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NOTES ON DYNAMICS—I

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NOTES ON DYNAMICS—I

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§1. This note is concerned with a point mass which starts from rest on the upper part of a smooth curve situated in a vertical plane, and the problem is to derive a number of theorems relating to the height at which the mass leaves the curve.

§2. The position at which the particle leaves the curve is given by the well-known theorem (Routh[1]) that the velcoity at the point is that due to one-fourth the chord of curvature in the direction of the resultant force, *i.e.*,

$$v^2 = g\rho \cos \theta, \tag{1}$$

where θ is the angle that the normal makes with the vertical, ρ the radius of curvature at the point, and gravity the only external force. If h be the initial height at which the particle starts, and y the height of leaving, we have

$$v^2 = 2\,g\,(h - y) \tag{2}$$

using which (1) reduces to

$$\rho \cos \theta = 2\,(h - y). \tag{3}$$

In this equation, the dynamical problem is reduced to a geometrical one, and special forms of the curve give different results for the height of leaving.

§3. Let us now consider conics, and first of all a parabola with its axis horizontal. In this case, the height of leaving is given by

$$y^3 + 3\,p^2 y - 2\,p^2 h = 0, \tag{4}$$

where p is the parameter. This result is known (Appell[2]). For the parabola with its axis vertical one has the trivial result $y = \infty$, where y is the depth below a fixed horizontal line, since the parabola is the natural trajectory for a particle under gravity.

A result of some interest can be obtained by considering the family of parabolas having a common horizontal axis, and passing through a common point P at a height h above this axis. Choosing the origin as the foot of the perpendicular from P on this axis which can be taken $y = 0$, the equation to the system of curves can be written as

$$y^2 = h^2 - 2\,px, \tag{5}$$

where p is the parameter of any parabola of the system. If we now consider particles started from P from rest and sliding on the different parabolas, the ordinate of the point at which the particle leaves the curve is given by equation (4), *i.e.*,

$$y^3/p^2 + 3\,y - 2\,h = 0. \tag{4'}$$

Eliminating p from (4′) and (5) we get the locus of the point of leaving as the quintic curve

$$4\,x^2y^3 = (2\,h - 3\,y)\,(h^2 - y^2)^2. \tag{6}$$

This curve has a very simple shape resembling the probability curve with the highest point at $(0,\ 2\,h/3)$ and having $y = 0$ for an asymptote. The points P, P′ $(0,\ \pm\,h)$ are acnodes on the curve. It might be noticed that it would not be right to consider P P′ as the limiting position for the system of parabolas, and take P as the point of leaving for this limiting case. This is also brought out by equation (4′) which gives $y = \frac{2}{3}\,h$ for $p \to \infty$.

For a central conic, say an ellipse, with its major axis horizontal, y is given by the equation

$$\frac{c^2\,y^3}{b^4} + 3\,y - 2\,h = 0, \tag{7}$$

where a, b are the semi-axes, and $c^2 = a^2 - b^2$. The coefficient of the first term in (7) is the reciprocal of the square of the perpendicular from the focus on the directrix. This leads to the theorem (Ionesco[3]) that for conics with the same focus and directrix, and masses with the same initial height, the height of leaving is independent of the eccentricity.

A theorem analogous to the above can be obtained by considering the ellipse to have its major axis vertical. If heights be measured above a horizontal line through the centre, it follows from (3) that y is given by

$$\frac{c^2\,y^3}{a^2} - 3\,y + 2\,h = 0. \tag{8}$$

The coefficient of y^3 in (8) is the reciprocal of the square of the perpendicular from the centre on the directrix. This leads to the theorem *that for conics with the same centre and directrix, major axis vertical, and masses with the same initial height, the height of leaving is independent of the eccentricity*.

As in the case of parabola, we might also consider the family of conics (ellipses) having a common horizontal major axis, passing through a common point P at a height h above the axis, and of given eccentricity. In this case the locus of the points of leaving of particles started from P is given by eliminating λ and b from the equations

$$\rho^2 (x-\lambda) + y^2 = b^2 \tag{9}$$

$$\rho^2 \lambda^2 + h^2 = b^2 \tag{10}$$

$$(\rho^2 - 1)\, y^3 = b^2 (2h - 3y), \tag{11}$$

where $\rho = a/b$ = ratio of major and minor axes, and (11) is the same as (7). The result of elimination is again a quintic curve of the same shape as (6) but without the acnodes, and its equation is

$$4\rho^2 (\rho^2 - 1)\, x^2 y^3 = (2h - 3y)\, \{(\rho^2 x^2 + y^2 - h^2)^2 + 4\rho^2 h^2 x^2\} \tag{12}$$

For the case $\rho = 1$ or $a = b$, *i.e.*, the family of circles, the locus is

$$y = \tfrac{2}{3} h \tag{13}$$

as is also evident from (7) with $c^2/b^4 = 0$.

§ 4. We shall now consider problems of the type where the height of leaving bears a constant ratio to the initial height. The simplest case is that of a *circle* for which (3) reduces to

$$y = 2(h - y)$$

$$\text{or } y = \tfrac{2}{3} h.$$

Equally simple is the case of a *catenary* having its axis vertical and vertex upwards. If h and y be now interpreted as the *depths* below the directrix of the points of starting and leaving, (3) becomes

$$\rho \cos \theta = 2(y - h) \tag{3$'$}$$

For the catenary ρ = length of the normal and $\rho \cos \theta = y$, *i.e.*,

$$y = 2h \tag{14}$$

We can therefore state the

THEOREM: *Particles start from rest on a system of catenaries having a common horizontal directrix and vertical axis, with vertex upwards: For a given initial depth below the directrix, the depth of leaving is independent of the parameter of the catenary and equal to twice the initial depth.*

The converse theorem is also true as can be seen by solving the differential equation

$$\rho \cos \theta = y \tag{15}$$

obtained by putting $h = \frac{1}{2} y$ in (3′), or the equation

$$\rho \cos \psi = y. \tag{15$'$}$$

Differentiating (15′) with respect to ψ

$$\frac{d\rho}{d\psi} \cos \psi - \rho \sin \psi = \frac{dy}{d\psi} = \rho \sin \psi$$

i.e., $\frac{d\rho}{d\psi} = 2 \tan \psi$, $\rho = c \sec^2 \psi$ or $s = c \tan \psi$, which is the intrinsic equation of the catenary.

An entirely similar thing is true for the *cycloid* in regard to heights of starting and leaving measured above its base. In this case $\rho \cos \theta = 2y$, and equation (3) gives

$$y = \tfrac{1}{2} h. \tag{16}$$

Thus *for particles started from a given height above the common base of a system of cycloids having vertex upwards, the height of leaving is independent of the radius of the generating circle.*

§5. A slightly different type of problem is that in which we require the curve for which the difference between the height of leaving, and the initial height is a constant. In this case

$$\rho \cos \theta = k \tag{17}$$

i.e., $$\rho \cos \psi = k.$$

Differentiating with respect to ψ, we get

$$\frac{d\rho}{d\psi} = \rho \tan \psi$$

i.e., $$\rho = a \sec \psi$$

and $$S = a \log (\sec \psi + \tan \psi) \tag{18}$$

which is the intrinsic equation of the *catenary of uniform strength*. This can also be seen otherwise for, in such a catenary, we know that $T \cos \psi =$ const., where T the tension varies as the mass per unit length which in turn varies as ρ. Thus we have the theorem that *for a particle moving from rest on a smooth curve in a vertical plane in the form of a catenary of uniform strength, the difference between the depth below the horizontal tangent of the point of leaving and the initial depth is a constant, and conversely.*

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[1] E. J. Routh .. *Treatise on the Dynamics of a Particle*, 1898, 114.

[2] P. Appell .. *Traité de Mécanique Rationelle*, **1**, 479.

[3] D. V. Ionesco .. *Bull. Soc. Sci. Cluj.*, 1939, **9**, 294–98 ; also *Math Rev.*, 1940, **1**, 27.

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