N (1642–1727)

(1) greatest physicist – Pope's lines – A, N & G three greatest maths – discordant voices – balance thru' Whiteside's 7 vols – finest ~~tribute~~ monument that could ~~ever~~ be raised to the memory of N.

(2) Development of the calculus thereby bringing about a revolution in maths. Arithmetic, algebra & geometry enriched by a new discipline viz analysis involving a new operation of limit.. Analysis with all its recent developments has plays a great part in the progress of maths & also other natural sciences

(3) Develop & not invent – Analysis before N & L consisted of scattered. piecemeal researches by many workers – Earliest Bhaskara (1144–85), $u = x - e \sin y \rightarrow u' - u = (x'-x) - e(y'-y)\cos y \rightarrow du = dx - e \cos y . dy$ – Kerala workers in Yuktibhasa, a Malayalam compendium (1639) based on 16th century Sanskrit original ~~Tantras~~ Tantrasangraha & due to Nilakantha (about 1500) re. series of Sin x, cos x, arc sin x, arc cos x. ~~Sin x ob~~ Series for Sin x (1st by Newton) obtained by repeated use of the identity $\sin x = \int [1 - \int \sin x \, dx]$ ~~dx – Series~~ – Also inverse tangent series obtained arc ~~tan x~~ (W. 2, 237 n) – Also inverse tangent series obtained by Nilakantha by exactly same method of Gregory (Jan. 1671) and Leibniz (late 1673) resply.

European mathcians before Newton – Stevin (c.g. of Δ) – Kepler $\left[\int_0^a \sin\phi = 1 - \cos\phi, \ \cancel{x = e + x +}\ x = m + e \sin x\right]$, Galileo [general observation], Cavalieri $\left(\int_0^a x\,dx = a^2/2 \text{ etc}\right)$, Fermat $\left[\int x^n dx \text{ \& max \& min of } x(a-x)\right]$, Pascal [Int. by parts], Roberval [cycloid], Huyghens [cycloid as Brachistochrone, evolute & involute] & last but not least Barrow (1630–1677) [some def. integrals] – Barrow Newton's nature. Kindness & encouragement to Newton, Lucasian post with Trinity Fellowship – freedom due to Royal mandate, 26 years of research – Tercentenary of Barrow's death

(4) Newton's work – assimilation & generalisation of earlier geometrical & analytical work & putting on a sound basis of gen. considerations – Fluxions –

3 periods (a) 1664–1666, prime age for invention, (b) ~~1666~~–1671
(1666 tract)

tract (C) 1687 Principia.

(a) Brilliant but immature – (lacking rigour & clarity) more of geometry than fluxions – fluents, fluxions, genita & moments – tabulations of derivatives & integrals in 11 cols (integrals as inverse fluxions) – Numerous geometrical examples using fluxions & inverse fluxions

(b) Fluxions & infinite series – great facility in latter – complete mastery of technical detail & of fundamental importance – general problems of finding ~~derivatives~~ rules for diffn, integrals by convergent ∞-series, max & min using $\dot{y}$, $\ddot{y}$, subt, sub n, curvature, polar coords – quadrature problems,
1671 ~~con~~ tract contains most of present day elem. diff & int. cal.

(c) Perhaps most important ~~but~~ contribution, first example of use of analysis applied to physics (dynamics & optics) – Laplace's opinion – ease & fertility of application of pure techniques in app science, ~~also~~ ~~optics~~ use of limits clarified.

(5) ~~Controversy with Leibniz~~ Controversies re. fluxions (a) Berkeley & Jonathan Swift of Gulliver's Travels (controversy not with Newton), ghosts of departed quantities (b) Controversy with Leibniz starting shortly after N (1666, & 1671) & L (1672–76) 1687 when N was Pres. Roy. Soc – F.R.S.'s indignant about a foreigner claiming credit for inventing calculus. Claim that he saw Newton's work, improved it but claims it as his own – cudgels on behalf of N & England, cudgels on behalf of Leibniz & Germany – British scientists' ~~pseudo~~ – Elephant on moon – Newton originally out but came later as British King was drawn into it – Letters between them

Examples of L's $\pi/4$ ~~$=1-\frac{1}{3}$~~ $= \int_0^1 \frac{dx}{1+x^2} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + -$ & Newton's (due to Nilakantha)

counterexample of $\int_0^1 \frac{(1+x^2)\,dx}{1+x^4} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} - \frac{1}{13} + -$

Ramanujan's two series for $4/\pi$

vehement & inconclusive – Newton to blame for not publishing 1671 tract,
reasons for the same – Final conclusion – Leibniz's notation superior & now followed

(6)

(6) ~~brief~~ Further work in ~~1727 – 1800~~ 18th century quite rapid growth of
analysis – Bernoullis systematised calculus followed by L – Euler & Lagrange
discovery Calc. of variations (brachistochrone). – ~~Daniel~~ Daniel Bernoulli (son of
Johann) & Euler ~~for~~ $c\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}$ from vibrating string – Monge's discovery
of the curvatures & diff. geom. On appl. math side hydrodynamics by D. B.,
Lagrange on analyt. dyn. – Taylor & McLaurin series, addn theorem on
integrals, proof of π being irrational.

ie [Bernoullis – Euler & Lag. (C.V) – D. B & Euler (partial diff eqns) –
Monge diff geom – Taylor & McLaurin – π irrational addn th.].

(7) 19th century – notion of function Fourier Cauchy, Dirichlet & ~~Riemann~~ &
Riemann (notion of integral & gen diff geom reasons) – Cauchy on
complex variable, use in App. math – numerous other developments (Abel &
As a sort of revolt against dominating influence of Analysis, subjects
like No. theory (Gauss & Eisenstein also worked in Analysis & App math
& Eisenstein), Groups & algebra (Galois,

Cauchy, Dirichlet, Weierstrass, Jacobi, Abel, Hermite, Bessel –
analysis, complex variable, Riemann, – Hamilton – Gauss & Eisenstein –
Galois, Boole, Cayley & Sylvester – De Morgan.

(8) 20th century – abstraction – unification – heaps of new disciplines –
18 topics of analysis – Analytic theory of numbers – $\zeta(s)$ – Analysis
(continuous) breaches No. theory (discrete).

(9) overall picture of Newton's math. work. – [illegible] 150, [illegible]
log tables – finite differences & interpolation – higher plane curves, cubic curves –
theory of eqns – annuities & algebra of quantities – computational maths
~~[illegible] fluxions, optics & Principia not [illegible]~~ to no. theory, also rational
Chinese remainder theorem
altitudes – rational cyclic quadrilateral – Diophantine eqn $x^2y^2+z^2=t^2$
Ramanujan –

Newton and the Calculus

It ia an undisputed fsct that Newton is one of the greatest physicists of all time, and according to the famous lines lines of the poet Alexander Pope

'' Nature and Nature's laws lay hid in night,

God said let Newton be and all was light''

There is also another assessment re. Newton viz. that Archimedes, Newton, and Gauss are the three greatest mathematicians of all time. There have, however, discordant voices about this judgment from some theoretical physicists, and abstract mathematicians. But even a mere hlance through the contentd of the recently published seven volumes of the collection of the mathematical papers of Newton edited by Whiteside

(10) motivation for his work – Math an art – Conclusion

1664–66 — I (N)
1667–70 — II (N)
1671–73 — III (N)
1674–82 — IV (mm)
1683–84 — V (Halley)
1684–1691 — VI (Pr)
1691–95 — VII (Pr. Taylor's cht)

(1) If $\xi = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\cdots$, ξ is transcendental if there exist infinitely many $\underline{n}$ such that $a_n = a_{n+1} = \cdots = a_{n+\lambda(n)+1}$
where $\lambda(n)$ is greater than a certain fn of q_n (p_n/q_n approximants)

(2) Measure of transcendency of π – Izv. Akad. Nauk SSSR. Ser. Mat. 24 (1960), 357–368 – N. I. Feldman.

For π it is shown that there exists an absolute constant Λ_0 such that

$$|\pi - \xi| > H^{-\Lambda_0 n \log(n+2)}$$

for all algebraic numbers ξ, of degree $\underline{n}$ and height H, provided $H > \exp\left[n^2 \log^4 (n+2)\right]$

(3) Is π a Liouville number? – Mahler proved in 1953 that $|\pi - p/q| > q^{-42}$ for any pair of integers p & q with $q \geq 2$

(4) Expansion of π in a regular continued fraction – done up to a_{400} – Verification of Khintchine's & Lévy's consts also extended

Reduce to half size

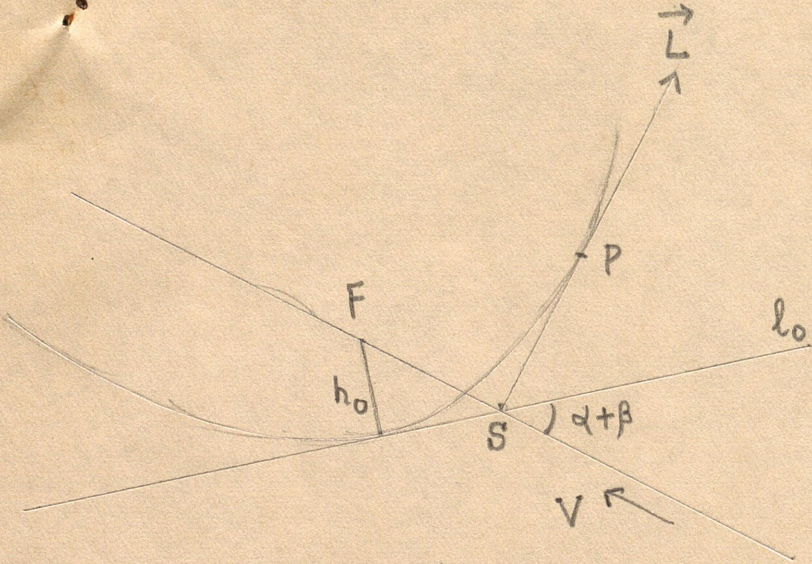

Fig 1. metacentric parabola, and lift force. ($h_0 > 0$).

B. S. Madhava Rao.

Reduce to half-size

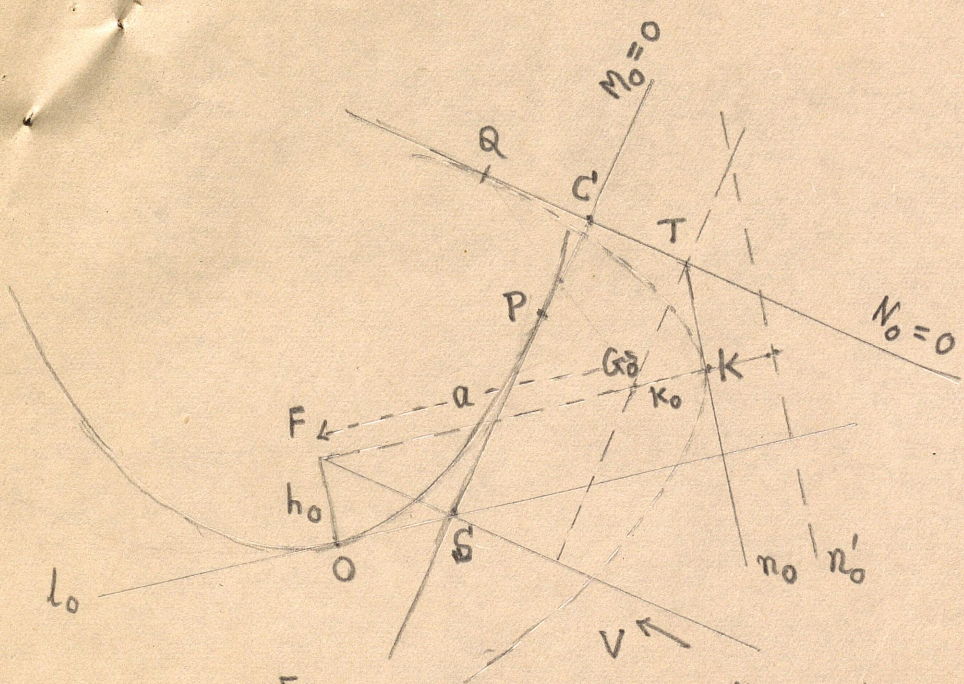

Fig. 2. Construction for Hamilton Centre
($h_0 > 0$, $k_0 > 0$)

B.S. Madhava Rao

Reduce half-size

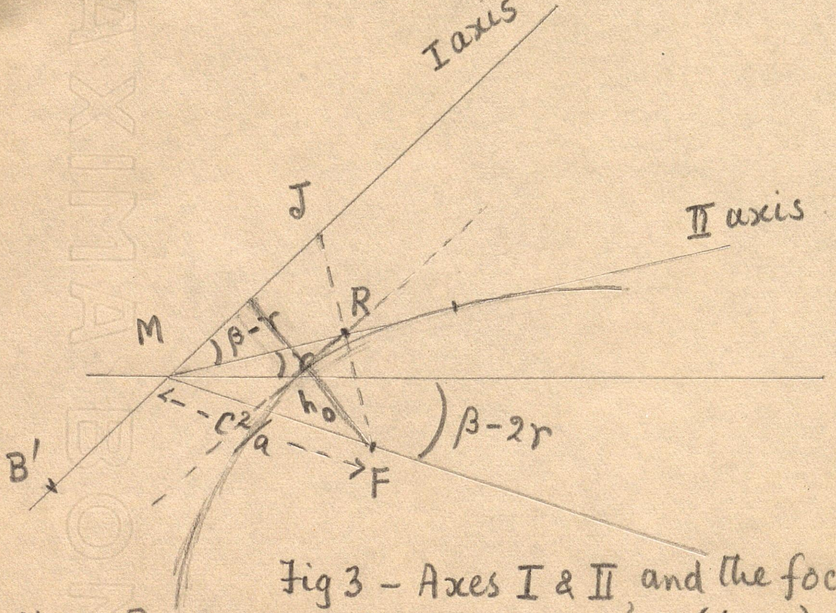

Fig 3 – Axes I & II, and the focus. ($h_0 < 0$)

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Reduce half-size.

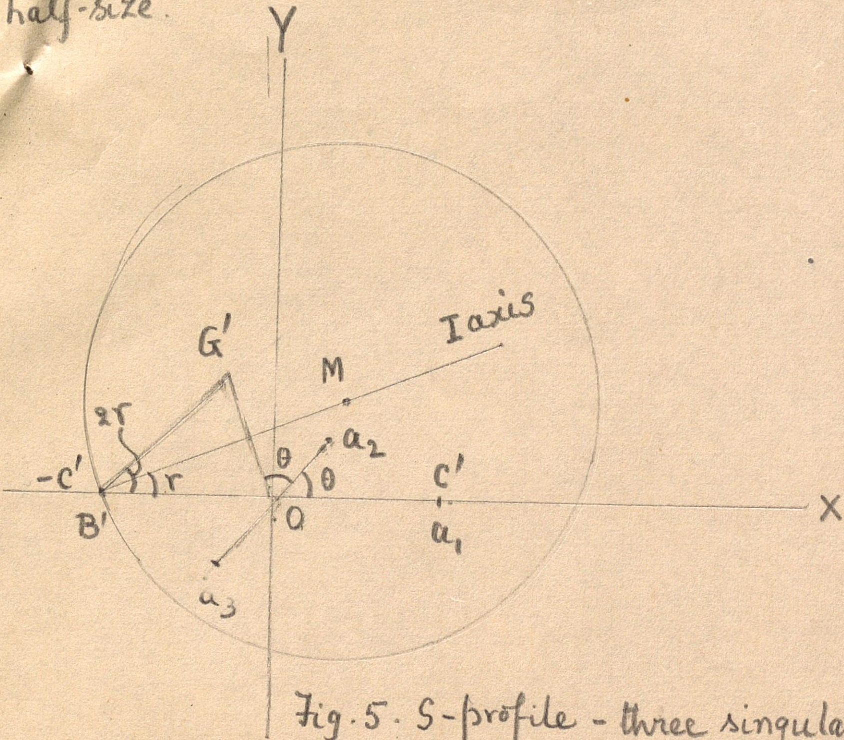

Fig. 5. S-profile - three singularities

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Reduce half size

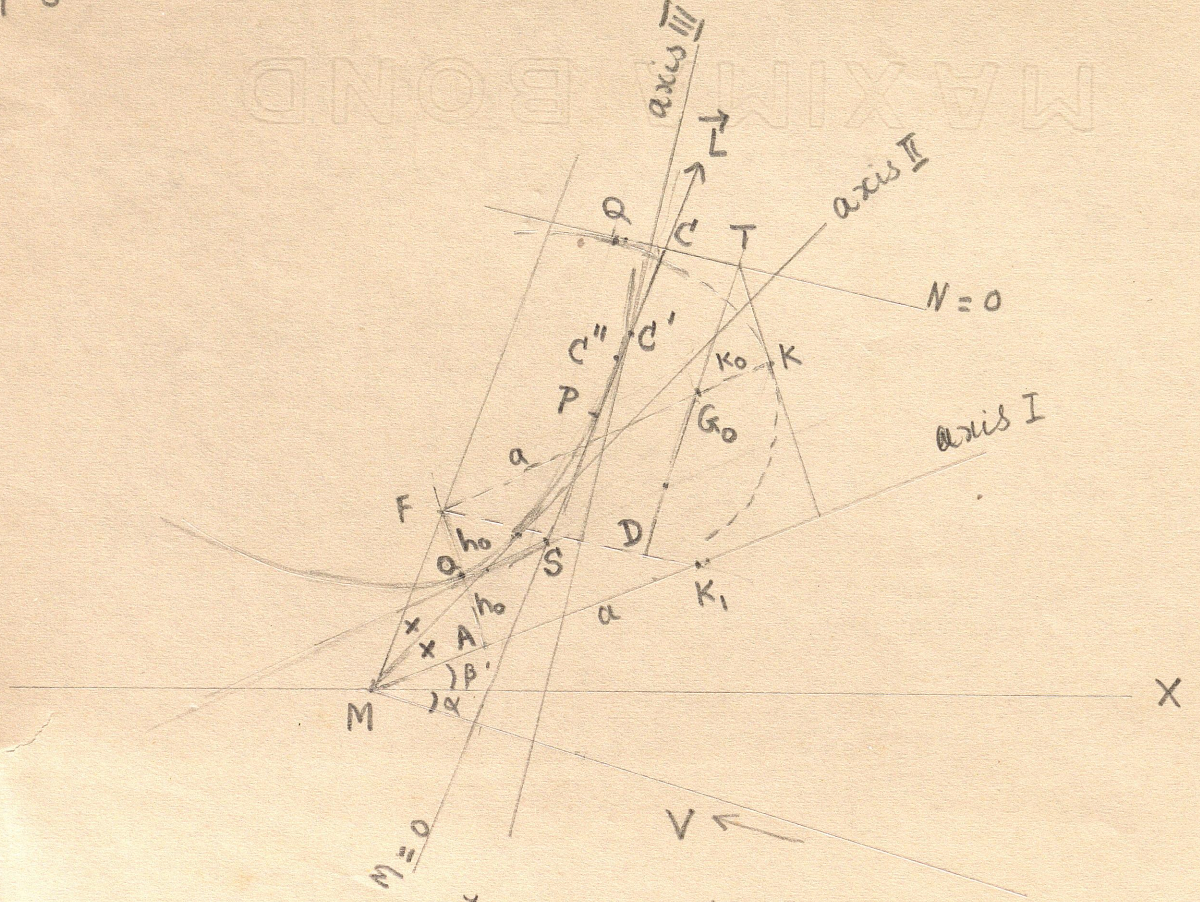

Fig. 4 – Showing the axes.

B. S. Madhava Rao.

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1967 - Math. of Computation

Center for Theoretical Studies

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

Dear Sir,

I am enclosing herein a note given to me by KV to be handed over to you. I am sorry that as I am not well, I could not ~~go~~ come myself.

KV also wanted to point out 2 misprints in your paper — omission of +1 in the degree of f_n in the 1st page & in the 2nd page +1 occurs in the exponent. He wants to point out that the deduction of (2) in the paper is not quite satisfactory. He has given an alternative argument.

Yours sincerely

[illegible]

BSM

8/11/69

Prof. B. S. Madhava Rao
4. Kankanahally Road

(1) Pope, – Archimedes, Newton & Gauss – Whiteside

(2) claim as a mathematician – ~~for~~ Calculus & Analysis

(3) Analysis before Newton

(4) Newton's work on fluxions & series – 3 periods

(5) controversy with Leibniz – Berkeley & Swift

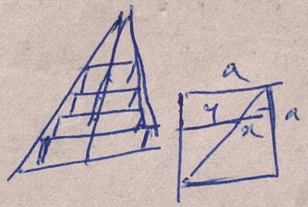
(6) 18^{th} century maths

(7) 19^{th} "

(8) 20^{th} "

(9) overall work

(10) Assessment.



Newton (1642–1727)

(1) $u = x - e \sin y$

W, 2, 237. n.

(2) $\sin x = \int \left[1 - \int \sin x \, dx \right] dx$ (IV)

Gregory Jan 1671 & Leibniz (Later) 1673.

arc tan x (N), Gregory Jan 1671 & Leibniz 1673.

(3) Kepler $\int_0^{\phi} \sin\phi = 1 - \cos\phi$, $x = m + e \sin x$.

(4) ~~Fermat~~ cf. Fermat – $x(a-x) = y$, max

(5) Cavalieri $\Sigma x + \Sigma y = \Sigma a$.

(6) Cycloidal vol $= 5/8$ circum cylinder (Roberval)

(7) $\int_0^1 dx/(1+x^2) = \pi/4$ – Wallis computing π; Barrow (~~163~~ 1630–1677)

(8) 1664–66, 1671, 1687. ($y = 3x - x^2$ Ex)

(9) Leibniz (1646–1716) – (1672–76) work on Calculus.

Wallis $\int_0^1 \frac{dx}{1+x^2} = \pi/4 = 1 - 1/3 + 1/5 - 1/7 + -$ also Nilakantha

$$\frac{\pi}{2\sqrt{2}} = \int_0^1 \frac{(1+x^2)\,dx}{1+x^4} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + -$$

(10)

(11) $$\frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 + \frac{19}{4^3}\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + -$$

$$\frac{4}{\pi} = \frac{1123}{882} - \frac{22583}{(882)^3} \cdot \frac{1}{2}\left(\frac{1 \cdot 3}{4^2}\right) + \frac{44043}{(882)^5}\left(\frac{1 \cdot 3}{2 \cdot 4}\right)\left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{4^2 8^2}\right) \dots$$

$x^3 + y^3 + z^3 = t^3 \quad (3, 4, 5, 6)$

$$\left.\begin{aligned} x &= 3a^2 + 5ab - b^2 \\ y &= 4a^2 - 4ab + 4b^2 \\ z &= 5a^2 - 5ab \cancel{+} - 3b^2 \\ t &= 6a^2 - 4ab + 4b^2 \end{aligned}\right\}$$

Constant-breadth Curves — Generalisation, (Details Here)?

Ramanujan's Conjecture.

$2^{n+2} - 7 = x^2$ has just

five solutions ~~n = 1, 3, 5,~~ $n = 1, 2, 3, 5, 12$

From this we prove the following —.

page 1, (3) & (4).

(3) $\longrightarrow$ $2^n - 1 \neq \frac{y(y+1)}{2} = \Delta_y$ for n odd and >1.

& (4)

If $2^n - 1 = \frac{y(y+1)}{2} \Longrightarrow 2^{n+1} = y^2 + y + 2$

or $2^{n+1} = (y + \frac{1}{2})^2 + \frac{7}{4} \Longrightarrow 2^{n+3} - 7 = (2y+1)^2$

from S. R.'s, this is possible only for n as given above. Hence 3 & 4 follow.

The deduction of (2) in the paper is not all right in the second paragraph.

$2^m + 1 = \Delta_y + 2 = \Delta_z$, This is not Correct.

What we have to show is

$2^m + 1 \neq$ any $\Delta_z = \frac{z(z+1)}{2}$.

Here it has been assumed for the Contrary that

$2^m + 1 = \Delta_z$ and $2^m + 1 = \Delta_y + 2$

The contradiction is done by assuming both these, but the proof of (2) is too trivial and does not S. R. Conjecture. for if $2^m + 1 = \frac{z(z+1)}{2}$, $2^{m+1} = (z-1)(z+2)$

This leads to $\left.\begin{array}{l} z-1 = 2^r \\ z+2 = 2^{r+s} \end{array}\right\} \Longrightarrow 3 = 2^r(2^s - 1)$

true only if $r = 0$, $s = 2$.

No Fermat no. $2^{2^n}+1 \neq \Delta y$.

2^m+1

S. R. Conjecture is

$2^m+1 = \Delta y + 2$ only for $2, 3, 5, 13$.

$2^{2^r}+1 \neq \Delta z$

$2^m+1 \neq \Delta y+2$ except for $m = 2 = \ldots$

Suppose $2^m+1 = \Delta z$

$2^m+1 = \Delta y + 2$

~~Suppose~~ $2^{\cancel{m}} \cancel{+1 =} \Delta z$

$2^n+1 \neq \Delta z$ for $\forall$ z, n

You are also assuming

$2^m+1 = \Delta y+2$ only if $\begin{cases} m_1 & m_2 \\ y_1 & y_2 \end{cases}$

$\cancel{2^{m+1}+2} = z(z+1)$

$(2^m+1)2^3+1 = \square$

$\cancel{1+2^3(2^m+1)} = \square$

$\cancel{2^{m+3}+2^4}+1 = \square$

$2^{m+3}+9 = \square = M^2$

$(M+3)(M-3) = 2^{m+3}$.

$M-3 = 2^r$

$M+3 = 2^{r+s}$ $\quad 6 = 2^r(2^s-1)$

true only if $r=1,\ s=2$

$2^m = \frac{z(z+1)}{2}-1$

$= \frac{z^2+z-2}{2}$

$2^m+1 = \frac{(2y+1)(2y+2)}{2} = (y+1)(2y+1)$

$2^m = 2y^2+3y \cancel{+1} = \cancel{(2y+1)(}$

$= y(2y+3)$ not true for any

$\cancel{2y(2y+}$

$\frac{(2y+1)(2y+2)}{2}$

$2^m = (2y+1)(y+1)-1$

$= y(2y+3)$

$\frac{2y(2y+1)}{2}$

$2^m = 2y^2+y-1$

$= (2y-1)(y+1)$

$y=1,\ m=1$

$2^{2m-1} = 1+\frac{y(y+1)}{2}$

$2^{2m} = y(y+1)+2 = y$

$2^{n+2} - 7 = x^2$

$= (2y+1)^2 = 4y^2 + 4y + 1.$

$2^{n+2} = 4y^2 + 4y + 8$

$2^{n-1} = \frac{1}{2}(y^2 + y + 2) = 1 + \Delta_y$

$\cancel{2^n = y^2 + y + 2}$ $\quad 2^m = 1 + \Delta_y$

$\cancel{2^n + 1 = \frac{1}{2}}$ $\quad \left. \begin{aligned} 2^m + 1 &= \Delta_y + 2 \\ &= \Delta_z \end{aligned} \right\}$ (true only for 0, 1.

$m = n - 1$ $\qquad m = 0, 1, 2, 4, 11.$

& $y = 0, 1, 2, 5, 90)$

$y = 1, m = 1$

$n^2 - 79n + 1601$

$n=2$ 1605 − 158 = 1447

1610 − 237 = 1373

1617 − 316 = 1301

1602 − 79 = 1523 $n=1$

1626
5

$79^2 - 79^2 + 1601.$

$78^2 - 78 \cdot 79 + 1601$

$-78(78-79) + 160$

$80^2 - 79 \cdot 80 + 1601$

$1681 = 41^2$

$80 + 1601$

~~155~~ 131

~~148~~ 201

164

~~157~~
161
[illegible]
231

Lecture
by
D. R.Kaprekar
on 8.6.78 at Indian Institute of Science, Bangalore

Mathematical short cuts, wonders, puzzles, tricks and mathematical magics

1. Properties and wonders of some big numbers
2. Short cuts in multiplications and divisions
3. Magic squares of 9 and 16 houses and Gandhi Shatabdi magic square and dated magic square
4. Puzzles of pin folded numbers
5. 6 big charts of square numbers cut in a certain way to catch 1 to 48 numbers
6. Vibgyor magic square with quick addition in any of 4950 ways; each group of 7 numbers
7. Properties of Ramanujam number 1729
8. Subtraction by addition
9. The Kaprekar's constant + 6174
10. Good bye magic squares for 1977
11. Periodic ~~affili~~ oscilating series of numbers
12. Wonders of finger/changing numbers
13. Magic on Harmonium number
14. Very big 110 numbers, how quickly answered
15. Newton W magic
16. Wonders of Demlo numbers and self numbers
17. Prove that a cat has 16 legs
18. Meaning of Hijklmno
19. 13 is a very good number, reasons
20. Palendromic numbers and unproved theory's about them
21. How many mangoes did you eat?
22. Magic of Ages of people.

The 25 books and 50 papers of Kaprekar can be had ~~at~~ on payment from the following address: D. R. Kaprekar
311, Devlali Camp
422401
(Maharashtra State)

(1) ~~Its~~ All the permutations of $\underline{n}$ elements form a group S_n the *symmetric group* on $\underline{n}$ letters. [v. diff] [S_n of degree $\underline{n}$ as in M.B.D]

Substitution group of degree $n \equiv S_n$
Permutation group of degree $n \equiv S_n$ [L]

(a) Since a permutation is a $\underline{\text{fn}}$ no need to prove associative property

(b) Unit element exists since $(1\,2 \ldots n) \to (1, 2 \ldots n)$ is the unit permutation

(c) For any two permutations $\underline{s}$ & $\underline{t}$, st also is a permutation

for eg, ~~for~~ $s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$ & $t \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

& $\underline{t}$ be 1st & then $\underline{s}$; $st = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$ & $ts = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$ [$st \neq ts$

(d) ~~If~~ For any s, s^{-1} exists such that $ss^{-1} = s^{-1}s = 1$

$1 \to 2$
$2 \to 4$
$3 \to 3$
$4 \to 1$

← for eg in above $s^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$

& $ss^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = s^{-1}s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = 1$

ie group property is proved.

(2) Alternating group ~~and~~ A_n a subgroup of S_n (as in B & M).
& elem. sym. polynomials. [A_n also a normal subgroup]

(3) Theorems on cyclic permutations, and any ~~product~~ permutation being a product of disjoint cycles ~~shown as in (B & M)~~, conjugate of any cycle is another cycle of the same length. [as in B & M]

[* $T^{-1} A_n T = A_n$ whether T be an odd or even permutation, for T is odd so is T^{-1} & if T be even so is T^{-1}]

[MBD] transpositions [MBD] (p. 16) - Any permutation = product of ~~substitutions~~ transpositions for eg

$$\left.\begin{aligned} a_1 a_2 a_3 \ldots a_n &= a_1 a_2 \cdot a_1 a_3 \cdot a_1 \ldots a_1 a_n \\ &= a_1 a_n \cdot a_n a_{n-1} \ldots a_3 a_2 \end{aligned}\right\}$$ true for any ~~transposed~~ permutation

$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$

$2431 = (2\cdot4)(2\cdot3)(2\cdot1)$

$(1234) = (12)(13)(14) = (14)(43)(32)$

$1234 \to 1432$
$\to 1423$
$\to 2431$

(12)(13)(14)

2134 4231
2314 3241
2341 = (1234) 2341 = (1234)

$1234 \to 2134 \to 2314 \to 2341$

$2431 = (24)(23)(21)$

$4231 \to 4321 \to 4312$

$$\begin{pmatrix}1\,2\,3\,4\\2\,4\,3\,1\end{pmatrix} \cdot \begin{pmatrix}1\,2\,3\,4\\2\,1\,4\,3\end{pmatrix}$$

$$= (1\,3\,4\,2) \times (4\,2\,1\,3)$$

1 2 3 4

$\left.\begin{matrix}2\,4\,3\,1\\3\,2\,4\,1\end{matrix}\right\}$

(2 4)(2 3)(3 1)

$$\begin{pmatrix}1\,2 & 3\,4\\ 2\,4 & 3\,1\end{pmatrix}\begin{pmatrix}1\,2\,3\,4\\ \cancel{2\,4\,3\,1}\\ 4\,1\,3\,2\end{pmatrix} = \begin{pmatrix}1\,2\,3\,4\\1\,2\,3\,4\end{pmatrix}$$

$$\begin{pmatrix}1\,2\,3\,4\\2\,4\,3\,1\end{pmatrix}\begin{pmatrix}1\,2\,3\,4\\4\,1\,3\,2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\,4\\1 & 2 & 3\,4\end{pmatrix}$$

$$\begin{pmatrix}1\,2\,3\,4\\4\,1\,3\,2\end{pmatrix}\begin{pmatrix}1\,2\,3\,4\\2\,4\,3\,1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\,4\\1 & 2 & 3\,4\end{pmatrix}$$

Points re Elem. particle Physics

(1) Atomic number Z = no. of electrons revolving round the nucleus = no. of protons in the nucleus

Atomic wt or mass number A – Structure of atomic nucleus clarified after discovery of neutron in 1932 by Chadwick – Also explanation of isotopes made possible – nucleus made up of Z protons and $A-Z$ neutrons, so that nuclear charge will be Ze & Z electrons revolve round the nuclei – Exs. of deuteron and triton which are isotopes of H with nuclei consisting of 1 proton & 1 neutron, and 1 proton and 2 neutrons respectively – protons & neutrons called nucleons

(2) Antiprotons & anti-neutrons – To materialise a proton-antiproton pair an energy greater than $2M_pc^2$ is required (For electron-positron pair $2mc^2$ ie 1 MeV or million electron-volts, an electron volt being the energy ~~req~~ acquired by an electron in falling through one volt potential-difference, or $1{\cdot}6\times10^{-12}$ times the K.E of a body of mass 1 gm travelling with a velocity of 1 cm/sec – one thousand million electron volts are denoted by GeV or BeV) ie nearly 2 BeV. This has been realised only during the last decade.

Re. anti-neutron, although the neutron has no electric charge, it has a magnetic-moment $e\hbar/2M_nc$, and an anti-neutron has a magnetic moment equal and opposite to that of a neutron.

(3) Masses & binding energies of nuclei – Masses of nuclei can be found by recent developments of the mass spectrograph, and it is now possible to compare the masses of two nuclei to about 1 part in 10^7. The binding energy of a nucleus is the energy required to break it up into individual free nucleons. If $M(Z,A)$ = mass of a nucleus of atomic & mass numbers Z and A, the binding energy is given by

$$E = \left\{ZM_p + (A-Z)M_n - M(Z,A)\right\}c^2$$

M_p can be obtained by modern precision measurements based on J. J. Thomson's work to find e/m, but this cannot be used to find M_n. Reactions involving deuterons have been employed to find M_n. Thus if M_d is the mass of the deuteron, its binding energy is

$$E_d = \left\{M_p + M_n - M_d\right\}c^2$$

M_p is known, M_d/M_p can be found by mass-spectrograph methods & hence M_d can be found. E_d is ~~measured~~ ~~be~~ found by determining the energy released as gamma radiation ($h\nu$) when a very low energy neutron combines with a proton to form a deuteron. Hence M_n can be determined. We have

$$M_p = (1836{\cdot}13 \pm 0{\cdot}04)\,m, \quad M_n = (1838{\cdot}66 \pm 0{\cdot}04)\,m$$

M_n exceeds M_p by more than m (mass of electron).

Binding energy ~~per nucleon (E/A)~~ per nucleon (E/A) as a fn of A. This rises rather irregularly for the light nuclei, then more smoothly to a maximum for $A \cong 60$ (iron, cobalt, nickel) & then falls gradually as the mass number increases, Vacancy particle ΔE is used in producing two particles of rest mass m_0 one with speed u & one with speed v. Least energy req^d to produce a pair in this way is obtained by putting $u = v = 0$ in above eq^n i.e. it is $2m_0c^2$. Pair creation by gamma rays.

Adjoint and charge conjugate wave eq^{ns}

For adjoint eq^n if $\tilde{\Psi}$ = hermitian conj of Ψ i.e. a 4-comp row vector (Ψ_ρ^*) $(\rho = 1, 2, 3, 4)$ and if $\overline{\Psi} = \tilde{\Psi}\gamma^4$, the adjoint eq^n becomes $\partial_\mu \overline{\Psi}\beta^\mu - \chi\overline{\Psi} = 0$ or with coup

$$\left(\partial_\mu - \frac{ie}{\hbar c}A_\mu\right)\overline{\Psi}\beta^\mu - \chi\overline{\Psi} = 0 \quad \text{— (a)}$$

Charge conjugation is operation of changing sign of all particles. If Ψ_C be the conjugate wave fn then $\beta^\mu\left(\partial_\mu - \frac{ie}{\hbar c}A_\mu\right)\Psi_C + \chi\Psi_C = 0$ — (b) & the charge conjugate adjoint wave eq^n obeys $\left(\partial_\mu + \frac{ie}{\hbar c}A_\mu\right)\overline{\Psi}_C - \chi\overline{\Psi}_C = 0$ (Ψ, Ψ_C col vectors $\overline{\Psi}$, $\overline{\Psi}_C$ row vectors a also O_T denotes transpose of a matrix; Ψ_T denotes Ψ written as a row vector & $\overline{\Psi}_T$ denotes $\overline{\Psi}$ written as a col. vector). The transpose of (a) is

$$\left(\partial_\mu - \frac{ie}{\hbar c}A_\mu\right)\beta_T^\mu\overline{\Psi}_T - \chi\overline{\Psi}_T = 0$$

Ψ_C defined by $\Psi_C = C\overline{\Psi}_T = C\beta_T^4\Psi^*$ satisfies (b) provided the 4×4 matrix C obeys $C^{-1}\beta^\mu C = -\beta_T^\mu$ $(\mu = 1, 2, 3, 4)$ — (c) $(-\beta_T^\mu)$ obey the same matrix rel^n defining β^μ & hence $(-\beta_T^\mu)$ and β^μ are equivalent matrices. Further since both β^μ & $(-\beta_T^\mu)$ are hermitian, C can be chosen unitary $(C\tilde{C} = 1)$. Because of (c) C has the further important property $C = -C_T$

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For deuteron ($1p + 1n$) $E = 2{\cdot}2$ MeV, for triton ($1p + 2n$) $8{\cdot}5$ MeV, for heli[um] isotope of mass 3 ($2p + 1n$) $7{\cdot}7$ MeV, for normal helium ($2p + 2n$) $28{\cdot}3$.

(4) Magnetic moments of nuclei – Can be measured with astonishing accuracy for eg

$$\mu_d/\mu_p = 0{\cdot}30701219 \pm 0{\cdot}00000003.$$

Knowing the spin A.M, the magnetic moment can be found. ~~The spin A.M is restricted to values $\sqrt{i(i+1)}\,\hbar$, with $i = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}$~~

The magnetic moment of the proton regarded as a particle of spin ½, the moment is $e\hbar/2M_pc$, but the measured value is $2{\cdot}79276 \pm 0{\cdot}0006$ times greater than this ie mag. moment of proton is anomalous. Similarly the mag. moment of the neutron which is $-0{\cdot}68500 \pm 0{\cdot}00003$ times that of a proton is also anomalous.

(5) Accelerators for speeding up charged particles to very high energies so they can be effective in producing nuclear reactions – cyclotron – Synchro-cyclotron – The proton Synchrotron – acceleration of electrons – Cosmotron & Bevatron for securing high energy particle beams of protons

(6) Detecting fast particles – Counters; the Geiger-Muller counter – Scintillation counter – Čerenkov counters – Coincidence & anti-coincidence counting systems – Wilson cloud chamber – Bubble chamber – Nuclear photographic emulsion.

(7) Some nuclear reactions

(a) Alpha particles (nuclei of helium atoms) bombarding nitrogen ie

Alpha particle $\begin{pmatrix}2\text{ protons}\\2\text{ neutrons}\end{pmatrix}$ + Nitrogen nucleus $\begin{pmatrix}7\text{ protons}\\7\text{ neutrons}\end{pmatrix}$

→ Proton + Nucleus of Oxygen isotope $\begin{pmatrix}8\text{ protons}\\9\text{ neutrons}\end{pmatrix}$

or ${}_2He^4 + {}_7N^{14} \rightarrow {}_1H^1 + {}_8O^{17}$ in modern notation (upper suffix = mass no, lower " = atomic no)

(b) Chadwick's discovery of the neutron (alpha particle & Beryllium nuclei)

$${}_2He^4 + {}_4Be^9 \rightarrow {}_6C^{12} + {}_0n^1$$

(c) Cockcroft & Walton's bombardment of Li by protons

$${}_3Li^7 + {}_1H^1 \rightarrow 2\,{}_2He^4$$

(d) Isotopes ${}_1H^3$ and ${}_2He^3$ discovered by reaction between 2 deuterons

$${}_1D^2 + {}_1D^2 \rightarrow {}_2He^3 + {}_0n^1 \rightarrow {}_1H^3 + {}_1H^1$$

also: ${}_1H^3 + {}_1D^2 = {}_2He^4 + {}_0n^1$

(e) neutrons used in reactions (for eg. with boron)

$${}_5B^{10} + {}_0n^1 \rightarrow {}_3Li^7 + {}_2He^4$$ (Even slow neutrons producing reaction)

(f) Artificial radioactivity achieved (Bombardment of Al by alpha particles by Curie & Joliot; among reaction products were not only protons & neutrons, but also positrons, thus showing that radio-active nuclei had been produced by

Magnetic moment. In the equation $\left(\Pi_\mu^2 + m^2c^2 + \frac{\hbar e}{2c}\sigma^{\mu\nu}f_{\mu\nu}\right)\psi = 0$,

~~using~~ where $\Pi_\mu = p_\mu + \frac{e}{c}A_\mu$, we have $\Pi_\mu^2 = (\vec{p} + \frac{e}{c}\vec{A})^2 + (p_4 + \frac{e}{c}A_4)^2$

$$= \,\text{''}\, + \left(\frac{iE}{c} + \frac{e}{c}i\phi\right)^2 = \,\text{''}\, - \left(\frac{E}{c} + \frac{e}{c}\phi\right)^2$$

Dividing the original eqn by $2m$, we can write

$$\left\{\frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 + \frac{e}{2mc}\cdot\frac{1}{2}\hbar\sigma^{\mu\nu}f_{\mu\nu}\right\}\psi$$

$$= \frac{1}{2m}\left\{\left(\frac{E}{c} + \frac{e}{c}\phi\right)^2 - \frac{m^2c^2}{2m}\right\}\psi = \frac{1}{2mc^2}\left\{(E + e\phi)^2 - m^2c^4\right\}\psi$$

$$= \frac{1}{2mc^2}(E + e\phi + mc^2)(E + e\phi - mc^2)\psi$$

Writing $E = E' + mc^2$, where E' is small and also for small ϕ, $\frac{1}{2mc^2}(E + e\phi + mc^2) \cong 1$, the R.H.S $= (E' + e\phi)\psi$ and the eqn $\frac{1}{2m}(\vec{p} + \frac{e}{c}\vec{A})^2\psi = (E' + e\phi)\psi$ is the Schrödinger eqn for a spinless particle in an e.m.f. $\left[\text{cf. } \frac{p^2}{2m}\psi = E'\psi\right]$. We can now show that the term $\frac{e}{2mc}\cdot\frac{1}{2}\hbar\sigma^{\mu\nu}f_{\mu\nu}$ has one part describing a magnetic moment.

The components $\mu = 4, \nu = 1, 2, 3$ of the term can be shown to vanish by using the representation $\beta^4 = \beta = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$, $\beta^k = -i\beta\alpha^k$ & $\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$, $\sigma^1, \sigma^2, \sigma^3$ being the Pauli spin-matrices ie $[\beta^4, \beta^1] = [\beta^4, \beta^2] = [\beta^4, \beta^3] = 0$. The components $\mu, \nu = 1, 2, 3$ give (apart from $\frac{e}{2mc}\cdot\frac{1}{2}\hbar$) $2(\sigma^{23}f_{23} + \sigma^{31}f_{31} + \sigma^{12}f_{12})$, and using

$$\sigma^{23} = \frac{1}{2i}(\beta^2\beta^3 - \beta^3\beta^2), \quad \beta^2 = -i\beta\alpha^2 = -i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix},$$

$$\beta^3 = -i\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad \beta^2\beta^3 = -\begin{pmatrix} \sigma_2\sigma_3 & 0 \\ 0 & \sigma_2\sigma_3 \end{pmatrix} = -\begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \quad \beta^3\beta^2 = -\begin{pmatrix} \sigma_3\sigma_2 & 0 \\ 0 & \sigma_3\sigma_2 \end{pmatrix} = +\begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}$$

$$\left[\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}\right]$$

$\beta^2\beta^3 - \beta^3\beta^2 = -2i\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$, we have $\sigma^{23} = -S_1$ & similarly σ^{31} & σ^{12}. Hence

$$2(\sigma^{23}f_{23} + \cdot + \cdot) = -2(S_1H_1 + \cdots) = -2(\vec{S}\cdot\vec{H}) \quad [H = \text{external m.f}]$$

& $\underline{S}$ the spin operator). Hence the components $\mu = \nu = 1, 2, 3$ give (apart from $\frac{1}{2}\hbar$)

$-\frac{e}{mc}(\vec{S}\cdot\vec{H})$ which (with $+^{ve}$ sign) is the energy of a magnetic dipole of moment $-(e/mc)\vec{S}$ in the external m.f. $\vec{H}$. Hence the intrinsic magnetic moment of the particle is $e\hbar/2mc$.

Hole Theory – Previous notes. $E_1 = -\frac{m_0c^2}{\sqrt{(1-v^2/c^2)}}$. If E = energy absorbed by the gamma ray, then speed u of electron which materialises in $+^{ve}$ energy is

$$\frac{m_0c^2}{\sqrt{(1-u^2/c^2)}} = E - \frac{m_0c^2}{\sqrt{(1-v^2/c^2)}} \quad \text{since energy is conserved}$$

or $$\frac{m_0c^2}{\sqrt{(1-u^2/c^2)}} + \frac{m_0c^2}{\sqrt{(1-v^2/c^2)}} = E \quad \text{ie ~~&~~ 2nd term = energy of the vacancy}$$

bombardment of Al)

$$_{13}Al^{27} + {}_2He^4 \rightarrow {}_{12}P^{30} + {}_0n^1$$

$_{12}P^{30}$ being a radioactive isotope of phosphorus, the stable one being $_{15}P^{32}$ (atomic wt. of P is in the neighbourhood of 31).

(8) Cosmic rays (a) – Very penetrating & could be analysed at sea level into two main components, the hard & soft components; the latter can be stopped by about 4" lead, whereas the former requiring nearly 20 times that thickness. Cosmic rays studies at present have lost much importance for work in this field since energy accelerators can be artificially produced up to 10,000 MeV or more (> cosmic ray energies of order 500 MeV of primary cosmic rays of beams of protons).

(b) Soft showers and positrons – Development of cosmic ray ~~shower~~ cascade showers of gamma rays & electron-positron pairs. (Fig)

(c) Yukawa's theory of nuclear forces, Mesons & the hard component – Yukawa's theory of 1935 of exchange forces between nucleons by particles of mass > m but < M called mesons; such particles were found in the hard component. Another suggestion by Yukawa that radio-activity is a 2 stage process viz mesons are not stable but transform into electrons i.e. a neutron first → a proton + -ve charged meson → $\cancel{p}\ p^+ + e^-$ – Further experimental work showed that there were 2 types of mesons of masses 200m & 300m, the former (or the secondary) decaying to an electron. Thus the primary 300m meson is identified with Yukawa's meson, and the 200m secondary as the sea level (or as seen in the hard component) meson – Eg. of a puzzle of brilliantly inspired but inaccurate guesses, apparent exptl. confirmation & later realisation that even more complicated and unexpected phenomena have been discovered – one further discovery of the neutral 300m meson (π^0) viz that if left alone it → 2 gamma rays in time < 10^{-14} sec – The correct picture appears to be that the primary cosmic ray protons entering the atmosphere irradiate Yukawa mesons (pions of 300m); +ve, -ve & neutral in collision with atmospheric nuclei. The +ve & some of the -ve mesons decay into the weakly interacting variety before reaching sea level constituting the hard component (muons of 200m). On the other hand, neutral pions transform very quickly into gamma rays which initiate the shower that constitutes the soft components.

(9) Properties of pions & muons: π^+, π^-, π^0 and μ^+, μ^- (produced artificially). Pions have no spin & hence are bosons like photons. Muons have intrinsic spin 1/2 and hence are fermions. Pions have no appreciable mag. moment, but muons have the expected mag. moment of a fermion. ($eh/2\mu c$)

Description of nuclear forces between nucleons at distances > about 0.5×10^{-13} cm is afforded by pion-exchange, ~~but~~ but for shorter distances complications due to creation of nucleons & antinucleons make it difficult even to make a theoretical approach.

Also anomalous mag. moments of proton & neutron can be attributed to the orbital motions of +ve & -ve pions when they are virtually dissociated into ~~their particles~~ π^+ & a neutron, and π^- & a ~~neutron~~ proton respectively for eg. the electromagnetic field strength can be written as an antisymmetric tensor

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \text{ where } A_\mu = (\vec{A}, i\phi) \text{ ie the 4-potential)}$$

and Maxwell's eqns can be written $\partial_\mu f_{\mu\nu} = \frac{4\pi}{c} j_\nu$ and $\partial_\lambda f_{\mu\nu} + \partial_\mu f_{\nu\lambda} + \partial_\nu f_{\lambda\mu} = 0$.

In classical mechanics, to bring in an external e.m.f. we have $p_\mu \to p_\mu + \frac{e}{c} A_\mu$ and this corresponds in quantum mechanics to $-i\hbar \frac{\partial}{\partial\mu} \to -i\hbar\frac{\partial}{\partial\mu} + \frac{e}{c} A_\mu$ or

$\frac{\partial}{\partial\mu} \to \frac{\partial}{\partial\mu} - \frac{e}{i\hbar c} A_\mu$ or $\frac{\partial}{\partial\mu} \to \frac{\partial}{\partial\mu} + \frac{ie}{\hbar c} A_\mu$. Hence Dirac's equation becomes

$$\beta^\mu \left(\partial_\mu + \frac{ie}{\hbar c} A_\mu\right)\psi + \varkappa\psi = 0 \quad \text{—— (1)}$$

If $\Pi_\mu = p_\mu + \frac{e}{c} A_\mu$ be a new operator ie $\Pi_\mu = -i\hbar\frac{\partial}{\partial\mu} + \frac{e}{c} A_\mu = -i\hbar\left(\partial_\mu + \frac{ie}{\hbar c} A_\mu\right)$, and multiplying (1) by $-i\hbar$, it can be written as $\beta^\mu \Pi_\mu \psi - \frac{mc}{\hbar} . i\hbar\psi = 0$ or $(\beta^\mu \Pi_\mu - imc)\psi = 0$ — (2)

The Π_μ's do not commute & in general $[\Pi_\mu, \Pi_\nu] = \left(-i\hbar\partial_\mu + \frac{e}{c}A_\mu\right)\left(-i\hbar\partial_\nu + \frac{e}{c}A_\nu\right)$

$$- \left(-i\hbar\partial_\nu + \frac{e}{c}A_\nu\right)\left(-i\hbar\partial_\mu + \frac{e}{c}A_\mu\right)$$

$$= -i\hbar\frac{e}{c}\partial_\mu A_\nu + i\hbar . \frac{e}{c}\partial_\nu A_\mu = \frac{\hbar e}{ic} f_{\mu\nu} \quad \text{—— (3)}$$

Operating on (2) with $\beta^\nu\Pi_\nu + imc$ gives $(\beta^\nu\Pi_\nu + imc)(\beta^\mu\Pi_\mu - imc)\psi = 0$, or

$(\beta^\nu\beta^\mu \Pi_\nu \Pi_\mu + m^2c^2)\psi = 0$ introducing $\sigma^{\mu\nu} = \frac{1}{2i}[\beta^\mu, \beta^\nu]$

or $(\beta^\mu\beta^\nu \Pi_\mu\Pi_\nu + m^2c^2)\psi = 0$ we have $\beta^\mu\beta^\nu + \beta^\nu\beta^\mu = 2\delta^{\mu\nu}$

$\beta^\mu\beta^\nu - \beta^\nu\beta^\mu = 2i\sigma^{\mu\nu}$

or $(\delta^{\mu\nu}\Pi_\mu\Pi_\nu + i\sigma^{\mu\nu}\Pi_\mu\Pi_\nu + m^2c^2)\psi = 0$ ie $\beta^\mu\beta^\nu = \delta^{\mu\nu} + i\sigma^{\mu\nu}$

or $\left\{\Pi_\mu^2 + \frac{1}{2}i\sigma^{\mu\nu}[\Pi_\mu, \Pi_\nu] + m^2c^2\right\}\psi = 0$, since $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$

or $\left(\Pi_\mu^2 + m^2c^2 + \frac{\hbar e}{2c}\sigma^{\mu\nu} f_{\mu\nu}\right)\psi = 0$ —— (4) using (3)

Spin. $\Lambda(\theta)$ describes the effect of a rotation θ about the O_1 axis on a fermion in a state of zero angular momentum. In general, the effect of a small rotation $\delta\theta$ about the O_1 axis is given by $\psi'(x_1, x_2, x_3) = \psi(x_1, x_2 + x_3\delta\theta, -x_2\delta\theta + x_3) = (1 + I_1\delta\theta)\psi$, where I_1 is the first component of the 3-vector operator $\vec{x} \times \text{grad}$ ie $I_1 = x_2\partial_3 - x_3\partial_2$. Replacing θ by $\delta\theta$ in $\Lambda(\theta)$ we get $\Lambda(\delta\theta) = \exp\left(\frac{1}{2}\beta^2\beta^3\delta\theta\right) \cong 1 + \frac{1}{2}\beta^2\beta^3\delta\theta$. & using $\beta^2\beta^3 - \beta^3\beta^2 = 2i\sigma^{23}$ and $\beta^2\beta^3 + \beta^3\beta^2 = 0$, we have $\beta^2\beta^3 = i\sigma^{23}$ & hence $\Lambda(\delta\theta) = 1 + \frac{1}{2}i\sigma^{23}\delta\theta$. Hence*

$S_1 = \frac{\hbar}{i} . \frac{1}{2}i\sigma^{23}\delta\theta. = \frac{1}{2}\hbar\sigma^{23}$, where $\vec{S} = (S_1, S_2, S_3)$ is the intrinsic A.M or spin.

$(\sigma^{23})^2 = -\frac{1}{4}[\beta^2, \beta^3][\beta^2, \beta^3] = -\frac{1}{4}(\beta^2\beta^3 - \beta^3\beta^2)(\beta^2\beta^3 - \beta^3\beta^2) = -\frac{1}{4}\{\beta^2\beta^3\beta^2\beta^3 + \beta^3\beta^2\beta^3\beta^2$

$$- \beta^2\beta^3\beta^3\beta^2 - \beta^3\beta^2\beta^2\beta^3\}$$

$= -\frac{1}{4}(-4) = 1$ ie $\sigma^{23} = \pm 1$. Hence eigenvalues of spin components in any axis are $\pm\frac{1}{2}\hbar$

* comp of A.M $= (\vec{x} \times \vec{p})_1 = -i\hbar(x_2\partial_3 - x_3\partial_2)$

$= \hbar/i \; I_1$

For <u>time reversal</u> if $\Lambda = \beta^1\beta^2\beta^3$, we have

(i) $\beta^1\Lambda = \Lambda\beta^1 \to \beta^1\beta^1\beta^2\beta^3 = \beta^1\beta^2\beta^3\beta^1$ ie L.H.S $= \beta^2\beta^3$ & R.H.S $= \beta^2\beta^3$ (2 interchanges)

(ii) $\beta^2\Lambda = \Lambda\beta^2 \to \beta^2\beta^1\beta^2\beta^3 = \beta^1\beta^2\beta^3\beta^2$ ie L.H.S $= -\beta^1\beta^3 =$ R.H.S (one interchange on each side)

(iii) $\beta^3\Lambda = \Lambda\beta^3 \to \beta^3\beta^1\beta^2\beta^3 = \beta^1\beta^2\beta^3\beta^3$ ie L.H.S $= \beta^1\beta^2$ (2 interchanges) $=$ R.H.S (using $\beta^3 = 1$)

(iv) $\beta^4\Lambda = -\Lambda\beta^4 \to \beta^4\beta^1\beta^2\beta^3 = -\beta^1\beta^2\beta^3\beta^4$ L.H.S $= -\beta^1\beta^2\beta^3\beta^4$ (3 interchanges) $=$ R.H.S.

We can also also verify the results

$$\Lambda^{-1}\beta^1\Lambda = \beta^1,\ \Lambda^{-1}\beta^2\Lambda = \beta^2,\ \Lambda^{-1}\beta^3\Lambda = \beta^3, \text{ and } \Lambda^{-1}\beta^4\Lambda = -\beta^4.$$ For this we

have to find Λ^{-1} which is obviously equal to $\beta^3\beta^2\beta^1$ & is easily shown to satisfy $\Lambda\Lambda^{-1} = \Lambda^{-1}\Lambda = 1$.

(i) L.H.S $= \beta^3\beta^2\beta^1\cdot\beta^1\cdot\beta^1\beta^2\beta^3 = \beta^3\beta^2\beta^1\beta^2\beta^3 = -\beta^3\beta^1\beta^3 = \beta^1 =$ R.H.S

(ii) L.H.S $= \beta^3\beta^2\beta^1\cdot\beta^2\cdot\beta^1\beta^2\beta^3 = -\beta^3\beta^2\beta^3 = \beta^2 =$ R.H.S

(iii) L.H.S $= \beta^3\beta^2\beta^1\cdot\beta^3\cdot\beta^1\beta^2\beta^3 = -\beta^3\beta^2\beta^3\beta^2\beta^3 = \beta^3\beta^3\beta^3 = \beta^3 =$ R.H.S

(iv) L.H.S $= \beta^3\beta^2\beta^1\cdot\beta^4\cdot\beta^1\beta^2\beta^3 = -\beta^3\beta^2\beta^4\beta^2\beta^3 = \beta^3\beta^4\beta^3 = -\beta^4 =$ R.H.S.

For both space inversion & time reversal, would $\Lambda = \beta^1\beta^2\beta^3\beta^4$ suffice?

ie $a_{11} = a_{22} = a_{33} = a_{44} = -1$.

we should have $\beta^1\Lambda = -\Lambda\beta^1,\ \beta^2\Lambda = -\Lambda\beta^2,\ \beta^3\Lambda = -\Lambda\beta^3,\ \beta^4\Lambda = -\Lambda\beta^4$

ie (i) $\beta^1\beta^1\beta^2\beta^3\beta^4 = -\beta^1\beta^2\beta^3\beta^4\beta^1$ L.H.S $= \beta^2\beta^3\beta^4$, & R.H.S $= -\beta^2\beta^3\beta^4$ (after 3 interchanges of β^1)

(ii) ~~β^2~~ L.H.S $= \beta^2\beta^1\beta^2\beta^3\beta^4 = -\beta^1\beta^3\beta^4$, R.H.S $= -\beta^1\beta^2\beta^3\beta^4\beta^2 = -\beta^1\beta^3\beta^4$ (2 of β^2)

(iii) L.H.S $= \beta^3\beta^1\beta^2\beta^3\beta^4 = \beta^1\beta^2\beta^4$, R.H.S $= -\beta^1\beta^2\beta^3\beta^4\beta^3 = \beta^1\beta^2\beta^4$ (1 of β^3)

(iv) L.H.S $= \beta^4\beta^1\beta^2\beta^3\beta^4 = -\beta^1\beta^2\beta^3$, R.H.S $= -\beta^1\beta^2\beta^3\beta^4\beta^4 = -\beta^1\beta^2\beta^3$

Yes $\Lambda = \beta^1\beta^2\beta^3\beta^4$ is the correct operator.

[In fact $c^2(1+\eta)^2 + v^2\eta^2 = c^2 + 2c^2\eta + \eta^2(c^2+v^2) = c^2 + 2c^2\eta + \dfrac{c^2(c^2+v^2)}{c^2-v^2}$

$$= 2c^2\eta + c^2\left\{1 + \frac{c^2+v^2}{c^2-v^2}\right\} = 2c^2\eta + 2c^4/v^2-c^2 = 2c^2\eta + 2c^2\eta^2$$

$$= 2c^2\eta(1+\eta)]$$

Simly $\Lambda^{-1}\beta^4\Lambda = -i\eta v\beta^1 + \eta\beta^4$ can be verified

Next $\Lambda^{-1}\beta^2\Lambda = \frac{1}{2}\left\{(1+\eta)^{\frac{1}{2}} - \dfrac{iv\eta}{c(1+\eta)^{\frac{1}{2}}}\beta^1\beta^4\right\}\left\{(1+\eta)^{\frac{1}{2}}\beta^2 + \dfrac{iv\eta}{c(1+\eta)^{\frac{1}{2}}}\beta^2\beta^1\beta^4\right\}$

$$= \frac{1}{2}\left[(1+\eta)\beta^2 + \cancel{\frac{iv\eta}{c}\beta^2\beta^1\beta^4} - \cancel{\frac{iv\eta}{c}\beta^1\beta^4\beta^2} + \frac{v^2\eta^2}{c^2(1+\eta)}\beta^1\beta^4\beta^2\beta^1\beta^4\right]$$

$$= \frac{1}{2}\left[(1+\eta)\beta^2 - \frac{v^2\eta^2}{c^2(1+\eta)}\beta^2\right] = \frac{1}{2c^2(1+\eta)}\cdot\beta^2\left\{c^2(1+\eta)^2 - v^2\eta^2\right\} = \frac{1}{2c^2(1+\eta)}\beta^2\cdot 2c^2(1+\eta) = \beta^2$$

Simly $\Lambda^{-1}\beta^3\Lambda = \beta^3$ can be verified.

Next to show that $\Lambda(v) = \exp\left(\frac{1}{2}i\beta^1\beta^4\phi\right)$, where $\sinh\phi = v\eta/c$, we have

expanding $\exp\left(\frac{1}{2}i\gamma^1\gamma^4\phi\right) = \left\{1 + \dfrac{(\frac{1}{2}\phi)^2}{2!} + \dfrac{(\frac{1}{2}\phi)^4}{4!} + \ldots\right\} + i\beta^1\beta^4\left\{(\tfrac{1}{2}\phi) + \dfrac{(\frac{1}{2}\phi)^3}{3!} + \dfrac{(\frac{1}{2}\phi)^5}{5!} + \ldots\right\}$

after suitably simplifying exprs like $\gamma^1\gamma^4\gamma^1\gamma^4$, $\gamma^1\gamma^4\gamma^1\gamma^4\gamma^1\gamma^4$, etc.

ie $\exp(\quad) = \cosh\frac{1}{2}\phi + i\beta^1\beta^4\sinh\frac{1}{2}\phi$.

with $\sinh\phi = \dfrac{v\eta}{c}$, $\cosh\phi = \sqrt{(1 + v^2\eta^2/c^2)} = \eta$

Using $2\cosh^2\frac{1}{2}\phi = 1 + \cosh\phi$, we have $\cosh\frac{1}{2}\phi = \dfrac{1}{\sqrt{2}}(1+\eta)^{\frac{1}{2}}$

& from $2\sinh\frac{1}{2}\phi\cosh\frac{1}{2}\phi = \sinh\phi = \dfrac{v\eta}{c}$, we have $\sinh\frac{1}{2}\phi = \dfrac{v\eta}{2c}\cdot\dfrac{\sqrt{2}}{(1+\eta)^{\frac{1}{2}}} = \dfrac{1}{\sqrt{2}}\cdot\dfrac{v\eta}{c(1+\eta)^{\frac{1}{2}}}$

Hence $\exp\left(\frac{1}{2}i\beta^1\beta^4\phi\right) = \dfrac{1}{\sqrt{2}}\left\{(1+\eta)^{\frac{1}{2}} + \dfrac{iv\eta}{c(1+\eta)^{\frac{1}{2}}}\gamma^1\gamma^4\right\}$

(iii) For the space inversion $x_1' = -x_1$, $x_2' = -x_2$, $x_3' = -x_3$, $x_4' = x_4$

we have $a_{11} = a_{22} = a_{33} = -1$, $a_{44} = 1$ & all others zero. Hence for Λ, we have

$\Lambda^{-1}\beta^1\Lambda = -\beta^1$, $\Lambda^{-1}\beta^2\Lambda = -\beta^2$, $\Lambda^{-1}\beta^3\Lambda = -\beta^3$, $\Lambda^{-1}\beta^4\Lambda = \beta^4$

The last gives $\beta^4\Lambda = \Lambda\beta^4$ & hence Λ can be taken $= \beta^4$ and then Λ^{-1} is also $= \beta^4$

and this satisfies the first three eqns

20/2/68 ? (iv) For the time-reversal $x_1' = x_1$, $x_2' = x_2$, $x_3' = x_3$, $x_4' = -x_4$, we have

$$a_{11} = a_{22} = a_{33} = 1, \quad a_{44} = -1 \text{ \& all others zero}$$

~~Hence for Λ, $\Lambda^{-1}\beta^1\Lambda = \beta^1$, $\Lambda^{-1}\beta^2\Lambda = \beta^2$, $\Lambda^{-1}\beta^3\Lambda = \beta^3$, $\Lambda^{-1}\beta^4\Lambda = -\beta^4$ (but $\Lambda^{-1}\beta^2\Lambda = \beta^2$ is false for $\Lambda = \beta^1$)~~

~~Hence from the first three, we can take $\Lambda = \beta^1$ or β^2 or β^3 such that $\Lambda^{-1} = \beta^1$ or β^2 or β^3~~

~~and these satisfy the last equation~~. It can be shown that $\Lambda = \beta^1\beta^2\beta^3$ [P.T.O]

(d) Spin & magnetic moment: First we have to introduce an external e.m.f in the Dirac eqn. To do this we note that e.m theory can also be put in relativistic form

(10) K-mesons & hyperons — 3 types of brick & 4 types cement ie

~~eta~~ (e, p, n) and (photon, π^+, π^-, π^0) with μ^+ & μ^- difficult to understand in the

role — Discovery of a track in an emulsion leaving 3 secondary tracks of pions lead to discovery

of a particle of mass $> 3\pi$ mass & this was first example of a K-meson of mass 966 m. —

Cloud chamber studies of hard component lead to detection of a primary whose secondaries are a proton

and a pion. ie a particle of mass $> M_p$ and led to discovery of a Lambda-nought hyperon Λ^0 of

mass 2181 m — +ve & -ve charged hyperons also have been found in cloud chamber & bubble

chamber expts, for eg. the Xi-minus Ξ^- which decays like $\Xi^- \to \Lambda^0 + \pi^-$ & the mode of

decay gives mass of Ξ^- as 2577 m — A further pair of charged hyperons known Σ (Sigma)

hyperons differing from the Ξ in that the Σ's decay into nucleons & pions instead of Λ^0 &

pion. Neutral counterparts of both charged Ξ & Σ hyperons have been noticed. For eg

$\Sigma^0 \to \Lambda^0 +$ gamma ray (in 10^{-20} secs). The very latest addition Ξ^0 and is given by the reaction

$K^- + p \to \Xi^0 + K^0$. Also $K^0 \to \pi^+ + \pi^-$ & $\Xi^0 \to \Lambda^0 + \pi^0$ — Hyperons are fermions

and K-mesons are bosons.

Investigations using bubble chambers, pion & proton beams from the cosmotron

& Bevatron give the following laws:—

(a) In no case is a hyperon materialised entirely from kinetic energy. When it is produced it is built on a nucleon already involved in the reaction

(b) A hyperon is always produced in association with one or more K-mesons: the

$$\pi^- + p \to \Lambda^0 + K^0, \text{ and } p + p \to p + K^+ + \Lambda^0 \text{ are observed, while}$$

$\pi^- + p \to \Lambda^0 + \pi^0$ and $n + p \to \Lambda^0 + p$ are never observed

(c) A K^+ is only produced in association with another K-meson or a hyperon

(d) A K^- is only produced in association with at least one K^+

(11) Baryon number & strangeness. — For weak interactions like decay of K-mesons & hyperons

the fundamental conservation rules (net e, total E, mv, A.M & spin) except the charge

conservation do not appear to hold — For strong interactions ie those between pions, K-mesons, nucleons

& hyperons for which rules (a) — (d) hold, further conservation laws have been deduced

viz. conservation of baryon number & strangeness.

Baryon is a term applied to nucleons & hyperons. All these are fermions & have

anti-particles. The baryon number of a group of reactants is the no. of baryons minus the

number of antibaryons. It is conserved in all reactions. This embraces rule (a); also

production of a nucleon-antinucleon pair by a reaction like

$$\pi^- + p \to \pi^- + p + p + \bar{p} \quad (\text{since baryon no. of } \bar{p} = -1)$$

Rules (b), (c)(d) can be covered under conservation of strangeness (where nucleons &

pions have strangeness no = 0, K^+ has 1, K^0 has ± 1, K^- has -1, Λ^0, Σ^+, Σ^0, Σ^- all

have -1, Ξ^- & Ξ^0 have 2)

$$= \left(\cos\tfrac{1}{2}\theta - \beta^2\beta^3 \sin\tfrac{1}{2}\theta\right)\left(\beta^2\cos\tfrac{1}{2}\theta + \beta^3 \sin\tfrac{1}{2}\theta\right) \quad [\text{using } \beta^2 = 1]$$

$$= \beta^2\cos^2\tfrac{1}{2}\theta + \beta^3\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta - \beta^2\beta^3\beta^2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta - \beta^2\sin^2\tfrac{1}{2}\theta \quad [\text{using } \beta^3 = 1)$$

$$= \beta^2\cos\theta + \beta^3\sin\theta \quad [\text{using } \beta^2\beta^3\beta^2 = -\beta^3 \text{ \& } \sin\theta = 2\sin\tfrac{1}{2}\theta\cos\theta,\ \cos\theta = \cos^2\tfrac{1}{2}\theta - \sin^2\tfrac{1}{2}\theta]$$

It can be similarly shown that $\Lambda^{-1}\beta^3\Lambda = -\beta^2\sin\theta + \beta^3\cos\theta$.

Lastly that the operator $\Lambda(\theta) = \exp\left(\tfrac{1}{2}\beta^2\beta^3\theta\right)$ can be seen by writing exp( ) as an infinite series viz

$$1 + \tfrac{1}{2}\beta^2\beta^3\theta + \frac{\frac{1}{4}\beta^2\beta^3\beta^2\beta^3\theta^2}{2!} + \frac{\frac{1}{8}\beta^2\beta^3\beta^2\beta^3\beta^2\beta^3\theta^3}{3!} + \cdots$$

$$= 1 + \tfrac{1}{2}\beta^2\beta^3\theta - \frac{(\frac{1}{2}\theta)^2}{2!} - \frac{\beta^2\beta^3(\frac{1}{2}\theta)^3}{3!} + \frac{(\frac{1}{2}\theta)^4}{4!} + \frac{\beta^2\beta^3(\frac{1}{2}\theta)^5}{5!} \cdots$$

$$= \cos\tfrac{1}{2}\theta + \beta^2\beta^3\sin\tfrac{1}{2}\theta.$$

(ii) For the simple L.T given by

$$x_1' = \eta(x_1 - vt),\ x_2' = x_2,\ x_3' = x_3,\ t' = \eta(t - vx_1/c)$$

i.e. $x_1' = \eta(x_1 - vx_4/ic)$, $x_2' = x_2$, $x_3' = x_3$, $x_4' = \eta(x_4 - ivx_1)$

These give $a_{11} = \eta$, $a_{14} = -\dfrac{\eta v}{ic} = \dfrac{i\eta v}{c}$; $a_{22} = a_{33} = 1$, and

$a_{41} = -i\eta v$, $a_{44} = \eta$, and Λ should be such that

$$\Lambda^{-1}\beta^1\Lambda = \eta\beta^1 + \frac{i\eta v}{c}\beta^4,\ \Lambda^{-1}\beta^2\Lambda = \beta^2,\ \Lambda^{-1}\beta^3\Lambda = \beta^3,\ \Lambda^{-1}\beta^4\Lambda = -i\eta v\beta^1 + \eta\beta^4.$$

Taking $\Lambda = \dfrac{1}{\sqrt{2}}\left\{(1+\eta)^{1/2} + \dfrac{iv\eta}{c(1+\eta)^{1/2}}\beta^1\beta^4\right\}$, we will first show that

$\Lambda^{-1} = \dfrac{1}{\sqrt{2}}\{\ \text{''}\ -\ \text{''}\ \}$ i.e. $v \to -v$ i.e. $\Lambda(-v)$.

$$\Lambda^{-1} = \sqrt{2}\cdot\frac{1}{(1+\eta)^{\frac{1}{2}} + \frac{iv\eta}{c(1+\eta)^{\frac{1}{2}}}\beta^1\beta^4}$$ & multiplying numerator & denominator by $(1+\eta)^{\frac{1}{2}} - \dfrac{iv\eta}{c(1+\eta)^{\frac{1}{2}}}\beta^1\beta^4$

we get $$\Lambda^{-1} = \frac{\sqrt{2}\left\{(1+\eta)^{\frac{1}{2}} - \frac{iv\eta}{c(1+\eta)^{\frac{1}{2}}}\beta^1\beta^4\right\}}{(1+\eta) + \frac{v^2\eta^2}{c^2(1+\eta)}\beta^1\beta^4\beta^1\beta^4}$$ and denominator $= \dfrac{c^2(1+\eta)^2 + v^2\eta^2\beta^1\beta^4\beta^1\beta^4}{c^2(1+\eta)}$

$$= \frac{c^2(1+\eta)^2 - v^2\eta^2}{c^2(1+\eta)}$$

$$= \frac{c^2 + 2c^2\eta + \eta^2(c^2 - v^2)}{c^2(1+\eta)} = \frac{2c^2(1+\eta)}{c^2(1+\eta)} = 2; \text{ using } \eta^2(c^2 - v^2) = c^2.$$

$$\therefore\ \Lambda^{-1} = \frac{1}{\sqrt{2}}\left\{(1+\eta)^{\frac{1}{2}} - \frac{iv\eta}{c(1+\eta)^{\frac{1}{2}}}\beta^1\beta^4\right\} = \Lambda(-v)$$

Now, $$\Lambda^{-1}\beta^1\Lambda = \frac{1}{2}\left\{(1+\eta)^{\frac{1}{2}} - \frac{iv\eta}{c(1+\eta)^{\frac{1}{2}}}\beta^1\beta^4\right\}\left\{(1+\eta)^{\frac{1}{2}}\beta^1 + \frac{iv\eta}{c(1+\eta)^{\frac{1}{2}}}\beta^4\right\} \quad (\text{using } \beta^{1\,2} = 1)$$

$$= \frac{1}{2}\left\{(1+\eta)\beta^1 + \frac{iv\eta}{c}\beta^4 - \frac{iv\eta}{c}\beta^1\beta^4\beta^1 + \frac{v^2\eta^2}{c^2(1+\eta)}\beta^1\right\} \quad (\text{using } (\beta^4)^2 = 1$$

$$= \frac{1}{2}\left[\frac{2iv\eta}{c}\beta^4 + \left\{\frac{c^2(1+\eta)^2 + v^2\eta^2}{c^2(1+\eta)}\right\}\beta^1\right] = \frac{1}{2}\left[\frac{2iv\eta}{c}\beta^4 + 2\eta\beta^1\right], \text{ after simplifying the coeff. of } \beta^1$$

$$= \eta\beta^1 + \frac{iv\eta}{c}\beta^4 \text{ as reqd}$$

and in any fast reaction the algebraic sum of the strangeness numbers must be conserved. No exception to strangeness rule has been so far observed in strong interactions. It is violated in decay interactions. Also no ~~reaction or~~ decay in which there is a change of ± 2 or more of strangeness has been observed.

The two notions of conservation of baryon number and strangeness number appear to be very deep seated, and appear to place modern physics on the threshold of revolutionary ideas comparable to those which resulted in relativity & quantum mechanics.

Strange atoms & strange nuclei – former produced by replacing electrons ~~or prot~~ ~~nucleons~~ by mesons, ~~hyperons & anti-particles~~, the latter produced by replacing nucleons by hyperons.

—

antiprotons & anti-neutrons

(1) neutron – binding energy – finding M_n from E_d ($M_n = 1838.66\, m$, $M_p = 1836.13\, m$, diffce $> m$)

Binding energy (E/A) as fn of A → knowledge of fission & fusion – Magnetic moments

(2) ~~Particle accelerators~~

Lecture for 13/2/68 (1)

(1) Clarification of points in last lecture re. Lorentz covariance of Dirac's eqn.

(a) The condition that $x'_\mu = a_{\mu\nu} x_\nu$ is a L.T. viz. $a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda}$ is not equivalent to $AA_T = 1$, but to $A_T A = 1$; for consider product of two matrices BA; the general term ~~is~~ $C_{\mu\nu}$ is $B_{\mu\lambda} A_{\lambda\nu}$ & if $B = A_T$, this leads to $A_{\lambda\mu} A_{\lambda\nu}$. Hence the condn can be written $a_{\lambda\mu} a_{\lambda\nu} = \delta_{\mu\nu}$ --- (1) or equivalently $a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda}$. Consider a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$A_T A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{21}^2 & a_{11} a_{12} + a_{21} a_{22} \\ a_{12} a_{11} + a_{22} a_{21} & a_{12}^2 + a_{22}^2 \end{pmatrix}$$ & using the above

condn, the non-diagonal elements are zero (Note that summation convention holds in (1)) & the diagonal elements are each unity.

Both $AA_T = 1$ & $A_T A = 1$ lead to $A_T = A^{-1}$

On the other hand $AA_T = 1$ leads to $a_{\nu\mu} a_{\lambda\mu} = \delta_{\nu\lambda}$ & is also a condn for L.T. for

(b) Re. the relation $\Lambda^{-1} \beta^\mu \Lambda = a_{\mu\lambda} \beta^\lambda$ satisfied by the operator given by $\psi'(x'_\mu) = \Lambda \psi(x_\mu)$, the Dirac eqn in the new frame is $\beta^\mu \partial_{x'_\mu} \psi' + \varkappa \psi' = 0$. Using $\partial_{x'_\mu} = a_{\mu\lambda} \partial x_\lambda$, this can be written as $\beta^\mu a_{\mu\lambda} \partial x_\lambda \Lambda \psi + \varkappa \Lambda \psi = 0$ or $\Lambda^{-1} \beta^\mu a_{\mu\lambda} \partial x_\lambda \Lambda \psi + \varkappa \psi = 0$. This is identical with $\beta^\lambda \partial x_\lambda \psi + \varkappa \psi = 0$ if $\Lambda^{-1} \beta^\mu a_{\mu\lambda} \Lambda = \beta^\lambda$. Multiplying both sides by $a_{\nu\lambda}$ and using on the L.H.S. $a_{\mu\lambda} a_{\nu\lambda} = \delta_{\mu\nu}$ (corresp. to $AA_T = 1$) and summing over μ on the L.H.S, it becomes $\Lambda^{-1} \beta^\mu \delta_{\mu\nu} \Lambda = \Lambda^{-1} \beta^\nu \Lambda$. Hence finally $\Lambda^{-1} \beta^\nu \Lambda = a_{\nu\lambda} \beta^\lambda$ or $\Lambda^{-1} \beta^\mu \Lambda = a_{\mu\lambda} \beta^\lambda$

(c) Re. derivation of $\Lambda(\theta)$, $A(v)$ etc —

(i) For the L.T, $x'_1 = x_1$, $x'_4 = x_4$, $x'_2 = x_2 \cos\theta + x_3 \sin\theta$, $x'_3 = -x_2 \sin\theta + x_3 \cos\theta$, we can easily show that $a_{11} = a_{44} = 1$, $a_{22} = a_{33} = \cos\theta$, $a_{23} = -a_{32} = \sin\theta$. and the relations $\Lambda^{-1} \beta^\mu \Lambda = a_{\mu\lambda} \beta^\lambda$ become

$$\Lambda^{-1} \beta^1 \Lambda = \beta^1,\ \Lambda^{-1} \beta^4 \Lambda = \beta^4 \text{ and } \Lambda^{-1} \beta^2 \Lambda = \beta^2 \cos\theta + \beta^3 \sin\theta,\ \Lambda^{-1} \beta^3 \Lambda = -\beta^2 \sin\theta + \beta^3 \cos\theta$$

and $\Lambda(\theta) = \cos\tfrac{1}{2}\theta + \beta^2\beta^3 \sin\tfrac{1}{2}\theta$ satisfies all these, for eg

$\Lambda^{-1} \beta^1 \Lambda = \beta^1$ means $\beta^1 \Lambda = \Lambda \beta^1$ ie $\beta^1 (\cos\tfrac{1}{2}\theta + \beta^2\beta^3 \sin\tfrac{1}{2}\theta) = (\cos\tfrac{1}{2}\theta + \beta^2\beta^3 \sin\tfrac{1}{2}\theta)\beta^1$ which holds since $\beta^1\beta^2\beta^3 = \beta^2\beta^3\beta^1$ (using $\beta^3\beta^1 = -\beta^1\beta^3$ & $\beta^2\beta^1 = -\beta^1\beta^2$).

Simly $\Lambda^{-1} \beta^4 \Lambda = \beta^4$ or $\beta^4 \Lambda = \Lambda \beta^4$ holds since $\beta^4\beta^2\beta^3 = \beta^2\beta^3\beta^4$.

Before taking up the other two cases, we note $\Lambda(-\theta) = \{\Lambda(\theta)\}^{-1}$ for

$$\{\Lambda(\theta)\}^{-1} = \frac{1}{\cos\tfrac{1}{2}\theta + \beta^2\beta^3 \sin\tfrac{1}{2}\theta} = \frac{\cos\tfrac{1}{2}\theta - \beta^2\beta^3 \sin\tfrac{1}{2}\theta}{(\cos\tfrac{1}{2}\theta + \beta^2\beta^3 \sin\tfrac{1}{2}\theta)(\cos\tfrac{1}{2}\theta - \beta^2\beta^3 \sin\tfrac{1}{2}\theta)}$$

$$= \frac{\Lambda(-\theta)}{\cos^2\tfrac{1}{2}\theta - \beta^2\beta^3\beta^2\beta^3 \sin^2\tfrac{1}{2}\theta} = \frac{\Lambda(-\theta)}{\cos^2\tfrac{1}{2}\theta + \sin^2\tfrac{1}{2}\theta} = \Lambda(-\theta)$$

Thus $\Lambda^{-1} \beta^2 \Lambda = (\cos\tfrac{1}{2}\theta - \beta^2\beta^3 \sin\tfrac{1}{2}\theta)\beta^2(\cos\tfrac{1}{2}\theta + \beta^2\beta^3 \sin\tfrac{1}{2}\theta)$

In greater detail:

1° The solution of (II), obtained by Von Neumann in the case of bicompact groups, was rediscovered
(1933), (1934) by Pontryagin the very next year while investigating (III) in the case of bicompact groups satisfying the 2nd Axiom of countability;

2° The solution of (II), obtained by Pontryagin in the case of commutative groups,
(1934) was a result of his own investigation of (III) for commutative locally bicompact groups;

3° The solution of (II), announced by Chevalley
(1941), (1946) for solvable groups, was rediscovered by Maltsev as a result of his investigation of (III) for solvable, connected, locally bicompact groups.

4° Finally, Gleason and Montgomery-Zippin
(1952), (1953) obtained the solution of (II) in the general case; and Yamabe, perfecting the methods of Gleason, proved:

Th 1: (Yamabe)

In every locally bicompact group, there exists an open subgroup which appears as a projective limit of Lie Groups.

This result, taken together with earlier results of Iwasawa (See [7], Th. 11) yields:

Th 2: (Yamabe-Iwasawa)

Any connected locally bicompact group is locally isomorphic to a direct product of a local Lie Group and a bicompact group.

In reality, the requirement of connectedness here is, as was observed by B.M. Glushkov, superfluous. The final result in this direction is thus:

Th 3: (Yamabe-Iwasawa-Glushkov)

For any locally bicompact group G, and any nbhd. U of its identity, there exists such an open nbhd. V of its identity, which is contained in U, and resolves into a direct product of a local Lie Group L and a bicompact group. Herein, if the group G is not totally disconnected, then the nbhd U may be so chosen that, in every resolution above described, the local Lie Group L will have positive dimension.

The Fifth Problem of Hilbert :

[1] D. Hilbert, Mathematische Probleme, Götting. Nachr. (1900), 253–297

[2] J. Neumann, Annals of Mathematics, 34, (1933), 170–190

[3] L.S. Pontryagin, Compt. Rend. (Paris), 198, (1934), 238–240

[4] C. Chevalley, Lectures in Topology, Univ. of Michigan Press, Ann Arbor, (1941)

[5] A. I. Maltsev, Matem. sbornik, 19, N°2, (1946), 165–174

[6] K. Iwasawa, On some types of topological groups, Annals of Math., 50, N°3, (1949), 507–538

[7] A. M. Gleason, Annals of Math, 56, N°2 (1952), 193–212

[8] D. Montgomery, L. Zippin, Annals of Math, 56, N°2 (1952), 213–241

[9] H. Yamabe, Annals of Math, 58, N°2 (1953), 351–365

(1900)	D. Hilbert	Statement of the Problems
(1933)	J. von Neumann	(Bicompact)
(1934)	L.S. Pontryagin	(Commutative, locally bicompact)
(1941)	C. Chevalley	Solvable
(1946)	A. I. Maltsev	
(1949)	* K. Iwasawa *	Investigation of general types of topological groups.
(1952)	A. M. Gleason	General case
(1952)	D. Montgomery, L. Zippin	
(1953)	H. Yamabe	Improved presentation

Among the 23 problems that Hilbert proposed, in 1900, at the International Congress of Mathematics at PARIS, the following was the fifth :

(I) To construct the theory of continuous groups of transformations (Theory of Lie Groups) without the assumption of the differentiability of the functions that enter into ~~the~~ definition of the group.

Later, when the concept of a «topological group» received its precise meaning, it was possible to reformulate the problem:

(II) Given an arbitrary locally euclidean group, is it possible to introduce analytic coördinates (i.e. such coördinates wherein the law of multiplication in the group is given by analytic functions) in a certain nbhd. of its unit element?

It is clear that such a formulation certainly renders the problem more concrete; but, at the same time, it narrows down the original problem proposed by Hilbert: for, in the field of vision of (II) fall only parametric groups, and not all continuous groups of transformations that were envisaged in (I). Therefore it is more natural to study all the questions raised by Hilbert in the case of «local groups» in general. However, according to established tradition, we mean, by the «Fifth Problem of Hilbert» the narrower problem (II) of the existence of analytic coördinates in locally euclidean groups in the large.

The development of Topological Algebra, in the last three decades (thanks to the Russian, Japanese, and French schools!) has raised the following more general and significantly more involved problem:

(III) What is the structure of arbitrary locally bicompact groups?

Both the problems (II) and (III) are very intimately tied up with each other; the solution of (II), for any special class of groups, has at once influenced progress towards partial solution of (III).