Mysore Summer School

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Lecture notes મેં પાડું [illegible]

1 May to 11 May 1939

1. Schwarzschild's metric.

1.1 The standard form:

We know that

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2 \qquad (1)$$

M = constant

satisfies $R_{ik}=0$. (1) is known as Schwarzschild's metric. As an exercise one may begin with (1) and write down g_{ik}. One may then calculate the determinant $g = \|g_{ik}\|$, then g^{ik}, Γ^i_{kl} and R_{ik} ~~&~~ to verify that R_{ik} ~~is 0~~ vanishes.

We know that ~~a~~ along ~~for~~ a ray of light $ds=0$. Assume that light rays are radial so that along a ray θ = const ϕ = const. (and of course $d\phi=0$) $\therefore$ For radially moving light

$$0 = \left(1-\frac{2M}{r}\right)dt^2 - \left(1-\frac{2M}{r}\right)^{-1}dr^2 \quad \text{or} \quad \left(\frac{dr}{dt}\right)^2 = \left(1-\frac{2M}{r}\right)^2$$

$\therefore$ Velocity of such a radial ray is $\boxed{\dot r = \frac{1}{1-\frac{2M}{r}}}$

$$\dot r = \left(1-\frac{2M}{r}\right)$$

Next consider light moving along ~~the transverse direction say~~ in the direction $r\,d\theta$. Then setting $ds=0$, $dr=0$, $d\phi=0$ we ~~get~~ get

$$0 = \left(1-\frac{2M}{r}\right)dt^2 - r^2 d\theta^2 \quad \text{or} \quad r^2\dot\theta^2 = 1-\frac{2M}{r}$$

or $\cancel{\boxed{r\dot{\theta} = \frac{1}{\sqrt{(1-\frac{2M}{r})}}}}$ $\qquad r\dot{\theta} = \sqrt{(1-\frac{2M}{r})}$

This shows ~~it in the radial~~ that velocity of light in radial direction is not the same as the velocity in a transverse direction. The coordinate (r, θ, ϕ, t) used in ~~one~~ (1) are such that they donot treat all the 3 directions at a point equally. They single out the radial direction as distinct from the two transverse directions at a point. Can we have another coordinate system (ρ, θ, ϕ, t) which will treat all the three directions at a point equally. Yes, this is possible and that is what we are going to do now

1.2 The isotropic form:

The problem is: Can we transform (1) to the form

$$ds^2 = e^{\nu} dt^2 - e^{\mu}[d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2] \qquad (2)$$

$\nu = \nu(\rho)$ $\mu = \mu(\rho)$?

If yes, then let us find the functions ~~ρ~~ μ, ν of ρ. We must also find a relation between the radial coordinates ρ of (2) and r of (1).

Since ds^2 is an invariant, equate the two values of ds^2 given by (1) and (2).

$$(1-\frac{2M}{r})dt^2 - (1-\frac{2M}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad (3)$$

$$= e^{\nu}dt^2 - e^{\mu}d\rho^2 - e^{\mu}\rho^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

Of the coordinates (r,θ,ϕ,t) and (ρ,θ,ϕ,t) only r and ρ are functions of each other, rest are ~~independent~~ independant of each other. Equating coeff. of dt^2, $d\theta^2$ ~~$d\rho^2$~~ and $d\phi^2$ ~~we~~ ~~finds~~ in (3) we find

$$e^{\nu} = 1-\frac{2M}{r} \quad (4) \qquad r^2 = e^{\mu}\rho^2 \quad (5)$$

And now (3) reduces to $(1-\frac{2M}{r})^{-1}dr^2 = e^{\mu}d\rho^2$ (6)

(4), (5), (6) are the three equations to determine three functions $\nu(\rho)$, $\mu=\mu(\rho)$ and $\rho=\rho(r)$.

Take (5). It $\Longrightarrow$ $r = e^{\mu/2}\rho$

$$\therefore dr = e^{\mu/2}\frac{1}{2}\frac{d\mu}{d\rho}d\rho\cdot\rho + e^{\mu/2}d\rho$$

$$= e^{\mu/2}d\rho\left(1+\rho\frac{1}{2}\frac{d\mu}{d\rho}\right)$$

Use this dr in (6)

$$(1-\frac{2M}{r})^{-1}e^{\mu}d\rho^2\left(1+\frac{\rho}{2}\frac{d\mu}{d\rho}\right)^2 = e^{\mu}d\rho^2$$

$$\therefore \left(1+\frac{\rho}{2}\frac{d\mu}{d\rho}\right)^2 = \left(1-\frac{2M}{r}\right)$$

$$= \left(1-\frac{2M}{\rho}e^{-\mu/2}\right)$$

We have to solve ~~the~~ this differential equation to find μ. We do it now.

$$1 + \frac{\rho}{2}\frac{d\mu}{d\rho} = \sqrt{\left\{1 - \frac{2M}{\rho}e^{-\mu/2}\right\}}$$

Put $1 - \frac{2M}{\rho}e^{-\mu/2} = Z^2$

$$\therefore \frac{2M}{\rho^2}e^{-\mu/2}d\rho - \frac{2M}{\rho}e^{-\mu/2}\left\{-\frac{1}{2}\frac{d\mu}{d\rho}\right\}d\rho = 2Z\,dZ$$

$$\therefore \frac{2M}{\rho^2}e^{-\mu/2}d\rho\left\{1 + \frac{\rho}{2}\frac{d\mu}{d\rho}\right\} = 2Z\,dZ$$

~~So~~ Turning back to our original diff. eqn. we find

$$\frac{2Z\,dZ}{\frac{2M}{\rho^2}e^{-\mu/2}d\rho} = Z \qquad \text{But } \frac{2M}{\rho}e^{-\mu/2} = 1 - Z^2$$

$$\therefore \frac{2\,dZ}{\frac{1}{\rho}(1-Z^2)d\rho} = 1 \qquad \text{or } \frac{2\,dZ}{1-Z^2} = \frac{d\rho}{\rho}$$

$$\text{or } \left[\frac{1}{1+Z} + \frac{1}{1-Z}\right]dZ = \frac{d\rho}{\rho} \qquad \text{or } \frac{1+Z}{1-Z} = A\rho, \quad A = \text{const.}$$

$$\therefore Z = \frac{A\rho - 1}{A\rho + 1} = \cancel{\frac{1 - \frac{1}{A\rho}}{1 + \frac{1}{A\rho}}}$$

~~But~~

$$\therefore 1 - \frac{2M}{\rho}e^{-\mu/2} = Z^2 = \frac{(A\rho-1)^2}{(A\rho+1)^2}$$

$$\therefore\ 1 - \frac{(A\rho - 1)^2}{(A\rho+1)^2} = \frac{2M}{\rho} e^{-\mu/2} \quad \text{or} \quad \frac{(A\rho+1)^2 - (A\rho-1)^2}{(A\rho+1)^2} = \frac{2M}{\rho} e^{-\mu/2}$$

$$\therefore\ \frac{4A\rho}{A^2\rho^2\left(1+\frac{1}{A\rho}\right)^2} = \frac{2M}{\rho} e^{-\mu/2}$$

$$\therefore\ e^{-\mu/2} = \frac{2}{AM} \frac{1}{\left(1+\frac{1}{A\rho}\right)^2}$$

But as $\rho \to \infty$ $e^{-\mu/2}$ must $\to 1$

$$\therefore\ \frac{2}{AM} = 1, \quad \frac{1}{A} = \frac{M}{2}$$

$$\therefore\ e^{\mu} = \left(1+\frac{M}{2\rho}\right)^4. \qquad e^{\nu} = 1 - \frac{2M}{r} = 1 - \frac{2M}{\rho} e^{-\mu/2}$$

$$= \frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2}$$

$\therefore$ Schwarzschild's solution in ~~isotropic~~ isotropic coordinates (ρ, θ, ϕ, t) is

$$ds^2 = \frac{\left(1-\frac{M}{2\rho}\right)^2}{\left(1+\frac{M}{2\rho}\right)^2} dt^2 - \left(1+\frac{M}{2\rho}\right)^4 \cancel{\left\{d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2\right\}} (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2) \quad (2a)$$

1.3 The retarded time

~~x ti~~ A light ray leaving O reaches P (OP = r) at time t. This ray started from O earlier at time $t - \frac{r}{c}$, c being the velocity of light. We call $t - \frac{r}{c}$, the ~~retarded~~ retarded time at P and denote it by u. $u = t - \frac{r}{c}$ or $du = dt - \frac{dr}{c} = dt - \frac{dr}{\text{light-vel}}$

In general relativity we use units in such a way that $c = 1$. So that $u = t - r$.

Now one consequence of Einstein's theory is that the gravitational field ~~is~~ reduces the velocity of light i.e. the velocity of light becomes less than 1. As a result the retarded time is not $t - r$ but a little shorter. Let us find this retarded time in the gravitational field described by Schwarzschild's metric (1).

In (1) we have already found the radial velocity of light. It is not 1 but $\frac{dr}{dt}$ where $\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right)^2$.

Now $du = dt - \frac{dr}{\text{light vel}} = dt - \frac{dr}{1 - \frac{2M}{r}}$

$$\therefore \; u = t - \int \frac{r\,dr}{r - 2M} = t - \int \frac{r - 2M + 2M}{r - 2M}\,dr$$

$$u = t - r - 2M \log(r - 2M) \qquad \text{~~for~~ for } r > 2M$$

This give the retarded time in Schwarzschild field

We shall now rewrite Schwarzschild metric replacing the coordinate t, therein by this retarded time u. (1) is

$$ds^2 = \left(1-\frac{2M}{r}\right)dt^2 - \frac{1}{1-\frac{2M}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Put $\quad u = t - r - 2M\log(r-2M)\quad$ or equivalently

$$du = dt - \frac{dr}{1-\frac{2M}{r}}$$

or $\quad dt = du + \dfrac{dr}{1-\frac{2M}{r}}$. $\therefore$ (1) becomes

$$ds^2 = \left(1-\frac{2M}{r}\right)\left[du + \frac{dr}{1-\frac{2M}{r}}\right]^2 - \frac{dr^2}{1-\frac{2M}{r}} - r^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

$$= \left(1-\frac{2M}{r}\right)du^2 + 2\,du\,dr - r^2(d\theta^2+\sin^2\theta\, d\phi^2) \qquad (2)$$

This form of metric (1) was first given by Eddington in 1924.

Exercises (1)

* 1. Show that ~~$d\theta^2+\sin^2\theta\, d\phi^2$ can be~~

$$\frac{dx^2+dy^2}{\left[1+\frac{1}{4}(x^2+y^2)\right]^2} \text{ can be transformed to } d\theta^2+\sin^2\theta\, d\phi^2$$

and find the equations of transformations.

[~~Hint~~ Hint: Use transformations $x=\rho\cos\phi$, $y=\rho\sin\phi$ and find $\rho=\rho(\theta)$]

2. For metric (7) ~~show~~ write down g_{ik}. Calculate g^{ik}, Γ^{i}_{kl} and R_{ik}. Verify that $R_{ik} = 0$

3. In metric (7) do not take M as a constant but choose $M = M(u)$. Show that if $M = M(u)$ $R_{ik} \neq 0$ but $R_{ik} = \sigma w_i w_k$; $g_{ik} w^i w^k = 0$.

[With $M = M(u)$ metric (7) represents the gravitational field of a radiating star]

2. Equilibrium of Fluid Spheres

~~1.~~

2.1 Introduction

Einstein's equations are

$$R_{ik} - \frac{1}{2} g_{ik} R = +8\pi T_{ik} \qquad (8)$$

Here the left hand side $R_{ik} - \frac{1}{2} g_{ik} R$ determine the geometry governing space-time measurements in a gravitational field. So we ~~a short~~ use a shortened notation for it and call it G_{ik}. Thus (8) can be written as $G_{ik} = +8\pi T_{ik}$ where G_{ik} stands for $R_{ik} - \frac{1}{2} g_{ik} R$.

The right hand side of (8) is determined by the physical properties of matter-distribution producing the gravitational field.

Let us begin with the geometric aspects of the gravitational fields ~~due~~ due to fluid spheres in equilibrium.

2.2 Static spherically symmetric geometry:

The most general metric showing spherical symmetry will be of the form

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 + 2a\,dr\,dt - e^{\mu}r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \qquad (9)$$

with ν, λ, a and μ being functions of r and t.

We have still two-fold arbitrariness left here, because we can replace the coordinates r and t by two arbitrary function R and T, $R = R(r,t)$, $T = T(r,t)$. We can use this arbitrariness to suitably adjust the four functions ν, λ, a and μ of (9). In this way we get the following three types of spherically symmetric line-elements which are easier to use mathematically.

Set $a = 0$, $\mu = 0$ in (9)

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\,d\phi^2 \qquad (10a)$$

Set $a = 0$, $\lambda = \mu$ in (9)

$$ds^2 = e^{\nu}dt^2 - e^{\mu}\left[dr^2 + r^2d\theta^2 + r^2\sin^2\theta\,d\phi^2\right] \qquad (10b)$$

Set $e^{\lambda} = 0$, $\mu = 0$

$$ds^2 = e^{\nu}dt^2 + 2a\,dr\,dt - r^2d\theta^2 - r^2\sin^2\theta\,d\phi^2 \qquad (10c)$$

It will be seen we have expressed Schwarzschild's metric in all these forms. (1) is of the form (10a), (2a) is of the form (10b) and (7) is of the form (10c).

We can use any one of the three forms ~~(10,~~ (10 a, b, c) to discuss the geometry of fluid spheres in equilibrium. As a result of equilibrium, we have the additional advantage that none of the coefficients $e^{\lambda}, e^{\mu}, e^{\nu}$ or r depend on t. They will be functions of r only.

We shall use the form (10a).

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \qquad (10a)$$

$$\nu = \nu(r), \quad \lambda = \lambda(r),$$

For this metric one can write down g_{ik}. One can also calculate g^{ik}, Γ^{i}_{kl}. R_{ik} and G_{ik} $(= R_{ik} - \frac{1}{2} g_{ik} R)$. We give ~~to~~ below the final form of G^{i}_{k}.

$$G^1_1 = -e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] + \frac{1}{r^2} \qquad (11a)$$

$$G^3_3 = G^2_2 = -e^{-\lambda}\left[\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right] \qquad (11b)$$

$$G^4_4 = +e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2}. \qquad (11c)$$

[Notation: $\lambda' = \partial\lambda/\partial r$ while $\nu'' = \frac{\partial^2 \nu}{\partial r^2}$ etc.]

Equations (11a, b, c) give us the left hand side of Einstein's equations (8). We now go to the consideration of the ~~right~~ Physics of equilibrium of fluid spheres to determine the right hand side of (8)

~~2.4)~~

2.3 The energy tensor for a perfect fluid

We know that in special relativity one can chose a set of Galilean coordinates such that the fluid is at rest at the point and the time of ~~interest~~ interest. In these coordinates, the energy-momentum tensor T_0^{ik} is given by

$$T_0^{11} = T_0^{22} = T_0^{33} = p \qquad T_0^{00} = \rho . \quad T_0^{ik} = 0 \text{ for } i \neq k . \tag{12}$$

Now principle of equivalence says that ~~we~~ in the Riemannian space-time of general relativity, one can chose proper coordinates $(x_0^0, x_0^1, x_0^2, x_0^3)$ such that in the neighbourhood of any given point special relativity holds good and so the energy-momentum tensor T_0^{ik} is given by (12).

If T^{ik} is the energy tensor in a general coordinate system $x^i = (x^0, x^1, x^2, x^3)$ we have by the law of transformation of a tensor

$$T^{ik} = \frac{\partial x^i}{\partial x_0^m}\frac{\partial x^k}{\partial x_0^n} T_0^{mn}$$

$$= \frac{\partial x^i}{\partial x_0^1}\frac{\partial x^k}{\partial x_0^1} p + \frac{\partial x^i}{\partial x_0^2}\frac{\partial x^k}{\partial x_0^2} p + \frac{\partial x^i}{\partial x_0^3}\frac{\partial x^k}{\partial x_0^3} p$$

$$+ \frac{\partial x^i}{\partial x_0^0}\frac{\partial x^k}{\partial x_0^0} \rho \tag{13}$$

Use the law of transformation of g^{ik} and note that in proper galilean coordinates

$$g_0^{11} = g_0^{22} = g_0^{33} = -1 \qquad g_0^{00} = +1 \qquad g_0^{ik} = 0, \; i \neq k$$

We thus find ~~$g^{\mu\nu} = \frac{\partial x^\mu}{\partial x_0^\alpha}\frac{\partial x^\nu}{\partial x_0^\beta} g_0^{\alpha\beta}$~~ $g^{ik} = \frac{\partial x^i}{\partial x_0^m}\frac{\partial x^k}{\partial x_0^n} g_0^{mn}$

$$\text{or } g^{ik} = -\frac{\partial x^i}{\partial x_0^1}\frac{\partial x^k}{\partial x_0^1} - \frac{\partial x^i}{\partial x_0^2}\frac{\partial x^k}{\partial x_0^2} - \frac{\partial x^i}{\partial x_0^3}\frac{\partial x^k}{\partial x_0^3} + \frac{\partial x^i}{\partial x_0^0}\frac{\partial x^k}{\partial x_0^0} \qquad (14)$$

Also if v^i is the four-velocity of the fluid ~~and v_0^α~~ in the general coordinates then ~~$v^\mu = \frac{\partial x^\mu}{\partial x_0^\alpha} v_0^\alpha$~~ $v^i = \frac{\partial x^i}{\partial x_0^m} v_0^m$.

But in proper coordinates the fluid is at rest at the point of interest so that

$$\frac{dx_0^1}{ds} = \frac{dx_0^2}{ds} = \frac{dx_0^3}{ds} = 0 \qquad \frac{dx_0^0}{ds} = 1 \qquad \{\text{Note that } ds^2 = -(dx_0^1)^2 - (dx_0^2)^2 - (dx_0^3)^2 + (dx_0^0)^2\}$$

$$\therefore \cancel{v_0^\alpha = (0,0,0,1)} \quad \therefore v_0^m = \cancel{(0,0,0,1)}\ (1,000)$$

$$\therefore \cancel{v^\mu = \frac{\partial x^\mu}{\partial x_0^0}} \quad v^i = \frac{\partial x^i}{\partial x_0^0} \qquad (15)$$

Using (14) and (15) in (13) we find

$$T^{ik} = (p+\rho) v^i v^k - p g^{ik} \qquad (16)$$

$$\cancel{T^{\mu\nu} = (p+\rho) v^\mu v^\nu - p g^{\mu\nu}} \qquad \cancel{(16)}$$

$$\cancel{g_{\mu\nu}}\ \cancel{g_{\mu\nu} v^\mu v^\nu = 1} \quad g_{ik} v^i v^k = 1.$$

(16) gives the expression for the energy-momentum tensor for a perfect fluid with pressure p, density ρ and 4-velocity ~~v^μ~~ v^i. We shall use this ~~tensor~~ tensor for the equilibrium of a perfect fluid.

~~2.4 Pressure and density of fluid spheres in~~

§ 2.4 Einstein's equations

We rewrite (16) after lowering the suffix ~~μ~~ k

$$\cancel{T^{\mu}_{\nu} = (p+\rho)v^{\mu}v_{\nu} - p\, g^{\mu}_{\nu}} \qquad T^{i}_{k} = (p+\rho)v^{i}v_{k} - p\, g^{i}_{k} \qquad (16a)$$

Now for an equilibrium ~~distribut~~ distribution

$$v^1 = v^2 = v^3 = 0 \qquad \text{and so}$$

$$v^k v_k = 1 \Longrightarrow v^4 v_4 = 1. \quad \text{We thus get}$$

$$T^1_{\ 1} = -p, \quad T^2_{\ 2} = -p, \quad T^3_{\ 3} = -p, \quad T^4_{\ 4} = (p+\rho) - p = \rho$$

Thus Einstein's equations

$$R^{i}_{k} - \tfrac{1}{2} g^{i}_{k} R \equiv G^{i}_{k} = +8\pi T^{i}_{k} \quad \text{gives}$$

$$G^1_{\ 1} = +8\pi T^1_{\ 1} = -8\pi p$$

$$G^2_{\ 2} = \qquad\qquad = -8\pi p$$

$$G^3_{\ 3} = \qquad\qquad = -8\pi p$$

$$G^4_{\ 4} = +8\pi T^4_{\ 4} = +8\pi\rho.$$

For the line-element (10a) we have already worked out G^{i}_{k} in (11a, b, c). Using them above we find that

$$8\pi p = e^{-\lambda}\left[\frac{\nu'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2} \qquad (17a)$$

$$8\pi p = e^{-\lambda}\left[\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{2r}\right] \qquad (17b)$$

$$8\pi \rho = e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + 1/r^2 \qquad (17c)$$

(17a) and (17b) give two different values for p. Equating them we find

$$e^{-\lambda}\left[\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} - \frac{\nu'+\lambda'}{2r} - \frac{1}{r^2}\right] + \frac{1}{r^2} = 0 \qquad (18)$$

This is one equation to determine the functions λ and ν. The other equation is supplied ~~to~~ by the physical property of the fluid like an equation of state i.e. in the form of a relation between p and ρ. As an illustration we take the simplest case of a homogeneous liquid with ρ = constant

2.5 Schwarzschild's interior solution

We take the perfect fluid to be a liquid of constant density ρ. So in (17c) ρ is a const.

$$\therefore\ e^{-\lambda}\frac{\lambda'}{r} - \frac{e^{-\lambda}}{r^2} + \frac{1}{r^2} = 8\pi\rho \qquad \rho = \text{const.}$$

$$\therefore\ e^{-\lambda}\lambda' + \frac{1}{r}(1-e^{-\lambda}) = 8\pi\rho r$$

Put $1-e^{-\lambda} = \psi(r)$ $\therefore\ e^{-\lambda}\lambda' = \psi'$

$$\therefore\ \psi' + \frac{\psi}{r} = 8\pi\rho r \qquad r\psi = \int 8\pi\rho r^2\,dr + c$$

$$= \cancel{\frac{8\pi\rho}{3}r^3 + c} = \frac{8\pi}{3}\rho r^3 + C$$

$$\therefore\ \psi = 1-e^{-\lambda} = 8\pi\rho\frac{r^2}{3} + \frac{c}{r}$$

$$\therefore\ e^{-\lambda} = 1 - 8\pi\rho\frac{r^2}{3} - \frac{c}{r}.$$

But at the centre of the liquid sphere ($r=0$)

we do ~~nararan~~ not want a singularity and

so we take $C=0$. Thus

$$e^{-\lambda} = 1 - \frac{8\pi\rho}{3}r^2. \quad \text{Write } \frac{8\pi\rho}{3} = \frac{1}{R^2} \quad R = \text{const.}$$

$$\therefore e^{-\lambda} = 1 - \frac{r^2}{R^2}. \qquad (19)$$

Thus the equation of state $\rho = \text{const}$ has led us to determine e^{λ}. We now find the corresponding e^{ν} ~~by~~ by solving equation (18). Putting $e^{-\lambda} = 1 - \frac{r^2}{R^2}$ and $e^{-\lambda}(-\lambda') = -\frac{2r}{R^2}$ in (18) we find

$$\left(1 - \frac{r^2}{R^2}\right)\left(\frac{\nu''}{2} + \frac{\nu'^2}{4}\right) - \frac{\nu'}{2r}\left(\frac{r^2}{R^2} + 1 - \frac{r^2}{R^2}\right) - \frac{1}{R^2} + \frac{1}{r^2}\left(1 - 1 + \frac{r^2}{R^2}\right) = 0$$

$$\therefore \left(1 - \frac{r^2}{R^2}\right)\left(\frac{\nu''}{2} + \frac{\nu'^2}{4}\right) - \frac{\nu'}{2r} = 0$$

Set $e^{\nu/2} = Z$ $\quad e^{\nu/2}\frac{\nu'}{2} = Z'$ $\quad e^{\nu/2}\frac{\nu''}{2} + e^{\nu/2}\frac{\nu'^2}{4} = Z''$

$$\therefore \left(1 - \frac{r^2}{R^2}\right)\frac{Z''}{Z} - \frac{Z'}{Z}\frac{1}{r} = 0$$

$$\therefore \frac{Z''}{Z'} = \frac{1}{r\left(1 - \frac{r^2}{R^2}\right)} = \frac{1}{r} + \frac{r/R^2}{\left(1 - \frac{r^2}{R^2}\right)}$$

$$\therefore Z' = A\frac{r}{\sqrt{\left(1 - \frac{r^2}{R^2}\right)}}.$$

$$\therefore Z = -AR^2\sqrt{\left(1 - \frac{r^2}{R^2}\right)} + \text{const.}$$

$$\therefore e^{\nu/2} = A + B + C\sqrt{\left(1 - r^2/R^2\right)} \qquad (20)$$

(19) and (20) give e^{λ} and e^{ν}.

~~3 Equilibrium of Fluid sphere (Contd)~~

~~3.1~~ Equations of ~~Etc~~

Putting these values of e^{λ} and e^{ν} in the line-~~element~~ element (10a) we find that the geometry inside the fluid sphere is given by

$$ds^2 = \left[B + C\sqrt{(1 - r^2/R^2)}\right]^2 dt^2 - \frac{dr^2}{1 - \frac{r^2}{R^2}} - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 . \qquad (21)$$

The density of the fluid is constant and its pressure can now be calculated from (17a). For that purpose we find ν' first.

$$e^{\nu/2} = B + C\sqrt{(1 - \frac{r^2}{R^2})} \quad \therefore e^{\nu/2}\frac{\nu'}{2} = \frac{C(-2r/R^2)}{2\sqrt{(1 - \frac{r^2}{R^2})}} = \frac{-Cr}{R^2\sqrt{(1 - r^2/R^2)}}$$

$$\therefore \frac{\nu'}{2} = \frac{-Cr}{R^2\sqrt{(1 - \frac{r^2}{R^2})}\left\{B + C\sqrt{(1 - r^2/R^2)}\right\}}$$

Now $\quad 8\pi p = e^{-\lambda}\left[\frac{\nu'}{r}\right] + \frac{1}{r^2}(e^{-\lambda} - 1)$

$$= \left(1 - \frac{r^2}{R^2}\right)\frac{(-2C)/R^2}{\sqrt{(1 - \frac{r^2}{R^2})}\left\{B + C\sqrt{(1 - \frac{r^2}{R^2})}\right\}} + \frac{1}{r^2}\left(1 - \frac{r^2}{R^2} - 1\right)$$

$$= \frac{-2C/R^2\sqrt{(1 - r^2/R^2)}}{B + C\sqrt{(1 - r^2/R^2)}} - \cancel{\frac{r^2}{r^2}}\frac{1}{R^2}$$

$$= -\frac{1}{R^2\left[B + C\sqrt{(1 - \frac{r^2}{R^2})}\right]}\left[2C\cancel{\frac{R^2}{R^2}}\sqrt{(1 - \frac{r^2}{R^2})} + B + C\sqrt{(1 - \frac{r^2}{R^2})}\right]$$

$$\therefore 8\pi p = \frac{-B - 3C\sqrt{(1 - r^2/R^2)}}{R^2[B + C\sqrt{(1 - r^2/R^2)}]} \qquad (22)$$

Now if the equilibrium sphere has radius a then on the boundary $r = a$ of the sphere the fluid pressure must vanish

$$\therefore -B - 3C\sqrt{(1 - a^2/R^2)} = 0 \qquad (23a)$$

This is one equation to determine unknown constants B and C.

Again outside the sphere $r = a$, there is no fluid, so the space is empty and so the geometry outside $r = a$ is given by metric (1) viz.

$$ds^2 = -(1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \qquad (1)$$

Since geometry has to be continuous at $r = a$

(1) must go over to (2) when $r = a$

$$\therefore 1 - \frac{2M}{a} = \left[B + C\sqrt{(1 - \frac{a^2}{R^2})}\right]^2 \quad \text{and} \quad 1 - \frac{2M}{a} = 1 - \frac{a^2}{R^2}$$

Second of these equation gives the total ~~+~~ mass M of the fluid sphere as ~~a~~ $M = \frac{a^3}{2R^2}$ and the first gives another equation for B and C as

$$B + C\sqrt{(1 - \frac{a^2}{R^2})} = \sqrt{(1 - a^2/R^2)} \qquad (23b)$$

Solving (23a) and (23b) for B and C we find

$$C = -\frac{1}{2} \quad B = \frac{3}{2}\sqrt{(1 - \frac{a^2}{R^2})}$$

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Thus we write Schwarzschild's ~~ext~~ interior solution in the form

$$ds^2 = \left[\frac{3}{2}\sqrt{\left(1-\frac{a^2}{R^2}\right)} - \frac{1}{2}\sqrt{\left(1-\frac{r^2}{R^2}\right)}\right]^2 dt^2$$

$$- \frac{1}{1-\frac{r^2}{R^2}} dr^2 - r^2(d\theta^2 + \sin^2\theta)$$

for $r \leq a$,

which at the boundary $r = a$ ~~to~~ becomes continuous with the exterior solution

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{1}{1-\frac{2M}{r}} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

for $r \geq a$

Here $M = \frac{a^3}{2R^2}$. Remember that $\frac{1}{R^2} = \frac{8\pi\rho}{3}$

$\therefore M = \frac{4}{3}\pi a^3 \rho$ a very satisfactory result for the mass of a liquid sphere of radius a and ~~an~~ uniform density ρ.

LECTURE NOTE ON GRAVITATIONAL COLLAPSE

PROF.P.C.VAIDYA

3. Gravitational ~~Contraction~~ Collapse

3.1. Introduction

~~3.1.~~ We now consider fluid spheres which are not in equilibrium. We know that gravitation tries to pull ~~in~~ the fluid inwards towards the centre. This inward pull is opposed ~~b~~ by the fluid pressure and ~~when~~ when the two balance each other we have equilibrium. To consider a non equilibrium case let us take the extreme case in which the ~~perfect~~ perfect fluid is not able to ~~e~~ ~~exert~~ exert ~~a~~ any pressure. In such a dust sphere, there is ~~n~~ nothing to ~~prevent~~ oppose ~~gravity~~ the gravitational pull and so the dust-sphere collapses under its own gravitation. ~~I~~ We are going to study the dynamics of such a gravitational collapse in this chapter.

Invariance of Einstein's equations under arbitrary coordinate transformations allows us to ~~choo~~ choose coordinates in such a way that in these coordinates even a dynamical system ~~exh~~ ~~exhibits~~ exhibits some of the properties of an equilibrium system. We go to describe such coordinates in the next section.

3.2 Co-moving coordinates

We have seen that metric (4) is the most general spherically symmetric metric. We rewrite it here

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - e^{\mu} r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) + 2a \, dr \, dt \qquad (5)$$

We noted ~~ther~~ there that there is two-fold arbitrariness still left in (4) which we ~~chose~~ chose there to get three different simple forms (10 a, b, c) of the metric. Now we shall use only one of the 2 arbitrariness to ~~simply~~ simplify the metric and use the other arbitrariness to simplify the velocity of fluid.

Now from symmetry of a sphere it is clear that the fluid will be moving in the radial direction and so θ and ϕ components of velocity will vanish. Thus ~~v~~ v^i will have components ~~$(v^1, 0, 0, v^4)$~~ $(v^0, v^1, 0, 0)$. Now we can replace (r, t) of (4) by new coordinates $\bar{r}$ and $\bar{t}$, $\bar{r} = \bar{r}(r, t)$, $\bar{t} = \bar{t}(r, t)$. We choose these two arbitrary functions in such a way that in the new coordinates

(i) $\bar{g}^{10} = 0$ i.e. there is no term in $d\bar{r}\, d\bar{t}$ in the resulting metric and (ii) $\bar{v}^1 = 0$ so that ~~$\bar{v}^i = (0, 0, 0, \bar{v}^4)$~~ $\bar{v}^i = (\bar{v}^0, 0, 0, 0)$.

Thus in the new coordinate system the fluid is ~~at~~ at rest and we call such coordinates comoving coordinates.

In a comoving coordinate system (4) can be simplified to the form

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - e^{\mu}r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad (24).$$

Since we are now working with a dynamical system the coefficients λ, μ and ν are functions of r as well as of t. In the next section we use the physics of the collapse situation to further simplify the metric (24)

~~3.2~~ 3.3 The energy tensor

We have seen that the energy tensor for a perfect fluid is

~~$T_\mu^\nu = (p+\rho)v_\mu v^\nu - p g_\mu^\nu$~~ $T_i^k = (p+\rho)v_i v^k - p\delta_i^k$

and in comoving coordinates, $v^k = (v^0, 0, 0, 0)$, $v_i = (v_0, 0, 0, 0)$ and so we find

$$T_1^1 = -p, \quad T_2^2 = -p, \quad T_3^3 = -p, \quad T_0^0 = \rho \quad (25)$$
$$T_1^0 = T_2^0 = T_3^0 = T_1^2 = T_1^3 = T_2^3 = 0.$$

Now consider the conservation equation $T_{i;k}^k = 0$ ~~$T^\nu_{\mu;\nu} = 0$~~. These are 4 equations. Take the equation corresponding to $\mu = 1$

$$T^k_{1;k} = 0 \quad \text{or} \quad \frac{\partial T_1^k}{\partial x^k} + \Gamma^k_{\alpha k}T_1^\alpha - \Gamma^\alpha_{1k}T_\alpha^k = 0$$

$$\therefore \frac{\partial T_1^1}{\partial x^1} + \Gamma^k_{1k}T_1^1 + \Gamma^\nu_{1\nu}T_1^1 - \Gamma^1_{11}T_1^1 - \Gamma^2_{12}T_2^2 - \Gamma^3_{13}T_3^3 - \Gamma^0_{10}T_0^0$$

$$\therefore -\frac{\partial p}{\partial r} + \Gamma^1_{11}(-p) + \Gamma^2_{12}(-p) + \Gamma^3_{13}(-p) + \Gamma^4_{14}(-p)$$
$$- \Gamma^1_{11}(-p) - \Gamma^2_{12}(-p) - \Gamma^3_{13}(-p) - \Gamma^4_{14}(\rho) = 0$$

$$\therefore -p' - (p+\rho)\,\Gamma^4_{14} = 0.$$

But for the metric (24) $\Gamma^4_{14} = \frac{1}{2} e^{-\nu} \left\{ \frac{\partial (e^{\nu})}{\partial r} \right\}$

$$= \frac{\nu'}{2}$$

$\therefore$ The conservation equation give

$$p' + (p+\rho)\frac{\nu'}{2} = 0 \qquad \cancel{(25)}$$

[Exercise: Show that ~~$\dot p + (p+\rho)(\dot\mu + \dot\lambda/2) = 0$~~ $\dot\rho + (p+\rho)(\dot\mu + \frac{\dot\lambda}{2}) = 0$]

Now we are considering a collapse situation in which there is no pressure to oppose the gravitational pull. So $p = 0$, hence $p' = 0$ and so the above equation ~~(25)~~ gives $\nu' = 0$, i.e ν is a function of t say $a(t)$. Now $e^{\nu} dt^2$ become $e^{a} dt^2$. One can use a new time-coordinate $\bar t$ such that $\bar t = \int e^{a/2} dt$ and then $e^{a} dt^2$ gets ~~becomes~~ replaced by $d\bar t^2$. So we say that we can incorporate ~~incorporate~~ $a(t)$ in the time coordinate and so set $e^{\nu} = 1$ or $\nu = 0$. Thus the metric giving the gravitational field of a dust ball in comoving coordinates ~~taken the~~ can be put in the form

$$ds^2 = dt^2 - e^{\lambda} dr^2 - e^{\mu} r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (26)$$

One can work out $G_i^k = R_i^k - \frac{1}{2} g_i^k R$ for this metric

$$G_1^{\,1} = -e^{-\lambda}\left[\frac{\mu'^2}{4} + \frac{\mu'}{r} + \frac{1}{r^2}\right] + \frac{e^{-\mu}}{r^2} + \cancel{e}\left[\ddot\mu + \frac{3}{4}\dot\mu^2\right] \qquad (26a)$$

$$G_3^{\,3} = G_2^{\,2} = -e^{-\lambda}\left[\frac{\mu''}{2} + \frac{\mu'^2}{4} - \frac{\lambda'\mu'}{4} - \frac{\lambda'}{2r} + \frac{\mu'}{r}\right] + \cancel{e}\left[\frac{\ddot\lambda}{2} + \frac{\ddot\mu}{2} + \frac{\dot\lambda^2}{4} + \frac{\dot\mu^2}{4} + \frac{\dot\lambda\dot\mu}{4}\right] \qquad (26b)$$

$$G_0^{\,0} = -e^{-\lambda}\left[\mu'' + \frac{3}{4}\mu'^2 - \frac{\lambda'\mu'}{2} + \frac{3\mu'}{r} - \frac{\lambda'}{r} + \frac{1}{r^2}\right] + \frac{e^{-\mu}}{r^2} + \cancel{e}\left[\frac{\dot\mu^2}{4} + \frac{\dot\mu\dot\lambda}{2}\right] \qquad (26c)$$

$$G_1^{\,0} = -e^{-\nu}\left[\dot\mu' + (\dot\mu - \dot\lambda)\frac{\mu'}{2} + (\dot\mu - \dot\lambda)\frac{1}{r}\right] \qquad (26d)$$

[Notation $\dot\mu = \frac{\partial\mu}{\partial t}$ $\quad \dot\mu' = \frac{\partial^2\mu}{\partial t\,\partial r}$ etc.]

3.4 Solution of Einstien's equations

Einstien's equations are $G^i_k = 8\pi T_i^k$. G_i^k for our problem are given above and T_i^k are recorded in (25). Put $p=0$, then and we find that $T^0_0 = \rho$ is the only surviving component all other components vanish. Thus our Einstein's field equation give

$$G_1^{\,1} = 0, \quad \cancel{G^k_2}\; G_2^{\,2} = 0 \quad G_1^{\,0} = 0$$

These are 3 equations for ~~2~~ 2 functions λ and μ and the system seems to be overdetermined. Let us try to solve it.

Begin with $G_1^{\,0} = 0$ $\qquad \dot\mu' + (\dot\mu - \dot\lambda)\left(\frac{\mu'}{2} + \frac{1}{r}\right) = 0$ $\qquad$ (26d')

This can be easily integrated. We write it as $\cancel{\left\{\frac{\mu'}{2} + \frac{1}{r}\right\}^{\cdot} + }$

$$\left\{\mu' + \frac{2}{r}\right\}^{\cdot} + \frac{\dot\mu - \dot\lambda}{2}\left(\mu' + \frac{2}{r}\right) = 0 \quad \text{or} \quad \left[\left(\mu' + \frac{2}{r}\right)e^{\frac{\mu-\lambda}{2}}\right]^{\cdot} = 0$$

$$\therefore \left\{\mu' + \frac{2}{r}\right\}e^{\frac{\mu-\lambda}{2}} = F(r) \qquad (27)$$

(27) is the general solution of $G_1^{\,0} = 0$. However a particular simple solution is immediately suggested from (26d').

$$\mu' = 0, \quad \dot\mu - \dot\lambda = 0 \quad \therefore \mu = \mu(t) \text{ and } \lambda = \mu + \psi(r) \text{ or } e^{\lambda} = e^{\mu} e^{\psi}$$

$$\mu = \mu(t) \qquad \psi = \psi(r)$$

Let us see what G_1^1 and G_2^2 look like for this particular solution ($G_1^1 = 0$, $G_2^2 = 0$)

$$G_1^1 = -e^{-\mu-\psi}\left[\frac{1}{r^2}\right] + \frac{e^{-\mu}}{r^2} + \ddot{\mu} + \frac{3}{4}\dot{\mu}^2 = 0 \qquad (28)$$

$$G_2^2 = -e^{-\mu-\psi}\left[-\frac{\psi'}{2r}\right] + \ddot{\mu} + \frac{3}{4}\dot{\mu}^2 = 0 \qquad (29)$$

Subtracting (28) from (29) we find

$e^{-\psi}\left(\frac{\psi'}{2r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0$. We can solve this for $\psi(r)$. Put

$e^{-\psi} = Z \quad e^{-\psi}\psi' = -Z'$ and so $-\frac{Z'}{2r} + \frac{Z}{r^2} - \frac{1}{r^2} = 0$

$\therefore Z' - \frac{2}{r}Z = -\frac{2}{r}$ or $\frac{Z}{r^2} = \int -\frac{2}{r^3}\,dr + k = \frac{1}{r^2} + k$

$Z = 1 + kr^2 \quad \therefore e^{-\psi} = 1 + kr^2$ or to get a more familiar expression set $k = -\frac{1}{R_0^2}$ so that $e^{-\psi} = 1 - \frac{r^2}{R_0^2}$ R_0 constant

Use this value of $e^{-\psi}$ in (28) and we shall get an equation for e^{μ}.

$$\frac{e^{-\mu}}{r^2}\left[1 - e^{-\psi}\right] + \ddot{\mu} + \frac{3}{4}\dot{\mu}^2 = 0$$

$$\therefore \frac{e^{-\mu}}{r^2}\left[1 - 1 + \frac{r^2}{R_0^2}\right] + \ddot{\mu} + \frac{3}{4}\dot{\mu}^2 = 0$$

$\therefore \ddot{\mu} + \frac{3}{4}\dot{\mu}^2 + \frac{e^{-\mu}}{R_0^2} = 0$ an equation which $\mu(t)$ satisfies

To put this equation in standard form we replace $\mu(t)$ by $R(t)$ defined to by $e^{\mu} = \frac{R^2}{R_0^2}$. The equation takes the form

$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{1}{R^2} = 0$. One integration of this equation is immediately available. It is $\dot{R}^2 = \frac{C}{R} - 1$ C undetermined constant.

So we have solved Einstein's equations. Our final solution is

$$ds^2 = dt^2 - \frac{R^2}{R_0^2}\left[\frac{dr^2}{1-\frac{r^2}{R_0^2}} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \qquad (28)$$

The pressure is zero and the density ρ is given by

$$8\pi\rho = G_0^0 = -e^{-\lambda}\left[-\frac{\lambda'}{r} + \frac{1}{r^2}\right] + \frac{e^{-\mu}}{r^2} + \frac{3}{4}\dot{\mu}^2$$

$$e^{-\lambda} = \frac{R_0^2}{R^2}\left(1 - \frac{r^2}{R_0^2}\right) \quad \therefore e^{-\lambda}(-\lambda') = \frac{R_0^2}{R^2}\frac{(-2r)}{R_0^2} \quad \therefore e^{-\lambda}\lambda' = \frac{2r}{R^2}$$

$$\therefore 8\pi\rho = \frac{2}{R^2} - \frac{1}{r^2}\frac{R_0^2}{R^2}\left(1 - \frac{r^2}{R_0^2}\right) + \frac{R_0^2}{R^2}\frac{1}{r^2} + \frac{3\dot{R}^2}{R^2}$$

$$8\pi\rho = \frac{3}{R^2} + \frac{3\dot{R}^2}{R^2}$$

where $\dot{R}^2 = \frac{C}{R} - 1$. $\therefore 8\pi\rho = \frac{3\left(1 + \frac{C}{R} - 1\right)}{R^2} = \frac{3C}{R^3}$.

It is clear that C has to be ~~great~~ ≥ 0.

3.5 **Equations of Fit**

In co-moving coordinates (i.e. in coordinates moving with the fluid), the fluid is at rest, and so the boundary of the fluid sphere is $r =$ constant say $r = a$. In other words an observer sitting on the surface of the sphere will find that his r-coordinate is always a. But how does a distant ~~obser~~ observer who is away from the sphere in empty space surrounding it ~~find the~~ decide the motion of the fluid sphere? ~~For~~ To find this we must fit this interior metric (28) with Schwarzschild's exterior metric.

$$ds^2 = \left(1 - \frac{2M}{\bar{r}}\right)d\bar{t}^2 - \frac{1}{1 - \frac{2M}{\bar{r}}}d\bar{r}^2 - \bar{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (29)$$

The coordinates t, r of (28) are clearly different from $\bar{t}, \bar{r}$ of (29), ~~and~~ but θ and ϕ are the same coordinates for both (28) and (29)

Then comparing coeff. of $d\theta^2 + \sin^2\theta d\phi^2$ in (28) and (29) we find

$$\frac{R^2}{R_0^2} r^2 = \bar{r}^2 \qquad (30)$$

Now the boundary of the sphere is $r = a$. The distant observer ~~assigns the val. coordina~~ calls this boundary as $\bar{r} = \bar{r}_b$ where $\frac{R^2}{R_0^2} a^2 = \bar{r}_b^2$

$$\therefore \bar{r}_b = \frac{a}{R_0} R$$

As $R = R(t)$, the distant observer finds the boundary as a ~~sphere~~ sphere of radius $\bar{r}_b$ which is a function of t. Again $\dot{\bar{r}}_b = \frac{a}{R_0} \dot{R}$

~~wh~~ where $\dot{R}^2 = \frac{C}{R} - 1 \qquad C \geq 0$

But then $2\dot{R}\ddot{R} = -\frac{C}{R^2}\dot{R}$ or $\ddot{R} = -\frac{C}{R^2}$ ~~or~~ $\ddot{R}$ is $-$ve

Thus ~~a~~ if $\dot{R}$ is $-$ve $\ddot{R}$ being $-$ve will always keep $\dot{R}$ $-$ve. Even if $\dot{R}$ is $+$ve, it will decrease and ultimately become $-$ve. So as observed by a distant observer, the sphere is contracting, the rate of at which its radius diminishes ~~big~~ being $\dot{\bar{r}}_b = \frac{a}{R_0}\dot{R} = -\frac{a}{R_0}\sqrt{\frac{C}{R} - 1}$

The undetermined constant C can be found by equations of fit

Since r, t of (28) are different from $\bar{r}, \bar{t}$ of (24) [as a matter of fact we have already ~~assumed~~ found that $\bar{r}_{tr} = \frac{R}{R_0} a$] to fit (28) with (24) over $r = a$ [$\bar{r} = \bar{r}_{tr}$] we must express both the metrics in the same coordinates. For that purpose we will transform (28) to $(\bar{r}, \bar{t})$ coordinates and let it take the form

$$ds^2 = e^{\alpha} d\bar{t}^2 - e^{\beta} d\bar{r}^2 - \bar{r}^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (30)$$

We already know $\bar{r} = \frac{R}{R_0} r$. We must find $\bar{t} = \bar{t}(r, t)$ and the function α and β by laws of transformation of (28) to (30). After that is done we shall equate the coefficients of $d\bar{t}$ and $d\bar{r}$ in (24) and (30) at $\bar{r} = \bar{r}_{tr}$

$$\bar{r} = \frac{R}{R_0} r \qquad \therefore d\bar{r} = r \frac{\dot{R}}{R_0} dt + \frac{R}{R_0} dr$$

Also $\bar{t} = \bar{t}(r, t)$ $\qquad \therefore d\bar{t} = \dot{\bar{t}}\, dt + \bar{t}' dr$

$\therefore$ (30) transforms to

$$ds^2 = e^{\alpha} (\dot{\bar{t}}\, dt + \bar{t}' dr)^2 - e^{\beta} \left(r\frac{\dot{R}}{R_0} dt + \frac{R}{R_0} dr \right)^2 - \frac{R^2}{R_0^2} r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= dt^2 \left[e^{\alpha} \dot{\bar{t}}^2 - e^{\beta} r^2 \frac{\dot{R}^2}{R_0^2} \right] - dr^2 \left[\frac{R^2}{R_0^2} e^{\beta} - e^{\alpha} \bar{t}'^2 \right] + 2\,dr\,dt \left[e^{\alpha} \dot{\bar{t}} \bar{t}' - e^{\beta} \frac{r R \dot{R}}{R_0^2} \right] - \frac{R^2}{R_0^2} r^2 (d\theta^2 + \sin^2\theta\, d\phi^2).$$

Compare this with the original metric (28).

$$1 = e^{\alpha}\dot{t}^2 - e^{\beta}r^2\frac{\dot{R}^2}{R_0^2} \qquad (31a)$$

$$\frac{R^2}{R_0^2(1-\frac{r^2}{R_0^2})} = \frac{R^2}{R_0^2}e^{\beta} - e^{\alpha}\dot{t}'^2 \qquad (31,b)$$

$$0 = e^{\alpha}\dot{t}\,t' - e^{\beta}\frac{r R\dot{R}}{R_0^2} \qquad (31,c)$$

These 3 equations determine the functions α, β and $\bar{t}$ so that the transformation from (28) to (30) is complete.

~~Now we make the transformed metric (30) continuous with the exterior metric (24) at $r = r_b$~~

First we find the transformed metric coeff. e^{β}. We can eliminate $e^{\alpha/2}\dot{t}$ and $e^{\alpha/2}t'$ from (31, a, b, c) as follows: (31a) gives $e^{\alpha}\dot{t}^2 = 1 + e^{\beta}r^2\frac{\dot{R}^2}{R_0^2}$. (31b) gives

$e^{\alpha}t'^2 = \frac{R^2}{R_0^2}\left(e^{\beta} - \frac{1}{1-\frac{r^2}{R_0^2}}\right)$ and (31c) gives $e^{2\alpha}\dot{t}^2 t'^2 = \frac{e^{2\beta}r^2\dot{R}^2R^2}{R_0^4}$

Elimination of $e^{\alpha}\dot{t}^2$ and $e^{\alpha}t'^2$ can be effected very easily and we get $\left(1 + e^{\beta}\frac{r^2\dot{R}^2}{R_0^2}\right)\frac{R^2}{R_0^2}\left(e^{\beta} - \frac{1}{1-\frac{r^2}{R_0^2}}\right) = \frac{e^{2\beta}r^2\dot{R}^2R^2}{R_0^4}$

$$\therefore e^{\beta} - \frac{1}{1-\frac{r^2}{R_0^2}} + \frac{e^{2\beta}r^2\dot{R}^2}{R_0^2} - e^{\beta}\frac{r^2\dot{R}^2}{R_0^2}\frac{1}{1-\frac{r^2}{R_0^2}} = \frac{e^{2\beta}r^2\dot{R}^2}{R_0^2}$$

$$\therefore e^{\beta}\left[1 - \frac{r^2\dot{R}^2/R_0^2}{1-\frac{r^2}{R_0^2}}\right] = \frac{1}{1-\frac{r^2}{R_0^2}}$$

$$\therefore e^{\beta}\left[1 - \frac{r^2}{R_0^2} - \frac{r^2\dot{R}^2}{R_0^2}\right] = 1 \qquad \therefore e^{-\beta} = 1 - \frac{r^2}{R_0^2} - \frac{r^2\dot{R}^2}{R_0^2}$$

$$= 1 - \frac{\bar{r}^2}{R^2} - \frac{\bar{r}^2\dot{R}^2}{R^2}$$

This give e^{β} of metric (30) in terms of comoving co-ordinates (r, t). When we make (30) continuous with

Schwarzschild's exterior solution over $\bar{r} = \bar{r}_b$ we find

$$1 - \frac{2M}{\bar{r}_b} = 1 - \frac{a^2}{R_0^2} - \frac{a^2 \dot{R}^2}{R_0^2}$$

$$\therefore \frac{2M}{\bar{r}_b} = \frac{a^2}{R_0^2} + \frac{a^2\dot{R}^2}{R_0^2} \qquad \text{But } \bar{r}_b = \frac{R}{R_0} a$$

$$\therefore \frac{2MR_0}{aR} = \frac{a^2}{R_0^2} + \frac{a^2\dot{R}^2}{R_0^2} \qquad \therefore \dot{R}^2 = \frac{2M}{a^3} R_0^3 \frac{1}{R} - 1.$$ But we have obtained $\dot{R}^2 = \frac{c}{R} - 1 \quad \therefore c = \frac{2MR_0^3}{a^3}$.

3.6 Motion of the boundary

The co-moving observer finds the boundary of the dust sphere as $r = a$, but a is not the radius of the boundary. The square of this ~~r~~ radius is the coeff. of $(d\theta^2 + \sin^2\theta\, d\phi^2)$ in (28) i.e. $\frac{R^2}{R_0^2} a^2$. $\therefore$ The boundary radius l is

$$l = \frac{R}{R_0} a \quad \therefore \dot{l} = \frac{\dot{R}}{R_0} a = -\frac{a}{R_0}\sqrt{\left(\frac{c}{R} - 1\right)}.$$

$\dot{l}$ vanishes when $R = c$. We can find the time taken ~~from~~ for the sphere to contract from the initial radius $l_1 = \frac{c}{R_0} a$ to ~~ϕ~~ any other smaller radius say $l = 0$ ~~as~~ i.e. $R = 0$ as

$$\text{time} = -\int_c^0 \frac{dR}{\sqrt{\left(\frac{c}{R} - 1\right)}} = \int_0^c \sqrt{\frac{R}{c - R}}\, dR$$

Put $R = C\sin^2\theta$ and we get the required time $= \int_0^{\pi/2} C\, 2\sin^2\theta\, d\theta = \int_0^{\pi/2} C(1-\cos 2\theta)\, d\theta$

$$= C\frac{\pi}{2},$$

So according to comoving observer the sphere collapses to a singularity in a finite time

To discuss the above situation as seen by an outside observer using $(\bar{t}, \bar{r})$ coord. of (29), we know r, t of (28) are functions of $\bar{r}, \bar{t}$

$$\therefore\ dr = \frac{\partial r}{\partial \bar{r}} d\bar{r} + \frac{\partial r}{\partial \bar{t}} d\bar{t}$$

But the boundary of the sphere is given by $r = a$ i.e. $dr = 0$

$\therefore$ On the boundary $0 = \frac{\partial r}{\partial \bar{r}} d\bar{r} + \frac{\partial r}{\partial \bar{t}} d\bar{t}$

$$\text{or}\ \left(\frac{d\bar{r}}{d\bar{t}}\right)_b = \frac{-(\partial r/\partial \bar{t})_b}{(\partial r/\partial \bar{r})_b}$$

a suffix b implies that the value is to be taken on the boundary.

The law of transformation from (28) to (30) will give $(\partial r/\partial \bar{t})$ and $(\partial r/\partial \bar{r})$. We have

$$\bar{r}^2 = \frac{R^2}{R_0^2} r^2 \qquad \bar{r} = \frac{R}{R_0} r \quad \text{Diff. w.r.t. } \bar{r} \text{ and w.r.t. } \bar{t}$$

$$1 = \frac{R}{R_0}\frac{\partial r}{\partial \bar{r}} + r\frac{\dot{R}}{R_0}\frac{\partial R}{\partial t}\frac{\partial t}{\partial \bar{r}} \qquad (a)$$

$$0 = \frac{R}{R_0}\frac{\partial r}{\partial \bar{t}} + r\frac{\dot{R}}{R_0}\frac{\partial t}{\partial \bar{t}} \qquad (b)$$

Next take law of transformation of g_{01}

$$\bar{g}_{01} = \frac{\partial x^i}{\partial \bar{x}^0}\frac{\partial x^j}{\partial \bar{x}^1} g_{ik} \qquad \text{But } g_{01}=0,\ \bar{g}_{01}=0$$

$$0 = \frac{\partial r}{\partial \bar{t}}\frac{\partial r}{\partial \bar{r}} g_{11} + \frac{\partial t}{\partial \bar{t}}\frac{\partial t}{\partial \bar{r}} g_{00}$$

$$= -\frac{\partial r}{\partial \bar{t}}\frac{\partial r}{\partial \bar{r}}\frac{R^2}{R_0^2}\frac{1}{1-\frac{r^2}{R_0^2}} + \frac{\partial t}{\partial \bar{t}}\frac{\partial t}{\partial \bar{r}} \qquad (c)$$

Put $\frac{\partial t}{\partial \bar{t}}$ from (b) and $\frac{\partial t}{\partial \bar{r}}$ from (a) in (c) and we shall get $\partial r/\partial \bar{r}$

$(b) \Longrightarrow \quad \partial t/\partial \bar{t} = -(R/\dot{R})(1/r)(\partial r/\partial \bar{t})$

$(a) \Longrightarrow \quad \partial t/\partial \bar{r} = (R_0/r\dot{R})[1-(R/R_0)(\partial r/\partial \bar{r})]$

Use these in (c)

$$0 = -\frac{(\partial r/\partial \bar{t})(\partial r/\partial \bar{r})R^2}{R_0^2(1-r^2/R_0^2)} - \frac{R}{\dot{R}}\frac{1}{r}\frac{\partial r}{\partial \bar{t}}\frac{R_0}{r\dot{R}}\left(1-\frac{R}{R_0}\frac{\partial r}{\partial \bar{r}}\right)$$

$$\therefore 0 = \frac{\partial r}{\partial \bar{r}}\left[\frac{R^2}{\dot{R}^2}\frac{1}{r^2} - \frac{R^2}{R_0^2(1-r^2/R_0^2)}\right] - \frac{R_0 R}{r^2\dot{R}^2}$$

$$\therefore \frac{\partial r}{\partial \bar{r}}\left[R^2\left(1-\frac{r^2}{R_0^2}-\frac{r^2\dot{R}^2}{R_0^2}\right)\right] - R_0 R\,(1-r^2/R_0^2) = 0$$

$$\boxed{\therefore \left(\frac{\partial r}{\partial \bar{r}}\right)_b = \frac{R_0}{R}\left(1-\frac{a^2}{R_0^2}\right)\Big/\left(1-\frac{2M}{r_b}\right)}$$

Next $\bar{g}_{00} = \frac{\partial x^i}{\partial \bar{x}^0}\frac{\partial x^j}{\partial \bar{x}^0} g_{ij} = \left(\frac{\partial r}{\partial \bar{t}}\right)^2 g_{11} + \left(\frac{\partial t}{\partial \bar{t}}\right)^2 g_{00}$

$$\therefore e^{\alpha} = -\left(\frac{\partial r}{\partial \bar{t}}\right)^2\frac{R^2}{R_0^2}\frac{1}{1-\frac{r^2}{R_0^2}} + \left(\frac{\partial t}{\partial \bar{t}}\right)^2$$

Put $\frac{\partial t}{\partial \bar{t}}$ from (b)

$$\therefore\ e^{\alpha} = -\frac{\dot{R}^2}{R_0^2}\frac{1}{1-(r^2/R_0^2)}\left(\frac{\partial r}{\partial \bar{t}}\right)^2 + \frac{R^2}{\dot{R}^2}\frac{1}{r^2}\left(\frac{\partial r}{\partial \bar{t}}\right)^2$$

$$\therefore\ e^{\alpha}\left\{\dot{R}^2 r^2(1-r^2/R_0^2)\right\} = \left(\frac{\partial r}{\partial \bar{t}}\right)^2 R^2\left(-\frac{\dot{R}^2}{R_0^2}r^2 + 1 - \frac{r^2}{R_0^2}\right)$$

on the boundary

$$\{e^{\alpha}\}_b = 1 - \frac{2M}{\bar{r}_b},\quad \left\{1 - \frac{r^2}{R_0^2} - \frac{\dot{R}^2}{R_0^2}r^2\right\}_b = \{e^{\beta}\}_b = 1 - \frac{2M}{\bar{r}_b}$$

$$\therefore\ \dot{R}^2 a^2(1 - a^2/R_0^2) = \left(\frac{\partial r}{\partial \bar{t}}\right)_b^2 R^2$$

$$\therefore\ \boxed{(\partial r/\partial \bar{t})_b = -\frac{\dot{R}}{R}a\sqrt{(1-a^2/R_0^2)}}$$

$$\therefore\ \left(\frac{d\bar{r}}{d\bar{t}}\right)_b = -\frac{(\partial r/\partial \bar{t})_b}{(\partial r/\partial \bar{r})_b}$$

$$= \frac{+\frac{\dot{R}}{R}a\sqrt{(1-\frac{a^2}{R_0^2})}\ \left(1-\frac{2M}{\bar{r}_b}\right)}{\frac{R_0}{R}\left(1-\frac{a^2}{R_0^2}\right)}$$

$$= \frac{a}{R_0\sqrt{(1-\frac{a^2}{R_0})}}\ \dot{R}\left(1-\frac{2M}{\bar{r}_b}\right)$$

But $\dot{R} = -\sqrt{\left(\frac{c}{R}-1\right)} = -\sqrt{\left(\frac{2MR_0^3}{a^3}\frac{a}{R_0 r_b}-1\right)} = -\sqrt{\left(\frac{R_0^2}{a^2}\frac{2M}{\bar{r}_b}-1\right)}$

$$\boxed{\therefore\ \left(\frac{d\bar{r}}{d\bar{t}}\right)_b = -\frac{a}{R_0\sqrt{(1-\frac{a^2}{R_0^2})}}\sqrt{\left(\frac{R_0^2}{a^2}\frac{2M}{\bar{r}_b}-1\right)}\ \left(1-\frac{2M}{\bar{r}_b}\right)}$$

This equation describes how an outside observer describes the contracting sphere $\frac{d\bar{r}_b}{d\bar{t}}$ was zero at $\bar{r}_b = 2M\frac{R_0^2}{a^2}$ then it becomes $-$ve and so it starts contracting. But as contraction proceeds, the rate of contraction initially increases after a while starts decreasing and ultimately contraction stops as the ~~[illegible]~~ fluid sphere goes into Schwarzschild's black hole $\bar{r}_b = 2M$.

~~As fluid sphere approaches the bh~~

As the boundary radius $\bar{r}_b$ approaches the black hole radius 2M, the contraction of the sphere slows down very rapidly and so the sphere will take an infinite time to merge into the black-hole.

Again the geometry ~~of~~ on the sphere $\bar{r}_b$ is Schwarzschild we know that ~~A~~ velocity light ~~o~~ coming from the sphere in the radial direction is $1 - \frac{2M}{\bar{r}_b}$. So as the radius $\bar{r}_b$ approaches 2M, the velocity of light is considerably diminished and so all information about the contracting boundary takes longer and longer to reach the outside observer.

~~Att~~ From the point of view of a distant observer, the dust-sphere sphere collapses under its gravitation into Schwarzschild's black hole ~~to~~ ~~for which~~ after an infinitely long time interval.

4. Geometry of Expanding Universes

4.1 Introduction:

Starting with simple geometrical considerations we shall derive metrics to describe an Einstein Universe, the Lemo Friedman universe and the de Sitter universe. Then considering physics of the Einstein universe we shall realise the necessity of introducing the cosmological constant. And on ~~our~~ consideration of the physical contents of the Friedmann universe we shall realise that introduction of cosmological constant was not that nessary.

At the end we shall try to transform the ~~Rw~~ Robertson-Walker metric to a form leading to a particular type of homogeneous but non-isotropic universes.

4.2 Spaces of Constant curvature:

We are familiar with the following ~~a~~ set of relations

I $\quad x = R\cos\phi \qquad x^2+y^2 = R^2 \qquad d\sigma^2 = dx^2+dy^2 = R^2 d\phi^2$

$\quad\; y = R\sin\phi \qquad R = \text{constant}$

The relations are equivalent and we say that they represent a circle. But we elaborate this and say that I gives defines a one-dimensional space of constant curvature immersed in the 2-dimensional flat space. The flat space is two dimensional because ~~because~~ we need two coordinates x and y to specify a point in it and the circle is one dimensional because $x^2+y^2=R^2$ is a relation between the two coordinates x and y.

We are also familiar with the following set of relations

$$\text{II} \quad \begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned} \qquad x^2+y^2+z^2=R^2. \quad d\sigma^2 = dx^2+dy^2+dz^2 = R^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

In common parlance we say that II represents the eqn a sphere of radius R and $d\sigma^2$ is the distance between two ~~n~~ neighbouring ~~p~~ points on the sphere, But we can also say that II defines a 2-dimensional space of constant curvature ~~in~~ immersed in a ~~b~~ 3-dimensional flat space.

It is now easy to generalize to higher dimensions. Note that in the equations defining x and y in I if we replace R by $R\sin\theta$ and ~~intro~~ introduce a third parameter $z=R\cos\theta$ we get II. Also in $d\sigma^2$ the term $R^2d\phi^2$ of I is obviously replaced by $R^2\sin^2\theta\, d\phi^2$ and introduction of the new parameter θ introduces the new ~~ter~~ term $R^2d\theta^2$. Following this procedure I write from II the following

$$\text{III} \quad \begin{aligned} x &= R\sin\chi\sin\theta\cos\phi \\ y &= R\sin\chi\sin\theta\sin\phi \\ z &= R\sin\chi\cos\theta \\ w &= R\cos\chi \end{aligned} \qquad x^2+y^2+z^2+w^2=R^2 \quad (2)$$

$$\text{and } d\sigma^2 = dx^2+dy^2+dz^2+dw^2 = R^2d\chi^2+R^2\sin^2\chi(d\theta^2+\sin^2\theta\, d\phi^2) \quad (3)$$

III can be is seen to be an immediate generalization of II. It can be readily verified that (2) and (3) follow from the equations defining (x, y, z, w) of III. We can therefore say the ~~so~~ set of equations III defines a three dimensional space of constant curvature immersed in four dimensional flat space.

~~In~~ ~~In 1~~ By 1915 Einstein had ~~so~~ discovered his general theory of relativity according to which a gravitating mass produces curvature in the space surrounding it and the geometry governing space-time measurements in a gravitational field has to ~~be~~ be Riemannian and not ~~Eucliad~~ Euclidean. In a famous paper in 1918, he observed that ~~our univ~~ there is so much gravitating matter present in our universe and so the space we live in, must be curved and not flat. Again taking the whole universe as a whole any ~~every~~ point of the universe must be as good as every other points and ~~see~~ at any one point any one direction must be as good as any other direction. In short the space of the universe must be homogeneous and isotropic. But this space has also to be curved. Thus in the universe the space must be of constant ~~curvation~~ ~~cu~~ curvature defined by equations III.

Thus Einstein wrote down the metric of the Riemannian geometry of space-time for the whole universe in the form

~~IV~~
$$ds^2 = c^2dt^2 - d\sigma^2$$
~~(4.1)~~
$$= c^2dt^2 - R^2d\chi^2 - R^2\sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

~~III~~ (32) describes the geometry of what is known now as Einstein's universe.

The four coordinates used in (32) are t, χ, θ & ϕ. ~~A~~ All the three space coordinates χ, θ, ϕ are angles. We would like to have a coordinate to represent radial distance in place of one of the angles. We can do it as follows: In (32) write $R \sin\chi = r$

$\therefore R\cos\chi \, d\chi = dr \quad \therefore d\chi = \dfrac{dr}{R\cos\chi}$ with $\sin\chi = \dfrac{r}{R}$

note that $\cos\chi = \sqrt{\left(1 - \dfrac{r^2}{R^2}\right)}$.

The metric (32) now transforms to

$$ds^2 = c^2dt^2 - \frac{R^2 dr^2}{R^2\cos^2\chi} - r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

$$= c^2dt^2 - \frac{dr^2}{1 - \frac{r^2}{R^2}} - r^2(d\theta^2 + \sin^2\theta \, d\phi^2) \qquad (33)$$

(33) is the standard form ~~in which~~ ~~Bn~~ of the metric of Einstein's universe.

4.3 Spaces of negative curvature:

On looking at the metric (33) it seen that ~~no~~ one can take the constant R^2 as positive or negative. But we know from III that $R^2 = x^2+y^2+z^2+w^2$ and so R^2 being negative needs a reconsideration of our strategy leading to III.

If R^2 is $-$ve R is to be replaced by iR. But $r = R\sin\chi$ implies that if R is replaced by iR χ needs be replaced by $i\chi$ ~~so sett~~ so that ~~r a~~ $r = -R\sinh\chi$ and continues to be real. This suggests the new strategy. In place of III take

$$\text{III}_A \quad \begin{aligned} x &= R \sinh\chi \sin\theta \cos\phi \\ y &= R \sinh\chi \sin\theta \sin\phi \\ z &= R \sinh\chi \cos\theta \\ w &= R \cosh\chi \end{aligned}$$

III_A will lead to

$$w^2 - (x^2+y^2+z^2) = R^2$$

and
$$d\sigma^2 = dx^2+dy^2+dz^2-dw^2$$
$$= R^2 d\chi^2 + R^2 \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Thus if III gives us the space as a three-space of uniform positive curvature $\left(\frac{1}{R^2}\right)$ III_A gives a 3-space of uniform negative curvature $\left(-\frac{1}{R^2}\right)$. We also see from III and $\text{III}A$ that the 3-space of constant positive curvature is a three-sphere, the 3-space of constant negative curvature is a 3-hyperboloid of one sheet.

Metric of the Einstein Universe with the 3-space of constant negative curvature will be

$$ds^2 = c^2dt^2 - R^2 d\chi^2 - R^2\sinh^2\chi\,(d\theta^2+\sin^2\theta\, d\phi^2). \quad (34)$$

Again on setting $R\sinh\chi = r$ we shall get

$$ds^2 = c^2dt^2 - \frac{dr^2}{1+\frac{r^2}{R^2}} - r^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

4.4 Expanding Universes

Let us turn back to the 3-space of positive curvature so that Einstein's universe is given by metric (32) viz

$$ds^2 = c^2dt^2 - R^2\left[d\chi^2 + \sin^2\chi\,(d\theta^2+\sin^2\theta\, d\phi^2)\right] \quad (32)$$

Here R is the radius of the spherical 3-space.

Now R is constant with regard to the equations III & — i.e. in relation to the variables x, y, z, w or χ, θ, φ. But in shifting from geometry to Physics we introduced a new time-parameter t, R need not be constant in relation to t. If we take R as a function of time, nothing changes in the set III and we get (32) with $R = R(t)$. But now the radius of the 3-sphere (which is our physical space) becomes a function of time and if this function is an increasing function, we have an "expanding universe.

$$ds^2 = c^2dt^2 - R^2(t)\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2)\right]$$

In order to introduce a radial coordinate r we rewrite $R^2(t) = \frac{R^2(t)}{R_0^2} R_0^2$, R_0 being a constant so that our metric takes the form

$$ds^2 = c^2dt^2 - \frac{R^2(t)}{R_0^2}\left\{R_0^2 d\chi^2 + R_0^2\sin^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2)\right\}$$

Now put $R_0 \sin\chi = r$ and we shall get

$$ds^2 = c^2dt^2 - \frac{R^2(t)}{R_0^2}\left\{\frac{dr^2}{1 - \frac{r^2}{R_0^2}} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right\} \quad (35)$$

~~(34)~~

~~Similary if R_0~~ Similarly beginning with metric (34) of Einstein universe of negative ~~constant~~ constant curvature we ~~ca~~ get the ~~corres~~ metric of the corresponding expanding universe as

$$ds^2 = c^2dt^2 - \frac{R^2(t)}{R_0^2}\left\{\frac{dr^2}{1 + \frac{r^2}{R_0^2}} + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)\right\} \quad (36)$$

In (35) and (36) one can replace r by a new coordinate $\bar{r}$ ~~as~~ $\bar{r} = \frac{r}{R_0}$ and then the two can be combined in the form

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$$ds^2 = c^2dt^2 - R^2(t)\left[\frac{d\bar{r}^2}{1-k\bar{r}^2} + \bar{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \qquad (37)$$

~~with k=+1 for 3-space to be of uniform +ve curvature~~

~~k=±1~~

with $k=+1$ for 3-space to be of uniform +ve curvature

$k=-1$ " " " " " " −ve "

and $k=0$ " " " " " " zero curvature

(37) is the well-known Robertson-Walker metric

4.5 <u>de Sitter Universe</u>

Let us revert to spaces of constant curvature. We shall use the single word 'Sphere' to denote these spaces of constant curvature irrespective of the dimensions. Thus we would say that I gives ~~a sphere of di~~ a 1-sphere immersed in a 2-flat; II gives a 2-sphere immersed in 3-flat and III gives a 3-sphere immersed in 4-flat. Can we extend this and have a 4-sphere immersed in 5-flat? Well, that is what we are going to do now.

In physics we use 4-dimensional space-time for which the metric is not +ve definite but indefinite. Therefore when we consider a 4-sphere our interest will be in a 4-sphere with an indefinite metric. This will ~~be~~ be reflected in the 5-flat in which it is immersed. We shall write a set IV of equations extending the set III in such a way that it will represent a 4-sphere in 5-dimensional Minkowskian flat space

$$x = R\sin\chi\sin\theta\cos\phi \qquad x^2+y^2+z^2 = R^2\sin^2\chi$$
$$\mathrm{IV} \quad y = R\sin\chi\sin\theta\sin\phi \qquad w^2-v^2 = R^2\cos^2\chi$$
$$z = R\sin\chi\cos\theta \qquad \therefore x^2+y^2+z^2+w^2-v^2 = R^2$$
$$w = R\cos\chi\cosh\psi$$
$$v = R\cos\chi\sinh\psi$$

and thus IV gives a 4-sphere in 5 dimensional Minkowskian flat space. Again in this Minkowskian flat space

$$ds^2 = dv^2 - dx^2 - dy^2 - dz^2 - dw^2$$

Using IV we shall ultimately get

$$ds^2 = R^2\cos^2\chi\, d\psi^2 - R^2 d\chi^2 - R^2\sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Setting $R\sin\chi = r$ we shall get

$$ds^2 = R^2\left(1-\frac{r^2}{R^2}\right)d\psi^2 - \frac{dr^2}{1-\frac{r^2}{R^2}} - r^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

On replacing $R\,d\psi$ by dt ~~we~~ we find

$$ds^2 = \left(1-\frac{r^2}{R^2}\right)dt^2 - \frac{dr^2}{1-\frac{r^2}{R^2}} - r^2(d\theta^2+\sin^2\theta\, d\phi^2) \qquad (38)$$

which is deSitter metric.

4.6 A ~~1~~ non-isotropic model:

We have derived above metrics ~~representing~~ representing various models of an isotropic and homogeneous universe. Which of these models will represent the actual universe we live in is a matter to be decided by observation. But before we do that let us ~~a~~ derive ~~the~~ another simple metric which is ~~isotropic~~ ~~but~~ ~~no~~ homogeneous but not isotropic-

We begin with Robertson-Walker metric for closed 3-space i.e. for which the surface $t=\text{const}$ is a 3-sphere of radius R. We write it as

$$ds^2 = c^2 dt^2 - R^2\left[d\chi^2 + \sin^2\chi\,(d\theta^2+\sin^2\theta\, d\phi^2)\right]$$

We shall put $c=1$

$$ds^2 = dt^2 - \frac{R^2}{R_0^2}\left\{R_0^2\, d\chi^2 + R_0^2 \sin^2\chi\,(d\theta^2+\sin^2\theta\, d\phi^2)\right\}$$

$$= dt^2 - \frac{R^2}{R_0^2}\left[\cancel{R_0}\frac{dr^2}{1-\frac{r^2}{R_0^2}} - r^2(d\theta^2+\sin^2\theta\, d\phi^2)\right]$$

$$= dt^2 - \frac{R^2}{R_0^2}\, dl^2$$

where
$$dl^2 = \frac{dr^2}{1-\frac{r^2}{R_0^2}} + r^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

$$= dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2) + dr^2\left\{\frac{1}{1-\frac{r^2}{R_0^2}} - 1\right\}$$

$$= dr^2 + r^2(d\theta^2+\sin^2\theta\, d\phi^2) + \frac{r^2\, dr^2}{R_0^2\left(1-\frac{r^2}{R_0^2}\right)}$$

One can now bring in 'usual' cartesian coordinates x, y, z through $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and get

$$dl^2 = dx^2+dy^2+dz^2 + \frac{(x\,dx+y\,dy+z\,dz)^2}{R_0^2-(x^2+y^2+z^2)}.$$

Now reintroduce three new angular coordinates α, β, ψ by a set of transformations first used by Schrödinger

$x = R_0 \sin\alpha \cos\beta$ We shall then get

$y = R_0 \sin\alpha \sin\beta$ $\cancel{dl^2 = R_0^2}$

$z = R_0 \sin\psi \cos\alpha$ $dl^2 = R_0^2[\cos^2\alpha\, d\psi^2 + d\alpha^2 + \sin^2\alpha\, d\beta^2]$

$\cancel{dl^2 = R_0^2[\cos^2\theta\, d\psi^2 + d\theta^2 + \sin^2\theta\, d\phi^2]}$

Replace the azimuthal angle β by ϕ

$\beta = \phi - \psi$. We shall then get

$\cancel{dl^2 = R_0^2[(d\psi - \sin^2\alpha\, d\phi}$

$$dl^2 = R_0^2\left[(d\psi - \underline{\sin^2\alpha}\, d\phi)^2 + d\alpha^2 + \sin^2\alpha\cos^2\alpha\, d\phi^2\right] \cancel{(34)}$$

Replace the ~~meridian~~ angle α by $\frac{\theta}{2}$

$$dl^2 = R_0^2\left[\left\{d\psi - \frac{1}{2}(1-\cos\theta)\, d\phi\right\}^2 + \frac{d\theta^2}{4} + \frac{1}{4}\sin^2\theta\, d\phi^2\right]$$

Replace ψ ~~ϕ~~ by χ such that $\psi - \frac{1}{2}\phi = \chi$ ~~$\psi - \frac{1}{2}\phi = \chi$~~

$$\therefore dl^2 = R_0^2\left[\left(d\chi + \frac{1}{2}\cos\theta\, d\phi\right)^2 + \frac{1}{4}\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \cancel{(34)}$$

$\therefore$ Original Robertson Walker metric has now taken the form

$$ds^2 = dt^2 - R^2(t)\left[\left(d\chi + \frac{1}{2}\cos\theta\, d\phi\right)^2 + \frac{1}{4}\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \quad (39)$$

This is homogeneous and isotropic. The isotropy can be ~~disn~~ disturbed ~~by making~~ ~~f~~ if we introduce another function of time $S(t)$ as follows

$$ds^2 = dt^2 - R^2(t)\left(d\chi + \frac{1}{2}\cos\theta\, d\phi\right)^2 - S^2(t)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

This metric of a form known in literature as Bianchi Type $\boxed{IX}$ metric

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Report

The thesis deals with the mathematical problem of finding exact solutions of Einstein's equations of general relativity. These are ten non-linear partial differential equations, which in general, are not easily amenable to standard methods of solution. no underline

In 1951, I wrote a paper describing a method of solving these equations to get solutions which would describe the gravitational fields of fluid-sphere radiating energy. But in those days all available observations of stars and galaxies could be explained by the Newtonian theory and so the solutions of complicated Einsteinian theory were not helpful in understanding any physical or astronomical phenomena. However the situation changed completely with the discovery of Quasars and pulsars in late sixties. Newtonian theory was quite inadequate to describe their strong gravitational fields. Thus the interest in deriving radiating fluid-solutions of general relativity is now revived and the area is being actively pursued. The subject matter of the present thesis is therefore very relevant to the present day applications of high-energy astrophysics.

The first chapter of the thesis begins with these new astronomical objects pointing out the importance of the study of mathematical model of a radiating fluid ball to understand certain phases in the evolution of a star. Choosing a general spherically symmetric metric $ds^2 = A^2\, dt^2 - B^2 dr^2 - C^2 r^2\, (d\Theta^2 + \sin^2\Theta d\phi^2)$ for describing the geometry in the interior of the fluid ball, the field equations are reduced to a single complicated differential equation for A, B, C as functions of r and t.

The geometry outside the radiating fluid ball is well known and in this introductory chapter the author describes the known junction conditions to be satisfied at the surface of the ball where the interior geometry will merge continuously with the known exterior geometry. Also physical parameters like, matter density, fluid pressure, density of flowing radiation, luminosity and mass-functions of the star have been worked out in terms of functions A, B, C.

Thus all necessary formulae to describe the physics of the gravitational field have been collected and presented in this chapter, so that in the later chapters when the differential equation for A, B, C has been solved, the forms of these functions can be plugged into those formulae to obtain the physical contents of the corresponding model.

In all, 6 specific mathematical models are presented in the following four chapters. One of these is based on a known earlier geometry while, the geometry of each of the remaining five have been derived here for the first time. For three of these models numerical calculations have been made and the march of the physical variables like p, e, l as we go outward from the centre of the model to the surface has been tabulated.

On going through the thesis, I got the impression that the author had studie~~s~~d the relevant ~~insisting~~ existing literature on his problem and has presented some original work ~~as~~ in the area of Exact Solutions of Einstein's Equations.

Recommendation

I recommend that the candidate be admitted to the degree.

P. C. Vaidya

CENTER FOR THEORETICAL STUDIES

INDIAN INSTITUTE OF SCIENCE

BANGALORE-560 012

Professor N. Mukunda 19 December 1988

Dear Colleague,

You would have received from the Executive Secretary, INSA, the papers relating to nominations for 1989 Young Scientists Award pertaining to our Sectional Committee. I write to request you to give me your assessments of the three nominations in the pattern described in the letter from INSA, and to ensure that this reaches me as soon as possible and in any case by February 15, 1989 at the latest. If in any of these three cases you feel it necessary to obtain the opinion of some Fellow of the Academy not on our Committee, kindly let me know.

With best regards,

Yours sincerely,

N. Mukunda

Dr. P.C. Vaidya
Gujarat University
Ahmedabad

CABLE: CTS 'SCIENCE' MALLESWARAM, BANGALORE, INDIA Telephone: 364411 Extns. 266, 268, 474
Telex: 0845-8349

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: 3318184 (निवास)
तार : नेटसाइंस
टेलेक्स : 31-61835 इन्सा इन

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: 3318184 (Res.)
Telegram : Natscience
Telex : 31-61835 INSA IN

भारतीय राष्ट्रीय विज्ञान अकादमी
INDIAN NATIONAL SCIENCE ACADEMY

Dr O N Kaul
Executive Secretary

बहादुरशाह जफर मार्ग
नई दिल्ली-110 002
Bahadur Shah Zafar Marg
New Delhi-110 002

No. CC/104/SC/01
Dated: 12/10/88

Dear Prof. Vaidya,

The Council of the Academy at its Meeting held on 5th October 1988 decided that the meetings of the Sectional Committees should be held at INSA, New Delhi on 19th and 20th April, 1989.

Kindly note the dates and convey your availability for the Sectional Committee Meetings to be held in April 1989. Please also inform by 15th November, 1988 if accomodation for your stay is to be arranged by INSA. Members of Sectional Committees who desire accomodation for their stay at the time of meetings are normally accomodated in double rooms at YWCA International Guest House, 10 Sansad Marg, New Delhi and YMCA Tourist Hostel, Jai Singh Road, New Delhi.

The Council is particular that all members of the Sectional Committees:

a) Should attend the meetings of Sectional Committees (if for any unavoidable reason, a Member is unable to attend, information should be conveyed atleast a month in advance to enable the President INSA to nominate a substitute for the meeting).

b) should send their well reasoned recommendations regarding election of Fellows at least a fortnight before the meeting of the Sectional Committee (the nomination papers of the eligible candidates will be sent in February - March 1989).

c) should send their assessment of candidates referred for consideration for the award of INSA Medal for Young Scientists to the Convener by the prescribed date (nomination papers of the candidates will be sent in February 1989).

With regards,

Yours sincerely,

(O N Kaul)

591 Prof. P.C. Vaidya FNA
Department of Mathemaics
School of Sciences
Gujarat University
Ahmedabad 380 009

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APPENDIX-V

INDIAN NATIONAL SCIENCE ACADEMY

Bahadur Shah Zafar Marg, New Delhi-110002

Programme of Sectional Committee/Advisory Board Meetings

April 19-20, 1989

Date	Time (in hours)	Meeting
19 April (Wednesday)	0900 - 1000	Meeting of the Conveners of Sectional Committees with President and other Officers of INSA.
	1000 - 1030	Joint Meeting of the Sectional Committees to be addressed by President.
	1030 - 1300	Meetings of 10 Sectional Committees (I & II, III, IV & V, VI & X, VII, VIII & IX) to consider INSA Medal for Young Scientists - presentation of work by candidates.
	1300 - 1400	Lunch Break
	1400 - 1600	Meetings (Contd.)
	1600	**Advisory Boards**
20 April (Thursday)	0830	**Advisory Boards**
	1000 - 1100	Joint meetings of Sectional Committees to consider cases referred to more than one Sectional Committee.
	1100 - 1300	Separate meetings of 10 Sectional Committees (I to X) to consider nominations for election of Fellows.
	1300 - 1400	Lunch Break
	1400 - 1700	Meetings (Contd.)
	1700	**Advisory Boards**

N. MUKUNDA, Ph.D.
Jawaharlal Nehru Fellow

Centre for Theoretical Studies
Indian Institute of Science
Bangalore–560 012, India

2nd February, 1989

Professors R.P.Bambah, A.S.Gupta, Phoolan Prasad, R.Sridharan, Gopal Prasad, P.C.Vaidya ✓

Dear Colleagues,

I am writing in connection with the three nominations for INSA Young Scientists Award under Sectional Committee I this year. Recently I received from Prof.S.K.Mitra (member of our committee) an assessment of one of the nominees, Dr.Debraj Ray, given by Prof.Kenneth J. Arrow, Nobel Laureate, at Prof. Mitra's request. Enclosed is a copy of Prof.Arrow's letter.

In case you have not already sent me your assessments of the three nominees, you may wish to take into account Prof.Arrow's comments. In case you have already sent me your assessment, please feel free to modify your comments if you so wish. It will be adequate if I receive all replies by about February 20, 1989, since I need to send in the final recommendation to INSA by March 1, 1989.

My apologies for the slight delay in sending you the enclosed letter.

With best regards,

Yours sincerely,

N.Mukunda

Telephone : Office : 344411 Ext. 266 Home : 368657 Telegram : CTS Care 'Science' Malleswaram, Bangalore-560 012, India
Telex : 0845-8349 BG

STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305-6072
FOURTH FLOOR, ENCINA HALL

KENNETH J. ARROW
JOAN KENNEY PROFESSOR OF ECONOMICS
AND PROFESSOR OF OPERATIONS RESEARCH

(415) 723-9165
TELEX 348402

14 December 1988

Professor S.K. Mitra
Head, Stat-Math Unit
Indian Statistical Institute
Delhi Centre
7, S.J.S. Sansanwal Marg
New Delhi - 110016
INDIA

Dear Professor Mitra:

This is in reply to your letter of 4 November, in which you requested an opinion of the work of Debraj Ray, for possible special recognition by the Indian National Science Academy in the field of Mathematical Sciences (sub-area, Applied Mathematics). I am happy to recommend Ray strongly, on the basis of both my past general knowledge of his high abilities and the five specific papers which you sent me.

Ray's work has been in mathematics applied to the fields of economics and game theory. In the past, he has applied probability theory (especially, stochastic processes) to credit against randomly fluctuating crops, studied savings and investment decisions over time, shown (in collaboration with Partha Dasgupta) the consequences of unequal income distribution for efficiency of economic equilibrium when efficiency depends on consumption, and introduced new solution concepts into cooperative game theory.

His papers sent to me continue and enrich these rather varied themes. His paper with Bhaskar Dutta, "Strongly Egalitarian Allocations," studies bargaining in the presence of egalitarian presumptions. That is, an allocation is rejected or "blocked" if it is dominated in the Lorenz sense (stochastically dominated in the language of probability theory). To be consistent, therefore, they do not accept blocking except by "acceptable" allocations. This theme of consistency, that the next stage in blocking must itself be acceptable or unblocked, is also carried through in a different way in the papers, "A Consistent Bargaining Set. (wtih Dutta, Kunal Sengupta, and Rajiv Vohra), and, "Collective Dynamic Consistency in Repeated GAmes," (with B. Douglas Bernheim). In indefinitely repeated games, there are of course many possible equilibria; the Pareto criterion is used to reduce the multiplicity.

The other two papers represent quite different applications of mathematics to economics. "On Perpetuation of Unemployment, Undernourishment, and Inequitable Land Ownership in Dynamic General Equilibrium," (with Peter Streufert), gives a dynamic extension of Ray's earlier work with Dasgupta (cited above). Capital accumulation is possible. They show that there are many equilibria and that these possess significant stability properties.

Finally, in, "On the Economics of Orchards," Ray, with Tapan Mitra and Rahul Roy, starts a new line in capital theory. They point out that the point-input flow-output case (as in fruit orchards) is not covered by the existing literature and indeed many of its properties are distinctly different.

Clearly Ray is a highly prolific mathematical economist, whose work has started a number of new lines of inquiry. I recommend him strongly.

Sincerely yours,

(Group C: I & II)

INSA Medal for Young Scientists-1989

Candidates invited for presentation of research work

SECTIONAL COMMITTEE-I — Convener - Professor N. Mukunda

S.N.	Name of candidate	Presentation title
1.	Dr. Nitin Nitsure, Research Associate, TIFR, Bombay.	Moduli space of semistable pairs on a curve.
2.	Dr. Debraj Ray, Professor, ISI, New Delhi.	Consistent Solution Concepts in the Theory of Games.
3.	Dr. Jagan Nath Sharma, Asstt. Professor, Regional Engg. College, Hamirpur.	Generalized thermoelastic waves in homogeneous anisotropic media.

To be returned (Ten copies) to INSA
latest by 8th April 1989.

INSA MEDAL FOR YOUNG SCIENTISTS - 1989

(To be completed in typewritten)

Full Name and address.

Nitin Nitsure
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005

Date of Birth	Marital Status	Present Post
9 November, 1957	Married	Fellow

Biodata in less than 100 words.

Passed M.Sc. (Mathematics), University of Poona, in 1980 with first rank, and joined TIFR as a research scholar. Held the National Science Talent Scholarship. Completed Ph.D. thesis in algebraic geometry in 1986.

Selected by the International Mathematical Union as one of the young mathematicians to receive complete financial support to participate in the International Congress for Mathematicians, 1986, in Berkeley. Visited the University of Oxford (Professor Atiyah) for six months in 1987-88 under INSA/Royal Society exchange programme and gave talks on own research at Oxford and Liverpool. Gave an invited address in the Indo-French Geometry Conference, Bombay 1989.

Field of research work.

Algebraic Geometry. Vector bundles on curves, their moduli, and related topological problems.

Institution (s) where research work carried out.

Most of my research was done at the Tata Institute of Fundamental Research, Bombay. A part of my work on the moduli of semi-stable pairs was done at the Mathematical Institute, Oxford.

Title of Research Presentation (Ten copies of the abstract to be attached).

Moduli space of semistable pairs on a curve.

P.T.O.

Highlights of Research Contributions (in less than 200 words).

The main results proved in my papers (see list below) are as follows:

1) In this paper I calculated the cohomology groups of the moduli space of parabolic bundles on a Riemann surface.

2) Here I determined the precise relationship between two cohomological invariants (Braner and Chern classes) associated to any conic bundle on a smooth variety, and give some applications of the result.

3) Here I prove that the third integral cohomology group of a desingularization of a certain moduli space of vector bundles on a curve is torsion free. It is unknown whether this moduli space is rational, hence the above result is important as the torsion part of the H^3 is a birational invariant for smooth projective varieties.

4) In this paper I construct a moduli scheme for semistable pairs $(E, \phi : E \longrightarrow L \otimes E)$ on a curve X in arbitrary characteristic. This completely generalizes the earlier construction of Hitchin which was only in the complex analytic category and was restricted to stable paris with rank $E = 2$ and $L = \Omega_X$. I prove in full generality that the characteristic polynomial morphism on this moduli scheme is proper.

List Five Important Publications of the candidate. (Title, Journal, Vol., Year, Page Nos., Author's Name) Reprints to be brought.

1) Cohomology of the moduli of parabolic vector bundles. Proc. Indian Acad.Sci. (math.Sci.) Vol. 95, (1986) 61-77.

2) Topology of Conic bundles. J.London Math. Soc. (2) 35 (1987) 18-28.

3) Cohomology of desingularization of moduli space of vector bundles. To appear in Compositio Math. in February 1989.

4) Moduli space of semistable pairs on a curve. TIFR preprint, Jan. 1989 (submitted to Proc. London Math.Soc.).

Signature Nitin Nitsure

Name Nitin Nitsure

Name written in Devanagari Characters (Hindi) नितिन नित्सुरे

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Moduli space of semistable pairs on a curve

(Abstract of research presentation by N. Nitsure)

Let X be a smooth projective curve over an algebraically closed field of arbitrary characteristic. A semistable pair (E,ϕ) over X is a vector bundle E on X together with an endomorphism ϕ of E which takes values in a fixed line bundle L on X, such that for any ϕ-invariant subbundle F of E, we have $\mu(F) < \mu(E)$. We construct a coarse moduli scheme $\mathcal{M}(r,d,L)$ for S-equivalence classes of semistable pairs of rank r, degree d. This completely generalizes the earlier complex analytic construction of Hitchin, which moreover was restricted to stable pairs of rank 2 and $L = \Omega_X$. In this full generality we show that the Hitchin Harmittonian morphism $\mathcal{M}(r,d,L) \longrightarrow \bigoplus_{0\leq i\leq r} \Gamma(X,L^i)$,

which associates to (E,ϕ) the characteristic polynomial of ϕ, is proper.

Reference

Hitchin, N.J. The self-duality equations on a compact Riemann surface. Proc. London Math. Soc. (3) 55 (1987) 59-126.

To be returned (Ten copies) to INSA
latest by 8th April 1989.

INSA MEDAL FOR YOUNG SCIENTISTS - 1989

(To be completed in typewritten)

Full Name and address. Debraj Ray
Apartment A-8, Indian Statistical Institute, 7,SJS Sansanwal Marg,New Delhi-16

Sept. 03, 1957	Married	Professor(from April 1989)
Date of Birth	**Marital Status**	**Present Post**

Biodata in less than 100 words.

Debraj Ray is currently Professor at the Indian Statistical Institute, where he joined as Associate Professor in 1986. He received his M.Sc. and Ph.D. degrees in Economics at Cornell University in 1983 and the B.A. degree in Economics at Presidency College, University of Calcutta in 1977. He was an Assistant Professor of Economics at Stanford University between 1982 and 1986.

Field of research work. Game theory, economic dynamics, development economics

Institution (s) where research work carried out. Indian Statistical Institute (1986 -)
Stanford University (1982-86)

Title of Research Presentation (Ten copies of the abstract to be attached). "Consistent Solution Concepts in the Theory of Games"

P.T.O.

Ray was nominated for his contribution in the following areas:

Highlights of Research Contributions (in less than 200 words).

1. **Game Theory** : Ray has contributed to the development of consistent solution concepts in the theory of games. In the area of noncooperative games, solution concepts have been explored by him that require the consistent application of the same individual and collective norms in every subgame of an extensive game tree. In the area of cooperative games, Ray pursues a similar methodological line, though different analytical techniques are required. Building on his earlier reformulation of the core as a consistent solution concept, Ray has formulated mixed ethical-positive solution concepts, and has developed the notion of a consistent bargaining set.

2. **Economic Dynamics** : Ray's research uses infinite-horizon dynamic programming and the theory of intertemporal general equilibrium, and has two important features from the mathematical point of view. The first feature concerns the question of relevant economic variables to a stationary distribution. Ray's research uncovers an anti-stability result in models of "Point-input, flow-output" capital theory. These results are related to the rate of discounting, a basic economic concept. The second feature is the analysis of dynamic models which possess multiple steady states. In a model designed to capture the emergence of income inequality from groups of near-identical agents, Ray provides a complete characterization of the different steady states that can arise.

Ray's presentation will concentrate only on Item 1.

List Five Important Publications of the candidate. (Title, Journal, Vol., Year, Page Nos., Author's Name) Reprints to be brought.

Bhaskar Dutta and Debraj Ray, "A concept of egalitarianism under participation constraints" forthcoming, Econometrica, Vol. 57, No. 2 (1989).

Bhaskar Dutta, Debraj Ray, Kunal Sengupta and Rajiv Vohra, "A consistent bargaining set" forthcoming, Journal of Economic Theory (1989) (exact issue unknown).

Douglas Bennheim and Debraj Ray, "Collective dynamic consistency in repeated games, "forthcoming, Games and Economic Behaviour (1989) (exact issue unknown).

Debraj Ray and Peter Streufert, "On the perpetuation of inequality, malnutrition and unemployment in dynamic general equilibrium" presently under review at Review of Economic Studies (1988).

Tapan Mitra, Debraj Ray and Rahul Roy, "The economics of Orchards : an exercise in point - input, flow-output capital theory, under review at Journal of Economic Theory (1988).

Signature Debraj Ray

Name DEBRAJ RAY

Name written in Devanagari Characters (Hindi): देबराज राय

Photograph : One copy of recent photograph (Glossy black and white : Size 6x4.5 cm. with name written on the back) should be attached.

CONSISTENT SOLUTION CONCEPTS IN THE THEORY OF GAMES

by

Debraj Ray

ABSTRACT

1. **Non-cooperative dynamic games** : Let G be a game in normal form, and G^T be the T-fold repetition of this game, $1 \leq T \leq +$. Non-cooperative behaviour leads to the set of subgame perfect Nash equilibrium as the possible solution outcomes. I study a consistent way to refine the set of subgame perfect equilibria, leading to the concept of <u>renegotiation proof equilibrium</u> (Bernheim and Ray [1987]). The existence of an internally consistent renegotiation proof set is demonstrated. The existence of externally consistent sets under a slightly different consistency notion is studied in Bernheim and Ray [1987].

2. **Cooperative games** : We study internally and externally consistent solution concepts, where the consistency is placed on <u>coalitions</u>, not on time (as in 1). (See Ray [1988], Dutta, Ray, Sengupta and Vohra [1988]). The unique internally consistent solution concept turns out to be the core; the externally consistent concept is either the core, the consistent bargaining set, or the von Neumann-Morgenstern solution depending on the consistency requirement. The consistency concept theorefore unifies a number of solution concepts in cooperative game theory. Internally consistent solutions with norms are studied in Dutta and Ray [1988a, 1988b].

REFERENCES

D.Bernheim and D. Ray (1988) "Collective dynamic consistency in repeated games", forthcoming, <u>Games and Economic Behaviour</u>.

B. Dutta and D. Ray (1988a), "A concept of egalitarianism under participation constraints", forthcoming, <u>Econometrica</u>.

B. Dutta and D. Ray (1988b), "Strong egalitarian allocations", under review for publication.

B. Dutta, D. Ray, K. Sengupta and R. Vohra (1988), "A consistent bargaining set", <u>Journal of Economic Theory</u>, forthcoming.

D. Ray [1988], "Credible coalitions and the core, forthcoming, <u>International Journal of Game Theory.</u>

To be returned (Ten copies) to INSA
latest by 8th April 1989.

INSA MEDAL FOR YOUNG SCIENTISTS - 1989

(To be completed in typewritten)

Full Name and address.

Dr. JAGAN NATH SHARMA,
ASSISTANT PROFESSOR,
REGIONAL ENGINEERING COLLEGE,
HAMIRPUR-177 001(HP).

10.02.1957	Married	Assistant Professor
Date of Birth	Marital Status	Present Post

Biodata in less than 100 words.

I did B.Sc. from Govt. College Hamirpur(HP) in 1977 and M.Sc. from GNDU, Amritsar in 1979 securing 70.5% marks. I completed my M.Phil in 1980 by obtaining 82.4% marks. I was registered for Ph.D with GNDU in March 1983, and submitted thesis in July, 1985 under the able guidance and supervision of Dr.Harinder Singh (Presently at Panjab University Chandigarh). I was awarded Ph.D. degree in May, 1986 on the topic "Wave Propagation in generalized thermoelasticity". I served GNDU, Amritsar as a Lecturer from 8.12.83 to 15.5.86 before I joined REC Hamirpur(HP). I joined as Assistant Professor on 2.1.1989.

Field of research work.

Applied Mathematics(Thermoelasticity/Elasticity).

Institution (s) where research work carried out.

1. Department of Mathematics, GNDU, Amritsar(Pb.)
2. Regional Engineering College, Hamirpur-177 001(HP).

Title of Research Presentation (Ten copies of the abstract to be attached).

Generalized thermoelastic waves in homogeneous anisotropic media.
(Copies of abstract attached)

P.T.O.

Highlights of Research Contributions (in less than 200 words).

A detailed investigation has been done on various aspects of wave propagation in dynamic coupled generalized thermoelastic waves in anisotropic media. The investigation reveals some new phenomena inherent in such systems. Some of the results of investigation find useful applications in structures operating at severe environmental conditions as in nuclear reactors and supersonic aircrafts and also in geophysical studies. Very little work has been done on different aspects of such materials though these materials are increasingly in use. So present work is expected to open a new chapter. It is hoped that this study assist investigators in interpreting their experimental results and provide some guidance for the selection of specimen and the design of experiments. Some more works on this line are being done and it is hoped that more useful results will be obtained.

The results of these investigations have been published in various national and international journals of repute. Major work has also been reviewed in review journals like Applied Mechanics Reviews.

List Five Important Publications of the candidate. (Title, Journal, Vol., Year, Page Nos., Author's Name) Reprints to be brought.

Of the 13 papers published in national and international journals, five important publications are listed below.

1. H.Singh and J.N.Sharma: Generalized thermoelastic waves in transversely isotropic media. J.Acoust. Soc.Am.77, 1046-1053(1985).
2. J.N.Sharma: On the low and high frequency behaviour of generalized thermoelastic waves Arch.Mech.38, 665-673(1986).
3. J.N. Sharma: Transient generalized thermoelastic waves in a transversely isotropic half-space. Bull.Polish.Acad.Sci.Tech.Sci.34, 631-646(1986)
4. J.N.Sharma: Transient generalized thermoelastic waves in transversely isotropic medium with a cylindrical hole. Int.J.Engng.Sci.25, 463-471(1987).
5. J.N.Sharma and H.Singh: Generalized thermoelastic waves in anisotropic media. To appear in J.Aconst.Soc.Am.85, March (1989).

Signature ____________ 16-3-89

Name Dr.JAGAN NATH SHARMA

Name written in Devanagari Characters (Hindi)

डा॰ जगन नाथ शर्मा

Photograph : One copy of recent photograph (Glossy black and white : Size 6x4.5 cm. with name written on the back) should be attached.

GENERALIZED THERMOELASTIC WAVES IN HOMOGENEOUS ANISOTROPIC MEDIA

BY

J.N.SHARMA

REGIONAL ENGINEERING COLLEGE, HAMIRPUR (H.P)

ABSTRACT

Propagation of generalized thermoelastic plane harmonic waves in homogeneous anisotropic media has been investigated. The low and high-frequency approximations for wave speeds and attendation coefficient have been obtained withthe help of algebraic functions of complex variables. Four types of waves namely, a quasi-longitudinal, two quasi-transverse and a thermal wave, are identified in such media. The thermal mode which is diffusive in nature in the non-thermal relaxation case, becomes wave-like with a finite but large wave speed in the system under study. Propagation of generalized thermoelastic waves in transversely isotropic media and cubic crystals have also been studied. It is found that the purely transverse(SH) wave gets decoupled from rest of the motion and vice-versa in such media. This wave travels without damping and dispersion. Rest of the motion contains three waves namely, quasi-longitudinal(QL), quasi-transverse(QT) and thermal waves which are dispersive in character and coupled with each other. The particle paths during the motion are found to be elliptic. Inclinations of the major axes of elliptical paths with the wave normal and their eccentricities have been determined. The lower limit on frequencies, above which the series expansions are valid, and the limitation on the values of thermal relaxation time for the validity of series approximations of propagation speeds at quite high frequencies have been obtained. The qualitative results obtained analytically have been verified numerically. They are presented graphically for single crystals of zinc and magnesium.

Distribution of temperature, displacement and stress in an infinite homogeneous transversely isotropic elastic solid

having a cylindrical hole has been investigated by taking (i) Unit step in stress and zero temperature change, and (ii) Unit step in temperature and zero stress at the boundary of the hole, by using laplace transform technique on time. As the "Second sound" effects are short lived so small time approximations have been considered. Displacement is found to be continuous at the waves fronts but temperature and stress are discontinuous at the wave fronts.

The disturbance due to a line source in a homogenaous transversely isotropic thermoelastic half space with thermal relaxations has been investigated using Cagniard's methods. Exact closed algebraic expressions for the displacements and temperature as functions of time and horizental distances valid for all epicentral distances have been obtained.

Some allied problems have also been investigated. Recently, the area of research has been stretched to magneto-thermoelastic problems.

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: 3318184 (निवास)
तार : नेटसाईंस
टेलेक्स : 31-61835 इन्सा इन

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3316440 } (Off.)
: 3318184 (Res.)
Telegram : Natscience
Telex : 31-61835 INSA IN

भारतीय राष्ट्रीय विज्ञान अकादमी
INDIAN NATIONAL SCIENCE ACADEMY

Dr O N Kaul
Executive Secretary

बहादुरशाह ज़फर मार्ग
नई दिल्ली-110 002
Bahadur Shah Zafar Marg
New Delhi-110 002

No.CC/104/
20 JAN 1989

Subject : Election as Fellows : Nomination under consideration by Sectional Committee- I

Dear Sir,

To facilitate scrutiny and assessment of the nominations received, a file containing the following is enclosed for your kind information:

i) List of members of your Sectional Committee **(Appendix-I).**

ii) Schedule of work **(Appendix-II).**

iii) List of persons proposed under your Sectional Committee for election as Fellows of the Academy and statements showing their research work as shown in the nomination paper and their list of publications **(Appendix-III).**

Members of the Sectional Committees are free to obtain information, from any scientist of high standing or expertise on the basis of strict confidentiality, on scientists under consideration for election to Fellowship of the Academy. As far as possible, this may be restricted to Fellows (including Foreign Fellows) of the Academy. The importance of all candidates getting a fair hearing in the Sectional Committees is paramount. Sectional Committee members should familiarize themselves, in advance, with the published work of the candidates allocated to them. Candidates' publications (reprints) can be studied in the Library of the Academy on any working day. These will also be made available to the members at the time of Sectional Committee meetings in April 1989.

Your recommendations along with appropriate reasons, without indicating order of preference, may kindly be sent in confidential cover in the enclosed proforma **(Appendix-IV)** to Convener of your Sectional Committee **at least a fortnight** before the meeting of the Sectional Committee scheduled to be held between **19th and 20th April 1989.** A **stamped envelope** for sending your recommendations to the Convener is also enclosed for your use.

Your as well as recommendations of the other members of the Committee will be considered at the meeting of the Sectional Committee on **20th April 1989.**

The report of the Committee should clearly specify the basis of the recommendations, by highlighting the importance of scientific contributions of recommended candidates, and also fully justify the order of priority.

Contd....on..::2::

Kindly make it convenient to attend the meetings of the Sectional Committees. A copy of the programme for the meetings is enclosed **(Appendix-V)**.

TA/DA for attending the meetings will be paid in accordance with the rules of the Academy.

With regards,

Yours sincerely,

(O.N. Kaul)

To

Dr. P.C. Vaidya
Member
Sectional Committee- I
Indian National Science Academy

Copy along with the file referred to above forwarded to:

Prof. N. Mukunda
Convener
Sectional Committee-I

It is requested that the recommendations of the Sectional Committee may kindly be forwarded in the enclosed proforma **(Appendix-VI)**. The Council desired that the report shall not only bring out the list of persons recommended for election in order of priority but also specify the reasons for their recommendations and also highlight the work of the persons recommended in order of priority.

(O.N. Kaul)
Executive Secretary

APPENDIX-I

Sectional Committee-I

Mathematical Sciences: Applied Mathematics Pure Mathematics, and Statistics

Name	Address	Specialization	To serve until 31 December
N Mukunda (Secretary & Convener)	Professor Centre for Theoretical Studies Indian Institute of Science Bangalore-560012	Mechanics and Particle Physics	1989
RP Bambah	Vice-Chancellor Punjab University Chandigarh-160014	Number theory Discrete Geometry	1989
SK Mitra	Professor of Statistics Indian Statistical Institute 7, SJS Sanswal Marg New Delhi-110016	Mathematical Statistics	1989
AS Gupta	Professor of Mathematics Indian Institute of Technology Kharagpur-721302	Theoretical Fluid Mechanics	1990
Phoolan Prasad	Professor Deptt. of Applied Mathematics Indian Institute of Science Bangalore-560012	Applied Mathematics Wave Propagation Partial Differential Equations	1990
R Sridharan	School of Mathematics Tata Instt. of Fundamental Research, Colaba Bombay-400005	Pure Mathematics	1990
Gopal Prasad	Professor Tata Instt. of Fundamental Research, Colaba Bombay-400005	Lie & Algebraic Group	1991
JK Ghosh	Director Indian Statistical Instt. 203 Barrackpore Trunk Road Calcutta-700035	Statistics	1991
PC Vaidya	Department of Mathematics School of Science Gujarat University Ahmedabad-380009	General Relativity and Gravitation	1991

INDIAN NATIONAL SCIENCE ACADEMY
NEW DELHI

PROGRAMME

Joint meeting of Sectional Committees to consider cases falling under more than one Sectional Committee.

Day : Thursday — Date : 20 April 1989

Time : 1000-1100 hrs

Venue : Conference Room

Sectional Committees	Candidates
III and IV	Agarwala, R.P.
III and IX	Pal, M.K.
	Venkatappa, M.P.
III and X	Ray, N.K.
VI and X	Singh, Randhir
VII and VIII	Anand Kumar, T.C.
VII and IX	Modak, S.P.
VII and X	Gupta, P.K.
VIII and IX	Chakravorty, Indira (Mrs)
	Ghosh, J.J.
	Ray, P.K.

INDIAN NATIONAL SCIENCE ACADEMY
NEW DELHI

PROGRAMME

Meetings of Sectional Committees
for consideration of
Nominations for election of Fellows 1989

Day : Thursday

Date : 20 April 1989

Time : 1100-1300 hrs
1400-1700 hrs

Separate Meeting of ten Sectional Committees (I to X)

Place	Sectional Committee	Convener
MAIN BUILDING		
Conference Room (GF)	III	Prof. U.R. Ghatak
	VII	Prof. P.N. Srivastava
	VIII	Dr. B. Ramamurthi
	IX	Dr. N.K. Notani
LIBRARY BUILDING		
Committee Room (GF)	V	Prof. P.K. Das
Auditorium (GF)	II	Prof. A.N. Mitra
Reading Room (2F)	IV	Dr. R. Narasimha
Lounge (2F)	VI	Prof. H.Y. Mohan Ram
	X	Prof. V.L. Chopra
Reading Room (1F)	I	Prof. N. Mukunda