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Temperature Distribution along a Wire Electrically Heated *in vacuo*

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The distribution is determined uniquely by three parameters, which for convenience may be taken to be (1) the temperature T_l at the centre of the rod, (2) the temperature T_m to which T_l tends as the length of the rod is increased indefinitely, the heating current being kept constant, and (3) a simple constant $a = \frac{2}{5}\frac{p\varepsilon\sigma}{\varkappa\omega}$, where $\varkappa$ is the thermal conductivity, σ is the Stefan constant of radiation, ε is the total (as distinguished from spectral) emissivity from the surface, and p and ω are the perimeter and the area respectively of the cross-section of the rod. The effect of the radiations from the surrounding walls is wholly accounted for by their effect on T_m.

The temperature distribution is given by an integral which can be readily formulated. It is found, however, that the integration cannot be done directly, and it has therefore been done numerically by previous workers. The purpose of this communication is to report that the integrand can be expanded as a power series, and the integration can be done term by term. The resulting series is found to be rapidly convergent, and may therefore be used conveniently for computation, the first few terms being quite adequate for this purpose. It is found that the rod divides itself naturally into two regions, region A in the middle corresponding to $0 \leqslant t \leqslant t_c$, and region B corresponding to $t > t_c$, where t is the drop in temperature as compared with that at the centre of the rod, and $t_c = (T_m^4 - T_l^4)/(2T_l^3)$. The power series into which the distribution integral may be expanded is different in the two regions, but the transition from one to the other is continuous, and hence either series can be used close to the boundary separating A and B.

The temperature variation in region A is found to be parabolic, and it is practically independent of the temperature coefficients of $\varkappa$, ρ and ε, where ρ is the specific resistance of the rod. When the rod is too short to have any B-region at all, the expressions become particularly simple, including the expression for T_l as a function of the length of the rod.

The distribution of temperature in an infinitely long rod is obtained readily. The distribution in the B-region of a *finite* rod is found to be practically the same as in the corresponding end region of an infinitely long rod, the discrepancy between the two increasing progressively as one approaches the upper limit of B in the finite rod, but remaining quite small at this limit, even for a short rod. The effect of the finite temperature coefficients of $\varkappa$, ρ and ε on the temperature distribution in the B-region can also be taken into account in a simple manner.

The well-known logarithmic formula for the temperature variation near the centre, usually derived on the basis $t \ll T_m$, is found to have a much wider applicability, extending far into the B-region even in short rods. The criterion for its validity is found to be $t/(2 \log t) \ll T_m$, which is a much less stringent condition than $t \ll T_m$. The additive constant in the logarithmic formula, which in the usual derivation has to be left undetermined, is also evaluated now.

Experimental results are available for the temperature distribution in tungsten filaments and in thin rods of Acheson graphite, but the lengths used are so great that T_l approximates closely to T_m in these measurements. Hence these measurements have been extended by us to shorter lengths, and also to wires of platinum and a few other metals. The observations verify in detail the results deduced theoretically.

An account of some of these investigations is in course of publication in the *Proceedings of the Royal Society*.

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The well-known logarithmic formula for the temperature variation near the centre, usually derived on the basis $t \ll T_m$, is found to have a much wider applicability, extending far into the B-region even in short rods. The criterion for its validity is found to be $t/(2 \log t) \ll T_m$, which is a much less stringent condition than $t \ll T_m$. The additive radio position; the arches are composed of systems of sharply defined filaments. The general distribution of the nebulosity in relation to the probable errors of the radio source is shown in Fig. 3.

The system is strikingly similar both in shape and structure to the larger double-loop system of *NGC* 6960 and *NGC* 6992 in Cygnus, and although no optical spectra of the Gemini loops are yet available to confirm this similarity, both a search for radio emission from the Cygnus loops and further optical studies of galactic nebulæ of this type might yield additional information about the nature of the galactic radio sources.

The radio observations were carried out as part of a programme of research supported in the Cavendish Laboratory by the Department of Scientific and Industrial Research, to which one of us (J. E. B.) is indebted for a maintenance allowance. We are also grateful to Mr. P. A. O'Brien for permission to use his unpublished results.

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Recombination in the *F*-Layers

SOME years ago, Bates and Massey[1], after showing the influence of negative ion reactions to be unimportant, suggested that dissociative recombination is the fundamental process responsible for removing electrons from the *E*-layer; and later they[2] suggested that the mechanism might also be effective in the *D*-layer. Recent experimental[3] and theoretical[4] work indicates that the recombination coefficient is probably of the required high magnitude. The *F*-layers are at least partially, and perhaps mainly, due to the photo-ionization of atomic oxygen, so it is not immediately obvious that dissociative recombination can be operative in them. However, Bates and Massey[1] tentatively proposed that the necessary molecular ions are formed through charge transfer collisions:

$$O^+ + XY \rightarrow O + XY^+, \quad (1)$$

XY being some unspecified atmospheric constituent.

In a communication in *Nature*, Banerji[5] has sought to identify XY as metastable nitrogen, which he supposes reacts through:

$$O^+ (^4S) + N_2 (A^3\Sigma_u) \rightarrow N_2^+ (X^2\Sigma_g) + O (^1S) + \sim 0{\cdot}01 \text{ eV}. \quad (2)$$

It may be noted that this process is rather more complicated than ordinary charge transfer, since a system in the ^1S state cannot be obtained by merely adding an electron to a system in the ^4S state. Arbitrarily assuming the rate coefficient to be as great as 10^{-11} cm.^{3}/sec., Banerji finds the required concentration [N_2 ($A^3\Sigma_u$)] is of order 10^7/cm.^{3}, which he points out is some 2×10^{-2} times the concentration of normal nitrogen molecules [N_2 ($X^1\Sigma_g$)].

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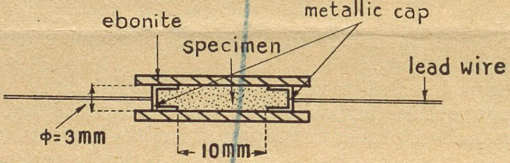

Fig. 1. Specimen cell for the measurement of electrical conductivity

conductance, (R_i), can be estimated by measuring the current which occurs by the electromotive force of the specimen cell itself compared with a standard resistor, without any applied potential from outside. Assuming the total resistivity (R), which is measured in the usual way, to be made up of the resistivity due to the electronic conductance (R_e) and that due to the ionic conductance (R_i) in parallel, then $1/R = 1/R_e + 1/R_i$, and the value of R_e can be estimated. In most cases, the value of R_i is $3 \sim 5$ times as large as R_e in freshly prepared specimen and thereafter increases to $10 \sim 20$ times. Hence,

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The distribution of temperature along a thin rod electrically heated *in vacuo*
I. Theoretical

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(*Received* 10 *July* 1953)

The paper describes a detailed investigation of the distribution of temperature along a thin rod or wire electrically heated *in vacuo*. The distribution is determined uniquely by three parameters, namely, (1) the temperature at the centre of the rod, (2) that at the centre of a similar infinitely long rod heated by the same current, and (3) a simple constant that involves directly the cross-section of the rod, its thermal conductivity and the emissivity of its surface. Though the integral that determines the temperature distribution cannot be evaluated directly, it is shown that it can be expanded as a convergent power series. The rod may be divided for this purpose into two well-defined regions, region A comprising a certain length in the middle of the rod, and region B consisting of the portions outside A, the appropriate series for the evaluation of the integral being different in the two regions. The temperature coefficients of the physical constants involved can also be taken into account in the evaluation.

The temperature distribution in the A-region is found to be parabolic. Hence when the rod is too short to have any B-region at all, the theoretical relations become particularly simple, and are also practically independent of the finite temperature coefficients of the physical constants involved.

The distribution in the B-region is found to be practically the same as in the corresponding end-portions of an infinitely long rod, except near the upper limit of the B-region when this region is very short.

An analytical expression is also obtained for the temperature at the centre of the rod as a function of its length.

The criterion for the validity of the well-known logarithmic formula for temperature distribution is shown to be much less stringent than is generally taken to be; the range of its applicability is thus extended far into the B-region even when the rod is not long. The additive constant in the formula, which previously had to be left undetermined, also gets evaluated now.

1. Introduction

The distribution of temperature along a thin rod or a thin-walled tube heated *in vacuo* by sending a heavy electric current through it is of interest experimentally. When the rod or the tube is sufficiently long, there is a considerable portion in the middle in which the temperature is sensibly constant. In this region practically the whole of the heat generated by the passage of the electric current is dissipated by radiation from the outer surface; the loss of heat by conduction from the centre outwards, and, in the case of the tube, also by radiative transfer occurring in the cavity, will then be negligible.

The occurrence of such a region of sensibly uniform temperature is utilized for various purposes experimentally, and it is therefore of some importance to know how long any actual thin rod or tube has to be in order that a specified length about the centre may be of sensibly uniform temperature, i.e. in order that the maximum deviation in temperature in this range may not exceed a specified small tolerance. More generally, it is desirable to know in detail the distribution over its whole length irrespective of whether it is long or short, and also to know the rate at which heat is conducted away at the ends. Such detailed information, apart from its intrinsic interest, will also be helpful in connexion with the design of sealed-in heating elements such as are used for thermionic or illumination purposes.

In this part (I) of the paper is given a theoretical investigation of the distribution of temperature along a thin rod. Extensive experimental data are available for tungsten, and some for Acheson graphite, which have been supplemented by some fresh measurements made by the present authors. Part II of the paper will be concerned with a discussion of these data in relation to the results deduced theoretically in part I.

The case of a *tube*, even when it is thin-walled, is slightly more complicated because of the radiative transfer of heat from the centre towards the ends that will occur in the cavity of the tube, which is more difficult to take into account.

2. Formulation of the problem and earlier work

Let κ be the thermal conductivity of the material, ρ its specific resistance and ϵ the total emissivity of the surface, i.e. the emissivity over the whole of the significant range of its emission spectrum. Neglecting to a first approximation the temperature coefficients of these quantities, and neglecting also, since the rod is taken to be thin, the radial gradient of temperature in the rod, the equation of conduction in the steady state may be written in the form (Carslaw & Jaeger 1947)

$$\frac{\mathrm{d}^2T}{\mathrm{d}x^2}-\frac{p\epsilon\sigma}{\kappa\omega}(T^4-T_0^4)+\frac{I^2\rho}{\kappa\omega^2}=0, \tag{1}$$

in which x is the distance of the point under consideration from the near end of the rod, ω is the cross-sectional area of the rod, p is the perimeter of the cross-section, T_0 is the temperature of the walls of the chamber and also of the ends of the rod, σ is Stefan's constant of radiation, and I is the heating electric current. The current is taken to be an alternating one, in which case the two halves of the rod will be symmetrical about its centre. The loss of energy due to thermionic emission is in general negligible and hence is not included here.

Following Carslaw & Jaeger and using the boundary conditions

$$\left.\begin{aligned} T=T_0 \quad &\text{when} \quad x=0,\\ \text{and} \qquad \frac{\mathrm{d}T}{\mathrm{d}x}=0 \quad &\text{when} \quad x=l,\end{aligned}\right\} \tag{2}$$

where $2l$ is the length of the rod, one obtains

$$x=\int_{T_0}^{T}\frac{\mathrm{d}T}{[a(T^5-T_l^5)-b(T-T_l)]^{\frac{1}{2}}}, \tag{3}$$

in which T_l is the temperature at the centre of the rod, i.e. at $x=l$,

$$a=\frac{2}{5}\frac{p\epsilon\sigma}{\kappa\omega}, \tag{4}$$

$$b=\frac{2I^2\rho}{\kappa\omega^2}+\frac{2p\epsilon\sigma T_0^4}{\kappa\omega}. \tag{5}$$

Evidently

$$l=\int_{T_0}^{T_l}\frac{\mathrm{d}T}{[a(T^5-T_l^5)-b(T-T_l)]^{\frac{1}{2}}}. \tag{6}$$

Now for a rod of given material and cross-section, and for a known heating current, (6) may be regarded as defining the temperature T_l at the centre of the rod when its length is $2l$. Knowing T_l, (3) enables one to obtain the temperature T at every other point x on the rod.

It does not seem to be possible to do the integration in (6) directly. Bush & Gould (1927) have done the integration with the help of a differential analyzer, and the temperature distribution in a heated tungsten wire obtained in this manner has been used by them for determining the distribution of thermionic emission along the wire. In using the differential analyzer for calculating the temperature distribution they found it possible also to include the known temperature variations of κ, ρ and ϵ. The integration of (3) has also been done numerically by Baerwald (1936). Using the known values of κ, ρ and ϵ for molybdenum at different temperatures, the temperature gradient $\mathrm{d}T/\mathrm{d}x$ is first evaluated at different points of the rod, and thence the distribution integral is evaluated by him numerically. Some of the steps in the integration have also been done graphically by Worthing (see Worthing & Halliday 1948). The distribution function has also been discussed by Langmuir & Taylor (1936) and by Cox (1943).

When the rod is very long there will be naturally a considerable region in the middle where the temperature will be sensibly constant. The loss of heat from the centre by conduction will in this case be negligible in comparison with the loss due to radiation. Confining attention to such a rod, and to a region where t, defined by

$$t = T_m - T, \tag{7}$$

is small in comparison with T_m, the equation of conduction (1) may be shown to reduce to the simple form

$$\mathrm{d}^2t/\mathrm{d}x^2 - At = 0, \tag{8}$$

which can be directly solved. The solution is

$$x\sqrt{A} = D - \ln t, \tag{9}$$

in which A is known, but not the integration constant D, since it cannot be determined directly from the boundary values.

The logarithmic formula—to be more precise the formula in its exponential form $t = \exp(D - x\sqrt{A})$—has been discussed by Worthing (1914, 1922), Stead (1920), Prescott & Hincke (1928), Bearwald (1936) and others. We may emphasize here an important result that emerges from the observations of Prescott & Hincke. Their plots of the experimental data for Acheson graphite show that the logarithmic formula (9) is verified even for regions where the fundamental basis for its derivation, namely, that t should be small in comparison with T_l, cannot be justified at all. Further, Baerwald has compared the values obtained by the numerical evaluation of the integral with the corresponding values given by the logarithmic formula. His curves show that the two sets of values practically coincide up to

$$t/T_l = 300°/1300°,$$

which is by no means a small fraction.

In view of these it is desirable to be able to discuss the temperature distribution analytically in order to follow more closely the implications of these and other findings. It is the main purpose of this paper to show that even in the general case of a rod of *any* length it is possible to expand the integrand in (3) as a convergent power series, and do the integration term by term. The convergence is found to be rapid, and one can therefore evaluate the integral to any desired degree of approximation in this manner. One also gets thus a proper estimate of the degree of approximation involved in any particular step that may be taken.

3. Expansion of the integrand in (3) as a series

At any point x the energy generated, and the energy radiated, per second per unit length of the rod are given by $I^2\rho/\omega$ and $p\epsilon\sigma(T^4-T_0^4)$ respectively. Neglecting as before the temperature variations of κ, ρ and ϵ—they will be taken into account in a later section—it is convenient to define a temperature T_m such that

$$p\epsilon\sigma(T_m^4-T_0^4) = I^2\rho/\omega. \tag{10}$$

Obviously (10) corresponds to the condition that the heat generated is dissipated away wholly by radiation, i.e. it corresponds to a region where the transfer of heat by conduction is negligible, and hence where the temperature is sensibly constant. This will evidently be the case near the centre of a very long rod. In other words T_m will be the value to which the temperature T_l at the centre tends when $l \to \infty$. For shorter lengths of the rod, however, T_l will be smaller than T_m, and hence

$$I^2\rho/\omega - p\epsilon\sigma(T_l^4-T_0^4) \tag{11}$$

will be finite, and quite large for short rods.

Now, in view of (4), (5) and (10),

$$b/a = 5T_m^4. \tag{12}$$

Hence (3) may be written in the form

$$x\sqrt{a} = \int_{T_0}^{T} [5T_m^4(T_l-T)-(T_l^5-T^5)]^{-\frac{1}{2}}\,\mathrm{d}T. \tag{13}$$

Putting as before $T_l - T = t$, one obtains from (13)

$$x\sqrt{a} = \int_{t}^{t_0=T_l-T_0} [5(T_m^4-T_l^4)\,t+10T_l^3t^2-10T_l^2t^3+5T_lt^4-t^5]^{-\frac{1}{2}}\,\mathrm{d}t. \tag{14}$$

The temperature distribution is thus determined uniquely by the three parameters T_l, T_m and a. Now the dependence on a (which includes the dependence on the dimensions p and ω of the cross-section of the rod) is particularly simple, and by expressing the lengths along the rod in terms of $1/\sqrt{a}$ (as has been done in plotting the curves in figure 1), the temperature distribution may be made to depend on the two parameters T_l and T_m only.

Before proceeding to consider the integration of (14) it is desirable to make a small generalization of the problem. Till now the ends of the rod had been taken to be at the same temperature T_0 as the walls of the chamber. This need not necessarily be the case. The effect of the temperature T_0 of the walls is wholly confined to determining the temperature T_m as defined in (10), and hence changing the temperature of the walls is merely equivalent to changing suitably the value of T_m appearing in equation (14).

In the discussion that follows, the temperature of the walls is taken to be T_0, and that of the two ends of the rod as zero. The results obtained therefrom can be readily adapted to the case where the ends are kept at any other temperature Θ, since it is equivalent merely to ignoring the portions in the theoretical rod whose temperatures are below Θ. It can also be adapted to the case when the two ends of the rod are at different temperatures.

4. Separation of the rod into two regions

For a given length $2l$ of the rod and given heating current, both T_l and T_m appearing in (14) may be regarded as constants independent of x. It is convenient to define a certain critical value of t, namely, t_c, such that

$$t_c = (T_m^4 - T_l^4)/(2T_l^3). \tag{15}$$

Depending on whether $t < t_c$ or $> t_c$, the first term inside the square brackets in (14), will be greater than, or less than, the second. The manner in which the integrand in (14) is to be expanded as a power series will therefore depend on whether $t < t_c$ or $> t_c$; there is, however, a continuity between the two regions which makes the two series overlap in a small range in the close neighbourhood of t_c.

One may accordingly divide the rod into two regions, one in the middle of the rod in which $0 < t < t_c$, which may be designated as region A, and the other outside this range, corresponding to $t > t_c$, which may be designated as region B. When l is not sufficiently great, t_c may exceed the total fall in temperature, namely, T_l over the whole length of the rod, in which case obviously the whole of the rod will be included in region A, and there will be no B-region at all.

5. Region A considered in detail

Taking up first the A-region, (14) may be expressed in the form

$$q[5a(T_m^4 - T_l^4)]^{\frac{1}{2}} = \int_0^t t^{-\frac{1}{2}}(1+y)^{-\frac{1}{2}}\,dt, \tag{16}$$

in which $q = l - x$ is the distance as measured from the centre of the rod, of the point under consideration whose temperature is t, and

$$y = \frac{t}{t_c}\left(1 - \frac{t}{T_l} + \frac{t^2}{2T_l^2} - \frac{t^3}{10T_l^3}\right). \tag{17}$$

The upper limit of the integral in (16), namely, the temperature t at the point under consideration, may extend, when the rod is long enough to have a B-region to $t = t_c$, and when the rod is too short to accommodate the whole of A, to $t = T_l < t_c$. It can be readily seen that for any position of the point in this range, y remains less than unity, so that the integrand in (16) may be expanded as a power series and the integration may be done term by term. Doing so, one obtains

$$q^2 = \frac{4t(1-S)^2}{5a(T_m^4 - T_l^4)}, \tag{18}$$

where
$$S = \frac{1}{2\sqrt{t}}\left[\int_0^t \tfrac{1}{2}t^{-\frac{1}{2}}y\,dt - \int_0^t \tfrac{3}{8}t^{-\frac{1}{2}}y^2\,dt + \int_0^t \tfrac{5}{16}t^{-\frac{1}{2}}y^3\,dt - \ldots\right]. \tag{19}$$

S is a number which can be seen to be small in comparison with unity, and may be readily evaluated from (19), since even when y is close to unity, which will be the case when the rod is very long and the point under consideration is close to the upper limit of the A-region, i.e. when $t = t_c \ll T_l$, the convergence of the series (19) is rapid enough for convenient numerical computation. This value of S, which naturally is an extreme one, is found to be about 0·13. But, in general, S will be much smaller than this, and when there is no B-region at all, actually less than 0·08. Hence, in general, whether the rod is short or long, S may be neglected in comparison with unity, and thus one obtains

$$q^2 = \frac{4t}{5a(T_m^4 - T_l^4)}. \tag{20}$$

In other words, as one moves out from the centre the observed temperature variation will be parabolic, and will continue to be practically so over the whole of the A-region.

In order to have some idea of the magnitude of the factor $T_m^4 - T_l^4$ appearing in (20), it may be mentioned that when the rod is just long enough to accommodate the whole of the A-region, $t_c = T_l$, and hence in view of (15)

$$T_l/T_m = (\tfrac{1}{3})^{\frac{1}{4}} = 0{\cdot}76. \tag{21}$$

As the length is increased, T_l will obviously approach closer to T_m. (The case of shorter rods will be considered in detail in the next section.)

Now the linear extent q_c of the A-region will be given by

$$q_c = \left(\frac{2}{5aT_l^3}\right)^{\frac{1}{2}}, \tag{22}$$

and it varies from $2^{\frac{1}{2}}/(5aT_m^3)^{\frac{1}{2}}$ for an infinitely long rod, to $3^{\frac{3}{8}}.2^{\frac{1}{2}}/(5aT_m^3)^{\frac{1}{2}}$ for a rod just long enough to include only the A-region; thus the length of the A-region remains of the same *order* of magnitude in spite of the enormous range of l.

The total drop in temperature over the whole of the A-region, namely lt_c, will naturally be the smaller the longer the rod.

6. Short rods, $l < q_c$

When the length of the rod falls short of q_c, equation (20), which gives the temperature distribution, will be an even closer approximation than before. Now the total drop in temperature over the whole length becomes T_l. Substituting this value for t in (20), one obtains

$$l^2 = \frac{4T_l}{5a(T_m^4 - T_l^4)}. \tag{23}$$

For such short rods, (23) determines the dependence of T_l on l, and on the heating current, which naturally determines T_m.

As the length of the rod is reduced, T_l^4 will at some stage become small enough to be negligible in comparison with T_m^4. At any given distance q from the centre, the temperature drop t, which will be seen from (20) to be proportional $T_m^4 - T_l^4$, will then become practically independent of T_l and therefore of l. In other words, the temperature distribution curves t against q for such short rods will be nearly parallel to one another, their relative separations being along the t axis and independent of q.

7. The series applicable to the B-region

Considering the B-region defined by $t > t_c$, the second term inside the square brackets in (14) now becomes larger than the first term, unlike in the A-region. Equation (14) may now be written in the form

$$x(10aT_l^3)^{\frac{1}{2}} = \int_t^{T_l} t^{-1}\left[1+\frac{t_c}{t}-\frac{t}{T_l}+\frac{1}{2}\frac{t^2}{T_l^2}-\frac{t^3}{10T_l^3}\right]^{-\frac{1}{2}} \mathrm{d}t. \tag{24}$$

All the terms inside the square brackets in (24) that involve t are separately and together less than unity, and hence the integrand can be expanded again as a power series

$$x(10aT_l^3)^{\frac{1}{2}} = \int_t^{T_l}\left[\frac{1}{gt}+\frac{1}{2T_l g^3}-\frac{t}{4T_l^2}\left(\frac{1}{g^3}-\frac{3}{2g^5}\right)+\dots\right]\mathrm{d}t, \tag{25}$$

where

$$g^2 = 1+t_c/t. \tag{26}$$

The integration in (25) can be done term by term. Doing so one obtains x as the sum of three convergent power series, namely

$$x(10aT_l^3)^{\frac{1}{2}} = [U+V+W]_t^{T_l}, \tag{27}$$

where

$$U = \ln t+\frac{t}{2T_l}\left(1+\frac{t}{8T_l}-\dots\right), \tag{28}$$

$$V = 2\ln(g+1)+\frac{t_c}{T_l}\left(\frac{1}{g}-\frac{3}{4}\ln\frac{g+1}{g-1}\right)$$
$$-\frac{t_c^2}{2T_l^2}\left(\frac{7}{2g}+\frac{1}{2g^3}+\frac{19}{16}\frac{g}{g^2-1}-\frac{75}{32}\ln\frac{g+1}{g-1}\right)+\dots, \tag{29}$$

$$W = \frac{(g-1)\,t}{2T_l}\left(1+\frac{t}{8T_l}-\dots\right). \tag{30}$$

Thus in both the regions A and B, the integral on the right-hand side of (14) can be evaluated to any desired approximation, though the series involved in the evaluation are naturally different for the two regions.

8. Infinitely long rod

Before considering (27), applicable to the B-region of a rod of any length $2l$, it is desirable to consider the special case when the rod is infinitely long, for which $T_l = T_m$ and hence $t_c = 0$ and $g = 1$. It will be seen from (29) and (30) that in this case both $[V]_t^{T_m}$ and W vanish, and (27) reduces to

$$x(10aT_m^3)^{\frac{1}{2}} = [U]_t^{T_m} = D-\left\{\ln t+\frac{t}{2T_m}\left(1+\frac{t}{8T_m}-\frac{t^2}{120T_m^2}+\dots\right)\right\}, \tag{31}$$

in which D is a constant given by

$$D = \ln T_m + \chi, \tag{32}$$

where
$$\chi = \tfrac{1}{2} + \tfrac{1}{16} - \tfrac{1}{240} + \dots. \tag{33}$$

Equation (31) gives the distribution over the whole length of an infinitely long rod.

This expression could also have been obtained directly from (3) by putting $T_l = T_m$. The integrand could then have been expanded in the form of a series and integrated term by term.

Consider now a region of the rod in which $t/(2\ln t) \ll T_m$. The temperature distribution in this region will be given by

$$x(10aT_m^3)^{\frac{1}{2}} = D - \ln t. \tag{34}$$

Thus one obtains the same logarithmic formula as was obtained in (9) from certain direct considerations. The constant A appearing in (8) and (9) can be readily shown, in view of (10) and (4), to be equal to $10aT_m^3$, which is identical with the corresponding value appearing in (34).

Further, the integration constant D appearing in the logarithmic formula, which previously had to be left undetermined, now gets determined uniquely, and is given by expression (32). This expression, like all the other expressions in this paper, refers to the case when the end-temperatures are zero. If the ends are kept at any other temperature Θ the additive constant D appearing in (34) may be shown to be

$$D = \ln \tau + \frac{\tau}{2T_m} + \frac{\tau^2}{16T_m^2} - \frac{\tau^3}{240T_m^3} + \dots, \tag{35}$$

where $\tau = T_m - \Theta$. It may be seen that (35) reduces to (32) when $\Theta = 0$.

Further the criterion for the validity of the logarithmic formula now comes out to be $t/(2\ln t) \ll T_m$, which is not quite so stringent as $t \ll T_m$, on the basis of which (9) was deduced. Taking, for example, $T_m = 1500°\,\mathrm{K}$ and a fraction $\frac{1}{150}$ of it as negligible in comparison with it, the present criterion will make the logarithmic formula valid over the range $0 < t < 100°$, whereas the corresponding range contemplated in the derivation of (9) is only a tenth of it.

At first sight it may seem a little surprising that the range of t over which the logarithmic formula (34) holds is much wider than that over which its *equivalent* differential equation (8) holds. That this should be so may be seen from the following argument. Retaining one more term on the right-hand side of (34), one obtains

$$x\sqrt{A} = D - \ln t - \frac{t}{2T_m}, \tag{36}$$

in which $t \ll T_m$. It can be readily seen by differentiating it twice that

$$\frac{d^2t}{dx^2} = At\left(1 - \frac{3t}{2T_m}\right). \tag{37}$$

Whereas in (36) the correction will become negligible when $t \ll 2T_m \ln t$, the corresponding correction term in (37) will be negligible only when $t \ll \frac{2}{3}T_m$.

The increased range of validity of the logarithmic formula in a long rod is in agreement with the results of the numerical calculations of Baerwald referred to in § 2, and also with the observations of Prescott & Hincke.

This extension of the range of applicability of the logarithmic formula has an important consequence, as we shall see in the next section, namely, that even in rods that will be normally regarded as not long enough, there is a considerable range in which the formula holds.

9. The B-region further considered

It will be seen from (27) which gives the temperature distribution over the B-region, and from (30), that W is practically negligible over the whole of this region. It is also found after a little calculation that *the temperature at any given distance x from the end of the rod is practically the same as in a similar infinitely long rod.* Indeed, the difference between the two becomes detectable only near the upper limit of the B-region of the finite rod, and even there only when the B-region is small. This is an important result, which may be readily verified from the expressions (27) and (31). It can also be seen to be the case from the following argument. From (13) one obtains for the gradient of the temperature along the rod

$$\phi = \mathrm{d}T/\mathrm{d}x = a^{\frac{1}{2}}T_m^{\frac{5}{2}}\left[\frac{5T_l}{T_m} - \frac{T_l^5}{T_m^5} - \left(\frac{5T}{T_m} - \frac{T^5}{T_m^5}\right)\right]^{\frac{1}{2}}. \tag{38}$$

As will be seen from (38) the difference in slope between a rod of finite length and one of infinite length, both at the same T, is determined by the defect of T_l from T_m. Denoting the gradients for the infinitely long rod and the finite rod, both at temperature T, by ϕ_∞ and ϕ_l respectively, it can be seen that $(\phi_\infty - \phi_l)/\phi_\infty$, i.e. the relative difference between the two slopes, will be a minimum at the end of the rod, and will be the largest near the upper limit of the B-region.

We shall now consider the two extreme cases (1) when the B-region is just present, and (2) when the B-region is extensive, in both cases at the upper limit of the B-region. Taking up (1) first, this corresponds to T being very small and $T_l^4/T_m^4 = \frac{1}{3}$. It can be readily seen that in this case

$$\left.\begin{aligned} \phi_\infty &= 2a^{\frac{1}{2}}T_m^{\frac{5}{2}}, \\ \phi_l &= (14^{\frac{1}{2}}/3^{\frac{5}{8}})\, a^{\frac{1}{2}}T_m^{\frac{5}{2}}, \end{aligned}\right\} \tag{39}$$

the difference between the two being about 6 % of either.

When the B-region is very extensive, we may regard $T_m - T_l$, equal to Δ, say, as small in comparison with T_m. At $T = T_l - t_c$ it can be seen from (38) that

$$\left.\begin{aligned} \phi_\infty &= 90^{\frac{1}{2}}a^{\frac{1}{2}}T_m^{\frac{3}{2}}\Delta, \\ \phi_l &= 80^{\frac{1}{2}}a^{\frac{1}{2}}T_m^{\frac{3}{2}}\Delta, \end{aligned}\right\} \tag{40}$$

the difference between the two being now about 5 %.

In other words, the slope at the upper limit of the B-region of a finite rod differs from that of an infinitely long rod at the same temperature by about 5 to 6 %, irrespectively of whether the finite rod has just a B-region present, or has an extensive B-region. The difference between the two slopes drops down rapidly as one moves inside the B-region.

Since the gradient of temperature ϕ_l at every temperature T inside the B-region is thus practically the same as ϕ_∞ at the same temperature in an infinitely long rod, the actual temperature T at any given distance x from the end will also be the same for the two rods. Since again $(\phi_\infty - \phi_l)/\phi_\infty$ has its largest value, namely, 5 %, at the upper limit of B, and as one moves into the B-region it drops down rapidly, the difference in temperature between the two rods at any given distance x from the end will be much less than 5 % even at the top of the B-region.

That the temperature distribution over practically the whole of the B-region of a finite rod is the same as in an infinitely long rod, apart from its intrinsic interest has also an important consequence. If the B-region is long enough, there will be a considerable range in it in which t is sufficiently small in comparison with $2T_m \ln t$ to make the distribution correspond to the logarithmic formula (34) characteristic of the infinitely long rod; t appearing in the formula will continue to denote $T_m - T$ in the case of the finite rod also.

Since the distribution of temperature in the B-region is nearly that of an infinitely long rod, the length of the B-region, equal to x_c, say, may be obtained from (31) by putting $t = t_c + T_m - T_l$. To a first approximation, which will be a fairly close one, one thus finds

$$x(10aT_m^3)^{\frac{1}{2}} = \ln \frac{2T_l^3 T_m}{T_m^4 + 2T_l^3 T_m - 3T_l^4} + \frac{3T_l^4 - T_m^4}{4T_l^3 T_m}. \tag{41}$$

Obviously

$$x_c + q_c = l, \tag{42}$$

which is an important relation defining T_l as a function of l. This relation has many useful applications.*

10. The temperature coefficients of κ, ρ and ϵ included

Till now the quantities κ, ρ and ϵ were regarded, for convenience, as temperature-independent. It is not, however, difficult to take into account their temperature variations too. The differential equation corresponding to equation (1) will then be

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\kappa_T \frac{\mathrm{d}T}{\mathrm{d}x}\right) + \frac{I^2 \rho_T}{\omega^2} - \frac{p\epsilon_T \sigma}{\omega}(T^4 - T_0^4) = 0. \tag{43}$$

* [*Note added in proof*, 28 *December* 1953.] Equation (1) may also be written in the form

$$\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} + \tfrac{5}{2}a(T_m^4 - T_l^4) + \tfrac{5}{2}a(T_l^4 - T^4) = 0.$$

In the close neighbourhood of the centre of a *finite* rod the third term on the left-hand side becomes negligible in comparison with the second. One then obtains the parabolic law (20).

On the other hand at *any* point of an infinite rod, or at a point sufficiently removed from the centre of a finite rod, the second term may become negligible in comparison with the third, in which case one obtains (13). The transition from the A- to the B-region in a finite rod corresponds to the third term growing to a magnitude comparable with the second.

Let κ_l, ρ_l and ϵ_l be the values of κ, ρ and ϵ at $T = T_l$. Further, let their temperature variations be linear,* and let the temperature coefficients be α, β and δ respectively. Equation (43) then reduces to

$$(1-\alpha t)\frac{d^2t}{dx^2} - \alpha\left(\frac{dt}{dx}\right)^2 = \tfrac{5}{2}a[T_m^4(1-\beta t) - (1-\delta t)(T_l - t)^4 + (\beta-\delta)\,tT_0^4], \tag{44}$$

where the physical constants involved in defining a are to be assigned values appropriate to $T = T_l$. Multiplying both sides by $2(1-\alpha t)$ and doing the first integration with respect to x one obtains

$$(1-\alpha t)^2\left(\frac{dt}{dx}\right)^2 = 5at\left[T_m^4 - T_l^4 + T_l^4\left\{\frac{2a_1 t}{T_l} - \frac{2a_2 t^2}{T_l^2} + \frac{a_3 t^3}{T_l^3} - \frac{a_4 t^4}{5T_l^4} + \frac{a_5 t^5}{6T_l^5} - \frac{a_6 t^6}{7T_l^6}\right\}\right]$$

$$= Q, \quad \text{say}, \tag{45}$$

where

$$\left.\begin{aligned}
a_1 &= \tfrac{1}{4}\{4 + (\alpha+\delta)\,T_l - (\alpha+\beta)\,T_m^4/T_l^3 + (\beta-\delta)\,T_0^4/T_l^3\},\\
a_2 &= \tfrac{1}{6}\{6 + 4(\alpha+\delta)\,T_l + \alpha\delta T_l^2 - \alpha\beta T_m^4/T_l^2 + \alpha(\beta-\delta)\,T_0^4/T_l^2\},\\
a_3 &= \tfrac{1}{4}\{4 + 6(\alpha+\delta)\,T_l + 4\alpha\delta T_l^2\},\\
a_4 &= 1 + 4(\alpha+\delta)\,T_l + 6\alpha\delta T_l^2,\\
a_5 &= (\alpha+\delta)\,T_l + 4\alpha\delta T_l^2,\\
a_6 &= \alpha\delta T_l^2.
\end{aligned}\right\} \tag{46}$$

From (45) one obtains

$$x = \int_t^{T_l} \frac{1-\alpha t}{Q^{\frac{1}{2}}}\,dt. \tag{47}$$

It will be seen that in the special case when $\alpha = \beta = \delta = 0$ each of the coefficients $a_{1,\ldots,6}$ defined by (46) reduces to unity, and (47) reduces to (14) as it should. Even otherwise a_1 will remain practically at unity.

Now let us consider first the case of an infinitely long rod, for which $T_l = T_m$. Equation (47) can in this case be shown to reduce to

$$x(10aa_1T_m^3)^{\frac{1}{2}} = \int_t^{T_m} \frac{(1-\alpha t)\,dt}{t\left(1 + \dfrac{b_1 t}{T_m} + \dfrac{b_2 t^2}{T_m^2} + \dfrac{b_3 t^3}{T_m^3} + \dfrac{b_4 t^4}{T_m^4} + \dfrac{b_5 t^5}{T_m^5}\right)^{\frac{1}{2}}}, \tag{48}$$

where

$$b_1 = -a_2/a_1, \quad b_2 = a_3/(2a_1), \quad b_3 = -a_4/(10a_1),$$
$$b_4 = a_5/(12a_1), \quad b_5 = -a_6/(14a_1),$$

and the values of $a_{1,2,\ldots,6}$ are now those corresponding to $T_l = T_m$.

Expanding the integrand in (48) as a power series in t/T_m and integrating term by term, one obtains

$$x(10aa_1T_m^3)^{\frac{1}{2}} = \ln T_m + \chi' - \left[\ln t + \frac{A_1 t}{T_m} + \frac{A_2 t^2}{T_m^2} + \frac{A_3 t^3}{T_m^3} + \ldots\right], \tag{49}$$

where

$$\left.\begin{aligned}
A_1 &= -(b_1/2 + \alpha T_m), \quad A_2 = \tfrac{1}{4}(3b_1^2/4 + \alpha b_1 T_m - b_2),\\
A_3 &= \tfrac{1}{48}(12b_1b_2 - 6b_1^2\alpha T_m + 8b_2\alpha T_m - 8b_3 - 5b_1^3),\\
\chi' &= A_1 + A_2 + A_3 + \ldots.
\end{aligned}\right\} \tag{50}$$

* This assumption considerably simplifies the expressions and hence is adopted here. But it is possible to take into account the actual temperature variations of these quantities even if they are not linear. In general, α is negative and β and δ positive.

Again it will be seen that when α, β and δ vanish, A_1, A_2 and A_3 tend to $\frac{1}{2}$, $\frac{1}{16}$ and $-\frac{1}{240}$ respectively, $\chi' \to \chi$, and (49) reduces to (31) as it should.

Now the effect of the finite temperature coefficients of κ, ρ and ϵ on the temperature distribution is twofold:

(1) the effect on the additive constant, which is now χ' instead of χ; this effect is naturally independent of t and is also independent of the length of the rod;

(2) the effect on the terms involving t, which will be the greater the larger the value of t.

The investigation in the A-region of a finite rod, which will be given presently, shows that even when the rod contains just the A-region, and t is of the order of t_c or T_l, the second effect is negligible. Hence the total effect of the finite temperature coefficients on the distribution in these regions, $t \lesssim t_c$, of an infinitely long rod, is to elongate x by

$$\Delta x = (\chi - \chi')/(10aT_m^3)^{\frac{1}{2}}. \tag{51}$$

The constant D appearing in the logarithmic formula will also be similarly increased by $\chi - \chi'$.

One can also investigate in the same manner the effect of the temperature coefficients of κ, ρ and ϵ in the case of a finite rod. It will suffice to quote here the final results. Taking up the A-region first, it is found that instead of (18), which holds when the temperature coefficients are zero, one now obtains

$$q^2 = \frac{4t(1-f)^2}{5a(T_m^4 - T_l^4)}, \tag{52}$$

where

$$f = \tfrac{1}{6}(a_1 + 2\alpha t_c)\frac{t}{t_c} - \frac{1}{5}\left(\frac{3a_1^2}{8} + \frac{a_2 t_c}{2T_l} + \frac{a_1 \alpha t_c}{2}\right)\frac{t^2}{t_c^2} + \dots. \tag{53}$$

In the case of temperature-independent κ, ρ and ϵ it was found in § 5 that $f \leqslant 0{\cdot}13$, and it was regarded as negligible in comparison with unity. The effect of the finite temperature coefficients is equivalent to multiplying f by a factor of the order of $1 + \alpha t_c$, which under normal working conditions will still be close to unity. Hence (20) will continue to represent the temperature distribution in the A-region and to the same degree of approximation, as with zero temperature coefficients. This will be the case even for very short rods.

Considering next the B-region of a finite rod, the temperature distribution in it will be given by an expression similar to (27) in which the right-hand side will now be

$$[U' + V' + W']_t^{T_l}, \tag{54}$$

and the temperature at a given distance x from the end will be the same as for an infinitely long rod, and differ appreciably from it only when the B-region is short and in addition t is close to t_c. Hence the temperature distribution in the B-region will be the same as for an infinitely long rod just as in the temperature-independent case. In particular, in regions close to t_c the effect of the finite temperature coefficients of κ, ρ and ϵ will be merely to elongate x by Δx defined by (51).

Since the temperature distribution in the A-region remains unaffected, this will be equivalent to extending by the same amount the total half-length l of the rod corresponding to a given T_l and T_m.

Comparing now two rods having the same T_l and T_m, one in which κ, ρ and ϵ have finite temperature coefficients and the other in which κ, ρ and ϵ are temperature-independent and have the same values as the former at $T = T_l$, the two curves diverge slightly near the origin, but soon settle to a constant displacement along the x axis, of the former curve with respect to the latter, by an amount given by (51). This will be the case also when the two rods are infinitely long.

11. Some relevant graphs

As was mentioned in an earlier section, when the physical constants involved are temperature-independent, the three parameters T_l, T_m and a determine uniquely the temperature distribution along the rod. Taking for example $T_m = 1500°$ K, and expressing the lengths in terms of $1/\sqrt{a}$, the temperature distribution curves have been plotted in figure 1 for different selected values of T_l. The values plotted were

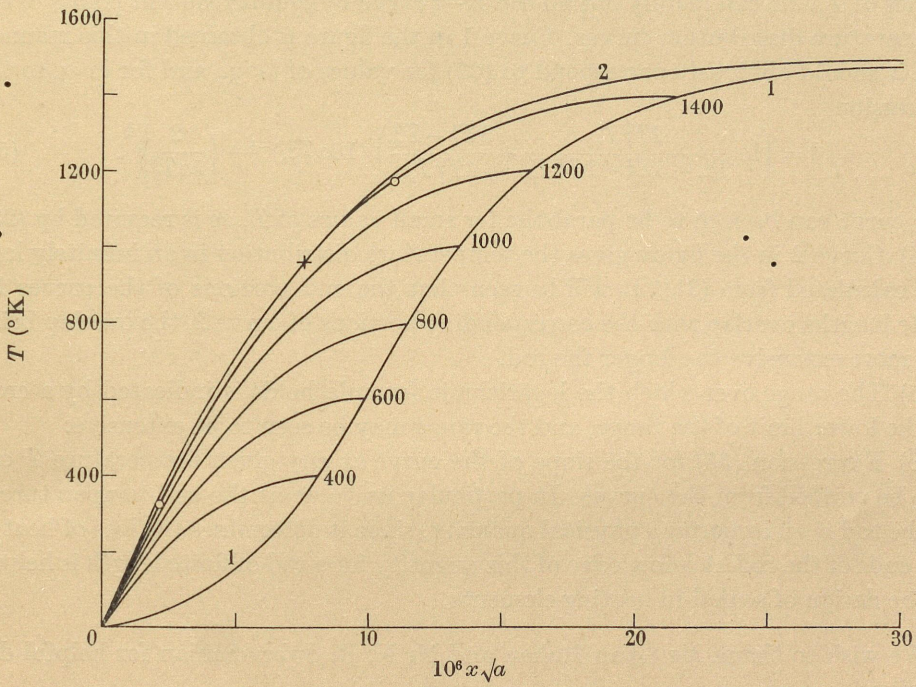

Figure 1. Temperature distribution curves for $T_m = 1500°$ K. and various values of T_l marked in the figure.

calculated on the basis of the relations (18) and (27) respectively in the A- and the B-regions, i.e. treating κ, ρ and ϵ as independent of temperature. *The same curves may also be utilized to give the temperature distribution in the case when the temperature coefficients of κ, ρ and ϵ are finite*, in the following manner. The abscissa should now be taken to represent not $x\sqrt{a}$, but

$$x\sqrt{a} - \left\{(\chi' - \chi) - (A_1 - \tfrac{1}{2})\frac{t}{T_m} - \dots\right\}(10T_m^3)^{-\frac{1}{2}}. \tag{55}$$

We shall draw attention here to some of the important features of the curves plotted in figure 1, and the corresponding results deduced in the previous sections which they help to illustrate.

(1) The limit of the A-region is indicated by a small circle on the curve. The horizontal distance from the circle to the end of the curve, corresponding to $T = T_l$ gives the linear extension q_c of the A-region. The variation of q_c with T_l or l obtained in (22) may be readily verified.

(2) The value of T_l corresponding to $l = q_c$ is given by $T_l = T_m/3^{\frac{1}{4}} = 1140°$. Hence curves for which $T_l < 1140°$ correspond to short rods, $l < q_c$, which have no B-region at all. These curves are all parabolic. From the curves of T against x plotted here the corresponding curves of t against q can be easily obtained by suitably displacing the former curves horizontally. These t against q curves for short rods may be seen to be parallel to one another (see § 6).

(3) Having plotted the temperature-distribution curves for different selected values of T_l, one can readily obtain the $T_l - l$ curve by joining the end points of the temperature distribution curves. Curve 1 in the figure is obtained in this manner, and it should obviously correspond to (23) for values of $l < q_c$, and for $l > q_c$ to the expression

$$l = \left(\ln \frac{2T_l^3 T_m}{T_m^4 - 3T_l^4 + 2T_l^3 T_m} + \frac{3T_l^4 - T_m^4}{4T_l^3 T_m}\right)(10aT_m^3)^{-\frac{1}{2}} + \left(\frac{2}{5aT_l^3}\right)^{\frac{1}{2}}. \tag{56}$$

The curve can be seen to be parabolic for small values of T_l, as is required by (23).

(4) Curve 2 in the figure gives the temperature distribution in an infinitely long rod calculated from (31). It will be seen that the end portions of the curves for finite lengths overlap with the corresponding portions of curve 2, the overlap being the more extensive the longer the rod.

(5) The range over which the logarithmic formula holds is indicated by a cross at the lower limit of the range, and the range may be seen to be extensive.

(6) Expression (38) for the slope of the curve at any given temperature T can also be verified from the curves. In particular its value at the end-temperature Θ of the rod is an important physical quantity, since it determines the loss of heat at the ends of the rod. A knowledge of this quantity for a rod of finite length is helpful in the design of sealed-in heating elements.

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Temperature Distribution in an Electrically Heated Filament

THE distribution of temperature in a filament electrically heated *in vacuo* has been studied by several previous authors. The differential equation defining the steady state is :

$$\frac{d^2T}{dx^2} + \frac{5}{2} a \, (T_m^4 - T^4) = 0, \qquad (1)$$

in which T is the temperature at a distance x from one of the ends, T_m is the value to which the temperature T_l at the centre tends as the length $2l$ of the filament is increased indefinitely, keeping the heating current the same, and a is a constant determined by the cross-section of the filament, its thermal conductivity and the emissivity of its surface. Using the boundary conditions $T = \Theta$ when $x = 0$, and $dT/dx = 0$ when $x = l$, one obtains[1]

$$x = \int_{\Theta}^{T} [5aT_m^4 \, (T_l - T) - a \, (T_l^5 - T^5) \,]^{-\frac{1}{2}} \, dT, \qquad (2)$$

the value of T_l occurring in (1) being determined by the condition that, when the upper limit of the integral is made equal to T_l, x should become l.

In a recent paper[2] it was shown that though the integral cannot be evaluated directly, it can be expanded as a convergent power series. The filament can be divided for this purpose into two distinct regions, region A comprising a certain length near the middle, and region B outside it, the power series referred to being different in the two regions.

Now, denoting $T_m - T$ by Δ, it can be shown that :

$$\Delta \exp f(\Delta) = \exp \pm x\sqrt{A} \qquad (3)$$

are solutions of (1), where

$$A = 10 \, a \, T_m^3 \qquad (4)$$

and

$$f(\Delta) = \frac{1}{2}\frac{\Delta}{T_m} + \frac{1}{16}\frac{\Delta^2}{T_m^2} - \frac{1}{240}\frac{\Delta^3}{T_m^3} - \ldots \qquad (5)$$

The effect of the finite temperature coefficient of the thermal conductivity, and of the other constants, can also be taken into account by replacing the factors $\frac{1}{2}$, $\frac{1}{16}$, $-\frac{1}{240}$, . . . by A_1, A_2, A_3, . . . which can be calculated. For a tungsten filament with $T_m = 2{,}400°$ K., $A_1 = 0{\cdot}75$, and the other A's are much smaller.

In the region where $\exp f(\Delta)$ is close to unity, a general solution for the finite filament can be obtained by suitably combining the two particular solutions given by (3). The corresponding two terms in the general solution are obviously such that one of them increases rapidly and the other decreases rapidly as we move away from the centre of the filament. If $T_m - T_l$ is sufficiently small, the latter term may become negligible in comparison with the former, while Δ still remains small enough to permit our combining the two particular solutions. When this has happened, the need to keep $\exp f(\Delta)$ close to unity disappears. We thus obtain a *practical* general solution that is applicable over the whole length of the filament, namely :

$$\begin{aligned}&\Delta \exp f(\Delta)\\&= \Delta_0 \exp f(\Delta_0) [\exp - x\sqrt{A} + \exp\{-(2l - x)\sqrt{A}\}]\\&= \Delta_0 \exp f(\Delta_0) \exp - l\sqrt{A} (\exp q\sqrt{A} + \exp - q\sqrt{A}),\end{aligned} \quad (6)$$

where $\Delta_0 = T_m - \Theta$, and $q = l - x$ is the distance measured from the centre.

Now for a filament for which $T_m - T_l$ is small in comparison with T_m, which we shall refer to for convenience as a 'long' filament, the limit of the A-region is given by $\Delta_c \sim 3(T_m - T_l)$. At this point it is found that the ratio of the second term inside the brackets in (6) to the first term has decreased to about $\exp(-3{\cdot}6)$, that is, below 3 per cent. Taking for convenience this fraction, namely $0{\cdot}03$, as small in comparison with unity, we can readily investigate how small $T_m - T_l$ should be in comparison with T_m in order that (6) may hold over the whole length of the filament to at least this degree of accuracy.

It may appear at first sight that the criterion for our being able to combine the two particular solutions, namely, that $\exp f(\Delta)$ should be close to unity, will become increasingly difficult to satisfy as Δ increases. But actually, owing to the disparity in the magnitudes of the two terms to be combined, which increases rapidly as one moves away from the centre, the working criterion becomes $f(T_m - T_l) < 0{\cdot}03$, or $(T_m - T_l) < 0{\cdot}04 T_m$. When this is secured, (6) will fit with observation to a much closer accuracy than 3 per cent, the accuracy increasing rapidly as one moves away from the centre of the filament.

The results obtained earlier for such a filament, on the basis of the series-expansion of the integral—namely, the parabolic variation of temperature in the A-region, the closeness of the distribution outside the A-region to that of a similar infinitely long filament heated by the same current and the logarithmic distribution at the top of this range—all come out as special cases of (6).

By putting $x = l$ in (6), one obtains a useful relation for the temperature T_l at the centre of a long filament, as a function of its length, namely :

$$T_l = T_m - 2\Delta_0 \exp f(\Delta_0) \times \exp - l\sqrt{A}, \quad (7)$$

in view of which (6) may also be written in the convenient form,

$$\Delta \exp f(\Delta) = \tfrac{1}{2} (T_m - T_l) (\exp q\sqrt{A} + \exp - q\sqrt{A}), \quad (8)$$

which is applicable over the whole length of a long filament.

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The distribution of temperature along a thin rod electrically heated *in vacuo* II. Theoretical (continued)

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For a thin rod electrically heated *in vacuo* the differential equation defining the distribution of temperature in the steady state has been formulated previously. If the rod is not too short, a solution can be obtained by suitably combining the two particular solutions for a similar infinitely long rod heated by the same current, which is applicable over the whole length of the rod. The solution is discussed in relation to some formulae that had been proposed by Stead. The solution also leads to a very useful expression for the temperature at the centre of a finite long rod as a function of its length.

1. INTRODUCTION

In part I (Jain & Krishnan 1954) of this paper was reported a theoretical investigation of the distribution of temperature along a thin rod electrically heated *in vacuo*. The integral defining the distribution cannot be evaluated directly, but it can be expanded as a convergent power series, the convergence being rapid in most parts of the rod. The rod is divisible for this purpose into two well-defined regions, region A comprising a certain length in the middle of the rod, and region B outside it, the appropriate series for the evaluation of the integral being different for the two regions. The main features of the distribution of temperature in the two regions are also different, and they have been investigated in detail in part I.

In the context of these investigations, an additive formula proposed originally by Stead (1920) connecting the temperature distribution in a finite rod, with that obtaining in a similar infinitely long rod heated by the same current, becomes significant. This formula of Stead's was derived by him as an approximation to a certain multiplicative formula. Though the latter formula, as we shall see in the present paper, is not justifiable, the additive formula is known to fit with observation. It suggests that it may be possible to formulate a solution of the differential equation defining the temperature distribution in a finite rod in terms of the appropriate particular solutions for an infinitely long rod. This method of approach to the problem has proved helpful, and the present part is an extension of the theoretical investigations given in part I, from this point of view.

2. STEAD'S FORMULAE

On the basis of some detailed measurements on the distribution of temperature along a very long tungsten filament electrically heated *in vacuo*, Worthing (1914) found that the temperature at any distance x from the cold end can be expressed in the form

$$T_x = T_m[1-\mathrm{e}^{-\mu(x+x_0)}]^n, \tag{1}$$

in which T_m is the temperature to which T_x tends as $x \to \infty$. In formulating (1) Worthing was guided partly by theory and partly by the requirements of the experimental data. For temperatures T_x close to T_m the terms inside the brackets could be justified theoretically, though even here the constant x_0 was not determinable theoretically. The index n was introduced in order to make a single formula applicable over a wider range of T_x.

But the most significant part of this expression, which has been utilized by Stead, is the proportionality of T_x to T_m. Had there been no conduction towards the cold end of the rod, the temperature at every point would have been equal to T_m, the value of T_m being determined by the balancing between the heat generated in the rod by the electric current and the radiation of energy from the surface of the rod. But owing to the loss of energy by conduction the actual temperature T_x is smaller. The proportionality of T_x to T_m observed by Worthing implies that in an infinitely long rod the effect of the cold end is to reduce the temperature at any distance x from the end by a factor F, where F is a function of the distance. Denoting by primed letters the temperatures along an infinitely long rod, in order to distinguish them from those relating to finite rods,

$$F(x) = T'_x/T_m. \tag{2}$$

Generalizing from this, Stead formulated that in a *finite* rod, since there are two cold ends through which heat is conducted away, the temperature at any given distance x from one of the ends should be given by

$$T_x/T_m = F(x)\,F(2l-x), \tag{3}$$

where $2l$ is the length of the rod.

Though in deriving (3) in this manner no restriction is imposed on the length of the rod, Stead finds that the formula is applicable to long rods only. Considering such a rod, for which $T_m - T_l$ is small in comparison with T_m, where T_l is the temperature at the centre, and confining attention to a region where $T_m - T_x$ also is small, it may be seen that $F(l)$, $F(x)$ and $F(2l-x)$ will all be close to unity, and that (3) then reduces to the additive formula

$$T_m - T_x = (T_m - T'_x) + (T_m - T'_{2l-x}). \tag{4}$$

It is in this form that the formula has been applied by Stead, and verified experimentally.

Any formula defining the temperature distribution has naturally to be a solution of the differential equation determining the steady state. As we shall see presently, an expression of the type (3) cannot be a solution, and hence the derivation of (4) in the manner indicated before may not be justifiable. On the other hand, Stead's additive relation (4) is known to fit with the experimental data for long rods. We proceed to investigate whether it would be possible directly to obtain a solution of the type (4) for the differential equation for such rods, irrespective of the validity of (3) on which Stead's derivation is based.

3. General solution of the differential equation of the steady state

The differential equation determining the steady state may be written in the form (see (1) and (10) of part I)

$$\mathrm{d}^2T/\mathrm{d}x^2 + \tfrac{5}{2}a(T_m^4 - T^4) = 0, \tag{5}$$

where the subscript x of T_x has been omitted for convenience. The equation applies naturally to the infinitely long rod also. The letter a appearing in (5) and the other letters which we shall use later in this part, have the same significance as in part I. Neglecting for the present the temperature coefficients of κ, ρ and ϵ, which have been taken into account in part I, but which are not quite relevant to the present discussion, and denoting $T_m - T$ by Δ, (5) may be written in the form

$$\mathrm{d}^2\Delta/\mathrm{d}x^2 = \tfrac{5}{2}a(4T_m^3\Delta - 6T_m^2\Delta^2 + 4T_m\Delta^3 - \Delta^4). \tag{6}$$

Multiplying both sides by $2\mathrm{d}\Delta/\mathrm{d}x$ and integrating with respect to x, and putting $10aT_m^3 = A$, one obtains

$$\left(\frac{\mathrm{d}\Delta}{\mathrm{d}x}\right)^2 = A\Delta^2\left(1 - \frac{\Delta}{T_m} + \frac{\Delta^2}{2T_m^2} - \frac{\Delta^3}{10T_m^3}\right), \tag{7}$$

whence

$$\pm x\sqrt{A} = \int \frac{\mathrm{d}\Delta}{\Delta\left(1 - \frac{\Delta}{T_m} + \frac{\Delta^2}{2T_m^2} - \frac{\Delta^3}{10T_m^3}\right)^{\frac{1}{2}}}, \tag{8}$$

or

$$\Delta\,\mathrm{e}^{f(\Delta)} = \mathrm{e}^{\pm x\sqrt{A}}, \tag{9}$$

where

$$f(\Delta) = \frac{\Delta}{2T_m} + \frac{\Delta^2}{16T_m^2} - \frac{\Delta^3}{240T_m^3} - \dots. \tag{10}$$

The *physical* solution applicable to the infinitely long rod is

$$\Delta\,\mathrm{e}^{f(\Delta)} = c\,\mathrm{e}^{-x\sqrt{A}}, \tag{11}$$

in which c may be eliminated from the boundary condition that when $x = 0$, $\Delta = \Delta_0 = T_m - \Theta$, where Θ is the temperature of the cold ends, which may have any value down to zero. One thus obtains

$$\Delta\,\mathrm{e}^{f(\Delta)} = \Delta_0\,\mathrm{e}^{f(\Delta_0)}\,\mathrm{e}^{-x\sqrt{A}}, \tag{12}$$

which can be seen to be identical with (31) of part I, remembering that the temperature coefficients of κ, ρ and ϵ are being neglected here. If desired, the temperature coefficients may be included by suitably modifying the expression for $f(\Delta)$ in the manner indicated in part I.

It can be readily seen from (12) that a multiplicative formula of the type (3) cannot possibly be fitted with (12). On the other hand, as will be shown in the next section, the additive formula (4) can be obtained by a suitable combination of the two particular solutions obtained in (9). We now proceed to demonstrate this, and to determine the conditions under which this can be done.

4. Solution for the finite rod as combination of particular solutions for the infinitely long rod

Now $\Delta \leqslant \Delta_0 \leqslant T_m$. Even when $\Delta = T_m$, which will be the case at the cold end of a rod when the end is kept at zero temperature,

$$f(T_m) = \tfrac{1}{2} + \tfrac{1}{16} - \tfrac{1}{240} - \dots = 0{\cdot}56. \tag{13}$$

For other values of $\Delta, f(\Delta)$ will be smaller, and when Δ is small in comparison with T_m, $\mathrm{e}^{f(\Delta)}$ will be practically unity.

In any case, in a finite rod and in a region where Δ is small, a general solution of the differential equation may be obtained by suitably combining the two particular solutions in (9), which is now permissible since $e^{f(\Delta)}$ is now practically unity. The condition that Δ be small implies first that the rod is long enough to make $T_m - T_l$ small, and secondly that the point concerned is not too far away from the centre of the rod. The general solution is

$$\Delta = c_1\,e^{-x\sqrt{A}} + c_2\,e^{x\sqrt{A}}, \tag{14}$$

which could also have been obtained directly as solutions of

$$d^2\Delta/dx^2 = A\Delta, \tag{15}$$

to which (6) reduces when Δ is small.

The two constants c_1 and c_2 appearing in (14) may be evaluated from the following conditions:

$$\left.\begin{aligned} d\Delta/dx &= 0 \quad \text{when} \quad x = l, \\ \Delta &= \Delta_1, \text{ say,} \quad \text{when} \quad x = x_1, \end{aligned}\right\} \tag{16}$$

where x_1 represents a distance which is considerably smaller than l, but is still within the range where Δ remains small in comparison with T_m. Using these conditions one obtains

$$c_1 = \frac{\Delta_1\,e^{l\sqrt{A}}}{e^{(l-x_1)\sqrt{A}} + e^{-(l-x_1)\sqrt{A}}}, \tag{17}$$

$$c_2 = \frac{\Delta_1\,e^{-l\sqrt{A}}}{e^{(l-x_1)\sqrt{A}} + e^{-(l-x_1)\sqrt{A}}}. \tag{18}$$

Now if it is possible to find a value of x_1 such that Δ at this point, namely, Δ_1, remains small in comparison with T_m, i.e. $e^{f(\Delta_1)}$ remains close to unity, and at the same time $e^{-2(l-x_1)}$ has become negligibly small in comparison with unity, then (14) will reduce to

$$\Delta = \Delta_1\,e^{x_1\sqrt{A}}\,[e^{-x\sqrt{A}} + e^{-(2l-x)\sqrt{A}}]. \tag{19}$$

If, further, Δ_1, which represents the temperature of the *finite* rod at a distance x_1, can also be shown to be the temperature of the infinite rod at the same distance x_1, then obviously

$$\Delta_1\,e^{x_1\sqrt{A}} = \Delta_0\,e^{f(\Delta_0)}. \tag{20}$$

The general solution (19) for regions $T_m - T_l < \Delta < \Delta_1$ will then reduce to

$$\Delta = \Delta_0\,e^{f(\Delta_0)}\,[e^{-x\sqrt{A}} + e^{-(2l-x)\sqrt{A}}]. \tag{21}$$

The two terms in (21) can then be seen to be separately the solutions for an infinitely long rod.

Equation (21) can be seen to be the same as Stead's additive relation (4), except that (21) gives also the actual values of the two particular solutions, which (4) does not.

It now remains to investigate whether the conditions imposed in the derivation of (21) can be actually secured, and the range over which this can be done. Before doing so we shall briefly indicate an alternative method of deriving (19), which is helpful in understanding the significance of the conditions imposed above. It also enables us incidentally to follow in detail the temperature variation in the A-region, which also is needed for our present purpose.

5. An alternative solution

Taking the case of a finite rod of *any* length, and confining attention to a region close to the centre, where $t = T_l - T$ is small in comparison with T_l, the differential equation defining the steady state can be written in the form

$$\mathrm{d}^2t/\mathrm{d}q^2 - Pt - Q = 0, \tag{22}$$

where
$$P = 10aT_l^3, \tag{23}$$

$$Q = \tfrac{5}{2}a(T_m^4 - T_l^4), \tag{24}$$

and $q = l - x$ is the distance measured from the centre. Solving (22) with the boundary conditions $t = 0$, and $\mathrm{d}t/\mathrm{d}q = 0$, when $q = 0$, one obtains

$$t = \frac{Q}{P}[\cosh(q\sqrt{P}) - 1]. \tag{25}$$

Expression (25) is applicable to *any* rod, short or long, provided that t is small, the linear extent of this region depending on how long the rod is, since the variation of t is the slower the longer the rod. When $q\sqrt{P}$ is sufficiently small that q^2P is negligible in comparison with unity, (25) reduces to

$$t = \tfrac{1}{2}Qq^2, \tag{26}$$

which is identical with the parabolic relation obtained in part I. As q increases, the temperature variation naturally deviates increasingly from the parabolic law. This deviation, which was investigated for the general case in part I with the help of a complicated series, comes out more elegantly, for small values of t, as just the deviation of $\cosh(q\sqrt{P})$ from $1 + \frac{1}{2}Pq^2$. For a given small t the deviation will be the smaller the smaller the value of P, i.e. the shorter the rod, which is in agreement with the result obtained in part I.

For a long rod for which $T_m - T_l$ is small, the deviation from the parabolic law is a maximum. In this case, and at the upper limit of the A region, where $t_c = 2Q/P$, the deviation of the actual value of q from that calculated from the parabolic law comes out as 10 %. This again is the same as the value calculated in part I.

For such a rod, $t_c = 2Q/P = 2(T_m - T_l)$. Thus even at the limit of the A-region, Δ is only thrice $T_m - T_l$, and therefore remains small in comparison with T_m.

Returning to (25) one can see that when $T_m - T_l$ is small in comparison with T_m, $Q/P = T_m - T_l$ and $P = A$. Remembering that $t = \Delta - (T_m - T_l)$, (25) can be readily seen to reduce to (19).

6. A more general expression applicable over the whole rod

Coming back to the general solution (19), it will be seen that the ratio of the two terms is equal to $\mathrm{e}^{-2q\sqrt{A}}$, and that it decreases rapidly with the increase of q. Even at the limit of the A-region, where Δ still remains small in comparison with T_m, and hence (19) remains valid, $2q\sqrt{A}$ has increased to 3·6. Hence this ratio has become as small as $\mathrm{e}^{-3\cdot 6}$, which is less than 3 %. Hence as one moves into the B-region, only one of the two particular solutions in (19) will be significant, the other having become relatively negligible.

It will be remembered that the condition that Δ also be small was invoked in order to enable us to combine the two particular solutions. Now that one of them has vanished, the necessity for this condition also vanishes. In other words the distribution of temperature over the rest of the rod will be practically the same as for a similar infinitely long rod heated by the same current, the agreement between the two becoming the closer the nearer one approaches the cold end. Incidentally it also shows that for such rods for which $T_m - T_l$ is small in comparison with T_m, the expression for Δ will reduce to the simple exponential form at the top of the B-region, which is in conformity with the result obtained in part I.

When we move further inside the B-region, since we have now to deal with only one solution, and it is characteristic of an infinitely long rod, one may use the more precise form (12).

Consider now the general expression

$$\Delta \, e^{f(\Delta)} = \Delta_0 \, e^{f(\Delta_0)} [e^{-x\sqrt{A}} + e^{-(2l-x)\sqrt{A}}]. \tag{27}$$

Though as such it may not satisfy the differential equation, it will be a solution in the following two special cases, as we have shown:

(*a*) if $e^{f(\Delta)} \approx 1$;

(*b*) even when $e^{f(\Delta)}$ differs much from unity, if one of the terms inside the brackets in (27) vanishes.

For a rod for which $T_m - T_l$ is small in comparison with T_m, we have shown that these two conditions are so related that one or the other of them is satisfied at *every* point of the rod. Indeed, there is a considerable range where (*a*) and (*b*) are *both* sensibly satisfied.* Hence (27) may be regarded *practically* as a solution of the differential equation over the whole length of the rod.

Now equation (27) may be seen to be the same as Stead's additive formula (4), and that it gives much more detailed information regarding the temperature distribution than (4). The necessary condition for its validity is that $T_m - T_l$ be small in comparison with T_m, and when this condition is satisfied, the formula will be valid over the whole length of the rod, not only in regions where $T_m - T$ is small in comparison with T_m, for which region (4) was supposed to be a good approximation to the correct formula, but also over regions where $T_m - T$ may have become as large as T_m itself. On the other hand, the multiplicative formula, of which (4) was regarded as an approximation, has no validity except in so far as it may approximate to (4).

* [*Note added in proof*, 16 *June* 1954.] It should be emphasized here that the condition for our being able to combine the two solutions, namely, that $e^{f(\Delta)}$ be close to unity, is indeed the most stringent near the centre, where the two terms to be combined are of the same magnitude. This is so in spite of the fact that Δ itself has the *minimum* value here. As we move away from the centre, because of the progressively growing disparity in the magnitudes of the two terms to be combined, the permitted tolerance, i.e. the permitted deviation of $e^{f(\Delta)}$ from unity increases progressively with the distance from the centre (Krishnan & Jain, *Nature, Lond.*, 1 *May* 1954). Putting $f(\tau_m - \tau_l) = \theta \ll 1$, equation (27) will represent the temperature distribution near the centre to an accuracy of θ in 1, and the accuracy will *increase* progressively as we move away from the centre.

From (27), which is applicable to a long rod over its whole length, one can readily obtain by putting $x = l$, an expression for T_l as a function of l. The expression is

$$T_l = T_m - B\,e^{-l\sqrt{A}}, \tag{28}$$

where

$$B = 2\Delta_0\,e^{f(\Delta_0)}. \tag{29}$$

Experimentally (28) is a very useful relation.

For the temperature distribution in short rods one has to fall back on the detailed solutions (25) and (11) respectively in the regions A and B, or the corresponding ones given in part I.

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The distribution of temperature along a thin rod electrically heated *in vacuo*
III. Experimental

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The present part is concerned with a detailed experimental verification of the main results obtained theoretically in the earlier parts. Thin rods of Acheson graphite were used for this purpose, and were found very suitable. In addition to the experimental data for the temperature distribution along rods of different lengths, and heated by alternating currents of different densities, one also needs for such a verification, data for the thermal and the electrical conductivities, and the spectral and the total emissivities of the surface, at different temperatures. These measurements also have been made.

The observed temperature variation along the rod is itself used to determine the thermal conductivity and its temperature variation. The thermal conductivity is found to decrease and tend to a constant value at high temperatures, as it should. The other physical constants are determined directly. At low temperatures the spectral emissivity is found to be much larger than the total emissivity, and they tend to approach each other at high temperatures. Further, the spectral emissivity is found to decrease rapidly with increase of wave-length, as in the case of metals.

Coming back to the temperature distribution, the main results that are verified here relate to the range of validity of the logarithmic formula in the B-region of a long rod, the constants involved in the formula, the deviation from the formula as one moves further into the B-region, the effect of the temperature coefficients, the parabolic law of variation near the centre, and the nature of the deviation from the parabolic law in long and short rods as one moves further away from the centre.

1. Introduction

In parts I and II (Jain & Krishnan 1954*a, b*) were reported some theoretical investigations on the distribution of temperature along a thin rod or filament electrically heated *in vacuo*. The present part is concerned with the experimental verification of the main results obtained there.

Extensive experimental data are available for the temperature distribution in electrically heated filaments of tungsten (Worthing 1914), and some for thin-walled tube of Acheson graphite (Prescott & Hincke 1928). The lengths used in these measurements, however, are such as to make the temperature over the whole of the A-region sensibly constant. The data for rods of medium and short lengths afford naturally a more stringent test of the theory. Such short rods do not seem to have been studied previously.

The experimental verification requires also a knowledge of some of the physical constants of the material, which are not available. Hence we have made detailed studies of the temperature distribution in thin rods of Acheson graphite, using a wide range of lengths and heating currents, and we have also made measurements on the physical constants required for the experimental verification of the results obtained in parts I and II. As we shall see later in this part, Acheson graphite is a very suitable material for this purpose.

2. The parameters involved

The distribution function is determined uniquely, as was found in part I, by three parameters, which can be taken to be conveniently the following: (1) the temperature T_l at the centre of the rod, or its semi-length l; (2) the temperature T_m to which T_l tends as l is increased indefinitely, keeping the heating current I constant; T_m is determined by the condition

$$p\epsilon\sigma(T_m^4 - fT_w^4) = I^2\rho/\omega; \qquad (1)$$

(3) a constant defined by $$a = \tfrac{2}{5}p\epsilon\sigma/(\kappa\omega); \qquad (2)$$

where ω is the cross-sectional area of the rod, p is the perimeter of the cross-section, ρ is the specific resistance and κ is the thermal conductivity, σ is Stefan's constant of radiation, and T_w is the temperature of the glass walls of the large vacuum chamber in which the measurements were made. The chamber was double-walled and between the two walls was circulated water at room temperature, and hence T_w could be taken to be practically the room temperature. Just as ϵ is the total emissivity at temperature T_m, $f\epsilon$ is the absorptivity of the same surface at the same temperature T_m for radiations from a black body at temperature T_w. As will be seen in the next section, the spectral emissivity ϵ_λ of graphite is a function of the wave-length, and it decreases as the wave-length is increased. Since the effective wave-length of the radiations from the walls at T_w is much longer than that of the radiations at T_m, f will be less than, but of the order of, unity. In order to avoid the uncertainty in our knowledge of f, most of our measurements were made at temperatures T greater than 1100° K, for which $T_w^4 \sim 300^4$ becomes small in comparison with T^4. There is a further advantage in using these high temperatures; the basis for the derivation of relation (1) is that the loss of energy from the central region of the rod towards the ends by conduction is negligible in comparison with the loss due to radiation. Though this will be true in any case for long rods, the length can be the shorter the higher the temperature T_m near the centre.

In investigating the temperature distribution, the known temperature dependence of κ, ρ and ϵ can also be taken into account. It was shown in part I that the temperature distribution in the A-region, expressed as a function of the distance from the centre, and relatively to the temperature at the centre, remains unaltered by the finite temperature coefficients of these quantities, while in the B-region the effect may be regarded as a small correction Δx to the distance x from the end of the rod, of any point having a temperature T, and Δx may be readily obtained from the known temperature coefficients. Thus the physical constants involved in the theoretical expressions are κ, ρ and ϵ at different temperatures in the range concerned. Now the measurement of the temperature at any point on the surface of the heated rod is most conveniently done with the help of an optical pyrometer, and for this purpose a knowledge of the *spectral* emissivity ϵ_λ of the surface, as distinguished from the *total* emissivity ϵ, is also needed.

We now proceed to describe briefly the experimental determination of these physical constants for Acheson graphite at high temperatures.

3. The spectral and the total emissivities

A thin-walled tube of Acheson graphite was heated *in vacuo* by sending a suitable *alternating* current through it. The tube used in the experiment was long enough to make the temperature in the middle of the tube sensibly constant over a considerable length of the tube. Under these conditions, the loss of heat from this region is due almost wholly to radiation from its surface, and very little of it is due either to the thermal conduction from the centre of the tube towards the ends, or to the radiative transfer occurring in the cavity of the tube. By puncturing a small hole in the wall of the tube near the centre, the temperature inside the cavity as viewed through this hole can be measured directly with the help of the optical pyrometer, since the temperature concerned is that of a *black body*.

By making the walls of the tube sufficiently thin, the temperature of the outer surface can be made to approximate to the temperature inside. The small difference in temperature between the inner and the outer surfaces, due to the finite thermal conductivity of the material of the wall, is readily obtained when κ is known. Even a very rough value of κ will serve this purpose. Knowing thus the real temperature T_m and the apparent temperature T_a of the surface as determined directly by the pyrometer,* the spectral emissivity ϵ_λ at this temperature, for the narrow spectral region transmitted by the glass filter of the optical pyrometer, namely, about $0{\cdot}655\mu$, is obtained with the help of the relation

$$\epsilon_\lambda = \exp\left[-\frac{hc}{k\lambda}\left(\frac{1}{T_a}-\frac{1}{T_m}\right)\right]. \tag{3}$$

Measurements on the spectral emissivity of Acheson graphite made in this manner were reported in a recent paper (Jain & Krishnan 1952) in the temperature range 1200 to 1700° K. The measurements have now been extended to 2100° K,

* A small correction is made for the loss of intensity of the radiation in its passage through the glass walls of the chamber (Jain & Krishnan 1952).

and the results are plotted in the upper curve in figure 1. Conversely, for any temperature T_a of the graphite surface obtained directly with the optical pyrometer, the corresponding true temperature T is obtained with the help of this curve and (3). Our measurements of the surface temperature of the graphite rod at different points along the rod were made in this manner.

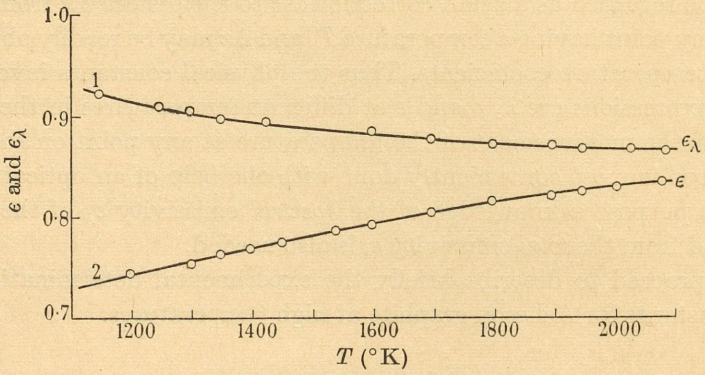

FIGURE 1. The spectral and the total emissivities of Acheson graphite.

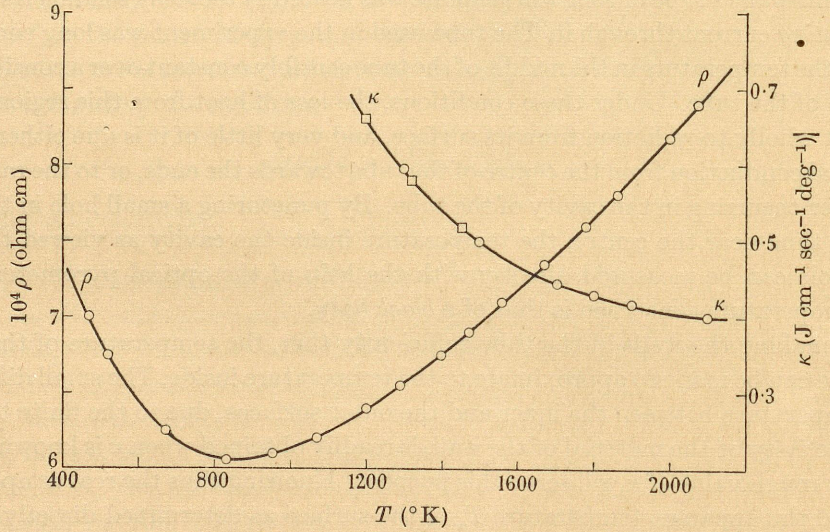

FIGURE 2. The electrical resistivity and the thermal conductivity of Acheson graphite.

It should be mentioned immediately that the emissivity of the surface naturally depends on the nature of the polish of the surface. For example, any rubbing of the graphite surface with the fingers reduces the emissivity appreciably. However, after maintaining the tube at a temperature of 2000° K for about an hour, the emissivity reaches a steady value that is reproducible. The data plotted in figure 1 refer to such a conditioned surface, which is also the surface of the thin rod along which the temperature distribution is to be studied.

Now the potential gradient V in the central region, where the temperature is sensibly constant, is obtained with the help of two electrical probes. Now, knowing the heating current I, the specific resistance ρ of Acheson graphite at temperature T_m is readily obtained. By suitably varying I, the value of ρ at any other desired high temperature can be obtained. The results are plotted in figure 2.

The restriction of the low-temperature limit in our other measurements to 1100° K, referred to in an earlier section, was to avoid the uncertainty in the theoretical expressions due to the radiation from the walls of the chamber, which will be appreciable at lower temperatures. In the measurement of resistance, however, one is not concerned with any such difficulty. The measurement of ρ has therefore been extended down to the temperature of the room, the measurement of temperature being now made with a calibrated thermocouple.

The temperature variation curve of ρ for Acheson graphite shows a minimum, as had been observed by previous workers (Powell & Schofield 1939), and explained (Bowen 1949) on the basis of the large anisotropy of the crystallites and their finite size.

At the high temperatures with which we shall be concerned hereafter in the paper, it will be seen from the curve that the variation of ρ with temperature is practically linear.

Knowing the energy generated per second per unit length of the long tube near the centre, which is also the energy radiated out from it per second, the total emissivity ϵ is obtained readily:

$$VI = p\epsilon\sigma(T_m^4 - fT_w^4). \tag{4}$$

The values of the total emissivity ϵ at different temperatures, obtained in this manner, are plotted in the lower curve of figure 1.

If the radiation from the surface followed the cosine law, i.e. if the radiation per unit solid angle along a direction making an angle θ with the normal to the surface were strictly proportional to $\cos\theta$, both ϵ_λ and ϵ would be independent of the direction of emission. Experimentally, however, the emission is known to deviate appreciably from the cosine law, especially when θ approaches $\frac{1}{2}\pi$. In view of this it is desirable to emphasize here that the spectral emissivity plotted in figure 1 is for the radiations normal to the surface, whereas the total emissivity plotted in the figure corresponds to an average value taken over all directions. The latter may differ slightly from the total emissivity along the normal to the surface.

4. The data for the two emissivities compared

It will be seen immediately from figure 1 that at the lower temperatures, ϵ is considerably smaller than ϵ_λ, and the two curves tend to approach each other at high temperatures. An obvious conclusion to be drawn from this is the following. There are two effects associated with the change in temperature. One is the direct effect of the temperature as such on the spectral emissivity in every spectral region. For the red region, as will be seen from the upper curve, the spectral emissivity falls off with the increase of temperature, and it may be presumed that this will be the case for the other spectral regions also. Hence the direct effect of increasing the temperature will be to reduce the total emissivity.

The second effect is indirect. As a consequence of increasing the temperature, the effective wave-length of the radiation, to which the total emissivity refers, will decrease. If ϵ_λ is a function of the wave-length also, as we should expect it to be, there will be an indirect effect of the change of temperature on the total emissivity due to the change in the effective wave-length accompanying the change of temperature.

We have seen that the direct effect of increasing the temperature is to decrease ϵ. But actually, as we can see from curve 2, ϵ is found to increase with rise of temperature, which shows that the predominant effect of the temperature on ϵ is the indirect one just mentioned, and that it more than compensates for the pure temperature effect. The obvious conclusion therefore is that ϵ_λ decreases rapidly with the increase in the wave-length.

Now, at the lowest temperature to which our measurements refer, namely, 1100° K, the effective wave-length will be in the near infra-red. From the magnitude of ϵ at this temperature, and from what was just stated regarding the dependence of ϵ_λ on the wave-length, it may be inferred that even in the near infra-red region ϵ_λ should have dropped below 0·7. Further, the progress of the curve indicates that it should be much lower in the farther infra-red.

This conclusion is of significance. Such a large drop in the value of ϵ_λ as we move to the far infra-red is known to be a characteristic of metals, and the observed similar behaviour of Acheson graphite therefore points to its pronounced metallic character. In this respect it differs essentially from a thin surface coat of lampblack, for which it is known that ϵ_λ continues to be high even for very long wave-lengths.

We have discused at some length the marked difference that is experimentally found between the total and the spectral emissivities of graphite, since frequently the two are taken for convenience to be equal.

5. The finite difference in temperature between the outer and the inner surfaces in relation to the thermal conductivity

The finite difference τ between the temperatures of the inner and the outer surfaces, *near the centre of a long tube* electrically heated *in vacuo*, as is well known (Angell 1911; see also Glazebrook 1922) is given by

$$\tau = \frac{V^2}{2\kappa\rho}\left[\frac{r_1^2 - r_2^2}{2} - r_2^2 \ln\frac{r_1}{r_2}\right], \tag{5}$$

where r_1 and r_2 are the outer and the inner radii respectively. It is assumed that the radial variations of the thermal and the electrical conductivities due to the finite radial gradient of the temperature are negligible. The error involved in this neglect is equivalent to neglecting $\alpha\tau$ in comparison with unity, where α is of the order of the temperature coefficient of either of the conductivities; $\alpha\tau$ will be of the order of 10^{-3}.

Relation (5) may be made the basis for a determination of κ from the observed temperature difference τ between the inner and the outer surfaces of the tube; τ may be made sufficiently large for this purpose by making the thickness of the walls large (Angell 1911). In practice, however, though it is easy to measure the inner temperature accurately, that of the outer surface cannot be measured so readily. It will therefore be more appropriate to make the tube thin-walled, and thus make τ small, and utilize any rough value of κ determined by other methods in order to calculate τ. In this case (6) reduces to the simple form

$$\tau = V^2(r_1 - r_2)^2/(2\kappa\rho) = VI(r_1 - r_2)/(4\pi\kappa). \tag{6}$$

As Prescott & Hincke (1928) first pointed out, the temperature variation *along* the tube itself in the logarithmic region provides the value of κ required for this purpose. (Actually the observed temperature variation in this region has been utilized by them to eliminate κ from (6) and thence to calculate τ.) Knowing τ the precise temperature T_m of the outer surface is obtained from the known temperature of the inner surface. It is this value of T_m that has been used by us in obtaining the spectral emissivity, and later the total emissivity, of graphite.

Indeed, as will be shown in the next section, the temperature variation along the rod, which can be measured accurately, may be made the basis for an accurate determination of not only κ, but also of its temperature coefficient.

It will be clear from the discussion given above that a thin-walled tube is the most convenient form for the determination of the temperature of the outer surface near the centre, and thence to determine ϵ_λ at this temperature. Graphite can be turned into a tube conveniently on the lathe. Without making the tube walls unduly thin, one can secure enough electrical resistance for heating it to high temperatures.

6. The B-region

We now proceed to describe the experimental studies on the distribution of temperature in a thin rod electrically heated *in vacuo*, and to utilize the experimental data thus obtained to verify the main results deduced theoretically in parts I and II.

We shall take up first the case of a rod long enough to include a considerable length of the B-region. For such a rod $T_m - T_l$ will be small in comparison with T_m. In part II was deduced a general expression for the temperature distribution in such a rod, which is applicable over its whole length. The expression is

$$\Delta\, e^{f(\Delta)} = \Delta_0\, e^{f(\Delta_0)}[e^{-x\sqrt{A}} + e^{-(2l-x)\sqrt{A}}], \tag{7}$$

in which

$$\Delta = T_m - T, \tag{8}$$

$$\Delta_0 = T_m - \Theta, \tag{9}$$

Θ is the temperature of the ends, and

$$f(\Delta) = \frac{\Delta}{2T_m} + \frac{\Delta^2}{16T_m^2} - \frac{\Delta^3}{240T_m^3} - \dots. \tag{10}$$

When the temperature coefficients of κ, ρ and ϵ are also taken into account, $f(\Delta)$ may be expressed in the form

$$f_1(\Delta) = \frac{A_1\Delta}{T_m} + \frac{A_2\Delta^2}{T_m^2} + \frac{A_3\Delta^3}{T_m^3} + \dots, \tag{11}$$

where A_1, A_2 and A_3 have the same significance as in part I. They are now functions of the temperature coefficients, and they naturally reduce to $\frac{1}{2}$, $\frac{1}{16}$ and $-\frac{1}{240}$

respectively when these coefficients become negligible. In order to give an idea of the magnitudes of these quantities it may be mentioned that for Acheson graphite and for $T_m \sim 2000°$ K, A_1 is of the order of unity, A_2 is about 0·2 and $|A_3|$ is still smaller. Since the highest value of Δ/T_m involved in our measurements is less than $\frac{1}{2}$, the second term in (11) is about 10 % of the first or less. Now, $f_1(\Delta)$ itself occurs as an additive term to $\ln \Delta_0 \sim 7$ (see (14) below). Even when $\Delta = \Delta_0$ the A_2 term in $f_1(\Delta_0)$ can be seen to be less than 1 % of $\ln \Delta_0$. Hence it would be ample if the A_1 term alone is retained in the expression for $f_1(\Delta)$.

Now $$A_1 = \tfrac{1}{2} - \alpha T_m + c. \tag{12}$$

The second term on the right-hand side of (12) involves the temperature coefficient of κ alone, namely α, whereas c involves the coefficients of κ, ρ and ϵ (see equation (50) of part I).

Returning now to (7) it was shown in part II that even at the upper limit of the B-region the ratio of the second term inside the square brackets to the first is $e^{-3\cdot 6}$, or less than 3 %. As we move inside the B-region the ratio will become still smaller. In other words, the distribution becomes independent of l, and is the same as for an infinitely long rod.

Equation (7) then reduces to

$$x\sqrt{A} = D - \ln \Delta - f_1(\Delta), \tag{13}$$

where D denotes the part that is independent of Δ, and is given by

$$D = \ln \Delta_0 + f_1(\Delta_0). \tag{14}$$

As was mentioned in part I, a simple logarithmic formula of the type

$$x\sqrt{A} = D - \ln \Delta \tag{15}$$

had been derived previously on the basis that Δ is small in comparison with T_m, and had been verified experimentally. The additive constant D, however, could not be determined from those simple considerations.

On the other hand, (13) and (14) not only enable us to determine the additive constant D, but also give the actual law of deviation from (15) when Δ is not sufficiently small. One therefore obtains further a criterion for the range of validity of (15), namely, that $\Delta/\ln \Delta$ should be small in comparison with T_m, which is a much less stringent criterion than that Δ should be small in comparison with T_m, on the basis of which (14) had been previously derived.

All these conclusions are verified experimentally. The temperature distribution was studied in thin long rods of different lengths and with different heating currents. In figure 3 are plotted the values of $\ln \Delta$ against x for a typical case, namely, for $T_m = 2100°$ K. The data for other rods are not plotted in the figure since they overlap much on one another. These data are, however, summarized in table 1.

It will be seen from figure 3 that there is a considerable portion in the lower ranges of Δ, where the plot is a straight line, from the slope of which $\sqrt{A}$, and hence the thermal conductivity κ, can be readily obtained. The values of κ obtained in this manner are entered in table 1, and they are also plotted in figure 2.

We should emphasize again that this method of determining the thermal conductivity and its temperature variation at these high temperatures,* is more reliable than the method based on measuring the difference in temperature between the inner and the outer surfaces in the middle of a long tube electrically heated *in vacuo*.

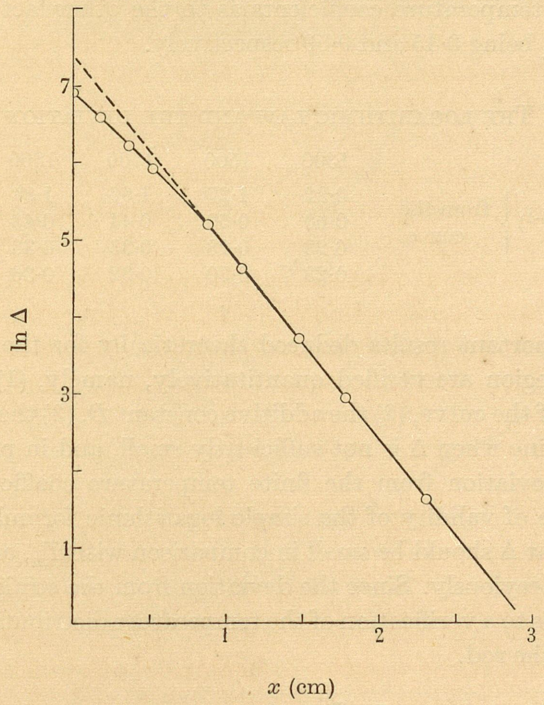

FIGURE 3. The temperature distribution in the B-region when $T_m = 2100°$ K.

Had the logarithmic formula been valid over the whole length of the rod, the additive constant D, which is given by the intercept of the straight line on the $\ln \Delta$ axis at $x = 0$, would also have been the value of $\ln \Delta$ at this point, i.e. D would have been equal to $\ln (T_m - \Theta)$. But actually D is considerably larger, since the curve can be seen to curve downwards, the deviation from the straight line increasing progressively with the increase of Δ. The difference according to (13) should be equal to $f_1(\Delta) \sim A_1 \Delta / T_m$. This is verified quantitatively.

In table 1 are given the experimental values for all the long rods studied by us, and the corresponding theoretical values for comparison. The values of α at different temperatures T_m needed for the calculation of A_1 were taken from the κ-T curve given in figure 2. Considering that the quantities compared in the last two rows in

* A graphical method of determining the thermal conductivity from the observed temperature distribution in a filament has been described by Worthing & Halliday (1948). The method is based on finding by graphical integration the total energy generated, and that radiated out, in the whole of the region on the hotter side of the cross-section under consideration, up to the centre of the filament. The difference between the two gives the heat conducted across this section. Knowing the gradient in temperature at this section the conductivity can be calculated.

the table refer to the small deviations from the logarithmic formula, these deviations also may be taken to be verified satisfactorily.

Incidentally it may be mentioned that at $T_m = 2100°$ K, $A_1 = 0{\cdot}95$, which is very much in excess of 0·50, which it would be if the temperature coefficients were neglected. This shows that the deviation from the straight line is due nearly as much to the finite temperature coefficients as to the other factors, their relative contributions to A_1 being 0·45 and 0·50 respectively.

TABLE 1. THE LOGARITHMIC LAW AND THE DEVIATION FROM IT

T_m (°K)		1300	1500	1700	1800	1900	2100
$\sqrt{A}$	from the curve	0·92	1·27	1·66	1·84	2·03	2·45
κ (J cm^{-1} sec^{-1} deg^{-1})	from the curve	0·60	0·50	0·44	0·43	0·42	0·40
$D - \ln \Delta_0$	from the curve	0·23	0·28	0·33	0·36	0·39	0·46
$A_1 \Delta_0 / T_m$		0·23	0·30	0·32	0·36	0·38	0·44

Thus all the important results deduced theoretically for the temperature distribution in this region are verified quantitatively, namely, (1) the slope of the straight-line part of the curve, (2) the additive constant D, (3) the precise deviation from the straight line when Δ is not sufficiently small, and in particular the contribution to the deviation from the finite temperature coefficients, and finally (4) the wider range of validity of the simple logarithmic formula than is implied by the criterion that Δ should be small in comparison with T_m, on which basis (15) had been derived previously. Since the deviation from the straight line is verified in detail, it amounts to a verification of the temperature distribution over the whole of the B-region of the rod.

7. THE A-REGION

Moving now into the A-region, and still confining attention to the long rod, it will be seen that $e^{f(\Delta)}$ will now be practically unity, and the temperature variation will be given by

$$\Delta = \Delta_0 \, e^{f(\Delta_0)} \, e^{-l\sqrt{A}} \, (e^{q\sqrt{A}} + e^{-q\sqrt{A}}). \tag{16}$$

This expression will hold not only in the A-region but over a considerable portion of B too, as long as $\Delta/\ln \Delta$ remains small in comparison with T_m, i.e. over the whole of the range in B where the simple logarithmic formula holds. When $q = 0$, Δ should be equal to $T_m - T_l$. Hence (16) may also be written in the form

$$t = \tfrac{1}{2}(T_m - T_l)\,(e^{q\sqrt{A}} + e^{-q\sqrt{A}} - 2), \tag{17}$$

which reduces, when q is small, to the parabolic law

$$t = \tfrac{1}{2}(T_m - T_l)\,Aq^2. \tag{18}$$

Curve 1, figure 4, in which t is plotted against q^2, relates to such a long rod, for which $T_m = 1500°$ K and $T_l = 1450°$ K. The thick line represents the theoretical curve (17), and the circles denote the experimental points. As should be expected for long rods (see next section), the straight-line portion of the curve corresponding to the parabolic variation of temperature near the centre is relatively short.

The slope of the straight-line portion, extended in the figure by the dotted line, will obviously be $\frac{1}{2}(T_m - T_l)A$, and it can be utilized as before to give us A, and thence κ. The value of κ at 1450° K plotted in figure 2 was obtained in this manner.

It should be mentioned here that the value of A needed in plotting the theoretical curve was taken from the straight-line portion of the experimental curve. It will be seen that all the experimental points lie almost exactly on the theoretical curve, which includes some part of the B-region too.

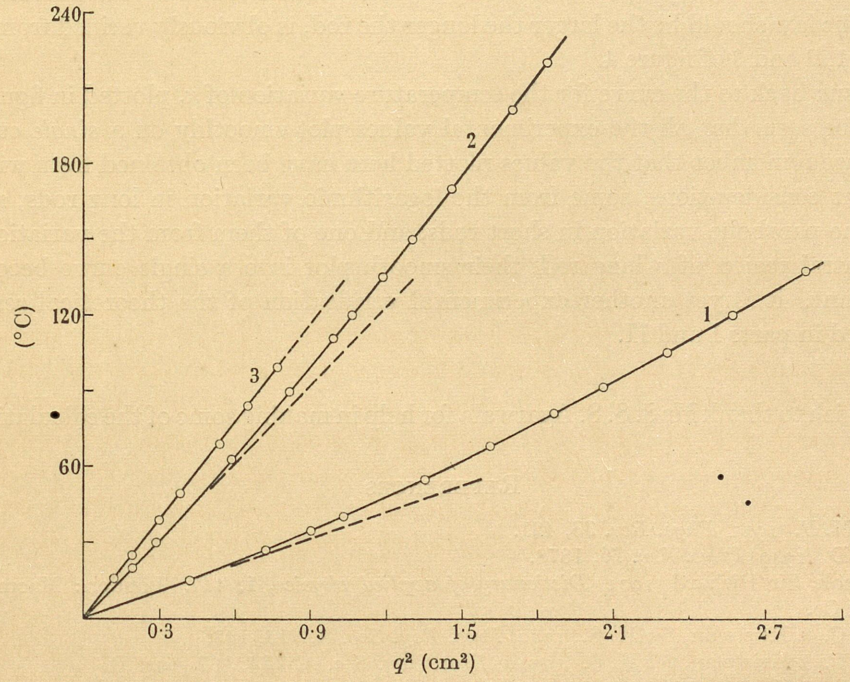

FIGURE 4. Temperature distribution in the A-region.

8. SHORT RODS

For short rods also the temperature variation near the centre is given by a parabolic law, namely,

$$t = \tfrac{1}{2}Qq^2, \tag{19}$$

where

$$Q = \tfrac{5}{2}a(T_m^4 - T_l^4), \tag{20}$$

and at larger distances from the centre by the more precise relation

$$t = \tfrac{1}{2}Qq^2/(1-S)^2, \tag{21}$$

where S is a small fraction which can be expressed as a power series in t/t_c (see (19) of part I), and can be easily calculated.

Curves 2 and 3 in figure 4 refer to such short rods for which $T_m = 1500°$ K, and $T_l = 1320$ and 1200° K respectively.

The slope of the straight-line portion can again be utilized to give Q and thence κ. The values of κ plotted in figure 2 for 1320 and 1200° K were obtained in this manner.

Now for a given q, which is not small, the deviation of t from the straight line will be given according to (21) by $2tS$. The full lines in curves 2 and 3 of figure 4 are the theoretical curves plotted on the basis of (21). The values of Q required for the theoretical plot were taken from the straight portions of the experimental curves, and the calculated values of $2tS$ gave the deviations from the straight line.

Here again the experimental values may be seen to lie almost exactly on the theoretical curves.

The result deduced in part II, namely, that for a given t the deviation from the parabolic law should be the larger the longer the rod, is obviously verified from the curves 1, 2 and 3 of figure 4.

Coming back to the curve for the temperature variation of κ, plotted in figure 2, it will be seen that all the experimental values plot smoothly on a *single* curve. When we remember that the values plotted here have been obtained from widely different considerations, some from the logarithmic variation in long rods, some from the parabolic variation in short rods, and one of them from the variation in the central region of a long rod, their smooth plot into a single curve becomes significant. It is yet another experimental verification of the theoretical results obtained in parts I and II.

We wish to thank Mr S. S. S. Agarwala for help in making some of the calculations.

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The distribution of temperature along a thin rod electrically heated *in vacuo*

IV. Many useful empirical formulae verified

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On the basis of extensive measurements made on the distribution of temperature, and on the distribution of other physical characteristics which are known functions of the temperature, along tungsten filaments heated *in vacuo*, several empirical formulae have been proposed by Worthing, and they have found wide practical application. Many of these formulae are shown in the present part to follow naturally from the theoretical expressions deduced in parts I and II, as special cases. The conditions for their validity also get determined.

The other formulae of Worthing are shown to be good approximations, and the degrees of their closeness to the corresponding theoretical formulae are estimated.

1. Introduction

Extensive experimental work has been done on the distribution of temperature, and of several temperature-dependent physical characteristics like thermionic emission, brightness, total radiation, electrical resistivity, etc., along tungsten filaments electrically heated *in vacuo* (Worthing 1914*a*, *b*, 1922; Langmuir 1916*a*, *b*, *c*, 1922, 1930; Stead 1920; Forsythe & Worthing 1925; Worthing & Halliday 1948). The experimental data have been analyzed in detail, and many useful empirical formulae have been deduced therefrom. These formulae have been used extensively in designing, and in studying the performance of, sealed-in heating elements, such as are used in thermionic tubes, filament lamps, optical pyrometers and the cathodes of X-ray tubes; and, in particular, in calculating the effect of the 'end-losses'.

In parts I and II (Jain & Krishnan 1954*a*, *b*) of this paper it was shown that a practical general solution of the differential equation defining the steady state of the temperature distribution can be obtained when the filament is long, and for shorter lengths the solution can be expanded as a power series which is found to be rapidly convergent in most parts of the filament. In part III (Jain & Krishnan 1954*c*) were described detailed measurements on thin rods of Acheson graphite, which verify the various theoretical results obtained in parts I and II. Knowing the distribution of temperature along the filament, the distribution of any other physical characteristic like thermionic emission, which is a known function of the temperature, can also be readily formulated.

In the light of these investigations many of the empirical formulae referred to acquire a new significance. The present part is devoted to an appraisal of the theoretical significance of some of the formulae that are widely used. One such formula of practical importance, namely, that due to Stead (1920), has been discussed in part II in connexion with obtaining the general solution of the differential equation of the steady state.

2. Temperature distribution in reduced co-ordinates

Consider a filament in which $T_m - T_l$ is small in comparison with T_m, where T_l is the temperature at the centre of the filament, and T_m is the value to which T_l tends as the length of the filament is increased indefinitely keeping the heating current constant. For convenience we shall refer to such a filament as a *long* one, though the actual length of the filament will depend on how low is the temperature T_0 of the cold ends, i.e. of the junction of the leads and the filament, as compared with T_l.

(1) Denoting $T_m - T$ by Δ, where T is the temperature at a distance x measured from one of the ends, it is well known that in regions where Δ is small in comparison with T_m, x varies linearly with $\ln \Delta$. Worthing defines a certain natural unit of length λ which is the distance over which Δ in this region drops to a fraction 1/e. Using the reduced lengths $X = x/\lambda$, the temperature distribution in this region may be written in the form

$$X = D - \ln \Delta. \tag{1}$$

(2) Using tungsten filaments of different cross-sections, in which, however, the heating currents are so adjusted as to give the same central temperature T_m in all of them, he finds that the corresponding values of λ, determined experimentally from the logarithmic region, vary as $\sqrt{r}$, as was expected, where r is the radius of the cross-section of the filament.

(3) The second set of measurements of Worthing were made on one such filament, using different heating currents, so that r remains constant in these measurements, but T_m varies over a wide range. Plotting the reduced temperature $\tau = T/T_m$ against X, Worthing finds that the temperature distribution in the filament for all the different heating currents can be represented by a single τ-X curve.

(4) Worthing gives also plausible reasons for expecting this result to hold in the logarithmic range.

(5) Incidentally, he finds that the logarithmic region extends considerably beyond the range over which Δ is small in comparison with T_m, which is the basis on which equation (1) is generally derived.

We now proceed to show that these experimental findings of Worthing follow naturally from the theoretical expressions deduced in parts I and II, which further enable us to formulate the conditions for the validity of these findings. Some of these findings are found to be particular cases of more general theoretical results.

(1) The natural unit of length λ chosen by Worthing can be readily identified with the length $1/\sqrt{A}$ deduced theoretically in part I, which is equal to $(10aa_1 T_m^3)^{-\frac{1}{2}}$, where $a = \frac{2}{5}p\epsilon\sigma/(\kappa\omega)$, κ is the thermal conductivity, ϵ is the total emissivity, σ is Stefan's radiation constant, and p is the perimeter and ω the area of cross-section of the filament. a_1 is a function of the temperature coefficients of ρ and ϵ, where ρ is the specific resistance. a_1 was evaluated in part I, and is found to be close to unity, and reduces to it when the temperature coefficients are negligible.

Now, equation (1) can be derived from certain simple theoretical considerations, which also enable us to calculate λ (Prescott & Hincke 1928) in the special case when the temperature coefficients referred to are negligible.

(2) Now, for a circular cross-section of radius r, p/ω is obviously equal to $2/r$, and hence the proportionality of λ to $\sqrt{r}$ is verified.

(3) The simple theoretical considerations referred to, on the basis of which (1) may be derived, do not, however, enable us to determine the integration constant D that appears in (1) (Jain & Krishnan 1954*a*). They do not also tell us in what manner D is dependent on T_m, or on the end temperature T_0, since the validity of (1) may not extend to the end of the filament.

Using now the reduced co-ordinates, (1) can be expressed in the form

$$\ln(1-\tau) = -(X+\xi), \tag{2}$$

where

$$\xi = \ln T_m - D. \tag{3}$$

Not knowing theoretically how D depends on T_m or T_0, it is not possible to infer the dependence of ξ on these quantities, and much less to demonstrate that ξ does not involve T_m (except probably through τ_0), which demonstration is essential for verifying theoretically, in the logarithmic region, Worthing's finding regarding the identity of the τ-X curves for the different heating currents.

We emphasize this point here since the appearance of ξ in (2) as an integration constant has been taken by Worthing to imply that it should be independent of T_m; and his observation that the τ-X curve for a filament is independent of the heating current, i.e. independent of T_m, is taken to follow from it as a natural consequence in this region. On the other hand, the appearance of D or ξ as an integration constant in (1) or (2) merely signifies that it is independent of T and x, but not necessarily independent of T_m.

(4) It has been shown in part I that the integration constant D, and hence also ξ, can be evaluated theoretically, though not from the simple considerations on which (1) and (2) had been derived previously. As we shall see presently, ξ depends on τ_0 only, and does not involve T_m separately.

(5) It has also been shown in part I that the criterion for the validity of the logarithmic formula is that $\Delta/\ln\Delta$ should be small in comparison with T_m, which naturally holds over a much wider range than the condition that Δ be small in comparison with T_m.

(6) As we shall show presently, the temperature distribution in all long filaments over their whole lengths, irrespective of their actual lengths or cross-sections, or the heating currents used, or even the materials of which the filaments are made, can be represented by a single τ-X curve, provided that the point $X = 0$ from which measurements of lengths are made is chosen to have the same reduced temperature τ_0 in all of them.

This result is a precise one when the temperature coefficients of κ, ρ and ϵ are neglected, but is otherwise a close approximation.

Worthing's observation on the identity of the τ-X curves for a long filament heated by different currents follows as a particular case of this general result since τ_0, though not kept constant, remains roughly so in his measurements.

(7) For shorter filaments the A-region becomes experimentally significant, and has to be differentiated from the B-region:

(*a*) In the B-region the temperature distribution can be represented by the same τ-X curve as for a similar long filament heated by the same current.

(*b*) The temperature distribution in the A-region also can be represented by a single reduced curve, but the units of length and temperature to be used for the reduction are different from those applicable to the B-region.

We shall take up the case of long filaments first, to which Worthing's measurements refer; and obtain a general expression for the τ-X curve, applicable to all long filaments.

3. A general expression for the τ-X curve

It was shown in part II that in the logarithmic region, and over the rest of the filament on the colder side, the temperature distribution in a long filament is given by

$$\Delta\, e^{f(\Delta)} = \Delta_0\, e^{f(\Delta_0)}\, e^{-x/\lambda}, \tag{4}$$

where

$$f(\Delta) = \frac{1}{2}\frac{\Delta}{T_m} + \frac{1}{16}\frac{\Delta^2}{T_m^2} - \frac{1}{240}\frac{\Delta^3}{T_m^3} - \dots \tag{5}$$

$$= \phi(\tau), \quad \text{say.} \tag{6}$$

In terms of the reduced co-ordinates that have been adopted, (4) can be expressed in the form

$$(1-\tau)\, e^{\phi(\tau)} = (1-\tau_0)\, e^{\phi(\tau_0)-X}, \tag{7}$$

which will hold over the whole of the filament, down to $\tau = 0$ if the ends are cold enough for it. Equation (7) is obviously *independent of the material and the length and the cross-section of the filament, and also independent of* T_m, *provided that* τ_0, *which denotes the temperature of the point from which the lengths are measured, is chosen the same for all the filaments.*

This condition is roughly secured in Worthing's second set of measurements, on which is based his finding that the τ-X curve is independent of T_m. The same long filament is heated to different central temperatures T_m. Since no special arrangement is made to control the end temperatures, these temperatures will be the higher the higher the value of T_m. Though, as we shall see in a later section, this increase in T_0 was not quite in proportion to the increase in T_m, which is needed to secure the constancy of τ_0, it was nearly so.

In the earlier parts the distances were measured for convenience from one of the ends of the filament. Since now the origin of X has to be chosen to correspond to the same temperature τ_0 in all the filaments, one has naturally to distinguish between this temperature and the temperatures $\tau_e = T_e/\lambda$ of the ends. Negative values of X also will now be significant, and will correspond to $\tau_e < \tau < \tau_0$.

The effect of the temperature coefficients of κ, ρ and ϵ can also be taken into account by expressing $f(\Delta)$ in the form (see (49) and (50) of part I)

$$f_1(\Delta) = \frac{A_1\Delta}{T_m} + \frac{A_2\Delta^2}{T_m^2} + \frac{A_3\Delta^3}{T_m^3} + \dots, \tag{8}$$

where $A_1, A_2, A_3, \dots$ are functions of the temperature coefficients and of T_m, and are readily calculated. They tend to $\frac{1}{2}, \frac{1}{16}, -\frac{1}{240}, \dots$, respectively, when the temperature coefficients vanish. Actually only the first term in (8) is significant, and further A_1 is found to be only slightly higher than $\frac{1}{2}$, even this small deviation from $\frac{1}{2}$ depending more on the material of the filament than on T_m. Hence (7) will hold to a close approximation even when the temperature coefficients of κ, ρ and ϵ are taken into account.

The expression for A_1 given in part I was on the basis that the temperature variations of κ, ρ and ϵ may be taken to be linear, which is justified when the temperature range involved is not large. In Worthing's measurements, however, this range is quite large. Hence in calculating A_1 for tungsten applicable to Worthing's measurements we have to take into account the known variation with temperature, which is not quite linear for ρ and ϵ, though it is for κ. The value of A_1 thus obtained for tungsten is 0·75.

In the region where Δ is small, $e^{\phi(\tau)} \sim 1$, and (7) can be seen to reduce to the logarithmic formula (2), in which, however, the integration constant ξ, and hence also D, now stand evaluated:

$$\begin{aligned} -\xi &= \ln(1-\tau_0)+\phi(\tau_0) \\ &= \ln(1-\tau_0)+A_1(1-\tau_0), \end{aligned} \tag{9}$$

which is known when the temperature τ_0 corresponding to $X = 0$ is known. ξ is thus a function of τ_0 alone, and does not involve T_m separately. This will be so even when the temperature coefficients of κ, ρ and ϵ are finite, except that in this case the magnitude of a trivial part of A_1 depends on T_m.

It may appear at first sight odd that (7) ultimately involves only *two* parameters, namely, T_m and λ, and is independent of the length of the filament. This is because we are dealing here with *long* filaments, and with regions not too close to the centre —actually outside the A-region as defined in part I. The complete expression for a finite long filament is (part II, equation (27))

$$\Delta\, e^{f(\Delta)} = \Delta_0\, e^{f(\Delta_0)}\, [e^{-X} + e^{-(2L-X)}], \tag{10}$$

where $2L = 2l/\lambda$ is the reduced length of the filament. Outside the A-region the second term inside the brackets becomes negligible.

Inside the A-region also, whether the filament is short or long, the temperature distribution can be represented, as we shall show in a later section, by a single curve, involving again just two parameters, which are, however, different from those applicable to the B-region of long filaments.

For long filaments of the type used by Worthing, the temperature variation in the A-region is negligible, and hence the exclusion of the A-region from the range of applicability of (7) is not significant in his experiments.

4. Corresponding distances in a filament heated to different central temperatures

It would be relevant to refer here to some curves plotted by Worthing to demonstrate the identity of the τ-X curves for a given filament heated to different T_m's. These curves also serve to emphasize the necessary condition for this identity, namely, that the origin of X should be so chosen that the corresponding temperatures τ_0 are the same for all of them.

As mentioned previously, a long filament is heated by different currents, so that the corresponding temperatures T_m at the centre can be made to have different values. He chooses the case when $T_m = 2400°$ K as reference. Let x be the distance from the cold end to the point whose reduced temperature is τ. Let x' be the corresponding distance from the cold end to the point τ when T_m has a different value. Worthing finds that x' plotted against x is a straight line, and its slope is given by λ'/λ, where λ' and λ are the units of length determined by the two values of T_m concerned.

In these measurements of Worthing no special precaution is taken to keep the reduced end temperature τ_0 the same when the heating currents are varied. Naturally Worthing could not have been aware of this requirement for securing the identity of the curves.

Now from (7) and (9) it will be seen that

$$(1-\tau)\,e^{\phi(\tau)} = e^{-(x/\lambda+\xi)}. \tag{11}$$

Since τ is the same for both x and x'

$$x' = (\lambda'/\lambda)\,x + \lambda'(\xi-\xi'). \tag{12}$$

One may draw from (12) the following conclusions:

(1) The plots of x' against x are straight lines, whose slopes are λ'/λ, which is also Worthing's experimental finding.

(2) They will not pass through the origin, since the necessary condition for it, namely, $\xi'-\xi = 0$, which is equivalent to $\tau_0' = \tau_0$, is not secured in Worthing's experiments.

(3) From the natural variation of τ_0 with T_m mentioned earlier, namely, that τ_0 should be expected to decrease slightly with the increase of T_m, it will be seen that the intercepts of these straight lines on the x'-axis at $x = 0$ should be negative for the curves corresponding to $T_m < 2400°$ K, and positive for the curves for which $T_m > 2400°$ K. This conclusion is verified in Worthing's curves.

(4) Indeed by adjusting the origin of x in the different experiments to correspond to the same τ_0, all the three straight lines can be made to pass through the origin. If, further, the distances plotted are the reduced ones, namely, $X' = x'/\lambda'$ and $X = x/\lambda$, then all the straight lines will obviously become identical. In other words, a single straight line, passing through the origin, and inclined at 45° to the axes, will represent the corresponding distances for all the different heating currents.

5. Distances corresponding to any given drop in temperature with reference to that at the centre

Worthing (Forsythe & Worthing 1925) has also determined experimentally, again for long filaments, the distance x_d at which τ has just approached unity to within 10^{-3}. Taking $\tau_0 = \frac{1}{4}$ he finds that x_d can be represented by the empirical formula

$$x_d/\lambda = 7{\cdot}0, \tag{13}$$

which is a very useful relation.

Actually Worthing's determination of the numerical constant appearing in (13) is equivalent to determining experimentally the integration constant ξ of the logarithmic formula (2), since the point corresponding to x_d lies in the logarithmic region. Obviously

$$X_d = x_d/\lambda = -(\xi + \ln 10^{-3}). \tag{14}$$

Since the integration constant ξ has already been evaluated theoretically (see (9)), one obtains therefrom, for the case $\tau_0 = \frac{1}{4}$, which was the end temperature in Worthing's measurements,

$$-\xi = \ln \tfrac{3}{4} + 0{\cdot}75 \times \tfrac{3}{4}, \tag{15}$$

from which one obtains

$$X_d = 7{\cdot}18, \tag{16}$$

which is in close agreement with Worthing's experimental determination,* namely, $X_d = 7{\cdot}0$.

When once the constant appearing in the logarithmic formula has been determined, one can evaluate X_d readily for other values of τ too. Worthing gives the values $X_d = 7{\cdot}0 \pm \frac{7}{3}$ for $1-\tau = 10^{-4}$ and 10^{-2} respectively. Now $7{\cdot}0$ is the value of X_d for $1-\tau = 10^{-3}$, as we have seen, and $\frac{7}{3}$ is close to $\ln 10$. His filaments are long enough for all the three points to lie in the logarithmic region.

Obviously similar calculations can also be made for end temperatures other than $\tau_0 = \frac{1}{4}$.

6. L as a function of T_l

Worthing has extrapolated these results to include the case of shorter filaments too, in which $T_m - T_l$ may be considerable, though still small in comparison with T_m. Specifying, for example, that $(T_m - T_l)/T_m$ should be less than 10^{-3}, his formula for the corresponding semi-length of the filament is as follows. The value of X_d in a long filament corresponding to $1-\tau = 10^{-3}$ is $7{\cdot}0$, as we saw in the last section. In order that $(T_m - T_l)/T_m$ in the shorter filament might be less than 10^{-3}, Worthing's criterion is that its reduced semi-length should be at least 13 % larger than $7{\cdot}0$.

We have shown in part II that the drop in temperature $T_m - T_l$ at the centre of a long filament of semi-length l will be double the value of Δ at the same distance l in a similar infinitely long filament heated by the same current. Hence $L = l/\lambda$ should be theoretically equal to $X_d + \ln 2$. Now $\ln 2 = 0{\cdot}7$, and it can be seen to be just 10 % of the value of $X_d = 7{\cdot}0$ appropriate to the particular tolerance specified, namely, 10^{-3}. This is not very different from the value of 13 % estimated by Worthing.

7. Temperature variation outside the logarithmic region

As was mentioned in a previous section, the criterion for the validity of the logarithmic formula is that $\Delta/\ln\Delta$ should remain small in comparison with T_m. Outside this region the deviation from the logarithmic formula increases progressively with the increase of Δ. In order to cover this deviation an empirical formula has been proposed by Worthing (1914*a*), which is found to extend the region

* In reducing his observed x_d to X_d Worthing uses an experimentally determined value of λ which is slightly greater than the theoretical value. When the latter value of λ is used, as it should be, in reducing his x_d to X_d, the small discrepancy between his $7{\cdot}0$ and our $7{\cdot}18$ disappears.

of fit with observation considerably beyond the logarithmic range, but not, however, over the whole length of the filament, even though the end temperature τ_0 in these measurements was kept as high as 0·62. The formula is

$$\frac{T}{T_m} = [1 - e^{-(X+c)}]^n, \tag{17}$$

which can be expanded in the form

$$\frac{\Delta}{T_m} = n\,e^{-(X+c)} - \tfrac{1}{2}n(n-1)\,e^{-2(X+c)} + \tfrac{1}{6}n(n-1)(n-2)\,e^{-3(X+c)} - \dots. \tag{18}$$

Substituting values for the constants n and e^{-c} appropriate to tungsten and to the boundary conditions in Worthing's measurements—these values will be determined presently—the series is found to be fairly rapidly convergent even at $X = 0$. The convergence is naturally the more rapid the larger the distance X from the end.

If now all the terms on the right-hand side of (18) after the first can be neglected, as they can be when X is large enough, the formula obviously reduces to the simple logarithmic formula (2), $\ln n - c$ being then equal to $-\xi$ of (2).

The corresponding theoretical expression is (see (4))

$$\Delta\,e^{f(\Delta)} = \Delta_0\,e^{f(\Delta_0)}\,e^{-X}, \tag{19}$$

which may be put in the form $\Delta\,e^{f(\Delta)} = T_m\,e^{-(X+\xi)}$. (20)

Where Δ is small, $e^{f(\Delta)} \sim 1$, and (20) reduces, as we have seen, to the simple logarithmic formula. The deviation of $e^{f(\Delta)}$ from unity, which increases progressively with the increase of Δ, accounts for the observed progressive deviation from the logarithmic formula. It may be mentioned immediately that (20) fits with Worthing's observational data over the whole range including the region near the leads, and also with other data of his which extend to $\tau = \frac{1}{4}$.

We now proceed to compare Worthing's empirical relation (17), of which (18) is an expansion, with (20). Obviously (17) cannot be identified with (20), but one can investigate the degree of approximation to which (17) may be made to conform to (20). We obtain from (20), for this purpose, an expression for Δ/T_m as a power series in $e^{-(X+\xi)}$ as before, which can be compared term by term with the series (18). Putting

$$e^{f(\Delta)} \approx e^{A_1\Delta/T_m} \approx 1 + A_1\Delta/T_m \tag{21}$$

to a first approximation, equation (20) reduces to the quadratic form

$$\frac{A_1\Delta^2}{T_m^2} + \frac{\Delta}{T_m} - e^{-(X+\xi)} = 0, \tag{22}$$

the solution of which is

$$\frac{\Delta}{T_m} = \frac{1}{2A_1}[\sqrt{\{1 + 4A_1 \exp -(X+\xi)\}} - 1]. \tag{23}$$

Now the quantity inside the square root in (23), and hence the expression for Δ/T_m also, can be expanded as a convergent power series in $e^{-(X+\xi)}$ if $4A_1\,e^{-(X+\xi)}$, which is known to be positive, is less than unity. In view of (9) this corresponds to the criterion

$$4A_1(1-\tau_0)\,e^{A_1(1-\tau_0)-X} < 1, \tag{24}$$

which can be secured at $X = 0$, and hence for all positive values of X, when $\tau_0 > 0{\cdot}61$. This condition is just satisfied in Worthing's measurements to which he applies his formula (17); the lowest temperature of measurement corresponds to $\tau = 0{\cdot}62$. Hence (23) may now be expanded in the form

$$\frac{\Delta}{T_m} = e^{-(X+\xi)} - A_1\,e^{-2(X+\xi)} + 2A_1^2\,e^{-3(X+\xi)} - \dots. \tag{25}$$

Like (18) this series also is fairly rapidly convergent even at $X = 0$. This being so, one can adjust the first two terms in (18) to be the same as the corresponding terms in (25) for all values of X, by putting

$$n\,e^{-c} = e^{-\xi}, \tag{26}$$

$$\tfrac{1}{2}n(n-1)\,e^{-2c} = A_1\,e^{-2\xi}, \tag{27}$$

from which the two constants n and e^{-c} can be determined in terms of the known quantities A_1 and ξ:

$$\left.\begin{aligned} n &= -\frac{1}{2A_1-1},\\ e^{-c} &= -(2A_1-1)\,e^{-\xi}.\end{aligned}\right\} \tag{28}$$

Using (28) the first three terms in Worthing's expansion (which is convergent even when $X = 0$) may be written in the form

$$\frac{\Delta}{T_m} = e^{-(X+\xi)} - \left(\frac{1}{2}-\frac{1}{2n}\right)e^{-2(X+\xi)} + \frac{(n-1)(n-2)}{6n^2}\,e^{-3(X+\xi)} - \dots, \tag{29}$$

in which the coefficient of the second term, namely, $\frac{1}{2}-\frac{1}{2n}$, takes the place of A_1 in (25). Having adjusted the first two terms in (18) to be the same as the corresponding terms in (25), naturally it will not be possible to secure equality of the succeeding terms in the two series. However, comparing the third term in (29), namely, $\frac{1}{6}(n-1)(n-2)/n^2$, with that in (25), namely, $2A^2$, it will be seen that they are of the same sign, though not of the same magnitude. Hence Worthing's formula (17) can be made to fit with the theoretical formula (25) to a slightly closer approximation than would correspond to the neglect of the third and the succeeding terms. Obviously the approximation depends on X too, and hence for any specified approximation one can estimate the lower limit of X for which the third and the succeeding terms may become negligible. At all distances greater than this the agreement of (17) with observation will naturally be closer than the tolerance prescribed.

Now the coefficients of the second terms, as we have seen, are $\frac{1}{2}-\frac{1}{2n}$ and A_1 respectively. In the special case when the temperature coefficients of κ, ρ and ϵ are negligible, $A_1 = \frac{1}{2}$. Hence $-1/(2n)$, which gives the difference between A_1 and $\frac{1}{2}$, is determined almost wholly by the finite temperature coefficients, and is therefore small. As we have seen, the deviation of A_1 from $\frac{1}{2}$ is determined more by the material of the filament than by T_m. Hence n should also be practically independent of T_m and the dimensions of the filament, which is also Worthing's observation.

For the same reason again, almost any value of n which does not change the sign or the order of magnitude of $\frac{1}{2}-\frac{1}{2n}$, for example, any positive value greater than unity, or any small negative value, will extend the fit with observation beyond the logarithmic region. It is significant in this connexion that Worthing uses the value $n = 1{\cdot}87$, whereas the theoretical value is negative and about $-2{\cdot}0$.

Even with the latter value for n, the validity of formula (17) does not extend much beyond $\tau = 0{\cdot}7$. The discrepancy becomes quite large when applied to other measurements of Worthing which extend to $\tau = \frac{1}{4}$. This is not surprising, since for such low values of τ (20) cannot even be expanded in a form that would permit comparison with (18).

Hence one has to fall back on (20) for this region. Taking $\tau_0 = 0{\cdot}25$, $e^{-\xi}$ comes out as 1·17. Using this value, and $A_1 = 0{\cdot}75$ and the known thermal conductivity of tungsten at $T_m = 2400^\circ$ K, namely, 0·96 W cm^{-1} deg^{-1}, the temperature distribution calculated on the basis of (20) is found to agree very closely with the distribution observed by Worthing over the whole length of the filament.

8. The distribution of other physical characteristics along the filament

Knowing the distribution of temperature along the filament, the distribution of any other physical characteristic which is a known function of the temperature can also be investigated theoretically. Some of these characteristics are listed in table 1. In the case of tungsten, the temperature dependence of all of them can be expressed, according to Langmuir (1930), in the form

$$F_i(T) = A_i e^{\gamma_i} e^{-\Theta_i/T}, \tag{30}$$

where $F_i(T)$ denotes the particular physical characteristic studied, and γ_i and Θ_i are constants whose values are known and are entered in table 1. For some of the characteristics like thermionic emission and the rate of evaporation, (30) can be deduced theoretically, and for others it is an empirical relation.

Table 1

property	electrical resistance	total radiation	brightness	thermionic emission	rate of evaporation
γ_i	1·2	5·1	0	2	0
Θ_i	0	0	25200	52600	94100

Worthing (1922) has made extensive measurements on the distribution of these various characteristics along heated tungsten filaments, except the rate of evaporation, and he has plotted $F_i(T)/F_i(T_m) = \psi_i$, say, against the distance from the end. Since all his measurements refer to the same filament, it is not material whether the plot is against the reduced distance or the actual distance. He finds that in the logarithmic region ψ_i can be represented by the formula

$$\psi_i = 1 - e^{-(X+\xi_i)}, \tag{31}$$

in which ξ_i is a constant independent of X and T. In other words, he finds that in this region the curves for the different characteristics are laterally displaced along the x-axis relatively to one another by constant amounts, that is by amounts that are independent of X. He observes further in a general way that this constancy of shift extends over a wider range than the logarithmic region. In an earlier paper (1914*b*) he has also found in the case of one of these properties, namely, the total radiation, that its observed distribution along the filament can be fitted over most parts of the filament by a formula very similar to his formula (17) for the distribution of temperature. This formula is

$$\psi_i = [1 - e^{-(X+c_i)}]^{n_i}. \tag{32}$$

This can be seen to reduce in the logarithmic range to (31) if $n_i e^{-c_i} = e^{-\xi_i}$.

We now proceed to demonstrate that these observations of Worthing can be verified theoretically, and that the constants appearing in them can also be evaluated, in the same manner in which the constants in (17) were evaluated earlier. Adopting Langmuir's expression (30) for $F_i(T)$ as an explicit function of T, it can be shown that

$$\psi_i = 1 - (\gamma_i + \Theta_i/T_m)\,\Delta/T_m + [\tfrac{1}{2}\Theta_i^2/T_m^2 + (\gamma_i - 1)(\tfrac{1}{2}\gamma_i + \Theta_i/T_m)]\,\Delta^2/T_m^2 - \ldots. \tag{33}$$

Since we are here concerned with regions not much beyond the logarithmic region, one may substitute for Δ/T_m from (25), and obtain

$$\psi_i = 1 - e^{-(X+\xi_i)} + B_{i1}\,e^{-2(X+\xi_i)} - \ldots, \tag{34}$$

where

$$\xi_i = \xi - \ln(\gamma_i + \Theta_i/T_m), \tag{35}$$

$$B_{i1} = [A_1(\gamma_i + \Theta_i/T_m) + \tfrac{1}{2}\Theta_i^2/T_m^2 + (\gamma_i - 1)(\tfrac{1}{2}\gamma_i + \Theta_i/T_m)]/(\gamma_i + \Theta_i/T_m)^2. \tag{36}$$

By securing

$$\left.\begin{aligned} n_i e^{-c_i} &= e^{-\xi_i}, \\ \frac{1}{2} - \frac{1}{2n_i} &= B_{i1}, \end{aligned}\right\} \tag{37}$$

the first three terms in series (34) can be made to be identical with the first three terms in the expansion of (32), which shows that an expression of the type (32) can be made to cover an appreciably wider range than the exponential formula (31), in the same manner in which (17) extends (2). Like (17), formula (32) also has serious limitations in the range of its applicability.

The differences in the value of ξ_i for the different curves represent naturally their relative shifts along the x-axis, and it can be seen from the expressions that these shifts are independent of X over the range over which formula (31) holds.

Worthing has given some numerical values for the shift along the x-axis, in the logarithmic region, of the distribution curves for the different characteristics with reference to the curve for the temperature distribution. These values are entered in table 2 along with the corresponding values calculated from (35). The agreement between the calculated and the observed values is quite satisfactory.

On the basis of the observation that the temperature distribution in a long filament heated by different currents can be represented by a single τ-X curve, and the further observation that in the logarithmic region the other distribution curves for the filament, namely ψ_i plotted against X, are displaced with reference to the τ-X curve along the x-axis by amounts independent of X, Worthing draws the following conclusion. Just as the distribution of temperature along a given long filament heated by different currents, can be represented by a single τ-X curve, the distribution of any of these other characteristics along the filament should also be representable by a single ψ_i-X curve. This conclusion, however, is not justified. What is implied by the observed relative shifts is that in the logarithmic region ξ_i is independent of X and T, whereas Worthing's inference would require that ξ_i be independent of T_m also. As will be seen from equation (35), this will be the case for only such characteristics for which $\Theta_i = 0$, like electrical resistance and total radiation, but not for example for thermionic emission, or for brightness. For any of the latter characteristics, for which $\Theta \neq 0$, the distribution curves for filaments heated by different currents, will obviously be shifted parallel to one another along the x-axis by amounts determined by their respective T_m's.

TABLE 2

property		electrical resistance	total radiation	brightness	thermionic emission	rate of evaporation
$\lambda(\xi - \xi_i)$ (cm)	obs.	0·04	0·39	0·60	0·82	—
	calc.	0·045	0·39	0·59	0·79	0·90

Having obtained an analytical expression for the distribution of these characteristics along the heated filament, namely (33), it is easy to integrate it over the whole length, or any part, of the filament, and in particular to calculate the correction due to the end-losses, for the purpose for which the filament is intended to be used.

The corrections due to the end-losses for many of these characteristics have been calculated by Worthing by graphical integration of the observed distribution curves for these characteristics. For such of the properties for which Θ_i is zero, and for which the distribution curve with reduced co-ordinates is independent of T_m, the values given by Worthing are naturally *generally* applicable. On the other hand for properties like thermionic emission for which $\Theta \neq 0$, the values given by Worthing hold only for filaments for which the temperature T_m at the centre has the particular value for which his calculations have been made, namely 2400° K. Naturally, not being aware of the significant difference between these two classes of physical characteristics, in being reproducible uniquely or not by a single curve in reduced co-ordinates, Worthing has not differentiated the data for these two classes.

Worthing has verified in detail in one case the correction calculated by him for the end losses, namely for the radiation output, with the help of the data obtained for different filaments and heated by different currents. He concludes therefrom that the end corrections given by him for the other characteristics also should be similarly valid. It so happens that the radiation output is one of those characteristics for which $\Theta_i = 0$, and this is true also of two others in his table, namely, heat content and energy input. On the other hand, the other two characteristics for which also he gives end losses, namely, thermionic emission and brightness, do not belong to this category.

Since the end-loss corrections are important practically, and an adequate treatment of it will be beyond the scope of the present part, we postpone the treatment of end losses to a later part.

9. Short filaments

Till now we have confined attention to long filaments in which $T_m - T_l$ is small in comparison with T_m, in which case the A-region is practically unimportant, since the temperature variation inside it is negligible. We shall now consider shorter filaments, in which the A-region becomes important. It was shown in part I that the A-region is defined by $0 < t < t_c$, where $t = T_l - T$, and

$$t_c = (T_m^4 - T_l^4)/(2T_l^3), \tag{38}$$

and the temperature distribution is given by (see (25) of part II)

$$t = \tfrac{1}{2}t_c[\cosh(q/\lambda_A) - 1], \tag{39}$$

where q is the distance measured from the centre, and $\lambda_A = (10aT_l^3)^{-\frac{1}{2}}$ is now the appropriate unit of length. Expressing t in terms of t_c and q in terms of λ_A, (39) takes the simple form

$$\tau_A = \tfrac{1}{2}(\cosh q_A - 1), \tag{40}$$

where τ_A and q_A are the reduced temperature and length respectively. The expression involves only these two quantities, and is even more general than (7) applicable to the B-region, since the latter requires at least τ_0 to be adjusted the same in all the filaments.

Outside the A-region (7) would be applicable, for a short filament also, with the appropriate value of T_m, which now will not be the temperature at the centre.

10. Summary

Many interesting empirical formulae relating to the distribution of temperature, and of some of the temperature-dependent physical characteristics, along heated filaments, which were originally proposed by Worthing, and have since found wide practical application, are discussed in the present part in relation to the theoretical expressions obtained in parts I and II. The following are some of the main results.

(1) Expressing temperatures in terms of the temperature at the centre, and lengths in terms of a certain natural unit, Worthing finds that the observed temperature distribution curves for a long tungsten filament heated by different currents become identical. This is shown to follow as a special case of the following general result. The temperature distribution in long filaments of different materials, of different lengths and cross-sections, and heated to different central temperatures, can all be represented by a single distribution curve in reduced co-ordinates, involving ultimately only two parameters, provided that the origin from which distances are measured is chosen to have the same reduced temperature in all the cases considered.

This general result is a precise one when the small temperature coefficients of the physical constants involved are neglected, but is otherwise a close approximation.

(2) It is further shown that the same reduced curve may be made to represent the distribution in the B-region of a short filament, provided that in reducing the temperature the unit used is not the actual temperature at the centre, but the temperature that would obtain at the centre of a similar long filament heated by the same current.

(3) The temperature distribution in the A-region of a filament, long or short, can also be similarly represented by a single curve, which however, is different from that applicable to the B-region. This curve also involves only two parameters.

(4) These two parameters, though different from those involved in the B-region, are not independent of them, the total number required for describing uniquely both the regions being three (in which number are not included temperature coefficients of the physical quantities involved).

(5) The distribution of the temperature-dependent physical characteristics cannot, in general, be so represented by single curves; this can be done only in the special case when the property is proportional to a constant power of the temperature. This is so for example for radiation output, but not for thermionic emission.

(6) Worthing's expressions for the distance from the end to the point where the temperature has just approximated to that at the centre to any given degree of approximation are verified.

(7) Similar expressions formulated by Worthing for the total length of a filament such that the temperature at its centre might approach to a given degree of closeness that at the centre of a similar long filament heated by the same current also verified.

(8) A simple generalization of the logarithmic formula was proposed by Worthing which extends its applicability considerably beyond the logarithmic region. The range of validity of this extended formula, and its limitations are discussed.

(9) Similar formulae for the distribution of certain physical characteristics which are known functions of the temperature, along the filament, are also considered in detail.

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Thermionic constants of metals and semiconductors
IV. Monovalent metals (continued)

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In a monovalent metal, in which the valency electrons may be regarded as forming a free-electron assemblage, the assumption of a temperature-independent energy barrier at the surface of the metal is shown to be equivalent to taking the free electrons in the condensed phase, namely, the metal, and the electrons in the gaseous phase in thermal equilibrium with it, as forming a homogeneous single component system. The temperature variation of the work function is then determined by the temperature variation of the thermodynamic potential of the electrons in the condensed phase, when the external pressure is kept constant at the value of the saturation vapour pressure of the electrons, which is equivalent to keeping the pressure of the electron assemblage in the condensed phase also constant, since the energy barrier at the surface is independent of temperature. It is further shown that for a degenerate, or nearly degenerate, electron assemblage the specific heat at constant pressure is the same as that at constant volume, and it is easily calculated. The temperature coefficient of the work function calculated therefrom corresponds to an apparent lowering of about 8 to 10 % in the value of the A coefficient in thermionic emission. This agrees with observation.

On the other hand, the thermal expansion of the lattice is found to be about 25 to 50 times that to be expected thermodynamically for the electron assemblage in the condensed phase. This result, when viewed against the nearly normal observed value of the A coefficient, shows that the energy barrier at the surface of the metal should decrease with increase of temperature by the same amount by which the thermodynamic potential of the electrons in the condensed phase decreases as a result of the thermal expansion of the lattice.

A detailed calculation is made of the effect of both the thermal expansion of the lattice, and the increased thermal oscillations of the atoms in the lattice, associated with the rise in temperature, on the energy of the barrier at the surface. The net effect is found to be a lowering of this energy of the required magnitude.

1. Introduction

In the early measurements on the thermionic constants of the monovalent metals the A coefficients in Richardson's equation for thermionic emission were found to be in general much less than the theoretical value of 120 amp deg.$^{-2}$ cm^{-2}. Two obvious explanations suggest themselves. One is the insufficient degassing of the metal surface. The adsorption of gases at the surface is known to depress the intensity of thermionic emission from the surface, and the low melting-points of these metals do not permit their being raised to temperatures high enough for efficient degassing. The other explanation is the thermal expansion of the metal (Herzfeld 1930; Reimann 1934). The conduction electrons in a monovalent metal form a degenerate assemblage whose thermodynamic potential is naturally a function of the density of the electrons—actually proportional to $n^{\frac{2}{3}}$, where n is the number of free electrons per unit volume. Hence the thermal expansion of the metal will lead to a lowering of the thermodynamic potential. *If the energy barrier at the surface of the metal may be regarded as remaining independent of the temperature*, the negative temperature coefficient of the thermodynamic potential will mean an equivalent positive temperature coefficient for the work function. Any such small linear increase of the work function with temperature will not be distinguishable in practice from a temperature-independent reduction in the value of the A coefficient.

In part III (Jain & Krishnan 1953) were described measurements on the thermionic constants of the monovalent metals copper, silver and gold, made by the effusion method, which is quite insensitive to the contamination of the surface of the metal by adsorbed gases. The measurements yield for all the three metals A values only slightly short of the theoretical value of 120 amp deg.$^{-2}$ cm^{-2}. The observed values of A were 110, 107 and 100 respectively for the three metals. Had the energy barrier been independent of temperature, the thermal expansion alone, as was shown in part III, would have reduced these values to an eleventh of the theoretical value in copper, to a fourteenth in silver, and to less than a fifth in gold.

In the light of these results it becomes desirable to re-examine the question of the temperature variation of the work function of a monovalent metal, which because of the simplicity of its electronic structure is somewhat amenable to theoretical treatment. The various factors that might contribute to the variation of the work function of a metal with temperature have been discussed by Herzfeld (1930), Reimann (1934), Wigner (1936), Seely (1941), Seitz (1940) and others. The present part is concerned with the detailed evaluation of the effects of these factors on the energy barrier at the surface of the metal, and on the thermodynamic potential of the electrons, and hence on the work function, which represents the difference between these two energies.

2. Statement of the problem

Any method of evaluating theoretically the rate of emission of electrons from the surface of a heated metal reduces *ultimately* to one of determining the saturation vapour pressure of the electron gas in thermal equilibrium with the metal. Since the saturation pressure of the vapour at all working temperatures is very small, the vapour can be regarded as a perfect gas obeying the classical laws. The number of electrons in this gas that cross a unit area from one side to the other per second, equal to N say, is therefore readily known. The number of electrons that enter the metal from the vapour phase per unit area per second will be the above number multiplied by a factor $1-r$, where r is the reflexion coefficient, and $1-r$ the transmission coefficient, of the surface for electrons incident on it. Since r should be the same whether the electrons are incident on the surface from the vapour phase or from the condensed phase, the condition for equilibrium between the electrons in the metal phase and in the vapour phase requires that the number of electrons emitted per second per unit area of the metal surface should be the same as the number $N/(1-r)$ crossing the surface in the opposite direction.

Consider the electron gas in thermal equilibrium with the metal in a chamber scooped out of it, and consider the electrons that *effuse* out of the chamber through a small aperture in a thin wall of the chamber. The number of electrons that effuse out per second per unit area of the aperture will be just N, the uncertain factor $1-r$ that appears in the expression for direct emission from the surface being now eliminated. We shall confine ourselves to this case, as it is also of interest experimentally (Jain & Krishnan 1952*a*, *b*, 1953), and the results obtained may be made applicable to direct emission also, by including the factor $1-r$.

Indeed, the difference between effusion from the saturated vapour and emission directly from the surface is analogous, as was shown in the earlier parts, to the difference between the electromagnetic radiations from a cavity and from the surface directly of a hot body.

The simplest way in which the saturation vapour pressure of the electrons in thermal equilibrium with the metal can be evaluated, as is well known, is to regard the system of electrons in the two phases as forming *a single component system* having a finite latent heat of evaporation, which represents the work done per electron in transferring it from the condensed phase to the gaseous phase at the saturation pressure of the latter. On this basis, and without any further approximations, the saturation vapour pressure may be evaluated readily by applying the well-known thermodynamic relation of Clapeyron and Clausius connecting the saturation vapour pressure at any given temperature with the latent heat of evaporation at this temperature and under this pressure.

We should mention immediately that the basic assumption that the *electrons in the metal and in the vapour phase form a single component system* implies the following. Though initially the number of free electrons per unit volume in the condensed phase, and the thermodynamic properties of this electron assemblage, are determined wholly by the density of the atoms in the lattice, any change in the volume or the pressure of the electron assemblage due to a change in the temperature is taken to be completely determined by the initial conditions of the electrons in the condensed phase, and not by the actual thermal expansion of the lattice, which, as we shall see in a later section, is 25 to 50 times larger.

On this model the work function of the metal at the absolute zero of temperature will be the same as the latent heat of evaporation of the electrons at this temperature, and it can be shown (see § 5) that the temperature variation of the work function with its sign changed, will then be just that of the thermodynamic potential of the electrons in the condensed phase *at constant pressure*, namely, the saturation pressure of the electron gas in equilibrium with it.

Now the occurrence of a large latent heat of evaporation implies also a large energy barrier between the electrons inside the metal and outside it, and hence one can also determine thermionic emission by finding the number of electrons in the metal that strike unit area of the surface per second whose components of momenta along the normal to the surface are large enough to enable them to cross the barrier. In a single-component system this energy barrier, which may be defined as the difference in energy between an electron at rest in the vapour phase, and an electron at rest in the condensed phase, i.e. at the bottom of the energy band of the free electrons in the metal, *can be shown to be independent of the temperature* (see §4). Conversely, the usual model in which the energy barrier as defined just now is taken to be independent of temperature can be shown to be equivalent to the model that regards the electrons in the two phases as constituting a single-component system. Hence on the postulate of a temperature-independent energy barrier also the temperature variation of the work function, except for its sign, will be that of the thermodynamic potential of the electrons in the condensed phase at *constant pressure*, and not at constant volume as it is generally taken to be. This distinction, as we shall see in a later section, is material.

Considering an actual monovalent metal, the thermal expansion of the ionic lattice is much larger than the expansion at constant pressure that we should expect for the electrons in the condensed phase from the initial conditions—for example, about 50 times in silver. That in spite of this large thermal expansion, the temperature variation of the work function remains small, as evidenced by the small deviation of the observed A coefficient from the theoretical value, shows that the energy barrier is by no means temperature-independent. Before discussing the effect of the thermal expansion of the lattice, and of the thermal agitations of the atoms in the lattice, on the energy barrier and on the thermodynamic potential of the electrons, it is desirable to investigate the full implications of treating the electrons in the two phases as forming a single-component system.

3. The saturation vapour pressure of the electron gas in equilibrium with a metal

Consider the electrons in the condensed and in the vapour phases as forming a *homogeneous single-component system*, and consider the thermodynamic potential, or the Gibbs free energy of the system, per electron, defined by

$$\zeta = u - Ts + pv, \tag{1}$$

where u, s and v are the internal energy, the entropy and the volume, per electron. Using the subscripts C and G to indicate that the quantity concerned refers to the condensed and the gaseous phases respectively, one can see that the thermodynamic potential *of the gaeous phase* is given by (see, for example, Planck 1926)

$$\zeta_{T,\mathrm{G}} = kT \ln p - c_{p,\mathrm{G}} T \ln T - kT \ln \Delta + \chi, \tag{2}$$

in which p is the saturation vapour pressure of the electron gas, and $c_{p,\mathrm{G}}$ is the specific heat per electron in the gaseous phase. Δ is the well-known Sucker-Tetrode constant for a monatomic gas, with an extra multiplying factor 2 to take into account the spin of the electrons (Fowler 1929), and is given by

$$\Delta = 2(2\pi m)^{\frac{3}{2}} k^{\frac{5}{2}}/h^3. \tag{3}$$

χ is the temperature-independent additive constant that appears in the expression for the heat function for the gas

$$h_{T,\mathrm{G}} = u + pv = c_{p,\mathrm{G}} T + \chi. \tag{4}$$

In other words, χ is the value of the heat function of the gas at $T = 0$.

Now the thermodynamic potential *in the condensed phase* will be given by

$$\zeta_{T,\mathrm{C}} = h_{T,\mathrm{C}} - Ts_{\mathrm{C}} = h_{T,\mathrm{C}} - T\int \frac{c_{p,\mathrm{C}}}{T}\mathrm{d}T. \tag{5}$$

Equating the expressions (2) and (5) for the thermodynamic potentials in the gaseous and in the condensed phases respectively, and treating the vapour as a perfect gas, for which $c_{p,\mathrm{G}} = \frac{5}{2}k$, one obtains

$$p = \Delta T^{\frac{5}{2}} \mathrm{e}^{-\phi_T/(kT)}, \tag{6}$$

in which

$$\phi_T = \chi - h_{T,\mathrm{C}} + T\int_0^T \frac{c_{p,\mathrm{C}}}{T}\mathrm{d}T. \tag{7}$$

Now the latent heat of evaporation L_0 at $T = 0$ will be given by

$$L_0 = \chi - h_{0,\mathrm{C}}. \tag{8}$$

Further,

$$h_{T,\mathrm{C}} - h_{0,\mathrm{C}} = \int_0^T c_{p,\mathrm{C}}\mathrm{d}T, \tag{9}$$

from which one obtains

$$\phi_T = L_0 + \epsilon_T = \phi_0 + \epsilon_T, \tag{10}$$

where

$$\epsilon_T = T\int_0^T \frac{c_{p,\mathrm{C}}}{T}\mathrm{d}T - \int_0^T c_{p,\mathrm{C}}\mathrm{d}T. \tag{11}$$

It will be seen from (5) and (7) that

$$\zeta_T = \chi - \phi_T. \tag{12}$$

Hence the temperature variation of the thermodynamic potential of either phase will be given by an expression similar to (10), namely,

$$\zeta_T = \zeta_0 - \epsilon_T. \tag{13}$$

Thus *for a single-component system* the temperature variation of both ϕ and ζ can be readily obtained in terms of the specific heat of the electrons in the condensed phase at constant pressure, namely, the saturation vapour pressure of the electron gas in thermal equilibrium with the condensed phase.

4. The energy barrier in a single-component system independent of temperature

It is desirable here to interpret the constants appearing in these thermodynamic equations in terms of the energy levels. We shall take, as before, the energy of an electron at rest inside the metal, i.e. the bottom of the energy band of the free electrons, as the zero level. From the expressions given in the previous section it will then be seen (1) that the constant χ appearing in the expression for the heat function $h_{T,G}$ in the vapour phase will represent the energy of an electron at rest outside the metal; i.e. χ will be the energy barrier at the surface as we have defined it in the previous sections; (2) that $\zeta_0 = h_{0,C}$ will be the energy of the Fermi level at $T = 0$, i.e. the Fermi level of the completely degenerate electron assemblage in the condensed phase; and (3) that at all ordinary temperatures $T \ll \zeta_0/k$, the difference between χ and the thermodynamic potential ζ_T, which obviously is a function of the temperature, represents the work function ϕ_T.

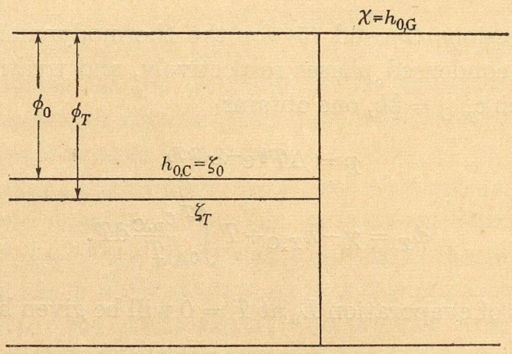

Figure 1. Energy levels of a single-component system.

Incidentally it may be mentioned that in equation (2) the first term tends to a finite value, namely, ϕ_0 as T tends to 0, and ζ then tends to $\chi - \phi_0$.

Obviously the identification of the energy barrier with the temperature-independent constant χ that appears in expression (4) for the heat function of the gas, leads to the following significant result, which needs emphasis here, namely, that in a single-component system, the energy barrier at the surface is independent of the temperature, and conversely that the usual model on which thermionic emission is calculated, in which the energy barrier at the surface is taken to be independent of temperature, is ultimately equivalent to taking the electrons in the metal phase and in the vapour phase as forming a homogeneous single-component system.

5. The thermionic constants of the metal

Consider a chamber scooped out of the metal, and consider the effusion of electrons from the saturated electron gas in the chamber through a small hole in a thin wall of the chamber. The number of electrons that effuse out per second per unit area of the hole in all directions, i.e. over the whole of the semi-solid angle 2π, is given by

$$N = p/\surd(2\pi mkT). \tag{14}$$

When these electrons are collected in a positively charged Faraday cylinder in the usual manner (see Jain & Krishnan 1952*a*) the corresponding saturation current, extrapolated to zero space charge, is given by

$$i = Ne = AT^2\,e^{-\phi_T/(kT)}, \tag{15}$$

where

$$A = 4\pi emk^2/h^3. \tag{16}$$

In other words, the expression for the rate of effusion of electrons from a chamber through a small hole has the same form as the expression for emission from a corresponding area of the surface of the heated metal, but with this difference; whereas the expression for emission from the surface involves the transmission coefficient $1-r$ as a multiplying factor, the expression for effusion from the chamber does not include it.

One can immediately draw the following conclusions for the single-component system that we have been considering:

(i) Equation (15) shows that ϕ_T defined by (7) is the thermionic work function of the metal as usually defined, at temperature T.

(ii) The temperature variation of the work function ϕ, and of the thermodynamical potential ζ, will be given by

$$\phi_T - \phi_0 = \zeta_0 - \zeta_T = \epsilon_T, \tag{17}$$

and can be calculated thermodynamically from (11) when the specific heat of the electrons in the condensed phase, at constant pressure, namely, the pressure of the saturated vapour, is known.

(iii) The latent heat of evaporation of the electrons from the metal at the absolute zero of temperature, namely, L_0, is the same as the work function ϕ_0 at this temperature, but at any other temperature T, $L_T \neq \phi_T$. Since by definition L_T is the heat per electron required to change to the vapour under the constant pressure of its saturated vapour,

$$L_T = L_0 + c_{p,\mathrm{G}}T - \int_0^T c_{p,\mathrm{C}}\,\mathrm{d}T, \tag{18}$$

whereas

$$\phi_T = L_0 + T\int_0^T \frac{c_{p,\mathrm{C}}}{T}\,\mathrm{d}T - \int_0^T c_{p,\mathrm{C}}\,\mathrm{d}T. \tag{19}$$

6. Thermionic emission on the basis of a temperature-independent energy barrier at the surface

Before proceeding to discuss the temperature variation of ϕ, which except for its sign is the same as that of ζ, in terms of the specific heat $c_{p,\mathrm{C}}$ of the electrons in the condensed phase, it is desirable to refer to an apparent discrepancy regarding the order of approximation, between the Richardson expression for thermionic emission obtained here and that obtained in the usual manner on the basis of a temperature-independent energy barrier at the surface of the metal, which, as we have seen, is equivalent to the present model. The number of electrons in the metal striking unit area of the surface per second whose components of momenta along the normal to the surface exceed $\surd(2m\chi)$ can be shown to be equal to (Sommerfeld & Bethe 1933; Fowler & Guggenheim 1939)

$$N = \frac{2}{h^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{E_0}^{\infty}\frac{\mathrm{d}p_y\,\mathrm{d}p_z\,\mathrm{d}E}{\mathrm{e}^{(E-\zeta)/(kT)}+1}, \tag{20}$$

where

$$E = (p_x^2+p_y^2+p_z^2)/(2m), \tag{21}$$

$$E_0 = \chi+(p_y^2+p_z^2)/(2m). \tag{22}$$

Neglecting unity in comparison with $\exp\{(E-\zeta)/(kT)\}$, which will be justified at all ordinary temperatures $T \ll \zeta/k < \chi/k$, one obtains immediately from (20)

$$i = Ne = AT^2\,\mathrm{e}^{-(\chi-\zeta)/(kT)},$$

which is the same expression as (15).

It may appear at first sight surprising that expression (15), which was obtained as a *precise* result thermodynamically, should now turn out as a first approximation only. The answer is that neglecting unity in comparison with $\exp\{(E-\zeta)/(kT)\}$ for the electrons whose energies are large enough to enable them to cross the barrier is equivalent to treating these electrons as conforming to the classical statistics of Boltzmann, which is precisely what we are doing already with the electron gas in thermal equilibrium with the metals, whose density is comparable with the density in the metal of electrons whose energies are greater than χ. Thus the discrepancy referred to is only an apparent one.

7. The internal and the external pressures of the assemblage of electrons in the metal

We now proceed to calculate the specific heat of the electrons in the condensed phase, which is required for the evaluation of the various expressions occurring in §§ 3 and 5, and in particular the expression for the temperature variation of ϕ. It should be emphasized here that the constant pressure to which the specific heat $c_{p,C}$ refers is the saturation pressure of the electron gas in thermal equilibrium with the condensed phase.

Since the assemblage of electrons in the condensed phase may also be treated as an electron gas, but highly degenerate, it will have its own pressure, which for convenience we designate as the *internal pressure*, in order to distinguish it from the pressure of the vapour with which we have been concerned till now, and which we shall refer to hereafter as the *external pressure*.

Now for a monovalent metal a simple calculation shows that the internal pressure should be of the order of a million atmospheres, whereas the pressure of the saturated vapour is a very small fraction of an atmosphere, of the order of 10^{-12} even at the highest working temperatures. This large difference in pressure is really maintained by the presence of the energy barrier. Now the difference between the internal and the external pressures is a function of χ, and for a single component system since χ remains independent of the temperature, the specific heat $c_{p,C}$ of the electrons in the condensed phase at constant external pressure, which we need to know in order to calculate the temperature variation of ϕ, will be the same as the specific heat of the electron assemblage in the condensed phase when its own pressure, i.e. the internal pressure, is kept constant.

It will be seen from equations (5) and (11) that the first integral in expression (7) for ϵ_T originates from the expression for the entropy, and the second from the expression for the heat function, of the electrons in the condensed phase, and that in both these integrals $c_{p,\mathrm{C}}$ may therefore be replaced by the specific heat c_p of these electrons at constant internal pressure. It is the latter quantity that is calculated in the next section.

8. The specific heat of a degenerate electron gas at constant pressure shown to be the same as that at constant volume

For simplicity we shall confine attention to the case when the assemblage of electrons in the condensed phase is nearly degenerate. If n is the number of electrons per unit volume at temperature T, the internal energy u per electron is given by the well-known expression (Sommerfeld & Bethe 1933)

$$u = \tfrac{3}{5}\zeta_0 + \tfrac{1}{2}\gamma T^2 + \dots, \tag{23}$$

where

$$\zeta_0 = \frac{h^2}{2m}\left(\frac{3n}{8\pi}\right)^{\frac{2}{3}}. \tag{24}$$

ζ_0 is a function of n, and therefore also of the temperature.

$$\gamma = \frac{\pi^2 k^2}{2\zeta_0}. \tag{25}$$

The specific heat at constant volume will obviously be given by

$$c_v = \left(\frac{\partial u}{\partial T}\right)_v = \gamma T. \tag{26}$$

Denoting the heat function of the assemblage per electron by

$$h_{\mathrm{C}} = u + pv, \tag{27}$$

where $v = 1/n$ is the volume per electron, and remembering that

$$pv = \tfrac{2}{3}u, \tag{28}$$

one obtains for the specific heat of the assemblage at constant pressure the expression

$$c_p = (\partial h_{\mathrm{C}}/\partial T)_p = \tfrac{5}{3}(\partial u/\partial T)_p = (\partial\zeta_0/\partial T)_p + \tfrac{5}{3}\gamma T. \tag{29}$$

Since ζ_0 is proportional to $n^{\frac{2}{3}}$, one obtains

$$(\partial\zeta_0/\partial T)_p = -\frac{2}{3}\frac{\zeta_0}{v}(\partial v/\partial T)_p. \tag{30}$$

Now

$$(\partial v/\partial T)_p = -(\partial p/\partial T)_v/(\partial p/\partial v)_T, \tag{31}$$

which in view of relation (28) is readily evaluated, since

$$(\partial p/\partial T)_v = \tfrac{2}{3}c_v/v, \tag{32}$$

and

$$(\partial p/\partial v)_T = -\tfrac{2}{3}\zeta_0/v^2. \tag{33}$$

Thus one obtains $$(\partial\zeta_0/\partial T)_p = -\tfrac{2}{3}c_v, \tag{34}$$

and hence from (32) the significant relation*

$$c_p = c_v = \gamma T. \tag{35}$$

That for a degenerate, or nearly degenerate, electron assemblage the specific heat at constant pressure comes out to be the same as that at constant volume need not occasion any surprise, since any increase in volume consequent on an increase in temperature at constant pressure brings down the degeneracy temperature and hence the zero point energy of the gas, and to the approximation aimed here, namely $kT \ll \zeta_0$, the energy gained thereby is just sufficient to do the work involved in the expansion of the gas against the constant pressure under which the gas is expanding.

At higher temperatures, when the assemblage of electrons in the condensed phase becomes less and less degenerate, c_p will naturally become greater than c_v, and they will tend to approach the classical values $\frac{5}{2}k$ and $\frac{3}{2}k$ respectively.

Incidentally we obtain from (31), (32) and (33) an expression for the coefficient of thermal expansion of the electron assemblage in the condensed phase at constant pressure, namely,

$$\beta = \frac{1}{v}(\partial v/\partial T)_p = c_v/\zeta_0. \tag{36}$$

9. The temperature variation of ζ in a single-component system

Considering again the single-component system, and the case when the electron assemblage in the condensed phase is highly degenerate, and using the result that $c_{p,\mathrm{C}}$ is equal to c_p derived in an earlier section, one obtains from (11) and (35)

$$\epsilon_T = T\int_0^T \gamma\,\mathrm{d}T - \int_0^T \gamma T\,\mathrm{d}T = \tfrac{1}{2}\gamma T^2, \tag{37}$$

where γ has the value given by (25). In view of (17), equation (37) gives the temperature variation of both ϕ and ζ for a single-component system

$$\mathrm{d}\phi/\mathrm{d}T = -\mathrm{d}\zeta/\mathrm{d}T = \gamma T. \tag{38}$$

The temperature variation of ζ can also be calculated alternatively from the following consideration. Consider the Fermi distribution function of the electrons in the metal,

$$f(E) = \frac{1}{\mathrm{e}^{(E-\zeta)/(kT)}+1}, \tag{39}$$

in which ζ is the thermodynamic potential defined by (1). Obviously the number of electrons in the condensed phase per unit volume is given by

$$n = 2\int_0^\infty f(E)\,N(E)\,\mathrm{d}E, \tag{40}$$

* For a preliminary report of the results embodied in equations (34) and (35) see Krishnan & Klemens (1952).

in which $N(E)\,dE$ gives the number of energy states between E and $E+dE$. In other words $N(E)$ is the density of the energy states at E. Taking as usual the energy distribution to be parabolic, i.e. taking the density of the energy states to be given by

$$N(E) = KE^{\frac{1}{2}}, \tag{41}$$

where K is a constant, and taking the electron assemblage to be highly degenerate, the thermodynamic potential will be given by (Sommerfeld & Bethe 1933)

$$\zeta = \zeta_0 - \tfrac{1}{6}\gamma T^2. \tag{42}$$

In (42) ζ_0 is not the actual value of ζ at $T = 0$, but the value which it would have at $T = 0$ *had the density remained constant.*

In the usual discussions on the temperature variation of ϕ on the model in which the energy barrier is taken to be independent of temperature, the whole of the variation is attributed to the variation of ζ, and the latter variation is taken to be given by (42), in which n and hence ζ_0 are regarded as remaining constant. One then obtains

$$d\phi/dT = -d\zeta/dT = \tfrac{1}{3}\gamma T, \tag{43}$$

which is only a *third* of the value obtained in (38) for the single-component system, which as we have seen, is equivalent to the model based on a temperature-independent energy barrier.

But actually, as we have shown, $d\phi/dT$ should be equal to $-d\zeta/dT$ at constant *pressure*, so that the variation of ζ_0 in (42) with temperature has also to be taken into account. Doing so one obtains

$$d\phi/dT = -(d\zeta/dT)_p = \tfrac{1}{3}\gamma T - (d\zeta_0/dT)_p. \tag{44}$$

The second term $-(d\zeta_0/dT)_p$ on the right-hand side of (44) can be seen, in view of (34) and (35), to reduce to $\frac{2}{3}\gamma T$, and hence the expression for the total variation of ζ with temperature will become identical with (38), as it should.

The first term on the right-hand side of (44) gives the variation of ζ with temperature as such, and the second the variation due to the change in the density of the electron assemblage accompanying the temperature change. The former accounts for one-third of the total change of ζ with temperature and the latter for the remaining two-thirds.

10. The monovalent metals treated as a single-component system

Having calculated the temperature variation of ζ, which, except for the sign, is also that of ϕ for a single-component system, we shall now extend the calculation to actual metals. We shall confine attention again to the monovalent metals copper, silver and gold. The number of electrons per unit volume in the condensed phase, which determines ζ, is practically the number of *atoms* per unit volume in these metals, and hence is known, as also the coefficient of thermal expansion β of this electron assemblage at constant pressure, namely the saturation vapour pressure. In table 1 are entered the values of the expected coefficient of thermal expansion β for the electron assemblage treated as a single-component system, as also the values of the actual coefficient of expansion δ of the lattice. It will be seen that δ is enormously greater than β. The obvious conclusion to be drawn from this is that the system cannot be regarded as a single-component one.

Treating it, however, *provisionally* as a single-component one, one can see from (17) and (37) that for these metals

$$\phi_T = \phi_0 + \tfrac{1}{2}\gamma T^2. \tag{45}$$

The temperature variation of ϕ is thus not linear, and hence the saturation current i cannot be represented by an equation of the Richardson type (see (15)) in which both A and ϕ are temperature-independent constants. However, since the second term on the right-hand side of (45) is much smaller than the first, and the actual thermionic measurements extend over a small temperature range only, we may still regard A as roughly constant over this range. One thus obtains

$$A/A_0 = e^{-\gamma T/k} \sim 1 - \gamma T/k. \tag{46}$$

The values of $\gamma T/k$ at 1300° K, which is roughly the middle of the temperature region in which the thermionic constants of these metals were measured, are given in table 1. As will be seen from the entries in the last column of the table, one should expect the observed A coefficients of these metals to be short of the theoretical value by about 10 %.

Table 1

metal	$10^{-4}\zeta_0/k$	$10^5\gamma/k$	$10^5\beta$	$10^5\delta$	$-\Delta A/A_0 = -\gamma T/k$ at 1300° K
Cu	8·1	6	0·10	5·0	0·08
Ag	6·4	8	0·16	5·8	0·10
Au	6·4	8	0·16	4·2	0·10

The actual values of A, as were reported in part III of this paper, are practically of this magnitude. One may therefore conclude that the temperature variation of the work function of a monovalent metal, to which is due the deviation of A from its theoretical value, *is as though it were a single-component system*. This result is particularly significant since we know from the observed large coefficient of thermal expansion of the lattice $\delta \gg \beta$, that there should be a marked deviation from the single-component behaviour; which means that the different thermal effects on the work function roughly balance one another.

11. Thermal effects on the work function

The various factors associated with a change in the temperature that might influence the work function of an *actual* metal, as distinguished from a single-component system, have been discussed by several authors, and in particular by Herzfeld, Wigner, Bardeen, Seitz and Seely, and the present position of the subject is reviewed critically by Herring & Nichols (1949). The two major factors are (1) the thermal expansion of the lattice, and (2) the thermal agitation of the atoms in the lattice. Since we have chosen the energy of an electron at rest in the metal as the zero energy level, the investigation of the temperature variation of ϕ would reduce to one of finding the effects of these two factors, namely, thermal expansion and thermal agitation, on ζ and on χ. Taking up first ζ, the effect of the thermal expansion is readily calculated on the basis of the free electron gas model, to which the electron assemblage in a monovalent metal closely approximates. On this basis, the extra coefficient of ζ is given by

$$\mathrm{d}\zeta/\mathrm{d}T = -\tfrac{2}{3}\zeta(\delta-\beta) \sim -\tfrac{2}{3}\zeta\delta. \tag{47}$$

The effect of the thermal agitation on ζ will be much smaller and will be to reduce only slightly the depression given by (47).

The observed value of A for the monovalent metal, which is known to be only slightly short of the theoretical value of 120 amp deg.$^{-2}$ cm^{-2}, would therefore imply that the energy barrier χ also is lowered by nearly the same amount as ζ, so that the difference between χ and ζ, which determines ϕ, remains practically unaltered.

12. The influence of the temperature on the energy barrier

We shall now proceed to show that the depression of the energy barrier χ accompanying a rise of temperature is of the required magnitude, i.e. just sufficient to compensate for the lowering in ζ.

The decrease of χ with temperature can be estimated roughly in the following manner. To a first approximation we may take χ to be inversely proportional to the interatomic distance (Seely 1941), i.e. proportional to $n^{\frac{1}{3}}$, in which case the effect of the thermal *expansion* of the lattice on χ will be given by

$$(\mathrm{d}\chi/\mathrm{d}T)_{\mathrm{expn}} = -\tfrac{1}{3}\chi\delta. \tag{48}$$

The effect of the thermal oscillations of the atoms, as distinguished from the simple expansion of the lattice with the atoms in their mean positions to which (48) refers, can be expressed roughly in the form

$$\left(\frac{\mathrm{d}x}{\mathrm{d}T}\right)_{\mathrm{oscn}} = -\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{2}{3}\frac{\pi}{V}e^2\overline{\xi^2}\right), \tag{49}$$

where V is the atomic volume, e is the electronic charge, and $\overline{\xi^2}$ is the mean square of the displacement of the atom, given by

$$\overline{\xi^2} = \frac{h^2T}{4\pi^2Mk\Theta^2}, \tag{50}$$

where M is the mass of the atom, and Θ is the Einstein characteristic temperature, which at the high temperatures that are being considered here, may be taken to be roughly three quarters of the Debye characteristic temperature (Mott & Gurney 1940), for which data are available from the standard tables. We thus obtain

$$\frac{\mathrm{d}\chi}{\mathrm{d}T} = -\tfrac{1}{3}\chi\delta - \frac{1}{6\pi}\frac{e^2h^2}{VMk\Theta^2}. \tag{51}$$

Table 2

metal	$-\frac{1}{k}\left(\frac{\mathrm{d}\chi}{\mathrm{d}T}\right)_{\mathrm{expn}}$	$-\frac{1}{k}\left(\frac{\mathrm{d}\chi}{\mathrm{d}T}\right)_{\mathrm{oscn}}$	$-\frac{1}{k}\frac{\mathrm{d}\chi}{\mathrm{d}T}$	$-\frac{1}{k}\frac{\mathrm{d}\zeta}{\mathrm{d}T}$
Cu	2·2	0·3	2·5	2·8
Ag	2·2	0·3	2·5	2·5
Au	1·6	0·3	1·9	1·8

In columns (2) and (3) of table 2 are entered the temperature coefficients of χ due to the thermal expansion and to the oscillations respectively, calculated from (48) and (49), and in the penultimate column the resultant temperature coefficient of χ. In the last column are entered the temperature coefficients of ζ calculated from (47). It will be seen from the values entered in the last two columns of the table that $d\chi/dT$ and $d\zeta/dT$ are of practically the same magnitude and their effects on the work function $\phi = \chi - \zeta$ practically cancel each other, and the whole of the observed temperature variation may be regarded as due to the electronic specific heat alone.

This result may have a wider significance. In the monovalent metals the thermal oscillations of the atoms in the lattice, and the accompanying thermal expansion (which as we have seen is much larger than that to be expected for the electron assemblage in the metal) are apparently so conditioned that maintaining the external pressure constant while varying the temperature becomes equivalent to maintaining the internal pressure of the electron assemblage also constant. Hence the observed temperature variation of ϕ is *as though* the electrons in the metal and in the vapour formed the two phases of a single-component thermodynamic system.

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Determination of thermal conductivities at high temperatures

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The usual electrical methods of determining thermal conductivities either cannot be used at high temperatures or their use is restricted to substances like carbon or graphite which satisfy certain special requirements. It is shown that the observed temperature distribution along a metal filament electrically heated in a vacuum can be used for the determination of the thermal conductivity at high temperatures. The temperature distribution near the centre of a short filament, which is known to be parabolic, or the distribution in regions slightly removed from the centre of a long filament, which is known to be logarithmic, are particularly suitable for this purpose. Measurements are reported on the thermal conductivity of platinum in the temperature region 1300 to 1800° K made in this manner.

THE DIFFERENTIAL EQUATION DEFINING THE STEADY STATE

Consider a filament electrically heated in a vacuum, so that the transfer of heat by convection in the surrounding atmosphere is avoided. The heat generated per unit length of the filament per second is $I^2\rho/\omega$, where I is the heating current, ρ is the specific resistance, and ω is the area of cross-section of the filament. The rate of loss of energy per unit length by radiation is $p\epsilon\sigma(T^4 - T_0^4)$, where p is the periphery of the cross-section, ϵ is the total emissivity from the surface, as distinguished from the spectral emissivity ϵ_λ, and σ is Stefan's constant of radiation. Owing to the conduction of heat towards the ends, the energy abstracted per second per unit length is $-\kappa\omega d^2T/dx^2$, where κ is the thermal conductivity. Hence the steady state is determined by the well-known differential equation[1]

$$\kappa\omega d^2T/dx^2 - p\epsilon\sigma(T^4 - T_0^4) + I^2\rho/\omega = 0 \quad (1)$$

Now the existence of a temperature gradient along the filament is a consequence of the finite thermal conductivity of the filament. Hence if we had an analytical solution of equation (1), we might utilize the observed distribution of temperature along the filament to determine experimentally the thermal conductivity. But an analytical solution of the differential equation in the general case was not available till recently. Hence some of the earlier experimental methods for the determination of κ, as for example those of Kohlrausch,[2] and of Callendar,[2] were so designed as to reduce equation (1) to a readily integrable form, namely to

$$d^2T/dx^2 = f \quad (2)$$

where f is an explicit function of κ. Obviously the solution is

$$T = \tfrac{1}{2}fx^2 + Bx + C \quad (3)$$

in which the constants B and C may be determined from the boundary conditions. The actual techniques used in these methods for securing equation (3), however, restrict the applicability of these methods to temperatures in the neighbourhood of the room temperature.

On the other hand Worthing and Halliday[3] have adopted a graphical method for the determination of the thermal conductivity from the observed distribution of temperature in heated filaments, which in essence is equivalent to solving the differential equation (1) graphically. Knowing all the physical constants involved in equation (1) except κ, κ then becomes known. Actually they integrate graphically the total heat generated in a selected finite length of the filament, and similarly also the total heat radiated out from this length. The difference between the two gives $\kappa\omega(G_1 - G_2)$, where G_1 and G_2 are the temperature gradients at the colder and the hotter ends respectively of the length selected. By making one end of the selected length coincide with the centre of the filament, G_2 can be made zero.

This method of determining κ is obviously applicable to high temperatures also.

In some recent papers[4,5,6] we have investigated in detail the analytical solution of equation (1). In the case of a long filament it is possible to obtain a general solution of equation (1), and in the case of shorter filaments too a solution can be obtained in terms of a power series which is rapidly convergent over most parts of the filament. It is the main purpose of this paper to show that the observed temperature distribution along such filaments, which can be readily measured with an optical pyrometer, can be made the basis of a convenient and direct method for determining thermal conductivities at high temperatures. This is of interest experimentally. The only other method that is available now, namely, that of Angell,[7] is based on the measurement of the difference in temperature between the inner and the outer surfaces in the middle portion of a long tube electrically heated in a vacuum. This method has been used by Powell and Schofield[8] to determine the thermal conductivity of carbon and of Acheson graphite; in which the walls of the tube can be made thick enough to secure a considerable difference in temperature between the inner and the outer surfaces, and at the same time the electrical resistance of the tube remains sufficiently large to permit our heating it to high temperatures without using inconveniently large currents. This cannot, however, be done with good metals.

It is significant that the only high temperature data for thermal conductivities available at present are those for tungsten, tantalum and carbon obtained by Worthing's graphical method, and those for carbon and graphite obtained by Angell's method.

THE METHODS OF KOHLRAUSCH AND CALLENDAR AS DEVICES FOR SIMPLIFYING THE TEMPERATURE DISTRIBUTION

In Kohlrausch's method, which has been developed experimentally by Jaeger and Diesselhorst,[2] the loss by radiation is practically eliminated by surrounding the conductor with a suitable wadding of insulating material. The differential equation (1) now reduces to

$$d^2T/dx^2 + I^2\rho/(\kappa\omega^2) = 0 \quad (4)$$

which is readily solved.

In Kohlrausch's method the distances x are not measured directly, but in terms of the corresponding potential drops

$$v = (I\rho/\omega)x \tag{5}$$

Since there is no loss by radiation, and therefore no gradient of temperature *across* the length, one may use, as these authors do, a rod or a bar of large cross-section, instead of a filament.

On the other hand, in Callendar's method a low heating current is used so as to make $T - T_0$ sufficiently small over the whole length of the filament that the radiation loss may be regarded as proportional to $T - T_0$ instead of to $T^4 - T_0^4$, that is, the radiation cooling follows Newton's law. The temperature range being now small, the variation of the specific resistance with temperature may be taken to be linear. Hence the resistance at every point may be split into a constant term characteristic of the temperature T_0, and an extra term proportional to $T - T_0$. Consider now the heating due to the latter part of the resistance, which will be proportional to I^2 and to $T - T_0$. By suitably adjusting the heating current I, this part of the heat generated can be made to balance the loss due to radiation, at every point of the filament, since both are proportional to $T - T_0$. When this condition is secured, the energy that is abstracted per unit length in the process of conduction becomes constant, i.e. the same at all points along the filament. This is essentially the condition for the temperature distribution to be parabolic. In the actual measurements the increase in resistance due to the heating, which can be readily calculated in terms of the parabolic distribution, is measured, rather than the distribution of temperature as such.

Both these methods, by the limitations imposed by the special techniques involved in securing the parabolic distribution, are applicable only to temperatures in the neighbourhood of the room temperature.

We now proceed to show that independently of any such special devices the temperature distribution near the centre of any finite filament is naturally parabolic, and can be utilized in the same manner as in Kohlrausch's and Callendar's special arrangements, to determine κ. Unlike these special arrangements, this method is applicable to high temperatures too.

ACTUAL DISTRIBUTION NEAR THE CENTRE SHOWN TO BE PARABOLIC AND UTILIZED TO DETERMINE κ

Let T_l be the temperature at the centre of the filament, and T_m the value to which T_l tends as $2l$, the length of the filament, is increased indefinitely, keeping the heating current constant. Obviously

$$p\sigma\epsilon(T_m^4 - T_0^4) = I^2\rho/\omega \tag{6}$$

Returning to equation (1), and eliminating T_0 with the help of equation (6), one obtains

$$\frac{d^2T}{dx^2} + \frac{p\epsilon\sigma}{\kappa\omega}(T_m^4 - T^4) = 0 \tag{7}$$

which in the case of a finite rod for which $T_l \neq T_m$ can be written in the form

$$\frac{d^2T}{dx^2} + \frac{p\epsilon\sigma}{\kappa\omega}(T_m^4 - T_l^4) + \frac{p\epsilon\sigma}{\kappa\omega}(T_l^4 - T^4) = 0 \tag{8}$$

If $T_m - T_l$ is considerable, as will be the case when the filament is not long, one may confine attention to a region close to the centre, where the temperature-dependent term in equation (8), namely, the third term on the left-hand side, becomes negligible in comparison with the second term. Equation (8) then reduces to

$$d^2T/dq^2 + f_1(\kappa) = 0 \tag{9}$$

where

$$f_1(\kappa) = \frac{p\epsilon\sigma}{\kappa\omega}(T_m^4 - T_l^4) \tag{10}$$

and $q = l - x$ is now the distance measured from the centre. Using the boundary conditions that when $q = 0$, $T = T_l$ and $dT/dq = 0$, one obtains

$$t = T_l - T = \tfrac{1}{2}f_1 q^2 \tag{11}$$

Thus from the observed temperature distribution in this region it is possible to determine f_1 and then κ in the same manner as before. Since the largest deviation of the temperature in the parabolic region from T_l, that is, the highest value of t involved in this region, is small, the quantity $p\epsilon/(\kappa\omega)$ appearing in equation (10) for f_1 may be assigned the value appropriate to the temperature T_l.

κ FROM THE LOGARITHMIC REGION

Consider a long filament in which $T_l \simeq T_m$. In this case, if one is not too close to the centre, that is, if T is not too close to T_l, the second term on the left-hand side of equation (8) may be neglected in comparison with the third. Consider a region sufficiently removed from the centre, but where $T_m - T = \Delta$ is still small in comparison with T_m. Equation (8) then takes the simple form

$$d^2\Delta/dx^2 = A\Delta \tag{12}$$

where

$$A = \frac{4p\epsilon\sigma T_m^3}{\kappa\omega} \tag{13}$$

Neglecting for the present the temperature variation of the physical quantities involved, one then obtains the well-known solution

$$x\sqrt{A} = D - \ln\Delta \tag{14}$$

in which $1/\sqrt{A}$ is obviously the distance over which Δ changes by a factor e. Hence A can be determined experimentally, and κ can be evaluated therefrom.

We should mention here that the temperature distribution along a thin-walled tube electrically heated in a vacuum is somewhat similar to that along a filament. When the tube is long, the temperature distribution in regions slightly removed from the centre is given by equation (14), in which A has the same significance as in a filament, except that the cross-sectional area ω occurring in the expression for A is now the area of the annular ring. A part of A, namely $\epsilon\sigma/\kappa$, occurs also in the expression for the difference in temperature between the inner and the outer surfaces of the tube. In determining this difference in temperature Prescott and Hincke[9] have used the observed temperature distribution in the logarithmic region, in order to eliminate $\epsilon\sigma/\kappa$ from the expression for this difference. Obviously the experimental data for the temperature distribution in the logarithmic region could have been used to determine κ also.

Coming back to the filament, we should mention also the following results which were derived in our earlier papers, and which are relevant to our present purpose.

(1) Though the condition under which equation (14) has been derived, namely that Δ be small in comparison with T_m, is a sufficient one, it is not necessary, the necessary condition being that $\Delta/\ln\Delta$ be small in comparison with T_m, which

obviously holds over a much wider range than the former condition.

(2) The region near the centre that is excluded corresponds to a total range of temperature equal to a few times $T_m - T_l$, and hence experimentally trivial.

(3) Now in formulating equation (12) we have neglected for convenience the temperature variations of the different physical quantities involved, namely $\kappa\omega$, ρ/ω and $p\epsilon$. The effect of the temperature coefficients of these quantities on equation (12) is essentially different from that in the parabolic region. As we shall see presently, even when attention is confined to a *narrow* region in the logarithmic range, the effect of the temperature coefficients will be appreciable, unlike in the parabolic range. This is due to the following circumstance. In the differential equation (9), which defines the temperature distribution in the parabolic range, the effect of the temperature coefficients is in the form of a contribution to the term proportional to t, which in any case may be neglected since t is chosen small. In other words equation (9) will hold even when the temperature coefficients are considerable, provided that the value of $p\epsilon/(\kappa\omega)$ occurring in the expression for f_1 is made to refer to the temperature T_l.

On the other hand in the logarithmic region the effect of the temperature variation of ρ/ω and $p\epsilon$ is to introduce an extra small term which varies linearly with temperature, which now cannot be neglected. Taking the values of $\kappa\omega$, ρ/ω and $p\epsilon$ to refer to the temperature T_m, the effect of the finite temperature coefficients of these quantities—the temperature coefficient of $\kappa\omega$ has very little effect in this region—is to introduce in equation (12) a small extra term proportional to Δ. In other words the expression for A in equations (12) and (13) will include a multiplying factor a_1 of the order of unity, whose deviation from unity is determined by the temperature coefficients of ρ/ω and $p\epsilon$, and by T_m.

The calculation of a_1 was given in a recent paper,[(4)] in which, however, the coefficient of thermal expansion, which normally is much smaller than the coefficients of κ, ρ and ϵ, was neglected. The expression for a_1 obtained in the paper can, however, be made to include thermal expansion also by replacing the temperature coefficients of κ, ρ and ϵ given in that paper by the coefficients of $\kappa\omega$, ρ/ω and $p\epsilon$ respectively.

We shall merely mention here the final result, namely, that for platinum, with which we shall be concerned in the present paper, a_1 varies from 0·97 for $T_m = 1800°$ K to 1·00 for $T_m = 1300°$ K.

TEMPERATURE DISTRIBUTION IN OTHER REGIONS ALSO UTILIZED TO GIVE κ

In the case of a long filament one can also obtain a general solution of equation (1) applicable over the whole length of the filament. Neglecting provisionally the temperature coefficients, the solution is

$$\Delta \exp[f(\Delta)] = \Delta_0 \exp[f(\Delta_0)]\{\exp(-x\sqrt{A}) + \exp[-(2l-x)\sqrt{A}]\} \quad (15)$$

where

$$\Delta_0 = T_m - T_0 \quad (16)$$

T_0 being the temperature of the point from which x is measured, and

$$f(\Delta) = \tfrac{1}{2}\Delta/T_m + \tfrac{1}{16}\Delta^2/T_m^2 - \ldots \quad (17)$$

Now in a long filament the region near the centre where $\exp[-(2l-x)\sqrt{A}]$ is significant, is not of experimental interest since the total variation of temperature in this region is a few times $T_m - T_l$, which is negligible. Outside this region, where $\exp[-(2l-x)\sqrt{A}]$ is negligible in comparison with $\exp(-x\sqrt{A})$, equation (15) reduces to the simple form

$$\Delta \exp[f(\Delta)] = \Delta_0 \exp[f(\Delta_0)] \exp(-x/A) \quad (18)$$

Now expressing all distances in terms of $1/\sqrt{A}$ and all temperatures in terms of T_m, that is, using the reduced co-ordinates $X = x\sqrt{A}$ and $\tau = T/T_m$, equation (18) may be seen to reduce to

$$(1-\tau)\exp[\phi(\tau)] = (1-\tau_0)\exp[\phi(\tau_0)]\exp(-X) \quad (19)$$

where $\phi(\tau) = f(\Delta)$. Obviously equation (19) is a very general expression, which implies that a *single* $\tau - X$ curve will represent the temperature distribution in all *long* filaments, independently of the nature of the filament, its length, its cross-section, or even the heating current used, provided that the reduced temperature τ_0 of the point from which the distances are measured is chosen the same in all the plots.

Hence, knowing T_m in any actual experiment, one can easily plot τ against x. This curve may be compared with the theoretical plot of τ against X, taking τ_0 from which x is measured, as the origin of distance in the $\tau - X$ curve too. The only difference between the two curves should be in the scales representing the abscissae. In other words it should be possible to transform the experimental $\tau - x$ curve into the theoretical $\tau - X$ curve by increasing the abscissae of the former by a *constant* factor F, that is, by a factor which is independent of x, and which can be experimentally determined. This factor should obviously be equal to $\sqrt{A}$, from which κ can be determined.

Here again we have neglected the temperature coefficients of the different quantities. Taking them into account would introduce a factor a_1 in the expression for A, as in the logarithmic region, and a factor a_2 in the expression for $\phi(\tau)$. For platinum a_2 varies from 1·4 for $T_m = 1800°$ K, to 1·8 for $T_m = 1300°$ K.

Confining now attention to the top of the region that we are considering, in which Δ is small enough to keep $\exp\phi(\tau)$ close to unity, equation (19) will obviously reduce to the logarithmic formula, as it should.

In reduced co-ordinates the temperature distribution in the logarithmic region is given by

$$\ln(1-\tau) = \ln(1-\tau_0) + \phi(\tau_0) - X \quad (20)$$

Hence, if the reduced temperature τ_0 from which distance measurements are made is taken to be the same in all the measurements, the plot of $\ln(1-\tau)$ against the distance will be a series of straight lines, all of them passing through the point τ_0, and having different slopes. The slopes give directly $\sqrt{A}$, and hence κ, taking into account the temperature coefficients of ρ/ω and $p\epsilon$.

When one moves out of the logarithmic region $\phi(\tau)$ grows fairly rapidly and the effect of the temperature coefficients on $\phi(\tau)$ also becomes significant. Now $\phi(\tau)$ involves in addition to the temperature coefficients of ρ/ω and $p\epsilon$ which are known, the temperature coefficient of $\kappa\omega$ too. Hence in the same manner in which temperature distribution in a long filament in the logarithmic range can be used to give κ, the distribution outside the logarithmic range can be made to yield both κ and its temperature coefficient α.

In practice, however, it is found more convenient to determine κ at different temperatures from the logarithmic range, by suitably varying the heating currents and varying the temperatures T_m to which κ refers. This has been done in the measurements described below.

EXPERIMENTAL

It has been shown that the thermal conductivity can be determined from the temperature distribution either in the parabolic region near the centre of any finite filament, preferably a short one, or in the logarithmic or any other region of a long filament. We have made measurements with a platinum wire, of spectroscopic purity (obtained from Johnson and Matthey and Co. Ltd.). The main part of the experiment is the determination of the temperature at different points along the heated wire. This was done with an optical pyrometer and for this purpose one needs to know the spectral emissivity ϵ_λ for the spectral region utilized in the pyrometer, namely the neighbourhood of $0{\cdot}655\ \mu$, at different temperatures.

This was determined in the following manner. In a recent paper we described some measurements on the spectral emissivity of Acheson graphite.[10] Using a thin-walled, long tube of graphite, electrically heated in a vacuum, the temperature of the inner surface of the tube near its centre is readily obtained by boring a small hole in the wall, and by measuring directly with the pyrometer the temperature of the cavity as viewed through the hole. The small difference in temperature between the inner and the outer surfaces is estimated easily from any rough value of the thermal conductivity. Thus the temperature of the outer surface near the centre is known. Knowing also the brightness temperature of the surface as measured with the optical pyrometer, the spectral emissivity ϵ_λ is obtained directly. By suitably adjusting the heating current, the spectral emissivity can be obtained at different known temperatures. Conversely, knowing ϵ_λ, the actual temperature of the surface corresponding to any observed brightness temperature can also be readily obtained.

For our present purpose, namely, the determination of the spectral emissivity of platinum at different temperatures, we used one of these thin-walled long graphite tubes, and wound round it in a thin groove cut round the middle of the tube, a single ring of platinum wire. The presence of either the groove or the platinum wire in the middle does not affect the temperature distribution. Knowing the actual temperature of the wire, which is that of the graphite surface adjoining it, and the brightness temperature of the wire as obtained with the optical pyrometer, its spectral emissivity is known. By suitably varying the heating current, values of ϵ_λ can be obtained at different temperatures. They are plotted in Fig. 1.

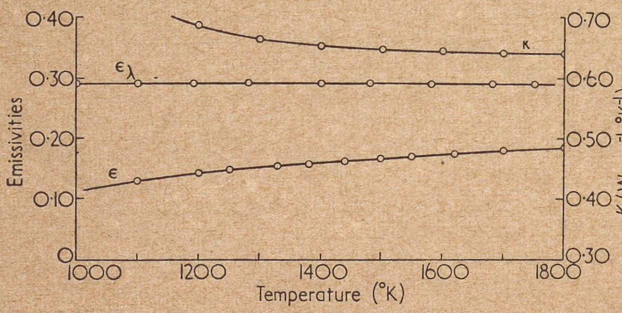

Fig. 1. The temperature variation of the thermal conductivity of platinum and of the spectral and the total emissivities

The first series of measurements that we made on the spectral emissivity of platinum was with a foil rolled into a cylinder, which just surrounded the graphite tube without touching it. It was fixed to the supporting block at one end. The external diameter of the graphite tube was 8 mm, and the clearance space between the graphite tube and the tube of platinum foil was about $\frac{1}{2}$ mm. The tube of platinum foil extended over nearly the whole length of the graphite tube, but not completely. This was to avoid passage of electric current through the platinum foil. Hence the temperature of the foil, particularly near the centre, where it is sensibly constant, will be just that of the graphite tube underneath, and that of the annular space between the two tubes. A small hole of about 1 mm diameter punctured in the platinum foil, and also in the wall of the graphite tube just underneath it, enabled the temperature in the cavity near the middle of the graphite tube to be measured directly.

Knowing this temperature, and the small difference between the inner and the outer surfaces of the graphite tube, the temperature of the platinum surface adjoining the hole is known. From this temperature and the brightness temperature of the platinum surface as measured with the pyrometer, the spectral emissivity of platinum is readily determined.

The spectral emissivity data for platinum obtained in this manner were found to agree closely with the measurements reported, which were made with a ring of platinum wire wound round the central portion of the graphite tube.

The available data for the spectral emissivity of platinum obtained by different authors vary widely, and the differences have sometimes been attributed to the probable differences of the surfaces of the specimens used. The close agreement between our measurements with the surface of the platinum foil and of the wire does not, however, support this view. Even so, since the specimen with which the thermal conductivity measurements were made, was in the form of a wire, we considered the measurements on ϵ_λ made with the same specimen to be more appropriate for our present purpose than the measurements made with the foil. The results of the latter measurements, which are not incorporated in the graph, are given in the following table.

T° K	1120	1250	1400	1525	1600	1690	1750
ϵ_λ	0·28	0·30	0·29	0·28	0·29	0·29	0·30

The single coil of the platinum wire wound in a small groove round the middle of the graphite tube involves negligible change in the heat capacity, and is therefore not likely to disturb the temperature distribution sensibly. The agreement between the values of ϵ_λ obtained by the two methods, supports this conclusion. Observations made with the optical pyrometer also confirmed that the temperature of the graphite surface in the close neighbourhood of the ring of platinum wire did not show any detectable variation.

The experimental error in the measurement of ϵ_λ is estimated to be less than $\pm 0{\cdot}02$.

Referring again to the temperature measurements, two sets of temperature measurements were made on platinum wires of $0{\cdot}5$ mm diameter (s.w.g. 25) heated in a vacuum by an alternating current. In the first set the wire was a medium long one, and the temperature measurements were made in the parabolic region near the centre. In Fig. 2 are plotted the values of t against q^2. In one series of measurements T_m and T_l were 1600 and 1400° K respectively, while in the other series 1500 and 1200° K respectively. The plot can be seen to be a straight line for small values of q^2, and its slope according to equation (11) gives $\frac{1}{2} f_1$, from which κ can be calculated. It can also be seen that t increases slightly more rapidly than in proportion to q^2, since the last one or two points are slightly above the straight line. This is to be expected theoretically. But the straight line portion is long enough to enable the slope to be determined accurately. The values of κ for 1200 and 1400° K plotted in Fig. 1 were obtained in this manner.

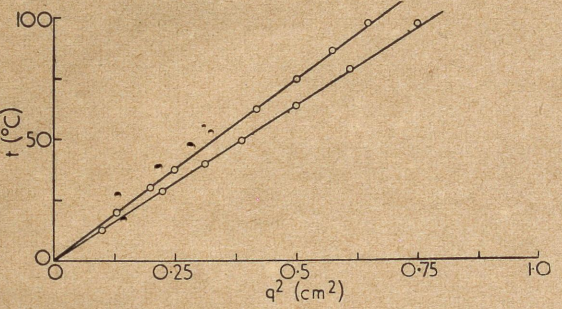

Fig. 2. The parabolic variation of temperature near the centre of the filament

In the second set measurements were made with a *long* platinum wire in the logarithmic region, using different heating currents. In Fig. 3 are plotted the values of ln (Δ/T_m) against the distance. The plot is a straight line over the whole of the temperature region included in the curves. The slope gives $\sqrt{A}$, from which again κ can be calculated.

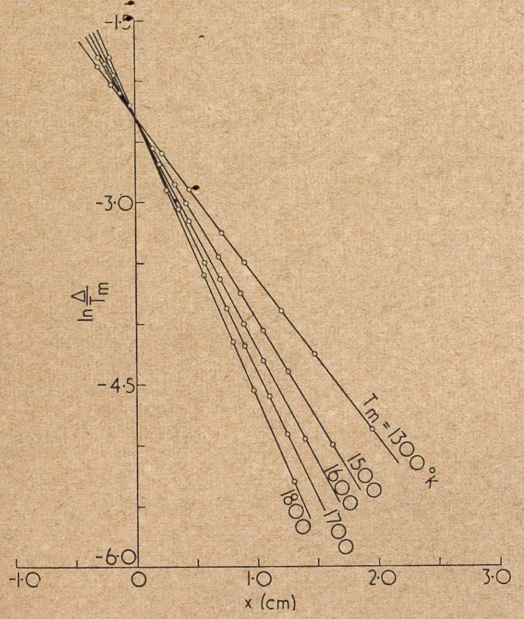

Fig. 3. Temperature variation in the logarithmic region

In order to obtain κ from f_1 or from A one needs to know also the total emissivity ϵ, as distinguished from the spectral emissivity. All that one needs for this purpose is the potential gradient g in the centre of a long wire heated by different currents, since

$$p\epsilon\sigma(T_m^4 - T_0^4) = Ig \qquad (21)$$

ϵ in equation (21) is the value of the total emissivity at temperature T_m.

The potential gradient g is easily measured with the help of two probes. The values of ϵ thus obtained are also plotted in Fig. 1. It is very significant that, whereas the spectral emissivity is almost independent of temperature, the total emissivity drops with the decrease of temperature.

DATA FOR THERMAL CONDUCTIVITY

The values of κ obtained from the parabolic and the logarithmic regions are also plotted in Fig. 1. The values for 1200 and 1400° K were obtained from the former region, and the remaining values from the latter. It will be seen that both the sets plot smoothly on a single curve.

It will also be seen that κ tends to a constant value at high temperatures, as should be expected at temperatures much above the Debye temperature.

THE LORENZ RATIO FOR PLATINUM

In the course of the determination of the total emissivity described in a previous section, one obtains incidentally from the potential gradient and the current, the electrical resistivity at different temperatures. Since we now have data for the thermal conductivities also, we can obtain directly the values of the Lorenz ratio $\kappa\rho/T$ at different temperatures. Since it is an important physical quality, for which at present data are not available for platinum at high temperatures, we have plotted in Fig. 4 this ratio.

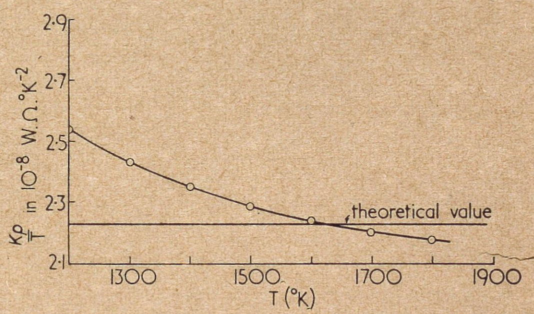

Fig. 4. Lorenz ratio for platinum

It will be seen from Fig. 4 that the Lorenz ratio for platinum decreases slightly with the increase of temperature, and remains close to the theoretical value of $2{\cdot}23 \times 10^{-8}$ W Ω °K^{-2}.

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