Syllabus on Elasticity

(1) Analysis of Stress & Strain

(2) Hooke's Law

(3) Strain-energy function

(4) Equations of ~~motion~~ Elasticity & uniqueness of soln

(5) Problems of plane stress & strain

(6) Extension, torsion & flexure of homogeneous beams.

(7) St. Venant's Problem.

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Monday — 2.10 P.M

Wednesday — 3 to 4 P.M

Friday — 12 to 1

Saturday — 1 to 2

Lecture Notes on Elasticity to M. Sc. Students, 1961.

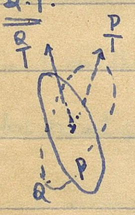
(Following Sokolnikoff)

~~First Class Lecture~~ First Lecture – Date ? – Before the floods – General Lecture

2nd Lecture – 19/8/61 – Stress – $\overset{\nu}{T}_i = \tau_{ji}\nu_j$ and $\tau_{ji,j} = -F_i$

3rd Lecture – 26/8/61 – $\tau_{ij} = \tau_{ji}$ – Soln. of eqns with boundary conditions – Reciprocity theorem – Tensor character of τ_{ij} – Invariance of τ_{ii}

4th Lecture – 2/9/61 – Problems of Qns 1, 2, 3, p. 44 of Sokolnikoff.



Q.1. $\overset{P}{T}$ is stress vector across plane P, and plane Q contains $\overset{P}{T}$, then to prove from the reciprocity theorem that stress vector $\overset{Q}{T}$ across plane Q lies in P.

If ν' be normal to plane Q, then it is given by $\overset{P}{T}\cdot\nu' = 0$ or $\overset{\nu}{T}\cdot\nu'$, where $\underline{\nu}$ is normal to the plane P. Hence from reciprocity theorem $\overset{\nu'}{T}\cdot\nu = \overset{Q}{T}\cdot\nu = 0$ i.e. $\overset{Q}{T}$ lies in P

Q.2. – To deduce symmetry of stress components viz $\tau_{ij} = \tau_{ji}$ from the reciprocity theorem

$$\overset{\nu'}{T}\cdot\nu = \overset{\nu'}{T}_j\cdot\nu_j = \tau_{ji}\nu_i'\nu_j$$

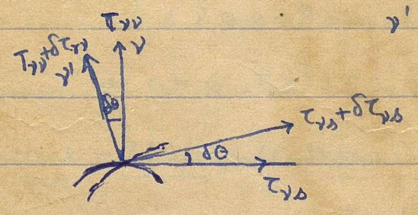
$$\overset{\nu}{T}\cdot\nu' = \overset{\nu}{T}_i\cdot\nu_i' = \tau_{ij}\nu_j\nu_i' \quad \text{\& hence } \tau_{ji} = \tau_{ij}$$

Q.3. $\overset{P}{T}$ & $\overset{Q}{T}$ are stress vectors across planes P & Q. To find stress vector $\overset{R}{T}$ on a plane R containing both $\overset{P}{T}$ & $\overset{Q}{T}$.

If normals to planes P, Q & R be ν, ν' & ν'' resply, we have, since R contains $\overset{\nu}{T}$ & $\overset{\nu'}{T}$,

$\overset{\nu}{T}\cdot\nu'' = \overset{\nu'}{T}\cdot\nu'' = 0$ i.e. $\overset{\nu''}{T}\cdot\nu = \overset{\nu''}{T}\cdot\nu' = 0$ i.e. $\overset{\nu''}{T}$ or $\overset{R}{T}$ lies along the line of intersection of planes P & Q.

Q.4. To show, using the reciprocity theorem, that the normal stress has a stationary value when the shear stress is zero.



ν' is inclined to ν at $\delta\theta$ – Components of $\overset{\nu}{T}$ along ν & $\perp^r$ to ν are $\tau_{\nu\nu}$ and $\tau_{\nu s}$

Components of $\overset{\nu'}{T}$ along ν' and $\perp^r$ to ν' are $\tau_{\nu\nu}+\delta\tau_{\nu\nu}$ and $\tau_{\nu s}+\delta\tau_{\nu s}$

$$\therefore \overset{\nu'}{T}\cdot\nu = (\tau_{\nu\nu}+\delta\tau_{\nu\nu})\cos\delta\theta + (\tau_{\nu s}+\delta\tau_{\nu s})\cos(90^\circ-\delta\theta)$$

$$= (\tau_{\nu\nu}+\delta\tau_{\nu\nu}) + (\tau_{\nu s} + \delta\tau_{\nu\nu})\sin\delta\theta$$

$$= \tau_{\nu\nu} + \delta\tau_{\nu\nu} + \tau_{\nu s}\cdot\delta\theta.$$

$$\overset{\nu}{T}\cdot\nu' = \tau_{\nu\nu}\cos\delta\theta + \tau_{\nu s}\cos(90^\circ+\delta\theta) = \tau_{\nu\nu}\cos\delta\theta - \tau_{\nu s}\sin\delta\theta$$

$$= \tau_{\nu\nu} - \tau_{\nu s}\delta\theta$$

Equating, $\overset{\nu'}{T}\cdot\nu = \overset{\nu}{T}\cdot\nu'$, $\tau_{\nu\nu}+\delta\tau_{\nu\nu} + \tau_{\nu s}\delta\theta = \tau_{\nu\nu} - \tau_{\nu s}\delta\theta$

i.e. $\delta\tau_{\nu\nu} = -2\tau_{\nu s}\delta\theta$ or $\dfrac{\partial\tau_{\nu\nu}}{\partial\theta} = -2\tau_{\nu s}$

i.e. when $\tau_{\nu s} = 0$, $\dfrac{\partial\tau_{\nu\nu}}{\partial\theta} = 0$ i.e. $\tau_{\nu\nu}$ is max or stationary for change of normal to plane.

5th Lecture – 9/9/61 – Working of Q.4, p. 44 of Sokolnikoff – Stress quadric of Cauchy: (i) Definition and obtaining the equation $NA^2 = \tau_{ij} x_i x_j = \pm K^2$, (ii) Invariance of $\tau_{ij} x_i x_j$ for $x_i \to x_i'$, (iii) Formulae for transformation of τ_{ij} viz $\tau_{\alpha\beta} = l_{i\alpha} l_{j\beta} \tau'_{ij}$, (iv) Construction of $\vec{T}$ geometrically from quadric / (v) Principal directions and principal stresses & eqn of quadric referred to principal directions.

6th Lecture – 30/9/61 – Item (v) left over from previous lecture (p. 47 of S) – If $\vec{\nu}$ be along an axis of the stress quadric $\tau_{ij} x_i x_j = \pm K^2$, $\vec{T}$ and $\vec{\nu}$ have the same direction & hence $\overset{\nu}{T}_i = \tau \nu_i = \tau \delta_{ij} \nu_j$ and since $\overset{\nu}{T}_i = \tau_{ij} \nu_j$, we get $(\tau_{ij} - \tau\delta_{ij})\nu_j = 0$ leading to the cubic equation $|\tau_{ij} - \tau\delta_{ij}| = 0$ in τ in the form of a determinantal equation giving three roots τ_1, τ_2, τ_3, which can be shown to be real; for if τ_1, τ_2, τ_3 be the roots and $\overset{1}{\nu}, \overset{2}{\nu}, \overset{3}{\nu}$ the corresponding directions, we have for $\tau = \tau_1$,

$$\tau_{ij} \overset{1}{\nu}_j = \tau_1 \overset{1}{\nu}_i$$

Multiplying both sides by $\overset{2}{\nu}_i$ and summing over i, $\tau_{ij} \overset{1}{\nu}_j \overset{2}{\nu}_i = \tau_1 \overset{1}{\nu}_i \overset{2}{\nu}_i$

Simly from $\tau_{ij} \overset{2}{\nu}_j = \tau_2 \overset{2}{\nu}_i$, applying the same procedure, $\tau_{ij} \overset{2}{\nu}_j \overset{1}{\nu}_i = \tau_2 \overset{2}{\nu}_i \overset{1}{\nu}_i$

leading to $(\tau_1 - \tau_2) \overset{1}{\nu}_i \overset{2}{\nu}_i = 0$ in view of $\tau_{ij} \overset{1}{\nu}_j \overset{2}{\nu}_i = \tau_{ji} \overset{1}{\nu}_i \overset{2}{\nu}_j = \tau_{ij} \overset{2}{\nu}_j \overset{1}{\nu}_i$

If roots be complex say $\tau_1 = T_1 + i T_2$, $\tau_2 = T_1 - i T_2$, τ_3 real, ~~the above eqn gives~~ we have from the equation $(\tau_{ij} - \tau\delta_{ij})\nu_j = 0$, the relations

$$\{\tau_{ij} - (T_1 + iT_2)\delta_{ij}\} \overset{1}{\nu}_j = 0$$

and $$\{\tau_{ij} - (T_1 - iT_2)\delta_{ij}\} \overset{2}{\nu}_j = 0$$

For these two be satisfied $\overset{1}{\nu}$ & $\overset{2}{\nu}$ have to be complex vectors & taking $\overset{1}{\nu}_j = a_j + i b_j$, it follows easily that $\overset{2}{\nu}_j = a_j - i b_j$ & hence $\overset{1}{\nu}_i \overset{2}{\nu}_i = (a_i + i b_i)(a_i - i b_i) = a_i^2 + b_i^2 \neq 0$ and there $\tau_1 - \tau_2 = 2 i T_2 = 0$ ie $T_2 = 0$ ie τ_1, τ_2, τ_3 are real. Also if $\tau_1 \neq \tau_2$, $\overset{1}{\nu}_i \overset{2}{\nu}_i = 0$ ie $\overset{1}{\nu}$ & $\overset{2}{\nu}$ are orthogonal – If coordinate axes are taken along the principal axes of the stress quadric its eqn is $\tau_1 x_1^2 + \tau_2 x_2^2 + \tau_3 x_3^2 = NA^2 = \pm K^2$

7th Lecture – 7/10/61 – (1) Invariants of the stress tensor.

writing $|\tau_{ij} - \tau\delta_{ij}| = \prod_i (\tau_i - \tau) = -\tau^3 + \Theta_1 \tau^2 - \Theta_2 \tau + \Theta_3 = 0$, we get

$\Theta_1 = \tau_1 + \tau_2 + \tau_3 = \tau_{11} + \tau_{22} + \tau_{33} = \Theta.$

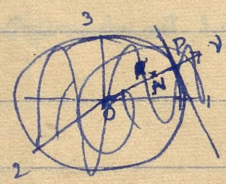
$$\Theta_2 = \text{coeff of } -\tau = \sum (\tau_{22}\tau_{33} - \tau_{23})^2 = \sum \begin{vmatrix} \tau_{22} & \tau_{23} \\ \tau_{23} & \tau_{33} \end{vmatrix}$$

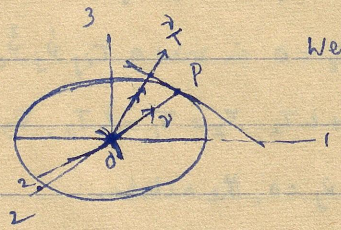
& $$\Theta_3 = \text{Constant term in } |\tau_{ij} - \tau\delta_{ij}| = \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix}$$

(2) Expression of τ_{ij} in terms of τ_α.

From $\tau_{ij} = l_{i\alpha} l_{j\beta} \tau'_{\alpha\beta}$ for change of axes, and from $\tau'_{\alpha\beta} = \tau_\beta \delta_{\alpha\beta}$ for principal axes, $\tau_{ij} = l_{i\alpha} l_{j\alpha} \tau_\alpha$,

(*)





We can proceed alternatively and find N first, for from the eq^n to the quadric

$$\tau_1 x_1^2 + \tau_2 x_2^2 + \tau_3 x_3^2 = N A^2 \quad \text{--- (1)}$$

If OP be radius vector, along the normal $\vec{\nu}$ to the surface element P^0 at O, of the stress quadric with $OP = A$, P is the point $(A\nu_1, A\nu_2, A\nu_3)$. Substituting in (1) for x_1, x_2, x_3, this gives at once $N = \tau_1 \nu_1^2 + \tau_2 \nu_2^2 + \tau_3 \nu_3^2$ -- (2)

But $N = \vec{T}_i \cdot \nu_i$ & from (2) therefore $\vec{T}_i = \tau_i \nu_i$ and $|\vec{T}|^2 = \tau_1^2 \nu_1^2 + \tau_2^2 \nu_2^2 + \tau_3^2 \nu_3^2$

(3) Character of dist^{n} of stress at P^{0} (i) $\tau_1, \tau_2, \tau_3 > 0$ - Ellipsoid - tensile stress

(ii) $\tau_1, \tau_2, \tau_3 < 0$ " - Compressive stress

(iii) $\tau_1 > 0, \tau_2 > 0, \tau_3 < 0$, $NA^2 = \pm K^2$, Hyperboloid of one sheet with $+K^2$ & tensile stress; Hyperboloid of two sheets with $-K^2$ & compressive stress

~~(iii)~~ $K^2 = 0$, ~~tanger~~ Asymptotic cone - no normal stress, only tangential

(iv) $\tau_1 < 0, \tau_2 < 0, \tau_3 > 0$, same as (iii) with differ in regions experiencing compression & tension

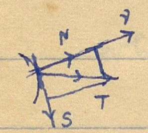
8th Lecture
(14/10/61) (4) Maximum normal & shear stresses - Mohr's diagram: Let $\tau_1 > \tau_2 > \tau_3$

$$N_{max} = \tau_1, \quad N_{min} = \tau_3 .$$

$S_{min} = 0$ ~~ie~~ for principal axes, as shown below, or as is otherwise obvious.

~~To find S_{max}~~, Using pr. axes. as ~~coordinates~~ axes), $\overset{\nu}{T}_i = \tau_{ij}\nu_j$ gives, (using $\tau_{ij} = \tau_i \delta_{ij}$) (*)

$$\overset{\nu}{T}_1 = \tau_1\nu_1, \quad \overset{\nu}{T}_2 = \tau_2\nu_2, \quad \overset{\nu}{T}_3 = \tau_3\nu_3$$



$$N = \overset{\nu}{T}_i \cdot \nu_i = \tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2$$

$$S^2 = (\tau_1^2\nu_1^2 + \tau_2^2\nu_2^2 + \tau_3^2\nu_3^2) - (\tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2)^2 \qquad [S^2 = (\overset{\nu}{T})^2 - N^2.$$

If $\nu_1 = \pm 1, \nu_2 = 0, \nu_3 = 0$
$\nu_2 = \pm 1, \nu_3 = 0, \nu_1 = 0$
$\nu_3 = \pm 1, \nu_1 = 0, \nu_2 = 0$
then $S = 0$ ie for principal dir^{ns} as otherwise obvious since for these dir^{ns} stress is purely tensile or compressive

To find Max of S, ~~write~~ let us first find its extremum values, and then using $\tau_1 > \tau_2 > \tau_3$, we can choose the real S_{max} & also the corresponding direction.

$$\text{Writing} \quad S^2 = \left\{\tau_1^2(1-\nu_2^2-\nu_3^2) + \tau_2^2\nu_2^2 + \tau_3^2\nu_3^2\right\} - \left\{\tau_1(1-\nu_2^2-\nu_3^2) + \tau_2\nu_2^2 + \tau_3\nu_3^2\right\}^2$$

and using $\partial(S^2)/\partial\nu_2 = 0$, $\partial(S^2)/\partial\nu_3 = 0$, we get, ~~if $\tau_1 \neq \tau_2$ and $\tau_1 \neq \tau_3$~~

$$-2\tau_1^2\nu_2 + 2\tau_2^2\nu_2 - 2\left\{\tau_1(1-\nu_2^2-\nu_3^2) + \tau_2\nu_2^2 + \tau_3\nu_3^2\right\}\left\{-2\tau_1\nu_2 + 2\tau_2\nu_2\right\} = 0$$

$$-2\tau_1^2\nu_3 + 2\tau_3^2\nu_3 - 2\left\{ \quad " \quad \right\}\left\{-2\tau_1\nu_3 + 2\tau_3\nu_3\right\} = 0$$

ie

$$(\tau_1^2 - \tau_2^2)\nu_2 = 2(\tau_1 - \tau_2)\left\{\tau_1(1-\nu_2^2-\nu_3^2) + \tau_2\nu_2^2 + \tau_3\nu_3^2\right\}\nu_2 = 0$$

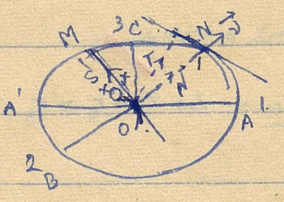
$$(\tau_1^2 - \tau_3^2)\nu_3 = 2(\tau_1 - \tau_3)\left\{ \quad " \quad \right\}\nu_3 = 0$$

Using $\tau_1 \neq \tau_2$ & $\tau_1 \neq \tau_3$, these can be written

$$(\tau_1 + \tau_2)\nu_2 - 2\nu_2\left\{ \quad " \quad \right\} = 0 \quad --- (1)$$

$$(\tau_1 + \tau_3)\nu_3 - 2\nu_3\left\{ \quad \right\} = 0 \quad -- (2)$$

~~From~~ these equations at least a solution should be either $\nu_2 = 0$ or $\nu_3 = 0$, for if neither ν_2 or ν_3 is $= 0$, then the eqns give $\tau_1 + \tau_2 = \tau_1 + \tau_3$ ie $\tau_2 = \tau_3$ which is not true in general. Hence, let us take soln of (1) to be $\nu_2 = 0$, but $\nu_3 \neq 0$, then (2) gives $(\tau_1 + \tau_3) = 2\left\{\tau_1(1-\nu_3^2) + \tau_3\nu_3^2\right\}$. $(\nu_2 = 0)$



(*) If ON ~~has~~ bisects AOC, dirn cosines of ON are

$\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

& S_{max} on plane thro' OM ($\perp^r$ to ON) and the O2 axis – ~~Plane~~ this plane bisects the angle between OA' & OC.

<u>21/10/61</u> – Still another method of finding S_{max} analytically without using Mohr's diagram.

$$S^2 = (\tau_1^2\nu_1^2 + \tau_2^2\nu_2^2 + \tau_3^2\nu_3^2) - (\tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2)^2$$

$$= (\qquad\qquad)(\nu_1^2 + \nu_2^2 + \nu_3^2) - (\qquad\qquad)^2$$

$$= \nu_2^2\nu_3^2(\tau_2 - \tau_3)^2 + \nu_3^2\nu_1^2(\tau_3 - \tau_1)^2 + \nu_1^2\nu_2^2(\tau_1 - \tau_2)$$

Being sum of squares, the three extrema are given by either term vanishing ie

$\nu_2^2\nu_3^2 = 0$, etc ~~ie (i) $\nu_2 = 0$ or $\nu_3 = 0$ → $\nu_2 = 0, \nu_3 \neq 0$~~,

If both $\nu_2 = \nu_3 = 0$, then $\nu_1 = \pm 1$ & $S = 0$ gives S_{min}. Similarly for $\nu_3 = \nu_1 = 0$, $\nu_1 = \nu_2 = 0$

Hence we might take three possibilities (i) $\nu_1 = 0$, $\nu_2, \nu_3 \neq 0$, (ii) $\nu_2 = 0$, $\nu_3, \nu_1 \neq 0$, (iii) $\nu_3 = 0$ $\nu_1, \nu_2 \neq 0$

For (i) $S^2 = \nu_2^2\nu_3^2(\tau_2 - \tau_3)^2$ & with $\nu_2^2 + \nu_3^2 = 1$, the extrema are given* by $\nu_2^2 = \nu_3^2 = \frac{1}{2}$ & $S^2 = \frac{1}{4}(\tau_2 - \tau_3)^2$

(ii) $\nu_3^2 = \nu_1^2 = \frac{1}{2}$ & $S^2 = \frac{1}{4}(\tau_1 - \tau_3)^2$, (iii) $\nu_1^2 = \nu_2^2 = \frac{1}{2}$ & $S^2 = \frac{1}{4}(\tau_1 - \tau_2)^2$

obviously (ii) gives the maximum for $S = S_{max} = \frac{1}{2}(\tau_1 - \tau_3)$.

corresponding value of $N = \frac{1}{2}(\tau_1 + \tau_3)$.

* ~~If~~ $4ab = (a+b)^2 - (a-b)^2$

ie ab is <u>max</u> when $a+b$ = const, when $a = b$

ie $(\tau_1 + \tau_3) = 2\tau_1 - 2\nu_3^2(\tau_1 - \tau_3)$ or $2\nu_3^2(\tau_1 - \tau_3) = \tau_1 - \tau_3$ ie $\nu_3^2 = 1/2$. Hence the corresponding dirn for this extremum is given by $\nu_3^2 = \frac{1}{2}$, $\nu_2^2 = 0$, $\nu_1^2 = 1/2$ & calling corresponding S^2 as S_2 we get $S_2^2 = \frac{1}{2}\tau_1^2 + \frac{1}{2}\tau_3^2 - (\frac{1}{2}\tau_1 + \frac{1}{2}\tau_3)^2 = \frac{1}{4}(\tau_1 - \tau_3)^2$ ie $S_2 = \frac{1}{2}|\tau_1 - \tau_3|$. From (1) & (2) we could have taken $\nu_3^2 = 0$ & got $\nu_2^2 = \frac{1}{2} = \nu_1^2$ giving another extremum leading to $S_3 = \frac{1}{2}|\tau_1 - \tau_2|$. Instead of (1) & (2) being eqns in ν_2^2 & ν_3^2, we might have taken eqns instead in ν_1^2 & ν_2^2 or ν_1^2 & ν_3^2 and obtained the third extremum, $\nu_1 = 0$, $\nu_2^2 = \frac{1}{2} = \nu_3^2$ leading to $S_1 = \frac{1}{2}|\tau_2 - \tau_3|$. Now using $\tau_1 > \tau_2 > \tau_3$ leads to $S_{max} = \frac{1}{2}(\tau_1 - \tau_3) = S_2$ with the dirns $\nu_2^2 = 0$, $\nu_1^2 = \nu_3^2 = 1/2$ leading to the result that S_{max} acts on the plane that bisects the angle between the largest & smallest principal stresses (*)

<u>Mohr's diagram</u>: Graphical representation in the N – S plane using

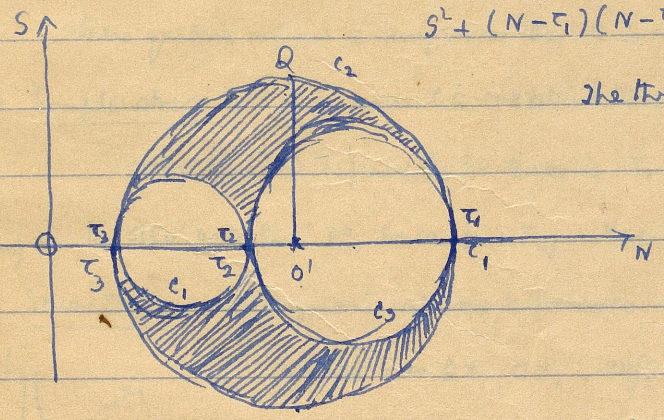
$$\left.\begin{aligned} N &= \tau_1\nu_1^2 + \tau_2\nu_2^2 + \tau_3\nu_3^2 \\ S^2 + N^2 &= \tau_1^2\nu_1^2 + \tau_2^2\nu_2^2 + \tau_3^2\nu_3^2 \\ \text{and} \quad 1 &= \nu_1^2 + \nu_2^2 + \nu_3^2 \end{aligned}\right\} \quad \text{-- (A)}$$

solving for $\nu_1^2, \nu_2^2, \nu_3^2$, getting

$$\left.\begin{aligned} \nu_1^2 &= \frac{S^2 + (N-\tau_2)(N-\tau_3)}{(\tau_1-\tau_2)(\tau_1-\tau_3)} \\ \nu_2^2 &= \frac{S^2 + (N-\tau_3)(N-\tau_1)}{(\tau_2-\tau_3)(\tau_2-\tau_1)} \\ \nu_3^2 &= \frac{S^2 + (N-\tau_1)(N-\tau_2)}{(\tau_3-\tau_1)(\tau_3-\tau_2)} \end{aligned}\right\} \quad \text{-- (B)}$$

Using $\tau_1 > \tau_2 > \tau_3$, these give

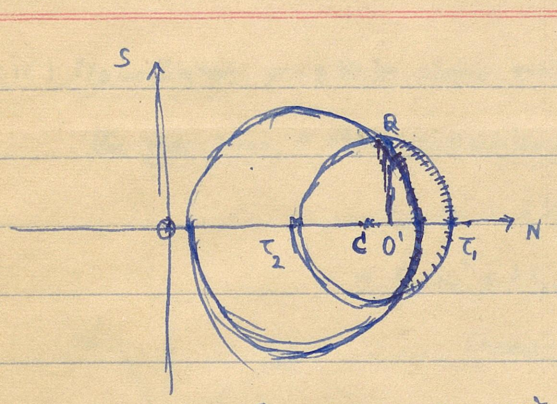
$$\left.\begin{aligned} S^2 + (N-\tau_2)(N-\tau_3) &= \nu_1^2(\tau_1-\tau_2)(\tau_1-\tau_3) \geqq 0 \\ S^2 + (N-\tau_3)(N-\tau_1) &= \nu_2^2(\tau_2-\tau_3)(\tau_2-\tau_1) \leqq 0 \\ S^2 + (N-\tau_1)(N-\tau_2) &= \nu_3^2(\tau_3-\tau_1)(\tau_3-\tau_2) \geqq 0 \end{aligned}\right\} \quad \text{-- (C)}$$



The three ⊙s are those having as diameters $\tau_2\tau_3$, $\tau_3\tau_1$ & $\tau_2\tau_1$, given by the equalities above (= 0), and taking the inequalities into consideration, the admissible values of S and N lie in the shaded crescent region & S_{max} is represented by the max. ordinate O'Q where O' is midpt of T_3T_1 ie the centre of circle on T_3T_1 as diameter & $S_{max} = \frac{1}{2}(\tau_1 - \tau_3)$ = radius of this ⊙le. Corresp. value $N = OO' = \frac{1}{2}(\tau_1 + \tau_3)$ ~~& then $S^2 + N^2 = \frac{1}{2}(\tau_1^2 + \tau_3^2)$, Substituting these values of N & S^2 in first two eqns of (A) gives~~ $\cancel{\frac{1}{2}\tau_1\nu_1^2 + \tau_2\nu_2^2 + \frac{1}{2}\tau_3\nu_3^2 = 0}$, Substituting these value of N and S^2 in (B)

21/10/61 (9th Lecture) $\cancel{\frac{1}{2}\tau_1^2\nu_1^2 + \tau_2^2\nu_2^2 + \frac{1}{2}\tau_3^2\nu_3}$ gives $\nu_2^2 = 0$, $\nu_1^2 = \nu_3^2 = 1/2$

<u>Problem on p. 53</u>: For the case $\tau_2 = \tau_3$, we have $S^2 + (N - \tau_2)^2 > 0$, $S^2 + (N-\tau_1)(N-\tau_2) = 0$, ie ⊙le c_1 → a pt.



$$S^2 + (N-\tau_1)(N-\tau_2) = 0$$

$$S^2 + (N-\tau_2)^2 = \nu_1^2(\tau_1-\tau_2)^2 > 0$$

The second is a circle with $(\tau_2, 0)$ as centre & radius less than $(\tau_1-\tau_2)$ ie a circle which does not pass thro' $(\tau_1, 0)$ & if this ⊙^{le} meets the first at Q, then O'Q gives the corresp S.

For different values of ν_1^2 we get different O'Q & this is obviously a max when O'Q = radius of the first circle of centre C & radius $\frac{1}{2}(\tau_1-\tau_2)$ & then the radius of the corresp. 2^{nd} circle is equal $= \left\{\frac{1}{4}(\tau_1-\tau_2)^2 + \frac{1}{4}(\tau_1-\tau_2)^2\right\}^{\frac{1}{2}} = \frac{1}{\sqrt{2}}(\tau_1-\tau_2)$ ie $\nu_1^2 = \frac{1}{2}$ ~~$\nu_3^2 = \frac{1}{2}$ ... $\nu_2 = 0$~~

and ν_2 & ν_3 may be anything such that $\nu_2^2 + \nu_3^2 = \frac{1}{2}$ ie there are infinitely many dirns for which S is S_{max}

but $S_{max} = \frac{1}{2}(\tau_1-\tau_2)$ - ~~No region can be marked here since~~ No region can be marked here since points must lie on the first ⊙^{le}; anyway these points can be shown ~~when~~ indented inwards.

since ν_2, ν_3 can take ∞ number of values, there are infinite dirns, the quadric being an ellipsoid of revolution, the axis of revolution being the τ_1-axis

Case $\tau_1 = \tau_2 = \tau_3$ - Here all circles become $S^2 + (N-\tau)^2 = 0$ $(\tau = \tau_1 = \tau_2 = \tau_3)$ ie a single point & $S_{max} = 0$

Examples of Stress

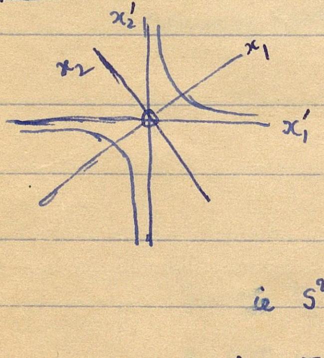
(i) The above case of $\tau_1 = \tau_2 = \tau_3$ ie for every plane thro' P^0, $\vec{T}$ is purely normal, Stress quadric referred to pr. axes is a sphere $x^2+y^2+z^2 = \pm K^2/\tau$ & hence referred to any axes.

Using $\tau_{ij} = l_{i\alpha} l_{j\alpha} \tau_\alpha$, we have $\tau_{11} = \tau_{22} = \tau_{33} = \tau$ & $\tau_{23} = \tau_{31} = \tau_{12} = 0$ ie stress quadric referred to any axes thro' O is also a sphere.

(ii) Simple stress - $\vec{T}$ is normal to one plane thro' the point & vanishes for every plane $\perp^r$ to it. ie stress quadric referred to pr. axes is $\tau_1 x_1^2 = \pm K^2$ ie a pair of $\parallel^l$ planes. Referred to other axes thro' P^0, we can find τ_{ij}. Here $\tau_{23}, \tau_{31}, \tau_{12}$ exist.

(10th Lecture)

28/10/61 (iii) Shear Stress - Stress quadric ~~referred to pr. axes~~ is of the form $2\tau x_1' x_2' = \pm K^2$. Rotating thro' 45° to make axes Ox_1, Ox_2, eqn becomes $\tau x_1^2 - \tau x_2^2 = \pm K^2$



Comparing with $\tau_1 x_1^2 + \tau_2 x_2^2 + \tau_3 x_3^2 = \pm K^2$, $\tau_{33} = 0$, $\tau_{11} = -\tau_{22} = \tau$

ie a case of pure shear [ie $\tau_3 = 0$, $\tau_1 = -\tau_2 = \tau$.

Taking pr. axes as x_1 & x_2, $N = \tau_1\nu_1^2 + \tau_2\nu_2^2$; $S^2 = (\tau_1^2\nu_1^2 + \tau_2^2\nu_2^2) - (\tau_1\nu_1^2 + \tau_2\nu_2^2)^2$

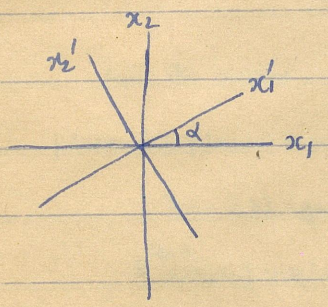
ie $S^2 = \tau^2(\nu_1^2+\nu_2^2) - \tau^2(\nu_1^2-\nu_2^2)^2$ [$S^2 = |\vec{T}|^2 - N^2$

ie $N = \tau_1(\nu_1^2-\nu_2^2)$, $S^2 = \tau^2(\nu_1^2+\nu_2^2) - \tau^2(\nu_1^2-\nu_2^2)^2$, for $\nu_1 = \pm\nu_2$ ie elements $\perp^r$ to x_1' & x_2'-axes, $N = 0$, $S = T$ ($\because \nu_1^2+\nu_2^2 = 1$ with $\nu_3 = 0$) ie case of pure shear

(iv) Plane Stress: one of the pr. stresses vanishing, stress quadric becomes cylinder whose base is a conic, called the stress conic & stress is called plane stress. If axis of cylinder be z-axis, the eqn of the conic is

$$\tau_{xx} x^2 + 2\tau_{xy} xy + \tau_{yy} y^2 = \pm K^2$$

$$\text{or } \tau_{11} x_1^2 + 2\tau_{12} x_1 x_2 + \tau_{22} x_2^2 = \pm K^2.$$



Changing axes (x_1, x_2) to (x_1', x_2'), $x_i = l_{\alpha i} x_\alpha'$

$$\left.\begin{aligned} \text{or } x_1 &= l_{11} x_1' + l_{21} x_2' \\ x_2 &= l_{12} x_1' + l_{22} x_2' \end{aligned}\right\} \text{ with } \begin{aligned} l_{11} &= l_{22} = \cos\alpha \\ l_{12} &= -l_{21} = \sin\alpha \end{aligned}$$

from $\tau_{ij} x_i x_j = \tau_{ij}' x_i' x_j'$

$$\tau_{ij} l_{\alpha i} l_{\beta j} x_\alpha' x_\beta' = \tau_{ij}' x_i' x_j' = \tau_{\alpha\beta}' x_\alpha' x_\beta'$$

$$\text{ie } \tau_{\alpha\beta}' = \tau_{ij} l_{\alpha i} l_{\beta j}$$

$$\left.\begin{aligned} \text{or} \quad \tau_{11}' &= \tau_{11}\cos^2\alpha + \tau_{12}\sin 2\alpha + \tau_{22}\sin^2\alpha \\ \tau_{12}' &= \tfrac{1}{2}\sin 2\alpha(\tau_{22} - \tau_{11}) + \tau_{12}\cos 2\alpha \\ \tau_{22}' &= \tau_{11}\sin^2\alpha - \tau_{12}\sin 2\alpha + \tau_{22}\cos^2\alpha \end{aligned}\right\} \quad \text{-- (A)}$$

Which give $\tau_{11}' + \tau_{22}' = \tau_{11} + \tau_{22} = \Theta_1$

Also $\tau_{22}' - \tau_{11}' + 2i\tau_{12}' = (\tau_{22} - \tau_{11} + 2i\tau_{12})e^{2i\alpha}$, a result due to Michell & Kolosov.

Dirn of principal axes is given by putting $\tau_{12}' = 0$ in (A) ie by

$$\tan 2\alpha = 2\tau_{12}/(\tau_{11} - \tau_{22})$$

If the original axes be principal axes (A) is still further simplified & if $\tau_1 = \tau_{11}$; $\tau_2 = \tau_{22}$, $\tau_{12} = 0$

$$\left.\begin{aligned} \tau_{11}' &= \tau_1\cos^2\alpha + \tau_2\sin^2\alpha \\ \tau_{12}' &= \tfrac{1}{2}\sin 2\alpha(\tau_2 - \tau_1) \\ \tau_{22}' &= \tau_1\sin^2\alpha + \tau_2\cos^2\alpha \end{aligned}\right\} \quad \text{-- (B)}$$

Which shows that $|\tau_{12}'|_{max} = \frac{1}{2}|\tau_1 - \tau_2|$ & is attained for $\alpha = \pm 45°$ ie dirns bisecting angle between the pr. dirns.

If τ_1, τ_2 & α be known, to find τ_{11}, τ_{12}, τ_{22}, we can use (B) by simply removing dashes on the L.H.S ~~giving~~ and changing α to $-\alpha$

$$\left.\begin{aligned} \tau_{11} &= \tau_1\cos^2\alpha + \tau_2\sin^2\alpha \\ \tau_{12} &= \tfrac{1}{2}\sin 2\alpha(\tau_1 - \tau_2) \\ \tau_{22} &= \tau_1\sin^2\alpha + \tau_2\cos^2\alpha \end{aligned}\right\} \quad \text{-- (B')}$$

giving $\tau_{11} + \tau_{22} = \tau_1 + \tau_2 = \Theta$ and $\tau_{22} - \tau_{11} + 2i\tau_{12} = -(\tau_1 - \tau_2)e^{-2i\alpha}$

the latter being given by Michell-Kolosov relation, by putting $\tau_{12} = 0$ & $\alpha \to -\alpha$ on its R.H.S & removing dashes on L.H.S.

— Recapitulation of results relating to stress.

<u>11th Lecture</u> — 25/11/61 — Analysis of strain — Rigid bodies & elastic bodies — translation & rotation — To separate out rigid body motions and pure deformations from genl. trn eqn $x_i' = x_i'(x_1, x_2, x_3)$, & inverse $x_i = x_i(x_1', x_2', x_3')$ —

Particular case of linear transformations or ~~part~~ special form of this where separation can be effected easily ie

$$x_i' = \alpha_{i0} + (\delta_{ij} + \alpha_{ij})\,x_j \quad (i = 1,2,3),\ \text{with ~~inverse existing~~ or } x_i' = \alpha_{i0} + x_i + \alpha_{ij}\,x_j \quad \text{—(1)} \quad \text{-- (2)}$$

with the inverse $x_i = \beta_{i0} + (\delta_{ij} + \beta_{ij})\,x_j'$; affine transformations — Affine transformations carries planes into planes & so P_0P to $P_0'P'$ ie $\vec{A} = \overrightarrow{P_0P}$ to $\vec{A'} = \overrightarrow{P_0'P'}$ or $A_i' = A_i + \alpha_{ij}A_j$ or $\delta A_i = \alpha_{ij}A_j$ -- (3)

From (3) equal vectors transform to equal vectors & ‖l vectors to ‖l vectors & equal & similarly oriented polygons to equal & similarly oriented polygons. Hence (1) is called <u>homogeneous deformation</u>. — Resultant of two affine trns

$$A_k'' = A_k' + \gamma_{kj}A_j' = A_k + \alpha_{kl}A_l + \gamma_{kj}(A_j + \alpha_{jl}A_l) = A_k + \alpha_{kl}A_l + \gamma_{kj}A_j \quad (\text{for inf. trn})$$

$$= A_k + (\alpha_{kj} + \gamma_{kj})A_j \quad \text{— Infinitesimal affine transformations} \quad \text{-- (4)}$$

<u>12th Lecture</u> — 16/12/61

Separation of rigid body motions and deformations — $A\,\delta A = A_i\,\delta A_i = A_i(\alpha_{ij}A_j) = \alpha_{ij}A_iA_j$

For rigid body motions vector length A^2 is const ie $A\,\delta A = 0$ ie $\alpha_{ij}A_iA_j = 0$ for all A ie

$$\alpha_{11} = \alpha_{22} = \alpha_{33} = 0\ ;\ \alpha_{12} + \alpha_{21} = \alpha_{23} + \alpha_{32} = \alpha_{31} + \alpha_{13} = 0 \text{ ie } \alpha_{23} = -\alpha_{32} \text{ etc}$$

ie α is a skew symmetric ~~mat~~ tensor say ω_{ij} & $A_i' = A_i + \omega_{ij}A_j$ means

$$A_1' = A_1 + \omega_{11}A_1 + \omega_{12}A_2 + \omega_{13}A_3 = A_1 + \omega_{12}A_2 + \omega_{13}A_3$$

If $\vec{\omega}(\omega_1, \omega_2, \omega_3)$ be rotation vector with $\omega_1 = -\omega_{23}$, $\omega_2 = -\omega_{31}$, $\omega_3 = -\omega_{12}$, this can be written as

$$A_1' = A_1 + (\vec{\omega} \times \vec{A})_1 \quad \text{or} \quad \vec{A'} = \vec{A} + (\vec{\omega} \times \vec{A}) \text{ ie trn + rotation.}$$

Separation of $\alpha_{ij} = \frac{1}{2}(\alpha_{ij} + \alpha_{ji}) + \frac{1}{2}(\alpha_{ij} - \alpha_{ji}) = e_{ij} + \omega_{ij}$

$$\delta A_i = \alpha_{ij}A_j = e_{ij}A_j + \omega_{ij}A_j \text{ , the reqd separation}$$

ie e_{ij} is the strain tensor characterising pure deformation.

<u>Geometrical interpretation of the components of the strain tensor</u>:

(i) $A\,\delta A = e_{ij}A_iA_j$ gives $\dfrac{\delta A}{A} = \dfrac{e_{ij}A_iA_j}{A^2}$. If all components of $e_{ij} = 0$ except e_{11} & vector has components $(A_1, 0, 0)$

$$\frac{\delta A}{A} = e_{11}$$

ie tension or compression per unit length

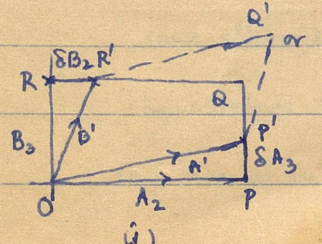
(ii) To interpret e_{23} etc — Consider vectors $\vec{A}_2$ & $\vec{B}_3$ along axes 2 & 3. or $\vec{A}(0, A_2, 0)$, $\vec{B} = (0, 0, B_3)$

Although $A_1 = A_3 = B_1 = B_2 = 0$, their δ's are not equal as seen from $\delta A_i = e_{ij}A_j$.

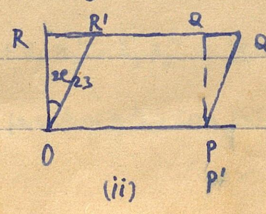
A' is $(\delta A_1, A_2 + \delta A_2, \delta A_3)$ & B' is $(\delta B_1, \delta B_2, B_3 + \delta B_3)$

and $\vec{A'}\cdot\vec{B'} \approx A_2\,\delta B_2 + B_3\,\delta A_3$ and $\cos\theta = \dfrac{\vec{A'}\cdot\vec{B'}}{A'B'} \approx \dfrac{\delta B_2}{B_3} + \dfrac{\delta A_3}{A_2}$.

or $\cos\theta = 2e_{23}$. & if $\delta\theta = \pi/2 - \theta$, $\cos\theta = \sin\delta\theta = \delta\theta = 2e_{23}$ = decrease in rt L^{le}



(i)



(ii)

ie $POP' = ROR' = \dfrac{\delta A_3}{A_2} = e_{23} = \dfrac{\delta B_2}{B_3}$

Hence by rotating □^{m} in (i) thro' angle e_{23}, we get □^{m} in (ii)

[Differce bet. OP' & $OP = \sqrt{A_2^2 + \delta A_3^2} - A_2$ is of second order]

e_{23} represents a shear in the 2-3 plane & a pure shear because in (ii) area of rect = area of □^{m}.

~~Strain quadric of Cauchy~~

13th Lecture – 6/1/62 – $^{(1)}$ Interpretation of e_{23} as shearing component of strain as in last lecture repeated and continued; angle of shear in x_2x_3-plane ∴ of … is $2e_{23}$ & shearing of elements $\parallel^{l}$ to the x_1x_2-plane ∴ $\perp^{l}$ to x_3 – $^{(2)}$ Strain quadric of Cauchy, locus of a point such that extension of $\vec{A} = OP$ is inversely ∴^{l} to square A^2; $\frac{\delta A}{A}$ … $\delta A_i = e_{ij}A_j \rightarrow A\,\delta A = A_i\,\delta A_i = e_{ij}A_iA_j$

$$\frac{\delta A}{A} = e = \frac{e_{ij}A_iA_j}{A^2} \qquad eA^2 = \pm K^2;\ e_{ij}A_iA_j = \pm K^2 \text{ or } e_{ij}x_ix_j = \pm K^2$$

is the eq^{n} of the quadric of Cauchy – Change of axes: $x_i = l_{\alpha i}x'_\alpha \rightarrow x'_i = l_{i\alpha}x_\alpha$

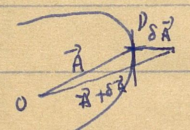
from $l_{i\alpha}l_{j\alpha} = \delta_{ij}$ & $l_{\alpha i}l_{\alpha j} = \delta_{ij}$ & quadric is $e'_{ij}x'_ix'_j = \pm K^2$ and $e_{ij}x_ix_j = e'_{ij}x'_ix'_j$

ie quadratic form is invariant & $e_{ij}l_{\alpha i}l_{\beta j}x'_\alpha x'_\beta = e'_{ij}x'_ix'_j = e'_{\alpha\beta}x'_\alpha x'_\beta$

$$\left.\begin{aligned} \text{or } e'_{\alpha\beta} &= l_{\alpha i}l_{\beta j}e_{ij} \\ \text{\& } e_{\alpha\beta} &= l_{i\alpha}l_{j\beta}e'_{ij} \end{aligned}\right\}$$

ie the tensor law 2G

From $2G(x_1,x_2,x_3) \equiv e_{ij}x_ix_j$, $\frac{\partial G}{\partial x_i} = e_{ij}x_j = \delta A_i$



ie $\delta\vec{A}$ is along normal to surface ie orientation or direction of $\vec{A}$ is changed.

(3) Principal strains. Invariants – If orientation is not changed $\delta\vec{A}$ & $\vec{A}$ have same direction

ie $\delta A_i = \lambda A_i$ & from $A\,\delta A = A_i\,\delta A_i$, $A\,\delta A = \lambda A_iA_i = \lambda A^2$ or $\lambda = \frac{\delta A}{A} = e$

$\delta A_i = eA_i$ ie $e_{ij}A_j = eA_i = e\delta_{ij}A_j$

$(e_{ij} - e\delta_{ij})A_j = 0.$

this has a non-vanishing soln if & only if $|e_{ij} - e\delta_{ij}| = 0$. ie $\begin{vmatrix} e_{11}-e & e_{12} & e_{13} \\ e_{21} & e_{22}-e & e_{23} \\ e_{31} & e_{32} & e_{33}-e \end{vmatrix} = 0$

ie cubic in $\underline{e}$ with roots e_1, e_2, e_3 say.

Roots are real.

14th Lecture – 20/1/62 – (missed Lecture on 13th on account of Annamalai meeting). – Continuation of last lecture from the stage left viz. determination of the principal axes of the strain quadric – mention here the work already done in the case of the stress quadric of Cauchy ~~$|\tau_{ij} - \tau\delta_{ij}| = 0$~~ $\tau_{ij}x_ix_j = NA^2 = \pm K^2$ & the secular eq^{n} $|\tau_{ij} - \tau\delta_{ij}| = 0$ – the principal dirns $\overset{1}{\vec{A}}, \overset{2}{\vec{A}}, \overset{3}{\vec{A}}$ or $\overset{i}{\vec{A}}$ are mutually orthogonal & referred to them eqn reduces to $e_1x_1^2 + e_2x_2^2 + e_3x_3^2 = \pm K^2$ (principal dirns of strain & principal strains), several cases ellipsoid (± 1), hyperboloids, surfaces of revolution, sphere, asymptotic cone to both the hyperboloids $e_1x_1^2 + e_2x_2^2 - e_3x_3^2 = \pm K^2$, viz $e_1x_1^2 + e_2x_2^2 - e_3x_3^2 = 0$, for this $eA^2 = 0$ ie elements whose normals are along generators of this cone experience pure shear ~~etc~~

† In this notation, $e_{xx} = \frac{\partial u}{\partial x}$; $e_{yy} = \frac{\partial v}{\partial y}$, $e_{zz} = \frac{\partial w}{\partial z}$ & $\vartheta = \operatorname{div} u$ //

$$e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad e_{zx} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right); \quad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

and $\omega_{xx} = \omega_{yy} = \omega_{zz} = 0$; $\omega_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)$, . . .

* From $x_i' = l_{i\alpha} x_\alpha$; $x_1' = l_{1\alpha} x_\alpha$ ie direc cosines of x_1' are $l_{1\alpha}$ ie l_{11}, l_{12}, l_{13}.
or simply taking a pure simple extension in dirn (l, m, n) & taking x_1' in this dirn, eqn $x_i' = l_{i\alpha} x_\alpha$ gives $x_1' = l_{1\alpha} x_\alpha'$ ie $l_{11} = l$, $l_{12} = m$, $l_{13} = n$. Then $e_{\alpha\beta} = l_{i\alpha} l_{j\beta} e'_{\alpha\beta}$ with only $e'_{11} = e$ & all other components zero gives $e_{11} = el^2$, $e_{22} = em^2$, $e_{33} = en^2$, $e_{23} = emn$, $e_{31} = enl$, $e_{12} = elm$.

(1) Taken aback - déconcerté
(2) abduct - enlever
(3) about - autour de, au sujet de
(4) abyss - abime
(5) accessible - abordable
(6) account - Compte
(7) accrue from - provenir de
(8) ache - douler, avoir mal
(9) acknowledge - reconnaître
(10) acquaint - faire connaître

1) come - Venir, arriver
(2) go - aller, marcher
(3) sit - etre assis
(4) stand - etre debout
(5) eat - manger, ronger
(6) drink - boire
(7) laugh - rire
(8) cry - crier
(9) wherefrom - où depuis
(10) news - nouvelles.
(11) newspaper - journal
(12) Today - aujourdhui
(13) Tomorrow - Demain
(14) yesterday. - hier
(15) when? - quand, lorsque
(16) where? - où
(17) why? - pourquoi
(18) how? - comment
(19) I can - pouvoir
(20) I must - devoir
(21) I know - savoir
(22) I forget - oublier
(23) I walk - éveiller
(24) meet - rencontre
(25) friend - ami
(26) enemy - ennemi
(27) Mango. - ?
(28) apple. - pomme
(29) wheat - froment
(30) sugar - sucre
(31) milk. - lait, traire
(32) paste. - pâte
(33) pencil - crayon
(34) paper - papier
(35) make - faire
(36) walk - se promener, marcher

(37) early - de bonne heure
(38) late. - lourde (slow)
(39) what is - Qu'est
(40) past - passé
(41) present - present
(42) future - futur
(43) Water - eau
(44) Air - air
(45) Food - nourriture
(46) Book - livre
(47) pen - plume
(48) pencil - Crayon
(49) Ink - encre
(50) Knife - Couteau
(51) Scissors - ciseaux
(52) Lens
(53) Ruler
(54) gum
(55) Razor
(56) Soap.
(57) drop.
(58) Fly
(59) Silk
(60) Sea
(61) ocean
(62) Sun
(63) Moon
(64) Planet
(65) Cloth
(66) Bed
(67) Colour
(68) Red - - Violet

(69) afternoon -
(70) get - obtenir
(71) whenever - toute les fois que
(72) whatever - tout ce que
(73) who - qui
(74) frog - ~~fourette~~ grenouille
(75) now - maintenant
(76) who - qui
(77) without - sans
(78) with - avec
(79) horse - Cheval
(80) mind - esprit - faire attention
(81) murder - meutre
(82) fortnight - quinze jour
(83) weep - pleurer
(84) Evil - Mauvais
(85) never - jamais
(86) always - toujours.
(87) In - dans
(88) what - qu'est-ce que? - ce qui
(89) whale - Baleine
(90) Show - montrer
(91) iron - fer
(92) much obliged - merci beaucoup.
(93) pen - plume
(94) wasp - guêpe
(95) Draw - tirer
(96) Dwarf - nain
(97) last - dernier
(98) Shame - Honte
(99) Sleep - sommeil
(100) acquaintance - connaissance
(101) Draw - tirer
(102) fire - feu
(103) fog - brume
(104) Head - Tête
(105) Lawyer - Avocat
(106) Knee - genou
(107) Short - Court
(108) Shoot - tirer
(109) Scoundrel - Coquin

<u>Invariants of strain</u>; from $|e_{ij} - e\delta_{ij}| = \cancel{-e^3 + \vartheta_1 e^2} = -e^3 + \vartheta_1 e^2 - \vartheta_2 e + \vartheta_3 = 0$, we have

$$\vartheta_1 = e_1 + e_2 + e_3 = e_{11} + e_{22} + e_{33} = \vartheta$$

$$\vartheta_2 = e_2 e_3 + e_3 e_1 + e_1 e_2 = e_{22} e_{33} + e_{33} e_{11} + e_{11} e_{22} - e_{23}^2 - e_{31}^2 - e_{12}^2$$

$$\vartheta_3 = e_{11} e_{22} e_{33} + 2 e_{23} e_{31} e_{32} - e_{11} e_{23}^2 - e_{22} e_{31}^2 - e_{33} e_{12}^2 = e_1 e_2 e_3$$

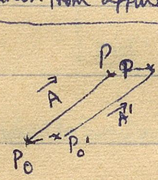
These are called invariants because for change of coord. axes do not alter them in view of geometrical meaning of e_1, e_2, e_3.

<u>Interpretation of $\vartheta_1 = \vartheta$</u> — A rect. $\square^d$ of ~~sides~~ edges with lengths l_1, l_2, l_3 along principal dirns $\rightarrow$ a rect. $\square^d$ of sides in the same dirns, but lengths changed by $l_1 + e_1 l_1 = l_1(1+e_1)$, $l_2(1+e_2)$, $l_3(1+e_3)$

$\therefore$ increase in volume $\delta V = l_1 l_2 l_3 (1+e_1)(1+e_2)(1+e_3) - l_1 l_2 l_3 = l_1 l_2 l_3 (e_1 + e_2 + e_3)$ to 1st order

$= V \cdot \vartheta$ ie $\vartheta = \dfrac{\delta V}{V}$ = Cubical <u>dilatation</u> or dilatation.

<u>Generalisation from affine to a general transformation</u>: ie $x_i' = x_i + u_i(x_1, x_2, x_3)$



$\delta\vec{A} = \vec{A}' - \vec{A} \Rightarrow \delta A_i = (x_i' - x_i^{0\prime}) - (x_i - x_i^0)$

$= (x_i' - x_i) - (x_i^{0\prime} - x_i^0) = u_i(x_1, x_2, x_3) - u_i(x_1^0, x_2^0, x_3^0)$

<u>15th Lecture</u> – <u>22/1/62 (Special class)</u>.

$= u_i(x_1^0 + A_1, x_2^0 + A_2, x_3^0 + A_3) - u_i(x_1^0, x_2^0, x_3^0)$

$= \left(\dfrac{\partial u_i}{\partial x_j}\right)_0 A_j$ + higher terms by Taylor's expansion

Since P_0 is any arbitrary chosen point drop suffix zero & if A be sufficiently small, we can drop higher order terms

and $\delta A_i = u_{i,j} A_j \quad (u_{i,j} = \partial u_i / \partial x_j)$

From here onwards, the procedure is the same as in the case of affine trns ie we introduce ω_{ij} & e_{ij} by $\underbrace{\tfrac{1}{2}(u_{i,j} - u_{j,i})}_{\omega_{ij}}$ & $\underbrace{\tfrac{1}{2}(u_{i,j} + u_{j,i})}_{e_{ij}}$, corresp. to rigid body & elastic displacement. The essential difference however is that the deformation is <u>not homogeneous</u> as in the affine case ie e_{ij} (as well as ω_{ij}) are fns of the point at which it is considered. – dilation $\vartheta = e_{11} + e_{22} + e_{33} = \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \dfrac{\partial u_3}{\partial x_3} = u_{i,i} = \operatorname{div}\vec{u}$ which also varies from point to point – Notation of Cartesian Coords †

<u>Examples of strain</u> (i) <u>Uniform dilatation</u> – strain quadric a sphere $e_{xx}x^2 + e_{yy}y^2 + e_{zz}z^2 = \pm K^2$ a sphere

$[\delta A_i = e A_i]$ with $e_{xx} = e_{yy} = e_{zz} = e$ & $e_{yz} = e_{zx} = e_{xy} = 0$ & $\vartheta = 3e$ or $e = \frac{1}{3}\vartheta$

(ii) <u>Simple extension: of magnitude e in dirn of x'-axis</u>: ie referred to (x', y', z') system strain quadric is $e x'^2 = K^2$ & using $e_{\alpha\beta} = l_{i\alpha} l_{j\beta} e'_{ij}$, where $e'_{11} = e$ & all other components of e'_{ij} are $= 0$ & hence $e_{\alpha\beta} = l_{1\alpha} l_{1\beta} \cdot e$ giving six components expressing simple extension ~~in~~ e in a direction whose direction cosines are $l_{1\alpha}$ (l_{11}, l_{12}, l_{13})*

& these components are $e_{11} = e_{xx} = e l_{11}^2$; $e_{22} = e_{yy} = e \cdot l_{12}^2$, $e_{33} = e_{zz} = e \cdot l_{13}^2$

$e_{23} = e_{32} = e_{yz} = e\, l_{12} l_{13}$, $e_{31} = e_{13} = e_{zx} = e\, l_{11} l_{13}$, $e_{xy} = e\, l_{11} l_{12}$

(iii) <u>Pure shear s</u> in x', y' plane ie $2s\,x'y' = \pm K^2$, Take x & y axes to bisect angles between x' & y' & $z' = z$.

(†) For simple extension:

$$e_{xx} = el^2,\ e_{yy} = em^2,\ e_{zz} = en^2$$

$$e_{xy} = elm,\ e_{yz} = emn,\ e_{zx} = enl.$$

& the quadric is $e_{ij}\, x_i\, x_j = \pm k^2$

$$e_{11} x^2 + \cdots + 2 e_{23} yz + \cdots = \pm k^2$$

$$l x^2 + \cdots + 2mn\, yz + \cdots = \pm k^2/e$$

$$\& \quad \mathscr{I}_1 = e(l^2+m^2+n^2) = e$$

$$\mathscr{I}_2 = e^2(lm + mn + nl) \neq 0 \ (*)$$

$$\mathscr{I}_2 = e^3 lmn. \quad (*)$$

✗ ~~e eqn~~ soln is wrong

BSM
5/8/73

Why? ~~lm+mn+nl = 0 (l, m, n~~ Soln is ~~not wrong~~. (*) These are not the expressions for $\mathscr{I}_2$ & $\mathscr{I}_3$. See Ex. 1. worked out on p. 10.

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The quadric becomes $s(x'^2 - y'^2) = \pm k^2$, ~~...~~ ie x, y, z axes are pr. axes and $e_3 = 0$, $e_1 = -e_2 = s$ ie equal extension in x ~~...~~ and contraction in y dirn constitute the shear.

(iv) Plane strain. given by say, e_1' & e_2' with $e_3' = 0$ ie quadric is $e_1' x'^2 + e_2' y'^2 = \pm k^2$. Transforming into the x-y plane with $z = z'$, the quadric is $e_{xx}x^2 + e_{yy}y^2 + 2e_{xy}xy = \pm k^2$.

Problems, p. 24.

(1) $\vartheta = e_{11} + e_{22} + e_{33} = e(l^2+m^2+n^2) = e$ ie invariant

$$\vartheta_2 = e_{22}e_{33} + \dots + \cdot - e_{23}^2 - \dots = e^2m^2n^2 + \dots + \cdot - e^2m^2n^2 - \dots = 0$$

$$\vartheta_3 = \begin{vmatrix} el^2 & elm & eln \\ eml & em^2 & emn \\ enl & enl & en^2 \end{vmatrix} = elmn \begin{vmatrix} l & m & n \\ l & m & n \\ l & m & n \end{vmatrix} = 0.$$

† (2) ~~dilatation~~ ϑ = dilatation $= e$; The secular eqn is $-\lambda^3 + \vartheta\lambda^2 = 0$ ie $-\lambda^3 + e\lambda^2 = 0$ ie roots are $e, 0, 0$ ie the principal strains are $(e, 0, 0)$ & strain quadric is $ex^2 = \pm k^2$ (ie same as $ex'^2 = \pm k^2$ ie a pair of ||el planes.

(3) (a) $e_{xx} = \frac{\partial u}{\partial x} = e$ ie $u = ex$ & sim $v = ey$, $w = ez$.

(b) $e_{11}' = e$ gives $\frac{\partial u'}{\partial x'} = e$, $u' = ex'$, $v' = w' = 0$.

(c) Here $e_{x'x'} = 0$, $e_{y'y'} = 0$, $e_{x'y'} = 2s$, and obviously $w' = 0$.

ie $\frac{\partial u'}{\partial x'} = 0$, $\frac{\partial v'}{\partial y'} = 0$, and $\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} = 2s$. Hence there are three possibilities

(i) $u' = 2sy'$, $v' = 0 = w'$, (ii) $u' = 0$, $v' = 2sx'$, (iii) $u' = sy'$, $v' = sx'$

(i) is the relation given in the book.

(d) $w = 0$ makes $e_{zz}, e_{zx}, e_{yz} = 0$ & u and v are general fns of x & y.

(4) (a) Components $\omega_{xx}, \omega_{yy}, \omega_{zz} = 0$ in general. We need only consider

$$\omega_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) = 0 \text{ from 3(a) \& sim } \omega_{zx} = \omega_{xy} = 0$$

(b) From 3(b), $\omega'_{xy} = \omega'_{zx} = \omega'_{xy} = 0$.

(c) Corresponding to 3(c) (i) $\omega'_{xy} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial x}\right) = s$, $\omega'_{yz} = \omega'_{zx} = 0$

(ii) $\omega'_{xy} = -s$, $\omega'_{yz} = \omega'_{zx} = 0$

(iii) $\omega'_{xy} = 0$, $\omega'_{yz} = \omega'_{zx} = 0$

(d) $\omega_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$ & $\omega_{yz} = \omega_{zx} = 0$.

Equations of Compatibility — St. Venant's condns in order that ~~...~~ u_i may be determined uniquely as continuous fns from $u_{i,j} + u_{j,i} = 2e_{ij}$ — Six eqns to be satisfied by the e_{ij}

$$\frac{\partial^2 e_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial e_{yz}}{\partial x} + \frac{\partial e_{zx}}{\partial y} + \frac{\partial e_{xy}}{\partial z}\right), \dots, \dots \text{ and } 2\frac{\partial^2 e_{xy}}{\partial x \partial y} = \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2}, \dots$$

Finite deformations – not in Syllabus.

16th Lecture – 27/1/62 – Remaining portions of previous lecture – Revision of results obtained for stress and strain – Hooke's law $T = Ee$) & generalisation $T = f(e)$ – Elasticity & plasticity – yield point – Generalised Hooke's law τ_{ij}

17th Lecture (Special) – 29/1/62 :– Revision of results completed – ~~Hoo~~ other portions left over from last lecture viz: Hooke's law, Young's modulus, Elastic limit, yield point stress, ultimate stress, plasticity illustrated by saw-tooth pattern curve [with $T = F(e)$ not merely $T = Ee$] – Generalised Hooke's Law, Linear theory of elasticity.

$\tau_{ij} = c_{ijkl}\, e_{kl}$ (homogeneous if c_{ijkl} are not f^ns of posn.) – 6 eqns written as $\tau_i = c_{ij}\, e_j$ $(i, j = 1, \ldots 6)$, $|c_{ij}| \neq 0$ &

$e_i = C_{ij}\, \tau_j$ – No. of consts $c_{ij} = 36$ reduces to 21 when strain energy density fn $W = \frac{1}{2} c_{ij}\, e_i e_j$ with $\tau_i = \partial W/\partial e_i$ exists, for then W is symmetric & $c_{ij} = c_{ji}$ – Elastic symmetry for dirns (i) w.r.t a plane, $21 \to 13$, (2) w.r.t. three mutually ⊥r planes, $21 \to 9$ (3) symmetry in all dirns ie isotropic $21 \to 2$. – Elastic symmetry determined by invariance of c_{ij} for change of coord. systems, (since both τ_i & e_j change when coord. systems are changed)

(i) Symmetry w.r.t $x_1 - x_2$ plane ie c_{ij} invariant for $x_1 = x_1'$, $x_2 = x_2'$, $x_3 = -x_3'$

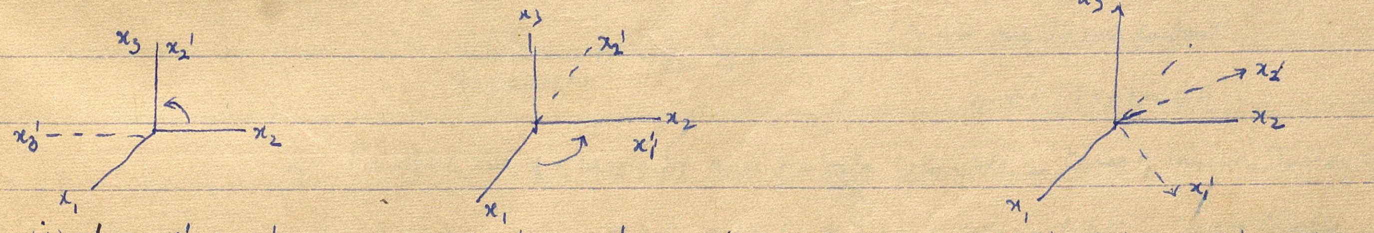
from $e'_{\alpha\beta} = l_{\alpha i}\, l_{\beta j}\, e_{ij}$ and $\tau'_{\alpha\beta} = l_{\alpha i}\, l_{\beta i}\, \tau_{ij}$ & transformation matrix as $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ & using first eqn of $\tau_i = c_{ij}\, e_j$ gives $c_{14} = c_{15} = 0$, & simily $c_{24} = c_{25} = c_{34} = c_{35} = c_{64} = c_{65} = 0$ ie 8 relns only ($c_{ij} = c_{ji}$) & no new reln) reduces 21 to 13. – Form of matrix, p. 63, Sok.

(2) Symmetry w.r.t. called orthotropic media – Besides symmetry w.r.t. plane $x_1 x_2$, use symmetry w.r.t. plane $x_2 \cdot x_3$ also ie trn matrix $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This gives $c_{26} = c_{36} = c_{16} = c_{45} = 0$ still further ie $13 \to 9$ (matrix c_{ij} in S)

No need to consider further symmetry in $x_3 - x_1$ plane since this follows from symmetry w.r.t $x_1 - x_2$ & $x_2 - x_3$ planes.

18th Lecture – 3/2/62 – Portions left behind from the previous lecture ~~(Q)~~ repeated unnecessarily – just started the consideration of the second trn matrix for $x_1' = -x_1$, $x_2' = x_2$, $x_3' = x_3$ – told the class that I would like to handle them on Wednesday the 7th.

19th Lecture – 17/2/62 – Consideration of homogeneous isotropic media – Reduction from orthotropic media to isotropic media – Further reduction of ~~13~~ 12 constants c_{ij} only to two achieved by considering invariance under (i) rotation of 90° about Ox_1, (ii) invariance under rotation of 90° about Ox_3 and (iii) rotation of 45° about Ox_3.



(i) $x_1' = x_1$, $x_2' = x_3$, $x_3' = -x_2$ (ii) $x_1' = x_2$, $x_2' = -x_1$, $x_3' = x_3$ (iii) $x_1' = \frac{1}{\sqrt{2}}(x_1 + x_2)$, $x_2' = \frac{1}{\sqrt{2}}(-x_1 + x_2)$, $x_3' = x_3$

The schemes of transformation are

	x_1	x_2	x_3
x_1'	1	0	0
x_2'	0	0	1
x_3'	0	−1	0

(i)

	x_1	x_2	x_3
x_1'	0	1	0
x_2'	−1	0	0
x_3'	0	0	1

(ii)

	x_1	x_2	x_3
x_1'	$\frac{1}{\sqrt2}$	$\frac{1}{\sqrt2}$	0
x_2'	$-\frac{1}{\sqrt2}$	$\frac{1}{\sqrt2}$	0
x_3'	0	0	1

(iii)

Using $e'_{\alpha\beta} = l_{\alpha i} l_{\beta j} e_{ij}$ and $\tau'_{\alpha\beta} = l_{\alpha i} l_{\beta j} \tau_{ij}$

and $\tau_{11} = \tau_1,\ \tau_{22} = \tau_2,\ \tau_{33} = \tau_3,\ \tau_{23} = \tau_4,\ \tau_{31} = \tau_5,\ \tau_{12} = \tau_6$

$e_{11} = e_1,\ e_{22} = e_2,\ e_{33} = e_3,\ 2e_{23} = e_4,\ 2e_{31} = e_5,\ 2e_{12} = e_6$

and the ~~scheme derive~~ $\tau_i = c_{ij} e_j$ with the ~~rate~~ matrix (c_{ij}) for orthotropic media given by

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}$$

invariance of $\tau_i = c_{ij} e_j$ under (i) gives $c_{12} = c_{13},\ c_{21} = c_{31},\ c_{23} = c_{32},\ c_{22} = c_{33},\ c_{55} = c_{66}$

(ii) gives $c_{11} = c_{22},\ c_{12} = c_{21},\ c_{13} = c_{23},\ c_{31} = c_{32},\ c_{55} = c_{44}$

(iii) gives $c_{44} = \frac{1}{2}(c_{11} - c_{12})$

Calling $c_{44} = \mu$ and $c_{12} = \lambda$, $c_{11} = 2\mu + \lambda$ & (c_{ij}) becomes

$\cancel{2\mu+\lambda \quad \lambda \quad \lambda \quad 0 \quad 0 \quad 0}$

$\cancel{\lambda \quad 2\mu+\lambda \quad \lambda \quad 0}$

$$\begin{pmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}$$

20th Lecture – Derivation of relation (iii) above & the matrix $\longrightarrow$

21/2/62 (Special)

Expn $\tau_{ij} = \lambda \delta_{ij} \vartheta + 2\mu e_{ij}$ .. Hooke's law (generalised)

$\Theta = (3\lambda + 2\mu)\vartheta$

$$e_{ij} = -\frac{\lambda}{2\mu}\delta_{ij}\vartheta + \frac{1}{2\mu}\tau_{ij}$$

$$= \frac{-\lambda\delta_{ij}}{2\mu(3\lambda+2\mu)}\Theta + \frac{1}{2\mu}\tau_{ij}$$

(λ & μ are Lamé's constants

$\mu \neq 0,\ 3\lambda + 2\mu \neq 0$.

– Principal axes of stress & pr. axes of strain coincide. – Cases of simple tension i.e. all comps. except $\tau_{11} = 0$ & introduction of E & σ, exprns for λ & μ in terms of E & σ – Case of simple shear $\tau_{23} \neq 0$ & all else zero, interpreting μ as modulus of rigidity

(*) To show these cond^ns to be necessary, we might point out Ex1. p-78 of Sokolnikoff, where the compatibility cond^ns in the τ_{ij} are not satisfied, although they satisfy eq^m cond^ns with $F_i = 0$. This would mean that the e_{ij} obtained from the τ_{ij} of this Example would be admissible for an equilibrium, but would not be admissible for finding the displacements since the compatibility cond^ns are not satisfied. In fact using $\varepsilon_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\Theta$ and rel^ns for λ & μ in terms of E & σ

we can show that $e_{xx} = cy^2/\lambda$

$$e_{yy} = cx^2/\lambda, \quad e_{zz} = 0, \quad e_{xy} = -\frac{c\sigma xy}{\mu}, \quad e_{yz} = e_{zx} = 0.$$

with these all the five cond^ns of compatibility are satisfied except one viz

$$2\frac{\partial^2 e_{xy}}{\partial x \partial y} = \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2}$$

which leads to $-\frac{2c\sigma}{\mu} = 4c/\lambda$ ie $\frac{\lambda}{\mu} = -2/\sigma$ which is not correct

for $\lambda/\mu = \frac{2\sigma}{1-2\sigma}$ and $\frac{2\sigma}{1-2\sigma} = -\frac{2}{\sigma}$ leads to imaginary values of σ.

23/2/62 – 21st Lecture (Special) – Case of hydrostatic pressure i.e. $\tau_{11} = \tau_{22} = \tau_{33} = -p$, $\tau_{23} = \tau_{31} = \tau_{12} = 0$

Equilibrium eqn on the surface is satisfied i.e. $\overset{\nu}{T}_i = \tau_{ij} \nu_j = -p\delta_{ij}\nu_j = -p\nu_i$

Equilibrium eqn in the interior is also satisfied i.e. $\tau_{ij,j} = 0$ for $(-p\delta_{ij})_{,j} = 0$ since p is constant.–

Finding e_{ij} & $\vartheta = -p/K$ $(K = \lambda + \frac{2}{3}\mu)$ or $K = -p/\vartheta$ = modulus of compression > 0.

K in terms of E & σ i.e. $K = \dfrac{E}{3(1-2\sigma)}$, $K > 0$ gives $\sigma < \frac{1}{2}$, $\sigma = 1/2$ incompressible, $\sigma \cong 1/3$

$$e_{ij} = -\frac{\sigma\delta_{ij}}{E}\Theta + \frac{1+\sigma}{E}\tau_{ij}$$

Examples: (i) Deriving $\Theta = 3K\vartheta$ from above eqn and given some more examples for H.W. – Mention of Problems

24/2/62 – 22nd Lecture – Equations of eqlm – ~~Problem I and II~~ Problems 1 and 2

(A) Problem 2: Putting $\tau_{ij} = \lambda\delta_{ij}\vartheta + \mu(u_{i,j} + u_{j,i})$ in $\tau_{ij,j} + F_i = 0$

we have $\tau_{ij,j} = \lambda\delta_{ij}\dfrac{\partial\vartheta}{\partial x_j} + \mu(u_{i,jj} + u_{j,ij})$ $\left[u_{j,ij} = u_{j,ji} = (u_{j,j})_{,i} = \dfrac{\partial\vartheta}{\partial x_i}\right.$

$$= \lambda\frac{\partial\vartheta}{\partial x_i} + \mu u_{i,jj} + \mu\,\partial\vartheta/\partial x_i$$

$$= \mu u_{i,jj} + (\lambda+\mu)\frac{\partial\vartheta}{\partial x_i}$$

i.e. $\mu\nabla^2 u_i + (\lambda+\mu)\dfrac{\partial\vartheta}{\partial x_i} + F_i = 0$ – Navier's eqn. $\vartheta = e_{ii} = u_{i,i} = \operatorname{div} u$

Solving Navier's eqns satisfying prescribed boundary condns $u_i = f_i(x_1, x_2, x_3)$ on boundary surface give u_i & hence e_{ij} & hence τ_{ij}. – Here compatability eqns not required because we go from u_i to e_{ij}

(B) Problem 1: Here the compatibility condns are necessary, for, we find τ_{ij}, then e_{ij} & from e_{ij} we find u_i. These condns are in the e_{ij} given by

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0 \quad (*) \qquad ---(1)$$

To obtain the compatibility condns in the τ_{ij}, substitute $e_{ij} = \dfrac{1+\sigma}{E}\tau_{ij} - \dfrac{\sigma}{E}\delta_{ij}\Theta$ and simplify. The actual simplification is complicated & may be omitted & only the final result assumed & it is to be noted that since $\tau_{ij,j} = -F_i$, the compatibility eqn will also involve expns like $F_{i,j}$. The final result is

(C) $$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} + \frac{\sigma}{1-\sigma}\delta_{ij}\operatorname{div}\vec{F} + (F_{i,j} + F_{j,i}) = 0 \qquad --(2)$$

called the Beltrami – Michell Compatibility ~~conditions~~ equations:

If the body force be conservative i.e. $\vec{F} = \operatorname{grad}\phi$ i.e. $F_i = \partial\phi/\partial x_i$, $\operatorname{div}\vec{F} = \nabla^2\phi$,

and $F_{i,j} = F_{j,i} = \phi_{,ij}$ and (2) becomes

$$\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} + \frac{\sigma\delta_{ij}}{1+\sigma}\nabla^2\phi + 2\phi_{,ij} = 0 \qquad ---(3)$$

Two particular cases (i) F = const. i.e. ϕ is a linear fn from $F_i = \partial\phi/\partial x_i$ and $\nabla^2\phi = 0$ & $\phi_{,ij} = 0$

$$\therefore \nabla^2\tau_{ij} + \frac{1}{1+\sigma}\Theta_{,ij} = 0 \qquad ---(4) \text{ Beltrami's eqn}$$

which gives $\nabla^2\tau_{ii} + \dfrac{1}{1+\sigma}\Theta_{,ii} = 0$ (This method is different from that of Sokolnikoff.

$$\nabla^2\Theta + \frac{1}{1+\sigma}\nabla^2\Theta = 0, \quad \frac{2+\sigma}{1+\sigma}\nabla^2\Theta = 0 \text{ i.e. } \nabla^2\Theta = 0 \text{ \& hence } \nabla^2\vartheta = 0 \quad -(5)$$

& hence from $\nabla^2\nabla^2\tau_{ij} + \frac{1}{1+\sigma}\nabla^2\Theta_{,ij} = 0$ ~~and~~ second term is $\frac{1}{1+\sigma}(\nabla^2\Theta)_{,ij} = 0$

$\therefore \nabla^4\tau_{ij} = 0$ & from $e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\Theta$, $\nabla^2 e_{ij} = \frac{1+\sigma}{E}\nabla^2\tau_{ij}$, since $\nabla^2\Theta = 0$

$\therefore \nabla^4 e_{ij} = 0$ also with

i.e. $\nabla^4\tau_{ij} = \nabla^4 e_{ij} = 0$ i.e. τ_{ij} & e_{ij} are biharmonic functions -- (6)

Case ii – ~~Φ~~ div $\vec{F} = 0$ or ~~div~~ $\nabla^2\phi = 0$, Again from (3) putting $i = j$ and summing

$$\frac{2+\sigma}{1+\sigma}\nabla^2\Theta = 0 \quad \text{since } \nabla^2\phi = 0 \text{ and } \phi_{,ii} = \nabla^2\phi = 0$$

Hence $\nabla^2\vartheta = 0$ i.e. again $\nabla^2\Theta = \nabla^2\vartheta = 0$ --- (7)

Theorem: If F is constant Θ & ϑ are harmonic, τ_{ij} and e_{ij} are biharmonic and if F is gradient of a harmonic potential function, Θ and ϑ are harmonic.

(D) Dynamical equations of isotropic elastic solid – Use D'Alembert's Principle & replace F_i by $(F_i - \rho\ddot{u}_i)$ in the eqns of equilibrium $\tau_{ij,j} + F_i = 0$ etc –

9/3/62 (Special class). (E) Strain energy function – ~~From~~ physical considerations we can assume that the work done $\mathcal{E}$ by the external forces in altering the configuration of the natural state to the state at time t is not merely equal to the K.E. K as in the case of a rigid body, but equal to K + strain energy U (i.e. energy used in producing the strain). We can assume U stored in the body when it is brought from natural configuration to state at time t i.e. $\mathcal{E} = K + U$. If at time t, there is equilibrium $K = 0$, $\mathcal{E} = U$. If we can now introduce a fn. W such that $U = \int W \cdot d\tau$ i.e. W = strain energy density or elastic potential, we can naturally assume

$W = W(e_1, e_2, \dots e_6)$ in the earlier notation – (8) Green's formula

and demand that a generalised Hooke's law $\boxed{\frac{\partial W}{\partial e_i} = \tau_i}$ be also satisfied. Expanding W in a power series

say $2W = c_0 + 2c_i e_i + c_{ij}e_ie_j + \dots$

and neglecting terms of ~~higher~~ order higher than two & also neglecting c_0 since we are interested in $\frac{\partial W}{\partial e_i} = \tau_i$ we can write to the order of approxn. made $2W = 2c_ie_i + c_{ij}e_ie_j$ or $W = c_ie_i + \frac{1}{2}c_{ij}e_ie_j$

~~gives $\tau_i = c_i + \frac{1}{2}c_{ij}e_ie_j$~~ & if $\tau_i = 0$ when $e_i = 0$, this gives $c_i = 0$

gives $\tau_i = c_i + \frac{1}{2}(c_{ij} + c_{ji})e_j$ i.e. $W = \frac{1}{2}c_{ij}e_ie_j$ leading to $\tau_i = \frac{1}{2}(c_{ij} + c_{ji})e_j$ – (8′)

If it be further assumed that $\tau_i = c_{ij}e_j$ -- (9) ~~Clapeyron's~~ then i.e. that $c_{ij} = c_{ji}$,

then we get $W = \frac{1}{2}\tau_ie_i$ -- (9′) – Clapeyron's formula.

or in the usual notation $(i, j = 1, 2, 3) = \frac{1}{2}\tau_{ij}e_{ij}$

from (9) solving for e_j we get $e_i = C_{ij}\tau_j$ and (9′) yields

$$W = \frac{1}{2}C_{ij}\tau_i\tau_j \quad \text{-- (10)}$$

so that $\frac{\partial W}{\partial \tau_i} = C_{ij}\tau_j = e_i$ – (10′) Castigliano's formula.

In the general case of anisotropic body, W is usually taken in the form (8′) i.e. $W = \frac{1}{2}c_{ij}e_ie_j$ $(c_{ij} = c_{ji})$

(*) $W = \frac{1}{2}\lambda\vartheta^2 + \mu(e_{11}^2 + \cdots + 2e_{23}^2 + \cdots)$

and $\vartheta_1 = \vartheta$ and $\vartheta_2 = e_{22}e_{33} + \cdots - e_{23}^2 - \cdots$

and $e_{11}^2 + \cdots + 2e_{23}^2 + \cdots = (e_{11} + e_{22} + e_{33})^2 - (2e_{22}e_{33} + \cdots - 2e_{23}^2 - \cdots)$

$$= \vartheta_1^2 - 2\vartheta_2$$

$$\therefore W = \left(\tfrac{1}{2}\lambda + \mu\right)\vartheta_1^2 - 2\mu\vartheta_2.$$

Similarly W can be expressed in terms of Θ_1 and Θ_2.

For $W = -\frac{\sigma}{2E}\Theta^2 + \frac{1+\sigma}{2E}(\tau_{11}^2 + \cdots + 2\tau_{23}^2 + \cdots)$

$$= \left(-\frac{\sigma}{2E} + \frac{1+\sigma}{2E}\right)\Theta^2 - \frac{1+\sigma}{E}(\tau_{22}\tau_{33} + \cdots - \tau_{23}^2 - \cdots)$$

$$= \frac{1}{2E}\Theta_1^2 - \frac{1+\sigma}{E}\Theta_2$$ (This is not done in Sokolnikoff)

i.e. instead of the 36 constants c_{ij} we now have only 21 constants. For case of symmetry about one plane we have now 13 instead of 20 constants and for orthotropic media 9 instead of 12 constants. For isotropic case however $c_{ij} = c_{ji}$ holds and there are only two constants.

Properties of W in isotropic case – From $W = \frac{1}{2}\tau_i e_i = \frac{1}{2}\tau_{ij} e_{ij}$ and $\tau_{ij} = \lambda\delta_{ij}\vartheta + 2\mu e_{ij}$

$$W = \frac{1}{2}\lambda\delta_{ij} e_{ij}\vartheta + 2\mu e_{ij} e_{ij} = \frac{1}{2}\lambda\vartheta^2 + \mu(e_{11}^2 + \cdots + 2e_{23}^2 + \cdots) \quad \text{--- (11)}$$

i.e. W is a positive definite quadratic form in the e_{ij} since λ & μ are both >0.

In terms of τ_{ij}, ~~we~~ using $e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\delta_{ij}\Theta$

$$W = -\frac{\sigma}{2E}\Theta^2 + \frac{1+\sigma}{2E}\tau_{ij}\tau_{ij}$$

$$= -\frac{\sigma}{2E}\Theta^2 + \frac{1+\sigma}{2E}(\tau_{11}^2 + \cdots + 2\tau_{23}^2 + \cdots) \quad \text{--- (12)}$$

i.e. W is a positive definite quadratic form in the τ_{ij}'s also, for

$$\frac{1+\sigma}{2E}(\tau_{11}^2+\tau_{22}^2+\tau_{33}^2) - \frac{\sigma}{2E}(\tau_{11}+\tau_{22}+\tau_{33})^2$$

$$= \frac{1}{2E}(\tau_{11}^2+\tau_{22}^2+\tau_{33}^2) - \frac{\sigma}{2E}(2\tau_{22}\tau_{33} + 2\tau_{33}\tau_{11} + 2\tau_{11}\tau_{22})$$

$$> \frac{1}{2E}(\tau_{11}^2+\tau_{22}^2+\tau_{33}^2) - \frac{1}{4E}(\quad\text{''}\quad) \text{ since } \sigma < \tfrac{1}{2}$$

$$> \frac{1}{4E}\{(\tau_{22}-\tau_{33})^2 + (\tau_{33}-\tau_{11})^2 + (\tau_{11}-\tau_{22})^2\} > 0$$

$\sigma < \frac{1}{2}$ follows from $K = \frac{E}{3(1-2\sigma)}$

10/3/62. (This is not shown in Sokolnikoff) // W in terms of invariants ϑ_1 and ϑ_2 *

(F) Superposition and uniqueness of solutions.

The eqns $\tau_{ij,j} = -F_i$, $\tau_{ij} = \lambda\delta_{ij}\vartheta + 2\mu e_{ij}$, $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ being all linear, principle of superposition holds. Thus if $u_i^{(1)}$ & $u_i^{(2)}$ be two solns corresp. to $F_i^{(1)}$ & $F_i^{(2)}$, then $u_i = u_i^{(1)} + u_i^{(2)}$ is a solution corresponding to $F_i = F_i^{(1)} + F_i^{(2)}$ ~ Similarly $\tau_{ij}^{(1)} + \tau_{ij}^{(2)}$ corresponds to $F_i^{(1)} + F_i^{(2)}$

As regards uniqueness of solution, a preliminary theorem has to be proved viz

Clapeyron's theorem: ~~If~~ Work done by F_i and $\overset{\nu}{T}_i$ through displacement u_i from unstretched state to state of equilibrium, ~~then~~ is equal to twice the strain energy of deformation i.e.

$$\int_\tau F_i u_i \cdot d\tau + \int_\Sigma \overset{\nu}{T}_i u_i \cdot d\sigma = 2\int W \cdot d\tau$$

To prove this consider the surface integral & transform it into a volume integral using $\int A_n\, d\sigma = \int \operatorname{div} A \cdot d\tau$

Hence $\int \overset{\nu}{T}_i u_i\, d\sigma = \int (\tau_{ij} u_i)\nu_j\, d\sigma = \int (\tau_{ij,j} u_i + \tau_{ij} u_{i,j})\, d\tau$

$$= \int -F_i u_i\, d\tau + \frac{1}{2}\int (\tau_{ij} u_{i,j} + \tau_{ji} u_{j,i})\, d\tau = \int -F_i u_i\, d\tau + \int \tau_{ij} e_{ij}\, d\tau$$

$$= \int -F_i u_i\, d\tau + 2\int W \cdot d\tau$$ & putting 1st term in the L.H.S. proves the theorem.

To prove uniqueness, suppose two possible solns $u_i^{(1)}$, $\tau_{ij}^{(1)}$ and $u_i^{(2)}$, $\tau_{ij}^{(2)}$ exist, then

$u_i = u_i^{(1)} - u_i^{(2)}$; $\tau_{ij} = \tau_{ij}^{(1)} - \tau_{ij}^{(2)}$ is also a solution & has $F_i = 0$. Hence using Clapeyron's theorem $\int \overset{\nu}{T}_i u_i\, d\sigma = 2\int W\, d\tau$ & since $u_i^{(1)} = u_i^{(2)}$ and $\overset{\nu}{T}_i^{(1)} = \overset{\nu}{T}_i^{(2)}$ since same boundary conditions hold for both the solns, u_i and $\overset{\nu}{T}_i$ are zero for the second and first boundary value problems respectively

Hence for both problems $2\int W\,d\tau = 0$, but W is a $+^{ve}$ definite quadratic form in the e_{ij} and so integral can vanish only when $e_{ij} = 0$ ie $e_{ij}^{(1)} = e_{ij}^{(2)}$ & consequently $\tau_{ij}^{(1)} = \tau_{ij}^{(2)}$ & $u_i^{(1)} = u_i^{(1)}$ everywhere ie solns are identical. Uniqueness of soln depends essentially on the $+^{ve}$ definite character of W.

(G) St. Venant's Principle :- This affords a simplification as regards the form of the boundary conditions which constitutes the greatest difficulty in problems of elasticity. The principle states that ~~of~~ two "statically equivalent" distributions can replace each other on the same portion S_0 of the body without altering the effects of the two distributions on parts of the body sufficiently far removed from S_0 - Statically equivalent means having same resultant & same moment.

12/3/62 (Special class) - Problems of plane stress and strain - (1) Problem of plane strain or plane deformation

Defn by $u_1, u_2 \neq 0$, $u_3 = 0 \longrightarrow e_{13} = e_{23} = e_{33} = 0$ and $\tau_{13} = \tau_{23} = 0$, $\tau_{33} \neq 0$ but $\tau_{33} = \sigma(\tau_{11} + \tau_{22})$ ie problem specified by 5 quantities $\tau_{\alpha\beta}, u_\alpha$ (Greek indices 1 & 2 only) . all fns of x_1 and x_2 only & $F_3 = 0$ and ~~F_α~~ $F_\alpha = F_\alpha(x_1, x_2)$

2nd B.V.P - ie soln of Navier eqns satisfying u_α = given values on boundary. Here Navier eqns viz

$$\mu\nabla^2 u_i + (\lambda+\mu)\frac{\partial \vartheta}{\partial x_i} + F_i = 0 \qquad \left[\vartheta = e_{11} + e_{22},\ \nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2\right.$$

becomes $\mu\nabla^2 u_\alpha + (\lambda+\mu)\,\partial\vartheta/\partial x_\alpha + F_\alpha = 0$

1st B.V.P ie solns of $\tau_{ij,j} = -F_i$ and Beltrami-Mitchell eqns satisfying $\vec{T}_\alpha = \tau_{\alpha\beta}\nu_\beta$ on boundary

~~with $T_3 = 0, \nu_3 = 0$~~ Here $\tau_{ij,j} = -F_i \longmapsto \tau_{\alpha\beta,\beta} = -F_\alpha$

for ~~F_3~~ $-F_3 = \tau_{3j,j}$ gives $0 = \tau_{33,3}$ which is also satisfied.

The B-M. eqns reduce to the single eqn $\nabla^2\Theta_1 = -\frac{1}{1-\sigma}F_{\alpha,\alpha}$ $\quad\left[\Theta_1 = \tau_{11} + \tau_{22}\right.$

$$= -\frac{2(\lambda+\mu)}{\lambda+2\mu}F_{\alpha,\alpha}$$

$\left.\sigma = \frac{\lambda}{2(\lambda+\mu)}\right]$

To show this we have:

$$\left.\begin{aligned} \nabla^2\tau_{23} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x_2 \partial x_3} &= -\left(\frac{\partial F_2}{\partial x_3} + \frac{\partial F_3}{\partial x_2}\right) \\ \text{\&}\quad \nabla^2\tau_{31} + \frac{1}{1+\sigma}\cdot\frac{\partial^2\Theta}{\partial x_3 \partial x_1} &= -\left(\frac{\partial F_3}{\partial x_1} + \frac{\partial F_1}{\partial x_3}\right)\end{aligned}\right\}\ \text{identically zero}$$

and $\nabla^2\tau_{12} + \frac{1}{1+\sigma}\cdot\frac{\partial^2\Theta}{\partial x_1 \partial x_2} = -\left(\frac{\partial F_1}{\partial x_2} + \frac{\partial F_2}{\partial x_1}\right)$ is identically satisfied, for

using $\Theta = (1+\sigma)(\tau_{11}+\tau_{22})$ ~~for~~, this can be written $\cancel{\tau_{12,11}} + \cancel{\tau_{12,22}} + \tau_{11,12} + \cancel{\tau_{22,12}} = \tau_{11,12} + \cancel{\tau_{12,22}} + \cancel{\tau_{21,11}} + \cancel{\tau_{22,21}}$

$$\left.\begin{aligned} \text{and}\quad \nabla^2\tau_{11} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x_1^2} &= -\frac{\sigma}{1-\sigma}\operatorname{div}\vec{F} - 2\,\partial F_1/\partial x_1 \\ \text{and}\quad \nabla^2\tau_{22} + \frac{1}{1+\sigma}\frac{\partial^2\Theta}{\partial x_2^2} &= -\frac{\sigma}{1-\sigma}\operatorname{div}F - 2\,\partial F_2/\partial x_2\end{aligned}\right\}$$

both reduce to $\tau_{11,22} + \tau_{22,11} = -\frac{\sigma}{1-\sigma}\operatorname{div}F + 2\tau_{12,12}$

$$= -\frac{\sigma}{1-\sigma}F_{\alpha,\alpha} - \left(F_{\alpha,\alpha} + \tau_{11,11} + \tau_{22,22}\right)$$

$$\tau_{33} = \lambda \vartheta + 2\mu e_{33} = \lambda(e_{11} + e_{22}) + (\lambda + 2\mu) e_{33}$$ ie $\tau_{33} = 0$ [illegible]

$$e_{33} = -\frac{\lambda}{\lambda+2\mu}(e_{11} + e_{22}) = -\frac{\lambda}{\lambda+2\mu}(u_{1,1} + u_{2,2})$$

$$\therefore \cancel{\Theta =} \; \vartheta = \frac{2\mu}{\lambda+2\mu}(u_{1,1} + u_{2,2}) = \frac{2\mu}{\lambda+2\mu}\bar{\vartheta}$$

or $\nabla^2\theta_1 = -\frac{1}{1-\sigma} F_{\alpha,\alpha}$

and finally the eqn $\nabla^2\tau_{33} + \frac{1}{1+\sigma}\frac{\partial^2\theta}{\partial x_3^2} = -\frac{\sigma}{1-\sigma}\,\mathrm{div}\,\vec{F} - \frac{2\partial F_3}{\partial x_3}$

becomes $\sigma\nabla^2\theta_1 = -\frac{\sigma}{1-\sigma}F_{\alpha,\alpha}$ ie $\nabla^2\theta_1 = -\frac{1}{1-\sigma}F_{\alpha,\alpha}$ ie the same eqn

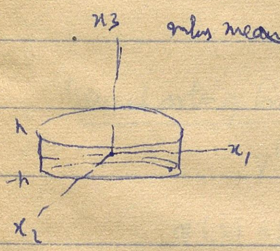
Hence the 2nd B.V.P is solved by finding $\tau_{\alpha\beta}$ and u_α ~~from~~ $\cancel{\tau_{33}}$ satisfying $\tau_{\alpha\beta,\beta} = -F_\alpha$ & the B.M. eqn and the boundary condn $T_\alpha = \tau_{\alpha\beta}\nu_\beta$.

The fact that $\tau_{33} \neq 0$ has the physical significance that for a ~~large~~ long cylinder strain of any section ⊥r to length may lead to a u_3 and to make this zero, a τ_{33} may be necessary.

(ii) <u>Plane stress</u>:- Defined by $\tau_{13} = \tau_{23} = \tau_{33} = 0$ (Cf. $e_{13} = e_{23} = e_{33}$ in case of plane strain), and other components are fns of x_1 & x_2 only; ~~Then~~ $\tau_{33} = \lambda\vartheta + 2\mu e_{33} \rightarrow \lambda\vartheta + 2\mu e_{33} = 0$ ie

$$e_{33} = -\frac{\lambda}{\lambda+2\mu}(e_{11}+e_{12}) \text{ ie } e_{33} \neq 0 \text{ just like } \tau_{33} \neq 0 \text{ in case (i)}$$

but this means $u_{3,3} \neq 0$ ie $u_3 \neq 0$ and also a fn of x_3 ie the problem is not really two dimensional. But we can make it two dimensional by using the case of a cylinder whose length is small cf. with the dimensions of the cross-section ie the opposite of case (i), ie really a plate of small thickness, ~~with no loads~~ say $2h$, with plane faces free of loads ie all external forces acting on lateral surface of the plate. One can introduce average ~~values~~ mean ~~values~~ u_i $\overline{u}_\alpha(x_1,x_2) = \frac{1}{2h}\int_{-h}^{h} u_i(x_1,x_2,x_3)\,dx_3$; $\overline{\tau}_{\alpha\beta}(x_1,x_2) = \frac{1}{2h}\int_{-h}^{h}\tau_{\alpha\beta}(x_1,x_2,x_3)\,dx_3$ & similarly $\overline{F}_\alpha$



and from the nature of the loads assumed $\overline{u}_3 = 0$ ($F_3 = 0$) and $\overline{\tau}_{33} = 0$.

Working with these average values in the case of a thin, one has the case of what is called <u>generalised plane stress</u> & is strictly two dimensional.

Even <u>formally</u> ie in the mathematical formulation we can get rid of e_{33} and u_3

$$\text{for, } \vartheta = \frac{2\mu}{\lambda+2\mu}(u_{1,1}+u_{2,2}) = \frac{2\mu}{\lambda+2\mu}\overline{\vartheta}, \text{ say}$$

$$\tau_{\alpha\beta} = \frac{2\lambda\mu}{\lambda+2\mu}\overline{\vartheta}\,\delta_{\alpha\beta} + \mu(u_{\alpha,\beta}+u_{\beta,\alpha})$$

So putting $\overline{\lambda} = \frac{2\lambda\mu}{\lambda+2\mu}$, using $\overline{\vartheta}$, $\overline{\tau}_{\alpha\beta} = \overline{\lambda}\delta_{\alpha\beta}\overline{\vartheta} + \mu(\overline{u}_{\alpha,\beta}+\overline{u}_{\beta,\alpha})$

<u>16/3/62</u> <u>(Special class)</u>. For generalised plane stress, we have, therefore

$$\overline{\tau}_{\alpha\beta} = \overline{\lambda}\delta_{\alpha\beta}\overline{\vartheta} + \mu(\overline{u}_{\alpha,\beta}+\overline{u}_{\beta,\alpha}), \quad \overline{\lambda} = 2\mu\lambda/(\lambda+2\mu)$$

$$\overline{\tau}_{\alpha\beta,\beta} + \overline{F}_\alpha = 0.$$

leading to Navier's eqn $\mu\nabla^2\overline{u}_\alpha + (\overline{\lambda}+\mu)\frac{\partial\overline{\vartheta}}{\partial x_\alpha} + \overline{F}_\alpha = 0$

Also the B-M eqn $\rightarrow$ $\nabla^2\overline{\theta}_1 = -\frac{2(\overline{\lambda}+\mu)}{(\overline{\lambda}+2\mu)}\overline{F}_{\alpha,\alpha}$

Boundary condn becomes $\overline{\tau}_{\alpha\beta}\nu_\beta = \overline{T}_\alpha(s)$ on C (C being the contour) ie both the plane deformation & generalised plane stress problems are identical and lead to a single type of

<u>plane-elastostatic problems</u> – These problems can be reduced to the case where body forces are absent, for

if $\tau^{(0)}_{\alpha\beta}$ be a set satisfying $\tau_{\alpha\beta,\beta} + F_\alpha = 0$, then putting $\tau_{\alpha\beta} = \tau^{(0)}_{\alpha\beta} + \tau^{(1)}_{\alpha\beta}$, then $\tau^{(1)}_{\alpha\beta,\beta} = 0$ for eg. consider

case of gravity force $F_1 = 0$, $F_2 = -g\rho$, the eqns $\tau_{\alpha\beta} = \lambda\vartheta\delta_{\alpha\beta} + \mu(u_{\alpha,\beta} + u_{\beta,\alpha})$ and Naviers eqn can be

shown to be satisfied by $\tau^{(0)}_{11} = \tau^{(0)}_{12} = 0$; $\tau^{(0)}_{22} = g\rho x_2$

$$u_1^{(0)} = -\frac{\lambda}{4\mu(\lambda+\mu)} g\rho x_1 x_2, \quad u_2^{(0)} = \frac{\lambda+2\mu}{8\mu(\lambda+\mu)} g\rho x_2^2 + \frac{\lambda}{8\mu(\lambda+\mu)} g\rho x_1^2 .$$

The new boundary condns will be $\tau^{(1)}_{\alpha\beta}\nu_\beta = T_\alpha^{(1)}$, $u_\alpha^{(1)} = f_\alpha^{(1)}$ on C.

17/3/62 (Special class).

<u>Airy's stress function</u> – $\tau_{\alpha\beta,\beta} = 0$ identically satisfied by $\tau_{22} = U_{,11}$, $\tau_{12} = -U_{,12}$, $\tau_{11} = U_{,22}$ [$U = U(x_1, x_2)$

& the B.M. eqns $\nabla^2(\tau_{11} + \tau_{22}) = 0 \rightarrow \nabla^4 U = 0$ .ie $U_{,1111} + 2U_{,1122} + U_{,2222} = 0$.

Choice of U also restricted by boundary condn which can be modified by using $\nu_1 = dx_2/ds$, $\nu_2 = -dx_1/ds$

ie boundary value problem is
$$\left.\begin{aligned} \nabla^4 U &= 0 \text{ in } R \\ U_{,\alpha} &= f_\alpha(s) \text{ on } C. \end{aligned}\right\}$$

or alternative
$$\left.\begin{aligned} &\nabla^4 U = 0 \text{ in } R \\ &\left.\begin{aligned} U &= f(s) + \text{Const} \\ \frac{dU}{d\nu} &= g(s) \end{aligned}\right\} \text{on } C. \end{aligned}\right]$$

<u>General solu of biharmonic equation</u> – Use of complex variables. From $\nabla^2\nabla^2 U = 0$ in R, putting $\nabla^2 U = P_1(x_1, x_2)$

$\nabla^2 P_1 = 0$ ie P_1 is harmonic and we can write $F(z) = P_1 + iP_2$ (P_2 is conjugate of P_1)

$\varphi(z) = \frac{1}{4}\int F(z)\,dz = p_1 + ip_2$; $\varphi'(z) = \frac{1}{4}(P_1 + iP_2)$, $p_{1,1} = p_{2,2} = \frac{1}{4}P_1$; $p_{1,2} = -p_{2,1} = -\frac{1}{4}P_2$

$$\nabla^2(U - p_1x_1 - p_2x_2) \equiv 0 \text{ in } R$$

ie $U = p_1x_1 + p_2x_2 + q_1(x_1, x_2)$, q_1 being harmonic in R

If $\chi(z) = q_1 + iq_2$, $U = \mathcal{R}[\bar{z}\varphi(z) + \chi(z)]$

or $2U = \bar{z}\varphi(z) + z\overline{\varphi(z)} + \chi(z) + \overline{\chi(z)}$.

<u>Formulas for stress & displacement</u>:

6/8/73 at Bangalore

Sokolnikoff – Ex. 4, p. 71.

<u>Defns</u>: $\sigma = \frac{\lambda}{2(\lambda+\mu)}$, $E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$; $k = \lambda + \frac{2}{3\mu}$.