$$= \frac{1}{a}\left[\left(1+\frac{a^2}{2}+\cdots\right)\left(1-\frac{a^2}{\lfloor 3}\right) \quad -1\right]$$

$$= \frac{1}{a}\left[\, a^2/3 \,\right] \quad = \quad \frac{a}{3} = \frac{\mu H}{3kT}.$$

Substituting for μ ~~the value~~ $gj\beta$

and for $\mu \cos\theta$ $\quad g m \beta$

we obtain ~~for~~ $\bar{\mu} = \dfrac{\sum m g \beta\, e^{mg\beta H/kT}}{\sum e^{mg\beta H/kT}}$

~~Let g~~ $\quad \dfrac{\bar{\mu}}{\mu} = \dfrac{1}{j}\,\dfrac{\sum m\, e^{m\xi}}{\sum e^{m\xi}}$

where $\xi = \dfrac{g\beta H}{kT}$ $\qquad m = -j, \ldots +j$

~~$\sum e^{m\xi} = e^{\xi}\left[e^{-j} + e^{-(j-1)} + \cdots + e^{j}\right]$~~

~~$= e^{-j\xi}\,[$~~

~~$\dfrac{\bar{\mu}}{\mu}$~~

p. 86-7

$= j$

Let $e^{\xi} = x$

$$\sum e^{m\xi} = \sum x^{m}$$

$$= x^{-j} + x^{-j+1} + \cdots + x^{+j}$$

$$= \frac{x^{j+1} - x^{-j}}{x-1}$$

$$\sum m e^{m\xi} = \sum m x^{m} = \sum x \frac{d}{dx}(x^{m})$$

(x indep of m)

$$= x \frac{d}{dx} \sum x^{m}$$

$$= x \cdot \frac{d}{dx} \left[\frac{x^{j+1} - x^{-j}}{x-1} \right]$$

$$= \frac{x}{x-1} \left[(j+1) x^{j} + j x^{-j-1} \right] - \frac{x}{(x-1)^2} \left[x^{j+1} - x^{-j} \right]$$

$$= \frac{(j+1) x^{j+1} + j x^{-j}}{x-1} - \frac{x^{j+2} - x^{-j+1}}{(x-1)^2}$$

$$\therefore \frac{\sum m e^{m\xi}}{\sum e^{m\xi}} = \underbrace{\frac{(j+1) x^{j+1} + j x^{-j}}{x^{j+1} - x^{-j}}}_{\mathrm{I}} - \underbrace{\frac{x^{j+2} - x^{-j+1}}{(x^{j+1} - x^{-j})(x-1)}}_{\mathrm{II}}$$

$$\mathrm{I} = \left(j + \tfrac{1}{2}\right) \frac{x^{j+\frac{1}{2}} + x^{-(j+\frac{1}{2})}}{x^{j+\frac{1}{2}} - x^{-(j+\frac{1}{2})}} + \frac{1}{2} - \frac{1}{2}$$

$$\mathrm{II} = \frac{x}{x-1}$$

$$\underline{\mathrm{I}} = \frac{(j+1)\, x^{j+\frac12} + j\, x^{-(j+\frac12)}}{x^{j+\frac12} - x^{-(j+\frac12)}}$$

$$= (j+\tfrac12)\, \frac{x^{j+\frac12} + x^{-(j+\frac12)}}{x^{j+\frac12} - x^{-j+\frac12}} + \frac12$$

$$= (j+\tfrac12)\, \coth.\ \xi(j+\tfrac12) + \frac12$$

$$\underline{\mathrm{II}} = \frac{x}{x-1}.$$

$$\therefore \frac{\sum m e^{m\xi}}{\sum e^{m\xi}} = \underline{\mathrm{I}} + \underline{\mathrm{II}}$$

$$= (j+\tfrac12)\, \coth\ \xi(j+\tfrac12) + \frac12 + \frac{x}{x-1}$$

$$= -\frac12 \cdot \frac{x+1}{x-1}$$

$$= -\frac12 \frac{x^{\frac12} + x^{-1/2}}{x^{1/2} - x^{-1/2}}$$

$$= -\tfrac12 \coth.\ \xi/2.$$

$$\therefore \underline{\mathrm{I}} + \underline{\mathrm{I}} = (j+\tfrac12) \coth \xi (j+\tfrac12) - \tfrac12 \coth.\ \xi/2$$

$\therefore \frac{\bar{\mu}}{\mu_0}$ = expn. (8) of Stoner p. 487.

I For large values of j

$$⑧ = \coth a - \frac{1}{2j} \times \frac{2j}{a} \times \left[1 + \frac{a^2}{12 j^2} + \cdots\right]$$

{The value of $\coth x$ for small values of $x = \frac{1}{x}\left[1 + \frac{x^2}{3} \cdots\right]$}

$$= \coth a - \frac{1}{a}.$$ Same as (6)

II For $j = 1/2$

$$\frac{\bar{\mu}}{\mu} = 2 \coth 2a - \coth a$$

$$= 2 \times \frac{e^{2a} + e^{-2a}}{e^{2a} - e^{-2a}} - \frac{e^{a} + e^{-a}}{e^{a} - e^{-a}}$$

$$= \frac{2\left(e^{2a} + e^{-2a}\right) - \left(e^{a} + e^{-a}\right)^2}{e^{2a} - e^{-2a}}$$

$$= \frac{e^{2a} + e^{-2a} - 2}{e^{2a} - e^{-2a}} = \frac{\left(e^{a} - e^{-a}\right)^2}{\left(e^{a} + e^{-a}\right)\left(e^{a} - e^{-a}\right)}$$

$$= \frac{e^{a} - e^{-a}}{e^{a} + e^{-a}}$$

$$= \tanh a.$$

Saturation Effects

From the curves on p. 288

for $j = \frac{1}{2}$ the deviation from linearity appears after

$a = \cdot 3$ eq. i.e. $\frac{9.17 \times 10^{-21} \times H}{1.37 \times 10^{-16} \times T} \sim 0.3$

i.e. $\frac{H}{T} \sim 4500$

i.e. with $H = 18{,}000$ gauss $T = 4^\circ K$.

On the other hand with $S = 5/2$ (Mn^{++})

Critical value $a \sim 1/2$ i.e. $\frac{H}{T} \sim \frac{7500}{\sqrt{35}} \sim 1300$.

With about 19,000 gauss $T = 15^\circ . K$

With Gd^{+++} $\frac{H}{T} \sim \frac{7500}{\sqrt{7 \times 9}} \sim \frac{2500}{\sqrt{7}}$

with $H = 20{,}000$ $T \sim 21^\circ K$

~~Stoner~~ p. 354 $\quad a \ll 1$ (~~to~~ derive 8b)

$$\frac{\sigma}{\sigma_0} = \frac{(2J+1)^2}{4J^2}\times\frac{a}{3} - \frac{1}{4J^2}\cdot\frac{a}{3} - \frac{(2J+1)^4}{(2J)^4}\frac{a^3}{45} + \frac{1}{(2J)^4}\cdot\frac{a^3}{45}$$

$$= \frac{a}{3}\times\frac{J+1}{J} - \frac{a^3}{90}\times\frac{(J+1)(2J^2+2J+1)}{J^3}$$

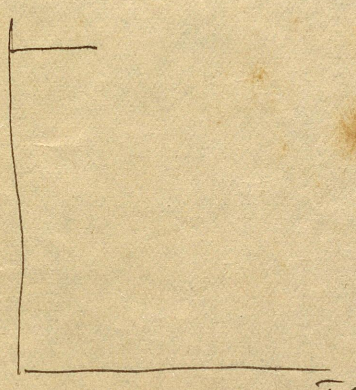
p. 355 to derive (12) $\quad$ From (10) $\quad a = \frac{3J}{J+1}\cdot\frac{\theta}{T}\cdot\frac{\sigma}{\sigma_0}$

From 8(b) wh is applicable to this case since $\frac{\sigma}{\sigma_0}$, hence a, is small

$$1\frac{\sigma}{\sigma_0} = \frac{\theta}{T}\frac{\sigma}{\sigma_0} - \frac{\cancel{(J+1)}(2J^2+2J+1)}{10\,\cancel{90}\,J^3}\times\frac{27J^3}{(J+1)^{3}}\left(\frac{\theta}{T}\right)^3\times\left(\frac{\sigma}{\sigma_0}\right)^{2}$$

$$\left(\frac{\sigma}{\sigma_0}\right)^2 = \frac{10}{3}\frac{(J+1)^2}{2J^2+2J+1}\left(\left(\frac{T}{\theta}\right)^2 - \left(\frac{T}{\theta}\right)^3\right) = \left(1-\frac{T}{\theta}\right)$$

since $\frac{T}{\theta}\sim 1$.



In the classical case $\left[d\left(\frac{\sigma}{\sigma_0}\right)\Big/ d\left(\frac{T}{\theta}\right)\right]_{T\to 0} = -\frac{1}{3}$

whereas in the quantum theory $= 0$.

$$\left[\frac{d\left(\frac{\sigma}{\sigma_0}\right)^2}{d\left(\frac{T}{\theta}\right)}\right]_{T\to\theta} = -\frac{10}{3}\frac{(J+1)^2}{2J^2+2J+1}$$

Magnetic energy per c.c. $= -\int H\,dI = -\int \alpha I\,dI = -\frac{1}{2}\alpha I^2$

~~Per gm mol $= \frac{1}{2}\alpha I^2\times\frac{M}{\rho} = \frac{1}{2}\alpha\cdot\frac{\sigma^2}{\sigma_0^2}\frac{\rho}{M}$~~ $\quad I^2 = \left(\frac{\sigma\times\rho}{M}\right)^2$

$$\alpha\times\frac{N J(J+1) g^2\beta^2}{3k} = \theta \qquad I^2 = \left(\frac{\sigma\rho}{M}\right)^2 = \left(\frac{\sigma}{\sigma_0}\right)^2\times\frac{\rho^2}{M^2}\times N^2J^2g^2\beta^2$$

Magc. energy per gm mol $U = -\frac{1}{2}\alpha I^2\times\frac{M}{\rho}$

$$= -\frac{1}{2}\times\frac{3k\theta\times NJ^2 g^2\beta^2}{N J(J+1) g^2\beta^2}\left(\frac{\sigma}{\sigma_0}\right)^2$$

$$= -\frac{3R\theta}{2}\cdot\frac{J}{J+1}\left(\frac{\sigma}{\sigma_0}\right)^2$$

Max. when $\sigma = \sigma_0$

i.e.

Stoner p. 354 $\quad a \ll 1$ (to derive 8b)

$$\frac{\sigma}{\sigma_0} = \frac{(2J+1)^2}{4J^2}\times\frac{a}{3} - \frac{1}{4J^2}\cdot\frac{a}{3} - \frac{(2J+1)^4}{(2J)^4}\frac{a^3}{45} + \frac{1}{(2J)^4}\cdot\frac{a^3}{45}$$

$$= \frac{a}{3}\times\frac{J+1}{J} - \frac{a^3}{90}\times\frac{(J+1)(2J^2+2J+1)}{J^3}$$

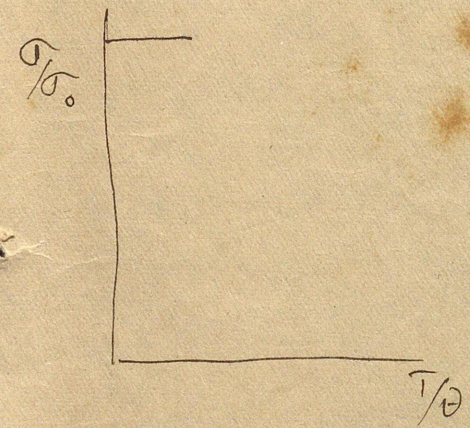
p. 355 to derive (12) $\qquad$ From (10) $\quad a = \frac{3J}{J+1}\cdot\frac{\theta}{T}\cdot\frac{\sigma}{\sigma_0}$

From 8(b) which is applicable to this case since $\frac{\sigma}{\sigma_0}$, hence a, are small

$$1\frac{\sigma}{\sigma_0} = \frac{\theta}{T}\frac{\sigma}{\sigma_0} - \frac{\cancel{(J+1)}(2J^2+2J+1)}{10\,J^3}\times\frac{27J^3}{(J+1)^2}\left(\frac{\theta}{T}\right)^3\times\left(\frac{\sigma}{\sigma_0}\right)^2$$

$$\left(\frac{\sigma}{\sigma_0}\right)^2 = \frac{10}{3}\frac{(J+1)^2}{2J^2+2J+1}\left(\left(\frac{T}{\theta}\right)^2 - \left(\frac{T}{\theta}\right)^3\right) = \left(1-\frac{T}{\theta}\right)$$

since $\frac{T}{\theta}\sim 1$.



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$$\left[\frac{d\left(\frac{\sigma}{\sigma_0}\right)^2}{d\left(\frac{T}{\theta}\right)}\right]_{T\to\theta} = -\frac{10}{3}\frac{(J+1)^2}{2J^2+2J+1}$$

Magnetic energy per cc. $= -\int H\,dI = -\int \alpha I\,dI = -\frac{1}{2}\alpha I^2$

~~Per gm mol~~ $\cancel{= \frac{1}{2}\alpha I^2 \times \frac{M}{\rho} = \frac{1}{2}\alpha\cdot\frac{\sigma^2}{\sigma_0^2}\frac{\rho}{M}}$ $\quad I^2 = \left(\frac{\sigma\times\rho}{M}\right)^2$

$$\alpha \times \frac{n\,J(J+1)g^2\beta^2}{3k} = \theta \qquad I^2 = \left(\frac{\sigma\rho}{M}\right)^2 = \left(\frac{\sigma}{\sigma_0}\right)^2\times\frac{\rho^2}{M^2}\times N^2J^2g^2\beta^2$$

Magn. energy per gm mol $U = -\frac{1}{2}\alpha I^2 \times \frac{M}{\rho}$

$$= -\frac{1}{2}\times\frac{3k\theta\times N J^2 g^2\beta^2}{N J(J+1)g^2\beta^2}\frac{\rho}{M}\left(\frac{\sigma}{\sigma_0}\right)^2$$

$$= -\frac{3R\theta}{2}\cdot\frac{J}{J+1}\left(\frac{\sigma}{\sigma_0}\right)^2$$

Max. when $\sigma = \sigma_0$ i.e.