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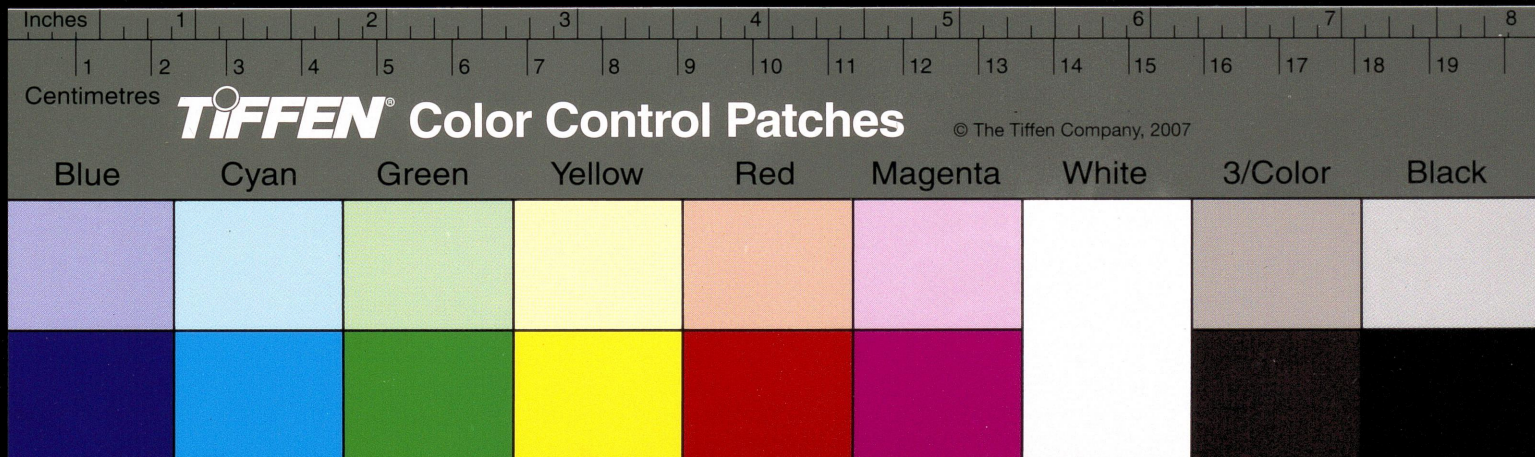
NOTES ON DYNAMICS—I

BY

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NOTES ON DYNAMICS—I

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§1. THIS note is concerned with a point mass which starts from rest on the upper part of a smooth curve situated in a vertical plane, and the problem is to derive a number of theorems relating to the height at which the mass leaves the curve.

§2. The position at which the particle leaves the curve is given by the well-known theorem (Routh¹) that the velocity at the point is that due to one-fourth the chord of curvature in the direction of the resultant force, *i.e.*,

$$v^2 = g\rho \cos \theta, \quad (1)$$

where θ is the angle that the normal makes with the vertical, ρ the radius of curvature at the point, and gravity the only external force. If h be the initial height at which the particle starts, and y the height of leaving, we have

$$v^2 = 2g(h - y) \quad (2)$$

using which (1) reduces to

$$\rho \cos \theta = 2(h - y). \quad (3)$$

In this equation, the dynamical problem is reduced to a geometrical one, and special forms of the curve give different results for the height of leaving.

§3. Let us now consider conics, and first of all a parabola with its axis horizontal. In this case, the height of leaving is given by

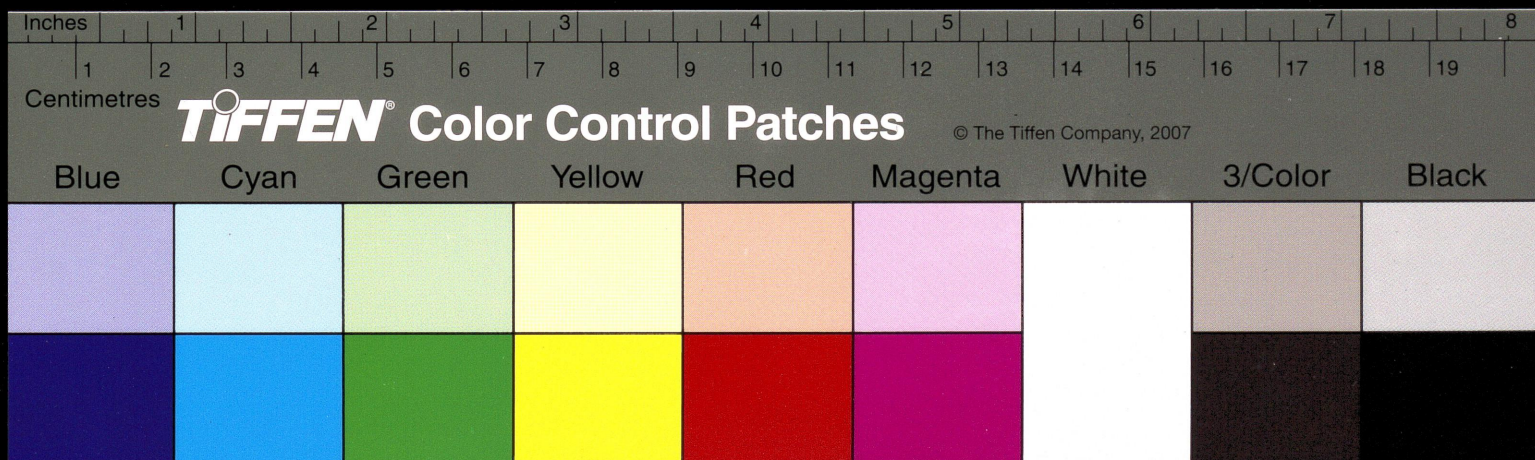
$$y^3 + 3p^2y - 2p^2h = 0, \quad (4)$$

where p is the parameter. This result is known (Appell²). For the parabola with its axis vertical one has the trivial result $y = \infty$, where y is the depth below a fixed horizontal line, since the parabola is the natural trajectory for a particle under gravity.

A result of some interest can be obtained by considering the family of parabolas having a common horizontal axis, and passing through a common point P at a height h above this axis. Choosing the origin as the foot of the perpendicular from P on this axis which can be taken $y = 0$, the equation to the system of curves can be written as

$$y^2 = h^2 - 2px, \quad (5)$$

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where p is the parameter of any parabola of the system. If we now consider particles started from P from rest and sliding on the different parabolas, the ordinate of the point at which the particle leaves the curve is given by equation (4), *i.e.*,

$$y^3/p^2 + 3y - 2h = 0. \quad (4')$$

Eliminating p from (4') and (5) we get the locus of the point of leaving as the quintic curve

$$4x^2y^3 = (2h - 3y)(h^2 - y^2)^2. \quad (6)$$

This curve has a very simple shape resembling the probability curve with the highest point at $(0, 2h/3)$ and having $y=0$ for an asymptote. The points $P, P' (0, \pm h)$ are acnodes on the curve. It might be noticed that it would not be right to consider PP' as the limiting position for the system of parabolas, and take P as the point of leaving for this limiting case. This is also brought out by equation (4') which gives $y = \frac{2}{3}h$ for $p \rightarrow \infty$.

For a central conic, say an ellipse, with its major axis horizontal, y is given by the equation

$$\frac{c^2 y^3}{b^4} + 3y - 2h = 0, \quad (7)$$

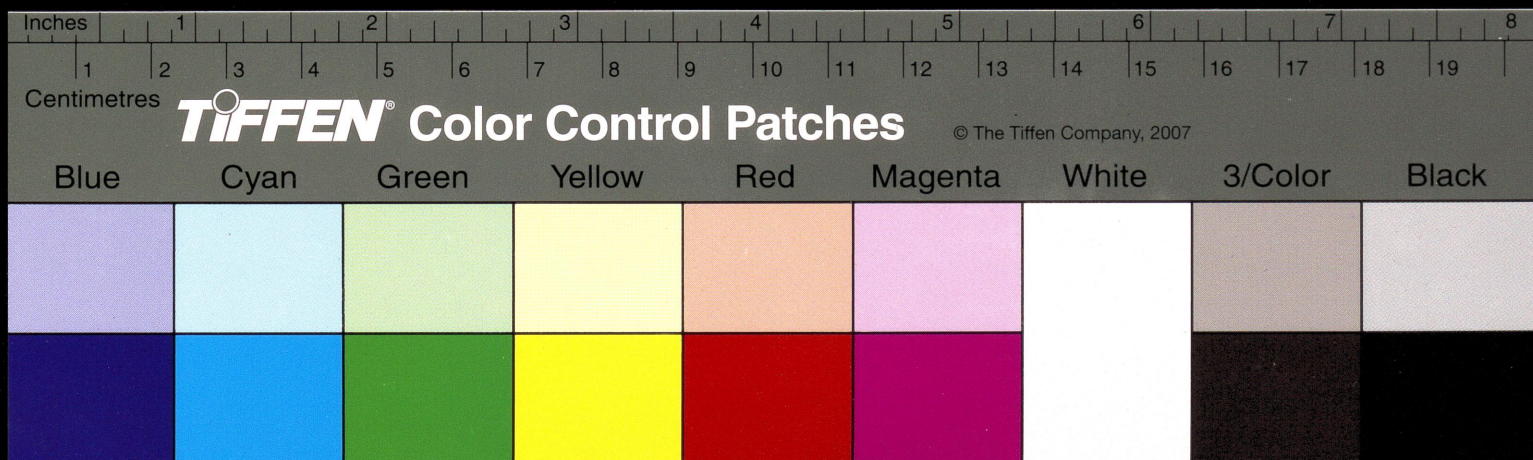
where a, b are the semi-axes, and $c^2 = a^2 - b^2$. The coefficient of the first term in (7) is the reciprocal of the square of the perpendicular from the focus on the directrix. This leads to the theorem (Ionesco³) that for conics with the same focus and directrix, and masses with the same initial height, the height of leaving is independent of the eccentricity.

A theorem analogous to the above can be obtained by considering the ellipse to have its major axis vertical. If heights be measured above a horizontal line through the centre, it follows from (3) that y is given by

$$\frac{c^2 y^3}{a^2} - 3y + 2h = 0. \quad (8)$$

The coefficient of y^3 in (8) is the reciprocal of the square of the perpendicular from the centre on the directrix. This leads to the theorem *that for conics with the same centre and directrix, major axis vertical, and masses with the same initial height, the height of leaving is independent of the eccentricity.*

As in the case of parabola, we might also consider the family of conics (ellipses) having a common horizontal major axis, passing through a common point P at a height h above the axis, and of given eccentricity. In this case the locus of the points of leaving of particles started from P is given by



i.e., $\frac{d\rho}{d\psi} = 2 \tan \psi$, $\rho = c \sec^2 \psi$ or $s = c \tan \psi$, which is the intrinsic equation of the catenary.

An entirely similar thing is true for the *cycloid* in regard to heights of starting and leaving measured above its base. In this case $\rho \cos \theta = 2y$, and equation (3) gives

$$y = \frac{1}{2} h. \quad (16)$$

Thus for particles started from a given height above the common base of a system of cycloids having vertex upwards, the height of leaving is independent of the radius of the generating circle.

§5. A slightly different type of problem is that in which we require the curve for which the difference between the height of leaving, and the initial height is a constant. In this case

$$\rho \cos \theta = k \quad (17)$$

i.e., $\rho \cos \psi = k$.

Differentiating with respect to ψ , we get

$$\frac{d\rho}{d\psi} = \rho \tan \psi$$

i.e., $\rho = a \sec \psi$

and $S = a \log (\sec \psi + \tan \psi)$ (18)

which is the intrinsic equation of the *catenary of uniform strength*. This can also be seen otherwise for, in such a catenary, we know that $T \cos \psi = \text{const.}$, where T the tension varies as the mass per unit length which in turn varies as ρ . Thus we have the theorem that *for a particle moving from rest on a smooth curve in a vertical plane in the form of a catenary of uniform strength, the difference between the depth below the horizontal tangent of the point of leaving and the initial depth is a constant, and conversely.*

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- ² P. Appell ... *Traité de Mécanique Rationnelle*, 1, 479.
- ³ D. V. Ionesco ... *Bull. Soc. Sci. Cluj.*, 1939, 9, 294-98 ; also *Math Rev.*, 1940, 1, 27.

