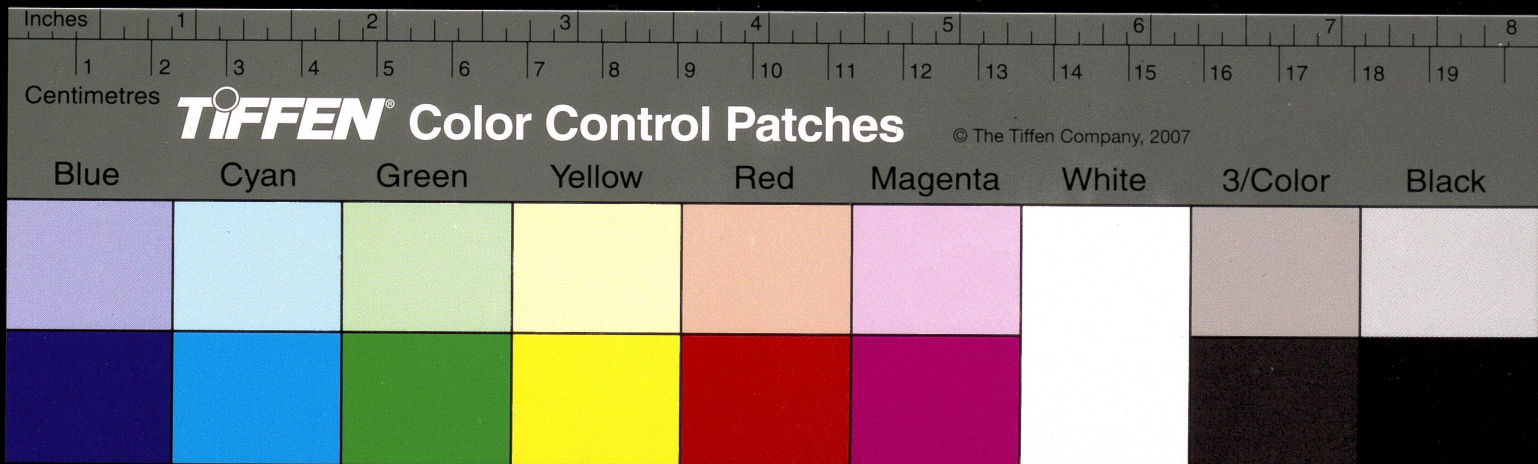


GENERALISED ACTION-FUNCTIONS IN  
BORN'S ELECTRO-DYNAMICS.

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B. S. MADHAVA RAO.



Reprinted from "The Proceedings of the Indian Academy of Sciences",  
Vol. VI, No. 3, Sec. A, 1937.

## GENERALISED ACTION-FUNCTIONS IN BORN'S ELECTRO-DYNAMICS.

BY B. S. MADHAVA RAO.

(University of Mysore.)

(From the Department of Physics, Indian Institute of Science, Bangalore.)

Received August 19, 1937.

### 1. Introduction.

INFELD has shown<sup>1</sup> that, in the development of Born's field theory, it is possible to choose an infinity of action-functions other than the one originally proposed by Born and Infeld<sup>2</sup> which, for simplicity, might be called Born's action-function. These action-functions of Infeld and the one recently introduced by Hoffmann and Infeld<sup>3</sup> have, in common with Born's action-function, the properties of self-conjugacy, of the existence of simple, algebraic relations between the  $f_{kl}$ - and  $p_{kl}$ -fields, and giving finite values for the energy for the electrical particle. They, however, differ from the latter in that the condition of the invariance of the action-integral itself is not insisted in them. As I have shown elsewhere<sup>4</sup> Born's function is unique if this condition as well as the condition of self-conjugacy are imposed on the action-function.

Laying aside for the present this condition of invariance of the action-integral, we might say that the action-functions of Infeld and Hoffmann and Infeld give rise to several systems of Born's non-linear electro-dynamics. These action-functions, however, are all confined to the case where the invariant  $G = (\vec{B} \vec{E})$  is neglected, as also the invariants  $Q = (\vec{D} \vec{H})$ , and  $S = (\vec{B} \vec{D}) + (\vec{E} \vec{H})$ . I have considered in this paper the general case where these invariants are not neglected and constructed a two-fold infinity of action-functions all having the same properties as Infeld's action-functions and reducing to them when  $G = 0$ . This generalisation has also enabled the deduction of close connections with the complex formalism developed by Weiss.<sup>5</sup> In view of the fact however, that the condition of "finite self-energy" has so far been discussed in the case of the point-singularity and

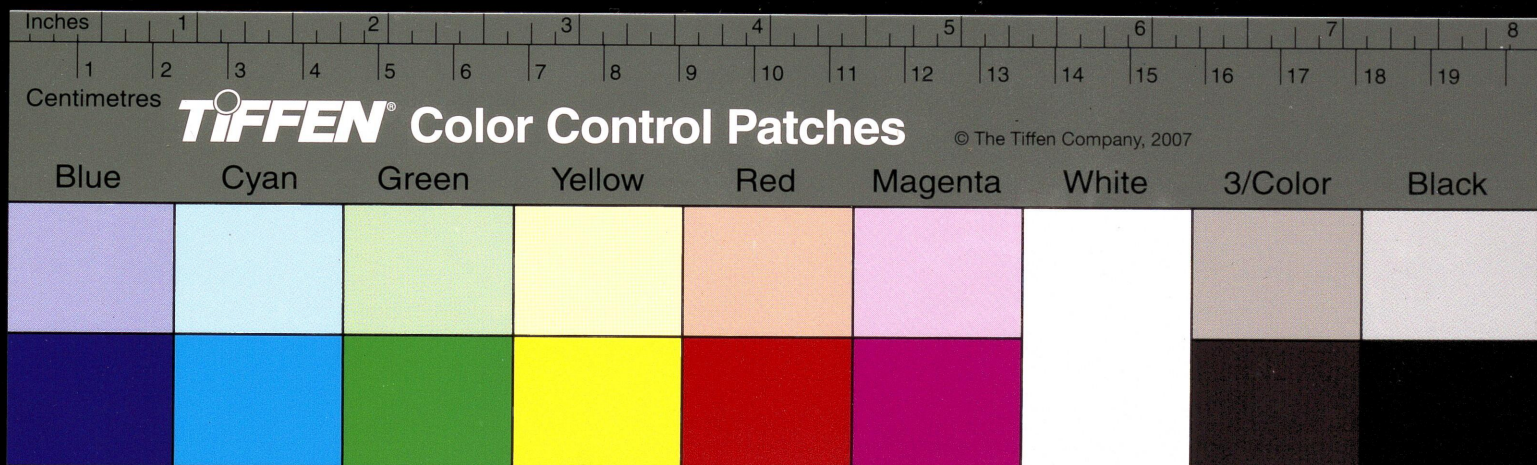
<sup>1</sup> Infeld, *Proc. Camb. Phil. Soc.*, 1936, 32, 127 ; 1937, 33, 70—hereinafter referred to as I and II.

<sup>2</sup> Born and Infeld, *Proc. Roy. Soc., (A)*, 1934, 144, 425.

<sup>3</sup> Hoffmann and Infeld, *Phys. Rev.*, 1937, 51, 765—hereinafter referred to as III.

<sup>4</sup> B. S. Madhava Rao, *Proc. Ind. Acad. Sci., (A)*, 1936, 3, 377.

<sup>5</sup> P. Weiss, *Proc. Camb. Phil. Soc.*, 1937, 33, 79—hereinafter referred to as IV.



the case of ring-singularity in the zero-approximation<sup>6</sup> where, in both cases, the invariant G does not appear, it seems difficult to set up a criterion limiting the choice of these action-functions. Nor is the regularity condition of Hoffmann and Infeld of any use since it is based on the character of a spherically symmetric electro-static solution.

An attempt has been made in this paper to construct an action-function which leads to a full coincidence of the second order terms when the Lagrangian of the field theory is compared with the Lagrangian that arises in the investigations of Euler and Köckel<sup>7</sup> on the phenomenon of scattering of light by light on the basis of Dirac's theory of holes. It is well known that such a coincidence does not exist<sup>8</sup> in the terms containing G<sup>2</sup> when Born's action-function is used. The result that I obtain is that even this condition does not restrict the choice of the action-function there being more than one such function wherefrom a coincidence with the Euler-Köckel Lagrangian could be obtained as well as the good determination of the fine-structure constant possible with the Hoffmann-Infeld action-function.

I have closely followed the method adopted by Infeld in I and II.

2. Condition of Self-conjugacy.

We introduce the  $f_{kl}$  and  $p^{kl}$  (or in the dual form  $f^{*kl}$  and  $p_{kl}^*$ ) tensors describing the electromagnetic field, the space-vector notation and also the several invariants in the usual manner,<sup>9</sup> with the modification introduced by Weiss,<sup>10</sup> which leads to a good lot of simplification by avoiding unnecessary numerical factors. The Lorentz frame of reference is used.

$$\left. \begin{aligned} (f_{23}, f_{31}, f_{12}) &= \vec{B}, & (f_{14}, f_{24}, f_{34}) &= \vec{E} \\ (p_{23}, p_{31}, p_{12}) &= \vec{H}, & (p_{14}, p_{24}, p_{34}) &= \vec{D} \end{aligned} \right\} \quad (2, 1)$$

$$\left. \begin{aligned} \frac{1}{4} f_{kl} f^{kl} &= -\frac{1}{4} f^{*kl} f_{kl}^* = F, & \frac{1}{4} f_{kl} f^{*kl} &= \frac{1}{4} f_{kl}^* f^{kl} = G, \\ \frac{1}{4} p_{kl}^* p^{*kl} &= -\frac{1}{4} p_{kl} p^{kl} = P, & \frac{1}{4} p_{kl}^* p^{kl} &= \frac{1}{4} p_{kl} p^{*kl} = Q, \\ \frac{1}{4} f_{kl} p^{kl} &= -\frac{1}{4} f^{*kl} p_{kl}^* = R, & \frac{1}{4} f_{kl} p^{*kl} &= \frac{1}{4} f^{*kl} p_{kl} = S \end{aligned} \right\} \quad (2, 2)$$

$$\left. \begin{aligned} F &= \frac{1}{2} (\vec{B}^2 - \vec{E}^2), & G &= (\vec{B} \vec{E}) \\ P &= \frac{1}{2} (\vec{D}^2 - \vec{H}^2), & Q &= (\vec{D} \vec{H}) \\ R &= \frac{1}{2} \{(\vec{B} \vec{H}) - (\vec{D} \vec{E})\}, & S &= \frac{1}{2} \{(\vec{B} \vec{D}) + (\vec{E} \vec{H})\} \end{aligned} \right\} \quad (2, 3)$$

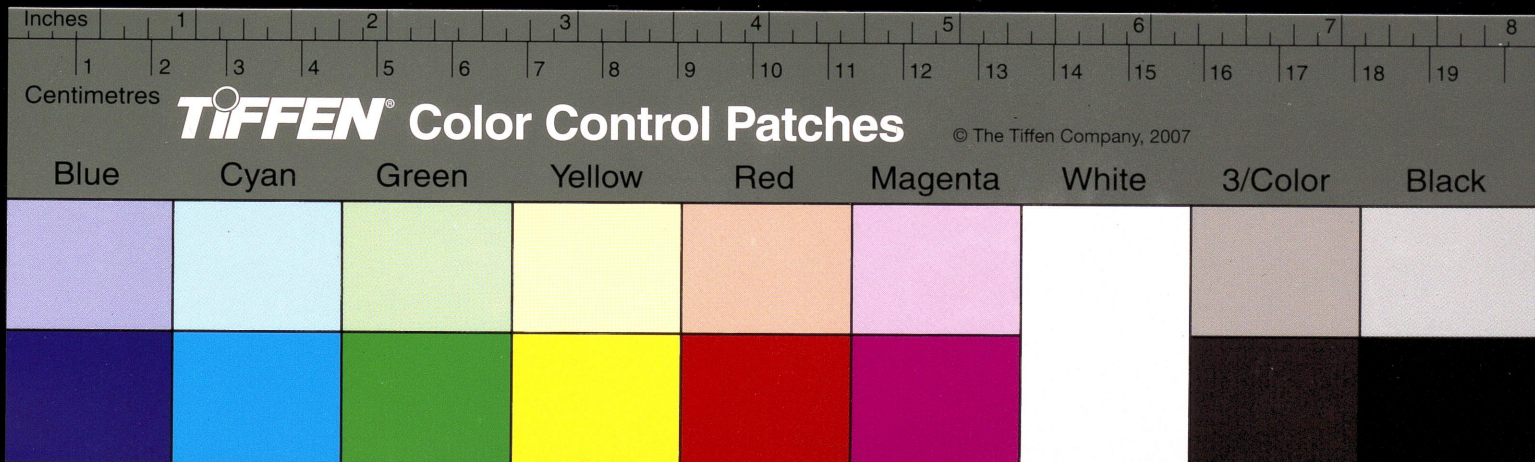
<sup>6</sup> B. S. Madhava Rao, *Proc. Ind. Acad. Sci.*, (A), 1936, 4, 355.

<sup>7</sup> Euler and Köckel, *Naturwiss.*, 1935, 23, 246.

<sup>8</sup> M. Born, *Proc. Ind. Acad. Sci.*, (A), 1935, 2, 560.

<sup>9</sup> B. S. Madhava Rao, *Proc. Ind. Acad. Sci.*, (A), 1936, 4, 578, §3—hereafter referred to as V.

<sup>10</sup> IV, foot-note p. 80.



We impose on our action-function  $T$ , which we assume to be a function of  $F, G, P$  and  $Q$ , that it should be self-conjugate. This means that the set of equations

$$p^{kl} = \frac{\delta T}{\delta f_{kl}} \quad (2, 4)$$

$$f^{*kl} = \frac{\delta T}{\delta p_{kl}^*} \quad (2, 5)$$

should be self-consistent, *i.e.*, that (2, 5) is a consequence of (2, 4), or *vice versa*. We can rewrite (2, 4) and (2, 5) in the form

$$p^{kl} = T_F f^{kl} + T_G f^{*kl} \quad (2, 6)$$

$$f^{*kl} = T_P p^{*kl} + T_Q p^{kl} \quad (2, 7)$$

where  $T_F, T_P, T_G, T_Q$  denote the derivatives with respect to  $F, P, G, Q$  respectively. We shall now find the restriction to be imposed on  $T$  to ensure the consistency of (2, 6) and (2, 7). We multiply (2, 6) respectively by  $\frac{1}{4}f_{kl}, \frac{1}{4}f_{kl}^*, \frac{1}{4}p_{kl}, \frac{1}{4}p_{kl}^*$  and sum up; similarly (2, 7) respectively by  $\frac{1}{4}p_{kl}^*, \frac{1}{4}p_{kl}, \frac{1}{4}f_{kl}^*, \frac{1}{4}f_{kl}$  and sum up. We get<sup>11</sup> after suitably grouping the terms

$$R = FT_F + GT_G \quad (2, 8)$$

$$-R = PT_P + QT_Q \quad (2, 9)$$

$$S = GT_F - FT_G \quad (2, 10)$$

$$S = QT_P - PT_Q \quad (2, 11)$$

$$-P = RT_F + ST_G \quad (2, 12)$$

$$F = RT_P - ST_Q \quad (2, 13)$$

$$Q = ST_F - RT_G \quad (2, 14)$$

$$G = ST_P + RT_Q \quad (2, 15)$$

The equations (2, 9), (2, 11), (2, 13) and (2, 15) must follow as consequences from (2, 8), (2, 10), (2, 12) and (2, 14) respectively.

Before proceeding to reduce the above number of eight equations we will directly obtain two conditions which follow from the self-consistency of (2, 6) and (2, 7). Taking the dual of (2, 6) we get

$$p^{*kl} = T_F f^{*kl} + T_G f^{kl} = T_F f^{*kl} - T_G f^{kl};$$

Substituting this value of  $p^{*kl}$  and the value of  $p^{kl}$  from (2, 6) in (2, 7) we should get an identical equation, *i.e.*,

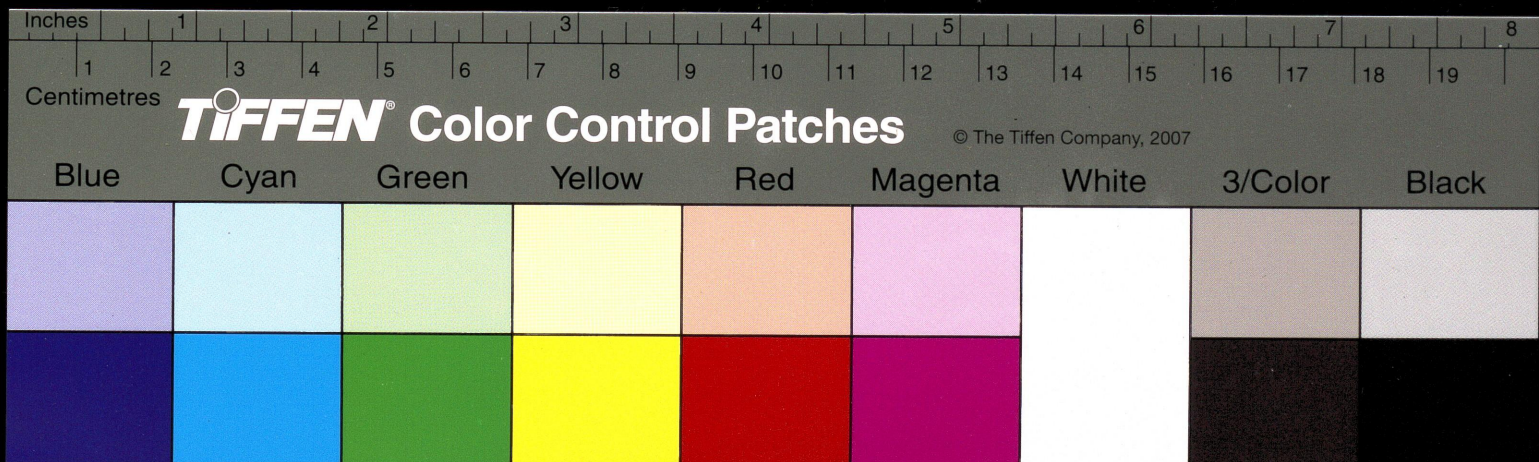
$$f^{*kl} = T_P (T_F f^{*kl} - T_G f^{kl}) + T_Q (T_F f^{kl} + T_G f^{*kl})$$

identically. This leads to the consistency conditions

$$T_F T_P + T_G T_Q = 1 \quad (2, 16)$$

$$T_F T_Q - T_P T_G = 0 \quad (2, 17)$$

<sup>11</sup> As in V, p. 579, where we replace both  $L$  and  $H$  by  $T$  and introduce the necessary numerical factors to correspond to Weiss's notation.





## 3. Solution of the System of Partial Differential Equations.

In the usual notation of the theory of partial differential equations we can write (2, 20)–(2, 21) as

$$F_1 \equiv x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 = 0 \quad (3, 1)$$

$$F_2 \equiv x_2 p_1 - x_1 p_2 - x_4 p_3 + x_3 p_4 = 0 \quad (3, 2)$$

and proceed to find out the most general functional form of the solution common to (3, 1)–(3, 2).

It is easy to verify that the Poisson bracket  $(F_1, F_2) = 0$ , so that the equations form a complete Jacobian system as they are. The most general solution of (3, 1) is any arbitrary function of  $(x_2/x_1, x_3/x_1, x_4/x_1)$ . Adopting the usual method we introduce new variables given by

$$x_1 = x_1, u = x_2/x_1, v = x_3/x_1, w = x_4/x_1.$$

With these substitutions, equation (3, 1), *i.e.*,

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} + x_4 \frac{\partial f}{\partial x_4} = 0$$

reduces to  $\frac{\partial f}{\partial x_1} = 0$ ; and the second equation (3, 2), *i.e.*,

$$x_2 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial x_2} - x_4 \frac{\partial f}{\partial x_3} + x_3 \frac{\partial f}{\partial x_4} = 0 \text{ reduces to}$$

$$x_2 \left( \frac{\partial f}{\partial x_1} - \frac{u}{x_1} f_u - \frac{v}{x_1} f_v - \frac{w}{x_1} f_w \right) - x_1 \left( \frac{1}{x_1} f_u \right) - x_4 \left( \frac{1}{x_1} f_v \right) + x_3 \left( \frac{1}{x_1} f_w \right) = 0,$$

$$\text{i.e.,} \quad -u^2 f_u - uv f_v - wf w - f_u - w f_v + v f_w = 0$$

$$\text{or} \quad f_u (1 + u^2) + f_v (uv + w) + f_w (uw - v) = 0 \quad (3, 3)$$

The Lagrangian subsidiary equations of (3, 3) are

$$\frac{du}{1+u^2} = \frac{dv}{uv+w} = \frac{dw}{uw-v} \quad (3, 4)$$

and we need only find two independent integrals of (3, 4).

From (3, 4) we can immediately deduce the relations

$$\frac{udu}{1+u^2} = \frac{vdv+wdw}{v^2+w^2}, \text{ and}$$

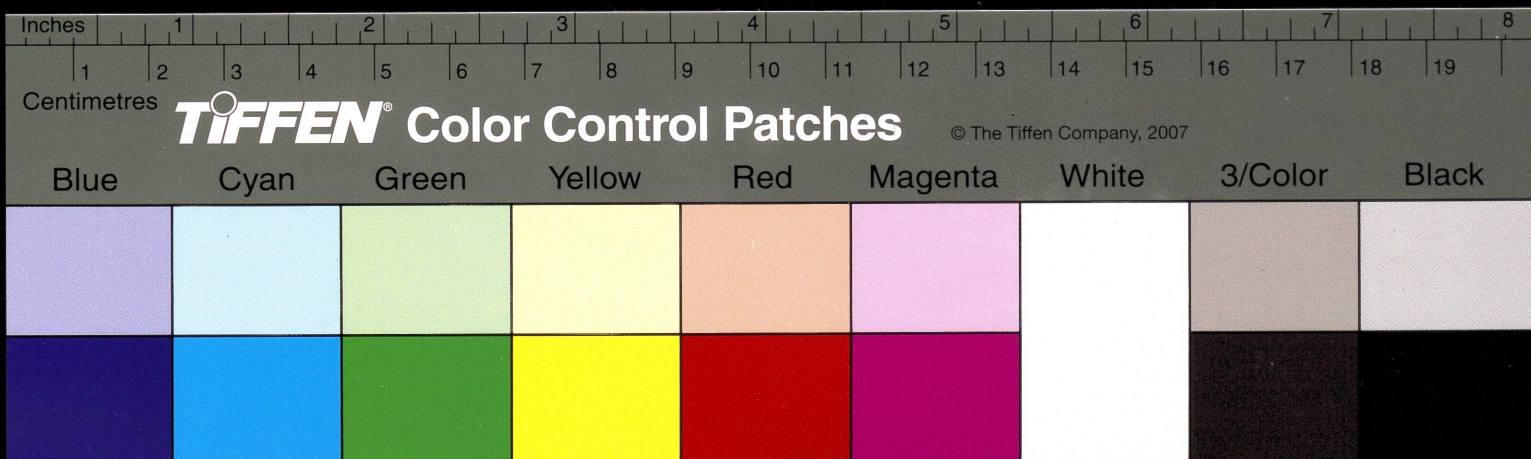
$$\frac{du}{1+u^2} = \frac{wdv-vdw}{v^2+w^2}.$$

These lead to the two integrals

$$\log(1+u^2) = \log(v^2+w^2) + \text{const.},$$

and  $\tan^{-1} u - \tan^{-1} \left( \frac{v}{w} \right) = \text{const.}$ , respectively

or,  $\frac{1+u^2}{v^2+w^2} = \text{const.}$ , and  $\frac{wu-v}{uv+w} = \text{const.}$



For reasons to be immediately given, we write,

$$\epsilon = \sqrt{\frac{1+v^2}{v^2+w^2}}; \quad \epsilon' = \frac{wv+w}{wm-v} \quad \left(\text{instead of } \frac{wm-v}{wv+w}\right)$$

or, introducing the invariants F, G, P, Q

$$\epsilon = \sqrt{\frac{F^2+G^2}{P^2+Q^2}}; \quad \epsilon' = \frac{FQ+PG}{FP-GQ} \quad (3, 5)$$

The most general form of the solution common to (3, 1), (3, 2) is any arbitrary function T ( $\epsilon, \epsilon'$ ) of  $\epsilon$  and  $\epsilon'$ . If T be chosen in this form equations (2, 20)–(2, 21) are identically satisfied.

The reason for choosing  $\epsilon$  and  $\epsilon'$  in the forms given by (3, 5) is that we want to have these quantities reduce themselves to the corresponding ones when we go over to Infeld's case where G and Q are neglected. If  $G = Q = 0$  then  $\epsilon' = 0$ , and  $\epsilon$  reduces to  $\pm (F/P)$ . We therefore further postulate that the negative sign is to be taken in order that our value of  $\epsilon$  should be the square of the corresponding  $\epsilon$  used in Infeld's investigations. We may also observe here that in the limiting Maxwellian case where  $G = Q$  and  $F = -P$ , we have  $\epsilon = 1$  and  $\epsilon' = 0$ .

Thus our generalisation has introduced the additional parameter  $\epsilon'$  and gives rise to a two-fold infinity of action-functions satisfying the condition of self-conjugacy.

4. T as the Sum of Lagrangian and Hamiltonian.

We will show that, in our general case, if (2, 4)–(2, 5) are satisfied the action-function T can be represented as the sum of a Lagrangian and a Hamiltonian

$$\text{Let } \left. \begin{aligned} L(F, G) &= \frac{1}{2} T + R \\ H(P, Q) &= \frac{1}{2} T - R \end{aligned} \right\} \quad (4, 1)$$

$$(4, 2)$$

where we have put the numerical factor  $\frac{1}{2}$  on the right-hand side for T in consonance with the notation of Weiss. From (4, 1)

$$2 dL = T_F dF + T_G dG + T_P dP + T_Q dQ + 2 dR.$$

From the relations (2, 18)–(2, 19)

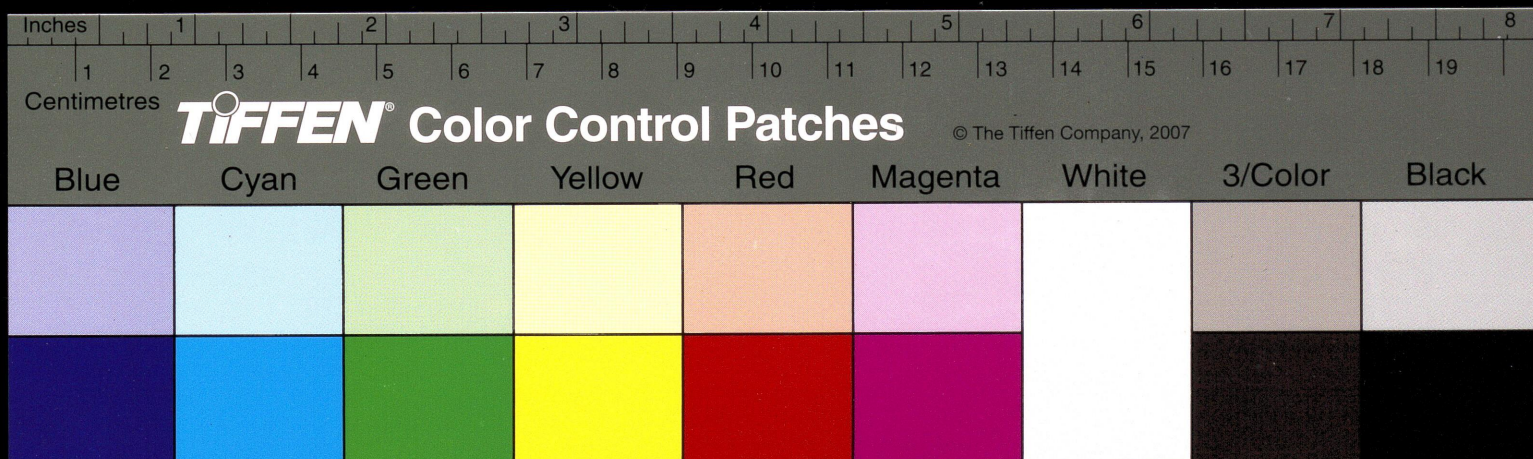
$$2 S dS - 2 R dR = F dP + P dF + G dQ + Q dG,$$

$$2 S dR + 2 R dS = F dQ + Q dF - G dP - P dG.$$

Eliminating dS from the above equations,

$$2 dR (R^2 + S^2) = dF (QS - PR) - dG (PS + QR) - dP (GS + FR) + dQ (FS - RG). \quad (4, 3)$$

We can simplify (4, 3) by obtaining the expressions for the derivatives of T in terms of the invariants. Directly solving (2, 8)–(2, 11) for these





(5, 3) and (5, 4) follow from (5, 1) and (5, 2) by interchanging F and G with P and Q, and thus determine P and Q as the same functions of F and G and conversely, guaranteeing in this way the self-consistency of (2, 4) and (2, 5).

The equations (5, 1)–(5, 4) express the derivatives  $T_F, T_G, T_P, T_Q$  in terms of F, G, P and Q. Since the action-functions we are considering can be taken as arbitrary functions of  $\epsilon$  and  $\epsilon'$ , we must be able to obtain  $T_\epsilon$  and  $T_{\epsilon'}$ , in terms of the invariants, in order that we might get for every special system of electrodynamics particular relations connecting F, G, P, Q. We proceed to determine these expressions for  $T_\epsilon$  and  $T_{\epsilon'}$ .

We can express  $T_F$  and  $T_G$  in terms of  $T_\epsilon$  and  $T_{\epsilon'}$  in the form

$$\left. \begin{aligned} T_F &= T_\epsilon \cdot \epsilon_F + T_{\epsilon'} \cdot \epsilon'_F \\ T_G &= T_\epsilon \cdot \epsilon_G + T_{\epsilon'} \cdot \epsilon'_G \end{aligned} \right\}$$

where  $\epsilon_F$ , for example, stands for  $\partial\epsilon/\partial F$ . Solving these for  $T_\epsilon$  and  $T_{\epsilon'}$ ,

$$\left. \begin{aligned} T_\epsilon (\epsilon_F \epsilon'_G - \epsilon_G \epsilon'_F) &= \epsilon'_G T_F - \epsilon'_F T_G \\ -T_{\epsilon'} (\epsilon_F \epsilon'_G - \epsilon_G \epsilon'_F) &= \epsilon_G T_F - \epsilon_F T_G \end{aligned} \right\}$$

From (3, 5),

$$\left. \begin{aligned} \epsilon_F &= \frac{1}{\epsilon} \cdot \frac{F}{P^2 + Q^2}; & \epsilon'_F &= -\frac{G(P^2 + Q^2)}{(FP - GQ)^2} \\ \epsilon_G &= \frac{1}{\epsilon} \cdot \frac{G}{P^2 + Q^2}; & \epsilon'_G &= \frac{F(P^2 + Q^2)}{(FP - GQ)^2} \end{aligned} \right\}$$

Substituting these in the above, and simplifying, we get

$$\left. \begin{aligned} \epsilon T_\epsilon &= FT_F + GT_G \\ -(1 + \epsilon'^2) T_{\epsilon'} &= GT_F - FT_G \end{aligned} \right\}$$

Using (2, 8) and (2, 10) these can be simplified to the important equations

$$\epsilon T_\epsilon = R \tag{5, 5}$$

$$(1 + \epsilon'^2) T_{\epsilon'} = -S \tag{5, 6}$$

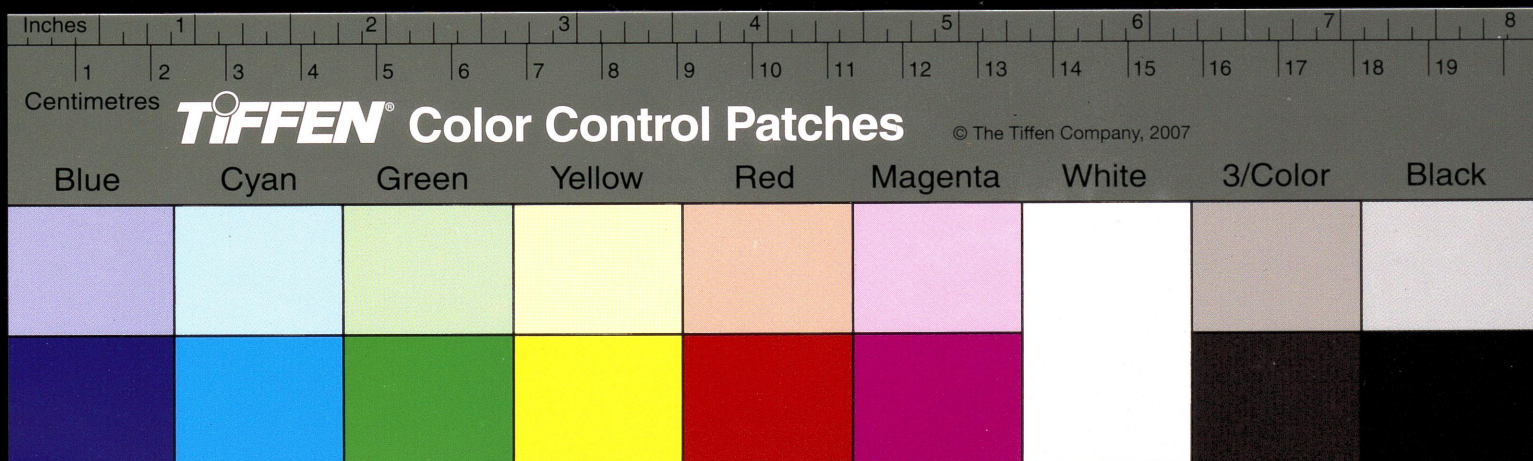
From these equations it can be seen that it is the derivatives  $T_\epsilon$  and  $T_{\epsilon'}$  that are relevant for the connection between F, G and P, Q, since we can assume in virtue of Weiss' conditions (2, 18)–(2, 19) that R and S are functions of the remaining invariants. This last procedure of finding R and S as functions of F, G, P, Q involves rather complicated square-root expressions and does not appear suitable for particular cases. We shall now find a slightly different method of procedure by a change of the parameters  $\epsilon, \epsilon'$ .

### 6. Introduction of New Parameters.

We can simplify considerations to a great extent by introducing two new functions  $\lambda$  and  $\mu$  of the invariants defined by

$$\epsilon = e^\lambda, \quad \text{or } \lambda = \log \epsilon \tag{6, 1}$$

$$\epsilon' = \tan \mu, \quad \mu = \tan^{-1} \epsilon' \tag{6, 2}$$





and,

$$\begin{aligned} Q &= -e^{-\lambda/2} (T_\mu \cos \frac{1}{2}\mu + T_\lambda \sin \frac{1}{2}\mu) \\ G &= -e^{\lambda/2} (T_\mu \cos \frac{1}{2}\mu - T_\lambda \sin \frac{1}{2}\mu) \end{aligned} \quad (6, 9)$$

Eliminating  $\lambda$  and  $\mu$  between the four equations (6, 8)–(6, 9) we get two relations between the invariants F, G, P, Q.

7. Relations with the Complex Formalism of Weiss.

The parameters  $\lambda$  and  $\mu$  are very closely connected with the complex invariants

$$\begin{aligned} \phi &= F + iG \\ \psi &= P - iQ \end{aligned}$$

introduced by Weiss.<sup>13</sup> Forming the complex parameter  $\lambda + i\mu$ ,  
 $\lambda + i\mu = \log \epsilon + i \tan^{-1} \epsilon'$

$$\begin{aligned} &= \log \sqrt{F^2 + G^2} - \log \sqrt{P^2 + Q^2} + i \tan^{-1} \left\{ \frac{(G/F) + (Q/P)}{1 - (G/F)(Q/P)} \right\} \\ &= \left\{ \log \sqrt{F^2 + G^2} + i \tan^{-1} \left( \frac{G}{F} \right) \right\} - \left\{ \log \sqrt{P^2 + Q^2} - i \tan^{-1} \left( \frac{P}{Q} \right) \right\} \\ &= \log (F + iG) - \log (P - iQ) \\ &= \log \left( \frac{\phi}{\psi} \right) \end{aligned}$$

i.e.,  $\frac{\phi}{\psi} = e^{\lambda + i\mu}$ . (7, 1)

From the above equation we see that  $\lambda + i\mu$  is an analytic function of both the complex variables  $\phi$  and  $\psi$ ; hence the Cauchy-Riemann conditions give

$$\left. \begin{aligned} \frac{\partial \lambda}{\partial F} &= \frac{\partial \mu}{\partial G}; \quad \frac{\partial \lambda}{\partial G} = -\frac{\partial \mu}{\partial F} \\ \frac{\partial \lambda}{\partial P} &= -\frac{\partial \mu}{\partial Q}; \quad \frac{\partial \lambda}{\partial Q} = \frac{\partial \mu}{\partial P} \end{aligned} \right\} \quad (7, 2)$$

Differentiating (6, 6) and (6, 7) respectively with respect to F and G,

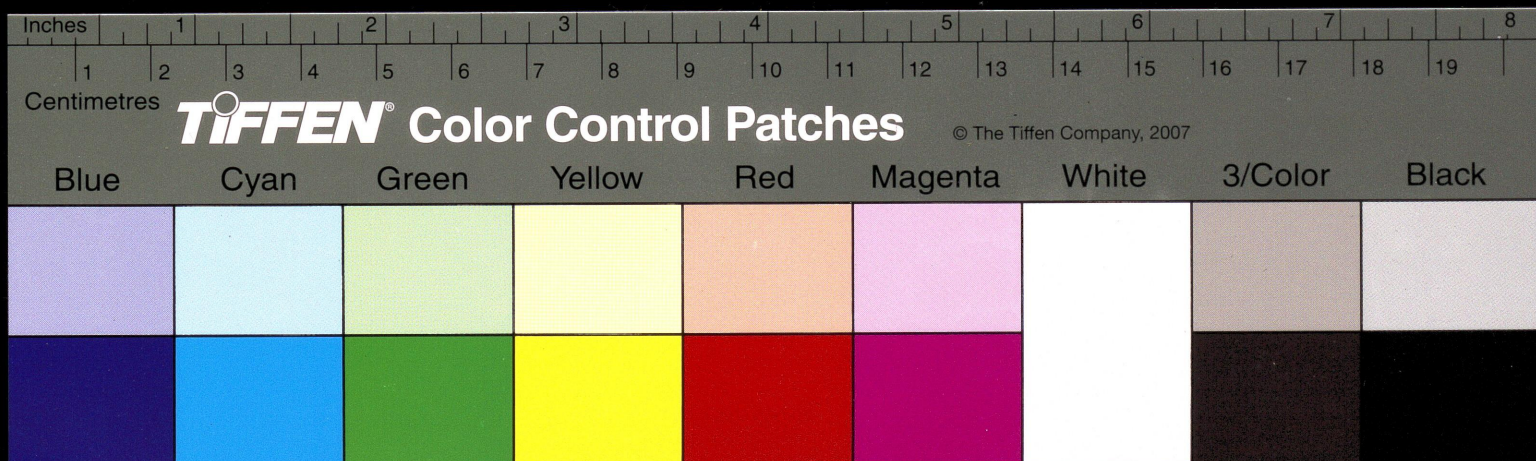
$$\left. \begin{aligned} \frac{\partial^2 T}{\partial F^2} &= -\frac{1}{2} T_F \frac{\partial \lambda}{\partial F} - \frac{1}{2} T_G \frac{\partial \mu}{\partial F} \\ \frac{\partial^2 T}{\partial G^2} &= -\frac{1}{2} T_G \frac{\partial \lambda}{\partial G} + \frac{1}{2} T_F \frac{\partial \mu}{\partial G} \end{aligned} \right\}$$

Adding these equations, and using (7, 2), we get the Laplacian equation

$$\left. \begin{aligned} \frac{\partial^2 T}{\partial F^2} + \frac{\partial^2 T}{\partial G^2} &= 0 \\ \frac{\partial^2 T}{\partial P^2} + \frac{\partial^2 T}{\partial Q^2} &= 0 \end{aligned} \right\} \quad (7, 3)$$

Similarly,

<sup>13</sup> IV, p. 86, equations (5, 7), (5, 8).







and (9, 2) can be written as

$$T = \sqrt{2\phi + 2\psi + 4} - 2 \quad (9, 6)$$

From (9, 6) it easily follows that  $T = \text{Lagrangian} + \text{Hamiltonian}$ . In fact,  $2\phi + 2\psi + 4 = 2F + 2P + 4$ , since  $G = Q$  in Born's case and from a known identity<sup>16</sup>

$$\sqrt{2F + 2P + 4} = L + H + 2.$$

The form (9, 6) is more general than the one given by Weiss<sup>17</sup> which holds when the invariant  $G$  is neglected. The complex Lagrangian and Hamiltonian given by him for this case are respectively  $(1 + 2\phi)^{\frac{1}{2}} - 1$ , and  $(1 + 2\psi)^{\frac{1}{2}} - 1$  and in this case only we have

$$-2 + \sqrt{1 + 2\phi} + \sqrt{1 + 2\psi} = \sqrt{2\phi + 2\psi + 4} - 2 \quad (9, 7)$$

in virtue of the relation

$$\phi + 2\phi\psi + \psi = 0.$$

While the left-hand side of (9, 7) is not Born's action-function in the general case where  $G$  is not ignored, we have shown that the right-hand side is so.

#### 10. Other Functions.

The one parameter group of action-functions given by Infeld, and the one by Hoffmann and Infeld are given in our notation by

$$T = A \cosh \frac{\lambda}{2} + B \sinh \frac{\lambda}{2} + C\lambda + D \quad (10, 1)$$

and following the analogy of Born's action-function we can use  $\cos \frac{1}{2}\mu$  or  $(1/\cos \frac{1}{2}\mu)$  which tend to unity with  $\mu = 0$  as a sort of gauge-factor to generalise (10, 1). It appears however that there is no criterion which leads to such a kind of generalisation.

In analogy with (9, 2) we can consider

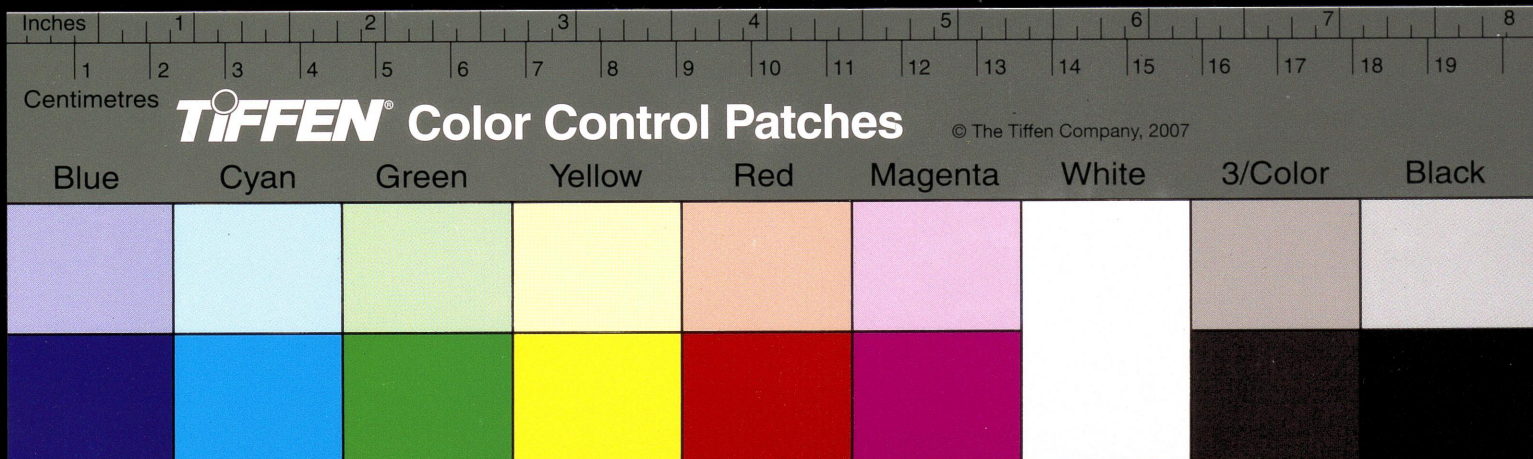
$$T = \frac{e^{\lambda/2}}{\cos \frac{1}{2}\mu} - \frac{1}{2}\lambda - 1 \quad (10, 2)$$

as a sort of generalisation of the Hoffmann-Infeld action-function. We shall use (10, 2) to obtain the corresponding Lagrangian, develop it in power series in  $F$  and  $G$  and compare it with the Lagrangian set up by Euler and Köckel as a correction to Maxwell's theory on the basis of the phenomenon of scattering of light by light according to Dirac's theory of holes. The reason for this investigation is that with the Hoffmann-Infeld action-function (with  $G = 0$ ) it is possible to make<sup>18</sup> a much better estimate of the

<sup>16</sup> V, p. 580, eqn. (2, 3).

<sup>17</sup> IV, p. 91, eqn. (8, 5).

<sup>18</sup> This has actually not been done with the Hoffmann-Infeld action-function, but with Infeld's action-function  $T = e^{-\lambda/2} + \frac{1}{2}\lambda - 1$  where  $L = \frac{1}{2}\log(1 + F)$ . It can however be shown that we get the same estimate  $1/\alpha \cong 130$  with the action-function  $T = e^{-\lambda/2} - \log \lambda - 1$  where  $H = \frac{1}{2}\log(1 + P)$ .



fine-structure constant than with Born's action-function. In the latter case there is no correspondence (as mentioned in the Introduction) in the terms containing  $G^2$  in the power series expansion. We shall try to find if this correspondence as well as the better estimate for the fine-structure constant could be secured from (10, 2).

The derivatives  $T_\lambda$  and  $T_\mu$  are given from (10, 2) in the form

$$\left. \begin{aligned} T_\lambda &= \frac{e^{\lambda/2}}{2 \cos \frac{1}{2}\mu} - \frac{1}{2} \\ T_\mu &= \frac{e^{\lambda/2} \sin \frac{1}{2}\mu}{2 \cos^2 \frac{1}{2}\mu} \end{aligned} \right\} \quad (10, 3)$$

Also  $T_\lambda = R$  gives

$$e^{\lambda/2} = (1 + 2R) \cos \frac{1}{2}\mu \quad (10, 4)$$

From (6, 8) and (6, 9)—(using (10, 3) and (10, 4)

$$\left. \begin{aligned} 2F &= (1 + 2R)^2 - (1 + 2R) \cos^2 \frac{1}{2}\mu \\ 2G &= -(1 + 2R) \sin \frac{1}{2}\mu \cos \frac{1}{2}\mu \end{aligned} \right\} \quad (10, 5)$$

Since our idea is to obtain the power-series expansion of the Lagrangian obtained from (10, 2) upto the second power in  $F$  and  $G$ , we assume<sup>19</sup>

$$\left. \begin{aligned} R &= a_0 F + a_1 F^2 + a_2 G^2 \\ \cos \frac{1}{2}\mu &= 1 + b_0 F + b_1 F^2 + b_2 G^2 \end{aligned} \right\} \quad (10, 6)$$

and since  $R \rightarrow 0$  as  $F$  and  $G \rightarrow 0$ , while  $\cos \frac{1}{2}\mu \rightarrow 1$ . Substituting (10, 6) in (10, 5) and equating coefficients of like powers after neglecting terms of order higher than two, we get,

$$\left. \begin{aligned} a_0 &= 1, a_1 = -2, a_2 = -2 \\ b_0 &= b_1 = 0, b_2 = -2 \\ \text{i.e.,} \quad R &= F - 2F^2 - 2G^2 \\ \cos \frac{1}{2}\mu &= 1 - 2G^2 \end{aligned} \right\} \quad (10, 6a)$$

The Lagrangian  $L$ , corresponding to (10, 2) is given by

$$\begin{aligned} L &= \frac{1}{2} T + R \\ &= 2R - \frac{1}{4} \lambda, \text{ using (10, 4)} \end{aligned} \quad (10, 7)$$

Again from (10, 4) the power-series expansion for  $\lambda$ , using (10, 6a), is given by

$$\begin{aligned} \lambda &= \log (1 + 4F - 4F^2 - 12G^2) \\ &= 4F - 12F^2 - 12G^2 \end{aligned}$$

and (10, 7) reduces to

$$L = F - F^2 - G^2$$

$$\text{or} \quad L = \frac{1}{2}(\vec{B}^2 - \vec{E}^2) - \frac{1}{4}\{(\vec{B}^2 - \vec{E}^2)^2 + 4(\vec{B} \cdot \vec{E})^2\} \quad (10, 8)$$

<sup>19</sup> The invariant  $G$  appears as square only, as pointed out by Weiss. See V, p. 89.

