A - D

Lectures on Analytical Dynamics to M.Sc. Students

1961 Session (Following Whittaker)

Lectures from 4/7/61 up to time before the colossal floods on 12/7/61: — Holonomic & non-holonomic systems — Derivation of Lagrange's eqns in generalized coordinates — natural systems — Impulsive motion — Conservative system — Lagrangian L — Integration of Lagrangian equations — Ignoration of coordinates and Routhian function — linear and angular momentum theorems as simple cases of ignoration — Integral of energy — Worked Example on p. 57 — Example 3, p. 69.

Ex. 3. p. 69: $T = \dfrac{\dot{q}_1^2}{2(a+bq_2)} + \frac{1}{2} q_2^2 \dot{q}_2^2$

$V = c + dq_2 \qquad (a, b, c, d \text{ const})$

Show that q_2 in terms of time is given by $(q_2 - k)(q_2 + 2k)^2 = h(t-t_0)^2$ (h, k, t_0 constants)

$$L = \frac{\dot{q}_1^2}{2(a+bq_2)} + \frac{1}{2} q_2^2 \dot{q}_2^2 - c - dq_2$$

q_1 is an ignorable coordinate, so that $\dfrac{\partial L}{\partial \dot{q}_1} = \text{const}$ ie $\dfrac{\dot{q}_1}{a+bq_2} = \text{const} = \beta$, say

& the Routhian fn $R = L - \dot{q}_1 \dfrac{\partial L}{\partial \dot{q}_1}$

$$= \frac{1}{2} q_2^2 \dot{q}_2^2 - c - dq_2 - \frac{\dot{q}_1^2}{2(a+bq_2)} = \frac{1}{2} q_2^2 \dot{q}_2^2 - c - dq_2 - \frac{1}{2}\beta^2 (a+bq_2)$$

$$= \frac{1}{2} q_2^2 \dot{q}_2^2 - (c + \tfrac{1}{2}\beta^2 a) - q_2 (d + \tfrac{1}{2}\beta^2 b)$$

and we have $\dfrac{d}{dt}\left(\dfrac{\partial R}{\partial \dot{q}_2}\right) - \dfrac{\partial R}{\partial q_2} = 0$

ie $\dfrac{d}{dt}\left(q_2^2 \dot{q}_2\right) - q_2 \dot{q}_2^2 - (d + \tfrac{1}{2} b\beta^2) = 0$

ie $q_2^2 \ddot{q}_2 + 2q_2 \dot{q}_2^2 - q_2 \dot{q}_2^2 - (d + \tfrac{1}{2}\beta^2) = 0$ or $q_2^2 \ddot{q}_2 + q_2 \dot{q}_2^2 = \tfrac{1}{2} h'$, say

ie $q_2 \dfrac{d}{dt}(q_2 \dot{q}_2) = \tfrac{1}{2} h'$ or $q_2 \dot{q}_2 \dfrac{d}{dt}(q_2 \dot{q}_2) = \tfrac{1}{2} h' \dot{q}_2$, which gives on integration

$\tfrac{1}{2}(q_2 \dot{q}_2)^2 = \tfrac{1}{2} h' q_2 + \text{const}$ or $q_2 \dot{q}_2 = \sqrt{h'(q_2 - k)}$, say.

ie $\dfrac{q_2 \, dq_2}{\sqrt{q_2 - k}} = \sqrt{h'}\, dt$ ie $\displaystyle\int \left\{ \sqrt{q_2 - k} + \frac{k}{\sqrt{q_2 - k}} \right\} dq_2 = \sqrt{h'}\,(t - t_0)$ say

ie $\tfrac{2}{3}(q_2 - k)^{3/2} + 2k(q_2 - k)^{1/2} = \sqrt{h'}\,(t - t_0)$

$\tfrac{2}{3}(q_2 - k)^{1/2}(q_2 + 2k) = \sqrt{h'}\,(t - t_0)$ or $(q_2 - k)^{1/2}(q_2 + 2k) = \sqrt{h}\,(t - t_0)$, say

and squaring we have the required relation $(q_2 - k)(q_2 + 2k)^2 = h(t - t_0)^2$.

~~17/8/61~~. Worked Ex. 7, p. 70. ie ~~Did §42 of Whittaker, p. 64 re reduction of a dynamical system to one of a~~

If $T = \frac{1}{2}(q_1^2 + q_2^2)(\dot{q}_1^2 + \dot{q}_2^2)$, $V = 1/(q_1^2 + q_2^2)$, show by (by use of Liouville's theorem, or otherwise) that relation between q_1 and q_2 is

$$a^2 q_1^2 + b^2 q_2^2 + 2ab\, q_1 q_2 \cos\gamma = \sin^2\gamma,$$

where a, b, γ are constants of integration.

17/8/61. – Solved Ex. 7, p. 70 without using Liouville's theorem.

18/8/61 – Separable Systems – Systems of Liouville type.

22/8/61 – Ex. 7, p. 70 solved by using Liouville's theorem. – Ex. 8, p. 70 just tried.

In the case of Liouville systems $T = \frac{1}{2} u \Sigma \dot{q}_r^2$, $V = \frac{1}{u} \Sigma w_r$ [$u = \Sigma u_r$

and the soln is $\dfrac{dq_1}{(hu_1 - w_1 + r_1)^{\frac{1}{2}}} = \dfrac{dq_2}{(hu_2 - w_2 + r_2)^{\frac{1}{2}}} = \dots$ with $r_1 + r_2 + \dots = 0$.

Here $u = q_1^2 + q_2^2$, $w_1 = w_2 = 1/2$

$$\therefore \frac{dq_1}{(hq_1^2 - \frac{1}{2} + r_1)^{\frac{1}{2}}} = \frac{dq_2}{(hq_2^2 - \frac{1}{2} - r_1)^{\frac{1}{2}}}$$

$$\frac{dq_1}{\sqrt{q_1^2 - d_1^2}} = \frac{dq_2}{\sqrt{q_2^2 - d_2^2}} \qquad \left[d_1^2 = \frac{\frac{1}{2} - r_1}{h} ; \; d_2^2 = \frac{\frac{1}{2} + r_1}{h} . \right.$$

$$\text{ie } \cosh^{-1}(q_1/d_1) = \cosh^{-1}(q_2/d_2) + K$$

$$\frac{q_1}{d_1} = \frac{q_2}{d_2} \cosh K + \sqrt{\frac{q_2^2}{d_2^2} - 1} \, \sinh K.$$

$$\text{ie } \left(\frac{q_1}{d_1} - \frac{q_2}{d_2} \cosh K \right)^2 = \sinh^2 K \left(\frac{q_2^2}{d_2^2} - 1 \right)$$

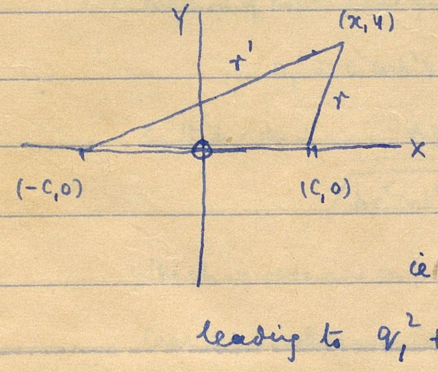
$$\text{ie } \frac{q_1^2}{d_1^2} + \frac{q_2^2}{d_2^2} - \frac{2 q_1 q_2}{d_1 d_2} \cosh K = -\sinh^2 K$$

and putting $d_1 = \frac{1}{a}$, $d_2 = -\frac{1}{b}$, $\cosh K = \cos \gamma$ & hence $-\sinh^2 K = \sin^2 \gamma$, this reduces to

$$a^2 q_1^2 + b^2 q_2^2 + 2ab\, q_1 q_2 \cos \gamma = \sin^2 \gamma.$$

25/8/61 – Ex. 8, p. 70 solved except for integration.

This example is a sort of a generalisation of § 53, p. 97 of Whittaker re. the problem of two centres of gravitation where $V = \frac{\mu}{r} + \frac{\mu'}{r'}$, while here V has three additional terms $A/x^2 + B/y^2 + C(x^2+y^2)$.



With $q_1 = \frac{1}{2}(r + r')$, $q_2 = \frac{1}{2}(r - r')$ as generalised coordinates

$r = q_1 + q_2$, $r' = q_1 - q_2$ and from

$r^2 = (x-c)^2 + y^2$, and $r'^2 = (x+c)^2 + y^2$

$$r^2 + r'^2 = 2(x^2 + y^2 + c^2); \; r^2 - r'^2 = -4cx$$

$$\text{ie } (r+r')^2 + (r-r')^2 = 4(x^2+y^2+c^2); \; (r+r')(r-r') = -4cx$$

leading to $q_1^2 + q_2^2 = x^2 + y^2 + c^2$; $q_1 q_2 = -cx$

or $x = -q_1 q_2 / c$; $y = \frac{1}{c}\{(q_1^2 - c^2)(c^2 - q_2^2)\}^{1/2}$

Using now the ξ and η of § 53 of Whittaker, by putting $q_1 = c \cosh \xi$, $q_2 = -c \cos \eta$, we have

$x = c \cosh \xi \cos \eta$, $y = c \sinh \xi \sin \eta$, $r = c(\cosh \xi - \cos \eta)$, $r' = c(\cosh \xi + \cos \eta)$

This gives $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}c^2(\cosh^2\xi \sin^2\eta + \sinh^2\xi \cos^2\eta)(\dot{\xi}^2 + \dot{\eta}^2) = \frac{1}{2}c^2(\cosh^2\xi - \cos^2\eta)(\dot{\xi}^2+\dot{\eta}^2)$.

and $V = \dfrac{A}{x^2} + \dfrac{A'}{y^2} + \dfrac{B}{r} + \dfrac{B'}{r'} + C(x^2+y^2)$

$$= \frac{1}{c^2}\left[A\,\mathrm{sech}^2\xi \sec^2\eta + A'\,\mathrm{cosech}^2\xi\,\mathrm{cosec}^2\eta + \frac{cB(\cosh\xi + \cos\eta) + cB'(\cosh\xi - \cos\eta)}{(\cosh^2\xi - \cos^2\eta)} + c^4C\left(\cancel{\mathrm{sech}^2\xi \sec^2\eta} + \cancel{\mathrm{cosech}^2\xi\,\sec^2\eta}\right)\right]$$

$$= \frac{1}{c^2(\cosh^2\xi - \cos^2\eta)}\Big[A(\sec^2\eta - \mathrm{sech}^2\xi) + A'(\coth^2\xi\,\mathrm{cosec}^2\eta - \mathrm{cosech}^2\xi\cot^2\eta).$$
$$+ c(B+B')\cosh\xi + c(B-B')\cos\eta$$
$$+ Cc^4\,(\cosh^2\xi - \cos^2\eta)(\cosh^2\xi\cos^2\eta + \sinh^2\xi\sin^2\eta)\ \cancel{(\sec^2\eta - \mathrm{sech}^2\xi + \coth^2\xi\,\mathrm{cosec}^2\eta - \mathrm{cosech}^2\xi\cot^2\eta)}\Big]$$

$$= \frac{1}{c^2(\cosh^2\xi - \cos^2\eta)}\Big[\{c(B+B')\cosh\xi - A\,\mathrm{sech}^2\xi + A'\coth^2\xi + Cc^4\sinh^2\xi\cosh^2\xi\}$$
$$+\{c(B-B')\cos\eta + A\sec^2\eta + A'\cot^2\eta + Cc^4\sin^2\eta\cos^2\eta\}.\Big] \quad \text{— (1)}$$

Hence we have the Liouville type & in the notation of Whittaker, p. 68 viz

$$T = \tfrac{1}{2}u(\dot{q}_1^2 + \dot{q}_2^2 + \cdots + \dot{q}_n^2)$$

$$V = \frac{1}{u}(w_1 + w_2 + \cdots + w_n) \qquad [w_r = w_r(q_r)$$

with $u = u_1 + u_2 + \cdots + u_n \qquad [u_r = u_r(q_r)$

we have here $u = c^2\cosh^2\xi - c^2\cos^2\eta$ ie $u_1 = c^2\cosh^2\xi$, $u_2 = -c^2\cos^2\eta$

w_1 = first term ~~of~~ $\{\ \}$ in the $[\ ]$ of (1) with $q_1, q_2 \to \xi, \eta$.

w_2 = second term $\{\ \}$ " "

Hence the soln is given by

$$\frac{d\xi}{\sqrt{hu_1 - w_1 + \gamma}} = \frac{d\eta}{\sqrt{hu_2 - w_2 - \gamma}} \quad \text{— (1)} \qquad \text{with } \tfrac{1}{2}u(\dot{\xi}^2 + \dot{\eta}^2) = h - V. \text{ (integral of energy).}$$

This can be shown to reduce to ~~elliptic functions~~ the form

[Note — In the simpler case of Whittaker § 53, p. 97, with $V = -\frac{\mu}{r} - \frac{\mu'}{r'}$, it is surprising to see that although Whittaker mentions that this problem of the Liouville type, he does not use the method of soln used for the Liouville type. In fact, in this case, we have

$$u_1 = c^2\cosh^2\xi,\ u_2 = -c^2\cos^2\eta,\ w_1 = -c(\mu+\mu')\cosh\xi;\ w_2 = -c(\mu-\mu')\cos\eta$$

so that soln is
$$\frac{d\xi}{\sqrt{hc^2\cosh^2\xi + c(\mu+\mu')\cosh\xi + \gamma}} = \frac{d\eta}{\sqrt{-hc^2\cos^2\eta + c(\mu-\mu')\cos\eta - \gamma}} \quad \text{-- (2)}$$

which is obtained on p. 99 of Whittaker after a long unnecessary simplification. Even this simple case (2) involves soln using elliptic fns and (1) is obviously much more complicated]

$$\frac{x_1\,dx_1}{(c_0 + c_1x_1^2 + c_2x_1^3 + c_3x_1^4 + c_4x_1^5 + c_5x_1^6 + c_6x_1^8)^{\frac{1}{2}}} = \frac{-x_2\,dx_2}{(d_0 + d_1x_2^2 + d_2x_2^3 + d_3x_2^4 + d_4x_2^5 + d_5x_2^6 + d_6x_2^8)^{\frac{1}{2}}} \quad \text{--- (3)}$$

where $x_1 = \cosh\xi$ and $x_2 = \cos\eta$.

29/8/61 – The second part of Ex. 5, p. 69 worked out, after proving the first part for the case of three degrees of freedom with one ignorable coordinate.

Let q_1, q_2, q_3 be the coordinates with q_3 ignorable. Writing $T = T_1 + T_2 + T_3$, where

$T_1 = \frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2)$; $T_2 = \frac{1}{2}a_{33}\dot{q}_3^2$, and $T_3 = \dot{q}_3(a_{13}\dot{q}_1 + a_{23}\dot{q}_2)$, the equation $\partial T/\partial \dot{q}_3 = p_3$ gives $p_3 = a_{33}\dot{q}_3 + a_{13}\dot{q}_1 + a_{23}\dot{q}_2$ .... (1)

$$\text{or } a_{33}\dot{q}_3 = p_3 - a_{13}\dot{q}_1 - a_{23}\dot{q}_2 \quad \text{.... (2)}$$

Using this equation, we can express T in terms of $\dot{q}_1, \dot{q}_2$ and p_3 only. The coefficients of $p_3\dot{q}_1$ and $p_3\dot{q}_2$ vanish in $T_2 + T_3$ and we are left with finally

$$T = \frac{1}{2a_{33}}\left(A_{22}\dot{q}_1^2 - 2A_{12}\dot{q}_1\dot{q}_2 + A_{11}\dot{q}_2^2\right) + \frac{1}{2a_{33}}p_3^2 = T' + K.$$

where A_{11}, A_{12} & A_{22} are cofactors of a_{11}, a_{12} & a_{22} in $\|a_{ij}\|$ $(i, j = 1, 2, 3)$

As regards the second, there appears to be something wrong in the problem as given in Whittaker, for the eqn contains $+\frac{\partial K}{\partial x}$ which should be $-\frac{\partial K}{\partial x}$ and if this should be compensated for, the last term $K\dot{y}\left\{\frac{\partial}{\partial x}\left(\frac{\partial \dot{\phi}}{\partial \dot{y}}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \dot{\phi}}{\partial \dot{x}}\right)\right\}$ should be equal to $-2\frac{\partial K}{\partial x}$. But this is equal to, using q_1, q_2 and q_3 instead of x, y, ϕ, and p_3 instead of K

$$p_3\dot{q}_2\left\{\frac{\partial}{\partial q_1}\left(\frac{\partial \dot{q}_3}{\partial \dot{q}_2}\right) - \frac{\partial}{\partial q_2}\left(\frac{\partial \dot{q}_3}{\partial \dot{q}_1}\right)\right\}$$

$$= p_3\dot{q}_2\left\{\frac{\partial}{\partial q_1}\left(-\frac{a_{23}}{a_{33}}\right) - \frac{\partial}{\partial q_2}\left(-\frac{a_{13}}{a_{33}}\right)\right\} \text{, using (2) above}$$

$$= -\frac{p_3\dot{q}_2}{a_{33}^2}\left\{\left(a_{33}\frac{\partial a_{23}}{\partial q_1} - a_{23}\frac{\partial a_{33}}{\partial q_1}\right) - \left(a_{33}\frac{\partial a_{13}}{\partial q_2} - a_{13}\frac{\partial a_{33}}{\partial q_2}\right)\right\} \quad \text{-- (3)}$$

From (1) $\frac{\partial a_{33}}{\partial q_1}\dot{q}_3 + \frac{\partial a_{13}}{\partial q_1}\dot{q}_1 + \frac{\partial a_{23}}{\partial q_1}\dot{q}_2 = 0$ -- (4)

& $\frac{\partial a_{33}}{\partial q_2}\dot{q}_3 + \frac{\partial a_{13}}{\partial q_2}\dot{q}_1 + \frac{\partial a_{23}}{\partial q_2}\dot{q}_2 = 0$ -- (5)

& from (1), (4), (5) we have $-p_3\frac{\partial a_{33}}{\partial q_1} = \dot{q}_1\left(a_{33}\frac{\partial a_{13}}{\partial q_1} - a_{13}\frac{\partial a_{33}}{\partial q_1}\right) + \dot{q}_2\left(a_{33}\frac{\partial a_{23}}{\partial q_1} - a_{23}\frac{\partial a_{33}}{\partial q_1}\right)$ -- (6)

and $-p_3\frac{\partial a_{33}}{\partial q_2} = \dot{q}_1\left(a_{33}\frac{\partial a_{13}}{\partial q_2} - a_{13}\frac{\partial a_{33}}{\partial q_2}\right) + \dot{q}_2\left(a_{33}\frac{\partial a_{23}}{\partial q_2} - a_{23}\frac{\partial a_{33}}{\partial q_2}\right)$ -- (7)

and if $\dot{q}_2\left(a_{33}\frac{\partial a_{13}}{\partial q_2} - a_{13}\frac{\partial a_{33}}{\partial q_2}\right) + \dot{q}_1\left(a_{33}\frac{\partial a_{13}}{\partial q_1} - a_{13}\frac{\partial a_{33}}{\partial q_1}\right) = 0$, ~~then the R.H.S.~~ —— (8)

then in view of (6), (3) would be equal to $\frac{p_3^2}{a_{33}^2}\frac{\partial a_{33}}{\partial q_1} = -2\frac{\partial K}{\partial q_1}$.

But if (8) holds $\frac{d}{dt}\left(\frac{a_{13}}{a_{33}}\right) = 0$ i.e. $\frac{a_{13}}{a_{33}}$ = const. is an integral & there is no reason why this should be so. Hence if (3) were itself, ~~on the other hand~~, of the form

$$-\frac{p_3}{a_{33}^2}\left\{\dot{q}_2\left(a_{33}\frac{\partial a_{23}}{\partial q_1} - a_{23}\frac{\partial a_{33}}{\partial q_1}\right) + \dot{q}_1\left(a_{33}\frac{\partial a_{13}}{\partial q_1} - a_{13}\frac{\partial a_{33}}{\partial q_1}\right)\right\} \text{ the result would follow}$$

ie of the form $p_3\left\{\dot{q}_2\frac{\partial}{\partial q_1}\left(-\frac{a_{23}}{a_{33}}\right)+\dot{q}_1\frac{\partial}{\partial q_1}\left(-\frac{a_{13}}{a_{33}}\right)\right\} = p_3\left\{\dot{q}_2\frac{\partial}{\partial q_1}\left(\frac{\partial\dot{q}_3}{\partial\dot{q}_2}\right)+\dot{q}_1\frac{\partial}{\partial q_1}\left(\frac{\partial\dot{q}_3}{\partial\dot{q}_1}\right)\right\}$

$$= K\left\{\dot{y}\frac{\partial}{\partial x}\left(\frac{\partial\dot{\phi}}{\partial\dot{y}}\right)+\dot{x}\frac{\partial}{\partial x}\left(\frac{\partial\dot{\phi}}{\partial\dot{x}}\right)\right\}$$

The other equation of motion would then be

$$\cancel{\frac{d}{dt}\left(\frac{\partial T'}{\partial}\right.}\ \frac{d}{dt}\left(\frac{\partial T'}{\partial\dot{y}}\right)-\frac{\partial T'}{\partial y}+\frac{\partial K}{\partial y}+\frac{\partial V}{\partial y}+K\left\{\dot{x}\frac{\partial}{\partial y}\left(\frac{\partial\dot{\phi}}{\partial\dot{x}}\right)+\dot{y}\frac{\partial}{\partial y}\left(\frac{\partial\dot{\phi}}{\partial\dot{y}}\right)\right\}=0.$$

So the form given in Whittaker would be correct only if $\frac{\partial\phi}{\partial\dot{x}}$ and $\frac{\partial\phi}{\partial\dot{y}}$ were integrals of the system.

31|8|61 – Proved the first part of Ex. 5, p. 69 for 3 coords ~~and two~~ of which two are ignorable, and did part of the general case of n coords with r ignorable coordinates

<u>Case of 3 coords, two ignorable</u>: Let q_1, q_2, q_3 be the coords of which q_1, q_2 are ignorable

writing $T = T_1 + T_2 + T_3$ where $T_1 = \frac{1}{2}a_{33}\dot{q}_3^2$, $T_2 = \frac{1}{2}(a_{11}\dot{q}_1^2 + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^2)$

and $T_3 = \dot{q}_3(a_{13}\dot{q}_1 + a_{23}\dot{q}_2)$, $\frac{\partial T}{\partial\dot{q}_1} = p_1$ and $\frac{\partial T}{\partial\dot{q}_2} = p_2$ give

$$\left.\begin{aligned} a_{11}\dot{q}_1 + a_{12}\dot{q}_2 + a_{13}\dot{q}_3 &= p_1 \\ a_{12}\dot{q}_1 + a_{22}\dot{q}_2 + a_{23}\dot{q}_3 &= p_2 \end{aligned}\right\}$$

Solving these for $\dot{q}_1$ and $\dot{q}_2$, we get $A_{33}\dot{q}_1 = a_{22}p_1 - a_{12}p_2 + A_{13}\dot{q}_3$

$$A_{33}\dot{q}_2 = -a_{12}p_1 + a_{11}p_2 + A_{23}\dot{q}_3$$

where A_{ij} is cofactor of a_{ij} in $\|a_{ij}\|$. Substituting these values of $\dot{q}_1$ and $\dot{q}_2$ in T_2 and T_3 we find

coefficient of $p_1\dot{q}_3$ in $T_2 = A_{13}/A_{33}$ and in $T_3 = -A_{13}/A_{33}$

& similarly " $p_2\dot{q}_3$ in $T_2 = A_{23}/A_{33}$ " $T_3 = -A_{23}/A_{33}$

so that these cross terms vanish

Coefficients of p_1^2, p_1p_2 and p_2^2 come only from T_2 and these coeffts are respectively

$\frac{1}{2}a_{22}/A_{33}$, $-a_{12}/A_{33}$ and $\frac{1}{2}a_{11}/A_{33}$ ie hence the quadratic in p_1, p_2 is

$$\frac{1}{2A_{33}}\left(a_{22}p_1^2 - 2a_{12}p_1p_2 + a_{11}p_2^2\right).$$

Coefft of $\dot{q}_3^2$ comes from T_1, T_2 and T_3, and is equal to

$$\frac{1}{2}a_{33} + \frac{1}{2A_{33}^2}\left(a_{11}A_{13}^2 + 2a_{12}A_{13}A_{23} + a_{22}A_{23}^2\right) + \frac{1}{A_{33}}\left(a_{13}A_{13} + a_{23}A_{23}\right)$$

$$= \frac{1}{2A_{33}^2}\left\{a_{33}A_{33}^2 + a_{11}A_{13}^2 + 2a_{12}A_{13}A_{23} + a_{22}A_{23}^2 + 2a_{13}A_{13}A_{33} + 2a_{23}A_{23}A_{33}\right\}$$

$$= \frac{1}{2A_{33}^2}\Big[A_{33}(a_{11}A_{13} + a_{23}A_{23} + a_{33}A_{33}) + A_{13}(a_{11}A_{13} + a_{12}A_{23} + a_{13}A_{33})$$
$$+ A_{23}(a_{12}A_{13} + a_{22}A_{23} + a_{23}A_{33})\Big]$$

$= \frac{1}{2A_{33}^2}\cdot A_{33}\cdot\Delta$, the last two brackets vanishing

$$= \frac{1}{2}\frac{\Delta}{A_{33}}$$

$$\therefore\quad T = \frac{1}{2}\cdot\frac{\Delta}{A_{33}}\dot{q}_3^2 + \frac{1}{2A_{33}}\left(a_{22}p_1^2 - 2a_{12}p_1p_2 + a_{11}p_2^2\right) = T' + K.$$

General case: Let $q_1, q_2 \ldots q_r$ be ignorable, and $q_{r+1}, \ldots q_n$ non-ignorable. Denote the former by Latin indices $i, j, k, l \ldots$ & the latter by the Greek indices $\alpha, \beta, \gamma, \delta \ldots$ Thus $i, j, k \ldots$ take values from 1 to r and $\alpha, \beta, \gamma, \ldots$ from $(r+1)$ to n. The summation convention is also to be used. Thus

$$T = \tfrac{1}{2}a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta + \tfrac{1}{2}a_{ij}\dot{q}_i\dot{q}_j + a_{i\alpha}\dot{q}_i\dot{q}_\alpha = T_1 + T_2 + T_3 \quad \ldots (1)$$

Eqns $\frac{\partial T}{\partial \dot{q}_i} = p_i$ give $\quad a_{ij}\dot{q}_j + a_{i\alpha}\dot{q}_\alpha = p_i \quad \ldots (2)$

Instead of considering the determinant of all the a's, as in the two particular cases considered, let us consider only the determinant $\| a_{ij} \|$ and let $A_{ij} = (\text{minor of } a_{ij} \text{ with proper sign})/\Delta\ [\Delta = \|a_{ij}\|]$

Multiplying (2) by A_{ik} and using the summation convention & $a_{ij}A_{ik} = \delta_{jk}$, we have

$$\dot{q}_k = A_{ik}p_i - a_{i\alpha}A_{ik}\dot{q}_\alpha \quad \ldots (3)$$

1/9/61 – Above continued:– Using (3)

$$T_2 = \tfrac{1}{2}a_{kl}\dot{q}_k\dot{q}_l = \tfrac{1}{2}a_{kl}A_{ik}A_{jl}(p_i - a_{i\alpha}\dot{q}_\alpha)(p_j - a_{j\beta}\dot{q}_\beta)$$
$$= \tfrac{1}{2}A_{ik}\delta_{jk}(\quad)(\quad) = \tfrac{1}{2}A_{ij}(\quad)(\quad)$$

and $\quad T_3 = a_{i\alpha}\dot{q}_i\dot{q}_\alpha = a_{k\alpha}\dot{q}_k\dot{q}_\alpha = a_{k\alpha}A_{ik}\dot{q}_\alpha(p_i - a_{i\beta}\dot{q}_\beta)$

Cross terms come only from T_2 and T_3 and are given ~~by~~ equal to

$$-\tfrac{1}{2}A_{ij}a_{j\beta}p_i\dot{q}_\beta - \tfrac{1}{2}A_{ij}a_{i\alpha}p_j\dot{q}_\alpha + A_{ik}a_{k\alpha}p_i\dot{q}_\alpha$$
$$= -\tfrac{1}{2}A_{ik}a_{k\alpha}p_i\dot{q}_\alpha - \tfrac{1}{2}A_{ki}a_{k\alpha}p_i\dot{q}_\alpha + A_{ik}a_{k\alpha}p_i\dot{q}_\alpha = 0, \text{ using } A_{ki} = A_{ik}$$

Quadratic in p_i's is given only by T_2 and $= \frac{1}{2}A_{ij}p_ip_j \quad \ldots (4)$

Quadratic in $\dot{q}_\alpha$'s is given by $T_1 + T_2 + T_3$ and

$$= \tfrac{1}{2}a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta + \tfrac{1}{2}a_{kl}A_{ik}A_{jl}a_{i\alpha}a_{j\beta}\dot{q}_\alpha\dot{q}_\beta - a_{k\alpha}A_{ik}a_{i\beta}\dot{q}_\alpha\dot{q}_\beta$$
$$= \tfrac{1}{2}a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta + \tfrac{1}{2}\delta_{li}A_{jl}a_{i\alpha}a_{j\beta}\dot{q}_\alpha\dot{q}_\beta - a_{j\beta}A_{ij}a_{i\alpha}\dot{q}_\beta\dot{q}_\alpha \quad \left(\text{i.e. in last term } k \to j \text{ \& } \alpha \rightleftarrows \beta\right)$$
$$= \tfrac{1}{2}a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta + \tfrac{1}{2}A_{ij}a_{i\alpha}a_{j\beta}\dot{q}_\alpha\dot{q}_\beta - A_{ij}a_{i\alpha}a_{j\beta}\dot{q}_\alpha\dot{q}_\beta$$
$$= \tfrac{1}{2}(a_{\alpha\beta} - A_{ij}a_{i\alpha}a_{j\beta})\dot{q}_\alpha\dot{q}_\beta \quad \ldots (5)$$

$$\therefore\quad T = \tfrac{1}{2}A_{ij}p_ip_j + \tfrac{1}{2}(a_{\alpha\beta} - a_{i\alpha}a_{j\beta}A_{ij})\dot{q}_\alpha\dot{q}_\beta, \quad \text{adding (4) \& (5)}$$
$$= K + T'. \quad \ldots (6)$$

– ~~Did the explicit~~ Using the final expn (6), the particular cases of $n = 3$, and $r = 1$ and 2 follows immediately.

– Next did the explicit form of Lagrange's equations as in Whittaker §28, p. 39.

5/9/61 – Reduction of degrees of freedom from n to $n-1$, using the integral of energy, W. §42, p. 64.

7/9/61 — Worked out the example on p. 67 of Whittaker in full.

Here $L = \frac{1}{2} f(q_2)\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 - \psi(q_2)$ -- (1)

Integral of energy gives $\dot{q}_1 \frac{\partial L}{\partial \dot{q}_1} + \dot{q}_2 \frac{\partial L}{\partial \dot{q}_2} - L = h$, ie $f(q_2)\dot{q}_1^2 + \dot{q}_2^2 - \frac{1}{2} f(q_2)\dot{q}_1^2 - \frac{1}{2}\dot{q}_2^2 + \psi(q_2) = h$

ie $f(q_2)\dot{q}_1^2 + \dot{q}_2^2 = 2\{h - \psi(q_2)\}$ or $\{f(q_2) + q_2'^2\}\dot{q}_1^2 = 2\{h - \psi(q_2)\}$

giving $\dot{q}_1 = (2h - 2\psi)^{1/2}/(f + q_2'^2)^{1/2}$ —— (2)

$$L' = \frac{\partial L}{\partial \dot{q}_1} + \frac{\partial L}{\partial \dot{q}_2}\cdot\frac{\dot{q}_2}{\dot{q}_1} = f(q_2)\dot{q}_1 + \frac{\dot{q}_2^2}{\dot{q}_1} = \frac{f(q_2)\dot{q}_1^2 + \dot{q}_2^2}{\dot{q}_1} = \frac{2h - 2\psi}{\dot{q}_1}$$

$\therefore L' = (2h - 2\psi)^{\frac{1}{2}}(f + q_2'^2)^{1/2}$ -- (3)

as required, and the relation between q_1 and q_2 is given by the reduced Lagrangian eqn

$$\frac{d}{dq_1}\left(\frac{\partial L'}{\partial q_2'}\right) - \frac{\partial L'}{\partial q_2'} = 0 \quad \text{-- (4)}$$

L' does not contain q_1 & hence an integral of energy exists & this integral is the same as the solution of the single Lagrangian equation (4). Integral of energy is given by

$$q_2'\frac{\partial L'}{\partial q_2'} - L' = h' \quad \text{-- (5)}$$

which on using (3) and simplifying reduces to

$$\frac{f(2h - 2\psi)^{1/2}}{(f + q_2'^2)^{1/2}} = f\dot{q}_1 = h' \quad \text{—— (6), using (2)}$$

We can now express q_2 and q_1 as functions of t & thus solve the system by quadratures. From (6)

$h'q_2' = \{f^2(2h - 2\psi) - fh'^2\}^{1/2}$ or $\cancel{h'\dot{q}_2}$ or $h'\dot{q}_2 = \{\ \}^{1/2}\dot{q}_1 = \{\ \}^{1/2}\cdot h'/f$

ie $\frac{dq_2}{d\cancel{q_1}t} = \frac{\{\ \}^{1/2}}{f}$ ie $dt = \frac{f\,dq_2}{\{\ \}^{1/2}}$, where $\{\ \}$ is a fn of q_2 only & hence

on integration $q_2 = F(t)$. Hence $\dot{q}_1 = h'/f = \psi(t)$ leading to $q_1 = G(t)$.

— Did theory of vibrations Chap VII of W, up to $\frac{d}{dt}\left(\frac{\partial T}{\partial q_r}\right) = -\frac{\partial V}{\partial q_r}$ with T & V as quadratic forms of which T is +ve definite

8/9/61 — Continued the same and did the reduction to normal coordinates up to the first step:

$$T' = T - \frac{1}{2}a_{11}\dot{q}_1^2, \quad V' = V - \frac{1}{2}\lambda a_{11}q_1^2 \quad (W, p. 180)$$

12/9/61 — Did the reduction to normal coordinates completely.

19/9/61 — Went to class, but did no work due to a bad throat.

21/9/61 — Did Sylvester's proof of the reality of the roots of $\|\lambda a_{ij} - b_{ij}\| = 0$, § 78 of W; p. 183

[Here the result $\frac{\partial D}{\partial \alpha_{11}}\cdot\frac{\partial D}{\partial \alpha_{22}} - \left(\frac{\partial D}{\partial \alpha_{12}}\right)^2 = D\cdot\frac{\partial^2 D}{\partial \alpha_{11}\,\partial \alpha_{22}}$ on p. 184, is nothing but

the relation between minors of a determinant $\|a_{ij}\|$ and its <u>adjoint</u> determinant $\|A_{ij}\|$ viz

If M^* be a minor of $\|A_{ij}\|$ of order $\underline{k}$, then $M^* = D^{k-1}\tilde{M}$, where $\tilde{M}$ is the corresponding cofactor of the $\|a_{ij}\|$ and $D = |a_{ij}|$ and for $k = 2$, $M^* = D\tilde{M}$ & this is what is used here]

<u>22/9/61</u> – Did condition for +ve roots of determinantal equation as on pp. 184–85 of Whittaker – Gave alternative proofs of reality of roots & condn. for +ve roots as in Ferrar's Higher Algebra, ~~adjoi~~ (Chapter on Simultaneous redn. of two quad. forms to normal form), adjusting it to the notation in Whittaker – Ferrar's proof is as follows:-

If λ be a root of the det. eqn. $\|\lambda a_{ij} - b_{ij}\| = 0$. This means the existence of n numbers $z_1, z_2, \ldots z_n$, which may be assumed to be complex in general, such that the set of n-homogeneous equations $(\lambda a_{ij} - b_{ij}) z_j = 0 \quad (i = 1, 2, \ldots n)$ are satisfied. – – (1)

Putting $z_j = x_j + i y_j$ & hence $\bar{z}_j = x_j - i y_j$, we easily find

$$z_j \bar{z}_j = x_j^2 + y_j^2, \quad z_i \bar{z}_j + \bar{z}_i z_j = 2(x_i x_j + y_i y_j) \quad \text{-- (2)}$$

Multiplying eqns (1) by $\bar{z}_i$ and summing

$$(\lambda a_{ij} - b_{ij}) z_j \bar{z}_i = \lambda\{a_{ii}(x_i^2 + y_i^2) + 2a_{ij}(x_i x_j + y_i y_j)\}$$
$$- \{b_{ii}(x_i^2 + y_i^2) + 2b_{ij}(x_i x_j + y_i y_j)\} = 0$$

using notation $A(x, x)$ for quadratic form $\frac{1}{2} a_{ij} x_i x_j$, we can write the above as

$$\lambda\{A(x, x) + A(y, y)\} = B(x, x) + B(y, y) \quad \text{--- (3)}$$

$B(x,x)$ & $B(y,y)$ are real and $A(x,x) + A(y,y) > 0$ i.e. λ is real. If for eg $A(x,x) + A(y,y)$ were $= 0$, this conclusion could not have been arrived at.

Further since $A(x,x) + A(y,y) > 0$ (Kin. energy being +ve def. quad. form), the condn for λ being +ve is that $B(x,x) > 0$ i.e. potential energy also a +ve def. quad. form.

This proof in Ferrar appears superficial, while proof in Whittaker goes deep into the matter.

<u>26/9/61</u> – Principle of superposition, §79, p. 185 – stability of eqm. configuration, p. 186 – Effect of a new constraint on periods of a vibrating system, §81, p. 191.

<u>28/9/61</u> – Case when $(n-1)$ constraints are introduced & the stationary value of the resulting single vibration – vibrations about steady motion.

Instead of following Whittaker's notation, we can use that used in solving 1st part of Ex 5, p. 69.

i.e. take $T = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j + \frac{1}{2} a_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta + a_{i\alpha} \dot{q}_i \dot{q}_\alpha$ (q_α's non-ignored, q_i's ignored) where a_{ij}, $a_{\alpha\beta}$ and $a_{i\alpha}$ are fns of q_α's only.

$\frac{\partial T}{\partial \dot{q}_i} = \text{const} = p_i$ ~~for~~ and we have final form of T as in eqn (6), p. 6 of these notes viz

$$T = \frac{1}{2} A_{ij} p_i p_j + \frac{1}{2}(a_{\alpha\beta} - A_{ij} a_{i\alpha} a_{j\beta}) \dot{q}_\alpha \dot{q}_\beta$$

using ~~$\dot{q}_j$~~ $\dot{q}_j = A_{ij} p_i - A_{ij} a_{i\alpha} \dot{q}_\alpha = A_{ij}(p_i - a_{i\alpha} \dot{q}_\alpha)$

To perform the ignoration, introduce the Routhian fn $R = L - p_i \dot q_i = L - p_j \dot q_j$

$$= T - V - p_j A_{ij}(p_i - a_{i\alpha}\dot q_\alpha)$$

$$= \frac{1}{2} A_{ij} p_i p_j + \frac{1}{2}(a_{\alpha\beta} - A_{ij} a_{i\alpha} a_{j\beta})\dot q_\alpha \dot q_\beta - A_{ij} p_i p_j + p_j A_{ij} a_{i\alpha} \dot q_\alpha - V.$$

To consider small oscillations about steady motion, we shall expand all fns A_{ij}, $a_{i\alpha}$ and V by Taylor's theorem & retain terms only up to second order in q_α & $\dot q_\alpha$. Also we might take without loss of generality $q_\alpha = 0$ as the configuration of steady motion from which vibration starts. We can write, therefore (also omitting purely constant terms).

$$-V = V^0 + V^0_\alpha q_\alpha + V^0_{\alpha\beta} q_\alpha q_\beta$$ the (0) denoting constant terms

$$A_{ij} = A^0_{ij} + A^0_{ij\alpha} q_\alpha + A^0_{ij\alpha\beta} q_\alpha q_\beta, \text{ etc.}$$

$\cancel{a_{\alpha\beta} =}$

Hence $R = -\frac{1}{2} A_{ij} p_i p_j + \frac{1}{2}(a_{\alpha\beta} - A_{ij} a_{i\alpha} a_{j\beta})\dot q_\alpha \dot q_\beta + p_j A_{ij} a_{i\alpha}\dot q_\alpha - V$

$$= -\frac{1}{2} p_i p_j (A^0_{ij\alpha} q_\alpha + A^0_{ij\alpha\beta} q_\alpha q_\beta)$$
$$+ \frac{1}{2}(a^0_{\alpha\beta} - A^0_{ij} a^0_{i\alpha} a^0_{j\beta})\dot q_\alpha \dot q_\beta + p_j (A^0_{ij} + A^0_{ij\beta} q_\beta)(a^0_{i\alpha} + a^0_{i\alpha\gamma} q_\gamma)\dot q_\alpha$$
$$+ \cancel{V^0} + V^0_\alpha q_\alpha + V^0_{\alpha\beta} q_\alpha q_\beta$$

Neglecting the purely constant terms,

$$R = -\frac{1}{2} p_i p_j (A^0_{ij\alpha} q_\alpha + A^0_{ij\alpha\beta} q_\alpha q_\beta) + \frac{1}{2}(a^0_{\alpha\beta} - A^0_{ij} a^0_{i\alpha} a^0_{j\beta})\dot q_\alpha \dot q_\beta$$
$$+ p_j A^0_{ij} a^0_{i\alpha}\dot q_\alpha + p_j (A^0_{ij\beta} a^0_{i\alpha} q_\beta \dot q_\alpha + A^0_{ij} a^0_{i\alpha\gamma} q_\gamma \dot q_\alpha)$$
$$+ V^0_\alpha q_\alpha + V^0_{\alpha\beta} q_\alpha q_\beta$$

Of these, the terms the terms linear in q_α alone not associated with $\dot q_\alpha$ can be omitted since we have assumed $q_\alpha = 0$. Also terms linear in $\dot q_\alpha$ alone not associated with q_α can be omitted since ~~in~~ they vanish in virtue of $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot q_\alpha}\right) - \frac{\partial R}{\partial q_\alpha} = 0$. Thus we are left with terms in $q_\alpha q_\beta$, $\dot q_\alpha \dot q_\beta$ and $\dot q_\alpha q_\beta$ with constant coefficients. These are

$$(V^0_{\alpha\beta} - \frac{1}{2} p_i p_j A^0_{ij\alpha\beta}) q_\alpha q_\beta, \text{ and } \frac{1}{2}(a^0_{\alpha\beta} - A^0_{ij} a^0_{i\alpha} a^0_{j\beta})\dot q_\alpha \dot q_\beta, \text{ and}$$

$$p_j (A^0_{ij\beta} a^0_{i\alpha} + A^0_{ij} a^0_{i\alpha\beta})\dot q_\alpha q_\beta \quad (\text{Changing } \gamma \to \beta)$$

Thus we can finally write $R = \frac{1}{2} b_{\alpha\beta} q_\alpha q_\beta + \frac{1}{2} c_{\alpha\beta}\dot q_\alpha \dot q_\beta + d_{\alpha\beta}\dot q_\alpha q_\beta$

where, from the way they are formed $b_{\alpha\beta} = b_{\beta\alpha}$, $c_{\alpha\beta} = c_{\beta\alpha}$ but $d_{\alpha\beta} \neq d_{\beta\alpha}$ and these last are called the gyroscopic terms. They don't appear in the expns for T and V for the case of vibns about eqm and their presence makes it impossible to transform R to normal coords by linear point transformations as in that case. It can however be shown that by actual integration of these eqns $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot q_\alpha}\right) - \frac{\partial R}{\partial q_\alpha} = 0$, which is rather complicated, redn to normal coords is possible & we can derive vibns about steady motion.

(*) Even when $\underline{t}$ is explicitly contained in L we have $\int \left(\frac{\partial L}{\partial t} . \delta t\right) dt$

$$= \left[\left\{\int \frac{\partial L}{\partial t}\, dt\right\} . \delta t\right]_A^B = 0 .$$

Hence ~~Action~~ Hamiltons Principle is also true when $\underline{t}$ is explicitly contained.

(†) In deriving (1) use $\frac{\partial \dot{x}_r}{\partial \dot{q}_k} = \frac{\partial x_r}{\partial q_k}$. from $m_r \dot{x}_r^2 = a_{kl} \dot{q}_k \dot{q}_l$

$$m_r \dot{x}_r \frac{\partial \dot{x}_r}{\partial \dot{q}_k} = a_{kl} \dot{q}_l$$

$$\text{ie } m_r \dot{x}_r \frac{\partial x_r}{\partial q_k} = a_{kl} \dot{q}_l$$

$$\& \therefore m_r \frac{\partial x_r}{\partial q_k} . \frac{\partial \dot{x}_r}{\partial \dot{q}_l} = a_{kl} \quad \text{ie } a_{kl} = m_r \frac{\partial x_r}{\partial q_k} . \frac{\partial x_r}{\partial q_l}.$$

– Also given the two following for tutorial work: – Write short notes on

(i) the process of ignoration of coordinates in a holonomic conservative system;

(ii) the theory of small vibrations of a dynamical system about a position of equilibrium

3/10/61 – Hamilton's Principle and Principle of least action in the general case with $\delta t \neq 0$ i.e. $L(q_r, \dot{q}_r, t)$.

5/10/61 – Simpler proofs of the two principles when t is not explicitly involved in L (*)

(1)

(i) $\delta \int L\,dt = \int \delta L \cdot dt = \int \left(\frac{\partial L}{\partial q_r}\delta q_r + \frac{\partial L}{\partial \dot{q}_r}\delta \dot{q}_r \right) dt$ [the term in $\frac{\partial L}{\partial t}$ does not appear since $\delta t = 0$ at A & B] (*)

$$= \int \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)\delta q_r + \frac{\partial L}{\partial \dot{q}_r}\frac{d}{dt}(\delta q_r) \right\} dt = \int \frac{d}{dt}\left\{ \frac{\partial L}{\partial \dot{q}_r}\delta q_r \right\} dt$$

$$= \left[\frac{\partial L}{\partial \dot{q}_r}\delta q_r \right]_A^B = 0 \text{ since } \delta q_r = 0 \text{ at } A \,\&\, B$$

(ii) $\delta \int \left(\dot{q}_r \frac{\partial L}{\partial \dot{q}_r} \right) dt$ for orbits with same end points & same total energy

using $\dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L = h$, the above $= \int \delta\left(\dot{q}_r \frac{\partial L}{\partial \dot{q}_r} \right) dt = \int \delta h \cdot dt + \int \delta L \cdot dt$

$$= 0 + \delta \int L\,dt = 0 \text{ from (i).}$$

(*) Action-Principle can also be written as $\delta' \int T\,dt = 0$ for $\dot{q}_r \frac{\partial L}{\partial \dot{q}_r} = \dot{q}_r \frac{\partial T}{\partial \dot{q}_r} = 2T$

since T is homogeneous & of second degree in $\dot{q}_r$

(2) Case of 2 degrees of freedom – orbits as geodesics on the surface $ds^2 = (h - V)(a_{11}dq_1^2 + 2a_{12}dq_1 dq_2 + a_{22}dq_2^2)$.

Ex 1, p. 254 : $T = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2)$, $V = mgq_2$

Line element of surface on which trajectories are geodesics, given by $ds^2 = (h - V)(a_{11}dq_1^2 + 2a_{12}dq_1 dq_2 + a_{22}dq_2^2)$ is here, $ds^2 = (h - mgq_2)\frac{1}{2}m(dq_1^2 + dq_2^2)$, and $ds^2 = E(du^2 + dv^2)$ is the line element of a ~~ruled~~ surface of revolution.

6/10/61 – Least curvature Principle of Gauss & Hertz proved using Cartesian coords – Expression for curvature of the path in terms of generalised coordinates done up to $\ddot{x}_r - \ddot{x}_{r0} = \frac{\partial x_r}{\partial q_k}(\ddot{q}_k - \ddot{q}_{k0})$, p. 257. W.

10/10/61 – For generalised coordinates, the curvature is defined in a different way using the relation

$$C = C_{min} + x^2, \left[\Sigma m_r \left\{ \left(\ddot{x}_r - \frac{X_r}{m_r}\right)^2 + .. \right\} = \Sigma m_r \left\{ \left(\ddot{x}_{r0} - \frac{X_r}{m_r}\right)^2 + .. \right\} + \Sigma m_r \{ (\ddot{x}_r - \ddot{x}_{r0})^2 + .. \right.$$

so that x^2 itself i.e. $\Sigma m_r \{ (\ddot{x}_r - \ddot{x}_{r0})^2 + \cdots + \cdot \}$ is defined as the curvature – obtained expression for this in generalised coordinates using $\Sigma m_r \dot{x}_r^2 = a_{kl}\dot{q}_k \dot{q}_l$ giving $a_{kl} = m_r \frac{\partial x_r}{\partial q_k} \cdot \frac{\partial x_r}{\partial q_l}$ --(1) (†)

from $\ddot{x}_r = \frac{\partial x_r}{\partial q_k}\ddot{q}_k + \frac{\partial^2 x_r}{\partial q_k \partial q_l}\dot{q}_k \dot{q}_l$ (which is obtained from $\dot{x}_r = \frac{\partial x_r}{\partial q_k}\dot{q}_k$), $\ddot{x}_r - \ddot{x}_{r0} = \frac{\partial x_r}{\partial q_k}(\ddot{q}_k - \ddot{q}_{k0})$ --(2)

Writing $S_k = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} - X_r \frac{\partial x_r}{\partial q_k}$ and noting $S_{k0} = 0$, $S_k = S_k - S_{k0} = a_{kl}(\ddot{q}_l - \ddot{q}_{l0})$ --(3)

from (3) $\ddot{q}_k - \ddot{q}_{k0} = \frac{1}{D}A_{kl}S_l$ [A_{kl} = Cofactor of a_{kl} in det. $D = \| a_{ij} \|$] & hence (2) gives

$$\ddot{x}_r - \ddot{x}_{r0} = \frac{1}{D}\frac{\partial x_r}{\partial q_k} A_{kl} S_l$$

finally leading to Curvature, $C = \Sigma m_r(\ddot{x}_r - \ddot{x}_{r0})^2 = \frac{1}{D} A_{ij} S_i S_j$ – Next did

Appell's eqns for holonomic case: From the eqns $m_i \ddot{x}_i = X_i$, $m_i \ddot{y}_i = Y_i$, $m_i \ddot{z}_i = Z_i$, one has

~~for generalised coordinates~~, $m_i\{\ddot{x}_i \frac{\partial x_i}{\partial q_r} + \dots + \dots\} = (X_i \frac{\partial x_i}{\partial q_r} + \dots + \dots) = Q_r$ --(1)

~~& the right hand side~~ Q_r is the R.H.S of Lagrange's eqns $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r$

For $x_i = x_i(q_r, t)$, it can be shown that

$$\frac{\partial x_i}{\partial q_r} = \frac{\partial \dot{x}_i}{\partial \dot{q}_r} = \frac{\partial \ddot{x}_i}{\partial \ddot{q}_r},$$ and hence (1) can be written

$$m_i\left(\ddot{x}_i \frac{\partial \ddot{x}_i}{\partial \ddot{q}_r} + \dots + \dots\right) = Q_r \quad \text{--(2)}$$

If $S = \frac{1}{2}\Sigma m_i(\ddot{x}_i^2 + \ddot{y}_i^2 + \ddot{z}_i^2)$, expressed in terms of q_r, $\dot{q}_r$ and $\ddot{q}_r$, then

$$\frac{\partial S}{\partial \ddot{q}_r} = \frac{\partial S}{\partial \ddot{x}_i}\cdot\frac{\partial \ddot{x}_i}{\partial \ddot{q}_r} + \dots + \dots = m_i\left(\ddot{x}_i \frac{\partial \ddot{x}_i}{\partial \ddot{q}_r} + \dots + \dots\right) = Q_r \text{ using (2)}$$

Hence we have Appell's eqns $\frac{\partial S}{\partial \ddot{q}_r} = Q_r$ $(r = 1, 2, \dots n)$

11/10/61 – Although I had no lecture, met the class for some time and discussed the two topics given for tutorial work.

12/10/61 – Examples worked

(1) Ex 1, p. 248. W. [Here prove the (*) part of both Principles being true even when t is explicitly contained in L]

To prove that $\delta\int\left(\dot{q}_r \frac{\partial L}{\partial \dot{q}_r} + t\frac{dh}{dt}\right)dt = 0$, where $\dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L = h \neq$ ~~not~~ constant $\neq$ ~~total~~ energy.

~~L.H.S. = δ~~ This is the case when L contains t explicitly. For, in this case

$$\frac{dL}{dt} = \frac{\partial L}{\partial q_r}\cdot\dot{q}_r + \frac{\partial L}{\partial \dot{q}_r}\ddot{q}_r + \frac{\partial L}{\partial t} = \frac{\partial L}{\partial \dot{q}_r}\ddot{q}_r + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right)\dot{q}_r + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\dot{q}_r \frac{\partial L}{\partial \dot{q}_r}\right) + \frac{\partial L}{\partial t}$$

ie $\frac{d}{dt}\left(\dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L\right) = -\frac{\partial L}{\partial t} \neq 0$ ie $\dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L = h \neq$ const. $\neq T+V$.

The

But Hamilton's Principle is true even for the case where L contains t explicitly

Now the L.H.S $= \delta\int\left(h + L + t\frac{dh}{dt}\right)dt = \delta\int L\,dt + \delta\int\left(h + t\frac{dh}{dt}\right)dt = \delta\int\left(h + t\frac{dh}{dt}\right)dt$

(the first term being zero from Hamilton's Principle)

$$= \delta\int\frac{d}{dt}(ht)\,dt = \delta(ht)\Big]_A^B = t\,\delta h\Big]_A^B + h\,\delta t\Big]_A^B$$

If $\delta h = 0$ at A & B and $\delta t = 0$ at A & B, this expn $= 0$. Hence $\delta\int\left(\dot{q}_r \frac{\partial L}{\partial \dot{q}_r} + t\frac{dh}{dt}\right)dt = 0$ is the generalisation of the Action Principle for this case, with $\delta h = 0$ at A & B.

(2) Ex 2, p. 248, W.

Reduction of system when integral of energy exists, as in §42, W, consists of introducing instead of L, the Lagrangian L′ defined by $L'(q_2', q_3', \dots, q_n', q_1, \dots q_n) = \sum_{r=1}^{n} \frac{\partial L}{\partial \dot{q}_r}\frac{\dot{q}_r}{\dot{q}_1}$, where in the

R.H.S, $\dot{q}_1$ is obtained as a function of $(q_2', \ldots q_n', q_1 \ldots q_n)$ from the equation of energy $\dot{q}_r \frac{\partial L}{\partial \dot{q}_r} - L = h$, after replacing therein $\dot{q}_r$ by $\dot{q}_1 q_r'$ $(r = 2, \ldots n)$ with $q_r' = dq_r/dq_1$. ~~Hence the Hamilton's Principle for the reduced system is given by $\delta \int L' dq_1 = 0$, the reducing system~~ The eqns of motion of the reduced system are given by $\frac{d}{dq_1}\left(\frac{\partial L'}{\partial q_r'}\right) - \frac{\partial L'}{\partial q_r} = 0$ $(r = 2, 3, \ldots n)$ ie q_1 plays the role of time.

Hence the Hamilton's Principle for the reduced system is given by

$$\delta \int L' dq_1 = 0 \text{ or } \delta \int L' \dot{q}_1 \, dt = 0$$

~~ie $\int \delta L' dq_1 = 0$ or $\int \dot{q}_1 \delta L' dt = 0 = \int \left(\frac{\partial L'}{\partial q_r'}\delta q_r' + \frac{\partial L'}{\partial q_r}\right)\dot{q}_1 dt = 0$~~

~~and using $L' = \frac{\partial L}{\partial \dot{q}_r}\frac{\dot{q}_r}{\dot{q}_1}$ this reduces to $\int \partial$~~

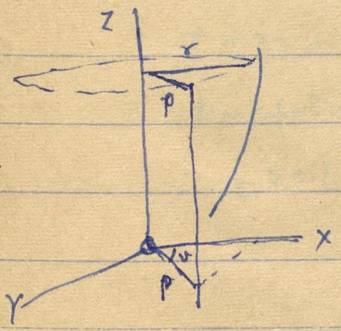
~~or $\int \delta L' dq_1 = 0$~~

& using $L' = \frac{\partial L}{\partial \dot{q}_r} \cdot \frac{\dot{q}_r}{\dot{q}_1}$, this reduces to $\delta \int \left(\frac{\partial L'}{\partial \dot{q}_r} \dot{q}_r\right) dt = 0$

which is the Action Principle for the original system.

(3) Ex 1, p. 256, W. – This is partly indicated in p. 10, where line-element of surface is given by

$$ds^2 = (h - mg\,q_2) \cdot \tfrac{1}{2} m (dq_1^2 + dq_2^2) = E\,(du^2 + dv^2),$$ and it is mentioned that this is line element of a surface of revolution. This is to be shown.



The eqn of the curve generating the rule surface in any position is $z = f(p)$

Let v = angle that the plane of curve makes with xz-plane in any posn.

$x = p\cos v$, $y = p \sin v$, $z = f(p)$ [Eisenhart, p. 107

$$ds^2 = [1 + f'^2(p)]\,dp^2 + p^2 dv^2$$

Putting $u = \int \frac{1}{p}\sqrt{1 + f'^2}\, dp$, the line-element becomes

$ds^2 = E\,(du^2 + dv^2)$, where E is a function of u

because $du = \frac{1}{p}\sqrt{1 + f'^2}\, dp$ & $(1 + f'^2)\,dp^2 = p^2 du^2$ and $ds^2 = p^2 (du^2 + dv^2)$

and $p^2 = f(u) = E$.

(4) Ex. 2, p. 256. W – For central force in a plane, $T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$, $V = \phi(r)$, because attractive force is given as $\phi'(r) = \partial V/\partial r$ $[-\phi'(r) = -\partial V/\partial r]$.

$ds^2 = (h - V)(a_{11} dq_1^2 + 2a_{12} dq_1 dq_2 + a_{22} dq_2^2)$, becomes in this case

$$ds^2 = \{h - \phi(r)\}(dr^2 + r^2 d\theta^2) \quad \text{—(1)}$$

To transform into line-element corresponding to s on a surface of revolution $z = f(p)$ viz. as given by Ex. above, $ds^2 = (1 + f'^2)\,dp^2 + p^2 dv^2$ —(2)

we make the transformations suggested viz $p^2 = r^2\{h - \phi(r)\}$ and $1 + f'^2 = \frac{p^2}{r^2}\left(\frac{dr}{dp}\right)^2$, which transform (1) into $ds^2 = \frac{p^2}{r^2}(dr^2 + r^2 d\theta^2) = \frac{p^2}{r^2}dr^2 + p^2 d\theta^2 = (1 + f'^2)\,dp^2 + p^2 d\theta^2$, which is exactly of the form (2) with $\theta = v$. Hence the result.

Ex. p. 259. If components of velocity of a particle of the rigid body along dirns of the principal axes of inertia at the fixed point O, be $\dot{x}, \dot{y}, \dot{z}$, then $\dot{x} = z\omega_2 - y\omega_3$, $\dot{y} = x\omega_3 - z\omega_1$, $\dot{z} = y\omega_1 - x\omega_2$

$$\therefore \ddot{x} = \dot{z}\omega_2 + z\dot{\omega}_2 - \dot{y}\omega_3 - y\dot{\omega}_3 = (y\omega_1 - x\omega_2)\omega_2 + z\dot{\omega}_2 - (x\omega_3 - z\omega_1)\omega_3 - y\dot{\omega}_3$$

$$= -x(\omega_2^2 + \omega_3^2) + y(\omega_1\omega_2 - \dot{\omega}_3) + z(\omega_3\omega_1 + \dot{\omega}_2)$$

A similar exprs can be found for $\ddot{y}$ and $\ddot{z}$. From these expressions we have to form $S = \Sigma \frac{1}{2} m(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2)$, the summation being taken over the whole body, and express it in terms of $\ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}_3$; $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$ and $\theta_1, \theta_2, \theta_3$ where ~~the~~ $\omega_1 = \dot{\theta}_1, \omega_2 = \dot{\theta}_2, \omega_3 = \dot{\theta}_3$. Obviously θ_1, θ_2 & θ_3 do not appear in the expression for S, but only $\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$ and $\ddot{\theta}, \ddot{\phi}, \ddot{\psi}$. Next, we have to write down Appell's eqns in the form $\frac{\partial S}{\partial \ddot{\theta}_1} = L$, $\frac{\partial S}{\partial \ddot{\theta}_2} = M$, $\frac{\partial S}{\partial \ddot{\theta}_3} = N$, or

$$\frac{\partial S}{\partial \dot{\omega}_1} = L, \quad \frac{\partial S}{\partial \dot{\omega}_2} = M, \quad \frac{\partial S}{\partial \dot{\omega}_3} = N$$

Hence in writing the above exprn for S we need only retain the terms which contain $\dot{\omega}_1, \dot{\omega}_2$ & $\dot{\omega}_3$, and discard the others. ~~Thus we can write~~ Further, in finding $\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2$, we can ~~$\ddot{x}^2 = y^2\dot{\omega}_3^2 + z^2\dot{\omega}_2^2 - 2yz$~~ discard terms in yz since Σmyz, Σmzx, & Σmxy being the products of inertia vanish since the axes are the principal axes. Hence we can retain only terms in x^2, y^2 & z^2. Thus

$$\frac{1}{2}\Sigma m(\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2) = \frac{1}{2}\Sigma x^2\{(\omega_2^2 + \omega_3^2)^2 + (\omega_1\omega_2 + \dot{\omega}_3)^2 + (\omega_3\omega_1 - \dot{\omega}_2)^2\}$$

$$+ y^2\{\quad\} + z^2\{\quad\}$$

$$= \frac{1}{2}\dot{\omega}_1^2 \Sigma m(y^2 + z^2) + \cdots + \cdots$$

$$- \dot{\omega}_1\omega_2\omega_3 \Sigma m(z^2 - y^2) + \cdots + \cdots$$, discarding terms not containing $\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3$

$$= \frac{1}{2}A\dot{\omega}_1^2 + \frac{1}{2}B\dot{\omega}_2^2 + \frac{1}{2}C\dot{\omega}_3^2 - \omega_2\omega_3\dot{\omega}_1(B - C) - \omega_3\omega_1\dot{\omega}_2(C - A) - \omega_1\omega_2\dot{\omega}_3(A - B).$$

Hence $\frac{\partial S}{\partial \dot{\omega}_1} = L$ gives $A\dot{\omega}_1 - (B - C)\omega_2\omega_3 = L$

$\frac{\partial S}{\partial \dot{\omega}_2} = M$ " $B\dot{\omega}_2 - (C - A)\omega_3\omega_1 = M$

$\frac{\partial S}{\partial \dot{\omega}_3} = N$ " $C\dot{\omega}_3 - (A - B)\omega_1\omega_2 = N$

which are Euler's well known eqns for rotation of a rigid body about a fixed point O under external forces giving moments about the pr. axes at O.

Examples on p. 261 (W) at end of Ch. IX.

(1) The orbits in the first case are geodesics on the surface whose line element is given by

$$ds^2 = (h - V)(E\,du^2 + 2F\,du\,dv + G\,dv^2)$$

and in the second case they are geodesics on surface whose line-element is given by

$$ds^2 = (h' - \tfrac{1}{V})V(E\,du^2 + 2F\,du\,dv + G\,dv^2) = (h'V - 1)(E\,du^2 + 2F\,du\,dv + G\,dv^2)$$

The ratio of the two $ds^2 = (h-V)/(h'V-1)$ ie ratio of the two first fundamental forms is a funct of u, v ie the two surfaces constitute a conformal transformation of one on the other. Since the geodesics depend only on the first fundamental form, knowledge of geodesics on one surface, leads to knowledge of geodesics on the other.

Ex(2) Using the Principle of Least Action, the trajectories are given by $\delta\int ds = 0$ where $ds^2 = (h-U)\,a_{ij}\,dq_i\,dq_j$ and $ds^2 = (h'-V)\,b_{ij}\,dq_i\,dq_j$ in the two systems ie they are geodesics given by the eqns

$$\frac{d^2 q_i}{ds^2} + \Gamma^i_{jk}\frac{dq_j}{ds}\frac{dq_k}{ds} = 0$$

where Γ^i_{jk} are the Christoffel three-index symbols formed out of the metric tensors $g_{ij} = (h-U)a_{ij}$ or $g'_{ij} = (h'-V)b_{ij}$ in the two systems. Hence if the trajectories are to be the same, we should have $\Gamma^i_{jk} = \Gamma'^i_{jk}$. This means that $(h-U)a_{ij}$ and $(h'-V)b_{ij}$ should be identical fns of the q's, or invariant forms. If for example a_{ij} & b_{ij} are constants, this would mean a linear transformation between U and V viz $V = \alpha U + \beta$, where α, β are constants, since h & h' are constants. In general, this invariance is obtained by $V = (\alpha U + \beta)/(\gamma U + \delta)$ & $b_{ij} = \sqrt{(\gamma U + \delta)}\,a_{ij}$ ~~An identical~~ Alternatively the same result would be obtained by postulating invariance of ds^2 ie $(h-U)\,a_{ij}\,dq_i\,dq_j = (h'-V)\,b_{ij}\,dq_i\,dq_j$ which is obtained by the bilinear or Mobius transformation $V = (\alpha U + \beta)/(\gamma U + \delta)$, and $b_{ij}\,dq_i\,dq_j = \sqrt{(\gamma U + \delta)}\,a_{ij}\,dq_i\,dq_j$.

Ex.3 The trajectories in the untransformed system are given by $\delta\int ds = 0$ where $ds^2 = (h-V)(dx^2+dy^2)$. Under the transformation, $dx = \frac{\partial\phi}{\partial X}dX + \frac{\partial\phi}{\partial Y}dY$ and $dy = \frac{\partial\psi}{\partial X}dX + \frac{\partial\psi}{\partial Y}.dY$, so that

$$dx^2 + dy^2 = \left[\left(\frac{\partial\phi}{\partial X}\right)^2 + \left(\frac{\partial\phi}{\partial Y}\right)^2\right](dX^2 + dY^2),$$ using the relations $\frac{\partial\phi}{\partial X} = \frac{\partial\psi}{\partial Y}$ and $\frac{\partial\phi}{\partial Y} = -\frac{\partial\psi}{\partial X}$ for conjugate functions. Hence the transformed ds^2 is given by

$$ds'^2 = (h-V)\left\{\left(\frac{\partial\phi}{\partial X}\right)^2 + \left(\frac{\partial\phi}{\partial Y}\right)^2\right\}(dX^2+dY^2)$$
$$= (0 - V')(dX^2 + dY^2) \text{ where } V' = \{V(\phi,\psi) - h\}\left\{\left(\frac{\partial\phi}{\partial X}\right)^2 + \left(\frac{\partial\phi}{\partial Y}\right)^2\right\}$$

which proves the result.

Ex. 4. We have $\frac{dV}{dt} = \Sigma\frac{\partial V}{\partial x}\dot{x}$, the Σ referring to terms in y & z,

$$\frac{d^2V}{dt^2} = \Sigma\frac{\partial^2 V}{\partial x^2}\dot{x}^2 + \Sigma\frac{\partial V}{\partial x}\ddot{x}.$$ Denoting by $S = \frac{1}{2}\Sigma m\ddot{x}^2$, as in Appell's eqns

$$2\frac{d^2V}{dt^2} + 2S = 2\Sigma\frac{\partial^2 V}{\partial x^2}\dot{x}^2 + 2\Sigma\frac{\partial V}{\partial x}\ddot{x} + \Sigma m\ddot{x}^2$$

$$\therefore 2\frac{d^2V}{dt^2} + 2S - \Sigma\frac{1}{m}\left\{\left(m\ddot{x} + \frac{\partial V}{\partial x}\right)^2\right\} = 2\Sigma\frac{\partial^2 V}{\partial x^2}\dot{x}^2 + \cancel{2\Sigma\frac{\partial V}{\partial x}\ddot{x}} + \cancel{\Sigma m\ddot{x}^2}$$
$$- \cancel{\Sigma m\ddot{x}^2} - \cancel{2\Sigma\frac{\partial V}{\partial x}\ddot{x}} - \Sigma\frac{1}{m}\left(\frac{\partial V}{\partial x}\right)^2$$

where the terms in $\ddot{x}$ cancel out;

Now $\frac{d^2T}{dt^2} = -\frac{d^2V}{dt^2}$ since $T + V = h$. Hence $2\frac{d^2V}{dt^2} + 2S = -\left\{2\frac{d^2T}{dt^2} - 2S\right\}$

* ~~terms like $\gamma_{rs}\dot{q}_s$ are constants for steady motion & can be neglected both from~~

$\cancel{\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_r}\right) \text{ and } \frac{\partial R}{\partial q_r}}$

For Steady motion non-ignorable coords have ~~steady~~ constant values

and velocities corresponding to ignorable coords have constant values. In the R

obtained after ignoration, we have the same posn as in the case of absolute eqm

problem with $\ddot{q}_r = 0$, $\dot{q}_r = 0$, $q_r = \alpha_r$ ($r = 1, 2 \ldots n$ for non-ignorable coords)

i.e. purely $\frac{\partial R}{\partial q_r} = 0$ where $\dot{q}_r$ is put $= 0$. just as in p. 177. W.

Thus in $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_r}\right) = \frac{d}{dt}(\dot{q}_s) + \frac{d}{dt}\left(\frac{1}{2}\beta_{rs} q_r q_s\right) + \frac{d}{dt}\left(\cancel{\gamma_{sr}\dot{q}_s}\right)(\gamma_{sr} q_s)$

all $\ddot{q}$'s & $\dot{q}$'s are to be put equal to zero & hence $= 0$.

& in $\frac{\partial R}{\partial q_r} = \frac{\partial}{\partial q_r}\left(\frac{1}{2}\beta_{rs} q_r q_s\right) + \gamma_{rs}\dot{q}_s$ $\cancel{= \frac{\partial}{\partial \theta}}$ with $\dot{q}_s = 0$

$\therefore \quad \frac{d^2T}{dt^2} - S = -\frac{1}{2}\left\{2\frac{d^2V}{dt^2} + 2S\right\} = -\frac{1}{2}\sum m\left\{\left(m\ddot{x} + \frac{\partial V}{\partial x}\right)^2\right\}$ + terms containing only x & $\dot{x}$

Hence for given h, and given x, $\dot{x}$, etc, $\frac{d^2T}{dt^2} - S$ has the maximum value when

$\sum m\left\{\left(m\ddot{x} + \frac{\partial V}{\partial x}\right)^2\right\} = 0$ ie $m\ddot{x} = -\partial V/\partial x$, etc ie for the actual motion.

Examples on vibrations – Chap VII.

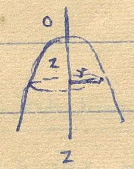
(A) Vibrations about steady motion.

(1) Ex.(ii) p. 204. W – Worked Example partly done.

24/10/61 – missed the class thanks to Rajasthan setting work.

26/10/61 – Ex.(ii) p. 204 of W worked in full – That eqm position is given by $\left(\frac{\partial R}{\partial r}\right)_{r=a} = 0$ is shown by considering the general case of $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0$ & for eqm $\dot{q}$'s & $\ddot{q}$'s $= 0$ ie given by $\frac{\partial L}{\partial q_r} = 0$ – For steady motion with $R = \frac{1}{2}(\alpha_{rs}\dot{q}_r\dot{q}_s) + \frac{1}{2}\beta_{rs}q_r q_s + \gamma_{rs}q_r\dot{q}_s$, $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_r}\right)$ gives terms in $\ddot{q}$ only* (α, β, γ's being const) & steady motion means constant values of $\dot{q}$ ie $\ddot{q} = 0$. Hence here also for eqm steady motion $\partial R/\partial r = 0$ –

For particular case of paraboloid of revolution; the eqn is taking OZ vertically downwards



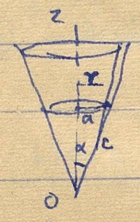
$4az = r^2$ (plane geometry eqn of $y^2 = 4ax$)

or $z = r^2/2l$ (l = semi-latus rectum)

$f(r) = r^2/2l$, $f'(r) = r/l$, $f''(r) = 1/l$. Substituting $f'(a) = a/l$, $f''(a) = 1/l$

in the general formula $\frac{2\pi}{\sqrt{g}}\left\{\frac{1 + f'^2(a)}{f''(a) + 3f'(a)/a}\right\}^{\frac{1}{2}}$ for the period, we get the period $= \pi\left(\frac{l^2 + a^2}{gl}\right)^{1/2}$.

(2) Ex. 28, p. 212 – Whittaker unworked



Using the same eqns for L, the integral corresponding to the ignorable coordinate is $r^2\dot{\theta} = k$ & for description of a circle of radius $\underline{a}$ condn is $k^2 = ga^3f'(a)$. For the cone $z = r\cot\alpha$, $f'(a) = \cot\alpha$, ie $k = \sqrt{ga^3\cot\alpha}$ ie ~~initial~~ value of $\dot{\theta}$ for description of circle $\underline{a} = k/a^2$ or required horizontal velocity ($v = r\dot{\theta}$) is $a\dot{\theta} = k/a$ ie reqd velocity $v = \frac{1}{a}\sqrt{ga^3\cot\alpha}$ or $v^2 = ga\cot\alpha$.

Using $f''(r) = 0$, the period, using the general formula of ~~Ex and~~ Ex.(ii) p. 204 of W

$= \frac{2\pi}{\sqrt{g}} \cdot \left\{\frac{1 + \cot^2\alpha}{3\cot\alpha/a}\right\}^{1/2}$ Putting $a = c\sin\alpha$, this reduces to $\frac{2\pi}{\sqrt{g}}\left(\frac{c\sec\alpha}{3}\right)$

ie period of a pendulum of length $\frac{1}{3}c\sec\alpha$.

(3) Ex. 1. generalised for the case where gravity is not the only force, but the force field is given by ~~V(~~ $V = \phi(r, z)$, and the particle is describing a circle $r = a$, $z = b$. & $\partial V/\partial z = 0$ for $r = a, z = b$. Take $m = 1$ for the particle

$$T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2), \quad L = T - V$$

θ is ignorable $\frac{\partial L}{\partial \dot{\theta}} =$ const gives $r^2\dot{\theta} = k$, $R = T - V - \frac{\partial L}{\partial \dot{\theta}} \cdot \dot{\theta}$ $\left[R = L - \sum \dot{q}_r \frac{\partial L}{\partial \dot{q}_r}\right.$

$$R = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\dot{z}^2 - \phi(r, z) - \frac{1}{2}k\dot{\theta} = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\dot{z}^2 - \phi(r, z) - \frac{k^2}{2r^2}$$

* ~~missed classes~~ missed class on 21st thanks to attending Prof. Martynov's lecture.

(f) Here $\frac{du}{dt} = -\mu x$, $\frac{dv}{dt} = -\mu y$, $u = \frac{dx}{dt}$, $v = \frac{dy}{dt}$.

It is enough if we prove $\frac{d}{dt}(u\delta x - x\delta u) = 0$

$$\text{L.H.S} = \frac{du}{dt}\delta x + u\,\delta\left(\frac{dx}{dt}\right) - \frac{dx}{dt}\delta u - x\,\delta\left(\frac{du}{dt}\right)$$

$$= -\mu x\,\delta x + u\,\delta u - u\,\delta u + x\cdot\mu\,\delta x = 0.$$

For motion being steady also for $r = a, z = b$, $\dfrac{\partial R}{\partial z} = 0$, $\dfrac{\partial R}{\partial r} = 0$ for $r = a, z = b$

$\dfrac{\partial R}{\partial z} = -\dfrac{\partial V}{\partial z}$ is assumed to be zero by hypothesis & $\dfrac{\partial R}{\partial r} = 0$ gives

$$-\frac{\partial \phi}{\partial r} + \frac{K^2}{r^3} = 0 \text{ for } r = a,\ z = b, \text{ or } K^2 = a^3 \partial\phi/\partial r = \cancel{a^3\phi_a} \text{ for } r = a$$

$$R = \tfrac{1}{2}\dot{r}^2 + \tfrac{1}{2}\dot{z}^2 - \phi(r, z) - \frac{a^3}{2r^2}\cdot\phi_a \qquad \left[\phi_a = \left(\frac{\partial \phi}{\partial r}\right)_{r=a}\right.$$

For vibrations about steady motion ~~given~~ about $r = a, z = b$, put $r = a + \rho$, $z = b + \zeta$

$$R = \tfrac{1}{2}\dot{\rho}^2 + \tfrac{1}{2}\dot{\zeta}^2 - \tfrac{1}{2}\rho^2\phi_{aa} - \rho\zeta\phi_{ab} - \tfrac{1}{2}\zeta^2\phi_{bb} - \frac{3\phi_a}{2a}\cdot\rho^2$$
$$= \tfrac{1}{2}\dot{\rho}^2 + \tfrac{1}{2}\dot{\zeta}^2 - \tfrac{1}{2}\rho^2\left(\phi_{aa} + \frac{3}{a}\phi_a\right) - \rho\zeta\phi_{ab} - \tfrac{1}{2}\zeta^2\phi_{bb}$$

[neglecting constant terms & terms in ρ & ζ which vanish in $\dfrac{d}{dt}\left(\dfrac{\partial R}{\partial \dot\rho}\right) - \dfrac{\partial R}{\partial \rho} = 0$, etc]

$\dfrac{\partial R}{\partial r} = 0 \Rightarrow \dfrac{\partial R}{\partial \rho} = 0$.

Condition for stability is that the quad. form

$$\rho^2\left(\phi_{aa} + \frac{3}{a}\phi_a\right) + 2\rho\zeta\phi_{ab} + \zeta^2\phi_{bb} \text{ should be }+^{ve}\text{ definite}$$

i.e.
$$\begin{vmatrix} \phi_{aa} + \frac{3}{a}\phi_a & \phi_{ab} \\ \phi_{ab} & \phi_{bb} \end{vmatrix} \text{ and } \phi_{bb} \text{ should both be } > 0.$$

(B) Examples on vibrations about position of equilibrium:

(1) Ex(ii) p. 187, Whittaker — worked.

14/11/61 (After the ~~vacat~~ Diwali vacation).

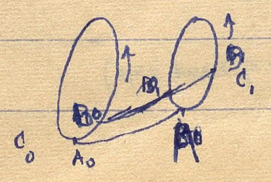
Hamilton's Principle – Principle of Least Action – Repeated the first part already done on 3/10/61, as pointed out by the Students almost at the end of the Lecture.

23/11/61* – Derivation of Hamiltonian equations – diff. form $\Sigma p_r\, dq_r - H dt$ – $H = h = T + V$, integral of energy from $L = L(q_r, \dot q_r)$

When L & hence H do not contain t explicitly – H for the simple pendulum $= p^2/2l^2 - gl\cos q$.

24/11/61 – Integral invariants, definition – Variational equations; To an integral invariant of order one of the given system corresponds an integral of the extended system & conversely – Condition for integral invariants of order one; Two Corollaries –

28/11/61 – Ex. on p. 205 on Vibrations worked out directly without using $z = f(r)$. This problem was given for the terminal examination – Ex. 2. on p. 268 on integral invariants worked out (†) – Relative integral invariants of order one reduced to absolute int. invariants of order two; integral of order p to ~~one~~ absolute one of order $p+1$ using Stokes theorem // Relative integral possessed by all Hamiltonian systems (§ 115, p. 272 – In setting $\delta\Omega$, the end pts of AB & CD coincide $A \equiv C$, $B \equiv D$ & hence $\delta\Omega = \Sigma p_r \delta q_r - \Sigma \beta_r \delta\alpha_r$



$$\int_{A_0A_1 + A_1B_1 + B_1B_0 + B_0A_0} \delta\Omega = 0 \longrightarrow \int_{A_0A_1} \delta\Omega = \int_{B_0B_1} \delta\Omega$$

(a) * Points to note in the proof :- To get $\iint \Sigma\, \delta p_r \delta q_r$ using Stokes's theorem from $\int \Sigma\, p_r \delta q_r$

put $p_r \delta q_r = p_1 \delta q_1 + \cdots + p_n \delta q_n + 0\cdot\delta p_1 + 0\cdot\delta p_2 + \cdots + 0\cdot\delta p_n$

& using $\iint \Sigma\Sigma \left(\frac{\partial M_i}{\partial x_j} - \frac{\partial M_j}{\partial x_i}\right)\delta x_i \delta x_j = \int \Sigma M_i \delta x_i$, consider the three cases

(i) $\delta x_i = \delta q_i,\ \delta x_j = \delta q_j$ then $\frac{\partial p_i}{\partial q_j}$ & $\frac{\partial p_j}{\partial q_i}$ are $= 0$ ∵ two sets of variables are independent

(ii) $\delta x_i = \delta q_i,\ \delta x_j = \delta p_j$, then $\Sigma\Sigma = \Sigma\Sigma\left(\frac{\partial p_i}{\partial p_j} - 0\right)\delta q_i \delta p_j = \Sigma\, \delta q_i \delta p_i$

~~$\delta x_i = \delta p_i,\ \delta x_j = \delta q_j$, then $\Sigma\Sigma = \left(0 - \frac{\partial p_j}{\partial p_i}\right)\delta p_i \delta q_j$~~

(iii) $\delta x_i = \delta p_i,\ \delta x_j = \delta p_j$, then obviously $\Sigma\Sigma = 0$.

(b) ~~$\delta p_i \delta q_i = \left(\frac{\partial p_i}{\partial \lambda}\delta\lambda + \frac{\partial p_i}{\partial \mu}\delta\mu\right)\left(\frac{\partial q_i}{\partial \lambda}\delta\lambda + \frac{\partial q_i}{\partial \mu}\delta\mu\right)$~~

(b) Transformation of $\iint \delta x \delta y$ to $\iint \frac{\partial(q_i, p_i)}{\partial(\lambda, \mu)}\delta\lambda\delta\mu$.

(c) The coeffts $\frac{\partial q_k}{\partial \lambda}\frac{\partial p_i}{\partial \mu}$, etc appear only in products & hence their coeffts equated to zero

& the eqns obtained are consistent if $Q_r = \partial H/\partial p_r$, $P_r = -\partial H/\partial q_r$ and only if,

we may put arbitrarily $Q_i = \partial H/\partial p_i$ which defines H & then the other eqn $P_i = -\partial H/\partial q_i$

follows from $\frac{\partial Q_i}{\partial q_k} + \frac{\partial P_k}{\partial q_i} = 0$; $\frac{\partial Q_i}{\partial p_k} - \frac{\partial Q_k}{\partial p_i} = 0$, $\frac{\partial P_i}{\partial q_k} - \frac{\partial P_k}{\partial q_i} = 0$.

Consider

(*) $\frac{\partial y_s}{\partial x_r}\cdot\frac{\partial^2 x_r}{\partial y_s \partial y_k}$ – ~~From $D = \sum_s A_{rs}\frac{\partial x_r}{\partial y_s}$ ∴ $D\frac{\partial y_k}{\partial x_r} = \sum_s A_{rs}\cdot\frac{\partial x_r}{\partial y_s}\cdot\frac{\partial y_k}{\partial x_r}$~~

~~$= \sum A_{rs}\delta_{sk} = A_{rk}$~~

~~From $D = \sum_k A_{rk}\frac{\partial x_r}{\partial y_k}$~~

~~∴ $\frac{\partial y_k}{\partial x_r} = \frac{1}{D}A_{rk}$ or $\frac{\partial y_s}{\partial x_r} = \frac{1}{D}A_{rs}$.~~

~~$D\frac{\partial y_s}{\partial x_r} = \sum_k A_{rk}\frac{\partial x_r}{\partial y_k}\frac{\partial y_s}{\partial x_r} = \sum A_{rk}\delta_{sk}$~~

~~$= A_{rs}$~~

~~i.e. $\frac{\partial y_s}{\partial x_r} = \frac{1}{D}A_{rs}$.~~

$D = \sum_l A_{rl}\,\partial x_r/\partial y_l$ & $D\cdot\frac{\partial y_s}{\partial x_r} = \sum_l A_{rl}\cdot\frac{\partial x_r}{\partial y_l}\cdot\frac{\partial y_s}{\partial x_r} = \sum_l A_{rl}\delta_{ls} = A_{rs}$

∴ $\frac{\partial y_s}{\partial x_r} = \frac{1}{D}A_{rs}$. (See reverse of next page)

30/11/61 – Portions not done on 28/11/61 – converse of last theorem i.e. systems having $\int \Sigma p_r \delta q_r$ as rel. int invariant are of the Hamiltonian type* // Corollary for eq^ns being K $> 2n$ in number; here the eq^ns for $r = n+1$ to K can be neglected – Exp^n for integral-invariants in terms of integrals; if $y_r = c_r$ $(r = 1 \ldots n)$ be $\underline{n}$ integrals of $\frac{dx_r}{dt} = X_r$ $(r = 1 \ldots n)$, $\int (N_1 \delta y_1 + \cdot + N_n \delta y_n)$ is an abs. int inv. because (N's are f^ns of y's)

$$\frac{d}{dt} \Sigma N_r \delta y_r = \Sigma \left\{ \frac{dN_r}{dt} \delta y_r + N_r \delta \left(\frac{dy_r}{dt} \right) \right\} = \frac{dN_r}{dt} \delta y_r = \Sigma \frac{dN_r}{dt} \delta y_r$$

$$= \sum_r \sum_s \frac{\partial N_r}{\partial y_s} \cdot \frac{dy_s}{dt} = 0.$$

$\int N_1 \delta y_1 + \cdot + N_n \delta y_n + \delta F$ is rel. integral invariant since $\int \delta F = 0$ for a closed curve and an abs. int. inv. is also a relative one, although converse is not true.

1/12/61 – Remaining portions of last lecture – Theorem of Lie & Koenigs (the proof in Whittaker appears quite wrong – Should be as follows:–

$$\text{For } \frac{dx_i}{dt} = X_i \quad (1) \quad (i = 1 \ldots K) \quad (K > 2n) \quad \ldots (1)$$

let $\int (\xi_1 \delta x_1 + - + \xi_K \delta x_K)$ be a rel. int. inv.

By the theorem of Pfaffian forms trns $q_i = q_i(x_1 \ldots x_K)$, $p_i = p_i(x_1 \ldots x_K)$ $(i = 1 \ldots n)$ exist so as to transform $\xi_1 \delta x_1 + \cdot + \xi_K \delta x_K$

to $p_1 \delta q_1 + \cdot + p_n \delta q_n$

Let q_i, p_i, u_j $(j = n+1, \ldots K)$ all f^ns of $x_1 - x_K$ transform (1) to

$$\frac{dq_i}{dt} = Q_i, \quad \frac{dp_i}{dt} = P_i, \text{ and } \frac{du_j}{dt} = U_j \quad \ldots (2)$$

Since $\int (p_1 \delta q_1 + - + p_n \delta q_n)$ is a rel. int. inv. of (1) & hence of (2) the first $2n$ integrals are of Hamiltonian form.

– The last multiplier (Theorem of the last multiplier); Proof of Lemma:

If $dx_r/dt = X_r$ $(r = 1 \ldots n)$

be → $dy_r/dt = Y_r$ $(r = 1 \ldots n)$

then $\Sigma \frac{\partial X_r}{\partial x_r} = \frac{1}{D} \Sigma \frac{\partial (D Y_r)}{\partial y_r}$ $\qquad D = \frac{\partial (x_1 \ldots x_n)}{\partial (y_1 \ldots y_n)}$.

$$\frac{\partial X_r}{\partial x_r} \equiv \frac{\partial}{\partial x_r} \left(\frac{dx_r}{dt} \right) = \frac{\partial}{\partial x_r} \Sigma \frac{\partial x_r}{\partial y_K} \cdot \frac{dy_K}{dt} = \frac{\partial}{\partial x_r} \Sigma Y_K \frac{\partial x_r}{\partial y_K}$$

$$= \sum_s \left(\frac{\partial}{\partial y_s} \sum_K Y_K \frac{\partial x_r}{\partial y_K} \right) \cdot \frac{\partial y_s}{\partial x_r}$$

$$\therefore \sum_r \frac{\partial X_r}{\partial x_r} = \sum_r \sum_s \sum_k \frac{\partial y_s}{\partial x_r} \left\{ \frac{\partial Y_K}{\partial y_s} \cdot \frac{\partial x_r}{\partial y_K} + Y_K \frac{\partial^2 x_r}{\partial y_s \partial y_K} \right\}$$

$$= \sum_s \sum_K \frac{\partial Y_K}{\partial y_s} \delta_{sK} + \Sigma \Sigma \Sigma Y_K \frac{\partial y_s}{\partial x_r} \cdot \frac{\partial^2 x_r}{\partial y_s \partial y_K} \circledast$$

$$= \sum_K \frac{\partial Y_K}{\partial y_K} + \frac{1}{D} \Sigma Y_K \cdot \frac{\partial D}{\partial y_K}.$$

$$\sum_r \sum_s \frac{\partial y_s}{\partial x_r} \cdot \frac{\partial^2 x_r}{\partial y_s \partial y_k} = \frac{1}{D} \sum \sum A_{rs} \frac{\partial^2 x_r}{\partial y_s \partial y_k}.$$

Now $= D = \sum_s A_{rs} \frac{\partial x_r}{\partial y_s}$ and $\frac{\partial D}{\partial y_k} = \sum_r \sum_s A_{rs} \frac{\partial^2 x_r}{\partial y_s \partial y_k}$ by formula of differentiation of a determinant

Hence the term $= \frac{1}{D} \cdot \frac{\partial D}{\partial y_k}$. $\quad \therefore \sum_r \frac{\partial \dot{x}_r}{\partial x_r} = \sum_k \frac{\partial \dot{y}_k}{\partial y_k} + \sum_k \frac{1}{D} \cdot \frac{\partial D}{\partial y_k}$

$$= \sum_k \frac{1}{D} \frac{\partial (D \dot{y}_k)}{\partial y_k}$$

(*) Several steps in this (i) Whittaker of L. M. gives $\int \frac{1}{\partial(V,H)/\partial(p_1,p_2)} \left\{ \frac{\partial H}{\partial p_2} dq_1 - \frac{\partial H}{\partial p_1} dq_2 \right\} =$ const as an integral

(ii) Expressing this in the form $\int \frac{\partial f_1}{\partial c} dq_1 + \frac{\partial f_2}{\partial c} dq_2 =$ const., with $p_1 = f_1(q_1, q_2, h, c)$

$p_2 = f_2(q_1, q_2, h, c)$

from $V = c$ & $H = h$

(iii) This would be equivalent to $\frac{\partial \theta}{\partial c} =$ const if $p_1 dq_1 + p_2 dq_2$ were a perfect differential $d\theta$,

~~$d\theta$~~ $= \theta(q_1, q_2, c)$.

(iv) To prove $p_1 dq_1 + p_2 dq_2 = d\theta(q_1, q_2, h, c)$

and the integrals are $\frac{\partial \theta}{\partial c} =$ const & $t = \frac{\partial \theta}{\partial h} +$ const.

(v) Consider V & H fns of q_1, p_1, p_2 only with p_1, p_2 fns of q_1 & q_2 & get $\frac{\partial V}{\partial q_1} = 0$, $\frac{\partial H}{\partial q_1} = 0$

& get $\frac{\partial f_2}{\partial q_1}$ & $\frac{\partial f_1}{\partial q_2}$

(vi) using $\frac{dV}{dt} = 0$, shows that $\frac{\partial f_2}{\partial q_1} = \frac{\partial f_1}{\partial q_2}$ which proves existence of $d\theta$.

& reexpressing (ii)

(vii) $dt = \frac{dq_1}{\partial H/\partial p_1} = \frac{dq_2}{\partial H/\partial p_2}$ get $dt = \frac{\partial V/\partial p_2 \, dq_1 - \partial V/\partial p_1 \, dq_2}{\partial(V,H)/\partial(p_2,p_1)}$

& from analogue of (ii) for $H = h$ instead of $V = c$, get $\partial f_1/\partial h$, $\partial f_2/\partial h$

& obtain $dt = \frac{\partial f_1}{\partial h} dq_1 + \frac{\partial f_2}{\partial h} dq_2$ i.e. $t = \frac{\partial \theta}{\partial h} +$ Const.

5/12/61 – Proof of the theorem of the last multiplier using the Lemma –

7/12/61 – The above completed – Also proof of Lemma repeated –

No classes from 8^{th} – 11^{th} and Sports Holidays

12/12/61 – Proved the theorem of the last multiplier completely – Also the theorem about quotient of two last multipliers being an integral – Gave Ex. 5. p. 287

14/12/61 – (1) Transformation of last multiplier under a change of coordinates (not in Whittaker)

Let under a trn. $(x_1, .. x_n, x) \longrightarrow (y_1, .. y_n, y)$, the system of eqns

$$\frac{dx_1}{X_1} = . = \frac{dx_n}{X_n} = \frac{dx}{X} = dt \longrightarrow \frac{dy_1}{Y_1} = .. = \frac{dy_n}{Y_n} = dt$$

and let the last multiplier $M \to N$

$$\frac{1}{M}\frac{dM}{dt} + \Sigma \frac{\partial X_k}{\partial x_k} = 0, \text{ and } \frac{1}{N}\frac{dN}{dt} + \Sigma \frac{\partial Y_k}{\partial y_k} = 0$$

$$\text{Now } \Sigma \frac{\partial X_k}{\partial x_k} = \frac{1}{D} \Sigma \frac{\partial (D Y_k)}{\partial y_k} = \frac{1}{D}\left\{ D \Sigma \frac{\partial Y_k}{\partial y_k} + \Sigma \frac{\partial D}{\partial y_k} Y_k \right.$$

$$= \Sigma \frac{\partial Y_k}{\partial y_k} + \Sigma \frac{1}{D} \cdot \frac{\partial D}{\partial y_k} \cdot \frac{dy_k}{dt} = \Sigma \frac{\partial Y_k}{\partial y_k} + \frac{1}{D}\frac{dD}{dt}.$$

$$\therefore \frac{1}{M}\frac{dM}{dt} + \Sigma \frac{\partial Y_k}{\partial y_k} + \frac{1}{D}\frac{dD}{dt} = \frac{1}{N}\frac{dN}{dt} + \Sigma \frac{\partial Y_k}{\partial y_k}$$

$$\text{ie } \frac{d}{dt}(\log MD) = \frac{d}{dt}(\log N) \quad \text{or } \underline{\underline{N = MD.}} \quad (N = MD).$$

(2) Ex. 4. Whittaker, p. 287: Here the change of variables is from $(x_1, .. x_n, x)$ to $(x_1, .. f, x)$

$$\therefore D = \frac{\partial(x_1, .. x_n, x)}{\partial(x_1, .. f, x)} \text{ or } \frac{1}{D} = \frac{\partial(x_1, .. f, x)}{\partial(x_1, .. x_n, x)} = \frac{\partial f}{\partial x_n}$$

Hence the L.M of the transformed eqn is given by $N = M \cdot D = M \Big/ \left(\frac{\partial f}{\partial x_n}\right)$ or $M' \Big/ \left(\frac{\partial f}{\partial x_n}\right)'$

(3) Ex 5. ibid: Here the trn is from $(x_1 .. x_n, x)$ to $(\theta_1 .. \theta_n, x)$ so that eqns

$$\frac{dx_1}{X_1} = . = \frac{dx_n}{X_n} = \frac{dx}{X} \longrightarrow \frac{d\theta_1}{0} = . = \frac{d\theta_n}{0} = \frac{dx}{X}$$

$$D = \frac{\partial(x_1 .. x)}{\partial(\theta_1 .. x)} = \frac{\partial(x_1 .. x_n)}{\partial(\theta_1 .. \theta_n)}.$$

Hence the last multiplier of the transformed system is given by N, where N satisfies the

$$\text{eqn } \frac{\partial}{\partial x}(NX) = 0 \text{ ie } NX = \text{const, say } N = 1/X$$

$$\therefore N = MD \text{ gives } M = N/D = \frac{1}{X}\frac{\partial(\theta_1 .. \theta_n)}{\partial(x_1 .. x_n)}.$$

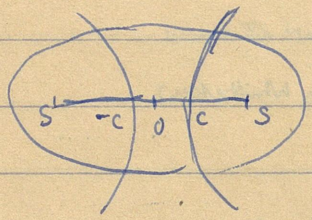
(4) Last multiplier of Hamiltonian system is unity ie $M = 1$ using $\Sigma \frac{\partial X_r}{\partial x_r} = 0$ & hence $\frac{1}{M}\frac{dM}{dt} = 0 \to M = \text{const} = 1$.

(5) (*) Use of this last result to solve problem of conservative holonomic systems of two degrees of freedom when in addition to integral of energy $H = h$, another integral $V(q_1, q_2, p_1, p_2) = c$ is known.

15/12/61 – Item (5) above – condn for $\iiint \cdots \int M \delta x_1 \delta x_2 .. \delta x_k$ being integral invariant of the system $\frac{dx_r}{dt} = X_r$ $(r = 1, .. k)$ is that M is a last multiplier ie $\frac{d}{dt}\left\{ M \frac{\partial(x_1 .. x_k)}{\partial(c_1 .. c_k)} \right\} = 0.$

From § 53, $x = c\cosh\xi\cos\eta$, $y = c\sinh\xi\sin\eta$ so that ξ = const & η = const give individual orbits

* Here $q_1 = c\cosh\xi$, $q_2 = c\cos\eta$.



$$V = -\frac{\mu_1}{\sqrt{(x-c)^2+y^2}} - \frac{\mu_2}{\sqrt{(x+c)^2+y^2}} = -\frac{\mu_1}{c(\cosh\xi-\cos\eta)} - \frac{\mu_2}{c(\cosh\xi+\cos\eta)}$$

$$= -\frac{\mu_1}{q_1-q_2} - \frac{\mu_2}{q_1+q_2}.$$

$$T = \frac{1}{2}(\dot{x}^2+\dot{y}^2) = \frac{1}{2}c^2(\cosh^2\xi-\cos^2\eta)(\dot{\xi}^2+\dot{\eta}^2)$$

$$= \frac{1}{2}c^2(q_1^2-q_2^2)\left(\frac{\dot{q}_1^2}{\sinh^2\xi}+\frac{\dot{q}_2^2}{\sin^2\eta}\right) \text{ using } q_1 = c\cosh\xi,\ q_2 = c\cos\eta.$$

$$\therefore p_1 = \frac{\partial T}{\partial \dot{q}_1} \ \& \ p_2 = \frac{\partial T}{\partial \dot{q}_2} \text{ give } p_1 = \frac{q_1^2-q_2^2}{q_1^2-c^2}\dot{q}_1,\ p_2 = \frac{q_1^2-q_2^2}{c^2-q_2^2}\dot{q}_2$$

$H = T+V$ in the form given & hence the Hamiltonian eq^ns of the two centres of gravitation.

† Note: From $\frac{dq_1}{dt} = \frac{\partial H}{\partial p_1}$, we get $\frac{dq_1}{dt} = q_1$ i.e. $\frac{1}{q_1}dq_1 = dt$ leading to $\log q_1 = t + \text{const}$ the second of the integrals.

Similarly $\frac{dq_2}{dt} = \frac{\partial H}{\partial p_2}$ gives $\frac{dq_2}{dt} = -q_2$ i.e. $\frac{1}{q_2}dq_2 = -dt \rightarrow \log q_2 = -t + \text{const}$

i.e. $\log q_1 + \log q_2 = \text{const}$ or $q_1 q_2 = \text{const}$, the first of the integrals. So that it looks as if the method suggested here is quite superfluous; something like taking a hatchet to kill a fly!

For Hamiltonian system $\iint \dots \int \delta q_1 \dots \delta q_n \, \delta p_1 \dots \delta p_n$ is an integral invariant

<u>19/12/61</u>. Use of L.M to determine the consts for a system of diff. eqns to reduce to the Lagrangian form for ~~the~~ a special case of $n = 1$ – Examples on p. 286 – Ex(1) on two centres of gravitation* (vide § 53, p. 97 of Whittaker & Liouville's theorem)

<u>21/12/61</u> Part of 2 centres of gravitation – Liouville's type. <u>W. Ex. 3, p. 287</u>: That $\frac{p_2 - bq_2}{q_1}$ = const is an integral follows from $\frac{d}{dt}\left\{\frac{p_2 - bq_2}{q_1}\right\} = 0$

$$\text{ie } \frac{d}{dt}\{\ \ \} = \cancel{\dot q} \, (p_2 - bq_2)\frac{dq_1}{dt} - q_1\left(\frac{dp_2}{dt} - b\frac{dq_2}{dt}\right) = (p_2 - bq_2)\frac{\partial H}{\partial p_1} - q_1\left(-\frac{\partial H}{\partial q_2} - b\frac{\partial H}{\partial p_2}\right)$$

$$= (p_2 - bq_2)q_1 - q_1(q_2 - 2bq_2 + bq_0) = 0.$$

Putting $\frac{p_2 - bq_2}{q_1} = c$, we have $p_2 = cq_1 + bq_2$

& from $H = h$, $q_1 p_1 = h + q_2 p_2 + aq_1^2 - bq_2^2 = h + q_2(cq_1 + bq_2) + aq_1^2 - bq_2^2$

gives $p_1 = \frac{h}{q_1} + cq_2 + aq_1$

$$\therefore p_1 dq_1 + p_2 dq_2 = \left(\frac{h}{q_1} + cq_2 + aq_1\right)dq_1 + (cq_1 + bq_2)dq_2$$

$$= d\left\{h \log q_1 + \tfrac{1}{2}aq_1^2 + \tfrac{1}{2}bq_2^2 + cq_1q_2\right\} = d\theta$$

The other two integrals are $\frac{\partial\theta}{\partial c}$ = const & $\frac{\partial\theta}{\partial h} = t$ + const

ie $q_1 q_2$ = const & $\log q_1 = t$ + const. †

22/12/62 – ~~AM~~ ~~Ex~~ <u>W. Ex. 2, p. 286</u> – defn of a C.T – Ex 1., p. 293

– Missed a number of classes consequent on attending Bombay Mechanics Conference & Ahmedabad Math. Conference.

<u>2/1/62</u> – $(q, p) \to (Q, P)$ a contact trn (C.T) if $\sum P_r dQ_r - \sum p_r dq_r = dW$ – Group property of C.T's – examples on pp. 293 – 294 –

Ex. 2, p. 294. & Ex. 3, p. 294.

<u>Ex. 2</u>: $P\,dQ - p\,dq = df(p, q)$ where $f = -q(p + \cot p)$

<u>Ex. 3</u>: $Q = \log(1 + q^{\frac{1}{2}}\cos p)$, $P = 2(1 + q^{\frac{1}{2}}\cos p)\,q^{\frac{1}{2}}\sin p$. $\Big\}$ $P\,dQ - p\,dq = (\sin p\cos p - p)\,dq - 2q\sin^2 p \, dp$

$= d\{q(\sin p\cos p - p)\}$

– Also eqns on p. 295 without the Σ's]

<u>Ex. p. 296</u>. $Q = (2q)^{\frac{1}{2}}K^{-\frac{1}{2}}\cos p$; $P = (2q)^{\frac{1}{2}}K^{\frac{1}{2}}\sin p$

$P\,dQ - p\,dq = (\sin p\cos p - p)\,dq - 2q\sin^2 p \, dp$, just as in above example.

ie $W = q(\sin p\cos p - p)$

To express this as a fn of q & Q, we have from the eqn giving Q as a fn of q & p, $\cos p = (2q)^{-\frac{1}{2}}K^{\frac{1}{2}}Q$

giving $\sin p = (2Kq - Q^2)^{1/2}(2q)^{-\frac{1}{2}}K^{-\frac{1}{2}}$ ie $\sin p\cos p = Q(2Kq - Q^2)^{1/2} \cdot (2q)^{-1}$

ie $W = \frac{1}{2}Q(2Kq - Q^2)^{1/2} - q\cos^{-1}\left\{K^{\frac{1}{2}}Q/(2q)^{\frac{1}{2}}\right\}$.

as given in the text.

4/1/62 — (1) General contact transformations with relns $\Omega_s(Q_i, q_i) = 0$ ($s = 1 \ldots k$) existing. In this case we can replace W by $W + \Sigma \lambda_s \Omega_s$ (λ_s being undetermined multipliers) since $d\Omega_s = 0$. This gives

$$\left.\begin{aligned} P_r &= \frac{\partial W}{\partial Q_r} + \sum_{s=1}^{k} \lambda_s \frac{\partial \Omega_s}{\partial Q_r} \\ p_r &= -\frac{\partial W}{\partial q_r} + \sum_{s=1}^{k} \lambda_s \frac{\partial \Omega_s}{\partial Q_r} \end{aligned}\right\} (r = 1, \ldots n)$$

giving along with $\Omega_s = 0$ ($s = 1 \ldots k$), $(2n+k)$ equations giving (P_r, Q_r, λ_s) in terms of (q_r, p_r) as an explicit C.T. characterised by (W, Ω_s) —

(2) Bilinear covariant of a general differential form, Pfaffian expression, $\delta\theta_d - d\theta_\delta = a_{ij}\, dx^i \delta x^j$, where $a_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}$; $a_{ij}\, dx^i \delta x^j = b_{ij}\, dy^i \delta y^j$, Bilinear covariant of an exact differential is zero.

(3) Conditions for a C.T. in ~~terms of~~ using the bilinear covariant $\Sigma(\delta P_r dQ_r - dP_r \delta Q_r) = \Sigma(\delta p_r dq_r - dp_r \delta q_r)$ — Example of $Q = (2q)^{1/2} k^{-1/2} \cos p$; $P = (2q)^{\frac{1}{2}} k^{1/2} \sin p$ ✓

5/1/62 — Lagrange Bracket expressions & condns ~~of~~ for a C.T. in terms of these — Poisson bracket expressions — Connection between L.B's & P.B's — condns for a C.T. in terms of P.B's.

9/1/62 — Portions left over from last lecture — The condns $\Sigma (u_t, u_r)(u_t, u_s) = \delta_{rs}$ show that the determinants formed by them are reciprocal for the r-sth element of the product determinant is obtained by multiplying the rth row of one by the sth column of the other ie in case of $a_{ij} \times b_{ij} = c_{ij}$ we have $c_{rs} = \sum_{k=1}^{2n} a_{rk} b_{ks} = \delta_{rs}$ ie only elements in the diagonal remain — From $[P_i, P_j] = 0$, $[Q_i, Q_j] = 0$, $[Q_i, P_j] = \delta_{ij}$ it follows using $\sum_{t=1}^{2n} (u_t, u_r)[u_t, u_s] = \delta_{rs}$ that (i) ~~if $u_r = Q_r$, $u_s = Q_s$~~ (i) if $u_r = Q_r$, $u_s = Q_s$ then the above reln leads to $(P_s, Q_r)[P_s, Q_s] = 0$ ie $(P_s, Q_r) = 0$ if $r \neq s$

(ii) $u_r = P_r$, $u_s = P_s \rightarrow (Q_s, P_r)[Q_s, P_s] = 0$ ie $(Q_s, P_r) = 0$ if $r \neq s$

(iii) $u_r = Q_r$, $u_s = P_s \rightarrow (Q_s, Q_r)[Q_s, P_s] = 0$ ie $(Q_s, Q_r) = 0$ if $r \neq s$ & obviously $= 0$ if $r = s$.

(iv) $u_r = P_r$, $u_s = Q_s \rightarrow (P_s, P_r)[P_s, Q_s] = 0$ ie $(P_s, P_r) = 0$ if $r \neq s$

From $(u_t, u_r)[u_t, u_r] = 1$ we have (i) $u_r = Q_r$; $(P_r, Q_r)[P_r, Q_r] = 1$ ie $(P_r, Q_r) = -1$

(ii) $u_r = P_r$, $(Q_r, P_r)[Q_r, P_r] = 1$ ie $(Q_r, P_r) = 1$. //

11/1/62 — W. Ex. 1. p. 300: $((f, \phi), \psi) + ((\phi, \psi), f) + ((\psi, f), \phi) = 0$.

Writing down the expansions in full, we find coeff of $\frac{\partial^2 f}{\partial q_r \partial q_s}$ is terms of the type, with double summations over r & s:— $\frac{\partial^2 f}{\partial q_r \partial q_s}\left(\frac{\partial \phi}{\partial p_s}\frac{\partial \psi}{\partial p_r} - \frac{\partial \psi}{\partial p_s}\cdot\frac{\partial \phi}{\partial p_r}\right)$; $\frac{\partial^2 f}{\partial p_r \partial p_s}\left(\frac{\partial \phi}{\partial q_s}\cdot\frac{\partial \psi}{\partial q_r} - \frac{\partial \psi}{\partial q_s}\cdot\frac{\partial \phi}{\partial q_r}\right)$

and $\frac{\partial^2 f}{\partial q_r \partial p_s}\left(\frac{\partial \phi}{\partial p_r}\cdot\frac{\partial \psi}{\partial q_s} - \frac{\partial \phi}{\partial q_s}\cdot\frac{\partial \psi}{\partial p_r}\right)$; $\frac{\partial^2 f}{\partial p_r \partial q_s}\left(\frac{\partial \phi}{\partial q_r}\cdot\frac{\partial \psi}{\partial p_s} - \frac{\partial \phi}{\partial p_s}\cdot\frac{\partial \psi}{\partial q_r}\right)$

Interchange of r & s changes the sign of the first two ~~terms~~ expns & hence they vanish. ~~Taking~~ Combining terms suitably terms in the 3rd & 4th expns ie $\left(\frac{\partial^2 f}{\partial q_r \partial p_s}\frac{\partial \phi}{\partial p_r}\frac{\partial \psi}{\partial q_s} - \frac{\partial^2 f}{\partial p_r \partial q_s}\cdot\frac{\partial \phi}{\partial p_s}\cdot\frac{\partial \psi}{\partial q_r}\right)$ and $\left(\frac{\partial^2 f}{\partial p_r \partial q_s}\cdot\frac{\partial \phi}{\partial q_r}\cdot\frac{\partial \psi}{\partial p_s} - \frac{\partial^2 f}{\partial q_r \partial p_s}\cdot\frac{\partial \phi}{\partial q_s}\cdot\frac{\partial \psi}{\partial p_r}\right)$, these two also change sign when r & s are interchanged & hence vanish.

* If $a = (q_r, p_r) \to (Q_r, P_r)$

$b = (Q_r, P_r) \to (R_r, S_r)$

$c = (R_r, S_r) \to (T_r, U_r)$

$$(ab)c = \{(q_r, p_r) \to (R_r, S_r)\}\{(R_r, S_r) \to (T_r, U_r)\} = (q_r, p_r) \to (T_r, U_r)$$

$$a(bc) = \{(q_r, p_r) \to (Q_r, P_r)\}\{(Q_r, P_r) \to (T_r, U_r)\} = (q_r, p_r) \to (T_r, U_r)$$

Similarly combining expressions containing second derivatives of ϕ, ψ, they can also be shown equal to zero.

Ex. 2, p. 300 $\quad (F,\phi) = \sum_r \left(\frac{\partial F}{\partial q_r}\cdot\frac{\partial \phi}{\partial p_r} - \frac{\partial F}{\partial p_r}\cdot\frac{\partial \phi}{\partial q_r}\right) = \sum_j \sum_i \sum_r \left(\frac{\partial F}{\partial f_i}\cdot\frac{\partial f_i}{\partial q_r}\cdot\frac{\partial \phi}{\partial f_j}\cdot\frac{\partial f_j}{\partial p_r} - \frac{\partial F}{\partial f_i}\cdot\frac{\partial f_i}{\partial p_r}\cdot\frac{\partial \phi}{\partial f_j}\cdot\frac{\partial f_j}{\partial q_r}\right)$

$$= \sum_i \sum_j \frac{\partial F}{\partial f_i}\cdot\frac{\partial \phi}{\partial f_j}(f_i, f_j)$$

Coeff of (f_i, f_j), since $(f_j, f_i) = -(f_i, f_j)$, is equal to $\frac{\partial F}{\partial f_i}\cdot\frac{\partial \phi}{\partial f_j} - \frac{\partial F}{\partial f_j}\cdot\frac{\partial \phi}{\partial f_i}$

& hence the result.

Ex 1. p. 301: The R.H.S $= \sum_i \sum_j \sum_r \left(\frac{\partial \phi}{\partial Q_i}\cdot\frac{\partial Q_i}{\partial q_r} + \frac{\partial \phi}{\partial P_i}\cdot\frac{\partial P_i}{\partial q_r}\right)\left(\frac{\partial \psi}{\partial Q_j}\cdot\frac{\partial Q_j}{\partial p_r} + \frac{\partial \psi}{\partial P_j}\cdot\frac{\partial P_j}{\partial p_r}\right) - (\quad)(\quad)$

$$= \sum_i \sum_j \left\{\frac{\partial \phi}{\partial Q_i}\cdot\frac{\partial \psi}{\partial Q_j}(Q_i, Q_j) + \frac{\partial \phi}{\partial Q_i}\cdot\frac{\partial \psi}{\partial P_j}(Q_i, P_j) + \frac{\partial \phi}{\partial P_i}\cdot\frac{\partial \psi}{\partial Q_j}(P_i, Q_j) + \frac{\partial \phi}{\partial P_i}\cdot\frac{\partial \psi}{\partial P_j}(P_i, P_j)\right\}$$

$$= \sum_i \sum_j \left(\frac{\partial \phi}{\partial Q_i}\cdot\frac{\partial \psi}{\partial P_j} - \frac{\partial \phi}{\partial P_i}\cdot\frac{\partial \psi}{\partial Q_j}\right)\delta_{ij} = \sum_i \left(\frac{\partial \phi}{\partial Q_i}\cdot\frac{\partial \psi}{\partial P_i} - \frac{\partial \phi}{\partial P_i}\cdot\frac{\partial \psi}{\partial Q_i}\right) = \sum_r (\quad).$$

Ex. 2. p. 301 – Requires theory of partial diff. eqns.

~~Sub~~ Mathieu subgroups – $\sum p_r\,dq_r = \sum P_r\,dQ_r$ – transformation fns $\{W=0$

$$\left.\begin{aligned} &\Omega_i(q_r, Q_r) = 0 \quad (i = 1..k) \\ &P_r = \lambda_i \frac{\partial \Omega_i}{\partial Q_r}\,;\quad p_r = -\lambda_i \frac{\partial \Omega_i}{\partial q_r} \end{aligned}\right\} \quad \mu p_r = -\mu\lambda_i \frac{\partial \Omega_i}{\partial q_r}\,.\ \text{ie } \lambda_i \to \mu\lambda_i$$

& hence $P_r \to \mu P_r$.

ie P_r's are homogeneous & of first degree as fns of p_r

Particular case where these are homogeneous and ~~d~~ integral ie linear in p_i's.

ie $P_r = \sum p_k f_{rk}(q_1 .. q_n) \to \sum f_{rk}(q_1 .. q_n)\,dQ_r = dq_k$ ie $f_{rk} = \partial q_k/\partial Q_r$

ie extended point transformations

16/1/62 – Remaining portions of previous lecture re. Mathieu subgroup of contact transformations, and the extended point transformations – Also mention of associative law for groups & ~~B~~ its validity for C.T's*.

Ex. p. 302: The second eqn is obvious since P_r is homogeneous and of degree one in the p_r's, for if $u = f(x,y,z)$ for ~~eg~~

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu \;\{\text{where } n \text{ is the degree of the homogeneous fn } u \text{ in } x, y, z$$

Thus $\sum p_k \frac{\partial P_r}{\partial p_k} = P_r$. Re. the first eqn we see from $\sum P_r\,dQ_r = \sum p_r\,dq_r$, that if $p_r \to \mu p_r$

$P_r \to \mu P_r$ and hence $dQ_r \to dQ_r$ and $dQ_r = \sum \frac{\partial Q_r}{\partial q_k} dq_k + \sum \frac{\partial Q_r}{\partial p_k} dp_k$ --(A)

& hence if when $p_k \to \mu p_k$, $dQ_r \to dQ_r$ it follows as evident from the second term

of (A) that $Q_r \to Q_r$ ie Q_r is homogeneous of degree zero in p_k ie $\sum p_k \frac{\partial Q_r}{\partial p_k} = 0$.

18/1/62.

Infinitesimal C.T's viz $Q_r = q_r + \Delta q_r = q_r + \phi_r \cdot \Delta t$ and $P_r = p_r + \Delta p_r = p_r + \psi_r \cdot \Delta t$ with ϕ & ψ as fn of (q_r, p_r) and Δt an infinitesimal independent of (q_r, p_r) – Derivation of $\phi_r = \dfrac{\partial K}{\partial p_r}$; $\psi_r = -\dfrac{\partial K}{\partial q_r}$ and of C.T. as

$Q_r = q_r + \dfrac{\partial K}{\partial p_r}\Delta t$; $P_r = p_r - \dfrac{\partial K}{\partial q_r}\Delta t$ – For any fn $df = (f, K)\Delta t$ i.e. P.B is symbol of C.T. –

Comparison of $\dfrac{\Delta q_r}{\Delta t} = \dfrac{\partial K}{\partial p_r}$; $\dfrac{\Delta p_r}{\Delta t} = -\dfrac{\partial K}{\partial q_r}$ with Hamilton's eqns i.e for a conservative holonomic systems, the H-eqns can be interpreted as values of (q_r, p_r) at time $\underline{t}$ to values $(q_r + dq_r, p_r + dq_r)$ at time $t + dt$ as given by an inf. C.T specified by H and $\underline{t}$ regarded as a parameter. – Whole course of a dyn. system as gradual self-unfolding of an inf. C.T; comparison with paths of rays in a pencil of light as wave propagation; cf. with particle & wave aspects of quantum mechanics (Rutherford & Planck and de Broglie & Schrödinger) – Interpretation of soln of H eqns viz $q_r = q_r(\alpha_r, \beta_r, t)$ and $p_r = p_r(\alpha_r, \beta_r, t)$ as a C.T from (α_r, β_r) at $t = t_0$ to (q_r, p_r) at $t = t$, where $\underline{t}$ plays role of a parameter. –

omit this

[Helmholtz's reciprocal theorem – $\Delta p_s \delta q_s = -\delta\beta_r \Delta\alpha_r$ – from the invariance of the bilinear covariant

$$\cancel{\delta p_r dq_r - \delta q} \Sigma(\delta P_r dQ_r - \delta Q_r dP_r) = \Sigma(\delta p_r dq_r - \delta q_r dp_r),$$ and putting this in terms of (q_r, p_r) & (α_r, β_r) at times $t = t$ & $t = t_0$ with Δ & δ as the variations from one orbit to neighbouring orbit.

$$\Sigma(\Delta p_i \delta q_i - \Delta q_i \delta p_i) = \Sigma(\Delta\beta_i \delta\alpha_i - \Delta\alpha_i \delta\beta_i)$$

~~Let~~ for orbits at $t = t_0$, let $\delta\beta_i = 0$ for all i except $i = r$ & also $\delta\alpha_i = 0$. & for orbits at time $t = t$, let $\Delta p_i = 0$ for all i except $i = s$ & $\Delta q_i = 0$. ($\underline{\delta}$ refers to orbits at $t = t_0$ & Δ to orbits at $t = t$), above becomes

$$\cancel{\Delta p_s \delta q_s - \Delta} \ \Delta p_s \delta q_s = -\Delta\alpha_r \delta\beta_r$$

$$\cancel{\delta q_s / \delta\beta_r = -\frac{\Delta\alpha_r}{\Delta p_s}} \text{ \& if } \Delta p_s = \delta\beta_r,\ \delta q_s = -\Delta\alpha_r$$

δq_s is due to $\delta\beta_r$ & corresponds to direct motion & $\Delta\alpha_r$ is due to Δp_s in reversed motion i.e $p_s \to -p_s$.]

19/1/62; (1) Conditions for transformed equations being again of the Hamiltonian form – This may be true for a particular dynamical system if $\int \Sigma P_r dQ_r$ be an integral invariant of the transformed system – To find condns which should be true for any dynamical system – Jacobi's theorem that contact transformations conserve the Hamiltonian form – In general Q_r, P_r are fns of q_r, p_r and t and the contact fn

$$\Sigma P_r dQ_r - \Sigma p_r dq_r = dW$$

has to be modified as $\Sigma P_r dQ_r - \Sigma p_r dq_r = dW + W' \cdot dt$

Since we want to associate rel. integrals with the Hamiltonian systems viz $\int \Sigma p_r \delta q_r$ & $\int \Sigma P_r \delta Q_r$ where $\underline{t}$ is not varied, we can consider C.T's such that $\Sigma P_r \delta Q_r - \Sigma p_r \delta q_r = \delta W$. For the original system $\int \Sigma p_r \delta q_r$ is an integral invariant, hence $\int \Sigma P_r \delta Q_r - \int \Sigma p_r \delta q_r = 0$ ($\int \delta W = 0$ for a closed ~~mou~~ curve) i.e $\int \Sigma P_r \delta Q_r$ is also an integral invariant since δ's refer to contemporaneous closed orbits in the two systems – In particular if Q_r are fns of q_r only, the fn is an extended pt. fn & the invariance of Hamiltonian form holds.

* Ex. p. 307 – See Ex. on p. 297, we can show as there that

$$dp\,\delta q - \delta p\,dq = dP\,\delta Q - \delta P\,dQ$$ ie trn is a C.T.

See also Ex. on p. 296

The form $\sum P_r\,dQ_r$ can be written in general with $Q_1, \ldots Q_n = x_1, \ldots x_n$ and $P_1 \ldots P_n = x_{n+1}, \ldots x_{2n}$

as $\quad x_{n+1}\,dx_1 + \ldots + x_{2n}\,dx_n + 0 \cdot dx_{n+1} + \ldots + 0 \cdot dx_{2n} + 0 \cdot dx_{2n+1}$

∴ The eqn $\sum_{i=1}^{2n+1} a_{ij}\,dx_i = 0$ for $j = 1, \ldots n$ ie $\sum_i \left(\frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} \right) dx_i = 0$ reduces to $-dx_{n+j} = 0$

for $j = n+1, \ldots n+k, \ldots 2n$, reduces to $dx_k = 0$

for $j = 2n+1$, reduces to $0 = 0$ ie the first Pfaffian system of eqns are

$$dx_{n+j} = 0 \quad (j = 1 \ldots n) \quad \text{and} \quad dx_k = 0, \; (k = 1 \ldots n)$$

ie $dP_r = 0$, $dQ_r = 0$.

Ex. p.307* – ~~H transforms into~~ $H = \frac{1}{2}(p^2 + k^2q^2) \rightarrow \frac{1}{2}\{2QK\sin^2 P + K^2 \cdot 2Q \cdot K^{-1}\cos^2 P\} = KQ$.

(2) Pfaffian expn θ_δ & its bilinear covariant $\delta\theta_d - d\theta_\delta = \sum\sum a_{ij}\, dx_i\, \delta x_j$ $\left[a_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i}\right]$ considered in $(2n+1)$ variables – First Pfaffian system of eqns associated with $\sum X_r\, dx_r$ is given by equating coeffs of $\delta x_j = 0$ in the bilinear covariant. These $(2n+1)$ eqns are compatible since a_{ij} is skew symmetric & odd order – System of eqns invariantively connected – Case of special form $\sum p_r\, dq_r - H dt$.

First Pfaffian system of eqns are the H-eqns & so if $(q, p, t) \rightarrow (x_1, \ldots x_{2n}, \tau)$, H eqns $\rightarrow$ 1st Pfaffian system for

$$X_1\, dx_1 + \ldots + X_{2n}\, dx_{2n} + T\, d\tau$$

$$\sum p_r\, dq_r - H dt = \sum P_r\, dQ_r - \left(H - \frac{\partial W}{\partial t} - \sum \lambda_s \frac{\partial \Omega_s}{\partial t}\right) dt - dW.$$ & neglecting dW in forming 1st Pfaffian system

giving $K = H - \frac{\partial W}{\partial t} - \sum \lambda_s \frac{\partial \Omega_s}{\partial t}$.

23/1/62 – Remaining portions of previous lecture – Transformations in which independent variable $\underline{t}$ is also changed ie $(q_r, p_r, t) \rightarrow (Q_r, P_r, T)$ which is equivalent to $(q_r, p_r, t, h) \rightarrow (Q_r, P_r, T, K)$ & solving $H + h = 0$ to get $K + k = 0$. – New formulation of integration problem. ie $q_r = \phi_r(Q_r, P_r, t)$ & $p_r = \psi_r(Q_r, P_r, t)$ such that $\sum p_r\, dq_r - H dt \rightarrow \sum P_r\, dQ_r - dT$ & for latter 1st Pfaffian eqns are $dQ_r = 0$, $dP_r = 0$.

~~Examples p.311~~

25/1/62 – Remaining portions of previous lecture – The results after Jacobi's theorem onwards ie §136 onwards in Whittaker apply to all C.T's, not merely inf. C.T's; these are only used to show that $(q_r, p_r) \rightarrow (q_r + \phi_r \Delta t, p_r + \psi_r \Delta t)$ is itself a C.T –

~~The~~ §140 may be formulated exactly on basis of §139, by taking the relation $K + 1 = 0$ in place of $K + k = 0$ (ie $k = -1$). From $t = f(Q_r, P_r, T, k)$ it follows $t = f(Q_r, P_r, T)$ ie ~~t =~~ $T = f(P_r, Q_r, t)$ hence the C.T may be taken as

$$\left.\begin{aligned} q_r &= \phi_r(Q_r, P_r, t) \\ p_r &= \psi_r(Q_r, P_r, t) \end{aligned}\right\} \quad \text{— (1)}$$

such that $\sum p_r\, dq_r - H dt \rightarrow \sum P_r\, dQ_r - dT$

The first Pfaffian system of the latter form are given by $dQ_r = 0$, $dP_r = 0$ * ie Q_r = const, P_r = const. & hence (1) constitutes the soln of the Hamiltonian system with Q_r & P_r constants ie Integration problem is to find a C.T such that last term of the diffl. form reduces to a total differential.

Examples. p.311.

(1) 1st method: using $\sum_{r=1}^{2}(dQ_r\, \delta P_r - dP_r\, \delta Q_r) = \sum_{r=1}^{2}(dq_r\, \delta p_r - dp_r\, \delta q_r)$

There is an error in this example as given in Whittaker P_1 and P_2 should have their signs changed. With this change, expressing dQ_r, dP_r, δQ_r, δP_r in terms of the corresp.

~~23~~ (60)

To show

(f) ie ~~X~~ $[Q_1, P_1] = 1$ & $[Q_2, P_2] = 1$. as can be shown directly.

(*) The eqns expressing q_1, q_2, p_1, p_2 in terms of Q_1, Q_2, P_1, P_2 are easily found to be

$$q_1 = -\sqrt{Q_1}\,\sin(2\lambda P_1 + P_2/\lambda)$$

$$q_2 = -\sqrt{2\lambda^2 Q_2 - Q_1}\,\sin(P_2/\lambda)$$

$$p_1 = \frac{\sqrt{Q_1}}{\lambda}\cos(2\lambda P_1 + P_2/\lambda)$$

$$p_2 = \frac{\sqrt{2\lambda^2 Q_2 - Q_1}}{\lambda}\cos(P_2/\lambda)$$

Putting
$$\left.\begin{aligned} \alpha' &= \sqrt{Q_1} \\ \beta' &= \sqrt{2\lambda^2 Q_2 - Q_1} \\ \theta' &= 2\lambda P_1 + P_2/\lambda \\ \phi' &= P_2/\lambda \end{aligned}\right\}$$
we can show $\dfrac{\partial(q_1, q_2, p_1, p_2)}{\partial(\alpha', \beta', \theta', \phi')} = \dfrac{\alpha'\beta'}{\lambda^2}$

and $\dfrac{\partial(\alpha', \beta', \theta', \phi')}{\partial(Q_1, Q_2, P_1, P_2)} = \dfrac{\lambda^2}{\alpha'\beta'}$

Hence $\dfrac{\partial(q_1, q_2, p_1, p_2)}{\partial(Q_1, Q_2, P_1, P_2)} = 1.$

increments in q_r & p_r and actual simplification gives the result at once.

(2) <u>Second method</u> :- Showing $\Sigma P_r dQ_r - \Sigma p_r dq_r = dW$ — with the correction made, it easily follows

that $$W = -\frac{1}{2\lambda}\left\{(q_1^2+\lambda^2p_1^2)\tan^{-1}(q_1/\lambda p_1) + (q_2^2+\lambda^2p_2^2)\tan^{-1}(q_2/\lambda p_2) + \lambda(p_1q_1+p_2q_2)\right\}.$$

(3) <u>Third method</u> :- Proving $(Q_i, Q_j) = (P_i, P_j) = 0$, and $(Q_i, P_j) = \delta_{ij}$, and showing directly that

$$(Q_1, Q_2) = (P_1, P_2) = (Q_1, P_2) = (Q_2, P_1) = 0 \text{ and } (Q_1, P_1) = (Q_2, P_2) = 1$$

(4) <u>Fourth method</u> :- Proving $\dfrac{\partial(Q_1 \ldots Q_n, P_1 \ldots P_n)}{\partial(q_1 \ldots q_n, p_1 \ldots p_n)} = 1$

(5) <u>Fifth method</u> — By Lagrange Brackets †

(6) <u>Sixth method</u> — change to polar coords (r_1, θ_1) & (r_2, θ_2)

That this Jacobian should be equal to unity for a C.T is obtained in general from the theorem that the Hamiltonian form is conserved under a C.T, and also using the theorem that under <u>any</u> ~~change~~ transformation, if system $dx_r/dt = X_r \rightarrow dy_r/dt = Y_r$, then $\Sigma \dfrac{\partial X_r}{\partial x_r} = \dfrac{1}{D}\Sigma \dfrac{\partial(DY_r)}{\partial y_r}$ (*). For a C.T both $\dfrac{dx_r}{dt} = X_r$ and $\dfrac{dy_r}{dt} = Y_r$ are of Hamiltonian form & for such a form $\Sigma \dfrac{\partial X_r}{\partial x_r} = \Sigma \dfrac{\partial Y_r}{\partial y_r} = 0$ & hence (*) gives $\Sigma \dfrac{1}{D} Y_r \dfrac{\partial D}{\partial y_r} = 0$ ie ~~$\frac{d}{dt}(\log D)$~~ ie $\Sigma \dfrac{1}{D}\cdot\dfrac{\partial D}{\partial y_r}\cdot\dfrac{dy_r}{dt} = 0$ or $\dfrac{d}{dt}(\log D) = 0$ or $D =$ const, which may be taken equal to unity without loss of generality ie $\dfrac{\partial(y_1 \ldots y_n)}{\partial(x_1 \ldots x_n)} = 1$ or in this case $\dfrac{\partial(Q_1 \ldots P_n)}{\partial(q_1 \ldots p_n)} = 1$.

In this particular problem, the actual writing of the determinant & expanding it gives the result.

or putting $\alpha = q_1^2 + \lambda^2 p_1^2$; $\beta = q_2^2 + \lambda^2 p_1^2$, $\theta = \tan^{-1}(q_1/\lambda p_1)$, $\phi = \tan^{-1}(q_2/\lambda p_2)$ (*)

$$Q_1 = \alpha,\quad Q_2 = (\alpha+\beta)/2\lambda^2;\quad -2\lambda P_1 = \theta - \phi,\quad -P_2 = \lambda\phi$$

we have $\partial\theta/\partial q_1 = \lambda p_1/\alpha$, $\partial\theta/\partial p_1 = -\lambda q_1/\alpha$ & similarly $\partial\phi/\partial q_2 = \lambda p_2/\beta$; $\partial\phi/\partial p_2 = -\lambda q_2/\beta$

Hence $$\frac{\partial(Q_1, Q_2, P_1, P_2)}{\partial(\alpha, \beta, \theta, \phi)} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1/2\lambda^2 & 1/2\lambda^2 & 0 & 0 \\ 0 & 0 & -\frac{1}{2\lambda} & 1/2\lambda \\ 0 & 0 & 0 & -\lambda \end{vmatrix} = \frac{1}{4\lambda^2}$$

Again $$\frac{\partial(\alpha, \beta, \theta, \phi)}{\partial(q_1, q_2, p_1, p_2)} = \begin{vmatrix} 2q_1 & 0 & 2\lambda^2 p_1 & 0 \\ 0 & 2q_2 & 0 & 2\lambda^2 p_2 \\ \lambda p_1/\alpha & 0 & -\lambda q_1/\alpha & 0 \\ 0 & \lambda p_2/\beta & 0 & -\lambda q_2/\beta \end{vmatrix} = \frac{4\lambda^2}{\alpha\beta}\begin{vmatrix} q_1 & 0 & \lambda^2 p_1 & 0 \\ 0 & q_2 & 0 & \lambda^2 p_2 \\ p_1 & 0 & -q_1 & 0 \\ 0 & p_2 & 0 & -q_2 \end{vmatrix}$$

$$= \frac{4\lambda^2}{\alpha\beta}(q_1^2+\lambda^2p_1^2)(q_2^2+\lambda^2p_2^2) \text{ after very easy simplification}$$

$$= 4\lambda^2$$

$$\therefore \frac{\partial(Q_1, Q_2, P_1, P_2)}{\partial(q_1, q_2, p_1, p_2)} = \frac{\partial(Q_1, Q_2, P_1, P_2)}{\partial(\alpha, \beta, \theta, \phi)}\cdot\frac{\partial(\alpha, \beta, \theta, \phi)}{\partial(q_1, q_2, p_1, p_2)} = \frac{1}{4\lambda^2}\cdot 4\lambda^2 = 1.$$

<u>Ex 2.</u> The L.H.S = the P.B $(u, v) = \sum_{r=1}^{n}\left(\dfrac{\partial u}{\partial x_i}\cdot\dfrac{\partial x_i}{\partial q_r}\cdot\dfrac{\partial v}{\partial x_j}\cdot\dfrac{\partial x_j}{\partial p_r} - \dfrac{\partial u}{\partial x_i}\cdot\dfrac{\partial x_i}{\partial p_r}\cdot\dfrac{\partial v}{\partial x_j}\dfrac{\partial x_j}{\partial q_r}\right)$ with double summation over i & j from $i, j = 1$ to $2n$

$$= \sum_{i=1}^{2n}\sum_{j=1}^{2n}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}(x_i, x_j). \quad \text{---(1)}$$

(†) In fact, $\tan P_1 = p_1'/q_1' = \dfrac{2(p_1 - p_2)}{\lambda_1 (q_1 - q_2)}$ and $Q_1 = p_1'^2 + q_1'^2 = \dfrac{1}{2\lambda_1}(p_1 - p_2)^2 + \dfrac{\lambda_1}{8}(q_1 - q_2)^2$.

which is required for the last part. & sim^y $\tan P_2$ and Q_2

Also, $\cos P_1 = q_1'/\sqrt{Q_1} = \sqrt{\dfrac{\lambda_1}{8Q_1}}\,(q_1 - q_2)$ or $P_1 = \cos^{-1}\left\{\sqrt{\dfrac{\lambda_1}{8Q_1}}\,(q_1 - q_2)\right\}$

and $Q_1 \sin P_1 \cos P_1 \cancel{= p_1' q_1' = \dfrac{(p_1 - p_2)}{\sqrt{2\lambda_1}}} = Q_1 \cdot \sqrt{\dfrac{\lambda_1}{8Q_1}}\,(q_1 - q_2) \cdot \sqrt{1 - \dfrac{\lambda_1}{8Q_1}(q_1 - q_2)^2}$

$$= \frac{\sqrt{\lambda_1}}{8}(q_1 - q_2)\left\{8Q_1 - \lambda_1 (q_1 - q_2)^2\right\}^{1/2}$$

i.e. $W = Q_1 \cos^{-1}\left\{\sqrt{\dfrac{\lambda_1}{8Q_1}}\,(q_1 - q_2)\right\} + Q_2 \cos^{-1}\left\{\sqrt{\dfrac{\lambda_2}{8Q_2}}\,(q_1 + q_2)\right\}$

$$+ \frac{\sqrt{\lambda_1}}{8}(q_1 - q_2)\left\{8Q_1 - \lambda_1 (q_1 - q_2)^2\right\}^{1/2} + \frac{\sqrt{\lambda_2}}{8}(q_1 + q_2)\left\{8Q_2 - \lambda_2 (q_1 + q_2)^2\right\}^{1/2}$$

* W can be expressed as a function $W(Q_1, Q_2, q_1, q_2)$ (†) & it can be verified that $P_r = \partial W/\partial Q_r$, $p_r = -\partial W/\partial q_r$ $(r = 1, 2)$ are ~~the actual eqns~~ equivalent to the eqns of transformation

From $\sum_{r=1}^{n} p_r dq_r = \sum_{i=1}^{2n} x_i dx_i$, — (2) invariance of ~~$d\theta_\delta - \delta\theta_d$~~ $\delta\theta_d - d\theta_\delta$ of a differential form leading to the bilinear covariant $a_{ij} = \dfrac{\partial X_i}{\partial x_j} - \dfrac{\partial X_j}{\partial x_i}$ ~~gives~~ leads to

$$\sum_r (dq_r \delta p_r - \delta q_r dp_r) = \sum_i \sum_j a_{ij}\, dx_i \delta x_j \quad - - - (3)$$

The L.H.S of (3) can be written $= \displaystyle\sum_i \sum_j \sum_r \left(\frac{\partial q_r}{\partial x_i}\frac{\partial p_r}{\partial x_j} - \frac{\partial q_r}{\partial x_j}\cdot\frac{\partial p_r}{\partial x_i} \right) dx_i \delta x_j$

$= \displaystyle\sum_i \sum_j [x_i, x_j]\, dx_i \delta x_j$; the $[x_i, x_j]$ being the Lagrange bracket

Hence from (3) we have $a_{ij} = [x_i, x_j]$, and from the relation between L.B & P.B's we get at once $(x_i, x_j) = A_{ij}$ and eq^n (1) becomes

$$(u, v) = \sum_i \sum_j A_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} \quad \text{reqd result}$$

[There are two errors in the Ex. given, summation for i & j should be ~~not~~ from 1 to $2n$ & not 1 to n; Also u & v should be interchanged on right hand side.

Ex. 3 – p. 312. For any transformation

$$\iint\cdots\int \delta Q_1 \cancel{\delta Q_2} \cdots \delta P_n = \iint\cdots\int \frac{\partial(Q_1, Q_2 \ldots P_1 \ldots P_n)}{\partial(q_1, q_2 \ldots p_1 \ldots p_n)} \cdot \delta q_1 \delta q_2 \ldots \delta q_n \delta p_1 \ldots \delta p_n$$

But for a C.T. as shown directly in the fourth method of Ex(1), the Jacobian $\dfrac{\partial(Q_1 \ldots P_n)}{\partial(q_1 \ldots p_n)} = 1$

& hence the equality of the two integral invariants follows.

Ex. 4. Putting $\frac{1}{2}(q_1 + q_2) = \sqrt{2/\lambda_2}\, q_2'$; $p_1 + p_2 = \sqrt{2\lambda_2}\cdot p_2'$

$\frac{1}{2}(q_1 - q_2) = \sqrt{2/\lambda_1}\cdot q_1'$; $p_1 - p_2 = \sqrt{2\lambda_1}\cdot p_1'$ — (1)

or

$$\left.\begin{aligned} q_1 &= \sqrt{2/\lambda_1}\, q_1' + \sqrt{2/\lambda_2}\, q_2' \\ q_2 &= -\sqrt{2/\lambda_1}\, q_1' + \sqrt{2/\lambda_2}\, q_2' \\ p_1 &= \sqrt{\lambda_1/2}\, p_1' + \sqrt{\lambda_2/2}\, p_2' \\ p_2 &= -\sqrt{\lambda_1/2}\, p_1' + \sqrt{\lambda_2/2}\, p_2' \end{aligned}\right\} \quad \text{— (2)}$$

the transformation can be written in the form:

$$q_1' = \sqrt{Q_1}\cos P_1 \; ; \; q_2' = \sqrt{Q_2}\cos P_2 \; ; \; p_1' = \sqrt{Q_1}\sin P_1 \; ; \; p_2' = \sqrt{Q_2}\sin P_2 . \quad - - (3)$$

From this it easily follows that $\dfrac{\partial(q_1', q_2', p_1', p_2')}{\partial(Q_1, Q_2, P_1, P_2)} = \dfrac{1}{4}$

Also from (2), $\dfrac{\partial(q_1, q_2, p_1, p_2)}{\partial(q_1', q_2', p_1', p_2')} = 4$ & hence $\dfrac{\partial(q_1, q_2, p_1, p_2)}{\partial(Q_1, Q_2, P_1, P_2)} = 4 \cdot \frac{1}{4} = 1$ ie tr^n is a C.T.

Otherwise: From (2), $p_1 dq_1 + p_2 dq_2 = 2(p_1' dq_1' + p_2' dq_2')$ — — (4).

From (3) $P_1 dQ_1 + P_2 dQ_2 - 2(p_1' dq_1' + p_2' dq_2') = (P_1 - \sin P_1 \cos P_1)\, dQ_1 + (P_2 - \sin P_2 \cos P_2)\, dQ_2$

$+ 2Q_1 \sin^2 P_1 \cdot dP_1 + 2Q_2 \sin^2 P_2 \cdot dP_2$

$= d(P_1 Q_1 + P_2 Q_2 - Q_1 \sin P_1 \cos P_1 - Q_2 \sin P_2 \cos P_2)$

$$\left.\begin{aligned} &\text{ie } (P_1 dQ_1 + P_2 dQ_2) - (p_1 dq_1 + p_2 dq_2) = dW \\ &\text{where } W = P_1 Q_1 + P_2 Q_2 - Q_1 \sin P_1 \cos P_1 - Q_2 \sin P_2 \cos P_2 \end{aligned}\right\} \quad - - (5)^{*}$$

ie tr^n is a C.T & since W is independent of t, $K = H$ & $H = p_1^2 + p_2^2 + \dfrac{\lambda_1^2}{8}(q_1 - q_2)^2 + \dfrac{\lambda_2^2}{8}(q_1 + q_2)^2$

* for $\frac{dx_r}{dt} = X_r$. $(r = 1 - n)$

$$\frac{d}{dt}(\delta x_r) = \sum_{s=1}^{n} \frac{\partial X_r}{\partial x_s} \delta x_s \qquad (r = 1 - n)$$

ie $H = \frac{1}{2}(p_1 - p_2)^2 + \frac{1}{2}(p_1 + p_2^2) + \frac{\lambda_1^2}{8}(q_1 - q_2)^2 + \frac{\lambda_2^2}{8}(q_1 + q_2)^2$

$= \lambda_1 p_1'^2 + \lambda_2 p_2'^2 + \lambda_1 q_1'^2 + \lambda_2 q_2'^2 = \lambda_1 (p_1'^2 + q_1'^2) + \lambda_2 (p_2'^2 + q_2'^2) = \lambda_1 Q_1 + \lambda_2 Q_2 = K.$

The solns of $\frac{dQ_r}{dt} = \frac{\partial K}{\partial P_r}$; $\frac{dP_r}{dt} = -\frac{\partial K}{\partial Q_r}$ $(r = 1, 2)$ with $K = \lambda_1 Q_1 + \lambda_2 Q_2$

are given by $\frac{dQ_r}{dt} = 0$ $(r = 1, 2)$, $\frac{dP_1}{dt} = -\lambda_1$, $\frac{dP_2}{dt} = -\lambda_2$

$$\left.\begin{array}{l} \text{ie } Q_1 = \text{const} = a_1,\ Q_2 = \text{const} = a_2 \\ P_1 = b_1 - \lambda_1 t,\ P_2 = b_2 - \lambda_2 t. \end{array}\right\}$$

and substituting these values of Q_r, P_r in the expr^n for q_r, p_r in terms of Q_r, P_r, solution of the original system is obtained directly (without introducing q_1', p_1'). Introducing p_1', q_1' and using $\frac{p_1'}{q_1'} = \tan P_1$, $\frac{p_2'}{q_2'} = \tan P_2$

and $Q_1 = q_1'^2 + p_1'^2$ and $Q_2 = q_2'^2 + p_2'^2$ we get

$$\frac{2(p_1 - p_2)}{\lambda_1(q_1 - q_2)} = \tan(b_1 - \lambda_1 t); \quad \frac{2(p_1 + p_2)}{\lambda_2(q_1 + q_2)} = \tan(b_2 - \lambda_2 t).$$

$$\frac{1}{2\lambda_1}(p_1 - p_2)^2 + \frac{\lambda_1}{8}(q_1 - q_2)^2 = a_1; \quad \frac{1}{2\lambda_1}(p_1 + p_2)^2 + \frac{\lambda_2}{8}(q_1 + q_2)^2 = a_2$$

30/1/62 – Sarma working Ex 1. on Board using 2^nd method

1/2/62 – Chalapathi Rao " 1^st "

2/2/62 – Lohokare " 3^rd "

6/2/62 – Pagare " 4^th "

8/2/62 " " " (contd)

9/2/62 C. N. Singh " 5^th "

13/2/62 – missed the class.

15/2/62 – C. N. Singh " completing – I also worked Ex. 2 & Ex. 3

16/2/62 – Repeated working of Ex. 2 – Direct proof of $\frac{\partial(Q_1 \ldots P_n)}{\partial(q_1 \ldots p_n)} = 1$ for a C. T – Working of Ex 1. by $\frac{\partial(Q \ldots)}{\partial(\alpha \ldots)} \cdot \frac{\partial(\alpha \ldots)}{\partial(p \ldots)} = 1$

~~working of Ex. 4~~.

20/2/62 – Working of Ex. 4 on p. 312

22/2/62 – Introduction of the H-J. equation. § 142 partly done

23/2/62 – Continuation of the same. – § 142 finished and Ex. under § 142 done.

27/2/62 – (i) $\int_{t_0}^{t_1} L\,dt$ as a soln of the H-J. eq^n from $\delta \int_{t_0}^{t_1} L\,dt = \Sigma p_r \delta q_r - \Sigma \beta_r \delta \alpha_r$ and also showing

$\frac{\partial W}{\partial t} + H = 0$ using $\frac{dW}{dt} = L$ & $H = \Sigma p_r \dot{q}_r - L$.

(ii) For integral $\phi(q_r, p_r, t) = $ const of the H-system, corresponds a soln $\delta q_r = \epsilon \frac{\partial \phi}{\partial p_r}$, $\delta p_r = -\epsilon \frac{\partial \phi}{\partial q_r}$ of the variational system* – Inf. contact transformations given by $\delta q_r = Q_r - q_r = \frac{\partial K}{\partial p_r} \delta t$, $\delta p_r = P_r - p_r$ $= -\frac{\partial K}{\partial q_r} \delta t$ & so alone are inf. C. T's changing system into itself ie integrals are to be interpreted

* The particular example to be done.

as giving rise to inf. C. T's changing the system to itself; the symbol of the C.T is (ϕ, f).

(iii) Poisson's theorem – Ex of pt (q_1, q_2, q_3) with vel. compts p_1, p_2, p_3; two integrals of A M: $q_2 p_3 - q_3 p_2 = $ const & $q_3 p_1 - q_1 p_3 = $ const lead to $p_1 q_2 - p_2 q_1 = $ const.

6/3/62 – Completion of (ii) of lecture of 27/2/62 and (iii) of same lecture* – Constancy of L. B. expns along a trajectory.

9/3/62 – Soln of Example left over from previous lecture – Constancy of L. B. expns along a trajectory – Ex. 2, p. 336 solved – Ex. 3, p. 337 (given to class, but not solved).

13/3/62 – Ex 3. p. 337: Since $\phi = $ const. & $\psi = $ const are integrals

$$\frac{\partial\phi}{\partial t} + A_i\frac{\partial\phi}{\partial q_i} + B_i\frac{\partial\phi}{\partial p_i} = 0 \quad \text{--- (1)}$$

$$\text{and,}\quad \frac{\partial\psi}{\partial t} + A_i\frac{\partial\psi}{\partial q_i} + B_i\frac{\partial\psi}{\partial p_i} = 0 \quad \text{--- (2)}$$

$(\phi, \psi) = $ const is also an integral ie

$$\frac{\partial(\phi,\psi)}{\partial t} + A_i\frac{\partial(\phi,\psi)}{\partial q_i} + B_i\frac{\partial(\phi,\psi)}{\partial p_i} = 0 \quad \text{--- (3)}$$

$$\text{ie}\ \left(\frac{\partial\phi}{\partial t}, \psi\right) + \left(\phi, \frac{\partial\psi}{\partial t}\right) + A_i\left(\frac{\partial\phi}{\partial q_i}, \psi\right) + A_i\left(\phi, \frac{\partial\psi}{\partial q_i}\right) + B_i\left(\frac{\partial\phi}{\partial p_i}, \psi\right) + B_i\left(\phi, \frac{\partial\psi}{\partial p_i}\right) = 0$$

Substituting for $\partial\phi/\partial t$ and $\partial\psi/\partial t$ from (1) & (2), this becomes

$$-\left(A_i\frac{\partial\phi}{\partial q_i} + B_i\frac{\partial\phi}{\partial p_i}, \psi\right) - \left(\phi, A_i\frac{\partial\psi}{\partial q_i} + B_i\frac{\partial\psi}{\partial p_i}\right) + \cdots = 0 \quad \text{--- (3')}$$

$$\text{ie}\quad -\left(A_i\frac{\partial\phi}{\partial q_i}, \psi\right) - \left(B_i\frac{\partial\phi}{\partial p_i}, \psi\right) - \left(\phi, A_i\frac{\partial\psi}{\partial q_i}\right) - \left(\phi, B_i\frac{\partial\psi}{\partial p_i}\right) + \cdots = 0 \quad \text{--- (3'')}$$

Now $(uv, w) = u(v, w) + v(u, w)$ and $(u, vw) = (u, v)w + (u, w)v$ & hence (3'') becomes

$$-A_i\left(\frac{\partial\phi}{\partial q_i}, \psi\right) - \frac{\partial\phi}{\partial q_i}(A_i, \psi) - B_i\left(\frac{\partial\phi}{\partial p_i}, \psi\right) - \frac{\partial\phi}{\partial p_i}(B_i, \psi) - \left(\phi, \frac{\partial\psi}{\partial q_i}\right)A_i - (\phi, A_i)\frac{\partial\psi}{\partial q_i} - \left(\phi, \frac{\partial\psi}{\partial p_i}\right)B_i$$
$$-(\phi, B_i)\frac{\partial\psi}{\partial p_i} + A_i\left(\frac{\partial\phi}{\partial q_i}, \psi\right) + A_i\left(\phi, \frac{\partial\psi}{\partial q_i}\right) + B_i\left(\frac{\partial\phi}{\partial p_i}, \psi\right) + B_i\left(\phi, \frac{\partial\psi}{\partial p_i}\right) = 0.$$

ie after cancelling terms with opposite signs ie all terms with A_i & B_i as factors of P. B. expns, we have

$$\frac{\partial\phi}{\partial q_i}(A_i, \psi) + \frac{\partial\phi}{\partial p_i}(B_i, \psi) + \frac{\partial\psi}{\partial q_i}(\phi, A_i) + \frac{\partial\psi}{\partial p_i}(\phi, B_i) = 0 \quad \text{--- (4)}$$

Writing down the expanded form of the P. B. expns

$$\frac{\partial\phi}{\partial q_i}\left(\frac{\partial A_i}{\partial q_j}\frac{\partial\psi}{\partial p_j} - \frac{\partial A_i}{\partial p_j}\frac{\partial\psi}{\partial q_j}\right) + \frac{\partial\phi}{\partial p_i}\left(\frac{\partial B_i}{\partial q_j}\cdot\frac{\partial\psi}{\partial p_j} - \frac{\partial B_i}{\partial p_j}\frac{\partial\psi}{\partial q_j}\right)$$

$$+ \frac{\partial\psi}{\partial q_j}\left(\frac{\partial\phi}{\partial q_i}\cdot\frac{\partial A_j}{\partial p_i} - \frac{\partial\phi}{\partial p_i}\cdot\frac{\partial A_j}{\partial q_i}\right) + \frac{\partial\psi}{\partial p_j}\left(\frac{\partial\phi}{\partial q_i}\cdot\frac{\partial B_j}{\partial p_i} - \frac{\partial\phi}{\partial p_i}\frac{\partial B_j}{\partial q_i}\right) = 0 \quad \text{--- (5)}$$

where in the last two terms i & j have been interchanged. Since this result is true whatever ϕ & ψ, ~~the result is that~~ we equate the coefficients of $\frac{\partial\phi}{\partial q_j}\frac{\partial\psi}{\partial q_i}$, $\frac{\partial\phi}{\partial q_j}\cdot\frac{\partial\psi}{\partial p_i}$, $\frac{\partial\phi}{\partial p_j}\frac{\partial\psi}{\partial q_i}$ and $\frac{\partial\phi}{\partial p_j}\cdot\frac{\partial\psi}{\partial p_i}$ to zero and obtain

$$\frac{\partial A_i}{\partial q_j} + \frac{\partial B_j}{\partial p_i} = 0, \quad \frac{\partial A_j}{\partial q_i} + \frac{\partial B_i}{\partial p_j} = 0$$

$$\frac{\partial A_j}{\partial p_i} - \frac{\partial A_i}{\partial p_j} = 0, \quad \frac{\partial B_j}{\partial q_i} - \frac{\partial B_i}{\partial q_j} = 0$$

ie a fn H exists such that $A_i = \partial H/\partial p_i$, $B_i = -\partial H/\partial q_i$

$$* \quad (v, w) = \sum_r \left(\frac{\partial v}{\partial q_r} \cdot \frac{\partial w}{\partial p_r} - \frac{\partial v}{\partial p_r} \cdot \frac{\partial w}{\partial q_r} \right) = \sum_i \sum_j \sum_r \left(\frac{\partial v}{\partial u_i} \cdot \frac{\partial u_i}{\partial q_r} \cdot \frac{\partial w}{\partial u_j} \cdot \frac{\partial u_j}{\partial p_r} - \frac{\partial v}{\partial u_i} \cdot \frac{\partial u_i}{\partial p_r} \cdot \frac{\partial w}{\partial u_j} \cdot \frac{\partial u_j}{\partial q_r} \right)$$

$$= \sum_i \sum_j \frac{\partial v}{\partial u_i} \frac{\partial w}{\partial u_j} (u_i, u_j) = 0$$

(†) Whittaker p. 324 bottom: That the 1st Pfaffian system for the form $-\sum \frac{\partial V}{\partial a_r} d\phi_r + dV$ is given by

$$d\left(\partial V/\partial a_r\right) = 0 \text{ and } d\phi_r = 0$$

is the result in § 140, p. 311 that this system for the diff. form $\sum P_r dQ_r$ is given by

$dP_r = 0$, $dQ_r = 0$. This can be seen by writing $Q_1, \ldots Q_n = x_1, \ldots x_n$ and $P_1 \ldots P_n = x_{n+1}, \ldots x_{2n}$

& then the diff. form becomes $x_{n+1} dx_1 + \cdots + x_{2n} dx_n + 0 \cdot dx_{n+1} + \cdots + 0 \cdot dx_{2n} + 0 \cdot dx_{2n+1}$

The Pfaffian system for $\sum_{i=1}^{2n+1} X_i dx_i$ is given by $\sum_{i=1}^{2n+1} a_{ij} \delta x_i = 0$ $(j = 1, \ldots 2n+1)$ with

the bilinear covariant $a_{ij} = \frac{\partial X_i}{\partial x_j} - \frac{\partial X_j}{\partial x_i} = 0$

In particular (i) if $j = 1$, the corresp. eqn is $\sum \left(\frac{\partial X_i}{\partial x_1} - \frac{\partial X_1}{\partial x_i} \right) dx_i = 0$ reduces to $-dx_{n+1} = 0$ since $X_1 = x_{n+1}$

(ii) if $j = n+k$ " $\sum \left(\frac{\partial X_i}{\partial x_{n+k}} - \frac{\partial X_{n+k}}{\partial x_i} \right) dx_i = 0$ reduces to $dx_k = 0$

(iii) if $j = 2n+1$, " reduces to $0 = 0$. Hence the system eqns are

$dx_{n+1} = 0$, $dx_{n+2} = 0$, $\ldots dx_{2n} = 0$ and $dx_1 = 0$, $dx_2 = 0$, $\ldots dx_n = 0$

ie $dP_r = 0$, $dQ_r = 0$.

– function group – involution system

28/3/62 – Portions of previous lecture – Involution systems (function groups) – if $u_i = 0$ be in invol. & if $v_j = 0$ be consequences of these $v_j = 0$ are in involution. ~~Solving dyn. system when half the integrals are known~~ –
To prove this, a direct proof of $(v, w) = 0$ as an alternative to that on p. 322 of Whittaker can be given * – Soln of a dynamical problem when half the number of integrals is known (+)