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GENERALISED ACTION-FUNCTIONS IN BORN'S ELECTRO-DYNAMICS.

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GENERALISED ACTION-FUNCTIONS IN BORN'S ELECTRO-DYNAMICS.

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1. *Introduction.*

Infeld has shown[1] that, in the development of Born's field theory, it is possible to choose an infinity of action-functions other than the one originally proposed by Born and Infeld[2] which, for simplicity, might be called Born's action-function. These action-functions of Infeld and the one recently introduced by Hoffmann and Infeld[3] have, in common with Born's action-function, the properties of self-conjugacy, of the existence of simple, algebraic relations between the f_{kl}- and p_{kl}-fields, and giving finite values for the energy for the electrical particle. They, however, differ from the latter in that the condition of the invariance of the action-integral itself is not insisted in them. As I have shown elsewhere[4] Born's function is unique if this condition as well as the condition of self-conjugacy are imposed on the action-function.

Laying aside for the present this condition of invariance of the action-integral, we might say that the action-functions of Infeld and Hoffmann and Infeld give rise to several systems of Born's non-linear electro-dynamics. These action-functions, however, are all confined to the case where the invariant $G = (\vec{B}\,\vec{E})$ is neglected, as also the invariants $Q = (\vec{D}\,\vec{H})$, and $S = (\vec{B}\,\vec{D}) + (\vec{E}\,\vec{H})$. I have considered in this paper the general case where these invariants are not neglected and constructed a two-fold infinity of action-functions all having the same properties as Infeld's action-functions and reducing to them when $G = 0$. This generalisation has also enabled the deduction of close connections with the complex formalism developed by Weiss.[5] In view of the fact however, that the condition of " finite self-energy " has so far been discussed in the case of the point-singularity and

[1] Infeld, *Proc. Camb. Phil. Soc.*, 1936, **32**, 127 ; 1937, **33**, 70—hereinafter referred to as I and II.

[2] Born and Infeld, *Proc. Roy. Soc.*, (A), 1934, **144**, 425.

[3] Hoffmann and Infeld, *Phys. Rev.*, 1937, **51**, 765—hereinafter referred to as III.

[4] B. S. Madhava Rao, *Proc. Ind. Acad. Sci.*, (A), 1936, **3**, 377.

[5] P. Weiss, *Proc. Camb. Phil. Soc.*, 1937, **33**, 79—hereinafter referred to as IV.

the case of ring-singularity in the zero-approximation[6] where, in both cases, the invariant G does not appear, it seems difficult to set up a criterion limiting the choice of these action-functions. Nor is the regularity condition of Hoffmann and Infeld of any use since it is based on the character of a spherically symmetric electro-static solution.

An attempt has been made in this paper to construct an action-function which leads to a full coincidence of the second order terms when the Lagrangian of the field theory is compared with the Lagrangian that arises in the investigations of Euler and Köckel[7] on the phenomenon of scattering of light by light on the basis of Dirac's theory of holes. It is well known that such a coincidence does not exist[8] in the terms containing G^2 when Born's action-function is used. The result that I obtain is that even this condition does not restrict the choice of the action-function there being more than one such function wherefrom a coincidence with the Euler-Köckel Lagrangian could be obtained as well as the good determination of the fine-structure constant possible with the Hoffmann-Infeld action-function.

I have closely followed the method adopted by Infeld in I and II.

2. *Condition of Self-conjugacy.*

We introduce the f_{kl} and p^{kl} (or in the dual form f^{*kl} and p_{kl}^{*}) tensors describing the electromagnetic field, the space-vector notation and also the several invariants in the usual manner,[9] with the modification introduced by Weiss,[10] which leads to a good lot of simplification by avoiding unnecessary numerical factors. The Lorentz frame of reference is used.

$$\left.\begin{aligned} (f_{23}, f_{31}, f_{12}) &= \vec{B}, \quad (f_{14}, f_{24}, f_{34}) = \vec{E} \\ (p_{23}, p_{31}, p_{12}) &= \vec{H}, \quad (p_{14}, p_{24}, p_{34}) = \vec{D} \end{aligned}\right\} \tag{2, 1}$$

$$\left.\begin{aligned} \tfrac{1}{4} f_{kl} f^{kl} &= -\tfrac{1}{4} f^{*kl} f_{kl}^{*} = F, \quad \tfrac{1}{4} f_{kl} f^{*kl} = \tfrac{1}{4} f_{kl}^{*} f^{kl} = G, \\ \tfrac{1}{4} p_{kl}^{*} p^{*kl} &= -\tfrac{1}{4} p_{kl} p^{kl} = P, \quad \tfrac{1}{4} p_{kl}^{*} p^{kl} = \tfrac{1}{4} p_{kl} p^{*kl} = Q, \\ \tfrac{1}{4} f_{kl} p^{kl} &= -\tfrac{1}{4} f^{*kl} p_{kl}^{*} = R, \quad \tfrac{1}{4} f_{kl} p^{*kl} = \tfrac{1}{4} f^{*kl} p_{kl} = S \end{aligned}\right\} \tag{2, 2}$$

$$\left.\begin{aligned} F &= \tfrac{1}{2}(\vec{B}^2 - \vec{E}^2), \quad G = (\vec{B}\,\vec{E}) \\ P &= \tfrac{1}{2}(\vec{D}^2 - \vec{H}^2), \quad Q = (\vec{D}\,\vec{H}) \\ R &= \tfrac{1}{2}\{(\vec{B}\,\vec{H}) - (\vec{D}\,\vec{E})\}, \quad S = \tfrac{1}{2}\{(\vec{B}\,\vec{D}) + (\vec{E}\,\vec{H})\} \end{aligned}\right\} \tag{2, 3}$$

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[6] B. S. Madhava Rao, *Proc. Ind. Acad. Sci.*, (A), 1936, **4**, 355.

[7] Euler and Köckel, *Naturwiss*, 1935, **23**, 246.

[8] M. Born, *Proc. Ind. Acad. Sci.*, (A), 1935, **2**, 560.

[9] B. S. Madhava Rao, *Proc. Ind. Acad. Sci.*, (A), 1936, **4**, 578, §3—hereafter referred to as V.

[10] IV, foot-note p. 80.

We impose on our action-function T, which we assume to be a function of F, G, P and Q, that it should be self-conjugate. This means that the set of equations

$$\left.\begin{aligned} p^{kl} &= \frac{\partial T}{\partial f_{kl}} \qquad (2,4) \\ f^{*kl} &= \frac{\partial T}{\partial p_{kl}^{*}} \qquad (2,5) \end{aligned}\right\}$$

should be self-consistent, *i.e.*, that (2, 5) is a consequence of (2, 4), or *vice versa*. We can rewrite (2, 4) and (2, 5) in the form

$$\left.\begin{aligned} p^{kl} &= T_F f^{kl} + T_G f^{*kl} \qquad (2,6) \\ f^{*kl} &= T_P p^{*kl} + T_Q p^{kl} \qquad (2,7) \end{aligned}\right\}$$

where T_F, T_P, T_G, T_Q denote the derivatives with respect to F, P, G, Q respectively. We shall now find the restriction to be imposed on T to ensure the consistency of (2, 6) and (2, 7). We multiply (2, 6) respectively by $\frac{1}{4}f_{kl}$, $\frac{1}{4}f_{kl}^{*}$, $\frac{1}{4}p_{kl}$, $\frac{1}{4}p_{kl}^{*}$ and sum up; similarly (2, 7) respectively by $\frac{1}{4}p_{kl}^{*}$, $\frac{1}{4}p_{kl}$, $\frac{1}{4}f_{kl}^{*}$, $\frac{1}{4}f_{kl}$ and sum up. We get[11] after suitably grouping the terms

$$\left.\begin{aligned} R &= FT_F + GT_G \qquad (2,8) \\ -R &= PT_P + QT_Q \qquad (2,9) \end{aligned}\right\}$$

$$\left.\begin{aligned} S &= GT_F - FT_G \qquad (2,10) \\ S &= QT_P - PT_Q \qquad (2,11) \end{aligned}\right\}$$

$$\left.\begin{aligned} -P &= RT_F + ST_G \qquad (2,12) \\ F &= RT_P - ST_Q \qquad (2,13) \end{aligned}\right\}$$

$$\left.\begin{aligned} Q &= ST_F - RT_G \qquad (2,14) \\ G &= ST_P + RT_Q \qquad (2,15) \end{aligned}\right\}$$

The equations (2, 9), (2, 11), (2, 13) and (2, 15) must follow as consequences from (2, 8), (2, 10), (2, 12) and (2, 14) respectively.

Before proceeding to reduce the above number of eight equations we will directly obtain two conditions which follow from the self-consistency of (2, 6) and (2, 7). Taking the dual of (2, 6) we get

$$p^{*kl} = T_F f^{*kl} + T_G f^{**kl} = T_F f^{*kl} - T_G f^{kl};$$

Substituting this value of p^{*kl} and the value of p^{kl} from (2, 6) in (2, 7) we should get an identical equation, *i.e.*,

$$f^{*kl} = T_P (T_F f^{*kl} - T_G f^{kl}) + T_Q(T_F f^{kl} + T_G f^{*kl})$$

identically. This leads to the consistency conditions

$$T_F T_P + T_G T_Q = 1 \qquad (2,16)$$

$$T_F T_Q - T_P T_G = 0 \qquad (2,17)$$

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[11] As in V, p. 579, where we replace both L and H by T and introduce the necessary numerical factors to correspond to Weiss's notation.

It is now easy to see with the aid of (2, 16) and (2, 17) that the equations (2, 12) – (2, 15) are consequences of (2, 8) – (2, 11). Hence the eight equations (2, 8) – (2, 15) can be reduced to the six equations (2, 8) – (2, 11) and (2, 16) – (2, 17).

In essence we have replaced the four equations (2, 12) – (2, 15) by the two equations (2, 16) and (2, 17). We can now find two other equations equivalent to the latter. From (2, 12) – (2, 15) and using (2, 16) – (2, 17) it is very easy to deduce the following two relations between the invariants, *viz.*,

$$\left.\begin{aligned} S^2 - R^2 &= FP + GQ \\ 2RS &= FQ - PG \end{aligned}\right\} \qquad \begin{aligned} &(2,\ 18) \\ &(2,\ 19) \end{aligned}$$

Further using (2, 8) – (2, 15) and (2, 18) – (2, 19) it is easy to deduce (2, 16) and (2, 17). Thus the original system of eight equations can be reduced to the six equations (2, 8) – (2, 11) and (2, 18) – (2, 19). These last two equations are relations between the invariants, F, G, Q, R and S which hold as a consequence of the condition of self-conjugacy. But they are exactly the same as those obtained by Weiss[12] by expressing the symmetry of the energy-impulse tensor. Thus the *relation between the invariants imposed by self-conjugacy is the same as that obtained by making the energy-impulse tensor symmetrical.*

Coming now to the conditions (2, 8) – (2, 11) we can rewrite them in the forms

$$\left.\begin{aligned} FT_F + GT_G + PT_P + QT_Q &= 0 \\ GT_F - FT_G - QT_P + PT_Q &= 0 \end{aligned}\right\} \qquad \begin{aligned} &(2,\ 20) \\ &(2,\ 21) \end{aligned}$$

and

$$\left.\begin{aligned} FT_F + GT_G - PT_P - QT_Q &= 2R \\ GT_F - FT_G + QT_P - PT_Q &= 2S \end{aligned}\right\} \qquad \begin{aligned} &(2,\ 22) \\ &(2,\ 23) \end{aligned}$$

(2, 20) – (2, 21) form a system of simultaneous partial differential equations for T in the homogeneous form—in particular (2, 20) shows that T must be a homogeneous function of degree zero in F, G, P, Q. If we, therefore, choose T so as to satisfy these relations (2, 20) – (2, 21) we will have reduced our original eight equations to four only in consonance with the consistency condition. The equations (2, 20) – (2, 21) would then be identically satisfied, (2, 18) – (2, 19) or (2, 16) – (2, 17) would give the relations between the invariants, and (2, 22) – (2, 23) would give P, Q, R, S as functions of F and G.

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[12] See IV, p. 85, equation (4, 5).

3. *Solution of the System of Partial Differential Equations.*

In the usual notation of the theory of partial differential equations we can write (2, 20) – (2, 21) as

$$\left.\begin{aligned} F_1 &\equiv x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 = 0 \\ F_2 &\equiv x_2 p_1 - x_1 p_2 - x_4 p_3 + x_3 p_4 = 0 \end{aligned}\right\} \qquad \begin{aligned}(3,\ 1)\\(3,\ 2)\end{aligned}$$

and proceed to find out the most general functional form of the solution common to (3, 1) – (3, 2).

It is easy to verify that the Poisson bracket $(F_1, F_2) = 0$, so that the equations form a complete Jacobian system as they are. The most general solution of (3, 1) is any arbitrary function of $(x_2/x_1,\ x_3/x_1,\ x_4/x_1)$. Adopting the usual method we introduce new variables given by

$$x_1 = x_1,\ u = x_2/x_1,\ v = x_3/x_1,\ w = x_4/x_1.$$

With these substitutions, equation (3, 1), *i.e.*,

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + x_3 \frac{\partial f}{\partial x_3} + x_4 \frac{\partial f}{\partial x_4} = 0$$

reduces to $\dfrac{\partial f}{\partial x_1} = 0$; and the second equation (3, 2), *i.e.*,

$$x_2 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial x_2} - x_4 \frac{\partial f}{\partial x_3} + x_3 \frac{\partial f}{\partial x_4} = 0 \text{ reduces to}$$

$$x_2 \left(\frac{\partial f}{\partial x_1} - \frac{u}{x_1} f_u - \frac{v}{x_1} f_v - \frac{w}{x_1} f_w \right) - x_1 \left(\frac{1}{x_1} f_u \right) - x_4 \left(\frac{1}{x_1} f_v \right) + x_3 \left(\frac{1}{x_1} f_w \right) = 0,$$

i.e., $$- u^2 f_u - uv f_v - uw f_w - f_u - w f_v + v f_w = 0$$

or $$f_u (1 + u^2) + f_v (uv + w) + f_w (uw - v) = 0 \qquad (3,\ 3)$$

The Lagrangian subsidiary equations of (3, 3) are

$$\frac{du}{1+u^2} = \frac{dv}{uv+w} = \frac{dw}{uw - v} \qquad (3,\ 4)$$

and we need only find two independent integrals of (3, 4).

From (3, 4) we can immediately deduce the relations

$$\frac{u\,du}{1+u^2} = \frac{v\,dv + w\,dw}{v^2+w^2}, \quad \text{and}$$

$$\frac{du}{1+u^2} = \frac{w\,dv - v\,dw}{v^2+w^2}.$$

These lead to the two integrals

$$\log (1 + u^2) = \log (v^2 + w^2) + \text{const.},$$

and $\tan^{-1} u - \tan^{-1} \left(\dfrac{v}{w} \right) = \text{const.}$, respectively

or, $$\frac{1+u^2}{v^2+w^2} = \text{const., and } \frac{wu - v}{uv+w} = \text{const.}$$

For reasons to be immediately given, we write,

$$\epsilon = \sqrt{\frac{1+u^2}{v^2+w^2}}; \quad \epsilon' = \frac{uv+w}{wu-v} \left(\text{instead of } \frac{wu-v}{uv+w}\right)$$

or, introducing the invariants F, G, P, Q

$$\epsilon = \sqrt{\frac{F^2+G^2}{P^2+Q^2}}; \quad \epsilon' = \frac{FQ+PG}{FP-GQ} \tag{3, 5}$$

The most general form of the solution common to (3, 1), (3, 2) is any arbitrary function T (ϵ, ϵ') of ϵ and ϵ'. If T be chosen in this form equations (2, 20)—(2, 21) are identically satisfied.

The reason for choosing ϵ and ϵ' in the forms given by (3, 5) is that we want to have these quantities reduce themselves to the corresponding ones when we go over to Infeld's case where G and Q are neglected. If $G = Q = 0$ then $\epsilon' = 0$, and ϵ reduces to $\pm$ (F/P). We therefore further postulate that the negative sign is to be taken in order that our value of ϵ should be the square of the corresponding ϵ used in Infeld's investigations. We may also observe here that in the limiting Maxwellian case where $G = Q$ and $F = -P$, we have $\epsilon = 1$ and $\epsilon' = 0$.

Thus our generalisation has introduced the additional parameter ϵ' and gives rise to a two-fold infinity of action-functions satisfying the condition of self-conjugacy.

4. *T as the Sum of Lagrangian and Hamiltonian.*

We will show that, in our general case, if (2, 4)—(2, 5) are satisfied the action-function T can be represented as the sum of a Lagrangian and a Hamiltonian

Let
$$\left.\begin{aligned} L(F, G) &= \tfrac{1}{2}T + R \\ H(P, Q) &= \tfrac{1}{2}T - R \end{aligned}\right\} \quad \begin{aligned}&(4, 1)\\&(4, 2)\end{aligned}$$

where we have put the numerical factor $\frac{1}{2}$ on the right-hand side for T in consonance with the notation of Weiss. From (4, 1)

$$2\,dL = T_F dF + T_G dG + T_P dP + T_Q\,dQ + 2\,dR.$$

From the relations (2, 18)—(2, 19)

$$2\,SdS - 2RdR = FdP + PdF + GdQ + QdG,$$
$$2\,SdR + 2\,RdS = FdQ + QdF - GdP - PdG.$$

Eliminating dS from the above equations,

$$2\,dR\,(R^2+S^2) = dF\,(QS-PR) - dG\,(PS+QR) - dP\,(GS+FR) + dQ\,(FS-RG). \tag{4, 3}$$

We can simplify (4, 3) by obtaining the expressions for the derivatives of T in terms of the invariants. Directly solving (2, 8)—(2, 11) for these derivatives we obtain,

$$\left.\begin{aligned} T_F &= \frac{FR+GS}{F^2+G^2}; \quad T_G = \frac{GR-FS}{F^2+G^2} \\ T_P &= \frac{QS-PR}{P^2+Q^2}; \quad T_Q = -\frac{QR+PS}{P^2+Q^2} \end{aligned}\right\} \tag{4, 4}$$

Also from (2, 16)—(2, 17), and (2, 20)—(2, 21)

$$\frac{T_P}{T_F} = \frac{T_Q}{T_G} = \left(\frac{F^2+G^2}{P^2+Q^2}\right)^{\frac{1}{2}} = \epsilon. \tag{4, 5}$$

From (2, 18)—(2, 19) by squaring and adding

$$R^2 + S^2 = \sqrt{(F^2+G^2)(P^2+Q^2)}. \tag{4, 6}$$

Now using (4, 4), (4, 6) and (4, 5) successively we can reduce (4, 3) to

$$2dR = T_F\, dF + T_G\, dG - T_P\, dP - T_Q\, dQ$$

and

$$2\, dL = 2\, T_F\, dF + 2\, T_G\, dG$$

i.e.,

$$\frac{\partial L}{\partial F} = \frac{\partial T}{\partial F}, \text{ and } \frac{\partial L}{\partial G} = \frac{\partial T}{\partial G}$$

i.e., L is the Lagrangian. Analogously we can show that H is the Hamiltonian.

5. *Expressions for the Derivatives of T.*

In (4, 4) above, we have obtained expressions for the derivatives of T in terms of all the six invariants, but since we assume T to be a function of only F, G, P, Q we need expressions for the derivatives which do not involve R and S. This is easily done by finding R and S from (2, 18) and (4, 6) and substituting in (4, 4). A simpler way is to use the equations (2, 12)—(2, 15). From these equations and using the consistency conditions (2, 16)—(2, 17) we get

$$\begin{aligned} FP - GQ &= -(R^2+S^2)(T_F T_P - T_G T_Q) \\ &= (R^2+S^2)(1 - 2\,\epsilon\, T_F^2), \text{ using (2, 16) and (4, 5)} \end{aligned}$$

and hence, using (4, 6) and (4, 5)

$$2T_F^2 = \sqrt{\frac{P^2+Q^2}{F^2+G^2}} - \frac{FP-GQ}{F^2+G^2} \tag{5, 1}$$

Similarly

$$2\,T_G^2 = \sqrt{\frac{P^2+Q^2}{F^2+G^2}} + \frac{FP-GQ}{F^2+G^2} \tag{5, 2}$$

In an entirely analogous manner we get

$$2\,T_P^2 = \sqrt{\frac{F^2+G^2}{P^2+Q^2}} - \frac{FP-GQ}{P^2+Q^2} \tag{5, 3}$$

$$2T_Q^2 = \sqrt{\frac{F^2+G^2}{P^2+Q^2}} + \frac{FP-GQ}{P^2+Q^2} \tag{5, 4}$$

(5, 3) and (5, 4) follow from (5, 1) and (5, 2) by interchanging F and G with P and Q, and thus determine P and Q as the same functions of F and G and conversely, guaranteeing in this way the self-consistency of (2, 4) and (2, 5).

The equations (5, 1)—(5, 4) express the derivatives T_F, T_G, T_P, T_Q in terms of F, G, P and Q. Since the action-functions we are considering can be taken as arbitrary functions of ϵ and ϵ', we must be able to obtain T_ϵ and $T_{\epsilon'}$, in terms of the invariants, in order that we might get for every special system of electrodynamics particular relations connecting F, G, P, Q. We proceed to determine these expressions for T_ϵ and $T_{\epsilon'}$.

We can express T_F and T_G in terms of T_ϵ and $T_{\epsilon'}$ in the form

$$\left.\begin{aligned} T_F &= T_\epsilon \cdot \epsilon_F + T_{\epsilon'} \cdot \epsilon'_F \\ T_G &= T_\epsilon \cdot \epsilon_G + T_{\epsilon''} \cdot \epsilon'_G \end{aligned}\right\}$$

where ϵ_F, for example, stands for $\partial\epsilon/\partial F$. Solving these for T_ϵ and $T_{\epsilon'}$,

$$\left.\begin{aligned} T_\epsilon\ (\epsilon_F\epsilon'_G - \epsilon_G\epsilon'_F) &= \epsilon'_G T_F - \epsilon'_F T_G \\ -T_{\epsilon'}\ (\epsilon_F\epsilon'_G - \epsilon_G\epsilon'_F) &= \epsilon_G T_F - \epsilon_F T_G \end{aligned}\right\}$$

From (3, 5),

$$\left.\begin{aligned} \epsilon_F &= \frac{1}{\epsilon}\cdot\frac{F}{P^2+Q^2}; \quad \epsilon'_F = -\frac{G(P^2+Q^2)}{(FP-GQ)^2} \\ \epsilon_G &= \frac{1}{\epsilon}\cdot\frac{G}{P^2+Q^2}; \quad \epsilon'_G = \frac{F(P^2+Q^2)}{(FP-GQ)^2} \end{aligned}\right\}$$

Substituting these in the above, and simplifying, we get

$$\left.\begin{aligned} \epsilon T_\epsilon &= FT_F + GT_G \\ -(1+\epsilon'^2)\,T_{\epsilon'} &= GT_F - FT_G \end{aligned}\right\}$$

Using (2, 8) and (2, 10) these can be simplified to the important equations

$$\left.\begin{aligned} \epsilon T_\epsilon &= R \\ (1+\epsilon'^2)\,T_{\epsilon'} &= -S \end{aligned}\right\} \qquad \begin{aligned}(5, 5)\\(5, 6)\end{aligned}$$

From these equations it can be seen that it is the derivatives T_ϵ and $T_{\epsilon'}$ that are relevant for the connection between F, G and P, Q, since we can assume in virtue of Weiss' conditions (2, 18) — (2, 19) that R and S are functions of the remaining invariants. This last procedure of finding R and S as functions of F, G, P, Q involves rather complicated square-root expressions and does not appear suitable for particular cases. We shall now find a slightly different method of procedure by a change of the parameters ϵ, ϵ'.

6. *Introduction of New Parameters.*

We can simplify considerations to a great extent by introducing two new functions λ and μ of the invariants defined by

$$\left.\begin{aligned} \epsilon &= e^\lambda, \quad \text{or } \lambda = \log \epsilon \\ \epsilon' &= \tan \mu, \quad \mu = \tan^{-1} \epsilon' \end{aligned}\right\} \qquad \begin{aligned}(6, 1)\\(6, 2)\end{aligned}$$

and consider the action-function T as any arbitrary function of λ and μ.

$$\left.\begin{aligned} T_\lambda &= T_\epsilon \cdot \frac{\partial \epsilon}{\partial \lambda} = \epsilon T_\epsilon \\ T_\mu &= T_{\epsilon'} \cdot \frac{\partial \epsilon'}{\partial \mu} = (1 + \epsilon'^2)\, T_{\epsilon'} \end{aligned}\right\}$$

Hence (5, 5)—(5, 6) reduce to the very simple and striking forms

$$\left.\begin{aligned} T_\lambda &= \ \ R \\ T_\mu &= -S \end{aligned}\right\}. \qquad \begin{matrix}(6,\ 3)\\(6,\ 4)\end{matrix}$$

We can also express (5, 1) and (5, 2) in terms of these new parameters λ and μ. Calculating the value of $(FP - GQ)/(F^2 + G^2)$ in terms of ϵ and ϵ', we readily find

$$\left(\frac{FP - GQ}{F^2 + G^2}\right)^2 = \frac{1}{\epsilon^2(1 + \epsilon'^2)}$$

so that

$$\frac{FP - GQ}{F^2 + G^2} = \pm \frac{1}{\epsilon\sqrt{1 + \epsilon'^2}}.$$

The choice of the + or − sign is decided by going to the limiting case of Infeld's action-functions. In this case the left-hand side reduces to P/F, and since we have already assumed (§3) that in this limiting case ($\epsilon' = 0$) we should take our ϵ as $(-F/P)$ we choose the negative sign and write

$$\frac{FP - GQ}{F^2 + G^2} = -\frac{1}{\epsilon\sqrt{1 + \epsilon'^2}} \qquad (6,\ 5)$$

Hence (5, 1)—(5, 2) reduce to

$$\left.\begin{aligned} 2T_F^2 &= \frac{1}{\epsilon} + \frac{1}{\epsilon\sqrt{1 + \epsilon'^2}} \\ 2T_G^2 &= \frac{1}{\epsilon} - \frac{1}{\epsilon\sqrt{1 + \epsilon'^2}} \end{aligned}\right\}$$

or, with the substitutions (6, 1) — (6, 2) to

$$\left.\begin{aligned} 2T_F^2 &= 2e^{-\lambda} \cos^2 \tfrac{1}{2}\mu \\ 2T_G^2 &= 2e^{-\lambda} \sin^2 \tfrac{1}{2}\mu \end{aligned}\right\}$$

or

$$\left.\begin{aligned} T_F &= e^{-\lambda/2} \cos \tfrac{1}{2}\mu \\ T_G &= e^{-\lambda/2} \sin \tfrac{1}{2}\mu \end{aligned}\right\} \qquad \begin{matrix}(6,\ 6)\\(6,\ 7)\end{matrix}$$

Making use of (4, 5)

$$\left.\begin{aligned} T_P &= e^{\lambda/2} \cos \tfrac{1}{2}\mu \\ T_Q &= e^{\lambda/2} \sin \tfrac{1}{2}\mu \end{aligned}\right\} \qquad \begin{matrix}(6,\ 6a)\\(6,\ 7a)\end{matrix}$$

Substituting (6, 6)—(6, 7) and (6, 3)—(6, 4) in (2, 12)—(2, 15) we get

$$\left.\begin{aligned} -P &= e^{-\lambda/2}\,(T_\lambda \cos \tfrac{1}{2}\mu - T_\mu \sin \tfrac{1}{2}\mu) \\ F &= e^{\lambda/2}\,(T_\lambda \cos \tfrac{1}{2}\mu + T_\mu \sin \tfrac{1}{2}\mu) \end{aligned}\right\} \qquad (6,\ 8)$$

and,

$$\left.\begin{aligned} Q &= -\ e^{-\lambda/2}\ (T_\mu \cos \tfrac{1}{2}\mu + T_\lambda \sin \tfrac{1}{2}\mu) \\ G &= -\ e^{\lambda/2}\ (T_\mu \cos \tfrac{1}{2}\mu - T_\lambda \sin \tfrac{1}{2}\mu) \end{aligned}\right\} \qquad (6,\ 9)$$

Eliminating λ and μ between the four equations (6, 8)—(6, 9) we get two relations between the invariants F, G, P, Q.

7. *Relations with the Complex Formalism of Weiss.*

The parameters λ and μ are very closely connected with the complex invariants

$$\phi = F + iG$$
$$\psi = P - iQ$$

introduced by Weiss.[13] Forming the complex parameter $\lambda + i\mu$,

$$\begin{aligned} \lambda + i\mu &= \log \epsilon + i \tan^{-1} \epsilon' \\ &= \log \sqrt{F^2 + G^2} - \log \sqrt{P^2 + Q^2} + i \tan^{-1} \left\{\frac{(G/F) + (Q/P)}{1 - (G/F)(Q/P)}\right\}. \\ &= \left\{\log \sqrt{F^2 + G^2} + i \tan^{-2} \left(\frac{G}{F}\right)\right\} - \left\{\log \sqrt{P^2 + Q^2} - i \tan^{-1} \left(\frac{P}{Q}\right)\right\} \\ &= \log (F + iG) - \log (P - iQ) \\ &= \log \left(\frac{\phi}{\psi}\right) \end{aligned}$$

i.e., $$\frac{\phi}{\psi} = e^{\lambda + i\mu}. \qquad (7,\ 1)$$

From the above equation we see that $\lambda + i\mu$ is an analytic function of both the complex variables ϕ and ψ; hence the Cauchy-Riemann conditions give

$$\left.\begin{aligned} \frac{\partial\lambda}{\partial F} &= \frac{\partial\mu}{\partial G};\ \frac{\partial\lambda}{\partial G} = -\frac{\partial\mu}{\partial F} \\ \frac{\partial\lambda}{\partial P} &= -\frac{\partial\mu}{\partial Q};\ \frac{\partial\lambda}{\partial Q} = \frac{\partial\mu}{\partial P} \end{aligned}\right\} \qquad (7,\ 2)$$

Differentiating (6, 6) and (6, 7) respectively with respect to F and G,

$$\left.\begin{aligned} \frac{\partial^2 T}{\partial F^2} &= -\ \tfrac{1}{2}T_F \frac{\partial\lambda}{\partial F} - \tfrac{1}{2}T_G \frac{\partial\mu}{\partial F} \\ \frac{\partial^2 T}{\partial G^2} &= -\ \tfrac{1}{2}T_G \frac{\partial\lambda}{\partial G} + \tfrac{1}{2}T_F \frac{\partial\mu}{\partial G} \end{aligned}\right\}$$

Adding these equations, and using (7, 2), we get the Laplacian equation

$$\left.\begin{aligned} \frac{\partial^2 T}{\partial F^2} + \frac{\partial^2 T}{\partial G^2} &= 0 \\ \text{Similarly,} \qquad \frac{\partial^2 T}{\partial P^2} + \frac{\partial^2 T}{\partial Q^2} &= 0 \end{aligned}\right\} \qquad (7,\ 3)$$

[13] IV, p. 86, equations (5, 7), (5, 8).

These conditions reduce to the Laplacian equations given by Weiss for L and H [see IV, p. 88, eq. (6, 3)] if we observe that, in accordance with our results in § 4.

$$T = L\,(F, G) + H\,(P, Q).$$

8. *The Maxwellian Case.*

This corresponds to $\epsilon = 1,\ \epsilon' = 0,\ \lambda = 0,\ \mu = 0.$

Also we have $F = -P = R$, and $G = Q = S$. From (6, 6) – (6, 7)

$$T_F = T_P = 1, \text{ and } T_G = T_Q = 0$$

Hence, $T = F + P = 0.$

9. *Born's Action-Function.*

The action-function $T\,(\lambda, \mu)$ corresponding to Born's case can be found by using the fact that one of the relations between invariants in this case is $G = Q$. Hence from (2, 14) – (2, 5)

$$S \cdot T_F - RT_G = ST_P + RT_Q$$

i.e., $$\frac{R}{S} = \frac{1-\epsilon}{1+\epsilon} \cdot \frac{T_F}{T_G}, \quad \text{using (4, 5)}$$

$$= -\tanh \tfrac{1}{2}\lambda \cos \tfrac{1}{2}\mu, \text{ using (6, 6)} - \text{(6, 7) and (6, 1)} - \text{(6, 2).}$$

i.e., $$\frac{T_\lambda}{T_\mu} = \tanh \tfrac{1}{2}\lambda \cot \tfrac{1}{2}\mu \qquad (9,\ 1)$$

Solving this partial differential equation we easily derive

$$T = \frac{\cosh \frac{1}{2}\lambda}{\cos \frac{1}{2}\mu}$$

as the simplest function satisfying (9, 1). We therefore write Born's action-function in the form

$$T = \frac{2 \cosh \frac{1}{2}\lambda}{\cos \frac{1}{2}\mu} - 2 \qquad (9,\ 2)$$

so that in the limiting case $\mu = 0$, this might reduce to Infeld's form.[14]

$$\left.\begin{aligned} \text{Also } R &= T_\lambda = \sinh \tfrac{1}{2}\lambda / \cos \tfrac{1}{2}\mu \\ -S &= T_\mu = (\cosh \tfrac{1}{2}\lambda \ \sin \tfrac{1}{2}\mu)/\cos^2 \tfrac{1}{2}\mu \end{aligned}\right\}$$

Substituting these values in (6, 8) – (6, 9), we get

$$\left.\begin{aligned} P &= \frac{e^{-\lambda} - \cos \mu}{1 + \cos \mu} \\ F &= \frac{e^{\lambda} - \cos \mu}{1 + \cos \mu} \end{aligned}\right\} \qquad (9,\ 3)$$

$$G = Q = -\tan \tfrac{1}{2}\mu \qquad (9,\ 4)$$

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[14] II, p. 73, equation (4, 1).

From the above equations we can easily deduce

$$\left.\begin{aligned} e^{\lambda} &= \frac{1+2F-G^2}{1+G^2} \\ \text{and} \qquad e^{-\lambda} &= \frac{1+2P-Q^2}{1+Q^2} \end{aligned}\right\} \qquad (9,\ 3a)$$

so that

$$\frac{1+2F-G^2}{1+G^2} = \frac{1+Q^2}{1+2P-Q^2} \qquad (9,\ 5)$$

which, besides $G = Q$, is the other relation between the invariants in Born's "new field theory".[15]

A much simpler method of deducing (9, 5) is from (9, 4) itself, for

$$\tan\mu = \frac{2\tan\frac{1}{2}\mu}{1-\tan^2\frac{1}{2}\mu} = \frac{2G}{1-G^2}$$

and,

$$\tan\mu = \epsilon' = \frac{FQ+PG}{FP-GQ} = \frac{G(F+P)}{FP-G^2}, \quad \text{since } G = Q$$

$$\therefore \quad \frac{2G}{1-G^2} = \frac{G(F+P)}{FP-G^2}$$

or,

$$(F+P)(1-G^2) = 2(FP-G^2)$$

which is only a simplified form of (9, 5).

From the form (9, 2) for Born's action-function we can immediately find the form of the Lagrangian. From (4, 1)

$$L = \tfrac{1}{2}T + R = \tfrac{1}{2}T + T_{\lambda}$$

$$= \frac{\cosh\frac{1}{2}\lambda}{\cos\frac{1}{2}\mu} - 1 + \frac{\sinh\frac{1}{2}\lambda}{\cos\frac{1}{2}\mu} = \frac{e^{\lambda/2}}{\cos\frac{1}{2}\mu} - 1$$

From (9, 3*a*) and (9, 4) this reduces to

$$L = \sqrt{1+2F-G^2} - 1$$

which is the standard form of Born's Lagrangian.

An alternative form for T can also be obtained in terms of the complex invariants ϕ and ψ of Weiss.

$$\phi = F + iG, \quad \psi = P - iQ$$

and using (9, 3) – (9, 4)

$$\phi(1+\cos\mu) = e^{\lambda} - e^{i\mu}$$

$$\psi(1+\cos\mu) = e^{-\lambda} - e^{-i\mu}$$

Hence $\cos^2\frac{1}{2}\mu\,(\phi+\psi) = \cosh\lambda - \cos\mu$, by addition

$$\text{i.e.,}\quad 1 + \cosh\lambda = \cos^2\tfrac{1}{2}\mu\,(\phi+\psi+2)$$

$$\text{i.e.,}\quad 2\phi + 2\psi + 4 = \frac{4\cosh^2\frac{1}{2}\lambda}{\cos^2\frac{1}{2}\mu}$$

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[15] See for *e.g.* V, p. 576, eqn. (8).

and (9, 2) can be written as

$$T = \sqrt{2\phi + 2\psi + 4} - 2 \qquad (9, 6)$$

From (9, 6) it easily follows that T = Lagrangian + Hamiltonian. In fact, $2\phi + 2\psi + 4 = 2F + 2P + 4$, since G = Q in Born's case and from a known identity[16]

$$\sqrt{2F + 2P + 4} = L + H + 2.$$

The form (9, 6) is more general than the one given by Weiss[17] which holds when the invariant G is neglected. The complex Lagrangian and Hamiltonian given by him for this case are respectively $(1 + 2\phi)^{\frac{1}{2}} - 1$, and $(1 + 2\psi)^{\frac{1}{2}} - 1$ and in this case only we have

$$-2 + \sqrt{1 + 2\phi} + \sqrt{1 + 2\psi} = \sqrt{2\phi + 2\psi + 4} - 2 \qquad (9, 7)$$

in virtue of the relation

$$\phi + 2\phi\,\psi + \psi = 0.$$

While the left-hand side of (9, 7) is not Born's action-function in the general case where G is not ignored, we have shown that the right-hand side is so.

10. *Other Functions.*

The one parameter group of action-functions given by Infeld, and the one by Hoffmann and Infeld are given in our notation by

$$T = A \cosh \frac{\lambda}{2} + B \sinh \frac{\lambda}{2} + C\lambda + D \qquad (10, 1)$$

and following the analogy of Born's action-function we can use $\cos \frac{1}{2}\mu$ or $(1/\cos \frac{1}{2}\mu)$ which tend to unity with $\mu = 0$ as a sort of guage-factor to generalise (10, 1). It appears however that there is no criterion which leads to such a kind of generalisation.

In analogy with (9, 2) we can consider

$$T = \frac{e^{\lambda)2}}{\cos \frac{1}{2}\mu} - \frac{1}{2}\lambda - 1 \qquad (10, 2)$$

as a sort of generalisation of the Hoffmann-Infeld action-function. We shall use (10, 2) to obtain the corresponding Lagrangian, develop it in power series in F and G and compare it with the Lagrangian set up by Euler and Köckel as a correction to Maxwell's theory on the basis of the phenomenon of scattering of light by light according to Dirac's theory of holes. The reason for this investigation is that with the Hoffmann-Infeld action-function (with G = 0) it is possible to make[18] a much better estimate of the

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[16] V, p. 580, eqn. (2, 3).

[17] IV, p. 91, eqn. (8, 5).

[18] This has actually not been done with the Hoffmann-Infeld action-function, but with Infeld's action-function $T = e^{-\lambda}/_2 + \frac{1}{2}\lambda - 1$ where $L = \frac{1}{2}\log(1 + F)$. It can however be shown that we get the same estimate $1/\alpha \simeq 130$ with the action-function $T = e^{-\lambda/2} - \log\lambda - 1$ where $H = \frac{1}{2}\log(1 + P)$.

fine-structure constant than with Born's action-function. In the latter case there is no correspondence (as mentioned in the Introduction) in the terms containing G^2 in the power series expansion. We shall try to find if this correspondence as well as the better estimate for the fine-structure constant could be secured from (10, 2).

The derivatives T_λ and T_μ are given from (10, 2) in the form

$$\left.\begin{aligned} T_\lambda &= \frac{e^{\lambda/2}}{2\cos\frac{1}{2}\mu} - \tfrac{1}{2} \\ T_\mu &= \frac{e^{\lambda/2}\sin\frac{1}{2}\mu}{2\cos^2\frac{1}{2}\mu} \end{aligned}\right\} \tag{10, 3}$$

Also $T_\lambda = R$ gives

$$e^{\lambda/2} = (1 + 2R)\cos\tfrac{1}{2}\mu \tag{10, 4}$$

From (6, 8) and (6, 9)—(using (10, 3) and (10, 4)

$$\left.\begin{aligned} 2F &= (1+2R)^2 - (1+2R)\cos^2\tfrac{1}{2}\mu \\ 2G &= -(1+2R)\sin\tfrac{1}{2}\mu\cos\tfrac{1}{2}\mu \end{aligned}\right\} \tag{10, 5}$$

Since our idea is to obtain the power-series expansion of the Lagrangian obtained from (10, 2) upto the second power in F and G, we assume[19]

$$\left.\begin{aligned} R &= a_0F + a_1F^2 + a_2G^2 \\ \text{and} \qquad \cos\tfrac{1}{2}\mu &= 1 + b_0F + b_1F^2 + b_2G^2 \end{aligned}\right\} \tag{10, 6}$$

since $R \to 0$ as F and $G \to 0$, while $\cos\frac{1}{2}\mu \to 1$. Substituting (10, 6) in (10, 5) and equating coefficients of like powers after neglecting terms of order higher than two, we get,

$$a_0 = 1,\ a_1 = -2,\ a_2 = -2$$

$$b_0 = b_1 = 0,\ b_2 = -2$$

i.e.,

$$\left.\begin{aligned} R &= F - 2F^2 - 2G^2 \\ \cos\tfrac{1}{2}\mu &= 1 - 2G^2 \end{aligned}\right\} \tag{10, 6a}$$

The Lagrangian L corresponding to (10, 2) is given by

$$\begin{aligned} L &= \tfrac{1}{2}T + R \\ &= 2R - \tfrac{1}{4}\lambda, \text{ using (10, 4)} \end{aligned} \tag{10, 7}$$

Again from (10, 4) the power-series expansion for λ, using (10, 6*a*), is given by

$$\begin{aligned} \lambda &= \log(1 + 4F - 4F^2 - 12G^2) \\ &= 4F - 12F^2 - 12G^2 \end{aligned}$$

and (10, 7) reduces to

$$L = F - F^2 - G^2$$

or

$$L = \tfrac{1}{2}(\vec{B}^2 - \vec{E}^2) - \tfrac{1}{4}\{(\vec{B}^2 - \vec{E}^2)^2 + 4(\vec{B}\cdot\vec{E})^2\} \tag{10, 8}$$

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[19] The invariant G appears as square only, as pointed out by Weiss. See V, p. 89.

whereas the Lagrangian of Euler and Köckel is

$$L = \tfrac{1}{2}(\vec{B}^2 - \vec{E}^2) - \frac{1}{90\pi m^4 a}\left\{(\vec{B}^2 - \vec{E}^2)^2 + 7(\vec{B}\vec{E})^2\right\} \tag{10, 9}$$

and Born's Lagrangian is

$$L = \tfrac{1}{2}(\vec{B}^2 - \vec{E}^2) - \tfrac{1}{8}\{(\vec{B}^2 - \vec{E}^2)^2 + 4(\vec{B}\vec{E})^2\}. \tag{10, 10}$$

From the above equations it is clear that while (10, 8) and (10, 10) agree in the second order terms in G, neither of them coincides with (10, 9). Also (10, 8) gives the estimate $1/a \simeq 130$, as done by Infeld's action-function $(\gamma = 0)$.

11. *Coincidence with the Lagrangian of Euler and Köckel.*

The question that now arises is whether it is possible to manipulate the action-function so as to obtain the coincidence with (10, 9) with the further restriction that it should reduce itself to the Hoffmann-Infeld action-function when $\mu \to 0$. Now, it can be shown that, if we alter the second term of (10, 2) by multiplying or dividing with the factor $\cos \frac{1}{2}\mu$, the relations between the invariants F, G, P, Q will not be algebraic. Since this is desirable on grounds of simplicity, we can try with

$$T = \frac{e^{\lambda/2}}{\cos \frac{1}{2}\mu} - \frac{\lambda}{2} - f(\cos \tfrac{1}{2}\mu), \text{ such that } f(x) = 1, \text{ for } x = 1$$

or

$$T = \frac{e^{\lambda/2}}{\cos \frac{1}{2}\mu} - \frac{\lambda}{2} - f(\nu), \quad \nu = \cos \tfrac{1}{2}\mu. \tag{11, 1}$$

We have $R = T_\lambda = \dfrac{e^{\lambda/2}}{2\cos\frac{1}{2}\mu} - \frac{1}{2}$; and $e^{\lambda/2} = (1 + 2R)\cos\frac{1}{2}\mu$ (10, 4)

$$T_\mu = \frac{e^{\lambda/2}\sin\frac{1}{2}\mu}{2\cos^2\frac{1}{2}\mu} + \frac{f'(\nu)\sin\frac{1}{2}\mu}{2}.$$

$$L = 2R - \tfrac{1}{4}\lambda + \tfrac{1}{2}\{1 - f(\nu)\} \tag{11, 2}$$

In consonance with (10, 6*a*), we assume the power-series expansions upto the second order

$$\begin{aligned} R &= F - 2F^2 + a\,G^2, \\ \cos\tfrac{1}{2}\mu &= 1 + \beta\,G^2, \\ f(\nu) &= 1 + l\,G^2, \\ f'(\nu) &= k + k'G^2. \end{aligned} \tag{11, 3}$$

From (6, 8)—(6, 9), the expressions for F and G are given by

$$\left.\begin{aligned} 2F &= (1 + 2R)^2 - (1 + 2R)\cos^2\tfrac{1}{2}\mu + (1 + 2R)f'(\nu)\sin^2\tfrac{1}{2}\mu\cos\tfrac{1}{2}\mu \\ -2G &= (1 + 2R)f'(\nu)\sin\tfrac{1}{2}\mu\cos^2\tfrac{1}{2}\mu + (1 + 2R)\sin\tfrac{1}{2}\mu\cos\tfrac{1}{2}\mu \end{aligned}\right\} \tag{11, 4}$$

Substituting (11, 3) in (11, 4) wherein we take the second equation after squaring both sides, and comparing coefficients, we have

$$\begin{aligned} \beta &= -2/(k + 1)^2 \\ a &= -2/(k + 1) \end{aligned} \tag{11, 5}$$

From (10, 4) we find λ, and substitute for λ, R and $f(\nu)$ in (11, 2) obtaining

$$L = F - F^2 + G^2(\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}l). \qquad (11, 6)$$

If (11, 6) should coincide in the factor of G^2 with (10, 9) we should have

$$\alpha - \tfrac{1}{2}\beta - \tfrac{1}{2}l = -7/4$$

$$ie., \quad 2\beta + 2i - 4\alpha = 7 \qquad (11, 7)$$

We now still further specialise (11, 1) by putting $f(\nu) = \nu^n$, and determine the value of n so as to satisfy (11, 7).

$$\text{If } f(\nu) = \nu^n = (1 + \beta G^2)^n = 1 + n\beta G^2, \text{ to the first approximation}$$

$$f'(\nu) = n\nu^{n-1} = n(1 + \{n-1) G^2\} \qquad ,,$$

Hence,

$$\left.\begin{aligned} l &= n\beta \\ k &= n \end{aligned}\right\} \qquad (11, 8)$$

Putting in (11, 5) and (11, 8) in (11, 7) we get the equation for n

$$\frac{8}{n+1} - \frac{4}{(n+1)^2} - \frac{4n}{(n+1)^2} = 7$$

or $\quad 7n^2 + 10n + 3 = 0 \qquad (11, 9)$

This equation gives $n = -1$ or $n = -3/7$; but $n = -1$ would make the values of α and β infinite, and consequently of R and $\cos\frac{1}{2}\mu$. Neglecting this value we can take $n = -3/7$ and obtain an action-function which has the desired coincidence of the Lagrangian.

In an exactly similar way, we can show that

$$T = e^{\lambda/2}\cos\tfrac{1}{2}\mu - \frac{\lambda}{2} - (\cos\tfrac{1}{2}\mu)^n,$$

where $n = 8/7$ also gives an action function of the desired type.

12. *Conclusion.*

It is practically certain that the method adopted in the previous article can be applied to several other action-functions so as to obtain the Lagrangian of the type searched for. It appears therefore that even this does not restrict the type of action-function; nor should this be surprising since we are merely comparing coefficients of particular terms in power-series expansions and not going on any general principle. Anyway it serves to bring out the possibilities of the field-theory.

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