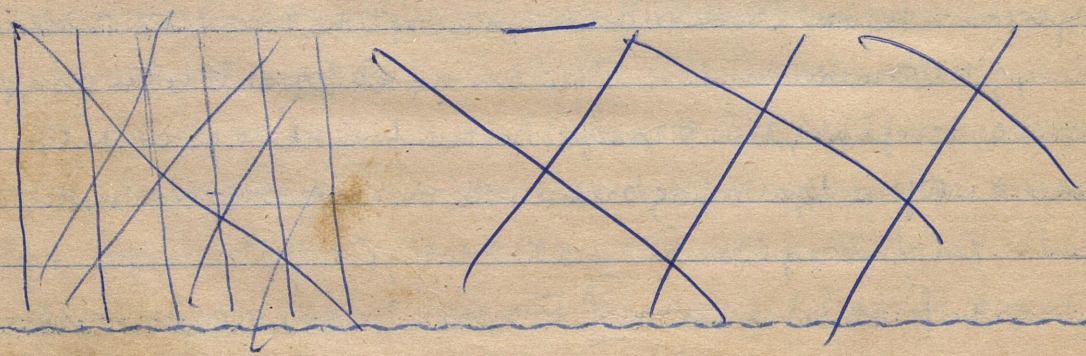
Ref. to Nasik squares (in England) or diabolic squares (in France) – To Schuldham's prime no magic square (see p. 48)



1/3/77 At Bangalore Univ Library

The American Peoples Encyclopaedia:— Vol. 12, p. 12-950.

Magic square. A set

1	11	17	29	43
29	43	1	11	17
11	17	29	43	1
43	1	11	17	29
17	29	43	1	11

Fig. 1.

Five numbers 1, 11, 17, 29, 43 are arranged in Fig. 1. It will be noticed that each number ~~occup~~ occupies cells connected by the Knight's move in chess [writing 1, 11, 17, 29, 43 in the first row, the other 1's in the other rows are obtained from this by a Knight's move like ⌐ in every case. The same is true of 11, 17, 29 & 43 also], and when the square is crossed the next cell may be taken in a contiguous square, and then the numbers may be moved to the corresponding cell in the original square; but other paths may also be taken* [not at all clear]. Squares with different numbers in each cell may be formed by making another square with four numbers, and adding the numbers in corresponding cells. This is De la Hire's method [again not clear]. Bachet arranged his numbers as in Fig 2, and then transferring those outside the square to the corresponding compartments within it, obtained a magic square as in Fig 2A. [For Figs 2 & 2A see page 65]

If one of the cols or rows in Fig 1 be moved to the other side of the square

* (other Knight moves like ⌐ or ⌈ or ⌉ may be taken)

the properties of the square will not be changed; and if the path of the knight in chess be followed from the corner containing 17, we get 11 in the second row, and 1, 43 and 29 in the other rows: such a square is called a Vasik square, after a town in India, where the inventor [doubtful; he must have learnt it from someone already in Vasik] Rev. A. H. Frost resided. These squares may be made so that the paths along (n-1) paths from a corner cell (n being no. of ~~cells~~ cells in the side of the square) may contain constant sums, and so that the sum of any two figures equidistant from the middle one may be any given multiple or fraction of the latter; and other combinations may be arranged [Not at all clear. Requires some more care].

Squares with an even number of cells require special adjustments; they also can be made Vasik

[The whole article (author not given) appears to be not quite that of an expert].

New Standard Encyclopaedia – Vol M, p. M-54.

Gives

2	7	6
9	5	1
4	3	8

and

1	16	11	6
13	4	7	10
8	9	14	3
12	5	2	15

& says that nos 1 to 25, 1 to 36, 1 to 49, 1 to 64 etc

[* This square is

3	44	38	13
24	27	33	14
35	16	22	25
36	11	5	46

For method see Frierson's method – Andrews, p. 139 or Note Book 12, pp. 177–78.

may all be arranged in magic squares.

Also nos. 3, 5, 11, 13, 14, 16, 22, 24, 25, 27, 33, 35, 36, 38, 44 and 46 may be arranged in four square cols. whose constant sum is 98 [to be looked into] *

Magic squares were probably invented by the Arabs during the Middle Ages, who appear to have worn them superstitiously as amulets to ward off evil influences [nonsense]. [Quite a trivial article]

The new world family Encyclopaedia – Vol. 12, p. 3241

6	1	8
7	5	3
2	9	4

Fig 1

gives the adjoining 3-square

The following shows how a 5×5 square may be formed: First write the five numbers 1, 6, 11, 16, 21 in five of the squares, putting 1 anywhere you please, 6 in the square two lines to right and one below from 1; then write 11 at the same distance from 6 and so on with 16 and 21, but going back or up five lines when a number falls in line outside of the diagram [Has not noticed the Knight's move]. Then fill in the numbers 2, 3, 4, 5 by continually counting two

1	24	17	15	8
20	13	6	4	22
9	2	25	18	11
23	16	14	7	5
12	10	3	21	19

Fig 2

* This is equivalent to primary move being the knight's move and the break move being an extended knight's move

lines below and one to the right of 1. Start from 6, 11, etc in the same way.

The same system may be applied to squares of any prime number of sides (Such squares were once believed to have magic properties)

[Constn for Fig 2 appears new* based purely on knight's moves to be looked into for other primes]

Chamber's Encyclopaedia – Vol VI, p. 794.

Magic squares are sets of different numbers each column of which, whether horizontal, vertical or diagonal, adds alike [simple defn ?]

1	1872	10	8
12	6	3	1870
7	9	1873	2
1871	4	5	11

Fig 1.

420	508	453	510
523	440	471	457
479	441	539	432
469	502	428	492

Fig. 2

The above are two examples of a magic square with the same summation (in either case 1891). Considering the difficulty with which a person without any previous knowledge could make even one such square, it may surprise many to hear that there are more than 700,000,000,000,000 (Seven Hundred billion) magic squares of this sort (4), with the summation of 1891,

each composed of different numbers, or with a different arrangement of the same numbers. Fig 1 is so constructed that a great variety of other squares may be made from it by altering the four highest numbers in it. Thus if 13, 14, 15, 16 be substituted for 1870, 1871, 1872, 1873 resp'ly; we get the smallest 4-square possible with the summation 34. It was at one time thought that magic squares could only be composed of arithmetical or other symmetrical series of numbers; but an examination of Fig 2 shows that that idea was erroneous.

[To be examined under the topic of magic ~~sq~~ series].

De la Hire's method of superposition most readily applies to odd squares [Is this correct. only odd squares?] more especially to those whose roots are prime numbers. We take the 5. square for example

1	3	5	2	4
5	2	4	1	3
4	1	3	5	2
3	5	2	4	1
2	4	1	3	5

Fig 3.

5	0	15	10	20
10	20	5	0	15
0	15	10	20	5
20	5	0	15	10
15	10	20	5	0

Fig 4.

Each row of the above squares contains the same numbers and in the same order relatively to the one another. But in Fig 3 the first number in each row is the same

as the third of the row above, whilst in Fig 4 it is the fourth. If now these two squares be combined by adding together the numbers in corresponding cells, the resulting square will be magic. In this case it will have the summation 65 & the top row will be 6, 3, 20, 12, 24. By altering the positions of the numbers in the top rows and making corresponding alterations in the other rows, 3600 distinct varieties of this magic square may be obtained [How does he get 3600?]

The literature on the subject is by no means easily accessible — Best known work is Hutton's Mathematical Recreations [I find that this book is not in the B.U. Lib] — Nasik squares [ref. to Frost] are those whose magicality is not destroyed by repeatedly removing the first column or row to the last ~~row~~ place ~~or~~ or vice versa [This defn to be examined in relation to the usual definition]. All squares with prime roots made by De la Hire's method are nasik. Even squares can also be made nasik. Fig 1, with the numbers 13, 14, 15, 16 substitute for the four highest, makes a nasik square.

Harmsworth Encyclopaedia — Vol ~~II~~ VII, p. 99

[This article is identical with that in American People's Encyclopaedia (see p. 59)]

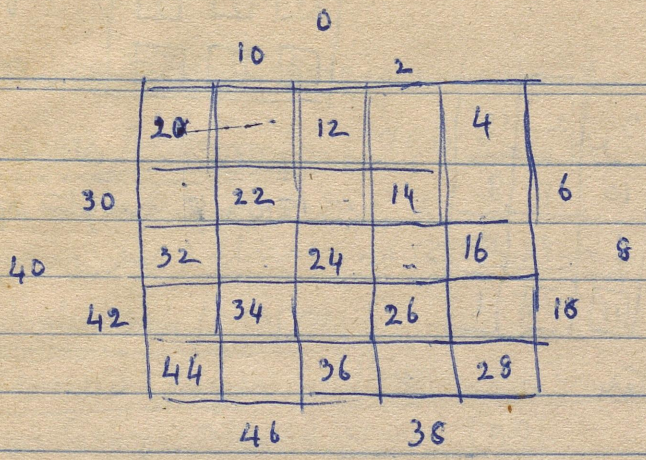


Fig 2 of Hoffmann. Peoples Encyclopaedia

20	46	12	38	4
6	22	48	14	30
32	8	24	40	16
18	34	0	26	42
44	10	36	2	28

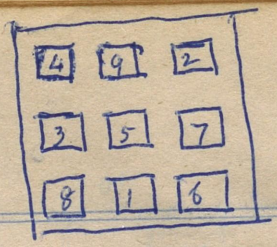
Fig 2A

The New Universal Library — Vol 8, p. 4721.

Pairs the 3-squares

4	9	2
3	5	7
8	1	6

and



Each small square with own edge

Dürer's square

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

1514 is the date of the square

& then the bordered 7-square

2	44	10	47	11	49	12
41	19	34	37	20	15	9
4	18	24	29	22	32	46
43	17	23	25	27	33	7
5	36	28	21	26	14	45
42	35	16	13	30	31	8
38	6	40	3	39	1	48

This magic const = $\frac{1}{2}n(n^2+1)$ - Number planets - Ref. to De la Hire

[Artistically the figures of magic squares are very elegant
Each no in a cell in the middle of a walk is made square painted
black This would be nice for the facing sheet of the

proposed book giving the Khajuraho square.

Encyclopaedia Brittannica, 1960 Edn – Vol. 14, pp. 625–628

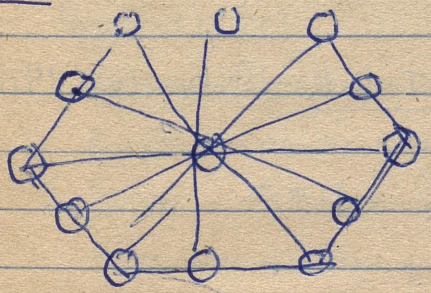
– Article by H. E. Du.; X

[Same as what I already at the Institute]

Bangalore Univ. Library (9/3/77)

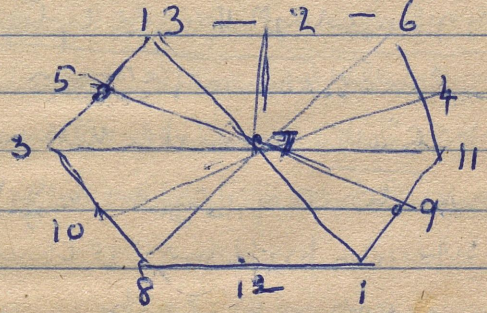
(1) Mathematical Puzzles and Perplexities by Claude Bertswistle – London George Allen & Unwin, 1971

p. 29



Fix nos 1 to 13 in the following space such that sum in any st. line is the same.

Soln p. 188 :–



p. 46. Find three nos. whose sum is 19, product is 144, and sum of squares is 149.

$a+b+c=19$, $abc=144$, $a^2+b^2+c^2=149$

from $(a+b+c)^2=361$, finds $ab+bc+ca=106$, using $a^2+b^2+c^2$

Hence eqn in x is $x^3 - 19x^2 + 106x - 144 = 0$

ie $(x-2)(x-8)(x-9) = 0$.

ie $a = 2, b = 8, c = 9$.

pp. 154 - 156 _ Something about latin squares - Little about magic squares

Some kind of an interesting book - References. p. 201

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(2) Mathematics & Logic by Kac & Ulam - Penguin, 1971 - Abridged Edition

(3) The magic of numbers by E. T. Bell - McGraw Hill, 1946

(4) History of Elementary Mathematics by Cajori

p. 221 - Something about magic squares

Ref to (i) Günther, Vermischte Untersuchungen Ch. IV

(ii) Article on magic squares in Johnson's Universal Cyclopaedia

p. 221 - Interesting problem of "15 Christians & 15 Turks - Other interesting verses on p. 222 also

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(5) De Morgan's Budget of Paradoxes

(*) Attended today a reception to G.N.R at the Microbiology Deptt on his election as an F.R.S – I wish I had said something at the meeting about Fermat's last theorem.

31/3/77 At Institute *

(1) Sc. American – Vol 235, No. 1, July 1976.

Math. Games – 71077345 – Inverter it spells SHLLOIL.

A mathematically interesting story: – 337 Arabs & 337 Israelis battle over a square property 8,424 metres on a side. Naturally we sum the squares of 337 & 8,424 and get 710777345. Is there another way to obtain this as the sum of two squares? Yes there is just one additional way: $5,324^2 + 6,537^2$. A paper back dealing with such inversions has been published: Calcu/Lette by Dan Steinbrocker (Pyramid Publications, 1975).

1/6 (2) Sc. Amer – Vol. 235, No. 2, August, 1976

p. 90 – The curvature of space in a finite universe by J. J. Callahan – A map is a representation of a curved surface. The map of the universe is the theory of relativity. – Quite an interesting article

Math. Games, p. 103 – talks of "Free planting puzzles" & refers to Dudeney's work & a lot of work on the subject – Refers to his own book "Mathematical Carnival"

(3) Ibid – No. 4, Oct. 76 (where is Sept. 76?)

Math. Games. p. 131 – Combinatorial problems, some old, some new and all newly attacked by computer.

(4) Ibid – No. 5, November 1976.

p. 48 - The Confinement of quarks by Yoichiro Nambu. - How is it that these particles that explain so much about other particles have not been isolated?

- Extremely interesting article giving up to date information on quarks, Charm, Color, flavour, gluons - ultraviolet freedom & infrared slavery - String model and bag model to explain quark confinement

p. 132 - Math. Games in which DM (Dr Matrix) is revealed as the guru of PM (Pentagonal Meditation) (!) - Very funny article - Describing wretched life in Calcutta - all sorts of rubbish.

(5) Universitas - Mai 1976

p. 533 - Neue Elementarteilchen mit ungewöhnlichen Eigenschaften by Prof. Dr. Gerd Buschhorn, München - Talks about quarks, Colorek - Article in German. - An English translation of this article is in Universitas Vol. 18, Ausg. E, No. 2, p. 163 - Small article starting with explanation of the (J or Psi) particle discovered as a particularly heavy elem. particle which has a lifetime about a thousand times as long as could be expected according to previous experience & theoretical conceptions - Cdis Color - quark a Charm - quark models to explain J or Psi.

(6) The Sciences - Vol. 16, No. 5, Sept/Oct. 1976

p. 17 - The shape of Change by James Gorman - The complex forms of Catastrophe theory - cusps, butterflies, hyperbolic umbilics - may help

to scientists understand sudden change - Very interesting article - References to

(1) Sc. Amer. April, 1976 - Article by E.C. Zeeman

(2) Structural Stability & Morphogenesis by Thom (W.A. Benjamin, 1975)

(3) Annals of the N.Y. Academy of Science, Vol. 231, 1974.

(4) New Scientist, 15/7/1976.

Reports applies to biology, civil engineering - based on topology.

(7) Scientia Sinica - October, 1976 - Number mourning with deepest grief the passing away of the great leader and teacher Chairman Mao Tsetung..

pp. 583 - 616 - Number of articles about Mao Tsetung & related topics about the cultural revolution etc

25/4/77 At Institute

New Books Section

(1) Intro to Linear Algebra by Irene Farkas & Miklós Farkas - Adam Hilger

A first year course for Physics & Engg. Students - Chaps Elementary Vector algebra - Complex Algebra - Matrix Algebra Systems of linear eqns - Linear & the Euclidean space - Linear operators

Quite an elementary Book

(2) Banach Algebra techniques in operator theory by Ronald C. Douglas – Acad. Press & Pure & Appl. Math. Series of Monographs – 1972

Chaps – Banach spaces – Banach algebras Geometry of Hilbert space – Operators on Hilbert space & C^*-algebras – Compact operators, Fredholm operators and Index theory – The Hardy spaces (new space?) – Toeplitz operators

(3) Conceptual foundations of quantum Mechanics by Barnard d'Espagnat

W. A. Benjamin Inc – Adv. Bk. Programme – 2nd Edn, 1976

– Philosophical, ~~but~~ and quite interesting

(4) Calculus – A modeling approach by Marvin L. Bittinger

Addison Wesley Pub. Co, 1976

Intuitive approach being intn to Calculus as applied to business, behavioral sciences, social sciences and biology – Prerequisite for course is Intermediate Algebra

– Leibniz (1646–1716) & Newton (1642–1727) are both credited with the invention of the calculus though each made the invention independent of the other. Newton used $\dot{y}$ for dy/dx

Quite an elementary book

(5) Partial Diff. Eqns by E. T. Copson (Paper Back) – Cambridge Univ. Press, 1975

Preface – This book is written in memory of my father-in-law the late Prof Sir Edmund Whittaker F.R.S. Today is the 100th anniversary of his birth (~~24/10~~ 24/10/73)

(6) Elements of Functional Analysis by J. J. Maddox. – © C.U.P. 1970.

Preface – It is my hope that the book may provide a really introductory non-trivial course on fnl. analysis – of the several excellent textbooks the master work of Banach "Théorie des Opérations Linéaires" (1932) must stand first, a remarkable germinal work

Contents. Basic set theory & analysis – Metric & Topological spaces – Linear & linear metric spaces – Normed linear spaces – Banach algebras – Hilbert space – Matrix transformations in sequence spaces

(7) Mathematics: a foundation for decisions by Growney, Fields, Robinson & Tabor – Addison-Wesley 1976 – Elementary analytical geometry & Calculus & matrix algebra.

(8) Differentiable germs and catastrophes by M. Brücker (Translated by L. Lander) – C.U.P. – 1975 & 76 – Lond. Math. Soc. Lecture Note Series No. 17.

About on Thom's Catastrophe theory

Foreword – Germs with constant rank – Regular values – construction of differentiable maps – Germs & jets – Symmetric germs – Mappings of the plane into the plane – Boardman – Thom singularities – The quadratic differential – Finitely determined germs – Some elementary algebraic geometry – Tougeron's theory – The universal unfolding of a singularity – The seven elementary Catastrophes – Proof of the main theorem on universal unfoldings – Pictures of the seven elementary Catastrophes –

Further reading – Literature on Catastrophe theory – Index of Symbols

– On p. 144 onwards are given many pictures of very interesting shapes & types and on p. 170 a picture of "a cat catches the mouse"!

– Ref. to A global dynamical scheme for vertebrate embryology by René Thom

– All told 56 references in Bibliography of which 1–15 are works by René Thom – 16 to 23 by E. C. Zeeman & 23–56 miscellaneous

(9) Integral transforms in App. Maths by John Mills by C.U.P with facing sheet having the picture

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(p)e^{px}\,dp.$$

Chapo – Integral transform pairs – Laplace transform (giving application) –

Fourier transforms – Hankel transforms – Finite Fourier transforms – meant for senior & graduate students in engineering & applied sciences

(10) The use of integral transforms by Ian N. Sneddon – (TMH) Tata Mc Graw-Hill Publishing Co, New Delhi, 1974 (Paper Back) – Like (9) but more comprehensive including maddelton Mellin, Kontorovich-Lebedev, Mehler-Fock transforms – Generalised functions & tables of the several transforms

(11) Mathematical methods for the Physical Sciences by K.F. Riley C.U.P – Paper back, 1974

Preliminary Calculus – Vector algebra – Calculus of vectors – Vector Operators – ord. diff eqns – Series solns of diff eqns – Integration methods – Fourier methods – Partial Diff. Eqns – Separation of variables – Numerical methods Calculus of variations – General eigenvalue problem – Matrices – Cartesian tensors, Complex variables – Text book for physics & Eng. students

(12) Groups of automorphisms of algebraic systems by B.I. Plotkin Wolters – Noordhoff Series of monographs and textbooks on pure and applied mathematics

The front cover has some Indian Sanskrit numerals & pictures of Rajasthani paintings – The first general and comprehensive

account of automorphisms

Part I - Three categories of abstract properties using novel ideas in math. logic & model theory - automorphic group of vector spaces and linear groups

Part II - Automorphic group of groups - Relns between a group & its normal sub-groups - abstract properties of linear groups.

(13) Same series as above by

Normed algebras by M. A. Naimark (translation by Dr. F. Boroms from Russian)

Quite an advanced book - The translation is dedicated to By. Dr. Mark Aronovich Naimark by L. F. B

Chapter I - Basic ideas from topology & functional analysis - 6 sections (1-6)

" II - Fundamental concepts and postulates in the theory of normed algebras - Contains 4 sections (7-10)

" III - Commutative normed algebras - 6 sections (11-16)

" IV - Representations of symmetric algebras - 6 sects (17-22)

" V - Some special algebras - 4 sects (23-26)

" VI - Group algebras - 6 sections (27-32)

" VII - Algebra of operators in Hilbert space - 7 sects (33-39)

" VIII - Decomposition of an algebra of operators into irreducible algebras

3 appendices - Colossal literature in alphabetic order of authors pp. 532 - 574.

Quite a wonderful book.

(14) Symmetry discovered by Joe Rosen (Zel Aviv Univ) - C.U.P, 1975

Foreword - Preface - Symmetry : what, where, how - Language of symmetry - Geometric symmetry - Other symmetries and approximate symmetry - Symmetry in nature - Uses of symmetry in science - Conclusion - Bibliography

Dedicated to the memories of Emmy Noether & Hermann Weyl

Bibliography contains 11 + 2 + 3 + 31 + 7 + 1 + 13 + 7 + 10 + 14 + 14 + 10 + 2 + 14 + 9 + 1 + 13 = 157 under the headings

Symmetry in art; Color symmetry; Conservation laws; Cosmology, astronomy & astrophysics - Crystals & Crystallography - Pauli exclusion Principle - Geometric symmetry - Group Theory - Symmetry in living things - Elementary particles - Symmetry in physics (Chemistry) - Polygons and polyhedrons - Reflection - Relativity - Symmetry in general - Time - Large no. of references to articles in Scientific American

- Funny foreword & Funny Preface - contains a lot of humorous references from A.A. Milne's Winnie-the-Pooh

- Quite an engaging book - Should be read at leisure.

(15) Problems of Life Research by W. Blasius — Springer-Verlag

Translated from German — Supposed to be a remarkable book. — 5 chapters.

(16) Abstract Set Theory by A. A. Fraenkel — North-Holland, 1976.

Fourth Revised Edition — Fraenkel was "Late Prof. of Maths in the Hebrew Univ. of Jerusalem" — quite a famous book

Defends Cantor's work —

(17) Principles of cosmology and gravitation by Michael Berry — C. U. P. 1976.

Supposed to be a textbook for a course on the subject — quite an [It would be interesting if one could try & write an alternative book of a more advanced type] interesting attempt by the author

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13/5/77 — Inst. World Culture

old issue of Eng. Brit of 1768 [Newton 1642 – 1727]

(1) Newton: In the early part of 1665 he discovered what is now known as the binomial theorem, and a little later came the elements of the diff. calculus which he called fluxions. In May of the following year he writes "I had entrance into the inverse method of fluxions (in modern terms the principles of the integral calculus and the method of calculating areas of curves or vols of solids) . . . All this was in the two years 1665 and 1666 for in those years, I was in the prime of my age

- His statement "Philosophy is such an impertinently litigious lady that a man had as good be engaged in law suits as have to do with her".

It is now generally admitted that Leibniz invented the Calculus later than, and independently of, Newton.

Article on Calculus

In algebra operations of addn, subtraction, multiplication, division, raising to powers, extractions of roots are the only ones - In calculus another important operation is stressed, viz. that of taking a limit (eg. from plane geometry). Two limits of vain derivative & definite integral.

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(2) Living biographies of great scientists by Henry Thomas and Dana Lee Thomas.

Re Newton: He had few students in his classes. When he tried to teach his latest discovery, the infinitesimal calculus, his class shuddered at the novelty and the complexity of the subject and stayed away from the formidable individual who had "foisted it upon the world". His fellow teachers were amazed at the facility of his computations - for the evaluation of infinity. By means of this or secret method" he had computed the area of a hyperbola to "two hundred and fifty figures". But he hadn't bothered to make his formula public -- Mathematics was but a game for his personal amusement and not an instrument

for practical use.

pp. 63–64 – Controversy with Leibniz – quite interesting

It had come to the ears of the Royal Academy, the learned Society of which Newton was now President, that the aggressive German philosopher, Leibniz, was claiming the sole credit for the invention of the Calculus. Newton's colleagues at the Royal Academy were outraged at the thought that a "foreigner" was trying to appropriate the discovery of a British mind – for it was Newton, they believed, who had first acquainted Leibniz with the possibilities of Calculus – "a method which Leibniz has later pretended, to be sure, but had never invented"

The members of the Royal Society took up the cudgels in behalf of Newton & England. The German savants, on the other hand, were equally vehement in their defense of Leibnitz & Germany. They lampooned the British as no scientists at all at all but as mere pseudo-scientists. "The British proclaim their discovery of an elephant on the moon when all they see is a fly on the ever glitter telescope"

Back & forth rages the international quarrel as to the priority of the invention of the Calculus. At first Newton tries to keep out of the quarrel. But finally, when even the British Knighthood been drawn into it, Newton undertook to prepare a defense of ——. But the controversy was as inconclusive as it was violent. Leibniz went to his eternal rest & Newton returned to his backgammon, and the world accepted the calculus with a grateful divided

not so much to the superiority of an Englishman or a German as to the genius of the human mind

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"If I have seen further, it is by standing on the shoulders of giants"

Nature & Nature's laws lay hid in night.
God said "Let Newton be!" — and all was light

16/5/77 — At Institute (New Books Section)

(1) Black holes, quasars & the Universe by Shipman — Houghton Mifflin Co Boston, 1976.
— Quite a beautiful elementary book

(2) Science & the Western domination by Rudi Mendelssohn — Thames & Hudson London, 1976

p. 18 — Remarks about Eastern regard for achievement of the individual by Eastern more sparingly except in case of great religious leaders or philosophers

p. 85 — In addition to fundamental contributions in mechanics & optics Newton also invented a basically new mathematical method; differential & integral Calculus. All these achievements were compressed into a time interval of less than eighteen months when Newton was twenty three. — Astonishing youthfulness of discovery

is a characteristic of Western Science.

(3) – (5) Mathematical Papers of Isaac Newton Vols 4, 5, 6.

Vol. 4 – 1674 – 1684
Vol. 5 – 1683 – 1684
Vol. 6 – 1684 – 1691

Edited by D. T. Whiteside
3 huge Vols (C. U. P)

A – Appears just what I wanted – I wonder where are Volumes 1 to 3.

Vol. IV (£. 30.00) – C. U. P. 1971.) (Also pp 1 – 600 & odd).

pp. 74 – 115 – Problems in elementary number theory – Some extracts might be made.

pp. 115 – 187 – Plane & Sph. Trigonometry.
Also geometry on other pages – Also geometry of curved lines.

~~Vol V~~ – on the reverse of the cover we find tributes to the edition from various sources

Vol V. pp 1 – 624

Newton's

pp. 501 – 516 Lectures on Algebra, 1673-83. (This professional lectures during the period).

pp. 517 – 621 – First book of Universal arithmetic. (What is this?)

Vol VI – pp. 1 – 609 – Mostly mechanical work.

Footnote on p. 523 gives an idea of Newton's notational language.

[$2T/SZ = \cos 2x$ · and $ZH/SZ = \cos 2(a+x)$ whence

omnia $2T$ / totidem SZ $\left(= \int_0^{2\pi} \cos 2x \cdot dx / 2\pi \right)$

$=$ omnia $2H$ / totidem SZ $\left(= \int_0^{2\pi} \cos 2(a+x)\, dx / 2\pi \right) = 0.$

Also omnia $(SZ^2 \mp HZ \times 2T)$ / totidem SZ^2

$$= \int_0^{2\pi} \left(1 \mp \cos 2(a+x) \cos 2x \cdot dx\right) / 2\pi$$

$$= \int_0^{2\pi} \left(1 \mp \tfrac{1}{2}(\cos 2a - \cos(2a+4x))\right) dx / 2\pi = 1 \mp \tfrac{1}{2}\cos 2a$$]

Saw a book on Liquid Crystals & Plastic Crystals – Vol. 2 – Gray & Winsor, 1974

Found only one reference to Chandrasekhar & K.N. Srinivasa Rao in Vol. 1 on "Optical properties of Liquid Crystals"

Vols I, II & III of Newton's Math. Papers ed. edited by D. T. Whiteside. (Ref. Section of Library).

Vol I – 1664 – 1666.

Preface – First para contains material which I might use.

pp xvi – xvii – Some useful material

p. xxxv – Bicentenary of Newton's death – anchor of his papers on 13-14/7/1936.

p. xxxvii – Foreword to Vol I

Vol II – 1667 – 1670
Vol III – 1670 – 1673
} Both Volumes

(2) Unpublished scientific papers of Isaac Newton by A. W. Turnbull, C. U. P, 1962.

– Explains Newton's notation like $\square$ for $\int dx$ – Leibniz – Looks as if Barrow was in no way inferior to Newton.

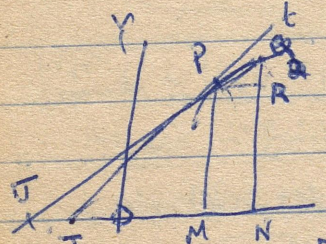
(3) Ency. Brit, 1973 Edn, Vol. 16, p. 418 on Newton
— Nothing much about Mathematical part.

17/5/77 Bangalore Univ Lib. C.C.B

(1) Newton's Principia - Sections I, II & III with notes and illustrations by Percival Frost (1863), Macmillan
Section I - on the method of prime & ultimate ratios.
Section II - Centripetal forces
Section III - on the motion of bodies in Conic Sections, under the action of forces tending to a focus
[It is not clear whether what all the author has given is a faithful reproduction from the Principia]
(2) Makers of Mathematics by Alfred Hooper (Faber
p. 274 - Newton
The lines "Nature & Nature's laws - - " are due to Pope.
Laplace says: Newton's ~~pace~~ work as assured of "a pre-eminence above all other productions of human genius".
Born on Christmas Day ~~1642~~ 1642 (the year of Galileo's death) - ~~His~~ He had perfect vision till his death at 85 years of age - his hair turned prematurely

From (1) example of Newton's geometric method used in Principia

(i) To find subtangent in common parabola



$\frac{PM^2}{2N}$ $PM^2 : QN^2 :: OM : ON$

$\therefore QN^2 - PM^2 : PM^2 :: ON - OM : OM$

and $QN - PM : PM :: ON - OM : MT$

[for $PM : MV :: QR : PR :: QN - PM : ON - OM$

Now as Q approaches P, the limiting position of QPV is that of tangent at P viz tPT, and $PM : MT$ is the limiting ratio of $QN - PM : ON - OM$]

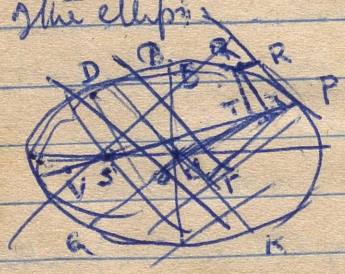
$QN + PM : PM :: 2 : 1$ ultimately

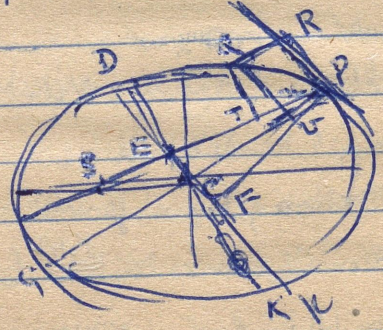
$\therefore QN^2 - PM^2 : PM^2 :: 2(ON - OM) : MT$

$\therefore MT = 2OM$

(ii) If a body is revolving in an ellipse, to find the law of force tending to a focus

The ellipse





Let S be the focus to which the force tends. P the posn of the body at any time, PCG and DCK conjugate diameters, Q a pt near P, QT & PF $\perp^{rs}$ on SP, DCK from Q, P resp^{ly}, PR a tangent at P, QR $\parallel^{l}$ to SP, Qxv $\parallel^{l}$ to PR meeting SP in x and PC in v, and let SP, DCK intersect in E

Then $F = \frac{2h^2}{SP^2} \cdot \frac{QR}{QT^2}$, ultimately when QT is indefinitely diminished

But by similar Δ^s QTx, PFE

$$\frac{QT^2}{Qx^2} = \frac{PF^2}{PE^2} = \frac{PF^2}{AC^2} = \frac{BC^2}{CD^2}$$

Now $\frac{Qv^2}{Pv \cdot vG} = \frac{CD^2}{CP^2}$ by the properties of the ellipse.

and $\frac{Px}{QR} = \frac{Pv}{Px} = \frac{CP}{PE}$ by similar Δ^s

$$\therefore \frac{Qv^2}{QR \cdot vG} = \frac{CD^2}{CP \cdot AC}$$

and $vG = 2CP$, $Qx = Qv$ ultimately

$$\therefore \frac{QT^2}{QR^2} = \frac{2BC^2}{AC^2} = L \text{ ultimately} \quad (L = \text{latus-rectum of ellipse})$$

$$\therefore F = \frac{2h^2}{L} \cdot \frac{1}{SP^2} \propto \frac{1}{SP^2}$$

[compare with method using Calculus]

grey when he was thirty – His interests were overwhelmingly intellectual – He never married, he cared nothing for dress – until the last few years of his life he was able to endure the intense mental & physical strain involved in working on the most profound subjects for as many as 18 or 19 hours on end. Like Archimedes he possessed the power of concentration to a remarkable degree – Meals ignored & getting up in the morning would forget to dress – Work Principia in 1685 of which Cajori says "envelops the name of Newton in a halo of perpetual glory" – Nervous breakdown & recovery – story of Brachistochrone problem after recovery – Incident of having a horse by the bridle & taking it up a hill, later finding only the bridle in his hand with no horse attached to it – entertaining guests, going to bring wine, thereby guests waiting & then going in search of him – Smeaton's anger with criticisms – Halley's soothing words made him write Principia Part III of which Halley corrected all the proofs – Fluxions discovered in 1665–66, matter left with Isaac Barrow & published only in 1693; this led to controversy with Leibniz in 1669

Fermat's geometrical method of finding derivatives in only a few particular cases for eg. $y = 3x - x^3$. Newton drew up a system of rules that applied to any function. When Newton said that he "has stood on the shoulders of giants" he was doubtless referring to men like Descartes, Fermat, Wallis & Barrow as far as his development of Calculus was concerned, and to scientists like Galileo & Kepler in connection with his work on gravitation.

Not until 1736 was Newton's Method of Fluxions made known to the general public in a translation of his manuscript made by J. Colson. From this book we find that Newton conceived of a line as being generated by a moving point. So he thought of a line as a ~~fluent~~ being a "flowing quantity" and called it a fluent. The velocity with which the ~~line~~ line "flowed" was called by him its fluxion. The "infinitely small length" by which a fluent increased in an "infinitely small time" was called by Newton the moment of the fluent. He designated the fluxion of x by the symbol $\dot{x}$, and when the fluxion $\dot{x}$ was "infinitely small" he would represent the moment of the fluent by the symbol $\dot{x}0$. This if he were dealing with two fluents x & y which were connected by the equation $y = 3x - x^2$, he would substitute $x + \dot{x}0$ for x & $y + \dot{y}0$ for y in the eqn $3x - x^2 - y = 0$. This would give him

$$3x + 3\dot{x}0 - x^2 - 2x(\dot{x}0) - (\dot{x}0)^2 - y - \dot{y}0 = 0$$

Ignoring $(\dot{x}0)^2$ as negligible & subtracting the original eqn $3x - x^2 - y = 0$ he would get

$$3\dot{x}0 - 2x(\dot{x}0) - \dot{y}0 = 0$$

$$\text{So } \frac{\dot{y}0}{\dot{x}0} = 3 - 2x$$

[average rate of change $\frac{\delta}{n}$ = $\frac{\text{difce in lengths of ordinates from } x \text{ to } x+h}{h}$. This was given by Fermat] - [I don't quite follow what is wonderful in

Newton's method] In 1669, Barrow sent Newton's work on fluxions and described it to Collins as work of "unparalleled genius", but Newton did not take Barrow's advice & failed to publish his papers. Had he done so the controversy with Leibniz would not have arisen

Newton made many geometric methods in all his proofs in the Principia since he realised that his fluxional calculus would be unknown to other mathematicians & thus might lead them to dispute results which were themselves opposed to many of the theories prevalent at the time. Another factor that probably weighed with Newton in reaching this decision to employ the familiar Greek geometry was that the calculus has not been fully developed when he wrote the Principia & consequently was not then as superior to Greek Geometry as it later became — Voltaire's views in 1736 made Principia immediately accepted — In 1691 the demand for the work was so great that it was impossible to purchase a copy of the book.

Controversy with Leibniz (a long account given) — Bishop Berkeley's criticism in 1734 in Analyst: "What are these fluxions? The velocity of evanescent increments. And what are these same evanescent increments? They are neither finite quantities nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities" — Maclaurin's work on fluxions. Later rigorous notions of limits

(3)

(3) A source book in Mathematics by D. E. Smith

p. 613 – Newton on Fluxions

(i) Interpolation by means of Infinite series: From Wallis' Algebra.

He doth therein, not only give us many such Approximations ... but he lays down general Rules and Methods ... And gives instances how those Infinite or Indeterminate Progressions may be accommodated, to the Rectifying of Curve lines ..., Squaring of Curve-lined Figures; finding the lengths of Arches ...

(ii) Newton's method of fluxions – from Quadratura Curvarum published with ..... Newton's Opticks, published 1704.

Therefore, considering that quantities, which increase in equal times, and by increasing are generated, become greater or less according to the greater or less velocity with which they increase and are generated; I sought a method of determining quantities from the velocities of the motions, or [of the] increments, with which they are generated; and calling those velocities of the motions or [of the] increments, fluxions, and the generated quantities fluents. I fell by degrees, in the years 1665 and 1666, upon the method of fluxions, which I have made use of here in the quadrature of curves.

Fluxions are very nearly as the augments of the fluents generated in equal, but very small, particles of time; and, to speak accurately, they are in the first ratio of the nascent augments, but they may be expounded by

any lines which are proportional to them. — —

It amounts to the same thing if the fluxions are taken in the ultimate ratio of the evanescent parts [Newton here gives some examples. He makes the tangent coincide with the limiting position of the secant by making the ordinate of the second point of intersection of the secant with the curve move up into coincidence with that of the first.

— — — —

Let the quantity x flow uniformly, and let it be proposed to find the fluxion of x^n.

In the time that the quantity x, by flowing, becomes $x + 0$, the quantity x^n will become $x+0]^n$, that is by the method of infinite series

$$x^n + n0x^{n-1} + \frac{n^2-n}{2}00x^{n-2} + \text{etc}$$

And the augments 0 and $n0x^{n-1} + \frac{n^2-n}{2}00x^{n-2} + \text{etc}$, are to one another as

$$1 \text{ and } nx^{n-1} + \frac{n^2-n}{2}0x^{n-2} + \text{etc.}$$

Now let these augments vanish, and their ultimate ratio will be as

$$1 \text{ to } nx^{n-1}.$$

From the fluxions to find the fluents is a much more difficult problem

— — indeterminate quantities z, y, x, v and their fluxions $\dot{z}, \dot{y}, \dot{x}, \dot{v}$, —

second fluxions $\ddot{z}$, — , third $\dddot{z}$, — fourth $\ddddot{z}$ —

The quantities z, y, x, v, may be considered fluxions of $\dot{z}, \dot{y}, \dot{x}, \dot{v}$, & these again as fluxions of $\ddot{z}, \ddot{y}, \ddot{x}, \ddot{v}$, — therefore $\dddot{z}, \dddot{y}, \dddot{x}, \dddot{v}$

$\ddot{z}, \dot{z}, z, \dot{z}, \ddot{z}$, — etc denominate a series of quantities whereof every one that follows is the fluxion of the preceding, and every one that goes before is a flowing quantity having the succeeding one as its fluxion.

— — —

And it is to be remembered that any preceding quantity in this series is as the area of a curvilinear figure of which the succeeding quantity is the rectangular ordinate, and [of which] the abscissa is z, . . .

Prop. I. Prob. I — An equation being given involving any number of flowing quantities, to find the fluxions.

Soln. Let every term of the equation be multiplied by the index of the power of every flowing quantity that it involves, and in every multiplication let a side [or root] of the power be changed into its fluxion, and the aggregate of all the products with their proper signs, will be the new equation

Explication $x^3 - xy^2 + a^2 z - b^3 = 0$, ($a, b, c, d$, — invariable quantities)

let the terms be first multiplied by the indices of the powers of x, and in every multiplication, for the root or x of one dimension, write $\dot{x}$, and the sum of the terms will be

$$3\dot{x}x^2 - \dot{x}y^2$$

Let the same be done in y, and it will produce

$$-2xy\dot{y}$$

Let the same be done in z, and it will produce $aa\dot{z}$ & sum placed $=0$ gives

$$3\dot{x}x^2 - \dot{x}y^2 - 2xy\dot{y} + aa\dot{z} = 0. \quad \text{—(1)}$$

I say that the relation of the fluxions is defined by this eqn

Demonstration. Calls $0\dot{z}$, $0\dot{y}$, $0\dot{x}$ the moments or the momentaneous synchronal increments of z, y, x; after a moment of time these become

$$z + 0\dot{z},\ y + 0\dot{y},\ x + 0\dot{x}.$$

Substituting these in the given eqn, simplifying & subtracting and dimensioning the quantity 0 infinitely & neglecting the terms which vanish we get (1)

(iii) The method of prime & ultimate ratios (~~From Quadratura Curvarum~~, 1704)

[From Principia 1714 Edition]

[A lot of nonsense containing hazy sentences suggesting that Newton must have been a fool to write all this rubbish. The following extracts have however some meaning & show that he had, however, a clear idea of the notion of limits]

There is no ratio of evanescent quantities . . . . By ultimate velocity I understand that with which the body is moved, neither before it arrives at the ultimate position and the motion ceases, nor thereafter, but just when it arrives, that is, that every velocity with which the body arrives at the

ultimate position and with which the motion ceases. Similarly for nascent or evanescent quantities — And the prime and ultimate amount is to be [that] with which they begin & cease (if you will, to increase & decrease). There exists a limit which the velocity may attain at the end of the motion, but [which it may] not pass. This is the ultimate velocity —

The ultimate ratios in which these quantities vanish, are not really the ratios of ultimate quantities, but the limits towards which the ratios of quantities, decreasing without limit, always approach; and to which they can come nearer than any given difference, but which they can never pass nor attain before the quantities are diminished indefinitely

[Last para appears the most sensible]

(IV) Bishop Berkeley's criticism in Analyst of 1734 of Fluxions.

Extracts only:

In finding fluxion of x^n, the last step was the increments are to one another as

$$1 \text{ to } nx^{n-1} + \frac{nn-n}{2}ox^{n-2} + \text{&c.}$$

Let now the increments vanish, and their last proportion will be 1 to nx^{n-1}. But it should seem that this reasoning is not fair or conclusive. For when it is said, let the increments vanish or let the increments be nothing or let there be no increments, the former supposition that the increments

were something or that there were increments is destroyed and yet a consequence of that supposition is an expression for by virtue thereof, is retained. Which, by the foregoing Lemma, is a false way of reasoning. Evidently when we suppose the increments to vanish, we must suppose their proportions, their expressions and everything else else derived from the supposition of their existence, to vanish with them.

The great author of the method of fluxions felt this difficulty, and therefore he gave in to those nice abstractions and geometrical metaphysics without which he saw nothing could be done on the received principles --- He used fluxions, like the scaffolding of a building, as things to be laid aside or got rid of as soon as finite lines were found proportional to them. But then these finite exponents were found proportional to them. But then these finite exponents are found by the help of fluxions. Whatever therefore is sought by such exponents and proportions is to be associated to fluxions: which must therefore be previously understood. And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities? --- Some queries.

Query 1. Is there any need at all of considering quantities either infinitely great or infinitely small?

Query 4 - 95 The method of fluxions a scientific method?

Query 31 - When there are no increments, can there be any ratio of increments? whether to talk of their proportions is not to talk nonsense?

Query 63 - Whether such mathematicians as cry out against mysteries have ever examined their own principles?

(4) A Source Book in Mathematics 1200 - 1800 edited by D. J. Struik.

Chapter II - Algebra.

A. 93-99 Newton - The roots of an equation (pp. 93-99)

Principia published 1687, his most famous work. Most of his other works were published many years after they were written, e.g. his Arithmetica Universalis dealing with Algebra. The Arithmetica contains investigations on the roots, including search for divisors of polynomials, common divisors of two polynomials, & a formulation of Descartes' Rule of signs. It finds upper limits of roots of an equation.

Chapter III - Geometry

Newton - Cubic curves pp. 168-170

This book Enumeratio linearum tertii ordinis was first

(9)

published in 1704, but it was written much earlier, perhaps
circa 1676. It represents a definite step ahead in the development of
analytic & algebraic geometry. Newton undertook classification of curves of the
third degree & he freely used (a novelty at the time) positive &
negative values of the coordinates. He was the first to study the
character of these curves by means of their points of intersection with
a straight line. Certain affine properties of conics like asymptotes &
diameters are extended to cubic curves by studying the curve
with respect to oblique axes — Hyperbolic & parabolic branches
and their directions — Reduction of all curves of the second genus
to four cases of equations — on latera recta & transversa — Names of
the types of curves, viz.: circumscribed hyperbola, circumscribed hyperbola,
ambigenous hyperbola, converging & diverging hyperbolas, contrary branched
hyperbolas, cruciform hyperbola, nodate, cusped, punctate and pure hyperbolas — Similarly
parabolas of the above types — Enumeration of curves, of the nine
redundant hyperbolas, having no diameters and three asymptotes making
a triangle — Generation of curves by shadows i.e. Newton's
theorem that every curve of the third order can be obtained from a

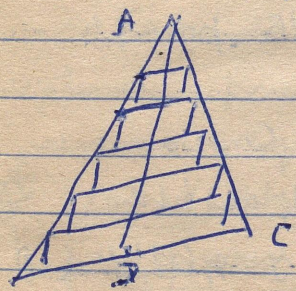
dividing parabola by central projection from one plane to another

Chapter IV – Analysis before Newton & Leibniz.

During the Middle Ages mathematical meditation on the infinitely great and infinitely small took the form of speculation – nature of the or incommensurable, the existence of the indivisible, & the potentially & the actually infinite – Men like Stevin, Kepler, Valerio, Galileo & later authors, although inspired by Archimedes, developed their own methods of integration (differentiation came somewhat later – [Mention here Bhaskara's differentiation & also Kerala mathematicians infinite series for $\arcsin x$ & $\arctan x$ as mentioned by K.V.] – The Renaissance mathematicians were mostly out for new results without laying stress on rigour.

(I) Stevin, Centre of gravity (1548–1620)

Finding C.G. of a triangle



Lines drawn $\parallel^{l}$ to BC & $\square^{rs}$ formed by drawing $\parallel^{l}$ to median AD. Each $\square^{m}$ has its C.G (using C.G = centre of symmetry) on AD & hence sum of all $\square^{ms}$ has also C.G on AD. Make the number of $\parallel^{l}$ to BC infinite. Hence C.G lies on AD. Simil. median thro' B, etc – Really an integration method

(2) Kepler (1571-1630) — Integration methods — with Tycho Brahe in Austria.

In announcing laws of Kepler, knows $\int_0^\varphi \sin\varphi\, d\varphi = 1 - \cos\varphi$ and attempts a solution of $x = e\sin x + m$ — Computation of volumes of solids obtained by the rotation of conics about a line in their plane using simple & direct Archimedean rigorous techniques.

(3) Galilei on infinites and infinitesimals (1564 – 1642)

Galilei never wrote a book on the purely mathematical aspects of his work, but his pupils Cavalieri & Torricelli gave the full elaboration to his ideas on problems relating to infinitesimals — Cantor's ideas found in Galilei's considerations — His works as Discorsi — 20 Vols, Barbera, Florence (1890-1909). — Proof of $v = at$ & $s = \frac{1}{2}at^2$ graphically in place of integration.

(4) Cavalieri (1598-1647) — Principle of Cavalieri — in his book "Geometria indivisibilibus continuorum" how to measure plane areas & solid volumes (in modern notation) This principle amounts to this:

Let two lines $x = a, x = b$ together with the continuous curves $y = f_1(x)$, $y = f_2(x)$ enclose an area A_1, and let the same lines together with the continuous curves $y = \phi_1(x)$, $y = \phi_2(x)$ enclose another area. Now if for all x, $a \le x \le b$

$$f_1(x) - f_2(x) = \varphi_2(x) - \varphi_1(x), \text{ then}$$

$$A_1 = \int_a^b [f_2(x) - f_1(x)]\,dx = A_2 = \int_a^b [\varphi_2(x) - \varphi_1(x)]\,dx$$

(ii) Cavilieri on integration — His method consisted in comparing the indivisibles of one figure with those of another. He however got the important result

$$\int_0^a x^n\,dx = \frac{a^{n+1}}{n+1} \quad (\underline{n} \text{ a +ve integer})$$

[He uses notation O.l instead of integration sign]

Archimedes had found this formula for $n = 2$ — By the time Cavilieri's work was published, the integral of x^n was known to Toricelli, Fermat, Roberval & other mathematicians, each finding the result in his own way & some generalising it to fractional and negative $\underline{n}$

Cavalieri's method of summation of lines into areas and of areas into volumes is paradoxical & looks funny, but his method was nevertheless equivalent to integration.

Fermat ~~Integration~~ (1601–1665).

(5) (i) Integration — Fermat generalised Cavalieri's method of finding $\int x^n\,dx$ for $\underline{n}$ fractional or negative & he, along (Done in 1640)

with Torricelli, succeeded in doing so by finding the area between an arc of the "hyperbola" $x^m y^n = k$ (m & n +ve integers) an ordinate and axis asymptote.

Divide the interval $0 \leq x < a$ into parts by the points $x_1 = a$, $x_2 = ar$, $x_3 = ar^2$, ... $r < 1$ which are separated by the intervals $l_1 = a(1-r)$, $l_2 = ar(1-r)$, $l_3 = ar^2(1-r)$, ... If $y = x^n$ ($n = p/q$, $p, q \geq 0$) is the equation "hyperbola" or "parabola", then the values of y corresponding to x_1, x_2, x_3 — are $y_1 = a^n$, $y_2 = a^n r^n$, $y_3 = a^n r^{2n}$, — Then the sum of the rectangles $l_1 x_1 + l_2 x_2 + l_3 x_3 + \ldots$ is

$$S = a(1-r)a^n + ar(1-r)a^n r^n + ar^2(1-r)a^n r^{2n} + \ldots$$
$$= (1-r)a^{n+1}\{1 + r^{n+1} + r^{2n+2} + \ldots\}$$
$$= \frac{1-r}{1-r^{n+1}}\, a^{n+1}.$$

When $r = s^q$ ($s < 1$) & $n \neq -1$, then

$$\int_0^a x^n\, dx = a^{n+1} \lim \frac{1-r}{1-r^{n+1}} = a^{n+1} \lim \frac{1-s^q}{1-s^{p+q}}$$
$$= \frac{q a^{n+1}}{p+q}$$

(i) on Maxima & Minima:

~~Fermat's method~~ (a)



To divide segment AC at E so that AE × EC may be a maximum
If AC = b, & a = one of the segments & the product for which maximum is to be found is $ba - a^2$. Let now $a+e$ be the first segment of b;
the second will be $b - a - e$ & product is $ba - a^2 + be - 2ae - e^2$;
this must be adequate with the preceding $ba - a^2$. Suppressing
common terms, $be = 2ae + e^2$. Suppressing e, $b = 2a$ is the solution.

[If AE = x, to find max. of $y = x(b-x) = bx - x^2$

$\frac{dy}{dx} = b - 2x$, $x = \frac{1}{2}b$ putting $\frac{dy}{dx} = 0$

$\frac{d^2y}{dx^2} = -2$, hence max.

so that Fermat's $e = dx$ in Leibniz's notation or o in Newton's]

(b) Similar method to find tangent of curves.

(c) C. of G. of paraboloid of revolution using same method

(6) Torricelli (1608–1647) successor of Galileo

$= \int_0^c 2\pi xy\,dx$ where $2xy = a^2/2$ giving $\pi a^2 c$.

(7) Roberval (1602–1675)

(a) the cycloid: — construction — constructing a tangent to it —

(c) $\int_0^{\pi} \sin\theta \cdot d\theta = \pi/2$ in a geometric way

– The Proof of the theorem that the volume of the solid generated by the cycloid as it revolves about its base line as an axis is equal to five-eighths of the volume of the circumscribed cylinder.

(b) on the lines of a quadrant of a circle – Prove geometrically the formulae:

(i) $$\int_{\varphi_0}^{\varphi_1} \sin^2\varphi \cdot d\varphi = -\int_{\varphi_0}^{\varphi_1} \sin\varphi \, d(\cos\varphi)$$

(ii) $$\int_{\varphi_0}^{\varphi_1} \sin^3\varphi \, d\varphi = -\int_{\varphi_1}^{\varphi_1} \sin^2\varphi \, d(\cos\varphi).$$

(8) Pascal (1623 - 1662)

(a) Partial integration

$$\int_{y=A}^{B} x \, dF(y) = x F(y)\Big]_{y=A}^{B} - \int_{y=A}^{B} F(y)\, dx$$

$$= \int_{x=C}^{D} F(y)\, dx$$

where $F(y) = \int_0^R f(y)\, dy$ & $x = 0, y = 0$ at A.

Putting $F(y) = y$,

$$\int_A^B x\, dy = \int_C^D y\, dx.$$

(9) Wallis (1616-1703).

Computation of π by interpolation — using analytical methods (came into vogue about 1650)

$$\int_0^1 \sqrt{1-x^2}\,dx = \pi/4$$

— use of B-functions & Γ-functions

(10) Barrow (1630-1677) (Isaac Barrow) — His most famous disciple was Isaac Newton who succeeded in the Lucasian Chair. $ds^2 = dx^2 + dy^2$; $\frac{d}{dx}\int^x y\,dx = y$ by geometrical methods — ie the fundamental theorem of the Calculus —

$$\int_0^{x_0} y\,\frac{dy}{dx}\,dx = \int_0^{y_0} y\,dy = \tfrac{1}{2}y_0^2,$$

$$\int_0^{x_0} y^2\,\frac{dy}{dx}\,dx = \int_0^{y_0} y^2\,dy = \tfrac{1}{3}y_0^3.$$

$$\int_0^{y_0} y\,dy \int_0^x y\,\frac{dy}{dx}\,dx = \int_0^{y_0} \tfrac{1}{2}y^3\,dy = \frac{1}{8}y_0^4.$$

(11) Huygens (1629-1695) — Evolutes & involutes — Search for accurate clocks — Cycloid as a tautochrone — many theorems on evolutes & involutes using geometrical methods.

Chapter V – Newton, Leibniz and their School.

The final discovery of the calculus required the assimilation of the geometric methods of Cavalieri & Barrow with the analytic methods of Descartes, Fermat & Wallis; it also required the understanding of the relation between the search for tangent constructions and quadratures. The fundamental step was taken by Isaac Newton (1642-1727) & Gottfried ~~Leibniz~~ Wilhelm Leibniz (1646-1716). Newton's "golden period" of discovery fell between 1664 & 1668 & Leibniz's from 1672 to 1676. In those years he discovered his form of the Calculus. Newton's discovery of the calculus thus came before that of ~~Leibniz~~ Leibniz, but Leibniz published it first (1684 & after). Newton, after hints in his Principia (1687), did not publish his calculus till 1704.

The development of Leibniz's calculus proceeded both through the work of Leibniz himself, the brothers Bernoulli, Euler and other continental mathematicians. Newton's calculus, the so-called theory of fluxions, had a much less spectacular progress in the works of Taylor, Maclaurin & other British mathematicians — Question of priority perhaps still alive.

Newton started to work on calculus in 1664 under Barrow at Cambridge & his work was presented in Analysis per aequationes (London 1711), Quadratura

Quadrature & Methodus Fluxionum (all translated by Whiteside in The mathematical works of Isaac Newton by Whiteside, 1964).
In 1676 Newton wrote two letters to Oldenburg concerning binomial series & fluxions, letters destined for Leibniz.
After sometime Newton's attention was drawn towards mechanics & astronomy resulting in the immortal work Principia.

(a) Newton in a letter to Collins in 1671 gives infinite series for $\tan^{-1}x$
[Kerala people 200 years earlier $\tan^{-1}x = x - x^3/3 + x^5/5 - x^7/7 + -$]
Also letting $x = y - y^2/2 + y^3/6 - y^4/24 + -$ from

$$y = x + x^2/2 + x^3/3 + x^4/4 + -$$

(b) Principia & ultimate ratios - given in Principia - theory of limits - gives 11 lemmas dealing with the subject & demonstrations are all in the ancient geometric way of the Greeks and are explained in a manner difficult for his contemporaries to understand. [perhaps even at the present day] for eg. some short Lemmas given here

Lemma V: All homologous sides of similar figures, whether curvilinear or rectilinear, are proportional and the areas are as the squares of the homologous sides

Lemma VII - The same things being supposed (as in Lemma VI), I say that the ultimate ratio of the arc, chord and tangent, any one

to any other, is the ratio of equality

Lemma X – The spaces which a body describes by any finite force urging it, whether that force be determined & immutable, or is continually augmented or continually diminished, are in the very beginning of the motion to each other as the squares of the times.

Lemma XI – The evanescent subtense of the angle of contact, in all curves which at the point of contact have a finite curvature, is ultimately as the square of the subtense of the coterminous arc

[eqn of a curve near a point can be written as $y = \alpha x^2 + \beta x^3 + \ldots$

(c) Genita and moments – In Principia differentials are introduced by the name of "moments" – "genita" was an approach to the conception of a "function".

Lemma II – The moment of any genitum is equal to the moments of each of the generating sides multiplied by the indices of the powers of those sides and by their coefficients continually.

Ex: If A, B, C, – be quantities in flux & of their moments or velocities of the mutations ∴ to them be called a, b, c, – the moment of $AB = aB + bA$, of ABC will be $aBC + bAC + cAB$, and in general of $A^{n/m}$ will be $\frac{n}{m} a A^{(n-m)/m}$, moment of A^3/B^2 for eg $= 3aA^2B^{-2} - 2bA^3B^{-3}$, and so on – Lemma is demonstrated.

In the text follows that famous Scholium which is different in the first and second editions of the Principia. In the first edition (1687) Newton mentions

the letters which he and "the very excellent G. W. Leibnizius" had exchanged ten years before, and praises him as ~~co~~ coinventor of the calculus. In the second Edition (1713) the Scholium mentions several men who had contributed to the invention but omits Leibniz. By this time the priority struggle between Newton & Leibniz was in full swing.

(d) Quadrature of curves – Introduces fluxions and fluents.

(i) Let the quantity x flow uniformly, to find the fluxion of x^n. Uses $x^n \to (x+0)^n$

To find fluents from fluxions – introduces $\ddot{z}, \dot{z}, z, \dot{z}$.

(ii) An eqn being given involving any number of flowing quantities, to find their fluxions. Uses Ex. $x^3 - xy^2 + a^2z - b^3 = 0$ [See earlier pages]

(iii) To find such curves as can be squared – Newton also shows interest in the convergence & divergence of series

(e) Method of fluxions – Vide Whiteside Vol I.

Later workers in Calculus.

(1) L'Hôpital (1661–1701). – a French nobleman & an amateur mathematician – gives the rule to evaluate 0/0.

(2) Jakob Bernoulli – Sequences & series

The collaboration of Leibniz with the Bernoulli brothers his earliest students led by 1700 to most of what we now call the elementary diff & int. Calculus along

with beginnings of diff. eqns.

(3) Johann Bernoulli – Integration

(4) Taylor (1685–1731) – The Taylor Series

(5) Berkeley – The Analyst.

Among the critics of Newton's theories (and of those of Leibniz) were Jonathan Swift (1667–1745) & George Berkeley (1685–1753) both deans in the Church of England in Ireland. Swift's criticism in Gulliver's travels (1726) & Berkeley's in Analyst – Lampooning of infinitesimals as ghosts of departed quantities – [See earlier pages] – Saying ~~[illegible]~~ first $0 \neq 0$ & then $0 = 0$ not allowed in Divinity!

(6) Maclaurin – on Series & extremes

(7) D'Alembert – on Limits

(8) Euler – Trigonometry – Famous book on Analysis – complex variable

(9) D'Alembert, Euler, Daniel Bernoulli – The vibrating string & its partial differential equation $C\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}$.

(10) (Daniel Bernoulli, son of Johann Bernoulli, founder of hydrodynamics) – Concept of function clarified in the nineteenth century by the work of Fourier, Cauchy, Dirichlet & Riemann

(10) Lambert – Irrationality of π

(11) Fagnano & Euler – Addition theorem on integrals

(12) Euler, Landen, Lagrange – Metaphysics of the Calculus

(13) Jakob & Johann Bernoulli – Brachistochrone

(14) Euler – Calculus of variations

Follows as a development of isoperimetric problems of the Bernoullis

(15) Lagrange – Calculus of Variations

(16) Monge – Metric curvatures of a curve surface

(1746 – 1818).

[Up to 1800 from Source Book]

[what about later on in 19th & 20th centuries

Find a few important land marks]

Whiteside (C.U. Lib) 28/5/77

Whiteside Vol I · (1664 – 1666) – C.U.P. 1967

(pp. 1–585)

Preface – His contributions to math. thought have always been less immediately understood – the depth & complexity of his scientific writings have for almost three centuries defied the combined efforts of his various editors to encompass them in an orderly fashion definitive form –

Editorial Note

General Introduction – Newton's creative mathematical researches continued with –

little break between the opening of his last undergraduate year in Cambridge, say from early summer of 1664, till he left University life before him more than thirty years afterwards in early 1696 for the Castle of London & the office of Warden of the Mint. Even then his inspiration flickered spasmodically on for another two decades, blazing up from time to time in answer to challenges from Leibniz & Johan Bernoulli – May the present edition be a small step toward that long overdue monument to a man who in so many areas of human thought himself took a giant's leap.

(Description of how his papers were scattered, lost & auctioned, not properly looked after or edited earlier)

Foreword – Illustration of the ease & fertility with which he was able to apply his pure techniques in appropriate areas of applied science

Part I – First Mathematical annotations

Part II – Researches in analytical geometry & Calculus (1664–1666)
(Analytical tables)

~~(The Calculus becomes an algorithm (mid 1665)~~

(1) Early notes on analytical geometry (autumn 1664)

(2) Work on the Cartesian subnormal (winter 1664)

(3) Miscellaneous problems in analytical geometry & Calculus (1664–65) – Attempts to calculate girths of algebraic curves

(4) Normals, curvature & general problem of tangents (winter 1664 – spring 1665) – quantity

of crookedness" of curves — Construction of subnormals & subtangents.

(5) The calculus becomes an algorithm (Mid-1665) — Miscellaneous observations on the general problem of quadrature — sketches a geometrical proof that the quadrature & tangent problems are inverse — First systematic tabulations of functions and their derivatives in parallel columns. fig (pp. 305–313 of text) mostly algebraic — Some examples chosen

$3xx = ay$ Parab: x^3/a

$7x^6 = ya^5$ —— x^7/a^5

$7x^6 + 3b^4x^2 = b^4ay$ —— $x^3/a + x^7/ab^4$

$x^4y = 3a^5$ —— a^5/x^3

$81x^8 + 144ax^7 + 64aax^6 = 4a^5xy^2 + 4a^6y^2$ —— $\frac{x^4}{a^3}\sqrt{ax+aa}$

$aaxx = aayy + xxyy$ —— $a\sqrt{aa+xx}$

$a^8 = aax^4yy + x^6yy$ —— $\frac{aa}{x}\sqrt{aa+xx}$

$a^8 = aax^4yy - x^6yy$ —— $a\sqrt{\frac{a^3-axx}{x}}$ —— A

A revised scheme of quadrature — Attempts to apply the rules of differentiation to the inverse problem of integration, seeking to discover the integrals of some simple rational functions by inspection — Tabulation of hyperbolic integrals for eg:

$\frac{1}{ax+b} = y$ — Its area (ie $\int y\,dx$) is □ (ie area) of $\frac{1}{ax+b}$

[$\int \frac{1}{a+bx}dx = \frac{1}{b}\log(a+bx)$ is to be derived as the area under the hyperbola $(a+bx)y = 1$ — Newton reduces other such integrals for eg.

$$\frac{xx}{ax+b}=y \longrightarrow \frac{xx}{2a}-\frac{bx}{aa}+\square\, q\,\frac{bb}{a^3x+aab}$$ ~~(ie)~~ of form $\frac{1}{ax+b}$.

$$\frac{x}{ax+b}=y \text{ —— } \frac{x}{a}-\square\, q\,\frac{b}{aax+ab}$$

$$\frac{1}{cx^3+ax^4}=y \longrightarrow \square\, q\,\frac{1}{cx^3}-\square\, q\,\frac{a}{ccxx}+\square\,\frac{aa}{c^3x}-\square\,\frac{a^3}{c^4+ac^3x}.$$

Systematic applns of integration techniques as an inverse method of fluxions.

For the first time uses fluxional notation. Specifically where $f(x,y)=0$ is a given (algebraic) function and t an independent variable of time, Newton denotes the fluxional "speeds" of x & y by p ($=dx/dt$) and q ($=dy/dt$), so that straightforwardly $p/q=dx/dy$.

This is used in systematic tabulation of elementary algebraic integrals for eg.

(i) $apx^{m/n}=q$, then $\frac{na}{m+n}x^{\frac{m+n}{n}}=y$ ie $y=\int q/p\,dx$ — Also gives derivative of $f(x)/g(x)$.

— with $z=f(x)$ & $v=g(x)$, $\int y\,dx$ and $\int v\,dz$ implies $y=v\frac{dz}{dx}$ for eg.

$2xx\sqrt{c+dxx}=y$, $\sqrt{cz+dzz}=v$, $xx=z$

(ie $\int y\,dx=\int v\,dz$ if $xx=z$).

$\frac{cb}{a+bx}=y$, $c+zv=0$, $\frac{1}{a+bx}=z$. (These things are from pp 348 to 363)

— First attempt (1665) at resolution of fluxional eqns

(b) General problem of tangents, curvature and limit-motion analysis by the

method of fluxions (Oct 1665 - May 1666)

(7) The October 1666 tract on fluxions.

(1665?)

p. 52, 2 of Whiteside - It is likely that Newton was influenced in the choice of these parametrisations (Related to Pells eqn) by reading the researches of Fermat on the eqn $x^2 - ny^2 = 1$ which Wallis printed in 1658. This problem ~~was~~ for given non-square integers n had been partially (?) solved by Bhaskara in the 12th century, but in ignorance of its previous history Fermat published the solution to the world in 1657 and on 11 Sept. of that year Brouncker communicated it to Wallis. Fermat noted that if $r^2 - ns^2 = 1$ is a solution then $(r^2 + ns^2)^2 - n(2rs)^2 = 1$ is a second as Bhaskara had already found (Brahmagupta's Rule). Newton's parametrisation is equivalent to this.

Whiteside Vol II (1667 - 1670).

Part I - Researches in pure and analytical geometry (cubic curves etc)

Part II - Researches in Calculus

(1) curve problems & further logarithmic computations

(2) Miscellaneous researches

(3) The "De Analysi per aequationes infinitas"

Part III - Researches in algebra and the construction of equations

⊗ [Sept

Extracts from Part II

This compendium 'De Analysi' is not in itself striking original. Its importance is due to the fact that it introduces Newton to John Collins & other friends outside Cambridge & it also paves the way to his composing the comprehensive 1671 tract on fluxions & infinite series (Whiteside Vol III).

[See p. 124 for continuation]

Whiteside Vol III (1670-1673)

General Introduction:

Part 1 – Researches into fluxions and infinite series (1670-1)

Part 2 – Miscellaneous researches

Part 3 – Researches in geometrical optics

Appendix – Newton's mathematical correspondence.

~~Part 1~~:

~~(1st Introduction~~:

General Introduction:

On 1670 in his 28th year, Newton teetered on the threshold between unappreciated youthful promise and acknowledged mature achievement, his mathematical & scientific discoveries of the previous half-dozen years – apart from the De Analysi – still mostly tightly to himself — in autumn 1669 Barrow

relinquished his Lucasian Professorship & on his recommendation Newton was appointed to the post on 29/10/1669. ~~and~~ also continued at Trinity – expressly liberated by royal mandate from all time-consuming responsibilities of academic teaching, tutoring and administration which would have been its normal accompaniment. To that freedom the quality of his future researches in mathematics, optics, dynamics, chemistry & astronomy manifestly owed much – Oldenburg sponsoring him for membership of Royal Society & his election. Newton at once began to communicate to Oldenburg the summaries of his optical experiments and hypotheses which were soon to make him internationally ~~famous~~ celebrated, and, together with the magisterial Philosophiae Naturalis Principia Mathematica (Principia) [not Mathematica] he was fifteen years ~~later~~ late, came ultimately to win him the Society's Presidency. – We are here concerned less with Newton's middling ability as a university teacher than with his surpassing intellectual genius.

Part I (a) – Introduction – Newton's idea of publishing his tract on fluxions & ∞-series along with his optical lectures ~~fell through~~ – no publisher to undertake its publication. Its sudden decision not to go ahead with printing his "Discourse of Lectures"

was due to wrangling disputes occasioned by the publication three months before of his "New Theory about Light & Colors" — Deciding not to publish his Optiks tract, he also decided to abandon work on his fluxional treatise since he thought the public may not receive it with enthusiasm. He however kept this tract with him & used it in 1676 to compile his two celebrated mathematical epistles for Leibniz & was not averse to revealing its contents privately to other interested party. Collins saw it & John Craige in 1684 and William Jones about 1710 were allowed to make copies of it and by the last years of Newton's life secondary manuscripts were circulating in London & elsewhere: With the passing of years, of course, the creation of more sophisticated Calculus procedures increasingly rendered Newton's early methods obsolete & substantial rewriting would have been necessary to bring the 1671 tract up to date: He is known to have actively contemplated making such revision on only one occasion — in autumn 1691 (tract not published) — In 1704 due to Gregory's intervention Newton decided to publish the "Tractus Quadraturae Curvarum" in an appendix to his "Opticks", a work very different and severely analytical treatise narrowly in line with the current

presentations of the subject — Thereafter, though its content was not wholly outdated, the 1671 treatise was essentially but of historical importance as a primary record of Newton's early achievements in calculus — The treatise finally appeared in print in 1736 in Colson's English translation, in 1740 in Buffon's French translation and a retranslation back into Latin in 1744 by Castiglione, but the original Latin text was published only in 1779 by Samuel Horsley — Joining the October 1966 fluxional tract to the De Analysi to write the 1671 tract in the winters of the years 1670 & 1671. — Analysis of this tract essential to decide the question of priority between Leibniz & Newton ~~also~~ on the discovery of higher analysis — In the 1671 treatise, Newton was led, under Barrow's guidance, to postulate a basic uniformly flowing "fluent" variable of "time" as a measure of the "fluxions" (instantaneous speeds of flow) of a set of dependent variables which continuously alter their magnitude.

In all these papers — above all the 1671 treatise — Newton's breadth of vision and his complementary mastery of technical detail are as impressive as ever, and they will remain always of fundamental importance to any understanding of his method.

of fluxions & infinite series. It is tragic that he himself could never feel an urgent necessity to publish them — if he had done so, scientific history might well subsequently have taken a different course.

Appendix - Draft preface to the "Commercium" - John Collins to James Gregory - John Collins to Jean Bertet - to Borelli - to Francis Vernon - Newton to Oldenburg for Leibniz (13/6/1676) - Newton to Oldenburg for Leibniz (24/10/1676). Here Newton refers to the inverse-tangent series & Note (26) p. 26 states that this method had exactly been anticipated by Nilakantha, though this was not then known in Europe.

p. 34, Note (5) Here Newton betrays his relative ignorance of the contemporary mathematical scene. In point of fact, Johannes Hudde on at least two occasions claimed priority in discovering the "Mercator" expansion of the logarithmic function which, of course, Newton had found for himself in the spring of 1665 in his Wallisian researches. We may forgive him for not knowing that a century and half before, the Hindu mathematician Nilakantha had derives had derived the inverse-tangent series by a "reductio per divisionem" exactly as James Gregory and Leibniz were to do in January 1671 and late 1673 respectively.

no hint of this reached Europe, it would appear, till 1835 (See A. R. A. Iyer & R. Tampuran's edition of the 1639 Malayalam compendium Yuktibhasa, Trichur, 1948; 113 ff; also C. T. Rajagopal and T. V. V. Aiyar "on the Hindu proof of Gregory's series" and "A Hindu approximation for π" Scripta Mathematica, 17, 1951, 65–74; 18, 1952, 25–30; and K. M. Marar and C. T. Rajagopal "on the Hindu Quadrature of the circle" Journal of the Bombay Branch of the Royal Asiatic Society, 20, 1944, 65–82. Both Gregory and Leibniz, of course, from Mercator's quadrature of the hyperbola.

(b) A preliminary scheme for a tract on fluxions: – Lists 23 problems – In the actual tract only 21 are discussed

(c) The tract on the methods of series and fluxions (The main part)

§1 The missing first leaf of Newton's autograph original – Talks of analogy between decimal numbers & algebraic terms – Examples for reductions by division ~~will become clear~~ & root extraction given.

§2 ~~The main autograph text~~

Gives $\frac{a^2}{b+x} = \frac{a^2}{b} - \frac{a^2x}{b^2} + \frac{a^2x^2}{b^3} - \frac{a^2x^3}{b^4} \ldots$ explaining the method of division

§2 The main autograph text

$$a^2/x+b \longrightarrow a^2/x - a^2b/x^2 + a^2b^2/x^3 - -$$

$$\frac{1}{1+x^2} \longrightarrow 1-x^2+x^4-x^6+x^8 \text{ or to } x^{-2}-x^{-4}+x^{-6}-x^{-8}+-$$

$$\sqrt{a^2+x^2} \to a + x^2/2a - x^4/8a^3 + -$$

$$\sqrt{x-x^2} = x^{1/2} - \tfrac{1}{2}x^{3/2} - \tfrac{1}{8}x^{5/2} - \tfrac{1}{16}x^{7/2} - -$$

$\sqrt{a^2+bx-x^2}$, $\sqrt{\dfrac{1+ax^2}{1-bx^2}}$, – & other complicated exp^ns

Finding roots of affected eq^ns for eg.

$\frac{1}{5}y^5 - \frac{1}{4}y^4 + \frac{1}{3}y^3 - \frac{1}{2}y^2 + y - z = 0$ gives on reversion $y = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 -$

~~Giving on reversion~~ $y = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 - -$

[$z = \log(1+y)$ giving $y = e^z - 1$]

Again $y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + - + \frac{63}{2816}y^{11} - z = 0$ gives on reversion

~~gives $y = z - \frac{1}{6}z^3 + \frac{1}{120}z^5$~~ [$z = \sin^{-1}y$ to $O(y^{13})$ yielding $y = \sin z$ to $O(z^{11})$

Problem 1 – Given the relation of the flowing quantities to find derivatives in simple cases – Fermatian "proof" of the technique of putting $x + 0\dot{x}$, $y + 0\dot{y}$ ($0\dot{x}$, – are called moments). Ex: $x^3 - ax^2 + axy - y^3 = 0$

Problem 2 – inverse problem: given a fluxional equation to determine its fluent form – use of infinite series expansions in integration –

Problem 3 – Determination of maxima & minima by equating the fluxion to zero – List of nine problems of geometric interest

Problem 4 – Determination of tangents to curves by considering ratios of fluxions of the coordinates – 7 per allied problems – different types of coordinates used. – Subtangent, Subnormal – one of the 7 problems is to determine the refraction of any ray when light falls on any curved surface you please.

Problem 5 – Determination of the curvature of a curve at any point. – use of polar coordinates – Numerous examples. – Formula $(\dot{x}^2 + \dot{y}^2)/\dot{x}\dot{z}$, $z = \dot{y}/\dot{x}$ –

Problem 6 – "Quality" of curvature at a point given by the measure $d\rho/ds$. – Related problems.

Problem 7 – Determination of areas of curves capable of exact algebraic quadrature – Fundamental theorem that the fluxion of the area of a curve is ∥l to the ordinate $D\left[\int y\,dx\right] = y$ – Examples

Problem 8 – General transformations which preserve ~~areas~~ area. Transform of $s = \int v\,dx \rightarrow t = \int y\,dz$ – Use of integration by parts to evaluate the area under mechanical curves – Examples – Catalogues of areas.

Problem 9 – Miscellaneous methods of determining areas of any curve proposed – use of convergent infinite series to ~~bring~~ reduce to

~~Cases~~ integrable forms – Simplified version of the 1665 calculations – evaluation of π (correct to 15 D) – interpolation methods – numerous examples.

Problem 10. To find arbitrarily many curves of finite (algebraic) arc-length – Evolute properties – nodes & cusps.

Problem 11 – Involute problems – Special cases – many examples.

Problem 12 – Rectification of curves defined in Cartesian coordinates $q = (\dot{t}) = \sqrt{[\dot{y}^2 + \dot{z}^2]}$ – Special curves as examples

(§2) Proofs by Fluxion & proofs by limit increments; says former is more "natural". – Fluxion of product of two magnitudes – Fluxion of x^n – Fluxional proofs of many ~~problems in~~ cases arising in Problems above – Rule of differentiation in fluxional language thus:

$AB \times fl(AC) + AC fl(AB) = AD fl(AE) + AE \cdot fl(AD)$ when $AB : AD = AE : AC$.

(§3) Quadrature of curves defined by equations of a finite number of terms.

Many general cases dealt with

Part II – Miscellaneous researches

I. H. M. induced in a Cycloid – Elementary geometry of curved surfaces; areas of curved surfaces, of a

spherical triangle & other cases.

Part 3 - Researches in geometrical optics (1670-72)

p. 469 - In dealing with extremes of dispersion in refractive media of varying density, Newton deals with the problem of maximising

$$\cos^{-1}\left(ks/\sqrt{x^4+x^2}\right) - \cos^{-1}\left(s/\sqrt{x^2+1}\right)$$

& putting derivative = 0, gets the quadratic eqn solved

$$x^4 - x^2 - 2(1-s^2) = 0$$

p. 481 - In dealing with refraction at curved surfaces, uses the expn for the subnormal $x + y\,dy/dx$, and finds the same for the ellipse $y^2 = rx - (r/q)x^2$

p. 485 - Uses expns for curvature & radius of curvature ρ.

p. 534 - In miscellaneous optical calculations, uses method of fluxions literally in the original form.

p. 567 (Appendix) - Evaluation of double integral

$$2\int_0^R \int_0^b \sqrt{\left[R^2 - x^2/r^2 - y^2/s^2\right]}\,dy\,dx$$, where the integrand is to be first expanded as a doubly infinite series of ascending powers of x^2 & y^2

p. 569 - Evaluates the first ten terms in the series expansion of

$2\int_0^a \int_0^b \sqrt{(r^2-x^2-y^2)}\cdot dy\,dx$, being the volume of the sleeved segment $(-a < x \leq a;\ 0 \leq y \leq b)$ of the sphere $x^2+y^2+z^2=r^2$

Vol II of Whiteside (some more points noted down)

Footnote:

P. 237 – Series for the sine & cosine appear for the first time in an European manuscript. These, however, had already been displayed in a 1639 Malayalam compendium, the Yuktibhasa, which itself professes to be based on an earlier sixteenth-century Sanskrit original Tantrasangraha, perhaps the work of the little-known Hindu mathematician Nilakantha (See A. R. A. Iyer's & R. Sampuran's modern Malayalam edition of the first part of Yuktibhasa (Trichur 1948); and its review by C. T. Rajagopal and A. Venkataraman in Mathematical Reviews, 12 (1951); 309–10. The latter manuscript is still unpublished): The Hindu approach, which depends, for example, on repeated integration of the identity

$$\sin x = \int \left[1 - \int \sin x\,dx\right]\cdot dx \qquad \left[\text{should be } \int_0^x \sin x\cdot dx\right]$$

is both wholly distinct from Newton's and was described in Europe only in the early nineteenth century (C. M. Whish "on the Hindu ...

quadrature of the circle Gives the Infinite series of the proportion of the circumference to the diameter Exhibited in the four Sastras ...'
Transactions of the Royal Asiatic Society of Great Britain and Ireland, 3 (1835): 509-23, so that there can be no question of any influence upon Newton in his independent rediscovery of the series.

[Newton finds $\sin^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots = z$, say
by $\int_0^x (1-x^2)^{-1/2}\,dx$, expanding $(1-x^2)^{-1/2}$ by Binomial Theorem
and then from this derives $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 \cdots$
which is the series for $x = \sin z$.]

p. 240 (7) - Newton expands $y = x \cos x$ as an infinite series in powers of x & then evaluates

$$\int y\,dx \left[= x \log \sin x + \int_{\frac{1}{2}\pi}^{\frac{1}{2}\pi - x} \log \cos z \cdot dz \right]$$

Also evaluates $\int_0^x \sqrt{\left[1 + (\cos x - x \operatorname{cosec} x)^2\right]}\,dx$

(126)

⊛ Ramanujan gave $x = 3a^2+5ab-b^2$, $y = 4a^2-4ab+6b^2$
$z = 5a^2-5ab-3b^2$, $t = 6a^2-4ab+4b^2$
a soln of this diophantine eqn. Newton gave some numerical solns for eg
(3,4,5,6) & (1,6,8,9)

Institute 31/5/77

Whiteside Vol. IV – 1674–1683

Interpolation by finite differences, number theory, trigonometry, pure & analytical geometry of conics & cubics, geometrical calculus & infinite series

Part I – Researches in Algebra, Number theory and Trigonometry

(1) Approaches to a general theory of finite differences
Tabulation of sines & cosines, table of logarithms to base 10, constant their difference, central difference interpolation formulas, Newton–Stirling formula & Newton–Cotes formulas

(2) Problems in elementary number theory
Solving simple diophantine eqns for eg: (i) To find three nos. in continued proportion whose sum shall be a square
(ii) To find two squares whose sum or difference shall be a square (cube/4th power
(iii) To find two cubes whose sum or differ is a (square, 4th power to sum or
(iv) $x^2+x=y^2$, $x^3+x^2=y^3$, $x^3+x=y^3$, $a^4+x^2=y^3$. $v^3+x^3+y^3=z^5$
$x^3+y^3+z^3=t^3$ (*)

21 such problems ~~last~~ 21st being to find perfect numbers, no remarks made.

Newton not perhaps interested in existence problems like all even perfect nos are given by $2^{n-1}(2^n-1)$ — Again numerical problems like

$x^2+xy+y^2=z^2$, z^3, z^4, z^5; $x^3+y^3=z^4$, $ax^2+by^2=c$ & $ax^3+by^3=c$

$x^3+y^3=a$, $x^3+y^3=x+y$ — To find ~~triangles~~ two isosceles Δ^s which shall be equal in area & perimeter, to find Δ whose $\perp^{rs}$ are rational. 26 such problems — Rational cyclic quadrilateral

(3) Codifications of elementary plane & spherical trigonometry

Stereographic projection — Lambert's formula in sph. trigonometry viz $\tan^2\frac{1}{2}(AB)/R^2 = -\cos S\cos(S-A)\cos(S-B)\cos(S-C)$ anticipated — Area of a sph. Δ — 12 lemmas for rt $\angle^d$ sph. Δ^s — Napier half angle theorem derived by stereographic projection — Ptolemy's theorem on the cyclic quadrilateral — Five figure tables of sines listed at half-degree intervals $0°$ to $90°$

(4) Miscellaneous notes on annuities and algebraic factorisation

Quadratic factorising of algebraic binomials — anticipation of Cotes' general quadratic factors of the binomial ~~[illegible]~~ — Newtonian series expansion for $\pi/2\sqrt{2}$ communicated to Leibniz — Leibniz's failure in 1702 to identify the (real) quadratic factors of x^4+1 & so could not earlier have penetrated in Newton's series for $\pi/2\sqrt{2}$

$$\left[\pi/2\sqrt{2} - \int_0^1 (1+x^2)/(1+x^4)\,dx = 1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+ \cdots\right]$$

Part II – Researches in pure & analytical geometry.

(1) Miscellaneous problems on elementary geometry – Euclid – Desargue & Pappus' theorems on line collinearities – inversion & homothety – Apollonius' problems of circle contact – "Centre of motion" of two bodies each moving uniformly in a st. line, Generalisation to 3 dimensions anticipating ~~Rosi~~ Boscovich's 1746 result – ~~Sketch of geometrical soln of the 4-line locus problem~~

(2) Researches into the Greek "Solid locus" – Conics through five given points & allied geometric constructions – Here Newton anticipates Projective geometry of conics for eg. locus of intersection of ~~the~~ two line pencils which are in 1-1 correspondence – Pascal's hexagram mysticus

(3) Miscellaneous topics in analytical geometry – Asymptotes & diameters in general cubic curve, Diametral hyperbola, miscellaneous calculations on cubics – Further work on cubics

Part III – The "geometria curvilinea" and "Matheseos universalis specimina"

(i) Geometry of curved lines

Book 1 – Geometric fluxions – Fluxions of a power, fluxion of sides etc etc.

Book 2 – Determination of Max & Minima

These books not published contain certain passages which significantly widen our insight into the subtlety of thought and immense technical skill of a man

on the eve encapsulating the mathematical principles underlying a wide range of natural phenomena in one of the most profound and enduring of scientific classics (a Principia)

The essence of "Geometries" (using fluxions) opening propositions published in the Principia (1687) was useful in his controversies with Leibniz

(2) Specimen of a universal system of mathematics

References to controversy with Leibniz

Appendix – Mathematical topics in Newton's correspondence

Lengthy "epistola posterior (Oct 1676) communicated to Leibniz, borrows from his 1671 tract, Leibniz's delay in receiving the letter (not till June 1677) and Oldenburg's death (Sept. 1677) terminate the interchange of insights and criticisms.

~~Whiteside – Vol V (1683–1684)~~

Reference to Indian workers & to Gauss

p. 90. N.7 Newton deals with the "Chinese problem of remainders" viz. to find a number which shall, after division by given numbers, leave given remainders. This problem was solved at least as early as the 1st century A.D in China by Suan-Ching of Sun-Tsŭ and apparently rediscovered independently by Brahmagupta 6 centuries later and introduced into Europe by Leonardo of Pisa

Problem 16 in No. theory of Newton viz. if $x^3+y^3=z^6$, then $e^3+f^3=z^3$. If this can be solved (ie Fermat's problem) the original problem can be solved. A watertight proof was finally given by Gauss about 1820.

p. 211 n — The Gregory ~~Leib~~ Leibniz series

$$\pi/4 = \int_0^1 1/(1+x^2)\,dx = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

derived by ~~Newton~~ Leibniz was first discovered ~~by~~ about 1500 by the Hindu Nilakantha, ~~was~~ and was communicated ~~by~~ to Newton by Leibniz in his letter (to Oldenburg) of 17/27 August 1676 with a somewhat highflung recommendation of it.

Whiteside V (1683–84)

General Introduction — Lucasian Professor lectures of the ten years from 1673 — Statutory obligation — Halley's first visit to Cambridge in summer of 1684 — Appearance Wallis' Algebra in 1685 — unworthy drudgery as other people called it — Halley's omit seeking to divert Newton's attention about 1684 towards the dynamics of Planetary motion — composing of various preliminary states of his mighty Principia (concerned with in next vol ie Vol VI) — Help of Humphrey Newton in secretarial work.

Part I — Deposited Lucasian lectures on algebra

Part II — The "Arithmetica universalis liber primus" — Eqns etc

Whiteside VI – ~~18~~ 1684 – 1691

Preface – Although Newton himself, half a century afterwards, preferred to look to the two "plague years" 1665 & 1666 as the "prime" of his "age for invention", the dozen & half months from August 1684, when Edmond Halley first travelled to Cambridge to seek his opinion on the currently vexing question of how dynamically to determine the closed orbits of the planets round the sun, have (so it seems to us) an overriding claim to be regarded as the most deeply fruitful annus mirabilis of Newton's life – over that year & a half he was encouraged to expand his immediate confident answer to Halley that the planetary orbits are exact ellipses having the sun at a focus, into a developed treatise and thereafter further to augment its scope to be the magesterial Principia which made its bow in 1687, the chef d'oeuvre of classical theoretical Physics – It must be reviewed here since otherwise we would ignore this supreme exhibition of Newton's scientific genius

General Introduction – Newton during this period missing his meals while at the table & when reminded he had forgotten would say "Have I?" & then eat some bits – His head scarcely combed – when going to dinner in the Hall for dinner he would turn to the left hand going to the street & finding his mistake would hastily return & then instead of going to the Hall would go back to his Chamber again. In the Chamber he walked – very much – & seldom sat by the fire – Further historical details – meeting with philosopher Locke at London with whom he kept contact for over 10 years and also Huyghens – Lot more details

Geometry & Dynamics of motion

Introduction – Kepler's 3rd law used to derive the planetary inverse square law of centripetal "endeavour" – Early thoughts on projectile motion – The classical facade of conceals an internal structure founded on infinitesimal limit increments – Computing of parabolic cometary paths – Newton's novel fluxional measure of force.

(1) Fundamental investigations "on the motion of bodies" – The three laws of motion propounded – Deduction of Kepler's laws – Problems of gravitational attraction – Number of Propositions – gravitational theory of optical refraction (corpuscular theory of light) – motion under resistance

(2) Approaches to the solution of particular problems

cometary path parabolic (very nearly) – hourly rate of motion of the moon's apogee – Calculation by integration of the advance of lunar perigee

(3) The "De Motu Corporum liber Primus" remodelled

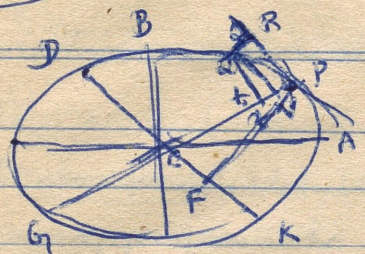
2nd Law of motion recast

use of Calculus in Principia, although hidden by the Grecian facade of Definitions, Axioms, Lemmas, Theorems, Problems and Scholia, is the cause for the rich harvest reaped by Newton

p. 45 – Problem 2 – A body orbits in a classical ellipse: there is required the law of centripetal force tending to the ellipse's centre.

Problem 3 – Do the centre being a focus (This is dealt with in this notebook on p. 84')

Problem 2



PG, DK conj. diams. Q near P. QV ∥ to tangent at P

Qt ⊥ to PG, PF ⊥ to DK. (QVPR a ||gm)

$$\frac{PV \cdot VG}{QV^2} = \frac{PC^2}{CD^2}; \quad \frac{QV^2}{Qt^2} = \frac{PC^2}{PF^2}$$

and on combining these proportions $\frac{PV \cdot VG}{Qt^2} = \frac{PC^2}{CD^2 \times PF^2/PC^2}$

Write QR in place of PV and BC × CA in place of CD × PF and in addition (with the points P & Q coalescing) 2PC in place of VG & when extremes and middles are multiplied into each other, there will come to be

$$Qt^2 \times PC^2/QR = 2BC^2 \times CA^2/PC.$$

The centripetal force is therefore reciprocally as $2BC^2 \times CA^2/PC$, thus that is, (because $2BC^2 \times CA^2$ is given) as $1/PC$, in other words directly as the distance.

Whiteside VII. (1691–1695). (with assistance of only Bray)

(1976).

Dedicated to Michael Hoskin

Preface – In autumn of 1695 Newton took his fateful decision to accept the Wardenship of the London Mint – further volume Newton recasting & refining the texts of the mighty Principia – Mathematical output from 1691 to 1695 consisted in extensions into general Taylor expansion of algebraic functions

and of the projective classification of curves, specially cubics — unpublished later proposations of his 1691 "De quadratura"

General Introduction — Many have it in them to be hard-headed business men, successful politicians, able organisers of people and administrators of government, even efficient Masters of the Mint & forceful Presidents of the Royal Society; only too few have ever possessed the intellectual genius, and surpassing capacity to stamp their image upon the thought of their age and that of centuries to follow

Part I — The ~~tract~~ first tract "De quadratura Curvarum" (early writer 1691-92)

~~Newton Mons Craige (1693)~~ Elaborates this tract in 1692 —

~~Introduction~~

Part II — Researches in pure geometry and quadrature of curves.

Part III — Cartesian analysis of higher plane curves and finite-difference approximations.

p. 207. Ref. to R. C. Gupta's article in the Indian Journal of the History of Science, 4, 1969, 86-98, specially 92-4. Gupta identifies some rules of computation of incremented sines and cosines attributed to the Hindu mathematician (14th century) Mādhava as equivalent to modern Taylor series approximations up to the second order of small quantities appears to us more than a little anachronistic

History of Maths by H Eve

Bhaskara (~~[illegible]~~ 1114 - ca 1185)

Brahmagupta (628 ca)

Mahavira (ca 850)

Narayana (1350)

Précis for the lecture.

(1) one of the greatest Physicists of all time, Pope's lines – Archimedes, Newton, & Gauss, doubt among modern theoretical physicists and modern abstract mathematicians, doubts dispelled by glancing through the 7 Vols of Whiteside – Tribute to this endeavour.

(2) Analysis before Newton – Bhaskara, Nilakantha, Sevin, ~~Gosdar~~ Kepler, Galileo Galilei, Cavalieri, Fermat, Roberval (cycloid), Pascal, Wallis & Barrows, Huygens

(3) Newton's fluxions, genita, moments, fluents, fluents from fluxions, prime & ultimate ratios, rules of diffn & integration by parts etc, tangents etc, quadrature, numerous cases, $\dot{y}$, $\dot{y}$ etc – use of infinite series of which he was a master – ~~more than all use in Principia – justification for being "inventor" of the calculus.~~

The Theer tracts on fluxions – use in Principia – justification for being "inventor" of the calculus. – criticism by Bishop Berkeley.

(4) Controversy with Leibniz – $\pi/4$, $\pi/2\sqrt{2}$, $\frac{4}{\pi}$ of Remmington – ~~Conclusion~~ not published by 1671 tract – conclusion of non-authorship.

(5) Principia - use of calculus in Principia - use of calculus in Opticks

- absent mindedness while composing Principia

(6) Role of analysis up to 1800 - ~~[illegible]~~

(7) ~~Analysis in modern times~~, Revolution brought about by Analysis, emergence of abstract algebra, Analysis & Th. Physics, allied topics.

(8) Remarks about other math. work of Newton - Motivation for working in Mathematics - Facility in applying mathes, Mathics as an Art. - Final tributes

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C.C. Lib 2Q (1/6/77)

(1) The scope of Mathematics by Holt & McIntosh (1966)

Ch. 9 Newton & the Calculus

Calculus led to an outstanding School of "Analysis" and traditional "pure maths" - Continuity idea solved by Cauchy in 1826

Later mathematicians (after Newton) - Euler (1707-83), Lagrange (1736-1813), 19th century dominated by Gauss (1777-1835) - Boole (1815-64), Cayley (1821-95)

(2) Fundamental Concepts of Mathematics by Langbehn, Lathrop, Martini (1972)

Nature of Mathematics - Elementary Logic, Algebra of propositions - Sets - Relations & functions - Probability - Number System - Numeration - Geometry

Historical retrospect – In 19th century, Non-Euclidean Geometry, acceptable foundation for calculus – logical reasoning

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(3) Books on Progress in Mathematics – Edited by ~~Gram~~ Gamkrelidze (USSR)

Vol. 10 – Math. Analysis

Systems of linear diff eqns – Fredholm operators – Representations of groups & algebras

Vol. 13 – Probability theory, Math. Statistics, Theoretical Cybernetics

Vol. 5 – Algebra

Abelian groups – Linear groups – Non associative binary systems – Ordered algebraic systems – Rings – Modules

Vol. 4 – Math. Analysis

Integral transforms (~~Neces~~ 25 of them – Fourier transform one of the most impt concepts in classical & modern analysis – Integration in function spaces – Measure transformation – Integration over distributions

Vol. 8 – Math. Analysis

Asymptotic methods in theory of diff eqns – Linear diff eqns – Non-linear – Best approximation theories – Dynamical systems with invariant measure.

Vol. 12 – Algebra & Geometry

Ring theory – Modules – Lattice theory – Integral geometry – Differential geometry

methods in the Calculus of Variations

Vol. 3. Probability etc

Markov processes in diff eqns – Discrete problems in Math. programming

Vol. 7 – Probability etc

Branching processes – Computer Systems

Vol. 6 – Topology & Geometry

Geometric topology of manifolds (knot theory inclusive) – Differential line geometry of three-dimensional space – Diff. geom of families of ~~surfaces~~ subspaces in multidimensional uniform spaces – Motions in generalised diff geometric spaces

(4)

Math. Reviews, Aug 76.

General	Number theory	Group theory & generalisation
History & Biography	Alg. No. theory, Field theory & Polynomials	Topological groups & Lie Groups
Logic & Foundations	Commutative rings & algebras	
Set theory	Algebraic geometry	
Combinatorics, Graph theory	Linear & multilinear Algebra, Matrix theory	Functions of a real variable
Orders, Lattices, ordered algebraic systems	Associative rings & algebras	Measure & Integration
Gen. Math. Systems	Non-associative rings & algebras	Fns of a complex variable
	Category theory, Homological Algebra	Potential theory

Several Complex Variables & Analytic Spaces

Special functions

ordinary Diff. eqns

Partial Diff. eqns

Finite differences & functional eqns

Sequences, series, summability

Approximations & Expansions

Fourier Analysis

Abstract Harmonic Analysis

Integral transforms, Operational Calculus

Integral eqns

Functional Analysis
Operator Theory

Calculus of Variations & Optimal Control

Geometry

Convex Sets & Geometrical inequalities

Differential geometry

General Topology

Manifolds & Cell Complexes

Global Analysis, Analysis on Manifolds

General Appl. Maths

Mechanics of Particles & Systems

Elasticity, Plasticity

Fluid Mechanics, Acoustics

Optics, Electromagnetic Theory

Classical Thermodynamics, Heat Transfer

Quantum Mechanics

Statistical Physics, Structure of matter

Relativity

Astronomy & Astrophysics

Geophysics

Economics, Operations Research, Programming, Games

Biology & Behavioural Sciences

~~General Appl. Maths~~ Probability Theory
& Stochastic Processes

Statistics

Numerical Analysis

Computer Science

Systems, Control

Information & Communication

Circuits

Automata

(5) Rouse-Ball: Short History of Mathematics

Newton 1665-66 - Fluxions, Optics & gravitation

1667-69 Researches?

1669 - Elder Lucasian Professor

1669-71 - Optics, Discoveries

1675 - Emission theory of light

1676 - Leibniz letters

1679 - Discoveries in gravitation

1673-83 - Discoveries & lectures on algebra

1684 - " " gravitation

1685-86 - Principia

Mathematics in the 19th century (Commencing a new period)

Gauss (1777-1855)	Kummer (1810-1893)	Riemann (1826-1866)
Dirichlet (1805-1859)	Abel (1802-1829)	Weierstrass (1815-1897)
Eisenstein (1823-52)	Jacobi (1804-1851)	Cayley (1759-1857)
		Hamilton (1805-1865)

Boole (1815–64)	Steiner (1796–1863)
De Morgan (1806–1871	Bessel (1784–1846)
Galois (1811–1832)	
Cayley (1821–1895)	
Sylvester (1814–1897)	
Hermite (1822–1901)	

(6) Developments of Mathematics by E.T. Bell, 1945

Ramanujan (1887–1920)

Arithmetic repaid its heavy debt to analysis by showing some of the foremost classical analysts of the twentieth century with an abundance of difficult problems.

Analytic methods applied to derive asymp. formulae for $P(n)$ (Partitions)

Many analyses of $\zeta(s)$ – Riemann's conjecture viz. that

$\zeta(s) = \sum n^{-s}$. All the zeros of $\zeta(s)$, except those at $s = -2, -4, -6, \ldots$ lie in the strip of the s-plane (Argand diagram for s) for which $0 \leq \sigma \leq 1$ where σ is the real part of s.

1920's – 1930's witnessed an era in ~~math~~ arithmetic comparable to that inaugurated by Gauss in 1801. Analysis, the mathematics of continuity, had at least breached outstanding problems in the domain of the discrete – Hilbert's work on integral eqns – Hilbert space – its physical significance abstract spaces.

19th century → 20th century – period of abstraction an inevitable stage –

Abstractionists & non-abstractionists – Metaphor compare the abstraction period as sterile & of the type existant at the time of the death of Greek mathematics – Early 20th century (1900 – 1945) – Mathematics as ideal truth expired somewhere on its journey from 1900 to 1945, and those who continue worshipping this ideal are attending the protracted obsequies in ignorance that their deity is dead

Abstraction in mathematics is evolved out of physical applications – mathematical description of nature either thro' analysis as at present or by Geometry revitalised in the future by topology etc.

15/6/77 At Institute – Books on Quantum Field Theory Collected

(1) Lectures on quantum field theory by Dirac, 1966
(2) Selected topics in field quantisation, 1973 (Pauli lectures on Physics Vol 6. ↓ by W. Pauli)
(3) Quantum Field Theory by A. Visconti, 1969 Vol I
(4)
(4) The quantum equation & theory of fields by H. T. Flint, 1966
(5) Mathematical aspects of the quantum theory of fields by K. O. Friedrichs, 1953
(6) Particles & fields by David Lurie, 1968

(7) Methods of quantum field theory in statistical physics by Abrikosov et al

(8) Introduction to quantum field Theory by F. Mandl, 1959

(9) Aspects of quantum Theory by Salam & Wyne, 1972 (Dirac 70th b-day)

(10) Introduction to quantum fields theory by Paul Roman, 1969

(11) Elementary quantum field theory by Henley & Thirring, 1962

(12) An introduction to field quantisation by Takahashi, 1969

(13) The method of second quantisation by Berezin, 1966

(14) Relativistic quantum fields by Bjorken & Dell, 1965.

No.

(5) Not much about quantum fields except an article on p. 61 by R. Jost - nothing striking.

(6) Has an introduction by Sudarshan

4/7/77 At Guindy

Summary of 1961 Engg. Brit. Article by H.E. Dudeney.

Constructing a 4×4 by moving

1	2	3	4
6	7	8	9
11	12	13	14
16	17	18	19

1	17	8	14
18	4	11	7
12	6	19	3
9	13	2	16

and this is a Nasik square:

Ex 2 8×8 square containing a knight's tour. The knight has to be placed once to every square of the chessboard in a complete tour or a non-recurrent path, numbering the squares visited in order so that when completed the square shall be magic with the constant 260. Recent approach to this due to Gaenisich (St. Petersburg, 1862) is given by

46	55	44	19	58	9	22	7
43	18	47	56	21	6	59	10
54	45	20	41	12	57	8	23
17	42	53	48	5	24	11	60
52	3	32	13	40	61	34	25
31	16	49	4	33	28	37	62
2	51	14	29	64	39	26	35
15	30	1	50	27	36	63	38

(Semi-magic)

unfortunately it is not a perfect magic square (only semi-magic) because the diagonals are incorrect, one adding to 264 and the other to 256 - requiring only the transfer of 4 from one diagonal to the other. Probably a perfect solution is impossible, but no rigid proof of this has been obtained

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Ref. Rouse & Walker - Brit. Math. Journal (1939)

(2) Scientific American, April, 1977, Vol. 236, No. 4

p. 30 – The collision between India & Eurasia by Peter Molnar and Paul Tapponnier.

The northward movement of the India plate has created spectacular land forms. For the last 40 million years the Indian subcontinent has been pushing northward against the Eurasian landmass, giving rise to the severest earthquakes and the most diverse land forms. – The theory worked out only in the last 10 years – Reference to Wegener's theory of continental drift – Ref. to Ashok Sahni & Vimal Kumar's work of Lucknow Univ. dating the oldest mammals of India to 45 million years ago – A fine picture of the northward drift from 71 million years ago to the present age (p. 35) – Beautiful pictures & wonderful work in Geology.

p. 116 – The theory of the Rainbow by H. Moysés Nussenzveig.

The phenomenon explained by Newton still challenges the resources of modern physics

The explanation has required all the resources of mathematical physics ie to explain why when sunlight is scattered by raindrops, colorful arcs appear in certain regions of the sky – Correct theory developed only a few years ago – Goethe's wrote that Newton's analysis of the rainbow's colors would ~~cripple~~

or cripple Nature's heart"! A similar sentiment was expressed by Charles Lamb & John Keats; at a dinner party in 1817 they proposed to toast to "Newton's health and confusion to mathematics." Descartes & Newton's explanation & Young's interference theory — what was lacking was a quantitative mathematical theory capable of predicting the intensity of the scattered light as a function of droplet size & scattering angle — Wave mechanical treatment of the atomic & nuclear rainbows formulated in 1959 by Kenneth W. Ford of Brandeis University & John A. Wheeler of Princeton Univ — Atomic rainbow first observed in 1964 by Hundhausen & H. Pauly of Bonn Univ — Scattering of atoms by atoms creates a particulate rainbow (Picture on p. 124); scattering of sodium atoms by mercury atoms — Such expts give information about interatomic forces — Use of Regge poles; contribution to the transformed partial-wave series — In 1975 Vijay Khare of Rochester made a detailed comparison of three theories of the rainbow (i) the Airy approximation (ii) the "exact" soln obtained by a computer summation of the partial-wave series & (iii) the rainbow terms in the complex angular momentum method associated with the collision of two saddle points — Another important set of contributions by Khare

p. 129 — Ref. Journal of Recreational Mathematics now in its 9th year; Subscription rates in the U.S. for four issues are $10 for individual subscribers, subscriptions are available from the Baywood Publishing Co., 43 Central Drive, Farmingdale N.Y. 11735.

(3) (i) New Scientist, Vol 74, No. 1046 (7/4/77) (English Journal)

p. 8. The spin-flip saga – How a tunable spin-flip Raman laser was tailored in New Castle – finds a simple theory

(ii) 3/2/77. Vol. 73, No. 1037

p. 292 – Review of Calder's book "The Key to the Universe" a report on the new physics dealing with quarks etc – Highly critical review also of Fabric of the Universe by Denis Postle – who uses the words "Vajrasattva in union with Viswatara ie embodiment of the union of wisdom with consciousness – Reviewer says it is neither religion nor science

(iii) Vol. 73, No. 1035 – 20/1/77.

p. 140 – More weight behind heavy leptons. – Ref. Phys. Rev. Letters, 38, p. 117 – About the likelihood of the existence of a new charged partner for the electron (besides the muon) in the region of mass between 1.6 & 2.0 GeV.

(iv) 27/1/77, 73, No. 1036

p. 207 – There is too little cosmic dust (matter) in the universe to reverse the big bang

(v) 2/9/76, 71, No. 1016

p. 478 – Soviets turn the lightest gas into a metal – Hydrogen is a metal at last Scientists at the Institute of High Pressure Physics, Moscow, have succeeded in cooling and compressing hydrogen until – at a pressure of few megabars and a

temperature of 4·2K – it becomes a conducting metal. – Determination of crystal structure of metallic hydrogen; only neutrons may perhaps give us a glimpse of the structure of the lattice; perhaps also indirect results using Mossbauer effect, but we shall be relying on neutrons – Report by Dr Ye Yakovlev of the Inst. Moscow

(vi) 6/1/77, 73, No. 1016. –

(vii) 24/2/77, 73, No. 1040 –

(viii) 10/2/77, 73, No. 1038

p. 329 – Monitor col: (1) Dinosaurs found in Tibet

(2) Changing protons to neutrons excites the nuclear the orists

Explains the "pion double-charge exchange" Phys. Rev Letters, 38, p. 149

These reactions show a way of making nuclei with the same structure & same total number of nucleons but different nos. of neutrons & protons, as the ground state of stable nuclei (which form the targets). – Originally one change of p to n was known, now a double charge exchange reaction has been observed

Inst. 12/8/77 – Ency. Brit. 1961 Article

4th order classified into nasilks 48, 384 semi nasiks (milling 48 associated), 448 simple totalling 880.

549, 504 fourth order magic squares in rows & cols.

3/9/77 At Surbiton – Qns for Calcutta paper on Q.M.

(1) Find the antisymmetrical matrices related to rotations round the Cartesian axes and the commutation rules satisfied by them (Gasiorowicz, p. 229, Trans. of QM)

(2) Define a space inversion in one-dimension, and the unitary operator Π associated with it. Consider space inversion as a special case of rotation through angle α & deduce that $\Pi = e^{-i(\alpha/2)(P^2+Q^2)}$. Q & P being the position & momentum operators. If ϕ be the operator Π when $\alpha = \pi/2$, show that

$$\phi Q \phi^{-1} = P, \quad \phi P \phi^{-1} = -Q \quad \text{and} \quad \phi^2 = \Pi. \qquad \text{(Ghosh p. 242)}$$

(3) Defining time reversal transformation by $Q \to Q$, $\dot{Q} \to -\dot{Q}$, show that it is not a unitary transformation, and that there exists an anti-unitary transformation operator T generating the time-reversal. Show that (i) T anticommutes with the angular momentum operator, (ii) T commutes with rotations & space inversions, such that $PT = -TP$, $QT = TQ$, & $T^2 = 1$. (Gasiorowicz pp. 244–45)

(4) Enumerate the physical basis leading to the unitary group $SU(3)$, and write down the matrices which generate the transformations of $SU(3)$ & show that this group has $SU(2)$ as a subgroup.

(5) Define the density operator W, and show that it is a self-conjugate operator satisfying the relations $W \geq 0$, $\mathrm{Tr}(W) = 1$, $W^2 \leq W$. Show that the state represented by W is pure if & only if $W^2 = W$; Denoting the pure state by p, show that there exists a projection p' of one dimension at most such that $p(P) = 1$ & vice-versa. (Gasiorowicz)

(158)

(6) obtain the relativistic Hamilton's eqns of motion for a charged particle

(7) Obtain the Liénard-Wiechert potentials for the general case of an accelerated electric charge, and ~~derive in express the electric~~ derive expressions for the resulting electric & magnetic fields. Deduce that in the case of uniform motion — —, ~~interpret the~~ Show that in general the line of $\underline{B}$ are perpendicular to the lines of $\underline{E}$.

(8) ~~method of partial waves~~ Find the differential and total scattering cross-sections, using the Born approximation, for a scattering ~~problem~~ potential represented by $V(r) = V_0 \exp(-r^2/a_0^2)$.

(9) Show that in the Born approximation the scattering at high energies is concentrated into small angles, and that the differential scattering cross-section takes the form $\sigma(\theta) = |f(v\theta)|^2$, where $\underline{v}$ is the velocity of the incident particle. Hence show that in these circumstances the total scattering cross-section varies inversely as the energy

(10) ~~Derive~~ Define the density matrix ρ of an ensemble of states, and deduce (i) $Tr(\rho) = 1$, (ii) ρ is Hermitian (iii) ρ' diagonal form of ρ can be obtained in view of (ii) by $\rho' = T^{-1}\rho T$ (T unitary), $Tr(\rho') = Tr(\rho)$ $Tr(\rho^2) \leqq Tr(\rho)$. If density matrix represents a pure state $Tr(\rho^2) = Tr(\rho)$. Eqn of motion of ρ is $\frac{\partial \rho}{\partial t} = \frac{i}{\hbar}[\rho, H]$ is the quantum-mechanical analogue of Liouville's theorem in classical statistical mechanics where D denotes the density of representative points in a small region of phase space

quantum mech. statistical dep of entropy is $S = -k \operatorname{Tr}(\rho \log \rho)$ - for polarisation of electron beams (non-relativistic) - pure states (with separate spin orientation) & can ρ have density matrices $P_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $P_\beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. For a completely polarised beam with $P = 1$, $\operatorname{Tr}(\rho^2) = 1$ & completely unpolarised beam ~~P(a)~~ $P = 0$, $\operatorname{Tr}(\rho^2) = 1/2$.

(11) This the uncertainty relation for the kinetic energy and potential energy of a 1-dim^{n} S.H.O

(23|9|77) - At Home: Excerpts from Hilbert by Constance Reid - Book left with me by K.V.

(1) Hilbert was one of the truly great mathematicians of his time. His work and his inspiring scientific personality have profoundly influenced the development of mathematical sciences up to the present time. His vision, his prognostic power and independent originality as a mathematical thinker, his versatility and breadth of interest make him a pioneer in many fields ... He was a unique personality, profoundly immersed in his work and totally dedicated to his science, a leader and leader of the very highest order and most fearless, tireless and persistent in all of his efforts.

(2) Mystery of the origin of great scientists in our Society.

(3) Like all boys in Koenigsberg, David grew up with the words of Kant in his mind and ears - Spelled out the words on the wall of Kant's crypt

"The greatest wonders are the starry heavens and the moral law within me".

(4) When the teacher failed to understand a mathematical problem on the board, the students chorussed "Minkowski, help!"

(5) Franz Neumann – many of whose discoveries were never published – had said simply "The greatest ~~discovery~~ luck is the discovery of a new truth; to that recognition can add little or nothing"

(6) Alexander complaining "Father will conquer everything and there will be nothing left for us to conquer". The world of mathematics is inexhaustible.

(7) The goal of full Professorship – a position of such eminence in the Germany of the day that professors were often buried with their title and subject of their speciality on their gravestones.

(8) "All human knowledge begins with intuitions, then passes to concepts and ends up with ideas" – Kant

(9) Hilberts condns on axioms – complete, independent, consistent – metamathematics

(10) Freedom from bias and tradition which is characteristic of truly great investigators.

(11) Astonishment at seeing how much information about nature can be obtained by the mathematical method. Profoundest reverence for theory would overcome me when it cast unexpected light on previously obscure facts – This man (Hilbert) lives in my memory as perhaps the greatest genius I ever laid eyes on. – von Laue.

(12) The general sections which dealt with bibliography & History & Teaching & methods

were considered of inferior rank to the mathematical sections — Paris Congress 1900.

(13) History teaches the continuity of the development of science. We know that every age has its own problems, which the following age either solves or casts aside as profitless & replaces by new ones. As long as a branch of science offers an abundance of problems, so long is it alive; a lack of problems foreshadows extinction or the cessation of independent development. — Lofty minds are led to strive for the advancement of science by nothing more than by laying before them difficult and at the same time useful problems

(14) Klein has devoted himself to activities which were not in themselves mathematics. In addition to his teaching & administrative duties, he was the moving force behind a number of projects: the plan for the 30-volume mathematical encyclopaedia; the newly organised International Schools commission, which has been set up to survey the development of teaching "methods" in all civilised countries" from kindergarten to graduate school; the attempt to improve and enlarge the scientific education offered in the German middle schools and to bring together at the ~~same time~~ university level technical and mathematical training; the dream, cherished since his visit to America, of fertilising technology itself with the methods of pure mathematics. Hilbert had scarcely any interest in these projects of Klein's

(15) Marx. Born on first visit to Göttingen – In any case, I still had not heard such frank, independent, free-ranging criticism of all possible proceedings of science, art, politics

(16) Re Klein – Even his daughter had to make an appointment to talk to her father

(17) Klein had once completely filled a very large blackboard with figures on the German middle schools (the scientific education of which he was also trying to reform), he asked his colleagues if they had any comments. "Don't it seem to you, Herr Geheimrat" Minkowski asked angrily "that there is an unusually high proportion of primes among those figures?"

(18) At the beginning of the 20th century mathematics students all over the world were receiving the same advice "Pack your suitcases and ~~go to~~ take yourself to Göttingen"

(19) Born was his personal assistant – Hilbert gave him time to see & talk to Hilbert every day

(20) It was common opinion in Göttingen, that there was no teacher who came close to Hilbert. In his classes mathematics seemed to the students to be still "in the making" and most felt preferred his lectures to those of Klein's "finished" encyclopaedic lectures.

(21) Which is the most important problem? Hilbert said "The problem of Zeros of the Zeta function, not only for mathematics, but absolutely most important"

(22) Unsuccessful Hilbert quite an interest in pedagogy. He did not have a very high opinion of the abilities of the ordinary student and believed that nothing was absorbed until it was heard several times. "Five times, Herrmann, five times!" was his

Minkowski's advice he gave to Weyl when that young man started his lecturing career "Keep computations to the lowest level of the multiplication table" and "Begin with the simplest examples" - These were his other favourite rules.

(23) lectures at the joint-seminar on maths & phys. conducted by Hilbert along with Minkowski & Born attending them speaks about electrodynamics, the Fitzgerald contraction, Lorentz local time, Michelson-Morley interference expts, all were thoroughly discussed and "we heard completely fantastic statements about electrodynamics - At that time Einstein as a patent clerk in Bern was developing his relativity theory." But of that" says Born" nothing was as yet known in Göttingen and in the Hilbert - Minkowski seminar the name of Einstein was never mentioned" - Born taking a topic in the field of ~~optics~~ for his dissertation, did not want geometry because he would be examined by Klein whom we dreaded. He therefore chose "astronomy", but still he would have to be examined by a mathematician, but now it would be Hilbert. Before the exam Born asked Hilberts advice on how he should prepare for the mathematical questions. "In what area do you feel yourself most poorly prepared" Hilbert asked. "Ideal theory". Hilbert said nothing more, and Born assumed that he would be asked no questions in this area; but when the day of the examination arrived, all of Hilbert's questions were on ideal theory. "Ja Ja" he said afterwards "I was just interested to find out what you know about things about which you know nothing".

(24) After 1905, the work of the Bern patent clerk became known in Göttingen & Minkowski

(156)

recalled his former student "Ach, der Einstein," he said sadly, "der Schwerenöter immer die Vorlesungen – dem hätte ich das nicht zugetraut" (Oh, that Einstein, always missing lectures – I really would not have believed him capable of it).

(25) There are periods in the life of a productive person when he appears to himself and perhaps actually is losing his powers. This comes as a great shock.

(26) Among the members of the audience (Minkowski's lecture on Space & Time) was Max Born whose interest in relativity had been aroused by the recent work of Einstein. Minkowski wanted Born to return to Göttingen as his colleague. He needed someone with the knowledge of optics which Born had – contacts between Minkowski & Born.

(27) Mrs Minkowski entrusted Born with editing Minkowski's work – He found his own work on the self-energy of the electron which so impressed Voigt that he offered him a position as Privatdozent in the Inst. of Th. Phys.

(28) Poincaré's remarks on the award of the 2nd Bolyai Prize to Hilbert "The variety of the investigations, the importance of the problems attacked, the elegance & simplicity of the methods, and the care for absolute rigor – Hilbert's influence on the progress of mathematics did not only come from his personal investigations, but also from his teaching, by the assistance which he gives to his students and which permits them in turn, utilizing the methods created by their teacher, to contribute to our understanding."

(29) Remarks of Feigmann, one of Hilbert's students: "What discomfort we mathematicians

felt in the lectures on theoretical physics when sometimes this, and sometimes that principle, without proof, was placed before us and all sorts of propositions and conclusions derived from it — "Physics" Hilbert announces " is much too hard for physicists"

(30) Ewald, Hilberts "Physics tutor" says " there was a long-standing controversy about the number of constants of elasticity in a crystal — I once took a report to Hilbert about the controversy & arguments of both sides. A few years later the whole problem ~~was solved~~ — which had held up crystal physics for more than fifty years — was solved by Max Born"

(31) Hilbert meeting Niels Bohr, Courant, Zermelo, Debye

(32) When people asked Hilbert why he did not have Fermat's last theorem, he says " Why should I kill the goose that lays the golden egg" — Reference to the interest derived on the Wolfskehl Prize for the theorem.

(33) Hilbert made four memorable trips to Weierstrass, Minkowski, Hurwitz & Derboux.

(34) The problem of a superior being having to live with lesser beings, which comes up quite often, and most people do not get it solved in time. They fail to recognise the existence of the problem, or else they need their superiority to overcome some complex they have. Hilbert avoided the difficulties very well.

(35) Weyl was convinced that mathematics — like art, music & literature — was a creative activity of mankind

(36) Hilbert's remark re - stopping inflation by issuing a new unit of currency called the Reichsmark " one cannot solve a problem by changing the name of the independent "variable"

(37) Harald Bohr & G.H. Hardy were frequent visitors ~~to~~ in Göttingen

(38) Transcendence of $2^{\sqrt{2}}$ being Hilbert's 7th Paris problem – Gelfond proved later the transcendence of $2^{\sqrt{-2}}$ & Siegel proved that of $2^{\sqrt{2}}$, utilising the Russian Gelfond's work.

(39) Born bringing as his assistants Pauli & Heisenberg in 1920 – Franck coming in 1922 – From 1922 on, the seminar on Structure of matter was carried on by Born & Franck. The members included during the twenties Heisenberg, Pauli, Oppenheimer, Compton, Jordan, Dirac, Pauling, Houtermans, Blackett & others. Hilbert rarely attended.

(40) Born had worked on matrix theory before Heisenberg when he worked on the lattice theory of crystals – Born's famous statistical interpretation getting the Nobel Prize later

(41) In 1927 Wigner became Hilbert's Physics Asst. after Nordheim left

(42) Hilbert's popular lectures – Enormous amount of labour put in by his assistants, making preliminary outlines of either new material or from old lectures, finally flavoured by Hilbert's own inimitable brand of logic & humor – ability to put things so that the "man in the street" might understand

(43) Hilbert had developed a taste for zoology & in genetics – His speech when Koenigsberg gave him the honour of making a citizen – Explanation of Kant's ideas – Kurt Gödel's paper of 1930 at end of Hilbert's career demolishing his specific epistemological objectives.

(44) He considered number theory as the most beautiful – At the international congress of mathematicians at Zurich (after his retirement), Hilbert stated that the theory of complex multiplication of elliptic modular functions which brings together number

theory and analysis, was not only the most beautiful part of mathematics, but also of all science (Cf. Ramanujan's work).

(45) Celebration of 70th birthday of Hilbert – Weyl wrote the birthday greeting that appeared in Naturwissenschaften. "The birthday of Hilbert", Weyl notes in his greeting, "was the high feast day for German mathematicians, celebrated year after year in warm personal veneration for the master, but also in personal affirmation of their beliefs and unity." Hilbert seems to me an outstanding example of a man through whom the immensely creative power of natural scientific genius manifests itself. .. I remember how enthralled I was by the first mathematical class I ever attended (at the Univ.) .. It was Hilbert's famous course on the transcendence of e and π."

(46) On the whole his lectures were a faithful reflection of his spirit; direct, intense; how could they fail to be inspiring?

Current Science _ Vol. 26, No. 10, Oct. 1957

p. 301 – The Centenary of the Lal Bagh by C.V.R.

p. 305 – Beautiful review of book by Randhawa on "Flowering trees of India"

[P.T.O

16/1/78 B.V.L.

Dictionary of Scientific Biography (Scribners) Vol. 9, p. 613

Narayana (fl. India, 1356) mathematics

Narayana the son of Nrisimha (or Narasimha) was one of the most renowned Indian mathematicians of the medieval period. His Ganitakaumudi, on arithmetic and geometry, was composed in 1356; in it he refers to his Bijaganitavatamsa, on algebra (see supplement). The Karmapradipika, a commentary on the Lilavati of Bhaskara II (b. 1115) is found in several south Indian libraries attributed to Narayana; but the author, a follower of Aryabhata I (b. 476) may be the Kerala astronomer & mathematician Madhava of Sangamagrama (ca 1340-1425)

The Ganitakaumudi consists of rules (sutras) and examples (udaharanas) which in the only edition, the two-volume one of P. Dvivedi (Benares, 1936-42) are given separate numberings that do not coincide with the division of the work into chapters (vyavaharas). In fact, the edition is based on a single manuscript which was evidently corrupt and perhaps incomplete. We do not ~~know~~ really know in detail the contents of the Ganitakaumudi. The Bijaganitavatamsa is preserved in a unique & incomplete manuscript at Benares; only the first part has been edited, by K.S. Shukla as a supplement to Rtam (1, pt. 2, 1969-71).

Bibliography: Various rules from the Ganitakaumudi are discussed by

B. Datta & A. N. Singh, History of Hindu mathematics 2 vols (Lahore 1935–38) Hassan and the section of their work devoted to magic squares is analysed by S. Gandmann "Islamic and Indian Magic Squares" in History of Religions 8 (1968–69) 181–209, 271–299, esp. 274 ff. — — Other references to Datta & Gandmann etc

28/2/77 At C.C.B.

(1) An operational approach to quantum probability — Davies & Lewis — Comm. Math. Phys. 17 (1970), 239–60

Making repeated measurements on continuous variables in a statistical system, a concept is used formalising the notion of observable as an operation by making a mathematical defn of an instrument. It is then possible to develop such notions as joint and conditional probabilities without any of the commutation conditions needed in the approach via observables. One of the crucial notions is that of repeatability, which we shall see is implicitly assumed in most of the axiomatic treatments of quantum mechanics, but whose abandonment leads to a much more flexible approach to measurement theory.

(2) Towards a general theory of measurement — I — D. R. Hoist — J. Math. Phys 11 (1970), 851–55

Derivation of intuitively plausible axioms for a general measurement theory — A slight generalisation of Birkhoff & von Neumann's approach to axiomatic quantum theory — To each physical system S is associated a complete orthocomplemented lattice (or logic) L_S and an order-determining set of states $\mathcal{L}_S$ closed under convex combinations. Thus a collection

$\mathcal{L}$ of such logics L_s is postulated. This collection is assumed to be directed i.e. any two logics L_1, L_2 in $\mathcal{L}$ are contained in some L_3 in $\mathcal{L}$ in such a way that the states on L_1 & L_2 are restrictions of states on L_3. Given a microscopic system $\underline{S}$ and an experimental apparatus $\underline{e}$ used to measure it, the composite system $\{\underline{S}, \underline{e}\}$ gives rise to a logic $L_{S,e}$ in $\mathcal{L}$. The sublattice of $L_{S,e}$ corresponding to the apparatus $\underline{e}$ is Boolean — Theorem — Every maximal logic in $\mathcal{L}$ is Boolean. Thus if there are non-Boolean logics in $\mathcal{L}$, the author concludes that there are systems for which there is no totality of knowledge

(3) Quantum families of Boolean algebras by M. D. Maczynski — (Look Russian summary)

A compact family $\mathcal{L}$ of σ-complete Boolean algebras is called a quantum family if ... — Ref. to Finch's work on quantum families

(4) Information-theoretical aspects of quantum measurement by A. S. Kholevo (Russian)

A generalized quantum measurement is defined as a generalized resolution of the identity in the Hilbert space of a quantum system, as introduced by Davis & Lewis (Ref. to paper (1) above) — This contrasts to the so-called "simple" measurement whose elements are, as in von Neumann's formulation, orthogonal projectors. Using Shannon's def^n (?) it is shown by an example that the information maximizes — — quantum system characterized by a non-commutative $*$-algebra of operators, to a classical system characterized by a commutative algebra, simple measurement suffice to acquire maximum information. The additivity of the information

for measurements on independent Quantum systems characterized by a product Hilbert space, is shown not to apply if measurements are restricted to simple measurements"

(4) Stable symmetric probability laws in quantum mechanics — Central limit theorem for quantum-mechanical canonical observables — Symmetric distribution (p, γ) in relation

(5) A mechanistic model for quantum measuring process — System S, a measuring apparatus A and an interaction coupling S S & A

(6) Respective roles of classical & quantum logic — Quantum mechanics & classical probability theory — Copenhagen interpretation of statistical probability

(7) orthomodular lattices & modal logic

(8) Quelques résultats dans la théorie non-commutative des probabilités — Conditional expectation of a self-adjoint operator A relative to another self-adjoint operator Q — Gleason measure $m(A)$ — Continuous of momentum observable P to the position-observable Q and a pure state $m(A) = (\psi, A\psi)$.

(9) Bemerkungen zur Quantenlogik by P. Jordan — The author believes that the difficulties of quantum theory cannot be solved without first generalizing the mathematical framework and that at present there is no better guide to this than to investigate mathematically interesting extensions. He chooses to begin with a lattice formulation of quantum logic & extend this to another (non-commutative) lattice. He considers doubly distributive

modular & distributive skew lattices, and notices that already in the last case difficulties arise as to the appropriate generalisation

(10)

19/5/78 At Institute

(1) Peter Havas I - J. Math. Phys 16 (1975), 1461-68

& II - " " No. 12, 2476-2489

Deals with separation of variables in the H.J., eg Schrödinger & related eqns

(2) R. Giesen - Helv. Phys. Acta 48 (1975), No. 3 - 261-286

Relativistische Gleichungen für Teilchen mit beliebigem Spin

- Ref. to ~~Paul~~ ~~Paul~~ Fierz-Pauli article - Proc. Roy. Soc. A. 173 (1939)

(3) Elem. particle physics & scattering theory Vol. 1] 211-232

Brandeis Univ. Summer Inst. July 19-28, 1967. - Sudarshan's paper on "Infinite component wave equation" presented at the ~~Institut~~ Summer Inst

(4) Bohr-Sommerfeld condns in geometric quantisation - Ref. Math. Phys - Ref to harmonic analysis of on homogeneous spaces - $ML(n, C)$ & $GL(n, C)$

Ref. to 2-dimn Kepler problem - one choice leads to integral A.M & to Sommerfeld formula for fine structure of H-atom; another choice leads to half-integral A.M & fine structure of π-mesonic atom.

(5) Onofri Enrico - J. Math. Phys 17 (1976) - Dynamical quantization of the Kepler manifold.

Refs. to "theory of geometric quantization" - Starts from an SO(n,2) homogeneous symplectic manifold & approximates

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(27/5/78) The much sought after "History of Religions, Vol. 8, was after all obtained from the Madras Univ. Library through the kindness of B. V. Raghavendra Rao. The article by Schyuler Cammann on "Islamic & Indian Magic Squares" consists of two parts - Part I devoted mainly to Chinese & Islamic work & Part II to Indian & European work.

Instead of copying out both the articles in full, we shall ~~put~~ make notes of important points from Part I relevant to Indian work & make more detailed notes from Part II

But the idea with which I searched for these articles viz. to find Narayana's method of doubly-even squares using sub-squares has not been fulfilled. In Part II the author just mentions these (just as in the Ency. article) but does not give any examples. I am afraid I have to go back Ganitha Kaumudi itself for reference

Part I ✓ pp. 181 – 209.

Modern mathematicians consider magic squares as mathematical curiosities or amusing games, and as such they have been relegated to the minor category of Mathematical Recreations. However, this is a rather recent development. Magic squares once enjoyed a much more important status, and they constitute a major preoccupation of many brilliant scholars in the Middle East and India.

Work on magic squares by different countries or different peoples was motivated by basic ideas from the philosophies or religions of the respective people who made them. Thus preferences for certain types of magic squares arose in each culture – only the common notion was that a magic square was a miniature diagram of the Earth or of the Universe, the nature of the Universe, and what took place within it, was very differently explained in each culture.

[I think the author's preoccupation with religious motivation has been responsible for his not explaining in greater detail the several techniques adopted by several earlier work. His writing in the Journal "History of Religions" is perhaps the justification for this attitude]

When we come to Indian magic squares, one authoritative early source, Narayanas Ganita-Kaumudi of 1356 has fewer garbled squares, but the text

is somewhat difficult. While the author refrains from direct statements about the ideas behind the squares, at times his choice of words affords a clue, as we shall see. Then too, the preference of Narayana and later Hindus for certain methods of constructing magic squares is so consistent and the religious or philosophical significance of those methods is so transparent, that one feels able to interpret them with some confidence.

[Ref. (n.1) Very little has been written about magic squares in India, but a number of short descriptions appear in the Punjab Notes and Queries I and II (1884-85). This is more useful for the examples illustrated in them than for the remarks about them. — For squares among Indian Muslims, see Jafar Shariff, Qanoon-e-Islam, trans G. A. Herklots, (London, 1832) esp. pp. 347-56.] — Ref. to Andrews. 1960 Edn]

[Ref. (n.7) Narayana's Ganita Kaumudi ed. Padmakara Dwivedi, was republished in the The Princess of Wales Sarasvati Bhavana Sanskrit series, No. 57, Part II of this containing the Chapter on magic squares, was published in Allahabad in 1942. I also have a microfilm

of an incomplete MS containing this section: India Office Library, London, No. 596 B (A similar MS is in the India Office Library, Cambridge, No. 77).]

Some remarks about secrecy being observed by earlier workers in China, Islamic countries & India in writing about magic squares — Many things by oral transmission

[Ref: (N-2) – Cammann. "Magic square of Three in old Chinese Philosophy and Religion" History of Religions I (1961), 37-80

(N-41 – 5. Cammann "Old Chinese magic squares" Sinologica. VII (1962), 14-53]

The Chinese background — Lo shu traced to 4th century B.C. although claimed to be ~~2th~~ two thousand years earlier -

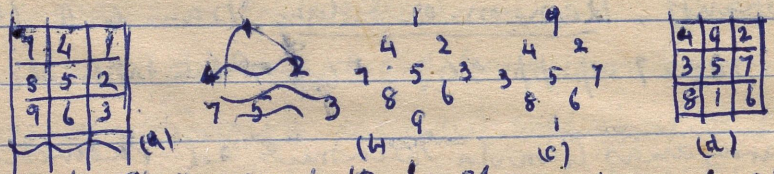

Fig. 1. How the Chinese made the Lo-shu - (a) natural square of three; (b) filling (a) to form like a diamond; (c) reversing centre row & centre column; (d) collapsing the diamond to form the magic square — Use of Lo-shu in constructing magic squares of higher numbers — Ch'ien Lo-Shing order nine - Lo-shu to make magic squares of size [Ref (N-15) – Cammann earlier than Yang-Hui's

magic square of eight & magic square of ten were both Chinese. I have since discovered that I was wrong; neither was of Chinese origin construction and they were undoubtedly borrowed from another land, probably India, as I shall explain in Part II]

[Yang Hui's book is 1275 A.D. So if he borrowed the methods for 8 and 10 from India, it cannot be from Narayana's book which is A.D. 1356. So this clearly indicates existence of these methods in India even before Narayana's book]

[The assumption that magic squares travelled from China to Persia, Arabia & India appears quite common with all Western scholars. This certainly requires more careful examination]

[p. 22) - Even in forming the natural squares, there were different procedures in the various cultures, due to difference in habits of writing. In China, the natural squares were constructed in vertical columns beginning at the upper right; in the Islamic world they were made in horizontal rows beginning at the upper right; in India, horizontal rows beginning at the upper left]

p. 191. for the magic square of six, the maker apparently began by halving the sequence from 1 to 36 and writing it in two parallel columns, reversing the direction of the second, to produce eighteen complementary pairs. Ten of these pairs he took to make four 4×4 blocks which he set out in a cross

then he split the last eight pairs and set the numbers in the opposing corners, doing this in such a way that the four numbers in each of the nine blocks gives the same sum of 74 (See Fig. 3a). This uniquely Islamic solution for the magic square of order six remained in use for several centuries.

11	22	32	5	23	18
25	16	7	30	13	20
27	6	35	36	4	3
10	31	1	2	33	34
14	19	8	29	26	15
24	17	28	9	12	21

p. 192 – The Hindus do not seem to have ever used bordered ~~squares~~ magic squares. (n. 25) The North Indian Muslims, however, being under strong cultural influence from Persia used bordered squares. The Qanoon-e-Islam shows bordered squares of six and ten, for example, although its magic square of six is done differently. – Magic squares of six by this deployment of numbers in 2×2 ~~squares~~ subsquares continued to be popular in the Islamic world for at least six centuries.

[Fig 3 (p. 192) – from the Rasa'il of the Ikhwān-as-Safā]

p. 193 – For the square n = 8, the Islamic world used the same technique as used for n = 6, but this did not work well. They did not realise the radical distinction between "singly even" & "doubly even" squares. The magic square of nine is also inept

p. 194 – Bordered squares are ascribed to the German author Stifel & are

Stifel called "Stifel squares", but they were known hundreds of years before his time, and were transmitted to Europe from the Islamic world, though the links are not yet all complete

(n. 30) Ahrens assumes that bordered squares were an Islamic invention. It would seem, however, that the earliest bordered squares for odd numbers were Chinese, while those for even numbers were probably invented by the Muslims. Anyway there is no question that both types were transmitted to Europe from the Islamic world.

p. 197 – Tables of the Islamic continuous magic square of five using the diamond method

p. 198 – There is no doubt about the priority of this continuous method of construction in the Islamic world.

p. 200 – Later Islamic books used "the Hindu continuous method", but this method was never popular with the Muslims.

The Knight moves method appears in Islamic books of the early thirteenth century A. D. It is quite likely that the Indians may have discovered it first.

p. 205 – Persian writers making 6 x 6 squares by break of symmetry. This system was characteristically Indian

p. 203 – They [the Christians] engraved magic squares on rings and amulets for protection

against sickness & other evils; they incised it in metal bowls and medicine cups to influence or purify the contents, and they inlaid it in gold on sword blades of fine Damascus steel to improve their cutting power (n. 49) The Metropolitan Museum of Art, in New York City, has a number of swords with magic squares on their blades in the Arms and Armor collection: for other Islamic uses of magic squares, see Ranson-e-Islam pp. 355-57)

Part II pp. 271-290. (pp. 290-299 devoted to oriental heritage in European magic squares).

Magic squares in India.

(p. 271) We have seen that in the Islamic world a magic square usually was only a preliminary exercise. It conveyed complex religious meanings. The same was true of the magic squares in ~~medieval~~ medieval India[1]. However, as the symbolism expressed

(n. 1) p. 271. Very little has been written about magic squares in India. The only early source is Narayana's Ganita-Kaumudi (hereafter known as G. K). microfilm translation by M.K. D. Joshi. I discovered construction methods myself by working through the squares to find out how they must have.

been made. Otherwise there are only a number of brief descriptions in the short-lived periodical Punjab Notes and Queries I and II (Allahabad, 1884-85). The Indian Muslim squares, which belong to in the Islamic tradition, are briefly described in Jafar Sheriff's Qanoon-e-Islam, trans G.A. Herklots (London, 1832)

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in the latter was described by ideas (p. 272) and beliefs from Indian thought and the Hindu Religion, the squares themselves were ~~invariably~~ inevitably different in form, and often in their basic construction, from those that were made by the ancient Chinese or the people of the Islamic nations. And yet some Chinese magic squares and perhaps some Islamic ones also, seem to have provided the stimulus for later developments in India.

In spite of the absence of concrete information from India — or perhaps because of that — many Western writers have asserted that magic squares were invented in India. When other ~~other~~ authors pointed out that there were no factual grounds for such assertions,[3] the pendulum swung to the other extreme, and some people claimed that India had nothing to do with the early development of magic squares[4]

(n. 3) p. 272. William Ahrens "Studien ---"
(n. 4) " — Florian Cajori — A History of Mathematics (1926), p. 92.

Actually, it now appears that the first definite magic squares come from China from about the fourth century B.C., and many other significant later developments were made by Arabic & Persian scholars, working from the tenth to the fourteenth centuries A.D. However, there is no doubt that some medieval Indians were deeply interested in magic squares, and they seem to have made significant innovations - though, as we shall see the full extent of their contributions is still unclear.

Probably the Indians' interest in magic squares began when they learned about the old Chinese Lo-Shu square of three. They could have obtained this directly from China, either through Hindu merchants or Buddhist missionaries returning from visits to China, or else through Chinese Buddhist monks who came to India as pilgrims [would it not well be that the Buddhist missionaries took the "so called Lo-Shu" invented in India to China?]. Alternatively, they might have acquired it by way of the Tibetans, who had been under strong influence from T'ang Dynasty China and had taken over the Lo-Shu for divination. Since the Indians had no feeling for dates in general, and never bothered to record borrowings from other nations, [is it not equally true that they did not record lending to other nations, and thus other nations cleverly missed out on knowledge of what they got from India?] it is now impossible to trace how or when this happened. As we have previously noted in discussing the earliest Islamic set of squares, it is remotely possible that Indians might have had some part in the first transmission of magic squares from China to the Islamic world;

(p. 273)

but we lack any direct evidence for the use of magic squares ~~in~~ India itself before the eleventh or twelfth century A.D., and even this earliest evidence suggests a counterinfluence from the Islamic world.

The earliest known magic square in India is a square of four. Examples of it were found by archeologists in two places: the first at Dudhai in the Jhansi district, and the other at Khajuraho. ~~The~~

The Dudhai square . . . . However this argument is ~~dubious~~ questionable & the dating is dubious. The other example from Khajuraho was found in a Jain inscription that is assumed to ~~be~~ date from the 12th or 13th century (7), and in this case the dating is probably more reliable. But is this square actually Indian? If we rotate it 180° we ~~see~~ can see that it is the

a

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

b

4	15	6	9
5	10	3	16
11	8	13	2
14	1	12	7

c

8	11	14	1
13	2	7	12
3	16	9	6
10	5	4	15

Fig. 1 – The earliest Indian magic square compared with an Islamic one; (a) the magic square from Khajuraho; (b) the same rotated 180° to show its relationship to the next; (c) the basic Islamic magic square of four (called Plato's square, See Part I, Fig 7, p. 202)

(11.7), p. 273. See the Jahresbericht der Deutschen Mathematiker Vereinigung XIII (Leipzig, 1904), 383

basic
familiar Islamic magic square of four (Fig 1(c)) with its sequences written in a variant order (3,1,4,2) commencing at the upper right corner in the usual Islamic fashion [I wonder what the author means by sequences in variant order. By sequences does he mean rows or cols? If so Fig 1(c) is not a variant of 1(b). For that matter 1(c) is a variant of 1(a). What was the need for Nasik? I think the author has messed up the whole thing]. As we have already noted the Islamic scholars were fond of experimenting (p. 274) with their favorite magic square of four, altering the sequences to get new variations & this seems a good example of that ... [some rubbish] (p. 8) see Punjab notes & queries I (1884), items 402, 537, 1017; and G.A. Grierson "An American Puzzle", The Indian Antiquary X (1881), 90.

The most authoritative source for early Indian magic squares is Narayana's Ganita-Kaumudi, written in A.D. 1356. An entire chapter of this book on Indian Mathematics, entitled Bhadra ganita is devoted to the construction of the squares. - - - Narayana seems to have prized one particular square of four above all others ...

Fig. 2

a

1	8	13	12
14	11	2	7
4	5	16	9
15	10	3	6

b

16	9	4	5
3	6	15	10
13	12	1	8
2	7	14	11

(a) Narayana's favorite square, (b) the square from Gwalior Fort and he later repeated this in other contexts.

At first glance Narayana's favorite magic square of four might seem to have

nothing to do with the Islamic basic square; yet actually this, too, was apparently merely another Indian version of the latter. Its numbering begins at the upper left in Indian style, but if we trace out on it the pattern of the Islamic square (in reverse because of beginning at the opposite side), we find double half-sequences: 1, 2 – 9, 10; 3, 4 – 11, 12; 5, 6 – 13, 14 7, 8 – 15, 16. This is a variation that the Muslims themselves sometimes used [This relationship makes no meaning. I wonder what the author is talking about. According to his argument every Indian 4-square could be related to his Islamic squares & all 4-squares are variants of each other. He is very much prejudiced against Indian work & is keen on showing that all of it was borrowed from Islamic work – most unlikely]

Another form of Narayana's favorite square was discovered in the last century in a sunken temple in the Gwalior Fort, bearing a date (p. 275) equivalent to 1483 A.D.[10]. Though this Gwalior example may look quite different, it is the

{(n. 10) – Capt. Shortreede "on an ancient Indian Magic square, cut in a Temple at Gwalior" J. Asiatic Soc. of Bengal (N.S.), XI, Part I (Calcutta, 1842), 292. The more recent literature contains many references to "the magic square on the Gate of Gwalior Fort" apparently referring to this one. I have personally twice climbed the citadel at Gwalior to examine

the fates of the Old Fort, but found no magic squares on them}

same square with its corner blocks interchanged diagonally. Both of the squares in Fig. 2 also embody another pattern which caused them to have a profound significance for the Hindu people. When one counts out the numbers in each in regular order (1, 2, 3, 4, ---), every pair is seen to alternate direction, in a manner that resembles the course of the threads in weaving. That was the reason for the popularity of this type of square among the Hindus. For they, too, considered the magic square as a universe in miniature: a microcosm corresponding to the greater Universe, and they conceived of the latter as a vast fabric woven by the Gods" In this respect, then, even

{Ref. n(11) - See Stella Kramrisch, Unknown India: Ritual Art in Tribe and Village (Philadelphia, 1968), p. 66. See also the Brihadaranyaka Upanishad II, 8, 3-8 and the Mundaka Upanishad, II, 2.5}

if their construction pattern may have borrowed from another culture, these squares were "typically Hindu"

[Are these interpretations of the author contained in Narayana's book, or is it a flight of fancy on his part? Perhaps he also wants to show off his knowledge of Indian Philosophy & Religion by referring to the Upanishads! Also the alternating direction he mentions is not true for Fig. 2 b. In fact the lines 7 to 8 and 9 to 10 are not alternate in direction but convey a straight. Also if one tallies the

"authors favourite" Islamic square, Fig 1(c), this alternating & converging in directions can be noticed. So all this nonsense is a figment of the authors imagination]

If the Hindu magic squares of four seem to echo earlier Islamic ones, with somewhat less versatility, the Indians made up for this with other kinds of squares [what a patronising attitude on the part of the author?]. In Part I we have already referred to the "Hindu continuous method" an example of which is shown here in Fig. 3b. Probably this ingenious method of construction was discovered through a careful ~~study~~ analysis of the Lo-Shu magic square of three in its inverted form, {Fig. 3. (a) The Chinese Lo-Shu with 1 in middle of top row, (b) Square n = 9 with usual R.M & B.M} since this square was much (~~Fig~~ p. 276) favoured by the Hindus and they doubtless subjected it to a thorough investigation [Here again the author wants to detract from Hindu work by referring to Lo-Shu].

Description of the method and reference to "potentially endless sound".

It would be tempting to assume that this might have occurred after some Indian had seen a Persian continuous magic square, and thus learned of the concept of "perpetual motion" within a square; because the first examples of this Hindu method do not appear until 1356 [according to the author no Indian work on odd squares appeared before this date], when they were presented in Narayana's book. However, his squares look highly sophisticated

suggesting a considerable familiarity with this method[12], and even though they are more

{ n(12), p. 276 – In the G-K (1942 ed), p. 393, there is a magic square of fine and one of seven, both made with augmented short sequences, skipping a number at each break move, before commencing the next short sequences. It is quite evident that Narayana tried to render them especially complex to conceal the relatively easy method of construction. As a further peculiarity, each of these squares begins in the center cell of the right edge, and ends in the center cell of the left edge }

[Here again the author appears unfair to Narayana, without trying to find exactly why a number was skipped at each break move. I think it is not for concealing but specifically to indicate the nature of the break move. Anyway the last sentence of n (12) shows that if 1 be taken at centre of right column, and 25 is at the center of the left col, the break move must be (−1, 0), the R M being (1, 1). This shows that Narayana knew the principle of the generalised Indian method for odd squares, dealt with in Chapter on this topic. This appears to be only new fact brought to light after reading Cammann's article].

than a century later than the earliest known Islamic continuous squares, we shall see that there is reason to believe that people in India must have known this method since the early thirteenth century at least. We shall also see that, although we (p. 277) lack any traces of the initial stages of magic squares in India itself, we do have some reasonable evidence from other countries which seem to indicate a family

early development there [A grudging acceptance or surmise by the author of early development of magic squares in India. In fact, if the Persian method be assumed later than the Hindu method, known very much earlier, facts appear to square up easily]

[By the Persian continuous method the author means the case where the R.M is (1, −1) & B.M is (0, −2) and 1 is below the middle cell. Hence in the endless round from 25 to 1 one has to pass through the middle cell 13 since 25 is above it. In the Indian method, one need not pass through the middle number 13 while passing from 25 to 1 since 1 is at top of the middle col. & 25 at the bottom]

Since the middle number is not even remotely involved ~~in the~~ with the renewal, the people of Islam took little interest in the Hindu method, after they had learned how to do it. The Muslims of Northern India (and modern Pakistan), for example, clung to the Persian continuous method and disregarded the Indian one completely

The Hindus were not interested in the middle number. Since in their method the numbering begins at the edge, and only resumes again by re-entering at the opposite side thus ~~be~~ the unending process being recurrently broken, the Indians saw the square as a perfect representation of the Hindu (or Buddhist) conception of Life as a process of "Endless Becoming", during which the individual dies, leaving the world for a brief time, to be reborn back into it; or — in another view involving a much

longest time span – the world itself is destroyed to be recreated.

~~The Persian method~~ [What a lot of philosophical meaning the author has put in this seemingly trivial construction of this magic square!]

Narayana was, however, not very much interested in it. – he was far more interested in a still more characteristically Indian technique, which involves constructing two preliminary squares, then using them to form a third, the actual magic square

[Mention in the book ~~that~~ at the appropriate place, Cammann's philosophical interpretations of the Islamic & Indian methods; also mention the Western method & make some funny remarks]

(p. 278)

Method explained by taking $n = 5$ – flipping as if to close a book – this could be used

4	5	1	2	3
5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2

a

15	20	0	5	10
20	0	5	10	15
0	5	10	15	20
5	10	15	20	0
10	15	20	0	5

b

14	10	1	22	18
20	11	7	3	24
21	17	13	9	5
2	23	19	15	6
8	4	25	16	12

c

Fig. 4 (p. 278) {The Hindu method of constructing preliminary squares; (a) the primary square; (b) the root square; (c) a lozenge type magic

square of five, made by flipping (b) upon (a) and adding the numbers in the corresponding cells } (15)

for any odd n (p. 279) — Remarks that in (c) there is an R.M which is an "extended Knight's move" of (−2, 2) and a B.M ... , y − ... R.M ... {7 n−1 n−1} & B.M (0, −1)

[quite an interesting ...

Another n... ...iminary squares of this ... ...ual square assume... ... all gathered in ... numbers separately ... may therefore be d... we can be ~~sure~~ reasona... ...bining squares.

{(n. 15) p. 279. ... ...umbers in the second squ... ... in the first square & towards the left in the second. ... see, the Hindus had a specific symbolic reason for insisting that the second (root) square ~~be~~ should be flipped over to face ~~the~~ down upon the first [I suppose the sexual intercourse business!]

[This gives an opportunity to introduce lozenge squares as Indian and to investigate such squares more fully & incorporate the results in the book — Also flipping business to be examined in relation to usual method]

(64)

$x + 6 = y + 8$

$y = x - 2$.

$y + 4 = x + 2$

$y = x - 2$.

2 33

12 19

23

31

2			10
	12	20	
	15	19	
21			33

2 7 9 10

11 12 14 15

17 18 19 20

21 26 30 33

54 32

43 34

23

~~5, 2, 1, 1, 2, 1, 2, 1, 1, 1, 1, 5, 4~~, 3

35

54 31

12

longer time span – the world itself is destroyed to be recreated.

~~The Persian method~~ [What a lot of philosophical meaning the author has put in this seemingly trivial construction of this magic ... 17

Nara... it. for he was far ... lly Indian techn... mary squares, the... i square

[Mentio...

philosophical ...

the Western met...

Method explained ... d.

274.00
111
163

28
52
74
110
264 (66)

2	7	9	16
11	12	14	15
17	18	19	20
21	26	30	33

29
38
38
0
50

2 − 3 = 8 − 9
4 − 7 = 14 − 17

2 + 7 + 2x + 13 = 50
2x = 28

a

4	5	1	2	3
5	1	2	3	4
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2

b

15	20	0	5	10
20	0	5	10	15
0	5	10	15	20
5	10	15	20	0
10	15	20	0	5

c

14	10	1	22	18
20	11	7	3	24
21	17	13	9	5
2	23	19	15	6
8	4	25	16	12

Fig. 4 (p. 278)

{The Hindu method of combining preliminary squares; (a) the primary square; (b) the root square; (c) a lozenge type magic

(70)
(185)

squared figure, made by flipping (b) upon (a) and adding the numbers in the corresponding cells }(15)

for any odd n (p.279) – Remarks that in (c) there is an R.M which is an "extended Knight's move" of (−2, 2) and a B.M (0, −1). For n > 5 (odd), he remarks that R.M is $\left\{-\frac{n-1}{2}, \frac{n-1}{2}\right\}$ & B.M (0, −1)

[quite an interesting remark]

An the more striking effect of this construction – using preliminary squares of this specific pattern – is that the numbers in the final square assume a particular formation, with the odd numbers all gathered in a lozenge or diamond, at the centre, and the even numbers separately clustered in the corners. Magic squares of this type may therefore be described as "lozenge squares, and whenever they appear we can be reasonably sure that they must have been made by this method of combining squares.

{(n.15) p.279. of course this can also be accomplished by initially writing the numbers in the second square in the opposite direction; working toward the right in the first square & towards the left in the second. But as we shall soon see, the Hindus had a specific symbolic reason for insisting that the second (root) square should be flipped over to face down upon the first [I suppose the sexual intercourse business!]

[This gives an opportunity to introduce lozenge squares as Indian and to investigate such squares more fully & incorporate the results in the book – Also flipping business to be examined in relation to usual method]

The method of combining preliminary squares remained the favourite Hindu technique for constructing odd-number magic squares, down through the centuries; and, as Narayana demonstrates, it was also extended to make squares of "doubly-even" order[18]. The

{(n.18), p.279 – See G-K. (1942 ed), pp. 375-78}

latter required a somewhat different (p.280) arrangement of numbers in the two initial squares, and it resulted in very different patterns of distributions. Instead of lozenge effects, we find squares of eight in which each row has an exact alternation of odd and even numbers – or even & odd – while in the columns the same kinds of numbers meet only at the centre; then, there is a second type in which the upper right and lower left quadrants consist entirely of even numbers, while the other two quadrants contain the odd numbers exclusively[19]. More characteristic patterns make the Hindu doubly even squares

{n.19, (p.280) Examples of both these types are shown in G-K, p.378. Those patterns which represent the contrast or balance of the odd and even numbers serve to emphasise the role of the magic square as a symbol of the coincidentia oppositorum}

just as distinctive as their lozenge squares, and can serve equally well as a clue to the construction method[20]

{n.20, p.280 – A study of these basic patterns can be most useful for classifying & identifying unfamiliar magic squares}

[In fact, the author has just not given what exactly I wanted. So it becomes essential that I should go to G-K itself for information. I wonder why the author did not give one example at least of the above two types of squares of order 8. Looking at the whole of Andrews' book, I haven't come across a single 8th order square of the second type i.e. where upper right & lower left quadrants consist entirely of odd nos, while the other two consist entirely of even nos. Anyway try & see if we can get this independently]

al-Buni refers to it as a "method of the Indians" in connection with a magic square; and although he was not specific, he must be referred to the IV-method. Similarly Yang-Hui's book of 1275 gives an example of the an 8-square of the first of the above two types which is foreign to Chinese methods.

This particular situation provides a good illustration of the fact that priority of publication does not necessarily establish priority of invention. [Hear, hear! The author's earlier attempts to give priority of invention to Islamic workers loses much of their force by this very fact] Narayana's method must have been passed down from teacher to disciple (p. 281) without being committed to writing; Narayana himself lists a number of sages or gurus, in the course of what must have been at least two centuries of transmission (24)

{(n. 24, p. 281 - Padmakara Dwivedi mentions this in his intn to G-K (1942 ed), iii-iv.

(~~87~~) (188) (*) [This statement is doubtful for as the author himself has pointed out Narayana used the root square with 5, 10, 15, 20, 25 (n. 14, p. 278), and the result was a magic square augmented by 5]

Persian manuscripts made by combining pieces refer only to odd squares, but no even ones. Moreover the single even number example mentioned by Yang Hui in 1275 was only an isolated case with no indication that he knew how to make it. Once again, we see that this technique must have been primarily an Indian development. [After all, some consolation].

This technique could not have been employed before the discovery of zero. (*) [This discovery appeared in S.E. Asia in the 7th century A.D. did not appear in India until the late 9th century [Of course, all according to Western Scholars, for eg. D.E. Smith & Joseph Needham's book "Science & Civilisation in China III (Cambridge, 1959), 186–88]. As we shall see this method is not likely to have been discovered until after the invention of the "Indian continuous method" which probably occurred around 1100. Thus the N-method must lie after 900 A.D & before 1200 A.D.

This sophisticated process of combining squares was probably not discovered by some great leap of intellect or by exceptionally brilliant mathematical analysis. There must have been a way by which it naturally suggested itself to someone.

(p. 182). The discovery of the "Indian continuous method" was a necessary first step. After this the Combination method could easily have been happened upon in the course of efforts on it.

The magic square of nine was a favourite one for use by the Continuous method. This was because (i) it was difficult to construct in the past & hence was a challenge, and (ii) it was not too unwieldy & hence interesting for experimenting. At any rate 9 × 9 squares became very popular in Islamic & Hindu circles about 12th century. They both knew the

trick of "casting out nines" (ie taking any large number, adding its digits, and continuing to do so until the number was reduced to a single digit[27])

{ n. 27, p. 282 – In fact the Hindus might have detected the method of casting out nines. See Datta & Singh, History of Hindu Mathematics, Part I (Lahore, 1935), pp. 186–86 }

Now of course one took a magic square of nine made by the Hindu continuous method (as shown in Fig. 36), and applied this "horizontal addition" to every number above 9, the entire square would have reduced to nine sequences of the digits from 1 to 9, producing a primary square as in Fig. 5a (p. 283). Later subtracting these numbers from the original numbers in Fig. 36 [not shown in this notebook], one would have got the root numbers 0, 9, 18, . . 72 (See Fig. 56, p. 283).

(a)

2	4	6	8	1	3	5	7	9
3	5	7	9	2	4	6	8	1
4	6	8	1	3	5	7	9	2
5	7	9	2	4	6	8	1	3
6	8	1	3	5	7	9	2	4
7	9	2	4	6	8	1	3	5
8	1	3	5	7	9	2	4	6
9	2	4	6	8	1	3	5	7
1	3	5	7	9	2	4	6	8

(b)

45	54	63	72	0	9	18	27	36
54	63	72	0	9	18	27	36	45
63	72	0	9	18	27	36	45	54
72	0	9	18	27	36	45	54	63
0	9	18	27	36	45	54	63	72
9	18	27	36	45	54	63	72	0
18	27	36	45	54	63	72	0	9
27	36	45	54	63	72	0	9	18
36	45	54	63	72	0	9	18	27

Fig. 5 (p. 283) – Analysis of the Hindu continuous magic square of nine: (a) primary square (Pattern B) made by "casting out nines"; (b) the root square (Pattern A) made by subtracting the numbers in the primary square from those in the original square

After finding this pair of analytic squares, the converse process of building original square by adding the two analytic squares.

Note that the derived squares have different basic patterns. The root square (Fig. 5b) follows the pattern shown in Fig. 4a (p. 283) with the numbering of each sequence starting one cell higher in each successive column, as it moves towards the right [for eg. in 1st row of 5b, the numbers are 45, 54 (45+9), 63 (54+9), 72 (63+9), 0 ≡ (72+9), and so on. Similarly in the all the other rows. In my notation starting from the middle row this would be end-step (8,1)]; where as the primary square (Fig. 5a) has its sequences beginning two cells higher each time [for eg. in first row of 5a, nos. are 2, 4 (2+2), 6 (4+2), 8 (6+2), 10 (8+2) ≡ 1, and 3, 5, 7, 9. In my notation this would be the middle step (4,5)]

Still further experimentation would have shown the discovery of this phenomenon that it was possible to use either of these patterns, individually, to make two uniform preliminary squares with the primary numbers and the root numbers arranged in exactly the same order. However, in this case, one of the pair would have to be flipped before adding in order to produce the required sums. The Hindus chose to flip over the second (root) square, for a symbolic reason, to be discussed later.

When they used two squares in Pattern A, they achieved a lozenge square of the type shown in Fig. 4c [In this case both the primary & root squares are formed by the same end step (8,1) in my notation. But the flipping of the root square is equivalent to reversing central row & changing the end step to (1,8). This is in consonance with general results. See my MSS. p. 96 or 97. The condition that middle row should be a central series follows from the method from which the two squares are split up from the Hindu continuous square of n = 9]

a rectangular lozenge property

However, when they used the two preliminary squares in Pattern B the result was a pandiagonal Knight's move square, as shown in Fig. 6c.

a

2	4	1	3	5
3	5	2	4	1
4	1	3	5	2
5	2	4	1	3
1	3	5	2	4

b

5	15	0	10	20
10	20	5	15	0
15	0	10	20	5
20	5	15	0	10
0	10	20	5	15

c

22	14	1	18	10
3	20	7	24	11
9	21	13	5	17
15	2	19	6	23
16	8	25	12	4

Fig 6, p. 283 – Magic square by the Knight's move in chess, made by the Hindu method of combining squares

(p. 282) [The author has taken two squares a & b of order 5 for illustration of Pattern B, thinking that the same thing holds good for $n = 9$, but unfortunately when applied to Fig 5.a, this pattern, which is equivalent in my notation to the middle step (4, 5) does not give a nasik square since the middle row of Fig. 5a does not form a nasik series. See the actual work on p. 78 of Bookno. 20. This is obviously a mistake on the part of the author]

(p. 284) The Hindu square is equivalent to an Indian continuous magic square made with the Knights' move in chess. [only true for $n = 5$ and not $n = 9$]; but if the Hindus had chosen to use this kind of square

(192)

⊛⊛ wrong statement.

(*) Also the very first magic square of order 4 is ~~found~~ in Granthā Kaumudī is made by using the Knight's move.

~~they would doubtless have reversed it, because they preferred progression to the right.~~

Actually, they do not seem to have liked the Knight's move square – none is shown in Narayana's book – although it was very popular among the Indian Muslims ⊛⊛ [These statements do not seem to make any sense since it does not mention Nilakantha's use of the Knight's move] ⊛

It would be deceptively easy to conclude ~~that~~ from this that the Knight's move squares were actually an Islamic invention; but the idea could have been ~~borrowed~~ discovered in India, then dropped. If so, we have again a clear example of the choice or rejection of a certain type of magic square on purely ideological grounds.

[At least he does not complete discard the idea that the Hindus might have used Knight's moves].

The preliminary squares of Pattern A have each digit or root number repeated along diagonal lines [because of end steps being used]. Thus when the second square is turned over to combine its numbers with the first, their diagonal lines intersect like the crossing of threads, to provide a symbolic interweaving of numbers, by means of which is created the magic square proper. This symbolic effect was fully in accord with the basic Hindu idea, previously alluded to, which conceived of the Universe as a woven fabric. Hence we can understand one reason for the great popularity of this method of construction (using end steps) among

the Hindus. However the preliminary squares of Pattern B [using middle steps] used to make the Knights' move square, have such a broad distribution of the numbers that no single digit appears more than once in the same diagonal (this is what makes the resulting magic square pandiagonal), and therefore the symbolism of an interwoven universe is totally absent [some meaningless remarks].

On the other hand, Pattern B is perhaps the most interesting and most practical pattern that can be used for magic squares by the combining method; and this is not the only cause it can be used to produce pandiagonal squares. Notice in Fig. 6 that the five 3's in the primary square, and the five 10's in the root square, fall into a very distinctive, equally spaced design. Due to this particular design, when one of these preliminary squares is reversed over the other, these two highly symmetrical, yet opposing patterns can be rotated in relation to each other, so the upper one can be turned over the lower one to produce four different (p. 285) Knights move magic squares, all pandiagonal (Because of this quality, one might appropriately speak of Pattern B as the "pin-wheel pattern" (30)). Since the Hindus did not appreciate

{ n. 30, p. 285. Still another pattern (Pattern C) we might call the "half-pinwheel pattern"

because the upper square can be rotated 180° on the lower one to produce a second Knight's move pandiagonal square. Thus there are six possible squares of this type – the two from pattern C and four from Pattern B – for any number of the sequence as the middle number, from unit-centre up to n^2 at the centre. [For explanation of pair-wheel etc see note book 20, p. 79] these squares, they never seem to have gone on to discover this striking effect; and since the Islamic square makers, once they learned about Knight's-move squares, simply took over that mode of progressing and made continuous square by the Knight's moves without bothering with preliminary squares, they apparently did not discover this either [What is this 'wonderful effect' the author is referring to? perhaps the pair-wheel business].

One might suggest that the men of Islam could have found how to make continuous magic squares by the Knight's move without having to see an example made by the method of combining squares; but it seems relatively safe to assume that the first Knight's move magic square was discovered by accident (in India or Persia) when someone happened upon it while experimenting with the Union method under discussion. [The author is arguing merely on the basis of his Fig. 6 where 6 (C) obtained by the combination method happens to be a Knight's move pandiagonal square which can also be obtained by the continuous method (in (C) R.M is the Knight's move $(-1, 2)$ & B.M is $(0, -1)$). But there are squares obtained by the M-method which cannot be obtained by the continuous method – to be sure, it is an accident that

[the author has nowhere mentioned this square being associated]

in 6C, a Knight's move appears. See pp. 94, 95 of main book where the N-method gives squares not amenable to continuous method. Here is the glaring example of a square of $n = 9$ which can be obtained as a result using the N-method, but not by the continuous method]. Because, in spite of the resemblance of the magic square's basic grid to an Oriental Chess-board (which usually has all its square "White", not in two colours like ours), and even allowing for the fondness for Chess among the Indians who had invented the game [at least some conversion], and among the Persians and Arabs who borrowed it from them, it does not seem very likely that anyone could have tried to use that somewhat erratic mode of progression (which sometimes entails awkward break-moves) unless he had seen - through an example like the one in Fig. 6C - that it was not only possible, but highly practical, as a means of constructing magic squares.[321] [In this connection, one should find out what exactly was the work of Vilakantha using Knight's moves referred to in the book Carrés magiques & show that without any hint from the N-method he should have used the Knight's moves in the construction method]

{note 32, p. 285 - Someone might consider that the Knight's move itself is an erratic type of progress which is not likely to have occurred to anyone unless he had first come upon it through casting out nines on a magic square of nine, as illustrated above [the author has certainly not illustrated it for the case $n = 9$ but only for the case $n = 5$. If he had started with square in Fig. 5a, and constructed the square using the technique used in Fig. 6, he would get a square (as possible) in which there is no Knight's move. See p. 789 of the book 20 in only a move (-2, 4) as the R.M]

(*) 

But there are two arguments against this. In the first place, the game of chess must have been invented long before these magic squares, and secondly, the knight's move is simply a logical combination of two other steps in early chess; the single step forward, and the single step diagonally* [a third argument is that the knight's move need not at all have appeared in a 9th order square made by the ~~co~~ N-method. These arguments confirm my conjecture that Nilakantha must have used the knight's move in constn method without any hint got from the N-method]. }

If the Hindus knew the swift and easy "Indian continuous method" for making magic squares, why were they so devoted to this method of combining squares which was more complex and time-consuming? We have already seen that one reason for liking this method was that – at least with preliminary squares of Pattern A – the technique involved a process of interweaving, which (p. 286) was symbolically important for squares that were to be used both as religious symbols & as magic charms. However, Narayana's book provides still another reason. According to him, the art of constructing magic squares was revealed to an Indian prince by the god Siva[33]: Siva was a generative god, and Narayana

{n. 33, p. 286 – See Dwivedi's intn to G.K (1942 ed), pp. iii–iv. For another reference to a connection between the god Siva and magic squares, see T. Aufrecht: Katalog der Sanskrit-Handschriften des Universitäts Bibliothek zu Leipzig (Leipzig, 1901) No 1331, pp. 439–40}

makes it clear that there must ~~be~~ have been an idea of symbolic generation in the creation of a magic square through the agency of two complementary auxiliary squares. This is

by the language with which he describes the method of construction.
The dominant or "leader" square, containing the larger multiples, which
we call the root square (34), Nārāyaṇa called the "coverer" (35), while the

{ n. 34, p. 286 - The second root square is "dominant" because of the
higher values of its numbers and their wider range, but more
especially because it determines the sequences in the finished square,
their order as well as the direction of their flow. Thus it has a
controlling masculine character in relation to the other auxiliary
square, which is numerically weaker and more "passive". In view of
all this, the sexual analogy seems rather logical and not so far-fetched
as it might first appear }

{ n. 35 - Chādaka }

more delicate primary square containing the lesser sequences, he
named the "covered" (36), and he spoke of the coverer" facing

{ n. 36 - Chādya }

down upon "the covered" to produce the magic square proper.
Further more, his name for this process, <u>saṃputa</u>, is also a term referring to
sexual intercourse. In these terms, the union of the preliminary squares
was a mathematical expression of <u>mithuna</u>, the sexual union
of a deity with his consort - which, in turn, symbolised the union

of the active and passive principles of the Universe (37)

{ n.37 – This was a coincidentia oppositorum, like the Chinese yin-yang symbol, but more graphic }

Because so many magic squares, and even different kinds of magic squares, can be made by the method of combining the two preliminary patterns, one occasionally finds an enthusiastic amateur stating that any magic square can be made by this method. However, this is too extravagant a claim: one cannot use this system to make the symmetric squares for simply even numbers (38), nor bordered squares nor those Chinese examples

{ n.38, p.286 – Philippe de la Hire claimed to be able to make squares of simply even order by this method. But actually he made a square by combining two others, and then how to make a great many transpositions before this square was really magic; so it was far from successful. See P. de la Hire "Sur les Quarrés magiques" in Memoires de l'Academie Royale des Sciences" 1705 (Paris, 1706) pp. 377-38 }

that were constructed on a Lo-Shu frame; although even (p.287) some of these

[Bordered & the Lo-Shu frame squares are non-Narayanean as remarked in the Chapter IV of my proposed book]

aberrant types may profitably be analysed by breaking them down into primary and root squares in order to investigate their construction. (99)

Since the Hindus had no trouble in magic squares ~~for~~ of ~~orders~~ odd orders, and since they could easily do those for doubly-even orders - either by combining Pacioli primary squares or by composite methods involving their favourite magic squares of four - this left only those of singly-even orders (6, 10, 14, etc), which have always given problems. For these they had a special technique - this involved taking

{ n. 39 - p. 287 - Such analysis often reveals striking patterns. For example, when one applies it to the Chinese "Grand Lo-Shu" of nine (made from the rows of a natural square), one finds a primary square composed of a very repetitious Lo-Shu pattern in which each subsquare contains the same digit nine times; but, when one applies it to the Islamic composite magic square of nine (made by columns of a natural square), one gets a primary square consisting of nine little Lo-Shu squares } [These 9×9 Chinese & Islamic squares to be worked into. The former is in the Ency. Brit. article of 1973 + the latter ?]

the natural square (simply made by filling an n × n square with the numbers 1 to n^2 in regular succession) - or another kind of preliminary square - and reversing the directions of certain groups of numbers, following

by a final exchanging of some pairs to break the symmetry. Thus it is generally called "the system of broken reversions"[40].

{n. 40, p. 287 – for a modern discussion of this system by C. Planck, see W. S. Andrews, Magic squares & cubes (Dover ed) New York, 1960, p. ~~255~~ 295 ff.}

Actually, the Hindus had three, rather different methods of procedure within that basic system. The first started by making a natural square (beginning, as usual, at the upper left), then reversing the diagonals – and for squares of singly-even order above six, reversing certain rows and columns as well – and finally exchanging in several pairs of numbers[41]. The second started rather differently recalling the Islamic

{n. 41, p. 287 – Narayana's third magic square of six was made by in this way. See C. K (1942 ed), p. 386}

or double sequence" method for making magic squares of eight (which required first ~~mark~~ marking the diagonals cells in ~~any~~ an empty frame, then filling these marked cells before returning to fill the rest). [This appears to be the usual one for 8 × 8 squares found for eg. in Heath's Mathemagics. This method ~~for~~ shows the general principle of broken reversions. It holds for a 4 × 4 square]. Taking the empty square, the Hindu mathematician would mark certain cells, following a definite pattern that usually showed the main diagonals and a large central block – and often

certain other sections as well — keeping a two-part symmetry. Then starting at the upper left and counting to the right across each row, he filled the marked cells; and when he reached the bottom right corner, he would reverse the process, beginning again with the same sequence, proceeding from right to left in each row as he moved upward to fill the cells (p. 288) that he has previously passed over. This, too, required, finally, some last exchanging of numbers (42)

{n. 42, p. 288, Narayana's second magic square of ten (ibid. p. 357) was basically made by this technique, but it also involved exchanging halves of the main diagonals}

The third variety, a more complicated form of the second, was illustrated by Narayana with several examples, squares of 6, 10 and 14. (43) After marking a regular pattern in the cells of

{n. 43, p. 288 — Narayana's first and second magic squares of size (the latter defective) are shown in ibid., pp. 384 and 386. The former is also shown in fig. 7 above. His first magic square of ten and his square of fourteen are shown in ibid., pp. 384 & 385}

an empty square, the maker began by counting out the sequence as he moved to the right across the top row, then back toward the left in the second, changing his direction with each line, as in

boustrophedon writing. When he reached the bottom left corner, he reversed the process, retracing his steps to fill the empty cells, again changing direction with each row, as he moved upward. This mode of proceeding is by far the cleverest ever devised for shuffling numbers to achieve a more balanced distribution [Very fine compliment to the Indians indeed]. Furthermore, it embodies physically an admixture of the aesthetic with the metaphysical; the latter being revealed here in the clear analogy to forming the warp and weft of the Universe. (614)

{NN.44, p.288 - See Kramrisch, Unknown India, p.66}

When the initial interweaving of the numbers has been completed, it is still necessary to exchange some of the numbers, - specifically those in the two central columns and a few in the two centre rows (as shown in Fig. 7). But these final steps recall the process of tying off the loose threads on removing a fabric from the loom, so this makes the symbolic correspondence to actual weaving seem even more complete.

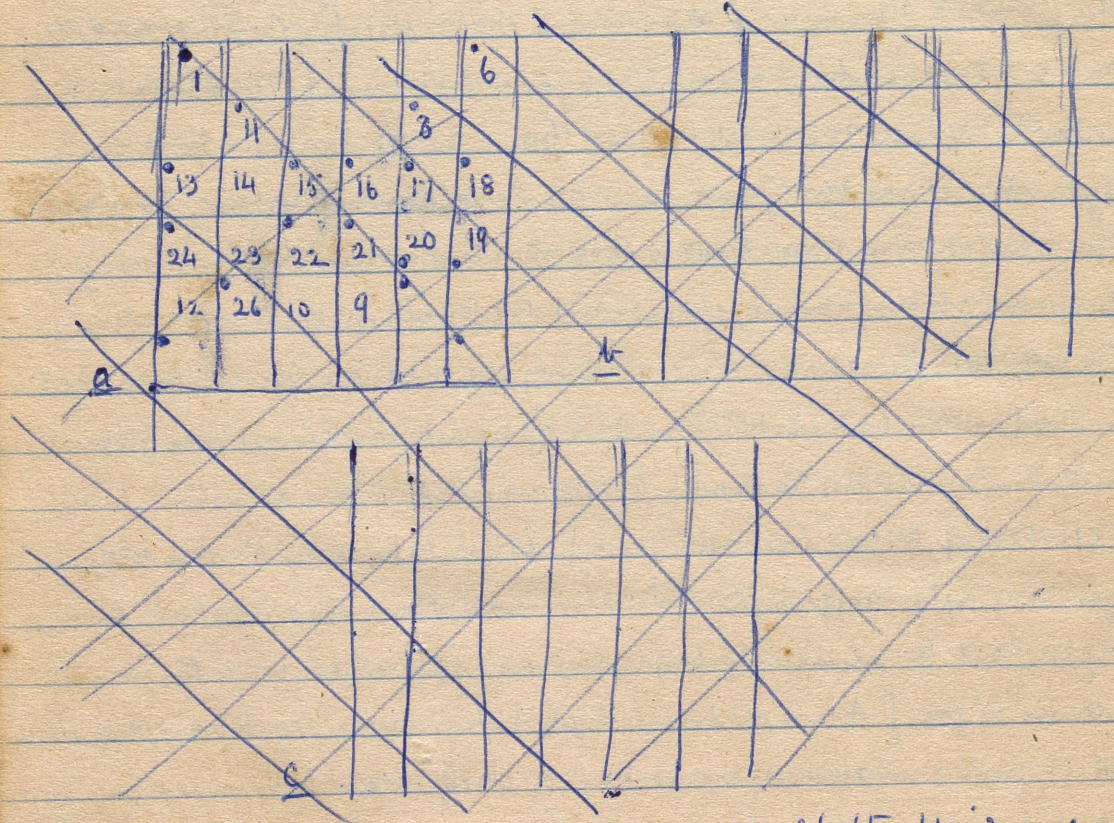
The general idea of making magic squares by a method of broken reversions must have been invented long before Narayana's time - perhaps by Islamic scholars, [obscure]

Cammanns is a pro-Pakistani!] in an extension of their method for magic squares of eight by writing double sequences. Because several magic squares of simply even order that were made by the "second Hindu variation" are illustrated in the Persian book in the British Museum (MS. Add. 7713, p. 112) which was done more than a century before Narayana wrote his. [The more natural explanation would be that this was known in India centuries before Narayana, and the Persians or Muslims borrowed them & put them in their Book. Also in the Persian book only illustration of the square is given & not the method of construction]

(p. 289) Ref. to semi-magic square of 10 made by Yang-Hui later than the Persian but earlier than C-K using a primitive form of the third Hindu variation method. This square did not originate with him. The fact that he has borrowed for his set a typically Hindu magic square of eight suggests that this also must have had an Indian origin. But its numbering beginning at upper right suggests an Islamic origin [A most natural explanation would be that Yang-Hui took it from an Islamic source which itself was borrowed from Indians.

Although some of the Islamic scholars knew the Hindu

methods of handling singly-even ~~squa ordese~~ order squares by "broken reversions", and may have even invented the idea [A very tall assumption], they seem to have (p. 290) deliberately avoided using them very often . . . . Symmetry breaking did not



appeal to them just as bordered squares did not appeal to the Hindus, even though the North Indian Muslims used them (Ref. to Ramanujan - Islam which gives bordered squares of ten & eight.

[In a the author has not marked the cells occupied by 14 and 23 which has been done below

a

1					6
	11			8	
13	14	15	16	17	18
24	23	22	21	20	19
	26			29	
36					31

b

1	35	34	33	32	6
25	11	27	28	8	30
13	14	15	16	17	18
24	23	22	21	20	19
12	26	10	9	29	7
36	2	3	4	5	31

Fig. 7 ~~ethindu~~ (p. 289) – Hindu magic square of six, by third method of broken reversions: (a) filling marked cells by boustrophodon numbering, (b) remaining cells filled by proceeding below by same kind of numbering

1	35	4	33	32	6
25	11	9	28	8	30
24	14	18	16	17	22
13	23	19	21	20	15
12	26	27	10	29	7
36	2	34	3	5	31

(c) completed square, after exchanging two pairs at bottom of middle columns (3, 4 and 9, 10), inverting the third column, and exchanging the circled pairs (24 & 13, 18 & 22, 19 & 15) [For sequence of working see p. 80, Book No. 20]

(204)

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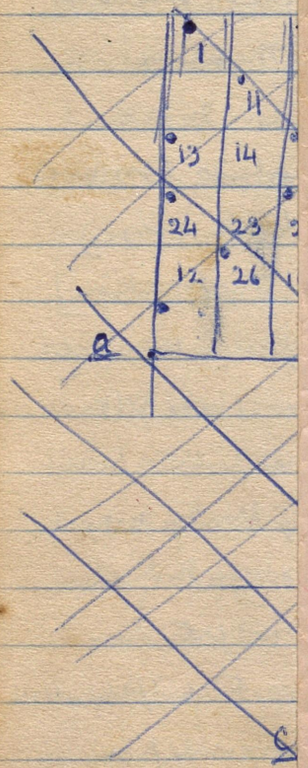

appeal to them just

though the North

sides bordered squares of ten Seight.

BOOK-POST

a, b, c are in H.P

Prove that $\sin^2\frac{A}{2}$, $\sin^2\frac{B}{2}$, $\sin^2\frac{C}{2}$ are also in H.P.



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Bangalore

560027.

THE KUPPUSWAMY SASTRI
RESEARCH INSTITUTE,
MYLAPORE. MADRAS-4

[In a the author has not marked the cells occupied by 14 and 23 (205)

32	6
8	30
17	18
20	19
29	7
5	31

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see p. 80, Book No. 20]

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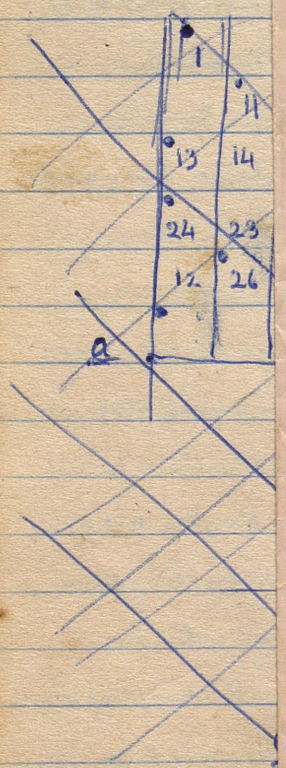

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[In a the author has not marked the cells occupied by 14 and 23

3	32	6
8	8	30
	17	18
	20	19
	29	7
	5	31

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see p. 80, Book No. 20]

BANGALORE 17.6.81

33	28	21	4	9	16
2	11	14	35	26	23
31	30	19	6	7	18
3	10	15	34	27	22
32	29	20	5	8	17
1	12	13	36	25	24

3	4	3	4	3	4
2	5	2	5	2	5
1	6	1	6	1	6
4 3	4 3	4 3	4 3	4 3	4 3
5 2	5 2	5 2	5 2	5 2	5 2
6 1	6 1	6 1	6 1	6 1	6 1

12	6	0	18	24	30
18	24	30	12	6	0
12	6	0	18	24	30
18	24	30	12	6	0
12	6	0	18	24	30
18	24	30	12	6	0

This further instance of marked cultural preferences provides still another example of the way in which the people of certain nations or cultures favoured the particular kinds of magic squares that contained elements which corresponded with their most cherished beliefs. - Thus even though some confusion may still remain as to when and where some of the construction methods were actually invented - due to certain overlappings of technique in the Islamic world and in India, which resulted largely from mutual interborrowings - it is still possible, as we have seen, to distinguish between certain typical Islamic magic squares and some typically Hindu ones. At the same time, the overlappings make it very difficult to discuss either the Islamic squares or the Hindu ones in isolation, because one tradition cannot be fully understood without the other. In fact, these two intermeshed traditions share something else in common, since they both contributed to the development of magic squares in Europe.

The oriental heritage in European Magic squares

The Cracow (Poland) set contains an exact Hindu magic square of 5 - Attempt to make square of 6 by Hindu method of broken reversions was a failure -

Sieur de la Loubère, returning from a diplomatic mission to Siam, was whiling away his time on ship board by constructing magic squares with the "method of Bachet de Mezeriac", when a fellow-passenger, a French physician named Vincent offered to show him a far better way. This was the "Hindu continuous method" which the latter had learnt on a visit to Surat. Loubère was delighted with this method, and in a two-volume account of his journey, he devoted one whole chapter to the subject of magic squares - Eudoxus discarded the Persian continuous method & used only the Indian method - He also got the idea of the IV-method from Dr Vincent - This new technique came to be known as the de la Hire method, and the Hindus received no credit - Loubère also made a splendid magic square of ten using the second Indian method of broken reversions, though he did not say whether he got it too from Dr. Vincent.

Bernard Frenicle de Bessy showed that there were exactly 880 magic squares of order 4 - Sauveur invented magic cubes and also the idea of Latin squares later taken up by Euler - The 1750 work of M. d'ons-le-Bray very comprehensive & made magic squares a recreation - Work of Benjamin Franklin published in London in 1769 (Franklin squares) - Last direct influence of Asia came through the English missionary A. H. Frost who spent some years in the old city of Nasik, sacred to Siva, also as a centre of study for magic squares (vide Narayana's ref. to Siva)

while there first learned the technique of making pandiagonal squares of four, and became much intrigued with them. When he returned to England, he wrote a number of learned articles about these pandiagonal squares which he called "Nasik squares" or simply "Nasiks". He had no idea of Islamic work on this which [according to Cammann] was probably carried over to India. Unfortunately his clumsy and misleading terms still persist among English & American writers for eg. the use of the adjective Nasical [How prejudices Cammann can be against Hindu work!] – First wrote the article for the Encyc. Britannica (11th Edn. XIII, 310–13). His work carried on the interest in magic squares up to the 20th century.

In spite of Frost's dedicated efforts to raise the study of magic squares to a higher plane of mathematical study, most people who worked with them did so as a form of hobby, or for simple recreation. And, of course, any idea of the ultimate origin of the European magic squares (except for the obviously-named "Nasik" ones) or of the metaphysical ideas that had surrounded them in the Islam world and in India, and even the strange and wonderful properties that made it possible for men to find and hold these ideas, had long since been completely forgotten

(Finished making notes from Cammann's articles on 31/5/78)

1/8/78 – At Institute

Proc. Roy. Soc. A, Vol 355, pp. 441-637, No. 1683, 25/7/1977

A discussion on new particles & new quantum numbers

Participants – R. H. Dalitz, B. Richter, Barkel et al, S. C. C. Ting, Buschhorn, D. Cline, G. Myatt, C. H. L. Smith, Feldman & P. T. Matthews. (3 Contributions by R. H. Dalitz) – A comprehensive survey

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Math. Reviews, 54, No. 5, Nov. 1977

(1) On palindromic squares of non-palindromic numbers by Simmons, Gustavus J. J. Recreational math. 5 (1972), no. 1, pp. 11-19.

Palindrome is an integer unaffected by the reversal of the order of its digits. Let P(m, k) be the set of integers in base m whose k^{th} powers are palindromes. It is known that P(10,2), P(10,3), P(10,4) are infinite. If $k \geq 4$, then it is not known whether P(10,k) is finite or infinite. In fact, it is not known whether P(10,K) is non-empty for $k \geq 4$. Let Q(m,K) be the set of integers in base m that are not palindromes but whose k^{th} power is a palindrome. It is known that Q(10,2) and Q(10,3) are non-empty since $26^2 = 676$ and $2201^3 = 10,662,526,601$. However it is not known whether Q(10,2) and Q(10,3) are finite nor whether Q(10,k) is non-empty for $k > 3$.

[Note $836^2 = 698896$. Found by Bell]

Review of Alan Baker's book (Ramachandra's Bombay) – Transcendental number theory

C. U. P., London, N. Y. 1975, pp. 1-147

$ 13.95

Review by R. Tijdeman (Leiden).

Another article on Baker's method in the theory of transcendental nos. in Russian.

Hurley & Sudarshan's article in Ann. Phys 85 (1974), 546-590 on the algebraic structure of a class of relativistic wave eqns (any reference to my work?)

Phys. Rev. Letters 38 (1977), no 8, 377-80. Newell et al — Breakdown of Zakharov-Shabat theory and soliton creation — Review by Brian De Facio

"Solitons are solitary waves with the remarkable property that two solitons each preserve their identity after propagating through one another. The most interesting solitons are solutions to one-space, one-time, two-dimensional non-linear dispersive partial differential eqns — a long standing problem is how to extend these non-linear modes to higher dimensional space-times

Another paper on solitons — "A system of particles equivalent to solitons" by W.R. Thickstun — J. Math. Anal. Appl. 55 (1976), no. 2, 335-46 — Ref. to book on "non-linear wave motion" — formulation for one soliton can be put in the Hamiltonian form.

Review of paper by Om Prakash & B. Rajput — on "Second Quantization and interaction of meson fields with nucleon" by Brian De Facio

in Acta Ci. Indica 2, no. 2, 205-14 (1976)

(A)

This is a strange paper which compelled the reviewer to check its date (1976) several times. The topic, style and omitted references all add up to an eerie feeling of having received a communication from a ten (or more) year time warp.

Ref. to an article by E.A. Lord in Inter. J. Th. Phys on massless field equations for arbitrary spin by J.S. Dowker who says that Lord's work is anticipated by him

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(2/8/78) At Institute again

In Journal Ganita, saw on the cover "Publications of the Dept of Maths & Astronomy, Lucknow Univ:-

Hindu astronomical & mathematical text series in

(1) Surya-Siddhanta with Sanskrit commentary of Parameswara (1430 AD)

(2) Pati-Ganita of Sridhara (900 A.D)

(3) Mata Bhaskariya by Bhaskara I (629 AD)

(4) Laghu Bhaskariya "

(5) Karana-Ratna by Deva (689 AD)

All edited with English translation by Dr K. S. Shukla

Write to him about ~~Madhava's~~ Narayana's book if available

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(14/8/78) At Institute

(~~105~~) (212)

New Books

(1) Lagrangian mechanics of non-conservative, non-holonomic systems by Dominic G. B. Edelen (under Mechanics, Dynamics systems) - Pub. Noordhoff International Publishing, 1976

(Edelen at Lehigh Univ., Bethlehem, Pennsylvania) - Dedicated to James F. Bell who first opened for me the pages of Whittaker. The preface refers to 2nd law of Thermodynamics. Lagrangian formulation is used throughout since it is the natural formulation, the Hamiltonian formulation being dependent on Legendre transformations whose very existence can fail to obtain in real problems. My one wish is that the reader derive as much pleasure from the reading as the writer derived from the writing

Chapt I - Kinematic space - The arena of Newtonian mechanics

Chap II - Newtonian motions in kinematic space - Lagrange's eqns - Trn properties

Chap III - Analysis of forces

Chap IV - Analysis of constraints

Chap V - Fundamental differential forms - Variational systems

Chap VI - Dr 2 - Non-conservative, non-holonomic dynamical systems

Chap VII - Integral forms of conservation & balance

Chap VIII - A transformation-theoretic approach

Chap IX - Field mechanics for systems with laws of balance

~~Chap X~~ - Appendix

Ref. to books of Whittaker, Goldstein, Abraham & Stehle, Liapounov, Pars & Winter

(2) Functional Analysis in modern app. maths by Ruth F. Curtain & A.J. Pritchard

Academic Press, Vol. 132 in a series, 1977

(3) Qualitative analysis of large scale dynamical systems by Michel & Miller –

Academic Press, 1971

17/8/78 At Institute

New Books

(1) Computer methods for mathematical computations by George E. Forsythe, Michael A. Malcolm & Cleve B. Moller – Prentice Hall Series in Automatic computation – Subroutines & exercises for the computer soln of problems involving matrices, integrals, diff. eqn, spline functions, zeros & extrema of fns, least squares & Monte-Carlo techniques

Journal of Recreational Mathematics – Found it is subscribed for by the Institute from 1977 i.e. from Vol. 9 (1976-77) – Editor – Harry A. Nelson. – Baywood

~~Vol. 9, No.~~ Publishing Co. Inc, 43, Central Drive, Farmingdale N.Y. 11735 – Subn rate ~~per vol.~~ for Vol. 9 (4 issues), $21.00. Add $2.00 per volume for postage outside U.S and Canada. Back volumes are available at $28.00. All prices are subject to change without notice. Subn is on volume basis only & must be prepaid.

Vol. 9 – No. 11 – 1976-77 – refers to another Journal Games & Puzzles, – Joe Madachy started the Journal & from Vol. 9, Nelson becomes Editor

p. 11 Four quickies (answer on p. 20)

Q(1) 777 is a multiple of 21. The only primes of the form 777xy are 77711, 77713,

77719, 77723, 77731, 77743, 77747, 77761, 77773, 77783 & 77797. In every case the last two digits form a prime as well. Is this really a surprising result?

(p. 20) A1 – All two-digit composite numbers are divisible by primes less than 11. Since 77700 is divisible by 2, 3, 5, 7, 777 + xy is composite if xy is divisible by 2, 3, 5 or 7. Since 77701 happens to be divisible by 13, 777xy can be prime only if xy is prime

[not quite clear]

Q. 2. – Building was complete in ..... It was a year equal to 111 times one more than the sum of its digits

A. 2. 1443 = 111 (1 + 1 + 4 + 4 + 3).

Q. 3 In this problem distinguish carefully between "number" and "number representation in a given base". What is the smallest 2-digit number representation which is never prime in any base?

A. 3. 20, which being twice the base, is always composite.

Q. 4 When can be sure that "two to the n minus one" is equal to "two ~~two~~ to the n minus one"? – – – – – –

A. 4 Both 2^{n-1} and $2^n - 1$ are read as stated, thus surely the same when $n = 1$.

~~p. 15 Reverse-transposable integers by Steven Kahan, Queens College, Flushing, N.Y~~

[Kahan is editor for Alphametics – ~~New~~ Solns & proposals for new puzzles in Alphametics to be sent to Steven Kahan, Journal of Recreational Mathematics, 150-38 B Union Turnpike, Flushing, New. Nelson's address

is Harry L. Nelson, Lawrence Livermore Laboratory, Livermore, California. –

Madachy's address is Joseph S. Madachy, Monsanto Research ~~Corporation~~ Corp. Kettering, Ohio – Address of Charles W. Trigg, San Diego, California]

p. 15 K-reverse-transposable integers.

A +ve integer x is K-transposable if and only if it has the property that when its leftmost digit is moved into the units place, the resulting integer is ~~Kx~~ equal to Kx. Thus 142857 is 3-transposable, since the product of $142857 \times 3 = 428571$. It was proved in [1] (Kahan's article on K-transposable integers, Mathematics Magazine, 49:1 January, 1976) that x is 1-transposable if and only if all of its digits are identical, and that for $K > 1$, K-transposable numbers occur only when $K = 3$. Finally it was shown that the only 3-transposable ~~numbers~~ integers are $\overline{142857}_m$ and $\overline{285714}_m$, when $\overline{N}_m$ denotes the integer obtained by concatenating N m-times if $m > 1$ and is just N itself when $m = 1$. [This result given by Troisky – Vide Ball, p. 13]

Now we discuss K-reverse transposable integers.

Defn – A x is K-reverse transposable if & only if it has the property than when its unit digit is moved into the left most place, the resulting integer is equal to Kx. Since $142857 \times 5 = 714285$, 142857 is 5-reverse transposable.

[Quite a complicated article – Find relation of this to Ball's example of reverse of ~~a number~~ x being equal to Kx].

pp. ~~45~~ 21–23 – Alphametics

(216)

p. 23 – Rules by Myron Tepper

1. Partial Differentiation: Marrying a rich girl who is ugly.

2. Tangent: Someone who has spent too much time in the sun.

4. Diameter: Roman god of hunting

5. Calculus: One of the seven + 6 deadly sins

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p. 24 – Problems & conjectures – Edited by David L. Silverman, West Los Angeles, California

Readers are encouraged to send solns, comments, and new problems to Problem Solution Editor, J. R. M, 4259 Emory Way, Livermore, CA 94550

[Also other instructions]

Problem 466 (p. 25). – Circumscribing with a square by Vincent J. Scally, Rome, Georgia

Given a triangle ABC, construct a square with two sides meeting at A and with the other two sides containing B & C respectively

22/8/78 At Institute

S. Chandrasekhar asked me to come over to Asoka Hotel at 8 A.M.? on Monday the 4th Sept, 1978 invitation or no invitation.

Listened to G. N. R's lecture & said I was not convinced.

Read Kaplan's article on R – reverse transpositions ie putting last digit in first place & result being K times original number. Has sophisticated approach

number $x = x_0 x_1 \ldots x_{n-1}$ (x has n digits, x_0 = first digit, x_{n-1} = last digit

is based on eqn $(10k-1)\sum_{i=1}^{n-1} 10^{i-1} x_i = (10^{n-1}-k)x_0$ which is exactly my eqn.

Problem 378 – A tribute to the U.S.A. bicentennial by Mc Leod, Glasgow Univ.

Prove that if x^{1776} and x^{1975} are both integers, then so also is x

Soln by Sidney Kravitz, Dover N.S.

Let x^{1776} = integer N and x^{1975} = integer M.

Then $\dfrac{N^{526}}{M^{473}} = \dfrac{(x^{1776})^{526}}{(x^{1975})^{473}} = \dfrac{x^{934176}}{x^{934175}} = x$

Thus x = rational number N^{526}/N^{473} & if it is not integer, then it cannot become an integer when raised to any integer power including 1776 & 1975

[Find a simple proof]

Problem 380 [Let x = four digit number ABCD & y the three digit number EFG]

Find a 7-digit number ABCDEFG such that half the square of ABCD plus twice the square of EFG is equal to ABCDEFG (Digits A, .. G are not necessarily distinct)

Soln: We have $\frac{1}{2}x^2 + 2y^2 = 1000x + y$.

ie $4(x-1000)^2 + (4y-1)^2 = 4000001 = 41 \times 97561$

Now a number that is the product of two primes of the form $4p+1$ can be the sum of three squares in precisely two ways as given by the identity

$(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2 = (ad+bc)^2 + (bd-ac)^2$

Obviously $4000001 = 2000^2 + 1^2$ leading to trivial soln ABCDEFG = 4000000

Thus we may set $ac+bd = 2000$, ~~$4d-5c=1$~~ $ad-bc=1$

$1+2\times 3 = 6.$
6^{n}

Since $41 = 4^2+5^2$, we may set $a=4, b=5$. Therefore

$$4c+5d = 2000, \quad 4d-5c = 1$$

Solving $c = 195$, $d = 244$. As a check we calculate $195^2 + 244^2 = 97561$. Therefore

$ad+bc = 1951$, $bd-ac = 440$, and $440^2+1951^2 = 4000001$.

This accordingly has $x = 780$ or 1220 and $y = 488$, as ABCDEFG = 0780488 or 1220488, but the former is not a 7-digit number in the generally accepted sense. So 1220488 is the only.

J. R. Math. – Vol 9, No. 2 (1976-77)

(1) Search for large weird numbers by Sidney Kravitz

If $\sigma(n)$ = sum of divisors of a nth integer n. define abundance of n as $a(n) = \sigma(n) - 2n$

If $a(n) > 0$, n is an abundant number, if $a_2(n) = 0$, then n is a perfect number

[obviously includes n itself as a factor of n], if $a(n) < 0$, n is deficient.

n is pseudo-perfect if it is equal to the distinct sum of some/all the proper divisors of n [e.g. $20 = 1+4+5+10$ is pseudo-perfect. The number n is weird if if it is abundant but not pseudo-perfect, and it is primitive weird if it is weird but none of its proper divisors is weird.

At Institute 25/8/78

(1) Some mathematical methods of Physics by Nunzio Tralli, 1960.

Suppose $\int_{-\infty}^{\infty} D(x)\,dx = 1$ ——— (1)

then for continuous $f(x)$

$$\lim_{a\to\infty}\int_{-\infty}^{\infty} a\,D[a(x'-x)]\,f(x')\,dx' = \lim_{a\to\infty}\int_{-\infty}^{\infty} D[a(x'-x)]\,f\left(\frac{ax'}{a}\right)d(ax').$$

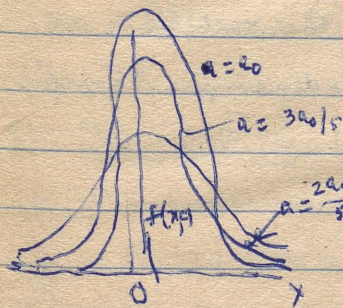
$$= \lim_{a\to\infty}\int_{-\infty}^{\infty} D(t)\,f\left(\frac{t}{a}+x\right)dt = f(x)$$

If one could interchange the limiting process with the integration

$$f(x) = \int_{-\infty}^{\infty}\left\{\lim_{a\to\infty} a\,D[a(x'-x)]\right\}f(x')\,dx'$$

or $\delta(x, x') = \delta(x'-x) = \lim_{a\to\infty} a\,D[a(x'-x)]$ ——— (2)

for any D satisfying (1). A sequence of fns $a\,D[a(x'-x)]$ is shown in fig.



It should be noted that all the curves have unit area with the understanding that the limit should be held off until after an integration (2) may be taken as defining $\delta(x'-x)$.

Alternately: consider a fn $\eta(x)$ such that

$\eta(x) = 1,\ x > 0$

$\eta(x) = 0,\ x < 0$

Then for $f(x)$ such that $\int_{-\infty}^{\infty} f(x)\,dx$ converges absolutely

$$\int_{-\infty}^{\infty} f(x')\,\eta(x-x')\,dx' = \int_{-\infty}^{\infty} f(x')\,dx' \quad \text{whence } f(x) = \frac{d}{dx}\int_{-\infty}^{\infty} f(x')\,\eta(x-x')\,dx'$$

This suggests writing, by differentiation, under the integral sign

$$\delta(x-x') = \frac{d}{dx}\eta(x-x') \quad \text{---} \quad (3)$$

which yields $f(x) = \int_{-\infty}^{\infty} \delta(x-x') f(x')\,dx'$. — (4)

$\delta(x-x')$ is known as Dirac's delta fn. It is defined equally well by (2), (3) or (4). From these it is evident that $\delta(x-x')$ must vanish for $x \neq x'$ and must be infinite for $x = x'$. Thus $\delta(x-x')$ is not well defined at $x = x'$ & is not a fn in the conventional meaning of the term. We therefore call it a symbolic function.

It is necessary to point out that the δ-fn has meaning only so long as an integration with argument is carried out. Thus any result such as

$$x\,\delta(x-y) = y\,\delta(x-y)$$

is true if & only if the left side of the eqn gives the same result when multiplied by any arbitrary continuous fn $f(x)$ & integrated over a suitable domain, as the right side.

Some relations listed below:

$$\delta(x-x') = \delta(x'-x)$$

$$\delta(ax) = \frac{1}{a}\delta(x)$$

$$f(x)\,\delta(x-x') = f(x')\,\delta(x-x')$$

$$\delta(x^2-a^2) = \frac{1}{2a}\left[\delta(x-a) + \delta(x+a)\right] \qquad a > 0$$

(2) Intro to Math. Phys by William Band

p. 16. Functionals are rules for assigning a number to each function in function space.

If Φ be the fn, F the rule, $F(\Phi)$ is the symbol for the functional. A very simple rule would be

$$F(\Phi) = \int_a^b f(x)\,\Phi(x)\,dx \qquad \text{——— (1)}$$

$f(x)$ is some other fn used to define the rule. A special functional is a distribution

For eg $D(\Phi) = \Phi(x_0)$ (2) is a rule assigning to the function $\Phi(x)$ its value at the fixed point x_0. But to be in form (1) above, the function fitting it should be the Dirac delta fn

$$D(\Phi) = \int^b \delta(x - x_0)\,\Phi(x)\,dx = \Phi(x_0)$$

Although $\delta(x - x')$ is not well-defined, the Dirac delta-functional (2) is well defined. Thus in functional analysis we can make rigorous many math. operations that were formerly non-rigorous & depended much on physical intuition.

(3) Intro to Math. Phys by C. Harper, 1976.

p. 217 – Generalised Fourier series & the Dirac δ-fn.

$\delta(x - x')$ is not a fn in the rigorous mathematical sense since the integral (if it exists) of a function that is zero everywhere except at one point must vanish.

[Ref. to rigorous math. treatment of the Dirac δ-fn in terms of the theory of distributions is given by M. J. Lighthill in Intro to Fourier Analysis & Generalised fns C.U.P. 1958]

Two widely used reps for the Dirac δ fn are (1) & (2) below

$$\delta(x-x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk \quad \text{—— (1)}$$

& $$\delta(x-x') = \frac{1}{2\pi}\lim_{l\to\infty}\int_{-l}^{l} e^{ik(x-x')}\,dk$$

$$= \lim_{l\to\infty}\frac{\sin[l(x-x')]}{\pi(x-x')} \quad \text{—— (2)}$$

In 3-dimensions $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{r}}\,d^3k.$

where $\delta(\vec{r}) = 0,\ r \neq 0$

and $\int_{-\infty}^{\infty}\delta(\vec{r})\,d^3\vec{r} = 1.$

At Institute 26/8/48

J. Rec. M-12 Vol. 9. No. 2

p. 95 – Busywork problems

bw. 1. For +ve integers a, b, j, k, p, q, r & t with j & $k \geq 2$ find all values of $t \leq 50$, for which $t = a^k$, $t = p+q+r$ ($p \geq q \geq r$ prime), & $p+r = b^j$

Soln on p. 113, $t = 16, 27, 27, 27, 32, 36, 49, 49, 49$

$p = 7, 23, 13, 11, 23, 23, 31, 29, 29$

$r = 2, 2, 3, 5, 2, 2, 5, 3, 7$

bw 2 - List all 2 digit integers which are divisible by the sum of their digits, but are not divisible by 3.

Soln p.113 - 10, 20, 40, 50, 70, 80

bw 3 - One is a non-prime integer whose cube is a palindrome; what is the next larger such integer?

Soln p.113 - 111 [Another 2201 which is non-prime as is divisible by 31, has 2201^3 = palindrome, but 2201 ≠ 111]

bw 4 It is known that all sufficiently large integers can be written as the sum of one prime & at most two squares. Which nos. "x" with $1 \le x \le 100$ cannot be so written?

Soln p.113 - 1.

bw 5. Defn - A "C.F" integer is an integer having a common factor with all non-prime integers smaller than itself. 1, 2, 3, 4 are considered to be "C.F" integers. Find the other six "C.F" integers.

Soln p.113 - 6, 8, 12, 18, 24, 30 (Prove that there are no more).

bw 6 - Find three sets of five integers each among the numbers 1 thru 9 besides the set (1, 2, 6, 7, 9) such that no three elements in a set are in A.P.

Soln (1, 2, 4, 8, 9), (1, 3, 4, 8, 9), (1, 2, 6, 8, 9)

bw 7 - Extend bw 6 such that one of the four sets in that problem is to be a set of "nine of the first 20" integers such that no three are in A.P.

Soln (1, 2, 6, 7, 9, 14, 15, 18, 20)

bw 9 – find all primes less than 100 which are not the sums of distinct primes in two or more ways

Soln: 2, 3, 5, 7, 11, 13, 17.

Solns on p. 113 are given for bw 10 – bw. 14, but these eqns are missing on p. 85 but found on p. 100

p. 86 – Const of ∞-magic squares by R. J. Lancaster.

Uses the geometric series $\sum_{n=1}^{\infty} 1/2^n$ as suitable for the const of such a square

The key to the const is in using

$$1/2 = \tfrac{1}{4} + \tfrac{1}{4} = 1/8 + 1/4 + 1/8 = 1/16 + 1/8 + \tfrac{1}{4} + 1/16.$$

Eg. 2 gives a square of magic sum = 2

[Is there something like this in Andrews?]

---

other bw. problems on p. 100:

bw. 11 – List all min primes ≤ 1000 whose sum is not a product of at least three (not necessarily distinct) prime factors

Soln. p. 113 – "2, 3" (or none, if you don't like to consider these twins). [Consider 1 to 1000]

bw 12 – 809 is sum of primes with all distinct digits involved, in more than one way, find them

Soln. $5 + 61 + 743 = 809 = 5 + 43 + 761$.

bw. 13 – $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$ – find all other examples where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and at least two of the three integers p, q & r are prime

Soln: $\frac{1}{2} + \frac{1}{2} = \frac{1}{1}$; $\frac{1}{2} + \frac{1}{-6} = \frac{1}{3}$

p. 101 – More fractions fractions by Steven Kahan – Ref. E Vol 7(4), p. 286.

Let F be a fraction whose numerator & denominator are distinct positive three-digit numbers. Define

$$Q_h = \frac{\text{hundreds digit of numerator}}{\text{hundreds digit of denominator}}$$

$$Q_t = \frac{\text{tens digit of N}}{\text{tens digit of D}}$$

$$Q_u = \frac{\text{units digit of N}}{\text{units digit of D}}$$

F is fractions by addition if $F = (Q_h + Q_t) + Q_u$

" subtraction if $F = (Q_h - Q_t) - Q_u$

" multiplication if $F = (Q_h \cdot Q_t) \cdot Q_u$

" division if $F = (Q_h / Q_t) / Q_u$.

Gives examples of three of these three sets called A_3, S_3, M_3 & D_3

[Has this anything to do with illegal cancellations? – Take an example from the set D_3 given by Kahan.

Ex: $\frac{232}{116} = F$, $F = (2/1)/(3/1)/(2/6)$

i.e. $\frac{2}{1} \times \frac{1}{3} \times \frac{6}{2} = 2$ & $F = 2$.

No, this has nothing to do with illegal cancellations]

p. 104 – disproving a conjecture on triconcatenated primes by Griffin & Washington –

Stanford, California

concatenation defined by $abc\ldots \| def\ldots = abc\ldots def\ldots$

If P & Q are primes & P || Q is also a prime, then P || Q is a triconcatenated prime

See [J. R. Math. 6 (2), p. 118–19)] Conjectured that for every P there is a Q such that P || Q is prime. Trey R[illegible], if P = 3, Q = 7 & 13 are two possibilities

After considering some numerical evidence Trey [Math. 8 (2), p. 123–25]

strengthens Lee's conjecture to the following:

Conjecture – for every prime P, there is a prime Q with $Q < 10^3$ such that P || Q is prime

This shows that this strengthened conjecture is false. In fact, we show that the conjecture is false even if Q is not required to be prime.

Theorem – There exists a prime P such that P || Q is composite for all positive integers $Q < 10^3$.

Proof – Write integers in arithmetic using Dirichlet's theorem re primes of the form

$an + b$ [Handy & Wright]

The prime P whose existence was proved in the author's demonstration is at least of order 10^{760}! Finding the first such P, however, seems much beyond the scope of present day computers!

Another conjecture of Trey – for every prime $P > 3$, there is a prime $Q < P$

such that P||Q is prime

Authors give a probability argument to show that this conjecture should be true at least for large enough P.

p. 106 – The spirit of '76 or who hid the Fibonacci – Some funny article not understandable easily by Norris Goodwin

[Author – born 4/7/1896 (4 years older than myself) & died 18/11/1969 – Chemist worker for various concerns – did business himself – Retired in June 64 & turned to Maths for recreation & mental pleasure]

p. 109. Michael R. Buckley

wants to evaluate $i!$

Newton discovered $\int_0^1 (1-x^2)^n dx = \frac{n!^2}{(2n+1)!} 2^{2n}$ & putting $n = \frac{1}{2}$ get $\frac{1}{2}! = \sqrt{\pi}/2$

Take the Gamma fn formula

$$(x-1)!\,(-x)! = \pi / \sin \pi x.$$

Putting $x = i$ & noting that $\sin(ix) = i \sinh x$ & $i! = i\,(i-1)!$ we get

$i!\,(-i)! = i\,(i-1)!\,(-i)! = \pi / \sinh \pi$. Now since $|i-n| = |-i-n|$ for all real n, then $|i!| = |(-i)!|$ & finally $|i!| = \sqrt{\frac{\pi}{\sinh \pi}} \cong 0.522$

This fixes $|i!|$ but leaves its argument elusive or undefined.

27/9/78 C.C. Lib

Elementary number theory by Uspensky & Heaslet (1939)

Appendix on p. 159 on magic squares

Since remote times magic squares have been known in China & India, but came to Europe in 15th century. Later men like Euler & Fermat did not deem it below their dignity to spend time on methods of construction of such squares. Despite innumerable efforts of men of science & amateurs, we do not possess methods for the construction of all possible magic squares, nor is their number known except for $n = 3$, & $n = 4$

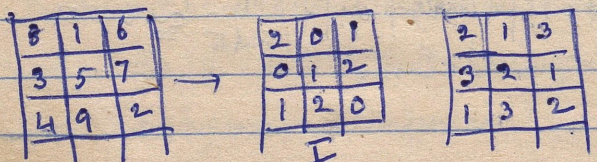
2. Auxiliary squares

Every number K from 1 to n^2 can be expressed uniquely in the form

$$K = nr + s \quad (r = 0, 1, 2, \ldots n-1) \quad \text{—(1)}$$

and $(s = 1, 2, \ldots n.)$

r & s are called 1st & 2nd components of K. Placing these components in two auxiliary squares I & II & using (1) we form the final magic square



8	1	6
3	5	7
4	9	2

$n = 3$

→

2	0	1
0	1	2
1	2	0

I

2	1	3
3	2	1
1	3	2

$(n \cdot I + II \rightarrow$ magic square$)$

1	14	15	4
12	7	6	9
8	11	10	5
13	2	3	16

$\longrightarrow$

0	3	3	0
2	1	1	2
1	2	2	1
3	0	0	3

I

1	2	3	4
4	3	2	1
4	3	2	1
1	2	3	4

Constn of auxiliary magic squares

(1) odd n (m) symbol for any number $0, 1, \ldots n-1$.

$$(m) = (m') \text{ if } m \equiv m' \pmod{n}$$

Horizontal rows from the top down will be denoted by $0, 1, 2, \ldots n-1$ & vertical columns of cells from left will be marked by same numbers. Then each cell is characterised by two integers i & j.

To form I fill the cell (i, j) with $(i+j)$ & make $(n-1) = \frac{n-1}{2}$ for a u.d

For $n = 5$, we might take

$(0) = 4$, $(1) = 3$, $(2) = 0$, $(3) = 1$, $(4) = 2$. Hence I is

4	3	0	1	2
3	0	1	2	4
0	1	2	4	3
1	2	4	3	0
2	4	3	0	1

I.

Similarly to form II use notation $\langle K \rangle$ (for sakey order)

$$\langle K \rangle = \langle K' \rangle \text{ if } K \equiv K' \pmod{n}$$

$\langle K \rangle$ from $0, 1, \ldots n-1$

for numbers $1, 2, \ldots n$.

To form II place in the cell (i,j) the number $\langle i-j \rangle$. To make descending diagonal magic take $\langle 0 \rangle = \frac{n+1}{2}$. The other symbols may be identified with the remaining numbers in any order. Taking (for $n=5$)

$\langle 0 \rangle = 3$, $\langle 1 \rangle = 1$, $\langle 2 \rangle = 4$, $\langle 3 \rangle = 2$, $\langle 4 \rangle = 5$, II will be

3	5	2	4	1
1	3	5	2	4
4	1	3	5	2
2	4	1	3	5
5	2	4	1	3

Combining cell (i,j) is occupied by

$$n(i+j) + \langle i-j \rangle$$

Shown that the numbers in cells are all different

combining I & II i.e. $n \cdot \mathrm{I} + \mathrm{II}$ we get the square

23	20	2	9	11
16	3	10	12	24
4	6	13	25	17
7	14	21	18	5
15	22	19	1	6

Case $n = 4m$. For I, fill (i,j) with $(i + 2mj)$

For II fill (i,j) with $\langle 2mi + j \rangle$

Ex. for I (0) = 0, (2) = 3, (1) = 2, (3) = 1, for II take ⟨0⟩ = 1, ⟨2⟩ = 4
⟨1⟩ = 3, ⟨3⟩ = 2

I is				II is			
0	3	0	3	1	3	4	2
2	1	2	1	4	2	1	3
3	0	3	0	1	3	4	2
1	2	1	2	4	2	1	3

Combining weights

Case of even-odd $n = 2m$ (m odd)

For I $\left(mi + j + m\left[\frac{i}{m}\right] \right.$ in cell $(i, j$

For II $\left\langle i + mj + m\left[\frac{j}{m}\right] \right\rangle$

Very complicated

Ex (0) = 0, (3) = 5 | ⟨0⟩ = 1, ⟨3⟩ = 6
(1) = 1 (4) = 4 | ⟨1⟩ = 2 ⟨4⟩ = 5
(2) = 2 (5) = 3 | ⟨2⟩ = 3 ⟨5⟩ = 4

28/9/78 Institute

J. Rec. Math

p. 110 – Modified digitadition by Charles F. Winans

Some modification of Kaprekar's self-numbers. Refers to Gardner's

Article in Math. Gams Sc. Amr. March 1975

p. 114 - Twin primes with polygonal number means by Harvey J. Hindin

Recently Trigg [Math. Mag. 47, p. 107, Prb. 899, 1974] asked if 6 was the only mean between a pair of twin primes which was triangular. The affirmative answer was given by Shanks [ibid. 48, pp. 120-21, Prb. 899. Soln 1975]. The problem may be generalized to finding the polygonal numbers which are means of twin primes

Book reviews

(1) Domino fans & Domino Puzzles by Leeflang, St Martin's Press N.Y. 1975, $ 8.95

(2) Mathematical Carnival by Martin Gardner - Collection of columns from Math. Gams

(3) Bibliography of recreational mathematics Vol. 3 by W. L. Schaaf - Contains also magic squares [National Council of Teachers of Maths, 1906 Association Drive, Reston, Virginia 22091, 1973, $ 5.00.

(4) 300 best word Puzzles by H. E. Dudeney - Charles Scribner's Sons N.Y, 1972, $ 2.65

Reproduction of the puzzle master's 1925 book, The "Worlds best word puzzles" augmented by 4 word puzzles from his A+ Puzzle mine, edited by Gardner who also provides an introduction. Puzzles classified as enigmas & riddles, rebusses, buried words, missing words, expanding words, cryptograms, word chains and trips, squares and stars, cross-words & miscellaneous puzzles. Having surmounted these hurdles, the devoted word puzzlers will enjoy the more ordinary

Recreational Aspects of Dmitri Borgmann's Language on vacation and Beyond Language

(5) Paper toy making by Margaret W. Campbell - Dover Publications, Inc. N.Y. 1975, $1.50
Reproduction of author's 1937 book - 79 toys
includes picture frame, rocking windmill, lily, duck, helmet, Chinese junk, life-boat, water-bomb, cockroach & flying bird. A delightful way to introduce children to geometrical relationships

(6) Geometry by Harold R. Jacobs - W.H. Freeman Co, $9.00, (1974)
This is a tremendous book! Jam-packed with a fantastic number of beautifully reproduced up-to-date photographs, etchings, drawings & cartoons pertinent to geometry. 1100 illustrations.

Letters to the Editor - p 124

(1) Some observations and queries concerning primitive roots by Leslie E. Card, Urbana, Illinois
If a is relatively prime to m, then the least positive exponent t such that $a^t \equiv 1 \pmod{m}$ is called the "order of a mod m" or also "the exponent to which "a belongs mod m". If p is a prime, and if "a belongs to $p-1$ mod p" then a is called a "primitive root" of p. The primitive roots of 7 are then 3 and 5 since $3^6 \equiv 1 \pmod{7}$ and $5^6 \equiv 1 \pmod{7}$ whereas {no powers of 3 or 5 or $5 < 6$} $\not\equiv 1 \pmod{7}$; while $1^1 \equiv 1 \pmod{7}$, $2^3 \equiv 1 \pmod{7}$, $4^3 \equiv 1 \pmod{7}$ & $6^2 \equiv 1 \pmod{7}$. Similarly the primitive roots of 17 are 3, 5, 6, 7, 10, 11, 12, 14

[3, 9, 27, 81, 243, 729, 2187, = where $3^{16} \equiv 1 \pmod{17}$. Same for 5, - 14]

Refers to list of 28,597 primitive roots of the first 167 odd primes & a second list of 600,000 roots for the first 668 odd primes. Observations based on these lists

pp. 125 - 126 - alphametics.

p. 127 - Problems & conjectures Edited by David L. Silverman.

General notice to solvers - Problems with (*) unsolved -

(1) * 58-2 + (formerly, problem 58, part 2) Magic squares by Rudolf Ondrejka.

Oldest unsolved problem of this Journal. Problem rerun in the hope that current knowledge in magic squares will enable some reader to settle the problem

8	1	6
3	5	7
4	9	2

Fig. 1

12	1	18
9	8	4
2	36	3

Fig. 2

Fig. 1 is a 3rd order magic square of constant 15

Fig. 2 " " with constant products of the integers in rows & cols. & diagonals the constant being 216.

Fig. 3 (~~[illegible]~~) gives a 9 x 9 square which is simultaneously ordinary and geometric in that the magic sum is 848 and the magic product is

5,804,807,833,440,000.

Two unanswered questions.

(1) Is there a geometric magic square with a smaller product using different +ve integers than the one in Fig. 1

(2) What is the smallest order magic square of the ordinary - geometric type

(or addition – multiplication type) with minimum sum and minimum product

Answer to first question is "no" – Proof given

" 2nd " is not known. There is however a 8th order square which however numbers appear more than once

[Terrific problem – Don't touch it!]

(2) APT numbers – An integer that can be the area of a Pythagorean (or Lt Δle) with rational sides – Ex. 15 is APT since it is the area of a $(4, 15/2, 17/2)$ right triangle. What is the smallest APT number?

(3) Artful numbers. A number that can be the area of a rational Δle. Thus 1, 2, 3 are rational being the areas resp'ly of a $(3/2, 5/3, 17/6)$, a $(5/6, 29/6, 5)$ & a $(5/2, 5/2, 3)$ Δle. What, if any, is the smallest artless number?

(4) Quadrilateral conjecture – of all quadrilaterals having sides of four stipulated lengths is the one with max. area necessarily inscribable in a circle?

(5) Square root of matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ given by Harry Nelson in explicit form

– Refers to an article in Math. Mag. 36 (1), pp. 487–88 where an author gives an incomprehensible soln.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \sqrt{(a-bc)/r^2} & b/r \\ c/r & \sqrt{(a-bc)/r^2} \end{pmatrix}^2 \text{ where } r^2 = a+d \pm 2\sqrt{ad-bc}$$

Ex (the square roots of $\begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$ are $\pm\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\frac{\pm 1}{\sqrt{33}}\begin{pmatrix} 9 & 10 \\ 15 & 24 \end{pmatrix}$

p. 129 – Solns to problems & conjectures:

All very complicated problems – many using computers.
only one Pillow problem is of some interest

J. Rec. Maths – Vol. 9 No. 3.

p p. 161–67 – Every planar map is 4-colorable – Article by Kenneth Appel and Wolfgang Haken – Univ. of Illinois – 29 references – soln by using computers – talks of "discharging procedures" – Complicated article which is an infn to the proof. A complete proof 56 pages of text and 114 pages of figures averaging more than 30 figures per page is the original.

p. 189 – Six little problems

No. 2. What is the smallest integer n for which neither $(6^n/2)+1$ nor $(6^n/2)-1$ is prime

Ans $n = 11$, for which $181398527 = 1187 \cdot 152821$ and

No. 5 – In what bases does $181398529 = 13 \cdot 19^2 \cdot 38653$

2×8 divide 8×2

Ans: 10, 11, 15, 22, 30.

p. 174 – A tale for mathematicians by Annette Sinclair – Pindus Univ. (A mathematicians version of John G. Saxe's "The Blind Men and the Elephant" based on a Hindu fable) – Lines of poetry – Quite interesting.

p. 178 - π_t by Underwood Dudley - De Pauw University

The value of π at time t (as calculated by people at that time). It now appears that π_t is decreasing with time. The best value, in the least square sense, is

$$\pi_t = 4.59183 - .000773t$$

where t is measured in years A.D. In particular $\pi_{1977} = 3.06361$. In 1876 (the year of U.S centennial year) π_t had the often-used approximation $3.14159265\ldots$ circumferences of circles will be particularly easy to calculate in 2059 when $\pi_t = 3$.

Quite a funny article

p. 181 - Description of a course on in Recreational Mathematics at Temple University by two people of that time

p. 188 - Stars on the U.S. Flag by Sidney Kravitz - Dover N.J.

Discussion of the geometric arrangement of the stars on past flags & the present flag (stars ~~added~~ increased from 13 to 50 in the 2 centuries of the national existence consequent on admission of each state to the Union). From this tradition, some guesses are hazarded about future flags

p. 195 - Lucas' solution to the non-attacking rooks problem by P.J. Campbell, St Olaf College - 24 references (Ahrens problem)

p. 201 - Book Reviews by C.W. Trigg (San Diego, California)

(1) The calculating book by James T. Rogers, Random House N.Y, 1975, $2.95

Statement is made (p. 46) "Starting with any number ___, add to it the same

number entered backward. Continue doing this & before long you will have a number that reads the same backward & forward" Hopefully, no neophyte will choose to start with 196, since Steven Hollingworth has carried this to 37,303 reversal-additions and Henry L. Nelson to our 40,000 reversal-additions without producing a palindrome.

Kaprekar's routine is also explained

(2) Number theory tables by Brother Alfred Brousseau - The Fibonacci Association, San Jose State Univ, San Jose, California, 1973, $12.50 (spirally bound)

Among the tables are primes to 10,000 (I have done this!), composites to 10,000, factorization primes, no. of divisors of n, sum of divisors of n, totient & summing totients, quadratic residues, primitive roots, etc - etc -

p. 205 - Gaussman's Corner - Silverman

pp. 206 - 207 - Alphametics

* No. 521 - Academy alphametic by R. S. Johnson, Montreal, Quebec

pp. 208 - 216 - Problems & conjectures - Edited by Silverman

* No. 112 - Does the Fibonacci ~~series~~ sequence 1, 1, 2, 3, 5, 8, 13, - ~~contain~~ contain an infinity of primes?

Comment - No answers received - No further light has been shed

No. 422 Let $A_n = n^{100} + 100^n$ ($n = 1, 2, 3, 4, -$

(a) ~~Are there~~ Prove that 3, 7, 11, and 13 are not factors of A_n for any value of n

(b) Are there infinitely many primes that never divide A_n? Are there

any greater than 13?

No. 522 – of all plane sections of a regular tetrahedron, which one has the maximum perimeter?

p. 216 – Solns to problems, conjectures and alphametics

No. 214 – Game of the game – Soln by Ashok Chandra, Stanford CA.

No. 242 – Turning Corners by J. S. Madachy, Kettering, Ohio

What is the longest piece of board one inch thick and three inches wide that can be carried around a right-angled corridor which is four feet wide and $8\frac{1}{2}$ ft. high? The board cannot be bent

Quite complicated

– No of solns for alphametical problems

p. 233 – Soln to Computer Challenge Corner.

Vol. 9. No. 4

24/10/78 At Inst. again

Listened to Dr Haar's lecture on Plasma Astrophysics – understood nothing, but was convinced that the lecturer must be a great ~~lecture~~ teacher

Vol. 9. No. 4

p. 241 – Perfect magic icosaperfectacles by B. W. Wynne – Simultaneous linear equations & magic cubes.

p. 268 – ~~Deriving~~ ~~perfectacles~~ divisibility tests by D. M. Burton

Reduces to solving diophantine eqns like $10x + 179 = 1$ for $\div$ by 17.

For 23 & 29 – At each step subtract twice the value of the number formed by the last two digits

Eg: 305688768 – Calculations are

$305688 - 2 \cdot 768 = 304152$

$304 - 2 \cdot 152 = 0$

$\therefore$ No is divisible by both 23 & 29.

p. 265 – Letter from a math. prof who fell in love with one of his students – "Closely, yet discreetly, yet continuously, yours"

p. 276 – Letter to the Editor by Dan P. Howard, York Univ, Toronto, Canada – Re-making a magic square of order 5 (a) Write 0, 1, 2, 3, 4 in top row &

(a)

0	1	2	3	4
2	3	4	0	1
4	0	1	2	3
1	2	3	4	0
3	4	0	1	2

(b)

04	13	22	31	40
21	30	44	03	12
43	02	11	20	34
10	24	33	42	01
32	41	00	14	23

(c)

4	8	12	16	20
11	15	24	3	7
23	2	6	10	19
5	14	18	22	1
17	21	0	9	13

5	14	18	22	1
17	21	0	9	13
4	8	12	16	20
11	15	24	3	7
23	2	6	10	19

Recursive method – In (d) increase each number by 1 & replace (24) by (P) in (P) & (24) by 0 in (0) and for $q = 23$ in (b)

~~multiply by~~ Add 2 & put the results (base 5) in successive rows below

(b) Reflect about middle vertical axis & put the reflected ones as second digits ~~as the~~

(c) convert nos to base 10

(d) Transpose the two bottom rows to bring the middle number to the ~~centre~~ centre to make it an associated nasik square.

Quite interesting

p. 278 – Extension of repunit tables

$$R_n = (10^n - 1)/9$$

(1) For all $n < 1009$, except $n = 2, 19, 23$ are composite

(2) $R_{47}/35121409$ is prime – & the prime is given

(3) Factorisation of R_{51} is given giving primitive prime divisors

(4) Do for R_{57}.

(5) Merson of England gives factorisations of R_{104}, R_{108}, R_{114} & R_{120}

(6) $(10^{53}+1)/11$ is prime

(7) Lehmer's result that $(10^{67}+1)/11$ is prime

(8) complete set of primitive prime factors given for $n = 104, 106, 108, 114, 120, 134$

Vol. 10, No. 1

p. 96 – Gakuho Abe, Japan.

The sequence, beginning with 3, of 73 odd primes ends with 373. These

73 primes can be arranged in a 3 × 3 square & an 8 × 8 square (9 + 64 = 73)

with S = 219 & S = 1460 resp. squares given – Quite interesting

p. 95 – Some interesting "Mathematical Show Me's" by C. W. Trigg. (10 of them) trig

No. 4 – Show me a nudist colony & I'll show you some finite differences

No. 9 – Show me a police state, and I'll show you the law of the mean.

No. 10 – Show me the checkbook balances during a long vacation & I'll show you a monotonically decreasing function.

[my & get some more]

Problem p. 128 (a) Prove that for a △le with sides 4, 5, 6 has a pair of angles one of which is twice the other

(b) Find triplets (a, b, c) (a, b, c integers) with such property

Problem p. 130 – Diophantine deluge

Solve the simultaneous diop^n divisibilities in +ve integers

$x/y + 5$; $y/x + 3$.

How many solns of wrt "+ve" is deleted?

p. 131 – To find a 5 × 5 magic square both additive & multiplicative

(Beyond recreational maths)

10/11/78 At Institute again to hear G. N. R's lecture.

J. R. Math - Vol. 10. No. 2.

(1) A practical approach to Egyptian fractions, i.e distinct fractions each with unit numerator

a is the sum of divisors of b. - More special is the situation of those b for which this is true for any a less than b. Numbers possessing this table property were dubbed "practical numbers" by Srinivasan [Curr. Sci 17, pp. 179-180, 1948] - Also ref. to Fibonacci

p. 95 - Mathematical Shorties by C. W. Trigg

(1) Show me a magic colony & I'll show you some finite differences!

(2) Show me a protest march & I'll show you some radical signs

(3) Show me a police state & I'll show the laws of the mean

(4) Show me the checkbook balances during a long vacation & I'll show you a monotonically decreasing function

p. 96 - Relative magic squares with prime elements by Gakuho Abe, Ogachi-gun Akita-Ken, Japan

The sequence beginning with 3, of 73 odd primes ends with 373. These 73 (the 21st a prime) consecutive primes can be arranged in a 3-by-3 array & an 8 by 8 array to form a third order magic square and an eighth order magic square with $S = 219$ and 1460.

They are

7,

109	7	103
67	73	79
43	139	37

~~7, 43, 67~~ 7, 37, 43
~~73, 79~~ 67, 73, 79
~~7, 37, 43~~ 103, 109, 139
~~109, 139~~,

3	367	97	5	281	263	173	271
137	19	151	179	269	347	257	101
359	239	373	41	227	61	71	89
31	313	349	353	107	167	127	13
241	113	29	193	59	283	211	331
197	53	191	307	163	83	317	149
311	199	47	131	17	233	293	229
191	157	223	251	337	23	11	277

The sum of the orders of these squares is 11, also a prime!

Are there 5th and 6th order magic squares the elements of which are 61 consecutive primes?

[Some work on magic cubes]

Ref. to W. S. Andrews, 1917, p. 64

p. 106, Analysis of a pattern by Samuel Yates

Refers to Prof. Berlekamp [Recreations in Theory of Numbers - The Queen of Mathematics entertains, Dover Publications Inc, p. 85, 1964]

who originated the word "repunit"

Another refers to numbers like

91, 9901, 999001, 99990001, ... 9999999999000000001

as Beiler numbers & shows connections with repunits. of Beiler's numbers

$\overbrace{99}^{2}01$, $\overbrace{9999}^{4}0001$, $\overbrace{999999}^{6}000001$, $\overbrace{99999999}^{8}00000001$ are

prime while the last one with 10 ten nines, 9 zeros & 1 is composite (unfortunately)

Author concludes that most Beiler numbers are composite

p. 113 – Thin books by Trigg

(1) odd perfect numbers ! by O. Yeah !

(2) The null class by M. I. T. Small

(3) Even prime numbers – Owen Lee Wan

Problem on 4, 5, 6 sides. Sol^n is repeated again – still no soln

Also problem 130 on Diophantine deluge is repeated

Vol. 10 – No. 3 – 1977 – 78.

p. 163 – K – transposable & K – reverse transposable rationals integers – Here integers are replaced by rationals

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CONVERGENT – DIVERGENT by C.W.T.

p. 183 – Amicable pairs of Euler's first form by Patrick J. Costello

Little problems (p. 189) with solns on p. 203

Problem – integer whose square root equals sum of its digits – Ans 0, 1, 81

p. 195 – Binary patterns by Doug Engel – appn to magic squares

p. 209 – Powerful integers by Stephen Karhan – a +ve integer called powerful if and only if ~~can~~ it can be expressed as a sum of +ve integral powers of its digits

Gives series of examples of the type I tried from Madachy who called them "narcissus numbers" Cf. with my lists.

Starts with $24 = 2^3 + 4^2$, $43 = 4^2 + 3^3$, — — $153 = 1^3 + 5^3 + 3^3$, —

& ends with $2062 = 2^3 + 0^* + 6^1 + 2^{11}$.

p. 220 APT numbers ie an integer that can be the area of a Pythagorean rational triangle ie a rt Δle with rational sides

Eg. 15 is APT = area of (~~[illegible]~~ 4, 15/2, 17/2) Δle. What is the smallest APT number. – Soln 5 is the smallest

p. 221 (*) – Artful numbers by R. W. Buckley (*) No valid solns received. The problem will be kept open for another year. [Artful = integer which is area of a rational Δle]

Buckley notes that the parametric formula for a rational Δle

$u = x(y^2+z^2)$, $v = y(x^2+z^2)$, $w = |(x+y)(xy-z^2)|$ yields a 4 parameter soln for the artful nos.: $A = |xyz(x+y)(xy-z^2)| / t^2$. It seems to be easy to get t, x, y, z values for any given A but no proof is available

Maths of Computation Vol. 31, No. 140,

p. 1030 – 17 primes in A.P by Sol Weintraub

Abstract – Two sets of primes in A.P. are listed. one is a set of 17 primes Δ.

the second is set of six consecutive primes

A computer search revealed the following 17 primes in A.P

1st term – 3,430,751,869

Last term – 4,827,507,229

Common difference = 87,297,210. = $2 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$

Earlier 16 were much were known.

Recent set of six consecutive primes in A.P

$999,900,067,719,989 + 30t$, $t = 0(1)5$

The smallest such result was found by Landeen & Parkin at

$121,174,811 + 30t$, $t = 0(1)5$

29/1/79

Next lecture

References: (Lecture by Yates on 23/1/79).

(1) Cyclotic primes by S. Yates J.R.M. 8:3

(2) Prime divisors of Repunits by S.Y. J.R.M. 8:1

(3) Cofactors of repunits by S.Y. J.R.M. 8:2

(4) Analysis of a pattern by S.Y. " 10:2

(5) The mythology of repunits by S.Y. Math. Mag. 51:1

(251) --

$\frac{4^2-1}{4-1} = \frac{15}{3} = 5$

Full period prime $- \frac{1}{7} = .142857$

$2 \cdot 10^n - 1$

If $1/p$ & $2/p$ have same cycle freq $\frac{1}{7} = 142857$

& $2/7 = .285714$

Every such p is of this form (primitive divisor) a factor

$n = lm + k$, $7 = 2 \cdot 10^4 - 1$, $2 \cdot 10^{10} - 1$ } ∓ 19999 - i with string of nines.

Smallest k is $2 \cdot 10^k - 1$.

Existence of primitive primes -

x	$(x^n-1)/(x-1)$ is prime when n is
2	2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2291, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701
3	3, 7, 13
4	2
5	3, 7, 11, 13
6	2, 3, 7
7	5, 13
8	3, 5
9	*
10	~~2, 923, 317~~ 2, 19, 23, 317 (1031?)

All these are repunits

$x^n - a^n$ has primitive solns in all cases except when $x = 2, n^6, a = 1$.

proved by Birkhoff & Vandiver in Ann of Math. 1904

x	B_{6x}	— c
1	91	— p
2	9901	— c
3	999001	— p
4	99990001	— c
5	9999900001	— p.
		c
10	99999999990000000001.	c

+ Repunits —

If $m | n$, → $R_m | R_n$.

for eg $2 | 6$, $R_2 | R_6$

$11 | 111111$

If $n = 2^{\alpha} p^{\beta}$ ie $n = 2p$ c.c not divisible by 2 or p. any prime other than

$n = 34$, $2 \times 17 \times 1$, $P_{34} = 90909090909091 = 103 \cdot 4013 \cdot 21993833369$

$n = 44$, $= 2 \times 11 \times 2$, $P_{44} = 99009900990099009901$

$n = 50$, $= 2 \times 5 \times 5$, $P_{50} = 99999000009999900001$.

23/1/79 A of Institute — Attended Prof Yates' lecture on repunits —

J.R.M. 10:4

p 279 Problem 569. Magic talisman squares by Greg Fitzgibbon, Valley Stream N.Y

A "talisman" square is a magic square in which any two neighboring elements (i.e. ringwise adjacent elements) differ by at least some constant. Ex:

3	5	14	12
10	16	7	1
8	2	9	15
13	11	4	6

having talisman constant 2

(a) Display all 24 mch 4×4 squares

*(b) the numbers 1 to n^2 to form a talismanic square. As n increases what is the maximum achievable talismanic constant?

Soln to (a) given (all 24 squares shown)

" (b) No solns received so far.

p. 291 An age problem

On the year 2000 if I live that long, my age will be a perfect cube, and my son's a perfect square. Not too many years ago, the situation was reversed. How old are we (in 1976).

Soln by Frank Rubin. Let ages of father & son be a^3 & b^2 in 2000 AD, and c^2 and d^3, n ("not too many") years ago. Then the difference between their ages is $D = a^3 - b^2 = c^2 - d^3$. The son presumably was born at least n years ago

so his age in 2000 will be at least $n+24$. The only ages for the father in 2000 are then 64 and 125, the latter would make him 101 now.

By examining a table of differences between squares & cubes in the range 0 to 125, we find 5 sets of solns with n and D positive

	A	B	C	D	E
a^3	64	64	125	125	125
b^2	36	49	25	100	121
c^2	36	16	100	25	4
d	8	1	0	0	0
n	4	24	1	76	97
D	28	15	100	25	4

Solns C & E seem implausible ~~impossible~~ with the father either 4, or 100 when the son was born. In solns A and B, though reasonable, the father is only 40 in 1976 that seems ~~to be~~ too young to be worried about living another 24 years. So soln D appears the best. After all to a man of age 101, 76 years ago would not mean too many.

Answers of colonial problems in Vol 10 : 3

9/6/79 At C.C.B

Science Today – April, 79 – Kaprekar's Demlo nos X

Sc. Amer – Jan 1979.

p.19 – Refers to a social security number (last month's question ie Dec 1978 issue) which is

381-65-4729. Adding 0 at the end gives the unique soln to the same problem with the ten digits from 0 to 9.

– do – Feb. 79 – on persistence of numbers. (multiplicative & additive).

– do March 79 – Again on persistence of nos (Ref. to Sloane's article which appeared in J. Recr. Maths Vol 6, No. 2, pp. 97-98, 1973)

Also refers to Hinden's Additive persistence – J. R. Math. Vol. 7, No. 2, pp. 134-35)

Amer. Math. Monthly

Vol. 86. No. 1 (1979).

pp. ~~40~~ 43-45 – Zetazeros on the critical line by H. L. Montgomery (Progress Report)

Simple article – Refers to Levinson's work

Elementary problem p. 55.

E. 2749. Show that neither of the eqns [L. J. Alex, Seney, Oneonta]

$$3^a + 1 = 5^b + 7^c, \quad 5^a + 1 = 3^b + 7^c$$

has a soln in integers other than $a = b = c = 0$.

(ii) Show that the only solns of

$$7^a + 1 = 3^b + 5^c$$

in integers are $(a, b, c) = (0, 0, 0), (1, 1, 1)$.

E. 2750 – Find all solns in integers of the eqn

$$9 + 5^a = 3^b + 7^c.$$

Vol. 81. No. 2. p. 132

Advanced problem No. 6153.

If $\pi(x)$ denotes number of primes $\leq x$. Are there infinitely many integers such as 2, 4, 6, 8, 30, 33, 100 with the property that $\pi(n)$ divides n?

Soln by an undergraduate Jon L. Levy, Univ. of Delaware

The answer is "yes". Let $f(n) = n/\pi(n)$, $n \geq 2$. Then $f(n)$ has discontinuities at and only at odd primes, is strictly increasing on each interval $[p, q]$ determined by successive primes, has "negative jumps" at odd primes < 4 is unbounded above (by virtue of the prime number theorem). It follows from these properties and $f(4) = 2$ that for every integer $K \geq 2$ there is an n/K

Let $\pi(x_k) = k$. Clearly $x_k = k\,\pi(x_k)$ is an integer & the x_k form an infinite set of distinct integers n such that $\pi(n) \mid n$.

p. 146 - Telegraphic reviews

Theoretical principles in Astrophysics & Relativity Ed. Norman R. Lebovitz etc. Proceedings of Symp held at Univ of Chicago in honour of S. Chandrasekhar on May 27-29, 1975

Vol. 85. No. 9

p. 737 - The Morley trisector theorem by Cletus O. Oakley and Justine C. Baker

Gives 150 references including M. T. N (No. 35) & K. V (No. 113) Supplementary references 117-150 by Clark, N. Trigg. Emeritus Professor Los Angeles City College.

p. 752 - Miscellanea

No. 16. $\frac{a}{b} = \frac{b}{a+b} > 0$ iff $\frac{a}{b} = \frac{-1+\sqrt{5}}{2}$

p. 770 - Advanced Problems

6344* - Let D_n & M_n denote the derangement number and the ménage number respectively (Z). Prove or disprove that the sequence

$\{M_n/D_n\}$; $n = 4, 5, 6$ is monotonically increasing

& $\lim_{n\to\infty} (M_n/D_n) = 1/e$

(For definitions see freq. H. J. Ryser. Combinatorial Maths, Chap. 2. Sec. 3 & Chap. 3, Sec. 2)

6266 – A conjecture of H. Gupta about Mobius fn $\mu(n)$

If $M(x) = \sum_{n \le x} \mu(n)$, Gupta conjectures that $\sum_{n \ge x} M(n) = O(x \log x)$. Prove or disprove

Proposed by Anderson & Ram Murty (?).

6138 – Abundant nos of the form $p_i p_{i+1} \cdots p_{i+n}$

Sol. given but very complicated

Vol. 85, No. 10.

p. 802 – Historical sketch of the Olympiads, national & International.

on p. 806 is the table of team rankings of countries participating in the IMO, 1959–1977

No single Asiatic ~~country~~ African country

E. 2671 – Labellings of binary trees?

Maths. Magazine Vol. 51. No. 1 Jan 1978

p. 18 – What is recreational mathematics by C. V. Trigg

Mathematics affords an ever-new never-boring avocation both before and after retirement where it is an offensive weapon against vegetating

The disdainful may say that in mathematics a little humour goes the wrong way others approach the topic from the standpoint of personal enjoyment

Recreational tastes are highly individualised, so no classification of particular mathematical topics as recreational or not is likely to gain universal acceptance.

[29 references given]

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p. 22 – The mystique of repunits by
Samuel Yates
104, Brentwood Drive
Mount Laurel, N.J. 08054.

to be read more carefully later ✓

[Sol this address verified on 15/6/79 at Inst from E.V.K of App. Math. Dept]

Ibid, Vol. 51. No. 3, May, 1978

p. 147 – Ramanujan's notebooks by Bruce C. Berndt. Very interesting to be read. (Facing sheet has Ramanujan's picture being India Postage stamp

Taxi cab no, $1729 = 1^3 + 12^3 = 9^3 + 10^3$ [50 references given in bibliography]

Other examples given by Ramanujan

$19^3 + 60^3 + 69^3 = 82^3$ & $133^3 + 174^3 = 45^3 + 196^3$

11|6|79 at C.C.B

[The third copy of R's notebooks was given to S.S. Pillai in 1950, Pillai lost his life in a plane crash in Egypt while enroute to the Int. Cong. of Mathematicians at Cambridge, Massachusetts - this copy of the note books was with him

p. 154 - defines highly composite nos as numbers with many small prime factors.

p. 156 $A_n = \int_0^\infty \frac{dx}{1+x^n}$ Examples of Ramanujan's formulae

(1) $\frac{\sqrt{3}-1}{1} - \frac{(\sqrt{3}-1)^4}{4} + \frac{(\sqrt{3}-1)^7}{7} - \&c = \frac{\pi}{4\sqrt{3}} + \frac{1}{3}\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)$.

(2) $A_n = \int_0^\infty \frac{dx}{1+x^n}$ worked for $n = 1-6, 8$ and 10. The formula

$$A_5 = \frac{1}{20}\log\frac{(1+x)^5}{1+x^5} + \frac{1}{4\sqrt{5}}\log\frac{1+x\frac{\sqrt{5}-1}{2}+x^2}{1-x\frac{\sqrt{5}-1}{2}+x^2} + \frac{1}{10}\sqrt{10-2\sqrt{5}}\tan^{-1}\left(\frac{x\sqrt{10-2\sqrt{5}}}{4-x(\sqrt{5}+1)}\right)$$

$$+ \frac{\sqrt{10+2\sqrt{5}}}{10}\tan^{-1}\frac{x\sqrt{10+2\sqrt{5}}}{4+x(\sqrt{5}+1)}$$

(3) $$\frac{\left(\frac{\sqrt[1]{1}\cdot\sqrt[3]{3}\cdot\sqrt[5]{5}\cdot\sqrt[7]{7}}{\sqrt{2}\cdot\sqrt[4]{4}\cdot\sqrt[6]{6}\cdot\sqrt[8]{8}}\,\&c\right)^{1/\log 2}}{\left(\frac{\sqrt[1]{1}}{\sqrt{3}}\cdot\frac{\sqrt[5]{5}}{\sqrt[7]{7}}\cdot\frac{\sqrt[9]{9}}{\sqrt[11]{11}}\cdot\frac{\sqrt[13]{13}}{\sqrt[15]{15}}\,\&c\right)^{4/\pi}} = \frac{\sqrt{2}}{\pi}\left(\Gamma\tfrac{1}{4}\right)^4$$

(Watson. Vol 2, Ch. 18, p. 217)

$\pi = \frac{355}{113}\left(1 - \frac{\cdot 0003}{3533}\right)$ very nearly and $\pi = \sqrt[4]{97\frac{1}{2} - 11}$ nearly

In this instance "very nearly" means approx. good to 15 decimal places and "nearly" means good to 9 decimal places

p. 162 He says that the number of integers between A & B that can be written as a sum of two squares is

$$C\int_A^B \frac{dx}{\sqrt{\log x}} \text{ nearly}$$

where $C = \cdot 764 -$. The statement is asymptotically true & a proof of this was first given by Landau.

Long intervals of composite nos for eg. 370,261 to 370,373 and 2,010,733 to 2,010,881.

As usual let $\pi(x)$ = no of primes not exceeding x. R. calculated $\pi(x)$ for several values of x. For eg. he claims that $\pi(10^8) = 5,761,460$; the correct value is ~~460~~ 5,761,455.

$$\left.\begin{aligned} &(A) \quad e^{\frac{\pi}{4}\sqrt{78}} = 4\sqrt{3}\,(75 + 52\sqrt{2}) \\ &(B) \quad e^{\frac{\pi}{4}\sqrt{130}} = 12\left(323 + 40\sqrt{65}\right) \end{aligned}\right\} \text{ R calls these are "approx. formulae"}$$

S. Wagstaff has computed these nos. & they are respectively & find the left & right hand sides of these are nearly.

LHS = 1029.1091087457.., R.H.S = 1029.1091087695.. for (A)

" = 7745.88371918324.., " = 7745.88371918330..

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on p. 375 of 3rd note book, we find that

$$e^{\pi\sqrt{58}} = 24591257751.99999982..$$

is "almost" an integer. Examples like these in this paragraph can be explained by means of the theory of complex multiplication & elliptic fns. A particular example of this type serves as the basis of an "April Fools" joke by Martin Gardner in Sc. Amer. 232 (1975), 126–133.

Examples of equalities between fourth & higher powers of integers are found on pp. 384–86 with some general rules on how to construct them

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p. 174. For a 2nd real no α, the sequence $S(\alpha)$ defined by

$$S(\alpha) = ([\alpha], [2\alpha], [3\alpha], \ldots)$$

where $[x]$ denotes the greatest integer $\leq x$, is called the spectrum of α

~~For~~ Ex. For any three spectra $S(\alpha_1), S(\alpha_2), S(\alpha_3)$ some pair of them must have infinitely many elements in common.

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p. 177 – ψ-fn $\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$

$\psi(1) = \Gamma'(1)/\Gamma(1) = \Gamma'(1) = -r$ (r = Euler's constant). Mathematicians have as yet been unable to determine whether r ($= 0.5772156649\ldots$) is an irrational or rational number.

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p. 183. A soln in integers of $x^x = a^b$: with $a = 4$, there is a soln in integers of $8^8 = 4^{12}$. It would be interesting to try & find other integral solns or even to find all of them

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Math. Magazine – Vol 51. No. 2 (March 78)

pp. 90–98 – Complementary rational numbers by Maurice Schrader-Frechette

Re repeating decimal reps for eg of $1/7 = .\overline{142857}$ – Very comprehensive article containing historical sketch. [20 references]

Ibid – Vol. 51. No. 4. Sep. 78.

pp. 211–219 – Square divisors & square free numbers by Karl Gregor, Univ. of Göteborg, Göteborg, Sweden

A topic from analytic number theory – Computer expts suggesting conjectures which may be confirmed probabilistically and then proved by exact analytical means

Interesting but complicated

p. 226 – Tabulating all Pythagorean triplets by Henry Klostergaard, Calif. State Univ., Northridge CA 91330.

General soln most cited of $x^2+y^2=z^2$

$$z = t\cdot(u^2+v^2),\ x = t\cdot(u^2-v^2),\ y = 2t\cdot u\cdot v$$

where $u > v$, u & v relatively prime & t assumes all integral values.

Gives a method for non-redundant listing using the two diophantine eqs

$$x\cdot y = n\cdot(x+y+z) \text{ \& } x^2+y^2=z^2.$$

Says the method lends itself smoothly, to execution on a programmable pocket calculator & the first 500 Pythagorean triplets were obtained in less than one hour. – ~~Quite~~ A

Also states that the classical 3:4:5 triangle has the least area per unit perimeter of any Pythagorean triple triangle

$$n = xy/(x+y+z) = tv(u-v)$$

p. 238 – A property of 70 by P. Erdos (Hungarian Academy of Sciences)

It is well known that 30 is the largest integer with the property that all smaller integers relatively prime to it are primes [Ref. R&T] ~~A~~ ~~such~~ An analogous special property of 70 is given here.

$a_0 = n$, take sequence $a_0, a_1, \ldots a_{k-1}$. If this sequence by known a_k is chosen

to be the smallest integer greater than a_{k-1} that is relatively prime to the product $a_0 a_1 \ldots a_{k-1}$. Clearly each prime greater than n is an a_k. Moreover each $a_k > n^2$ is a prime.

(12/6/79) At Raman Inst. — Science Today May 79 p. 55 — Kaprekar

$6^2+1^2+7^2+4^2 = 3^2+8^2+2^2+5^2 = 102$

(6174) with (3825) to give 63, 18, 72, 45

& get the pattern 63 18 72 45 54 72 81 36

$63+18+72+45 = 54+72+81+36 = 195$

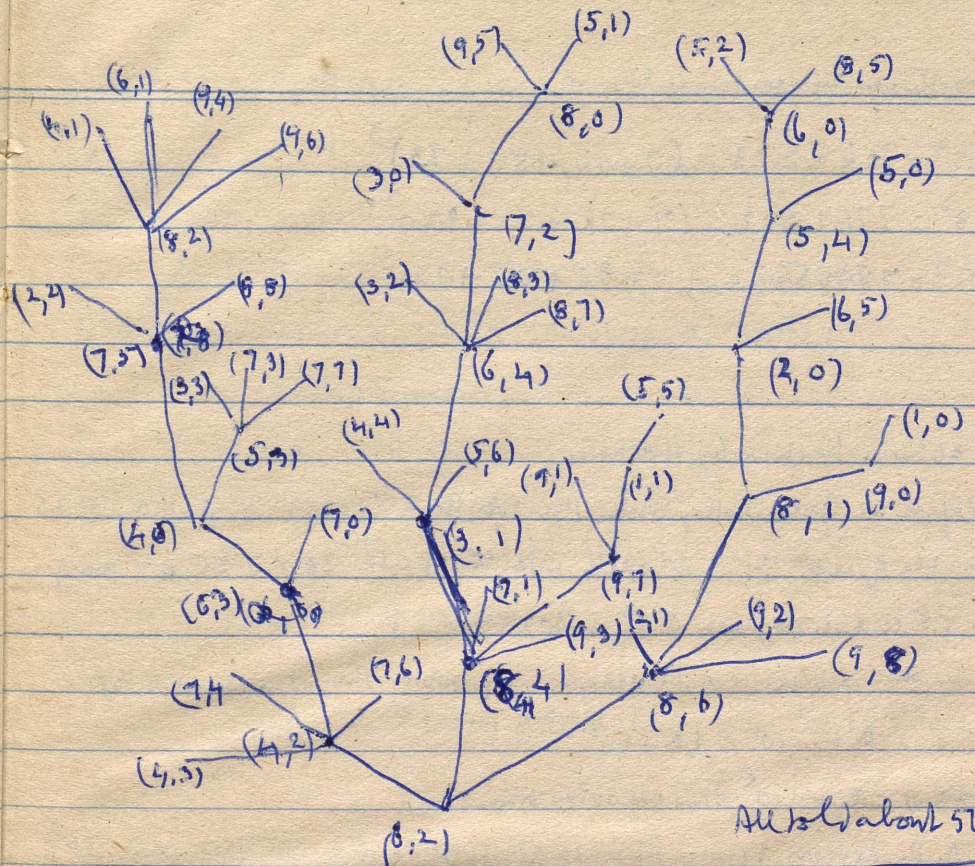
$63^2 + \quad = 54^2 + \quad = 11502$

Magic square with ivs 199, 409, 619, 829, 1039, 1249, 1459, 1669 & 1879 are all prime with (.) = 210. Makes a 3 x 3 square out of them

Science Today — Feb. 1959 — Found in Staff Room at the Inst.

Zeroing in on 6174

by S. Sowmosan



All told about 50 (x,4)'s

A very complicated tree

Amer. Math. Monthly Vol. 85, No. 2

Greenwald: $(1+29+50+29)^3 = 1295029$

$(2+92+42+07)^3 = 2924207$

$(3+65+32+64)^3 = 3652264$

$(3+58+15+77)^3 = 3581577$

He also gives three consecutive cubes

$$197^3 = 7645373 = (7+64+53+73)^3$$
$$198^3 = 7762392 = (7+76+23+92)^3$$
$$199^3 = 7880599 = (7+88+05+99)^3.$$

E. Problem E. 2624.. Proposed by Hashway

J. T. Rogers in Calculating Book, Random House (1976) states that 3025 has "remarkable quirk". If it is "split into two parts" 30 and 25 the sum of those parts is 55 and its square $55^2 = 3025$

Such numbers can be obtained by solving the Diophantine eqn

$$a + b \cdot 10^k = (a+b)^2$$

where $0 < a, b < 10^k$. Find all solns when $k \leq 5$

Soln given by O. P. Lossers Jr.

with $x = a+b$, the given eqn gives

$$x(x-1) = b(10^k - 1) \quad \text{—(1)}$$

For a given k, x determines uniquely an ordered pair (u,v) of positive relatively prime ~~numbers~~ integers such that

$$10^k - 1 = uv \quad \text{—— (2)}$$

and $u \mid x$, $v \mid (x-1)$

Conversely such a pair (u,v) satisfying (2) determines a unique x, $1 \le x \le 10^k - 1$ such that $x \equiv 0 \pmod{u}$ & $x \equiv 1 \pmod{v}$ by the Chinese remainder theorem

It follows that the no. of solns for fixed $k = 2^i - 1$ where i is the number of prime divisors of $10^k - 1$.

Solns tabulated

$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
81	2025	088209	↓	↓
	3025	494209	7 in 20	7 in 20
	9801	998001	↓	↓
			(Larson) 99980001	9999800001

Math. Gazette - Vol. 62, No. 419, March 1978

p. 41 - Obituary of J. E. Littlewood - Died last Sep. - age of 92.

Ref. to other nos. of Gazette re. his work

15/6/79 Pr. Dashtak

Amer. Math. Monthly, Vol 85, No. 4

p. 293 – Review of book "Mathematical Gems" by Ross Honsberger, M.V.I. Assn of America, Vol 1, 1973 $11.00 & Vol 2.

If p is a prime, then p divides $a^p - a$. Is it true conversely that if an integer $n (>1)$ divides $a^n - a$ for every integer a, then n is a prime? Answer is no: for $561 = 3 \times 11 \times 17$ is an "absolute pseudoprime" i.e. it divides $a^{561} - a$ for every a.

The two vols under review treat of that kind of mathematics that mathematicians on a business holiday, like to talk about.

Napoleon's theorem – If an equilateral Δ^s are erected outwards on the sides of an arbitrary triangle, then their centres form an equilateral triangle. This is one sample of a few. And you know it is called Napoleon's theorem?

Does it please you to know that for each of the 80 integral values x between 1460 and 1539 inclusive, the polynomial $x^2 - 2999x + 2248541$ is a prime? If it does, consider the function

$$f(x,y) = 2 + (y-1)\,\mathrm{neg}\left(x(y+1) - (y!+1)\right)$$

where neg is the "negative part" function which assigns to each number Z the value of $Z \geq 0$ and the value $-Z$ if $Z < 0$; explicitly

$$\mathrm{neg}(z) = \tfrac{1}{2}(|z| - z)$$

The range of the function is the set of all prime numbers & moreover, it takes every odd prime value just once.

(*) Tables of potential Archimedeses, Gausses & Ramanujans (270)

Housberger's two vols. are recreational maths. There is already a large library of such books, rang; from "How many 3 cent stamps in a dozen" by Herman Hover (Price-Stern-Sloan, 1976 - the answer is 12) - (*)

My favorites are Steinhaus, Rademacher & Toeplitz, Rouse-Bale, Lewis Carroll, George Polya & Martin Gardner.

Why is the recreational library growing? Why do authors write such books? to attract the amateur & to amuse the professional *

Authors hunt far & wide for new material, go out of their way to reformulate the old, and often lean over backward much too far to avoid boring the blasé

Something good, something bad. These are minor cavils withal. I'd be glad to have written this book, and the next few times I talk to lay audiences, I shall happily steal from it.

Reviewed by P. R. Halmos, Univ. of California - Santa Barbara

Math. Reviews March 1978

Saburô Uchugama: Shows that the diophantine eqn

$$2^x * = 3^y + 13^z$$

has the only solns in non-negative integers given by

$(x, y, z) = (1, 0, 0), (2, 1, 0), (4, 1, 1)$ & $(8, 5, 1)$

Sc. Amr. Dec. 79 – Math. Games, p. 16.

"Persistence" of a number is the number of steps required to reduce it to a single digit by multiplying all its digits to obtain a second number & pon.

Ex (1) $77 \to 49 \to 36 \to 18 \to 8$ (persistence 4)

(2) $10 \to 0$ (Persistence 1)

(3) $25 \to 10 \to 0$ ($p = 2$)

(4) $39 \to 27 \to 14 \to 4$ ($p = 3$)

What is the maximum number of persistences ?

Sloane determined by computer that no number less than 10^{50} has a persistence greater than 11. He conjectured that there is a persistence number c such that no number has a persistence $> c$.

Little is known that persistence in base notations other than 10.

Sc. Amr. July 1977, p. 123 – Article on Gauss by Ian Stewart.

p. 131: Elsewhere Gauss conjectured that prime factorisation was unique for numbers of the form $p + q\sqrt{-D}$ where D is a +ve integer, only when D is equal to one of the numbers 1, 2, 3, 7, 11, 19, 43, 67, and 163. This result which he inferred from numerical evidence has recently been proved by Harold M. Stark & Alan Baker and leads to important new theorems in number theory

Amr. Math. Monthly. Vol 55, No. 5, May 1948, pp. 308–309.

The sieve of Eratosthenes & the Mobius strip by H. T. McAdams.

Amer. Math. Monthly, Vol. 86, No. 2, pp. 89 – 108

An introduction to Ramanujan's "lost" note books by G. E. Andrews, Deptt of Maths, Pennsylvania State University, University Parks, PA. 16802.

Notebooks get from Cambridge out of what was left with G. N. Watson – comprehensive article to be read more carefully & referred to K. V – 28 references given

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18/6/79 at B.U. Lib

Earlier available volumes of Math. Magazine

Vol. 49, No. 1 – Jonnes, 1976.

p. 19 – Factoring integers whose digits are all ~~primes to~~ ones by Elmer K. Hayashi, Wake Forest Univ.

Some factors of a special class of integers which includes the Mersenne primes ~~Based~~ Unfortunately uses base b (>1).

$I_k = b^{k-1} + b^{k-2} + \dots + b + 1$ ie a b-repns consisting of k ones

If $(n, b) = 1$ ie n & b relatively prime, define $C(n)$ to be the smallest ~~number~~ positive integer k such $n \mid I_k$ – In particular if $b = 10$, then $C(n) = n$ if and only if n is a power of 3 (Ref. to Andrew Sobczyk – Factorisation of certain decimal integers

Notices Amer. Math. Soc. 20 (19 Nov. 1973), A-656.

An interesting unanswered question is "For which primes p is I_p prime?" This is a difficult question because when $b=2$ this is exactly the problem of determining the Mersenne primes. Main results:

Prop 11: Let n be a +ve integer with $(n,b)=1$, then

(i) If $b \not\equiv 3 \pmod 4$, then $C(n) = n$ if & only if every prime factor of n is also a factor of $b-1$.

(ii) If $b \equiv 3 \pmod 4$, then $C(n) = n$ if & only if $4 \nmid n$, and every prime factor of n is also a factor of $b-1$

Prop 12 - If $(n,b)=1$, then $C(n) = n-1$ if and only if n is a prime which does not divide $b-1$, and b is a primitive root of n

[Unfortunately the article relates to base b - Otherwise we could have had results regarding repunits - Anyway translate these for $b=10$ & OK]

p. 27 K-transposable integers by Steven Kahn, Queens College, Flushing N.Y.

Let K be a +ve integer. Then the +ve integer x is K-transposable if and only if it has the property that when its leftmost digit is moved into the units place, the resulting integer is equal to Kx. It is immediately apparent that x is 1-transposable if & only if all its digits are identical. We hence assume $K>1$.

Notation (i) If $t>1$, $\overline{N}_t$ is the integer obtained by concatenating N t times

(ii) If $t=1$, $\overline{N}_1 = N$

(iii) If $t=0$, $\overline{N}_0$ is the statement that N is entirely absent

It is shown in this paper that for $K > 1$, K-transposable integers occur only for $K = 3$, and that the only 3-transposable numbers are

$$\overline{142857}_m \text{ and } \overline{285714}_m \text{ where } m \geq 1.$$

[Some kind of a proof. I remember this chap writing in J. R. M. 12]

p. 30 — Some Scherk sequence & ref. to Fibonacci numbers

p. 34. Search for a set of distinct ~~primes~~ odd integers whose reciprocals total up to 1. by E. J. Barrean.

~~Note~~ The number of reciprocals must be odd (otherwise we get on adding a fraction with an even numerator & odd denominator). Five are too few since $1/3 + \cdots + 1/11 < 1$. It is straightforward to verify that seven are also too few. So at least 9 numbers are needed. Tedious calculation shows that if nine is possible, five of them must be $(3, 5, 7, 9, 11)$. It is not known if a repn involving only 9 reciprocals is possible or, if perhaps, the minimum number of required reciprocal is 11 or 13.

The author gives a chain reaction method giving 13 reciprocals

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{21} + \frac{1}{45} + \frac{1}{55} + \frac{1}{63} + \frac{1}{105} + \frac{1}{231} + \frac{1}{315} + \frac{1}{1155} = 1$$

p. 42 — Elementary example of a transcendental p-adic number by Glen H. Sutter, Albany State College, Georgia.

Liouville proved that $\sum_{i=1}^{\infty} (-1)^i 2^{-i!}$ is transcendental in the real numbers, Author's

Theorem: $\sum_{i=1}^{\infty} p^{i!}$ is a transcendental p-adic number

Vol. 49. No. 2 — absent

Ibid. Vol. 49. No. 3

p. 150. Solns

Ex. 937 — Which is greater. e^{π} or $(e^e . \pi^e . \pi^\pi)^{1/3}$?

Proved by number of people that $(e^e . \pi^e . \pi^\pi)^{1/3} > e^\pi$.

p. 152 — Bottin 940 — old problem of Dudeney on hour & minute hands of a clock being of same length & working ambiguously

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Ibid. Vol. 49. No. 4, Sept. 76.

pp. 163–180 — Search for finite simple groups by Joseph A. Gallian

(Simple group if its only normal subgroups are the identity group & the group itself.) — Very scholarly paper with 79 references

on p. 180 — a poetic musical symphony on the development of the subject

p. 187 — The "sales tax" theorem by Solomon W. Golomb

p. 200 — A solvable diophantine eqn by Norman Wildberger, Student, Univ of Toronto

Soln of $5x^6 + 2y^3 = 3z^5$ is surely $(3.7^4, 3^2.7^6, 3.7^5)$

p. 214 Charlemagne's magic square by Charles W. Trigg, San Diego, California

Early in his reign as Emperor, Charlemagne ordered a fortress to be built at a strategic point of his domain. As good luck charms, he had a third order magic square with all prime numbers engraved on each wall. The four magic squares were different from each other, but they had the same magic constant — the year in which the fortress

completed. That fort would be able to withstand attack midway through his reign.

on this evidence, reconstruct the magic squares.

Since Charlemagne reigned as Emperor of the West from 800 to 814, a fort would be able to withstand attack midway through his reign, $800 \leq S \leq 807$ and so $267 \leq S/3 \leq 269$. But $A = S/3$ is a prime. Thus $A = 269$, $S = 807$.

There are 13 triples of prime numbers with 269 as one element and having a sum of 807. They are 257, 269, 281; 227, 269, 311; 191, 269, 347; 179, 269, 359; 149, 269, 389; 137, 269, 401; 107, 269, 431; 89, 269, 449; 71, 269, 467; 59, 269, 479; 47, 269, 491; 29, 269, 509 and 17, 269, 521. Inspection of a listing of the twenty-seven distinct numbers which appear immediately above, ordered from smallest to largest, and their differences yields the following five sets of trios of numbers [$v =$

(i) 47, 59, 71; 257, 269, 281; 469, 479, 491 ($d = 12, v = 186$)

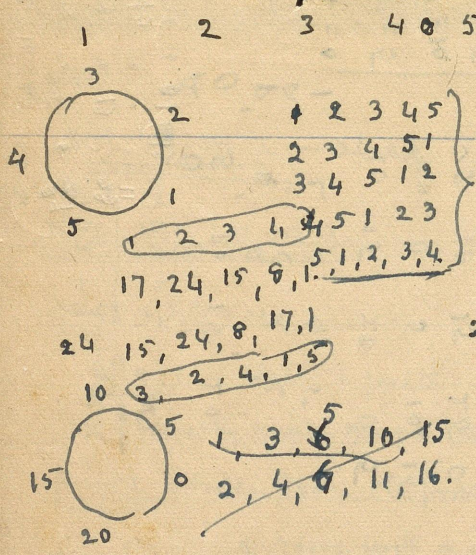
(ii) 107, 149, 191; 227, 269, 311; 347, 389, 431 ($d = 42, v = 36$)

(iii) 71, 149, 227; 191, 269, 347; 311, 389, 467 ($d = 78, v = 36$)

(iv) 47, 137, 227; 179, 269, 359; 311, 401, 491 ($d = 90, v = -48$)

(v) 17, 137, 257; 149, 269, 389; 281, 401, 521 ($d = 120, v = -108$)

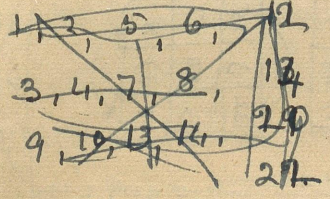
From these triples the following 3rd order magic squares are obtained



A=0	1	2	3	4	5
B=5	6	7	8	9	10
C=10	11	12	13	14	15
D=15	16	17	18	19	20
E=20	21	22	23	24	25

0	1	2	5	6
2	3	4	7	8
8	9	10	13	14
10	11	12	15	16

17, 19, 21, 22, 24, 25



1	3	5	7	...
2	4	6	8	.
9	11	13	15	.
10	12	14	16	..
.	.	.	.	.

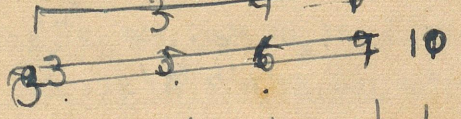
17, 18, 19, 20, 21, 22, 23, 24, 25

11, 12, 13, 14, 15

4, 6, 8, 10, 12

21, 22, 23, 24, 25

0	1	3	5	7	9
1	2	4	6	8	10
10	11	13	15	17	19
11	12	14	16	18	20
	17	19	21	23	25



18, 18, 20, 22, 24

nothing available.

22, 24.

A=0	1	b	C	d
B	B+1	B+b	B+C	B+d
C	C+1	C+b	C+C	C+d
D	D+1	D+b	D+C	D+d

1	D+C	B+d	C+b
C+d	B+b	D+1	C
D+b	d	C+C	B+1
B+C	C+1	b	D+d

21	18	15	4	7
10	2	24	13	16
19	11	8	22	5
3	25	17	6	14
12	9	1	20	23

12	9	1	20	23
10	2	24	13	16
19	11	8	22	5
3	25	17	6	14
21	18	15	4	7

5, 2, 4, 1, 3
4, 1, 3, 5, 2.

15+5, 20+2, 0+4, 5+1, 10+3
5+4, 10+1, 15+3, 20+5, 0+2

1 + . 20

$\frac{20 \times 21}{2}$

22
23
24
25
26

210
120
330 = 66
5

1	24	20	12	9
A + a	E + c	D + e	C + b	B + d
15	7	4	22	18
C + e	B + b	A + d	E + a	D + c
25	16	13	10	2
E + d	D + a	C + c	B + e	A + b
8	5	23	19	11
B + c	A + e	E + b	D + d	C + a
17	14	6	3	26
D + b	C + d	B + a	A + c	E + e

15 20 0 5 10
5 10 15 20 0.

2 3 4 5
4 2 5 3
3 5 2 4
5 4 3 2

17	12	31
39	20	1
9	28	23

A = 0
B = 5.
C = 10
D = 15
E = 21.

~~a = 1, b = 2, c = 3, d = 4, e = 5~~

15	13	32
37	20	3
8	27	25

17	12	31
34	20	6
9	28	23

17
18
19
20

20, 22, 23, 24, 25

0	1	2	3	4
2	3	4	7	8	10
10	11	12	15	16	19
12	13	14	17	18	21.

0	1	3	6	8
1	2	4	7	9
8	9	11	14	16
9	10	12	15	17

28	17	6
31	20	9
34	23	12

10+3, 15+2, 20+1, 0+5, 5+4

~~0, 1, 2, 3, 4~~

1 ~~1 2 3 4~~ 5

17, 18, 19, 20, 21, 22, 23

~~24, 25~~

4 14 25 20 2
15 . . . 7
3 . 13 . 23
19 . . . 11
24 6 17 12 22

$\frac{10x+y}{10y+z} = \frac{kx}{kz}$

A B C
0, 5, 10, 15, 20

P Q R S T
a b c d e

	1	2	3	4	5
A = 0		10	15	20	0 5
B = 5		20	0	5	10 15
C = 10		✓5	10	15	20 0
D = 15		15	20	0	5 10
E = 20		0	5	10	15 20

$10xz + yz = 10xy + xz$

$10kxz + kyz = 10kxy + kxz$

10 (1, 2, 3, 4, 13) 22, 23, 24, 25

1 2, 3, 4, 5, 6, 7, 8, 9.

10+5, 15+3, 20+1, 0+4, 5+2
20+4, 0+2, 5+5, 10+3, 15+1
8, 11, 19, 22, 25
17, 25, 3, 6, 14
1, 9, 12, 20, 23

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

4 25 2
3 23
24 22

5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18

15	20	0	5	10	19
20	0	5	10	15	20
0	5	10	15	20	21
5	10	15	20	0	
10	15	20	0	5	

2	4	1	3	5
3	5	2	4	1
4	1	3	5	2
5	2	4	1	3
1	3	5	2	4

15	20	0	5	10
0	5	10	15	20
5	10	15	20	0
20	0	5	10	15
10	15	20	0	5
0	5	10	15	12

34
26
78

15+3, 20+4, 0+5, 5+1, 10+2
5+5, 10+1, 15+2, 20+3, 0+4
20+2, 0+3, 5+4, 10+5, 15+1
10+4, 15+5, 20+1, 0+2, 5+3
0+1, 5+2, 10+3, 15+4, 20+5

5	3	1	4	2
4	2	5	3	1
✓3	1	4	2	5
2	5	3	1	4
1	4	2	5	3

3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3
1	2	3	4	5

(i) 479 71 257 (ii) 389 191 227 (iii) 389 227 191
47 269 491 107 269 431 71 269 467
281 467 59 311 347 149 347 311 149

(iv) 401 227 179 (v) 401 257 149
47 269 491 17 269 521
359 311 137 389 281 137

Contrib Bob Prielipp
Univ. of Wisconsin – Oshkosh.

Another set using the numbers by Scott Smith, Bellingham, Washington

Re p. 274 – Problem 945 – Find the smallest Pythagorean triangle in which a square with integer sides can be inscribed so as that an angle of the square coincides with the right angle of the triangle [Alan Wayne, Pasco Hernando Community College, Florida]. In

Ans by Gleason Bartel – Sides of Pythagorean triangle are 21, 28, 35.

pp. 219–222 – News & Letters – Solution of the 4-color problem

Solved by Computer by Kenneth Appel & Wolfgang Haken of the University of Illinois – Took 1000 hours of computer calculation – Proof contains several hundred pages of what even the authors call "ridiculous detail"

Impossibility of computer proofs of problems like Fermat's Last Theorem because of the lack of theoretical maneuver by the reduction from an infinite to a finite number

of cases. This is possible in the 4-color problem because of the intricate geometry of maps: the behaviour of graphs of size n very strongly influences the behaviour of graphs of size $n+1$. With luck & insight, it is possible to develop a finite number of cases that cover all infinitely many possible maps.

In Fermat's theorem the behaviour of prime numbers appears to be much more loosely knit than is the geometry of graphs. So the finitisation of problems in number theory will either be very difficult or perhaps impossible, and no computer assault can work until the finitisation theory is complete.

The Haken-Appel result points therefore to the existence of a new class of mathematical theorems that are true, but for which no simple proof exists. Exploration of this realm by mathematician-computer team work is, according to Haken & Appel, a major challenge for mathematicians in the final quarter of this century. "This work has changed my view of what mathematics is" said Haken "I hope it will do the same for others"

— Lynn Arthur Steen

On back of closing sheet is given ~~an~~ an account of contents of Honsberger's book (Ross Honsberger, Univ. of Waterloo) "Mathematical Gems" of which I saw a review by P. R. Halmos the other day.

Contents of Chapter titles: — An old Chinese theorem and Pierre de Fermat,

Louis Pósa; Equilateral triangles; The Orchard Problem, Δ-curves; It's Combinatorics that counts!; The Kozyrev-Grinberg Theory of Hamiltonian Circuits, Morley's theorem, A problem of combinatorics; Multiply-perfect, Super-abundant, and Practical numbers; Circles, squares & Lattice points; Recursion; Poulet, Super-Poulet & Related numbers; Solns to selected exercises

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