Review for Zentralblatt.

358 An elementary proof of one of Ramanujan's simple conjectures in number theory that ~~the eq~~ $2^{n+2} - 7 = \square$ has just five solutions given by $n = 1, 2, 3, 5, 13$, has been given here based on the expressions for the convergents of the continued fraction for $\sqrt{2}$. ~~Since x is an odd~~ Putting $\square = (2y+1)^2$, ~~and~~ the conjecture is reduced to the form $1 + \Delta_y = 2^{n-1}$ (*). ~~by putting $x = 2y+1$.~~ ~~as is made~~ The proof is based on a result due to ~~A.~~ Boutin (Dickson's History, vol 2, p. 31) that the relation $1 + \Delta_y = x^2$ ~~(**)~~ can be reduced to the Pellian equation $p^2 - 2q^2 = -1$ by setting $2x = 3q \mp p$ (**), $y = \frac{1}{2}(k-1)$; $k = 3p \mp 2q$. The solution of the Pellian equation is then carried out, by using the classical method, obtaining p and q in terms of the odd convergents of the continued fraction. ~~for $\sqrt{2}$~~. ~~The A detailed exa~~ A detailed examination of the two sets of values of x ~~given by~~ in (**) thus ~~obtained~~ derived shows that there are only ~~two~~ four such values ~~of~~ of x which are powers of 2, viz $x = 1, 2, 2^2, 2^6$, thereby giving the four odd values 1, 3, 5, 13 of n for which (*) holds. A further elementary reasoning is given to show that the only even value of n for which (*) holds is given by $n = 2$, thus completing the proof of the conjecture.

In order to show how apparently different results in elementary number theory are inter-related, three known results relating to the expressions of Fermat and Mersenne numbers as Δ-numbers are derived from Ramanujan's conjecture. ~~Pro Proof of (2) on p. 44 448 448 of the paper might perhaps have been made more explicit.~~

Unfortunately, a number of misprints have occurred in the paper. Thus on p. 447 the last value of n given in the third line should be 13 instead of 12. The expression 2^{m+1} in line 7, p. 448 should be $2^m + 1$. The page numbers 355 and 553 in the Reference at bottom of p. 449, and in the first Reference on p. 450 should both be 535. Finally ~~in~~ Reference 2 at the end of the paper ~~should be~~ is to Dickson instead of Dickens!

For eg. (i) p. 447, ~~third line~~ line 3, for last value of n ~~in the~~ read 13 ~~instead of~~ for 12.;

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(ii) p. 448, line 7, read $2^m + 1$ for 2^{m+1};

(iii) p. 449, ~~last reference~~ Ref. at bottom, & p. 450, Ref. 1 ~~(2)~~ should both read p. 535

(iv) p. 450, Ref. 2, read Dickson for Dickens.

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$$\phi_n(x) = \frac{1}{\{2^n \cdot n! \, \pi^{\frac{1}{2}}\}^{1/2}} e^{-x^2/2} H_n(x).$$

~~Plotting~~ of $\phi_0(x)$

In Russell's tables $\phi_n(x)$ is merely $e^{-x^2/2} H_n(x)$ i.e. without any normalising factor. This makes the use of Russell's tables for higher values of n rather doubtful. Also in view of errata pointed out on p. 71 of Math. Aids to Computation Vol 13 (1950) No. 65-68, p. 71 given by the three Sextons viz. that $\phi_n(2.50)$ and $\phi_n(6.60)$ are suspect for $n > 3$ and results for $3 < n < 7$ are suspect, I wonder how Russell's tables can be relied upon. Anyway let us try

(1) $\phi_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2} \cdot H_0(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2}$. $\phi_0^2 = \frac{1}{\sqrt{\pi}} \cdot e^{-x^2} H_0^2(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$.

x	ϕ_0^2
.00	$1/\sqrt{\pi}$
0.04	

But fig. in Durkman shows $\phi_0^2(0) > 1$ (?)

Take classical case $P = \frac{1}{\pi \sqrt{2n+1-x^2}}$

gives for $n = 0$, $P = \frac{1}{\pi\sqrt{1-x^2}} = \frac{1}{\pi}$ for $x = 0$

i.e. $P(0) < \phi_0^2(0)$ all right

but the above $\phi_0^2(0) > 1$ is still a riddle

For $x = 0.04$, $\phi_0^2 = \frac{(.99920)^2}{\sqrt{\pi}}$

Pauling & Wilson gives $N_n^2 = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}} \cdot \frac{1}{2^n \cdot n!}$ which is $\frac{1}{N_n^2}$ of Durkman

$$\phi_0^2(0) = \frac{\phi_0^2}{\left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}} \quad \frac{\sqrt{2}}{\sqrt{\pi}}$$

$$2^{n+2} - 7 = (2y+1)^2$$

$$2^{n+2} - 7 = 4y^2 + 4y + 1$$

$$2^{n+2} = 8 + 4y(y+1)$$

$$2^{n-1} = 1 + \tfrac{1}{2} y(y+1)$$

$$1 + \Delta_y = 2^{m}$$

Review of my article for Zentralblatt

An elementary proof of Ramanujan's conjecture that the equation

$$2^{n+2} - 7 = x^2 \qquad \cancel{(1)}$$

has just five solutions given by $n = 1, 2, 3, 5, 12$ ~~with the corresponding values of~~ ~~$x = 1, 3, 5, 11, 181$ respectively~~ is ~~presented~~ proved here in an elementary way, ~~by using use of properties~~ based on the expressions for ~~based only on~~ the convergents of the continued fraction for $\sqrt{2}$. Since x is obviously odd, (1) ~~can be~~ is reduced to the form

$$1 + \Delta_y = 2^{n-1} \; (*) \qquad \cancel{(2)}$$

by putting $x = 2y + 1$, ~~where~~ Δ_y ~~is the triangular number $\frac{1}{2}y(y+1)$~~ being. ~~Next,~~ ~~Use is~~ made of a remark due to A. Boutin quoted on p. 31, ~~of Dickson~~ Vol 2, of Dickson's History ~~of the theory of numbers~~

that the relation

$$1 + \Delta_y = x^2 \quad (*) \qquad \cancel{(3)}$$

can be reduced to the Pellian equation

$$p^2 - 2q^2 = -1 \qquad \cancel{(4)}$$

by setting $2x = 3q \mp p$; $y = \frac{1}{2}(K-1)$; $K = 3p \mp 2q$ $\quad (**) \qquad \cancel{(5)}$

and (ii) ~~the well known result~~ that the ~~solutions of (4) are~~ given by the odd convergents ~~of the continued frac~~ ~~(p_{2n+1}, q_{2n+1}) $(n = 0, 1, 2, \ldots)$ of the continued fraction for $\sqrt{2}$~~ [P.T.O]

~~$$\sqrt{2} = 1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2+}\cdots \qquad (6)$$~~

~~of which the general convergents are given by~~

~~$$q_n = (\alpha^n - \beta^n)/(\alpha - \beta); \; p_n = \frac{1}{2}(\alpha^n + \beta^n) \ldots (7)$$~~

~~where with $\alpha = 1 + \sqrt{2}, \beta = 1 - \sqrt{2}$.~~

By a detailed examination of the two cases ~~two sets of~~ expressions for ~~of obtaining~~ x ~~values for x in $(*)$ expressed~~ thus ~~obtained~~ in terms of ~~the expansions of the~~ these odd convergents, it is shown that there are only four values of $x = 1, 2^1, 2^2, 2^6$, of $(**)$ giving the four odd values of n for which $(*)$ holds. A further elementary reasoning ~~is~~ is given to show that the only even value n for which $(*)$ holds is given by $n = 2$, thus completing the proof of the conjecture.

~~Since $(*)$ involves~~ In order to show ~~results~~ many how apparently ~~of an apparently~~ different ~~nature~~ results in elementary number theory are ~~related~~ interrelated, three known ~~properties of prime~~ results relating to expr. of Fermat and Mersenne numbers as Δ-numbers are derived from Ramanujan's conjecture. Proof of eqn (2) given on p. 3 four might ~~have~~ perhaps have been made more explicit. ~~There is a~~ There are ~~two~~ three misprints, one on line 7, p. 2 where 2^{m+1} should be $2^m + 1$, and another in ~~the~~ References 2 at the end of the paper, where Dickens should be Dickson. Also in Ref. 1, page no. should be 535 instead of 553

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2^{14}

soln of the
The Pellian equation ~~is not then solved~~ by the classical method ~~which gives~~ using the continued fraction for $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}$, ~~which gives~~ gives

$2^{n+2} - 7 = \square$

$n = 1 \rightarrow 8 - 7 = 1 = 1^2$

$n = 2 \rightarrow 16 - 7 = 9 = 3^2$

$n = 3 \rightarrow 32 - 7 = 25 = 5^2$

$n = 5 \rightarrow 128 - 7 = 121 = 11^2$

$n = 12 \rightarrow 16384 - 7 = 16377$

$$
\begin{array}{r}
181 \times 181 \\
\hline
181 \\
1448 \\
181 \\
\hline
32761
\end{array}
$$

$32761 + 7 = 32768$

$32768 = 4 \times 8192$

16384

$2^{14} = 2^5 \times 2^5 \times 2^4$

$= 32 \times 32 \times 16$

$= 1024 \times 16$

$= 16384$

6 $\quad 2^{15}$