A SIMPLE RESULT IN QUADRATURE

To

The Editor of "NATURE"

Dear Sir,

In calculating the intensity of light scattered by a transparent homogeneous medium one comes across the infinite series[1)]

$$\alpha\sum_{n=-\infty}^{+\infty}\frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

γ

in which θ is a constant and ~~n an integer.~~ α is a positive number, which under the conditions under which light-scattering is generally studied is very small, and hence the sum of the above series is usually replaced by the corresponding integral

$$\int_{-\infty}^{+\infty}\frac{\sin^2 x}{x^2}\,dx, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

which evidently is equal to π.

It can be shown,[1)] however, that the equality of (1) and (2) holds not only in the limit when $\alpha\to 0$, but for any value of α in the range $0<\alpha\leqslant\pi$. An obvious interpretation of this result is that the area subtended between the curve $y = \sin^2 x/x^2$ and the x-axis can be obtained by taking the sum of the ordinates at equal intervals α, $0<\alpha\leqslant\pi$, and multiplying by α, i.e. by simple rectangulation, just as well as by integration. In erecting these ordinates at equal intervals we may start from any value θ of x.

The main purpose of this note is to draw attention to this property, namely

$$\alpha\sum_{n=-\infty}^{+\infty} f(n\alpha+\theta) = \alpha\sum_{n=-\infty}^{+\infty} f(n\alpha) = \int_{-\infty}^{+\infty} f(x)\,dx, \quad \ldots\ldots\ldots (3)$$

where $0<\alpha\leqslant$ ~~a certain constant~~ α_0, which characterizes $f(x) = \sin^2 x/x^2$, and to show that numerous other functions can be constructed that have this property. A method of evaluating (1) given us by Professor Wiener, and quoted in the paper referred to,[1)] suggests the criterion by which to construct such functions.

α_0 being a certain constant

Consider an even function $F(x)$ $[= F(-x)]$ whose Fourier transform

$$g(v) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{+\infty} F(x)\, e^{ivx}\,dx \quad \ldots\ldots\ldots\ldots (4)$$

has non-zero values when $|v|$ is less than a certain constant

1) Bhatia, A.B., and Krishnan K.S., Proc. Roy. Soc. A 192 181 (1947)

~~then~~ α_0 is a certain constant

v_0 , and is zero otherwise. Then by Poisson's summation formula[2]

$$\alpha \sum_{n=-\infty}^{+\infty} F(n\alpha + \theta) = \surd(2\pi) \sum_{N=-\infty}^{+\infty} e^{-i2\pi N\theta/\alpha} . \; g(2\pi N/\alpha), \;\ldots\ldots(5)$$

where n ~~and N are integers,~~ and $\alpha > 0$. If now α ~~is chosen to be~~ $\leq 2\pi/v_0$, then all the terms on the right hand side of (5) except the one corresponding to $N = 0$ vanish, and we obtain

$$\alpha \sum_{n=-\infty}^{+\infty} F(n\alpha + \theta) = \surd(2\pi) g(0),$$

which can be seen from (4) to be equal to $\int_{-\infty}^{+\infty} F(x)dx$, and hence the function F satisfies (3).

Since the Fourier integrals of a large number of functions have been tabulated by Campbell and Foster[3] and by others, it is easy to select or construct, examples of functions F whose Fourier transforms satisfy the above criterion, and which satisfy (3). $F(x) = \sin^m x/x^n$, where m and n are positive integers, $m \geqslant n$, both of them odd or both of them even, is one such function, with[4] $v_0 = m$. A few other examples are given below.

$F(x)$	$\surd(2\pi).g(v)$	v_0
$\dfrac{\sin\left[a(x^2+\lambda^2)^{\frac{1}{2}}\right]}{(x^2+\lambda^2)^{\frac{1}{2}}}$	$\pi J_0\left[\lambda(a^2-v^2)^{\frac{1}{2}}\right]$	a
$\cos\left[a(x^2+\lambda^2)^{\frac{1}{2}}\right] - \cos ax$	$-\dfrac{\pi a\lambda J_1\left[\lambda(a^2-v^2)^{\frac{1}{2}}\right]}{(a^2-v^2)^{\frac{1}{2}}}$	a
$\dfrac{1}{\Gamma(\alpha+x)\Gamma(\beta-x)}$	$\dfrac{(2\cos\frac{1}{2}v)^{\alpha+\beta-2} . e^{-\frac{1}{2}iv(\alpha-\beta)}}{\Gamma(\alpha+\beta-1)}$	π [5]

A more detailed account will be published shortly in the Journal of the Indian Mathematical Society.

Yours truly,

National Physical Laboratory of India
Delhi, June 6, 1948.

(K.S.Krishnan)

2. See E.C.Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford, 1937, p.60.
3. Bell Telph. Sys. Tech. Publns. Monograph B - 584(1931).
4. When m=1, the range of v over which $g(v) \neq 0$ includes $v = 1$.
5. The last example is from Ramanujan, Collected Papers p.216.

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We trust you will be interested in the attached cutting from our issue for 7 AUG 1948

The correction of the numerator in the last entry of column 2 of table published in ~~Aug~~ Sept 25th issue p. 485.

A Simple Result in Quadrature

In calculating the intensity of light scattered by a transparent homogeneous medium, one comes across the infinite series[1]

$$\alpha \sum_{n=-\infty}^{+\infty} \frac{\sin^2 (n\alpha + \theta)}{(n\alpha + \theta)^2}, \qquad (1)$$

in which θ is a constant and n an integer. α is a positive number, which under the conditions under which light-scattering is generally studied is very small, and hence the sum of the above series is usually replaced by the corresponding integral

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\, dx, \qquad (2)$$

which evidently is equal to π.

It can be shown[1], however, that the equality of (1) and (2) holds not only in the limit when $\alpha \to 0$, but for any value of α in the range $0 < \alpha \leqslant \pi$. An obvious interpretation of this result is that the area subtended between the curve $y = \sin^2 x/x^2$ and the x-axis can be obtained by taking the sum of the ordinates at equal intervals α, $0 < \alpha \leqslant \pi$, and multiplying by α, that is, by simple rectangulation, just as well as by integration. In erecting these ordinates at equal intervals, we may start from any value θ of x.

The main purpose of this note is to direct attention to this property, namely,

$$\alpha \sum_{n=-\infty}^{+\infty} f(n\alpha + \theta) = \alpha \sum_{n=-\infty}^{+\infty} f(n\alpha) = \int_{-\infty}^{+\infty} f(x)\, dx, \qquad (3)$$

where $0 < \alpha \leqslant$ a certain constant α_0, which characterizes $f(x) = \sin^2 x/x^2$, and to show that numerous other functions can be constructed that have this property. A method of evaluating (1) given by Prof. Wiener, and quoted in the paper referred to[1], suggests the criterion by which to construct such functions.

Consider an even function $F(x)$ [$= F(-x)$] the Fourier transform of which,

$$g(v) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{+\infty} F(x) \exp\,(ivx)dx, \qquad (4)$$

has non-zero values when $|v|$ is less than a certain constant v_0, and is zero otherwise. Then by Poisson's summation formula[2]

$$\alpha \sum_{n=-\infty}^{+\infty} F(n\alpha + \theta) = \sqrt{(2\pi)} \sum_{N=-\infty}^{+\infty} \exp\,[-i2\pi N\theta/\alpha] . g(2\pi N/\alpha), \qquad (5)$$

where n and N are integers and $\alpha > 0$. If now α is chosen to be $\leqslant 2\pi/v_0$, then all the terms on the right-hand side of (5) except the one corresponding to $N = 0$ vanish, and we obtain

$$\alpha \sum_{n=-\infty}^{+\infty} F(n\alpha + \theta) = \sqrt{(2\pi)}\, g\,(0),$$

which can be seen from (4) to be equal to $\int_{-\infty}^{+\infty} F(x)\, dx$; and hence the function F satisfies (3).

Since the Fourier integrals of a large number of functions have been tabulated by Campbell and Foster[3] and by others, it is easy to select, or construct, examples of functions F the Fourier transforms of which satisfy the above criterion, and which satisfy (3). $F(x) = \sin^m x/x^n$, where m and n are positive integers, $m \geqslant n$, both of them odd or both of them even, is one such function with $v_0 = m$; when $m = 1$, the range of v over which $g(v) \neq 0$ includes $v = 1$. A few other examples are given below.

$F(x)$	$\sqrt{(2\pi)} . g(v)$	v_0
$\dfrac{\sin [a(x^2 + \lambda^2)^{1/2}]}{(x^2 + \lambda^2)^{1/2}}$	$\pi J_0 [\lambda(a^2 - v^2)^{1/2}]$	a
$\cos [a(x^2 + \lambda^2)^{1/2}] - \cos ax$	$-\dfrac{\pi a\lambda J_1 [\lambda(a^2 - v^2)^{1/2}]}{(a^2 - v^2)^{1/2}}$	a
$\dfrac{1}{\Gamma(a + x)\, \Gamma(\beta - x)}$	$\dfrac{(2 \cos 1/2v)^{\alpha + \beta - 2} \{-\frac{1}{2} \exp iv(\alpha - \beta)\}}{\Gamma(\alpha + \beta - 1)}$	π[4]

A more detailed account will be published shortly in the *Journal of the Indian Mathematical Society*.

K. S. Krishnan

National Physical Laboratory of India, Delhi. June 6.

[1] Bhatia, A. B., and Krishnan, K. S., *Proc. Roy. Soc.*, A, **192**, 181 (1947).

[2] See Titchmarsh, "Introduction to the Theory of Fourier Integrals", 60 (Oxford, 1937).

[3] Bell Tel. Sys. Tech. Pub. Monograph B584 (1931).

[4] From Ramanujan, "Collected Papers", 216.

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$$\alpha \sum_{n=-\infty}^{+\infty} F(n\alpha + \theta) = \sqrt{(2\pi)} \sum_{N=-\infty}^{+\infty} \exp[-i2\pi N\theta/\alpha].\, g(2\pi N/\alpha), \quad (5)$$

where n and N are integers and $\alpha > 0$. If now α is chosen to be $\leqslant 2\pi/v_0$, then all the terms on the right-hand side of (5) except the one corresponding to $N = 0$ vanish, and we obtain

$$\alpha \sum_{n=-\infty}^{+\infty} F(n\alpha + \theta) = \sqrt{(2\pi)}\, g(0),$$

which can be seen from (4) to be equal to $\int^{+\infty} F(x)\, dx$.

o, densities ... kind may afford an indication for linking more intimately than has been done so far the mechanism of the various physical periodicities with the periodic system of the electrons constituting the shells and subshells of the atom.

G. HERDAN

C/o Paint Research Station,
Teddington, Middlesex.
June 6.

[1] Kendall, M. G., "The Advanced Theory of Statistics", vol. 2 (1946).

X-Ray Scattering by Thermal Vibrations in Crystals

SIR C. V. RAMAN, in his letter on "X-Rays and the Eigen-Vibrations of Crystal Structure"[1], claims that the elastic vibrations cannot "give rise to any geometric diffraction pattern exhibiting an observable relationship to the structure of the crystal". The reasons given for this statement make it appear that he has fallen into Debye's original error, and supposes that the atomic vibrations are essentially independent, or are markedly dependent on the form of the crystal and on external boundary conditions. This would perhaps be the case for very small crystallites; for crystals of ordinary size the boundary conditions would involve second-order effects only, as they do in the case of Bragg reflexion. It should also be clearly understood that the change of frequency due to the X-ray 'Doppler effect' is so small that the modified wave-lengths still lie well within the width of the monochromatized $K\alpha_1$ line, assuming that to be the incident radiation; the change is of the order of 1 in 10^5.

That the elastic vibrations do indeed give rise to a geometric diffraction pattern has been most conclusively proved by the excellent agreement between theory and experiment for metals such as sodium, lead and tungsten. The theory of eigen-vibrations can offer no explanation whatever of these experimental results, whereas the elastic vibration theory predicts them correctly in every detail[2].

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July 15.

[1] Raman, C. V., *Nature*, 162, 23 (1948).

[2] Jahn, H. A., *Proc. Roy. Soc.*, A, 179, 320 (1942). Lonsdale, K., and Smith, H., *Nature*, 148, 628 (1941); 149, 21 (1942). Lonsdale, K., *Proc. Phys. Soc.*, 54, 314 (1942).