Reprinted from the "Proceedings of the Indian Academy of Sciences," Vol. XIII, 1941

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Received December 3, 1940

IT was shown by Bhabha (1939) that the scattering of neutral mesons due to the mesic charge of the heavy particles ($g_1$ interaction) as given by the quantum theory has a complete correspondence with the scattering of mesons as given by the classical theory. The classical scattering is the analogue of the Thomson formula as recently extended by Dirac, remaining approximately constant up to meson energies comparable with the rest energy of the heavy particles, after which it decreases due to the effects of radiation reaction. The quantum cross-section like the Klein-Nishina formula also decreases with increasing energy due to the appearance of quantum effects, and the introduction of radiation reaction into the quantum theory would decrease it still more for high energies. On the other hand, the scattering of charged mesons on the usual theory differs completely from the classical and quantum mechanical scattering of neutral mesons, first in being larger by a factor $(M/\mu)^2$, M and $\mu$ being respectively the neutron and meson masses, and secondly in having an entirely different dependence on energy. The scattering of charged mesons due to the $g_1$ interaction alone was proportional to $p^4/E^2$, $p$ being the momentum and E the energy of the meson. It was shown in the paper quoted above that this difference in the scattering of neutral and charged mesons is entirely due to the fact that whereas a positive meson can only be absorbed by a neutron and emitted by a proton, a neutral meson may be absorbed or emitted by either a neutron or a proton. To avoid this difference, Bhabha put forward the idea that the heavy particles might exist in states of all integral charge, positive and negative with different rest energies, of which only the two of lowest rest energy, namely the proton and neutron, occur normally in nature. This assumption puts the scattering of charged mesons due to the $g_1$ interaction on the same footing as the scattering of neutral mesons, and establishes correspondence with the classical theory. (*Cf.* Heitler, 1940.)

It is our purpose in this paper to investigate the modifications which the above idea of allowing the heavy particles to exist in states of all integral charge introduces in the scattering of charged mesons by the mesic dipole of the heavy particles ($g_2$ interaction). The dipole interaction leads even in the classical theory to a scattering of mesons which increases with energy as $p^4/E^2$ in the region of low energies in which the effects of radiation reaction are negligible, and this coupled with the fact that only total scattering cross-sections have been compared, has concealed the very considerable difference in the dependence on scattering angle and polarisation of the meson which exists between the scattering on the classical theory and the usual quantum theory. It will be shown in this paper that if the heavy particles be allowed to exist in states of all integral charge, then complete correspondence is established in the scattering due to the $g_2$ interaction as given by the classical and quantum theories, both in its dependence on the scattering angle and on the directions of polarisation of the incident and scattered meson. This complete correspondence seems to us on the one hand to add further support to the correctness of the assumption that the heavy particles can exist in states of all integral charge, and on the other to the correctness as far as scattering phenomena are concerned of a classical theory of the spin even when applied to a particle with a spin $\hbar/2$ (with the exception of a constant numerical factor which will be discussed in detail in the last section). On the other hand the hypothesis put forward by Heitler (1940) of allowing the heavy particles to exist in states of higher spin possesses no correspondence with any classical theory. Moreover the need for it has disappeared since a complete classical theory as put forward for a Maxwell field by Bhabha (1940, *a*), and Bhabha and Corben* and for a meson field by Bhabha* (1940, *c*) has shown that the effect of radiation reaction is ultimately to make the scattering due to the spin diminish as $E^{-2}$ instead of increasing as $E^2$.

### *Quantum Mechanical Scattering*

We will first consider the scattering of charged mesons due to the $g_2$ interaction on the basis of the assumption that the heavy particles can exist in states of all integral charge positive and negative with different rest masses (Bhabha, 1940, *b*). Neglect the mass difference between a proton and a neutron, and in the notation of the above paper let $\triangle M_2$ and $\triangle M_{-1}$, denote the mass excesses of protons of charge $2e$ and $-e$ respectively over that of an ordinary proton. Since we are interested in investigating the effect of the above assumption on the scattering by the spin, we put $g_1 = 0$. For the same reason we will put $M = \infty$ as has been done in the classical calculations. This is equivalent to treating the heavy particles as fixed in space. The correction due to the motion of the heavy particles is of the order $(\mu/M)^2$

* *Proc. Roy. Soc.* (A), in print.

for the case of low momenta $p \ll Mc$ to which we restrict ourselves, and is hence negligible.

In this paper we adopt the notation used by Bhabha (1938) in the formulation of meson theory. $\mathbf{p}$ denotes the momentum of a meson, and $E = c\sqrt{\mu^2 c^2 + p^2}$, its energy. $\epsilon_{1p}$, $\epsilon_{2p}$ and $\epsilon_{3p}$ are three mutually perpendicular unit vectors, $\epsilon_{3p}$ being in the direction $\mathbf{p}$. As usual $\alpha$, $\beta$ denote the four Dirac matrices, and we define $\alpha_{1p}$, $\alpha_{2p}$, $\alpha_{3p}$ as in the above paper by

$$\left.\begin{aligned} \alpha_{1p} &= (\alpha, \epsilon_{1p}) \\ \alpha_{2p} &= (\alpha, \epsilon_{2p}) \\ \alpha_{3p} &= (\alpha, \epsilon_{3p}) = (\alpha, \mathbf{p})/p \end{aligned}\right\} \tag{1}$$

Since the mass of the heavy particles is treated as infinite, we may proceed as in the usual non-relativistic approximation and replace products of two $\alpha$'s, by the Pauli matrices $\sigma_k$, $\sigma_l$, $\sigma_m$, thus

$$\sigma_k \sigma_l = i\sigma_m. \tag{2}$$

In this approximation $\beta = -1$ and terms containing an odd number of $\alpha$'s are negligible. Using (1) and (2)

$$\left.\begin{aligned} (\alpha_{3p}\, \alpha_{1p}) &= i(\sigma, \epsilon_{2p}) = i\sigma_{2p} \\ (\alpha_{3p}\, \alpha_{2p}) &= -i(\sigma, \epsilon_{1p}) = -i\sigma_{1p} \end{aligned}\right\} \tag{3}$$

The interaction of mesons with the heavy particles is given by $\mathcal{I}_0$ in the paper mentioned above (formula 58 *a*). $\tau_{NP}$ and $\tau_{PN}$ are there the operators which convert a proton into a neutron and *vice versa*. On the basis of the new idea, we have to replace these as in the previous paper (Bhabha, 1940, *b*) by $\tau_-$ and $\tau_+$ which respectively decrease and increase the charge of the heavy particles by one unit. Putting $g_1 = 0$, and remembering (3), the interaction of mesons with the heavy particles in the non-relativistic approximation of this paper is then

$$\mathcal{I}_0 = -\frac{g'_2}{\sqrt{2EV}} \sum_p \frac{p}{\mu c} \left[ \left\{ \sigma_{2p} (\bar{a}_{1p} - b_{1p}) - \sigma_{1p} (\bar{a}_{2p} - b_{2p}) \right\} \tau_- e^{-\frac{i}{\hbar}(\mathbf{p}, \mathbf{X})} + \left\{ \sigma_{2p} (a_{1p} - \bar{b}_{1p}) - \sigma_{1p} (a_{2p} - \bar{b}_{2p}) \right\} \tau_+ e^{\frac{i}{\hbar}(\mathbf{p}, \mathbf{X})} \right] \tag{4}$$

As usual $g'_2 = g_2 \mu c/\hbar$. Here V is the volume of some very large box in which the wave functions are assumed to be periodic and at the end of the calculation V is made to tend to infinity. $\bar{a}_{1p}$ and $a_{1p}$ are the operators which give the creation and annihilation of positive mesons of momentum $\mathbf{p}$ polarised in the direction $\epsilon_{1p}$, and $\bar{b}_{1p}$ and $b_{1p}$ are the corresponding operators for negative mesons. $\mathbf{X}$ represents the co-ordinates of the heavy particles.

Now consider the scattering of a negative meson of momentum $\mathbf{p}$ by a proton. The momentum of the scattered meson will be denoted by $\mathbf{p}'$. Denote the angle between $\mathbf{p}$ and $\mathbf{p}'$ by $\theta$. It can be shown as usual that the differential cross-section $dQ$ for the scattering of the meson into the solid angle $d\Omega$ is given by

$$dQ = 4E^2\,V^2 \left| \sum_m \frac{(i|\mathcal{J}_0|m)\,(m|\mathcal{J}_0|f)}{E_i - E_m} \right|^2 d\Omega \tag{5}$$

where $i$, $m$ and $f$ denote the initial, intermediate, and final states of the whole system, $E_i$ and $E_m$ being the total energies of the initial and intermediate states of the whole system. There are four intermediate states $m$ by which the process takes place.

$$U^- + P + [N] + [P_2] \rightarrow \left\{ \begin{array}{c} N + [N] + [P_2] \\ \hdashline U^- + P + U'^- + P' + [P_2] \\ \hdashline U^- + P_2 + U'^- + [N] + [P_2] \\ \hdashline P + [N] + P' \end{array} \right\} \rightarrow U'^- + P' + [N] + [P_2] \tag{6}$$

The square brackets denote a virtual neutron and a proton of charge $2e$ in their negative energy states. For the four intermediate states, $(E_i - E_m)$ is respectively $E$, $(-2M - E)$, $(-E - \triangle Mc^2)$, and $(-2M - \triangle Mc^2 + E)$, where for brevity we write $\triangle M$ in place of $\triangle M_2$. Since M is taken as infinite, the second and fourth intermediate states give no scattering and the scattering comes only from the first and third intermediate states. On the old theory there would be no intermediate state like the third.

Take the vectors $\epsilon_{2p}$, $\epsilon_{2p'}$ to lie in the $\mathbf{p}$, $\mathbf{p}'$ plane. Then $\epsilon_{1p} = \epsilon_{1p'}$ are perpendicular to this plane. There are four cases to be considered depending on the polarisation of the incident and scattered meson. The case in which the meson is initially polarised in the direction $\epsilon_{2p}$, the scattered meson being polarised in the direction $\epsilon_{2p'}$ we denote schematically by $(2p) \rightarrow (2p')$. The other three cases are $(2p) \rightarrow (1p')$, $(1p) \rightarrow (2p')$ and $(1p) \rightarrow (1p')$. We notice at once that the interaction (4) does not involve longitudinally polarised mesons, *i.e.*, those polarised along $\epsilon_{3p}$. These are therefore not scattered at all by the spin of the heavy particles in the non-relativistic approximation.

Denote the two possible spin states of the heavy particles by

$$a_{\mathrm{I}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad a_{\mathrm{II}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{7}$$

First consider the case $(2p) \rightarrow (2p')$. For the transition *via* the first intermediate state in (6) we get, using (4)

$$\frac{(i \mid \mathcal{I}_0 \mid m)(m \mid \mathcal{I}_0 \mid f)}{\mathrm{E}_i - \mathrm{E}_m} = \frac{g'^2_2}{2\mathrm{E}^2\,\mathrm{V}} \cdot \frac{p^2}{\mu^2 c^2} \sum_{m = \mathrm{I, II}} (\bar{a}_f \sigma_{1p'} a_m)(\bar{a}_m \sigma_{1p} a_i).$$

The summation over the two spin orientations of the intermediate state can be carried out at once, and gives the unit matrix **1**, thus

$$\sum_{m = \mathrm{I, II}} (\bar{a}_f \sigma_{1p'} a_m)(\bar{a}_m \sigma_{1p} a_i) = (a_f \sigma_{1p'} \mathbf{1} \sigma_{1p} a_i) = (a_f \sigma_{1p'} \sigma_{1p} a_i),$$

whence

$$\frac{(i \mid \mathcal{I}_0 \mid m)(m \mid \mathcal{I}_0 \mid f)}{\mathrm{E}_i - \mathrm{E}_m} = \frac{g'^2_2}{2\,\mathrm{E}\,\mathrm{V}} \cdot \frac{p^2}{\mu^2 c^2} \frac{(a_f \sigma_{1p'} \sigma_{1p} a_i)}{\mathrm{E}}. \tag{8}$$

The transition *via* the third intermediate state similarly gives

$$\frac{(i \mid \mathcal{I}_0 \mid m)(m \mid \mathcal{I}_0 \mid f)}{\mathrm{E}_i - \mathrm{E}_m} = \frac{g'^2_2}{2\,\mathrm{E}\,\mathrm{V}} \cdot \frac{p^2}{\mu^2 c^2} \cdot \frac{(a_f \sigma_{1p} \sigma_{1p'} a_i)}{-\mathrm{E} - \Delta \mathrm{M}c^2} \tag{9}$$

Thus, substituting (8) and (9) into (5), the differential cross-section for the transition $(2p) \rightarrow (2p')$ is

$$d\mathrm{Q}\,\{(2p) \rightarrow (2p')\} = \frac{g'^4_2\, p^4}{\mu^4 c^4} \left| \frac{(a_f \sigma_{1p'} \sigma_{1p} a_i)}{\mathrm{E}} - \frac{(a_f \sigma_{1p} \sigma_{1p'} a_i)}{\mathrm{E} + \Delta \mathrm{M}c^2} \right|^2 d\Omega \tag{10}$$

The summation over the two final spin states of the heavy particle and the averaging over the two initial spin states can be carried out as usual and we get

$$\tfrac{1}{2} \sum_{f = \mathrm{I, II}} \sum_{i = \mathrm{I, II}} \left| \frac{(a_f \sigma_{1p'} \sigma_{1p} a_i)}{\mathrm{E}} - \frac{(a_f \sigma_{1p} \sigma_{1p'} a_i)}{\mathrm{E} + \Delta \mathrm{M}c^2} \right|^2$$

$$= \tfrac{1}{2}\,\mathrm{Spur} \left[ \frac{\sigma_{1p} \sigma_{1p'} \sigma_{1p'} \sigma_{1p}}{\mathrm{E}^2} + \frac{\sigma_{1p'} \sigma_{1p} \sigma_{1p} \sigma_{1p'}}{(\mathrm{E} + \Delta \mathrm{M}c^2)^2} - 2 \frac{\sigma_{1p'} \sigma_{1p} \sigma_{1p'} \sigma_{1p}}{\mathrm{E}\,(\mathrm{E} + \Delta \mathrm{M}c^2)} \right]$$

$$= \left[ \frac{1}{\mathrm{E}^2} + \frac{1}{(\mathrm{E} + \Delta \mathrm{M}c^2)^2} - \mathrm{Spur}\, \frac{\sigma_{1p'} \sigma_{1p} \sigma_{1p'} \sigma_{1p}}{\mathrm{E}\,(\mathrm{E} + \Delta \mathrm{M}c^2)} \right] \tag{11}$$

It can easily be shown that if **k** and **l** be any two vectors

$$\mathrm{Spur}\,(\sigma \mathbf{k})(\sigma \mathbf{l})(\sigma \mathbf{k})(\sigma \mathbf{l}) = 4\,(\mathbf{k}\,\mathbf{l})^2 - 2k^2 l^2 \tag{12}$$

Applying this formula to the spur in (11) and remembering that $\epsilon_{1p}$ and $\epsilon_{1p'}$ lie in the same direction we get

$$d\mathrm{Q}\,\{(2p) \rightarrow (2p')\} = g'^4_2 \frac{p^4}{\mu^4 c^4} \left[ \frac{1}{\mathrm{E}^2} + \frac{1}{(\mathrm{E} + \Delta \mathrm{M}c^2)^2} - \frac{2}{\mathrm{E}\,(\mathrm{E} + \Delta \mathrm{M}c^2)} \right] d\Omega \tag{13 a}$$

Now $\Delta \mathrm{M}c^2$ lies between 15 and 20 M.e.V. (*cf.* Heitler, 1940), while E must be greater than $\mu c^2 \approx 85$ M.e.V., the rest energy of a meson. Hence $\Delta \mathrm{M}c^2/\mathrm{E}$

is less than or of the order 1/5. To a rough approximation we may expand (13 *a*) in powers of $(\triangle Mc^2/E)$ and get

$$dQ\,\{(2p)\rightarrow(2p')\} = g'^4_2 \frac{p^4}{\mu^4 c^4 E^2}\left\{\left(\frac{\triangle Mc^2}{E}\right)^2 - 2\left(\frac{\triangle Mc^2}{E}\right)^3 + \cdots\right\} d\Omega \quad (13\ b)$$

Now consider the transition $(2p)\rightarrow(1p')$. The cross-section for this is just got by replacing $\sigma_{1p'}$ in formulæ (10) and (11) by $\sigma_{2p'}$. Using (12) and remembering that $\epsilon_{2p'}$ and $\epsilon_{1p}$ are perpendicular, we get

$$dQ\,\{(2p)\rightarrow(1p')\} = g'^4_2 \frac{p^4}{\mu^4 c^4}\left[\frac{1}{E^2} + \frac{1}{(E+\triangle Mc^2)^2} + \frac{2}{E\,(E+\triangle Mc^2)}\right] d\Omega \quad (14\ a)$$

$$\backsimeq 4\,g'^4_2 \frac{p^4}{\mu^4 c^4 E^2}\left\{1 - \left(\frac{\triangle Mc^2}{E}\right) + \frac{5}{4}\left(\frac{\triangle Mc^2}{E}\right)^2 - \cdots\right\} d\Omega \quad (14\ b)$$

Next consider the transition $(1p)\rightarrow(2p')$. We have now to replace $\sigma_{1p}$ in (10) and (11) by $\sigma_{2p}$. Since $\epsilon_{2p}$ and $\epsilon_{1p'}$ are perpendicular, we get as before

$$dQ\,\{(1p)\rightarrow(2p')\} = g'^4_2 \frac{p^4}{\mu^4 c^4}\left[\frac{1}{E^2} + \frac{1}{(E+\triangle Mc^2)^2} + \frac{2}{E\,(E+\triangle Mc^2)}\right] d\Omega \quad (15)$$

Finally for the transition $(1p)\rightarrow(1p')$ we have to replace $\sigma_{1p}$ and $\sigma_{1p'}$, in (10) and (11) by $\sigma_{2p}$ and $\sigma_{2p'}$ respectively. Using (12) and remembering that $\epsilon_{2p}$, $\epsilon_{2p'}$, both lie in the **p p′** plane and that the angle between them is in consequence the scattering angle $\theta$, so that Spur $(\sigma_{2p}\,\sigma_{2p'}\,\sigma_{2p}\,\sigma_{2p'}) = 4\cos^2\theta - 2$ we get

$$dQ\,\{(1p)\rightarrow(1p')\} = g'^4_2 \frac{p^4}{\mu^4 c^4}\left[\frac{1}{E^2} + \frac{1}{(E+\triangle Mc^2)^2} - \frac{2\cos 2\theta}{E\,(E+\triangle Mc^2)}\right] d\Omega \quad (16\ a)$$

$$\backsimeq 4g'^4_2 \frac{p^4}{\mu^4 c^4 E^2}\sin^2\theta\left\{1 - \left(\frac{\triangle Mc^2}{E}\right) + \left(\frac{\triangle Mc^2}{E}\right)^2\left(1 + \frac{1}{4}\operatorname{cosec}^2\theta\right) - \cdots\right\} d\Omega \quad (16\ b)$$

Adding (13), (14), (15), and (16), and dividing by 2 we find that the differential cross-section for the scattering of a meson through an angle $\theta$, averaged over the two directions of transverse polarisation of the incident meson is

$$dQ = 2g'^4_2 \frac{p^4}{\mu^4 c^4}\left[\frac{1}{E^2} + \frac{1}{(E+\triangle Mc^2)^2} + \frac{\sin^2\theta}{E\,(E+\triangle Mc^2)}\right] d\Omega \quad (17)$$

The total cross-section is then

$$Q = 8\pi \frac{g'^4_2\,p^4}{\mu^4 c^4}\left[\frac{1}{E^2} + \frac{1}{(E+\triangle Mc^2)^2} + \frac{2}{3}\,\frac{1}{E\,(E+\triangle Mc^2)}\right] \quad (18\ a)$$

$$\backsimeq \frac{64\pi}{3}\,\frac{g'^4_2\,p^4}{\mu^4 c^4 E^2}\left\{1 - \left(\frac{\triangle Mc^2}{E}\right) + \frac{11}{8}\left(\frac{\triangle Mc^2}{E}\right)^2 - \cdots\right\} \quad (18\ b)$$

To get the scattering of neutral mesons we remember that when a neutral meson is emitted or absorbed, a proton remains a proton and a neutron a neutron. The calculation is exactly the same, and the cross-sections for the scattering of neutral mesons in the four cases considered above are given by formulæ (13) to (16) if we put $\triangle Mc^2 = 0$ in them. Thus the total scattering cross-section for neutral mesons is got by putting $\triangle Mc^2 = 0$ in (18) and is

$$Q \text{ (neutral)} = \frac{64\pi}{3} \frac{g'^4_2 p^4}{\mu^4 c^4 E^2}. \tag{19}$$

As stated before, the scattering process on the old theory of charged mesons took place *via* the first intermediate state in (6) only. Hence, on the old theory the second term in (10) would be absent. As a result, the last two terms in (11) would drop out, and the scattering cross-section for each of the four cases considered above would be just

$$dQ \text{ (old theory)} = \frac{g'^4_2 p^4}{\mu^4 c^4 E^2} d\Omega. \tag{20}$$

### *Classical Scattering*

The general classical theory of the scattering of mesons by spinning particles taking radiation reaction into account has been given by Bhabha in a recent paper* (1940, *c*). For low energies the effects of radiation reaction are negligible, and it is only in this region that the classical and quantum theories can be compared. In this paper, therefore, we will neglect the terms expressing the effects of radiation reaction in the formulæ given in the paper mentioned. We will follow the notation of this paper except for one minor alteration.

In the classical theory neither $\mu$ nor $\hbar$ appear separately, but only in the combination $\chi = \mu c/\hbar$. As in A, we consider the scattering of a meson wave of frequency $c\omega_0$ with a magnetic force **H** perpendicular to the direction of propagation.

As before, the heavy particle which does the scattering is taken as fixed ($M = \infty$) and its spin is represented by a unit vector $\mathbf{M}$. The angular momentum of the spin is denoted by I. The magneto-mesic force acting on the spin due to the incident meson wave can be written in the form

$$\mathbf{H} = \mathbf{H}_0 \cos c\omega_0 t \tag{21}$$

* *Proc. Roy. Soc.* (A), in print—referred to here as A.

where $\mathbf{H}_0$ is a constant. The average rate of flow of energy in the incident wave is then

$$\frac{H_0^2}{8\pi} \frac{c\omega_0}{(\omega_0^2 - \chi^2)^{\frac{1}{2}}}. \tag{22}$$

The scattering will be calculated for small oscillations of the dipole, and corresponding to (21) we write

$$\mathbf{M} = \mathbf{M}_0 + \mathbf{M}_1 \sin c\omega_0 t \tag{23}$$

where $\mathbf{M}_0$ is the initial direction of the dipole, $\mathbf{M}_0^2 = 1$, and $\mathbf{M}_1$ is a direction perpendicular to $\mathbf{M}_0$, that is $(\mathbf{M}_0 \, \mathbf{M}_1) = 0$. Denote the angle* between $\mathbf{M}_0$ and $\mathbf{H}_0$ by $\phi$. (23) is the expression to which the oscillation of the dipole given in A reduces if radiation reaction be neglected. Then as shown in A,

$$|\mathbf{M}_1| = \frac{g_2 \, H_0 \sin \phi}{I \, c\omega_0} \tag{24}$$

and $\mathbf{M}_1$ lies in the direction $[\mathbf{M}_0 \times \mathbf{H}_0]$, so that $(\mathbf{M}_1 \, \mathbf{H}_0) = 0$.

The potentials at a large distance $\mathbf{r}$ due to a vibration of the dipole given by (23) are

$$\left.\begin{aligned} U_0^{ret} &= 0 \\ \mathbf{U}^{ret} &= \frac{g_2 (\omega_0^2 - \chi^2)^{\frac{1}{2}}}{r^2} [\mathbf{r} \times \mathbf{M}_1] \cos \{c\omega_0 t - r \sqrt{\omega_0^2 - \chi^2}\} \end{aligned}\right\} \tag{25}$$

Put the dipole at the origin of co-ordinates and take the $z$-axis along the direction of the incident meson wave. This is the direction of the vector $\mathbf{p}$ of the last section. Choose the $y$-axis so that the direction $\mathbf{r}$ lies in the $y$-$z$ plane and as in the last section denote the angle which $\mathbf{r}$ makes with the $z$-axis by $\theta$. The direction $\mathbf{r}$ is the direction of the scattered wave $\mathbf{p}'$ of the last section.

(25) shows that the potential $\mathbf{U}$ is perpendicular to $\mathbf{r}$, so that the scattered wave is always transverse. Further, a longitudinally polarised wave has no magneto-mesic force $\mathbf{H}$, so that no scattering of longitudinally polarised waves by the dipole takes place. We have then only to calculate the dependence of the scattering on the two transverse directions of polarisation of the incident and scattered waves.

First consider the case $(2p) \rightarrow (2p')$. Here the potential of the incident wave is along the $y$-axis, so that $\mathbf{H}_0$ must lie along the negative $x$-axis. Then

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* This angle was called $\theta$ in A, but since $\theta$ has been used to denote the scattering angle in this paper, we have changed its symbol to $\phi$.

since $\mathbf{M}_1$ is perpendicular to $\mathbf{H}_0$, it lies in the $y$-$z$ plane, and hence $[\mathbf{r} \times \mathbf{M}_1]$ in (25) is perpendicular to the $y$-$z$ plane. Thus the scattered wave has no component polarised along the direction $\epsilon_{2p'}$, and hence

$$dQ\,\{(2p) \rightarrow (2p')\} = 0 \qquad (26)$$

When $\mathbf{H}_0$ lies along the $x$-axis, therefore, $[\mathbf{r} \times \mathbf{M}_1]$ lies along the $x$-axis, and hence the scattered wave is polarised in the direction $\epsilon_{1p'}$. The average rate of energy transfer corresponding to the potentials (25) of a wave polarized in the direction $\epsilon$ perpendicular to $r$ is

$$\frac{1}{8\pi}\, g_2{}^2\, \frac{c\omega_0\,(\omega_0{}^2 - \chi^2)^{3/2}}{r^4}\, (\epsilon,\, [\mathbf{r} \times \mathbf{M}_1])^2 \qquad (27)$$

Denoting the angle between $\mathbf{r}$ and $\mathbf{M}_1$ by $\theta_1$, using (24) and dividing by the rate of energy transfer (22) of the incident wave, we get the differential cross-section for the case $(2\,p) \rightarrow (1\,p')$

$$dQ'\,\{(2p) \rightarrow (1p')\} = \frac{g_2{}^4}{\mathrm{I}^2}\, \frac{(\omega_0{}^2 - \chi^2)^2}{c^2\,\omega_0{}^2}\, \sin^2\phi\, \sin^2\theta_1\, d\Omega. \qquad (28)$$

Denote by $\theta_0$ the angle between $\mathbf{r}$ and $\mathbf{M}_0$. The relations between the various directions are shown in Fig. 1. We shall have to make repeated use of the well-known theorem that if ABC be a spherical triangle, then

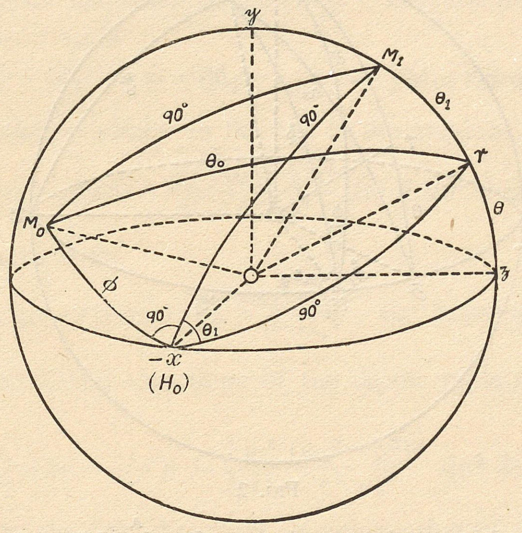


FIG. 1

$$\mathrm{Cos\,AB} = \mathrm{Cos\,AC.\ Cos\,BC} + \mathrm{Sin\,A\,C.\ Sin\,BC} \cdot \mathrm{Cos}\,\mathrm{A}\hat{\mathrm{C}}\mathrm{B}. \qquad (29)$$

Now since $r\,\mathrm{H}_0$ and $\mathrm{M}_1\,\mathrm{H}_0$ are right angles, the angle $\mathrm{M}_1\,\hat{\mathrm{H}}_0\,r$ is equal to the angle $\mathrm{M}_1\,r$, *i.e.*, $\theta_1$. Since $\mathrm{M}_1\,\mathrm{M}_0$ is also a right angle, $\mathrm{M}_1$ is the pole of the great circle $M_0 H_0$ and hence the angle $M_0 \hat{H}_0 M_1$ is a right angle. Thus the angle $M_0 \hat{H}_0 r$ is $\frac{\pi}{2} + \theta_1$. Now applying the equation (29) to the triangle $M_0 H_0 r$ it follows at once that

$$\cos \theta_0 = \cos M_0 r = \sin M_0 H_0 \sin H_0 r \cos M_0 \hat{H}_0 r = - \sin \phi \sin \theta_1.$$

Thus (28) simplifies to

$$dQ' \{(2p) \to (1p')\} = \frac{g_2^4}{I^2} \frac{(\omega_0^2 - \chi^2)^2}{c^2 \omega_0^2} \cos^2 \theta_0 \, d\Omega \qquad (30\,a)$$

Averaging over all initial directions of the dipole replaces $\cos^2 \theta_0$ by 1/3, so that

$$dQ \{(2p) \to (1p')\} = \frac{1}{3} \frac{g_2^4}{I^2} \frac{(\omega_0^2 - \chi^2)^2}{c^2 \omega_0^2} d\Omega \qquad (30\,b)$$

Next consider an incident wave polarised along $\epsilon_{1p}$, so that $\mathbf{H}_0$ now lies along the $y$-axis. $\mathbf{M}_1$ must now lie in the $x$-$z$ plane. Denote the angles made by $\mathbf{M}_0$ and $\mathbf{M}_1$ with the $x$-axis by $\psi_0$ and $\psi_1$ respectively, and the angle between $\mathbf{M}_0$ and the $z$-axis by $\eta_0$. The angles are represented in Fig. 2. Now

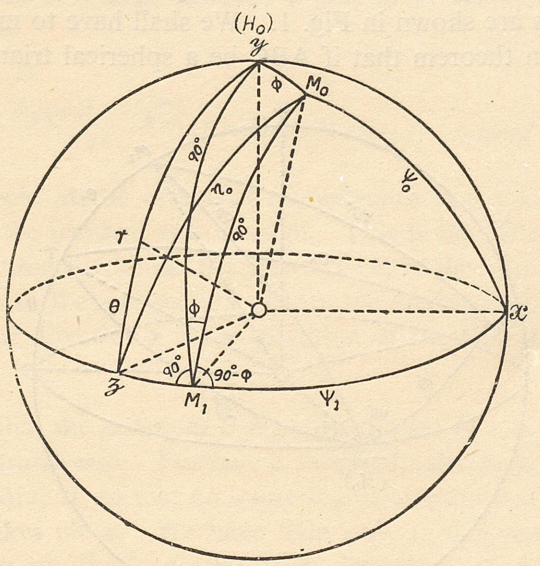


FIG. 2

since $y M_1$ and $M_0 M_1$ are right angles, angle $y \hat{M}_1 M_0 =$ the angle $y M_0 = \phi$, and hence the angle $M_0 \hat{M}_1 x$ is $\frac{\pi}{2} - \phi$. Applying the identity (29) to the triangle $M_0 M_1 x$ it follows that

$$\cos \psi_0 = \cos M_0 x = \sin M_0 M_1 \cdot \sin M_1 x . \cos M_0 \hat{M}_1 x = \sin \psi_1 \cdot \sin \phi \qquad (31)$$

Applying the same identity to the triangle $M_0\, z\, M_1$ it follows that

$$\cos \eta_0 = \cos M_0\, z = \sin z\, M_1 \cdot \sin M_1\, M_0 \cdot \cos z\, \hat{M}_1\, M_0 = -\cos \psi_1 \cdot \sin \phi. \quad (32)$$

Now consider the transition $(1\,p) \to (2\,p')$. The component of the vector potential (25) in the direction $\epsilon_{2p'}$ is

$$(\epsilon_{2p'} \cdot [\mathbf{r} \times \mathbf{M}_1]) = (\mathbf{M}_1 \cdot [\epsilon_{2p'} \times \mathbf{r}]) = |\, r\,| \cdot \mathbf{M}_{1x}$$

$M_{1x}$ being the $x$ component of $\mathbf{M}_1$, since $\epsilon_{2p'}$, and $\mathbf{r}$ both lie in the $y$-$z$ plane. This is just $r\, |M_1| \cos \psi_1$, and hence, using (22) and (27) and remembering (24), the differential cross-section for the case $(1\,p) \to (2\,p')$ can be deduced as before to be

$$dQ' \{(1p) \to (2p')\} = \frac{g_2^4}{I^2} \frac{(\omega_0^2 - \chi^2)^2}{c^2\, \omega_0^2} \sin^2 \phi \cos^2 \psi_1\, d\Omega$$

$$= \frac{g_2^4}{I^2} \frac{(\omega_0^2 - \chi^2)^2}{c^2\, \omega_0^2} \cos^2 \eta_0\, d\Omega \quad (33\,a)$$

in view of (32). Averaging over the initial orientations of the dipole, *i.e.*, over $\cos^2 \eta_0$, we get

$$dQ \{(1p) \to (2p')\} = \frac{1}{3} \frac{g_2^4}{I^2} \frac{(\omega_0^2 - \chi^2)^2}{c^2\, \omega_0^2}\, d\Omega \quad (33\,b)$$

Finally consider the case $(1p) \to (1p')$. The component of $[\mathbf{r} \times \mathbf{M}_1]$ in the direction $\epsilon_{1p'} = \epsilon_{1p}$ is

$$(\epsilon_{1p} \cdot [\mathbf{r} \times \mathbf{M}_1]) = (\mathbf{r} \cdot [\mathbf{M}_1 \times \epsilon_{1p}]) = r\, |\, M_1 \,| \sin \psi_1 \sin \theta.$$

Hence, by a calculation similar to the one in the previous case we find

$$dQ' \{(1p) \to (1p')\} = \frac{g_2^4}{I^2} \frac{(\omega_0^2 - \chi^2)^2}{c^2\, \omega_0^2} \sin^2 \theta \sin^2 \phi \sin^2 \psi_1\, d\Omega$$

which, by (31) reduces to

$$dQ' \{(1p) \to (1p')\} = \frac{g_2^4}{I^2} \frac{(\omega_0^2 - \chi^2)^2}{c^2\, \omega_0^2} \sin^2 \theta \cos^2 \psi_0\, d\Omega \quad (34\,a)$$

Averaging over all initial directions of the dipole replaces $\cos^2 \psi_0$ by 1/3, so that finally

$$dQ \{(1p) \to (1p')\} = \frac{1}{3} \frac{g_2^4}{I^2} \frac{(\omega_0^2 - \chi^2)^2}{c^2\, \omega_0^2} \sin^2 \theta\, d\Omega \quad (34\,b)$$

Summing over the final directions of polarisation of the scattered wave and averaging over the two transverse polarisations of the initial wave, *i.e.*, adding (26), (30 *b*), (33 *b*) and (34 *b*), and dividing by 2, we find

$$dQ = \frac{1}{6} \frac{g_2^4}{I^2} \frac{(\omega_0^2 - \chi^2)^2}{c^2\, \omega_0^2} (2 + \sin^2 \theta)\, d\Omega \quad (35)$$

The integration of this over all angles gives the total cross-section

$$Q = \frac{16\pi}{9} \frac{g_2^4}{I^2} \frac{(\omega_0^2 - \chi^2)^2}{c^2 \omega_0^2}. \tag{36 a}$$

The exact classical formula given in A was

$$Q = 4\pi (\omega_0^2 - \chi^2)^2 \frac{\{a^2 \omega_0^2 + (\omega_0^2 - \chi^2)^3 + \chi^6\}}{\{a^2 \omega_0^2 + (\omega_0^2 - \chi^2)^3 + \chi^6\}^2 - 4a^2 \omega_0^2 \chi^6} \tag{36 b}$$

where $a = 3\, Ic/2g_2^2$,

which may be written correctly to within 2% in the simpler form

$$Q \backsimeq 4\pi \frac{(\omega_0^2 - \chi^2)^2}{a^2 \omega_0^2 + (\omega_0^2 - \chi^2)^3} \tag{36 c}$$

The formula (36 *a*) results from (36 *c*) if we neglect the effects of radiation reaction, *i.e.*, the second term in the denominator of (36 *c*).

Writing $E = c\hbar\, \omega_0$, $p = \hbar\, (\omega_0^2 - \chi^2)^{\frac{1}{2}}$, $g_2 = g_2'\, \hbar/\mu c$ and specialising the problem by putting $I = \hbar/2$ in order to apply it to the spin of a heavy particle (proton or neutron) we find that the expression which occurs in (30), (33) and (34) as a common factor is just

$$\frac{g_2^4}{I^2} \frac{(\omega_0^2 - \chi^2)^2}{c^2 \omega_0^2} = 4 \frac{g_2'^4}{\mu^4 c^4} \frac{p^4}{E^2}. \tag{37}$$

With this specialisation (36 *c*) may be written in the form

$$Q = 4\pi \frac{p^4}{\left(\frac{3}{4} \frac{\mu c^2}{g_2'^2}\right)^2 \mu^2 E^2 + p^6 \hbar^{-2}} \tag{36 d}$$

*Discussion*

We notice first that the quantum and classical theories agree in giving no scattering of longitudinally polarised mesons by the spin of the heavy particle in the approximation in which the heavy particle is taken as fixed.

The results of the previous two sections on the scattering of transversely polarised mesons are collected in the following table for the four cases considered.

| | Usual Theory (quantum) charged mesons | New Theory (quantum) charged mesons | Quantum Theory neutral mesons | Classical Theory |
|---|---|---|---|---|
| | $(g'^4_2 p^4/\mu^4 c^4 E^2)\, d\Omega$ | $(4g'^4_2 p^4/\mu^4 c^4 E^2)\, d\Omega$ | $(4g'^4_2 p^4/\mu^4 c^4 E^2)\, d\Omega$ | $(4g'^4_2 p^4/3\mu^4 c^4 E^2)\, d\Omega$ |
| $dQ\,\{(2p)\to(2p')\}$ | 1 | $\frac{1}{4}(\triangle Mc^2/E)^2 + \cdots$ | 0 | 0 |
| $dQ\,\{(2p)\to(1p')\}$ | 1 | $1-(\triangle Mc^2/E)+\cdots$ | 1 | 1 |
| $dQ\,\{(1p)\to(2p')\}$ | 1 | $1-(\triangle Mc^2/E)+\cdots$ | 1 | 1 |
| $dQ\,\{(1p)\to(1p')\}$ | 1 | $\text{Sin}^2\theta\left\{1-\left(\frac{\triangle Mc^2}{E}\right)+\cdots\right\}$ | $\text{Sin}^2\theta$ | $\text{Sin}^2\theta$ |

The first column gives the scattering of charged mesons on the usual quantum theory in which only the ordinary proton and neutron are assumed to exist. This is just given in each of the four cases by (20).

The next column gives the scattering on the basis of Bhabha's idea of allowing the heavy particles to exist in states of all integral charge. For a better comparison with the classical theory, we give the approximate expressions got by expanding in powers of $(\triangle Mc^2/E)$; namely (13 *b*), (14 *b*) and (16 *b*).

For scattering by a proton, a proton of charge $2e$ is virtually formed during the scattering process, so that $\triangle M$ has to be put equal to $\triangle M_2$, the mass excess of a proton of charge $2e$ over an ordinary proton. For scattering by a neutron $\triangle M$ has to be put equal to $\triangle M_{-1}$ the mass excess of a negative proton over an ordinary proton. The two mass-excesses need not be the same. The scattering of positive and negative mesons is however exactly the same in every case.

The third column gives the quantum mechanical scattering of neutral mesons. This is derived from the second column by putting $\triangle M = 0$.

The last column gives the classical scattering as given by (26), (30 *b*), (33 *b*) and (34 *b*), and we have specialised the expressions by the substitution (37).

A comparison of the third and fourth columns shows that the quantum and classical theories give the same dependence on scattering angle and on the directions of polarisation of the incident and scattered meson for the scattering of neutral mesons. The expressions at the top of both columns are the same, except that the classical expression is smaller by a factor 3. This difference is not difficult to understand. It is a consequence of the averaging over the initial orientation of the spin of the heavy particle.

Consider the expression (30 *a*). An averaging over the initial direction of the spin of the heavy particle means that we have to average $\cos^2 \theta_0$, where $\theta_0$ is the angle between the initial direction of the spin of the heavy particle and the direction *r*. This averaging results in a factor 1/3. In this it must differ from the averaging over the initial orientation of the spin in the quantum theory, since, the spin being $\hbar/2$, the cosine of the angle it makes with any given direction can only be $\pm 1$. Thus an averaging over the initial directions of the spin of the heavy particles gives a factor 1 in the quantum theory, instead of 1/3 as in the classical theory.* The same argument applies in the other two cases. This completely explains the only difference between the quantum mechanical and classical scattering of neutral mesons. The identity of the scattering in its dependence on energy, the scattering angle and the polarisation of the incident and scattered mesons shows that we may correctly use the classical formula to get the scattering of neutral mesons by a particle of spin $\hbar/2$ in the region of high energies where it differs from the quantum formula due to the effects of radiation reaction. The classical formula (36 *b*) would only go wrong seriously due to the appearance of quantum effects at energies comparable with or higher than the rest energy of the heavy particles, but for such high energies the scattering is already a small fraction of its value at the maximum due to the effects of radiation reaction.

A comparison of the second and third columns shows that the scattering of charged mesons on the assumption that the heavy particles can exist in states of all integral charge only differs from the scattering of neutral mesons by quantities of the order $\triangle Mc^2/E$. To a first approximation it is smaller by a factor $(1 - \triangle Mc^2/E)$. Thus, on the basis of this assumption we see that the quantum mechanical scattering of charged mesons by the spin of the heavy particles shows complete correspondence with the classical scattering in its dependence on scattering angle and polarisation of the incident and scattered mesons. The dependence on energy is also very nearly the same, differing only by the correcting factor $(1 - \triangle Mc^2/E)$ which varies little with energy. Due to this factor there will be a slight difference in the scattering by neutrons and protons since $\triangle M$ has to be put equal to $\triangle M_{-1}$ and $\triangle M_2$ respectively in the two cases.

This complete correspondence is a further argument in favour of the assumption that the heavy particles can exist in states of all integral charge.

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* Had the heavy particle spin been $\hbar$, $\cos \theta_0$ could have taken on the values 1, 0 and $-1$, and hence the quantum mechanical average over the initial directions of the spin would have given 2/3 in this case instead of 1. For higher spins the quantum mechanical average is still smaller, and for larger and larger spin gradually approaches the classical average.

It also shows that the classical formula for the scattering of mesons multiplied by a factor 3 will give the scattering of *charged* mesons correctly to within 20% up to energies comparable with the rest energy of the heavy particles, as has already been suggested by one of us in a recent note.* Perhaps a more accurate formula could be obtained by multiplying the classical formula (36 *d*) by the factor $3\,(1 - \Delta \mathrm{M} c^2/\mathrm{E})$ for the reasons mentioned above, thus

$$\mathrm{Q}\ (\text{charged}) = 12\pi \frac{p^4}{\left(\dfrac{3}{4}\dfrac{\mu c^2}{g'^2_2}\right)^2 \mu^2 \mathrm{E}^2 + p^6 \hbar^{-2}} \left(1 - \frac{\Delta \mathrm{M} c^2}{\mathrm{E}}\right). \qquad (36\ e)$$

A comparison of the first and last columns shows that there is no correspondence between the scattering on the usual quantum theory and the classical theory either in its dependence on scattering angle or in its dependence on the polarisation of the incident and scattered meson.

The scattering of charged mesons due to the spin of the heavy particles as calculated by Heitler and Ma (1940 formula 23 *c*) with Heitler's further assumption of allowing the heavy particles to exist in states of higher spin shows no correspondence with any classical theory. As the authors themselves have pointed out, the scattering vanishes if the mass excesses of the states of higher spin and charge are put equal to zero. In contrast to this, the results of the present paper show that the correspondence with the classical scattering by the spin becomes even greater when the mass excesses of the higher charge states are put equal to zero.

*Summary*

It is shown that the scattering of neutral mesons *by the spin of the heavy particles* ($g_2$ interaction) on the quantum theory agrees completely in its dependence on energy, scattering angle, and polarisation of the incident and scattered meson with the scattering on the classical theory (neglecting radiation reaction), except for being larger by a constant factor 3, which is due to differences in the averaging over the initial directions of spin of the heavy particle.

On the basis of the assumption put forward by Bhabha that the heavy particles can exist in states of all integral charge, it is shown that the scattering of charged mesons by the spin of the heavy particles only differs from the scattering of neutral mesons by factors $(1 - \Delta \mathrm{M}_2 c^2/\mathrm{E})$ and $(1 - \Delta \mathrm{M}_{-1} c^2/\mathrm{E})$ for scattering by protons and neutrons respectively, $\Delta \mathrm{M}_2$ and $\Delta \mathrm{M}_{-1}$ being

* Bhabha: *Physical Review*, letter to the Editor.

the mass excesses of protons of charge $2e$ and $-e$ over an ordinary proton, and E being the meson energy. *Thus on this assumption the scattering of charged mesons shows complete correspondence with the classical theory*, and hence the previously given classical formula (36 *d*) taking radiation reaction into account multiplied by a factor 3 will give the scattering of charged mesons up to energies of roughly $10^9$ e.v., correctly to within about 20%.

A new formula (36 *e*) is given tentatively which should be more accurate.

The scattering of charged mesons on the usual quantum theory shows no correspondence with the classical scattering. The scattering on the basis of Heitler's idea of allowing the heavy particles to exist in higher spin states also shows no correspondence with the classical scattering.

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