Inaugural address delivered before the

St. Joseph's College Math & Science Assn.

on

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Subject – Topology.

Mr President, Ladies & Gentlemen,

I am deeply sensible of the honour you have done me in asking me to deliver the inaugural address of your Association. On a previous occasion I had the privilege of addressing you on an astronomical topic, and when the subject of this evening's talk was left to my choice I ~~immediately~~ thought I had better change my subject and say something about one of the most recent branches of ~~mod~~ pure mathematics viz topology.

Topology is one of the youngest branches of mathematics, and research in that branch ~~are~~ is proceeding at a tremondous pace. If you take any mathematical journal of some standing the odds are ten to one against not finding an article or contribution on topology. I hope you will realise the importance attached to the subject by mathematicians when I tell you that the year before last there was an international Conference on topology at Warsaw at which the present position of the subject and the possible lines of its future development were discussed and taken stock of. In fact, a prominent mathematician - one who is naturally an enthusiastic topologist - declared that if you want to see a problem in all its generality you must always topologise. Topology has been a great unifying force in mathematics connecting as it does several branches like ~~algebra~~, group theory, analysis and geometry - in a rough manner of speaking one could divide all mathematics into algebra and topology, the symbols so to say of the discontinuous and continuous entities.

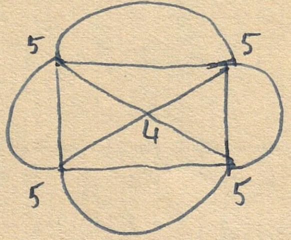
Having duly emphasised the importance of the subject, it is only right that I should tell you at once what it is about. But this is really the most difficult thing to do from a layman's point of view, or even from the point of view of men who are eminent in fields of science other than mathematics. When I once mentioned the subject to a friend of mine who is an accomplished Physicist he at once told me that he was very familiar with the subject, and this was very widely used by surveyors and geographers! – obviously a confusion with topography and topo-sheets! The statement that topology is the subject which deals with the properties of sets and invariants under continuous one-one transformations is utterly meaningless to a non-mathematician. To amplify the statement by saying that topology deals with duality theorems, general properties of manifolds, indecomposable continua, homeomorphic transformations in abstract spaces, and so on, only heighten the obscurity still further. Fortunately there is an alternative method by means of which I could introduce you to some elementary aspects of topology. This is through the recreational methods of mathematics, since many problems in topology are entremely enchanting as ~~problems~~ recreations, but frightfully difficult ~~of~~ to ~~solution~~ solve.

<u>Problem 1</u>: Problem of Seven Bridges of Königsberg:

[Drawing on Board; trace the corresp. graph in dotted lines]

Sunday stroller wishes to plan his walk so as to pass over each bridge once and only once. ~~Replace~~ This is impossible. Replace bridge XY by bridge WV; the walk can be planned & further return to same point is possible. Add an eighth bridge at WV, the walk can still be planned but no return possible. The solution of this problem is based on a branch of topology called the theory of graphs. A graph is a configuration consisting of a finite number of points called vertices which are endpoints of arcs, no two arcs having any common points except perhaps a common vertex. Traversing a graph means finding a path which passes through all the arcs exactly once, and, of course, a vertex may be passed several times along different arcs. It is a theorem in topology that traversing with return to same point is possible if graph has only even vertices, and mere traversing if there are at most two odd vertices. This is exemplified in the dotted curve which is the graph for the Konigsberg Bridge problem. The corresponding graphs of the other two variations of the problem could also be shown to conform to the above theorem.

* I might mention here a problem which was pointed out to me by my wife yesterday as a common one in Indian homes, and carrying ~~with it~~ a story behind it. It appears that a damsel proposed the traversing of the following graph as a challenge to any suitor to her hand



offering to marry anybody who could solve the problem successfully. If you observe that the graph has four odd vertices and one even vertex, you will be convinced that the poor damsel must have lived and died a maid.

The same could be extended to a problem of fifteen bridges where the corresponding graph is shown with only two odd vertices at D and E, and hence whose traversing is possible. Lest I ~~fall~~ fumble into a pitfall trying to trace the actual path, I had better leave it to your ingenuity, and go on to something which is much more complicated literally a labyrinth shown in the figure above which is to be traced continuously from A to B. This is also possible since A and B are the only odd vertices, but I shall not attempt the actual tracing of the path. To the same order of ideas belongs the famous signature of Mahomet which he ~~designed~~ could trace at one stroke with the point of his ~~a~~ scimitar, and formed by two opposing crescents. This contains only even vertices and on account of its simplicity it is also possible to follow the actual course of the signature which I have pointed out in the figure.*

Problem 2 : I might now consider a different type of traversing a graph which leads exactly once through all the vertices, and there might be, in general, arcs of the graph which are not used in the path. This might be

*

tetrahedron — $\alpha_0 = 4$, $\alpha_1 = 6$, $\alpha_2 = 4$

cube — $\alpha_0 = 8$, $\alpha_1 = 12$, $\alpha_2 = 6$

octohedron — $\alpha_0 = 6$, $\alpha_1 = 12$, $\alpha_2 = 8$.

dodecaheoron — $\alpha_0 = 20$, $\alpha_1 = 30$, $\alpha_2 = 12$

icosahedron — $\alpha_0 = 12$, $\alpha_1 = 30$, $\alpha_2 = 20$

called <u>vertex-traversing</u> to distinguish it from the <u>arc-traversing</u> problems. One such problem is Hamilton's problem of the dodecahedron (Show Jumping model of Steinhaus) with all its faces regular pentagons. It has twelve such faces, 20 vertices and 30 edges. I might perhaps mention at once here the famous theorem of Euler on ~~polyhedra viz~~ convex polyhedra viz that if $\alpha_0, \alpha_1, \alpha_2$ be respectively the number of ~~faces~~ vertices, edges and faces, one has

$$\alpha_0 - \alpha_1 + \alpha_2 = 2 \quad (20 - 30 + 12 = 2)$$

* {show five polyhedra verify relations & mention only 5 of them}

I mention this because it is a topological theorem belonging to the branch of ~~combinatorial~~ topology called <u>combinatorial topology</u>. We might consider the ~~the~~ polyhedron as a manifold of points, lines and planes or as having been divided into 0, 1 and 2-cells which are straight. This could be generalised in a topological way by replacing the straight elements by curved ones which are related to them ~~continu~~ point by point continuously - or, in technical language by their <u>homeomorphs</u>. Thus a simple closed curve is homeomorphic with the circumference of a circle, surface of a sphere with that of an ellipsoid (but not torus - show this), a

hemisphere with a circular disc (orth. projection). ~~Such~~
The sphere itself could be treated as such a generalised manifold of two hemispheres (2 cells), two semi-circles (1 cells) and two points (zero-cells). It can be shown that for a general two-dimensional fold, one has

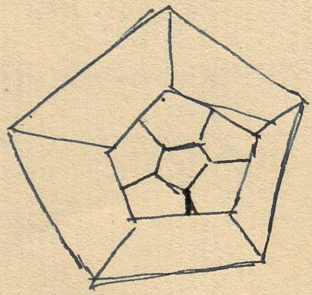
$$\alpha_0 - \alpha_1 + \alpha_2 = 3 - k$$

where k is something which is a topological characteristic of the surface called its connectivity and is equal to 1 in the case of polyhedra where the cells are all straight. In other words we might say that $\alpha_0 - \alpha_1 + \alpha_2$ (called the Euler number) is itself a topological invariant and from this view point all polyhedra are equivalent topologically. In the same sense squares, circles, ellipses, triangles and polygons are all equivalent being homeomorphic, which has led some people to define plane topology as "rubber sheet" geometry" (expt. with rubber sheet (ellipse → circle, ball → ellipsoid)). It is one of the major results of combinatory topology that Euler's relation could be generalised to the statement that

$$\sum_{i=0}^{n} (-1)^i \alpha_i$$

is an invariant for n-dimensional manifolds.

After all this relevant topological digression let me ~~return~~ now return, to the problem of vertex-traversing on the dodecahedron. This is the question of finding a path along the edges of the solid passing through all the vertices but once. This problem can be solved by considering a plane homeomorphic map of the dodecahedron, and



reducing it to a problem of graphs. I shall not dwell for long on the details of the solution but shall merely state that the problem may be solved in a number of ways so as to return each time to the starting point [Show one such path]

Problem 3 – Four Colour problem: Another example of the above type but one which is unsolved up to this day is the four colour problem (on plane or sphere) – 3 are not sufficient (Show figure). only partial solutions are known.

(i) Any map can be coloured if any cubic map can be coloured, a cubic map being one in which at most three regions meet in a point

(ii) Any map of less than 32 regions may be coloured in four

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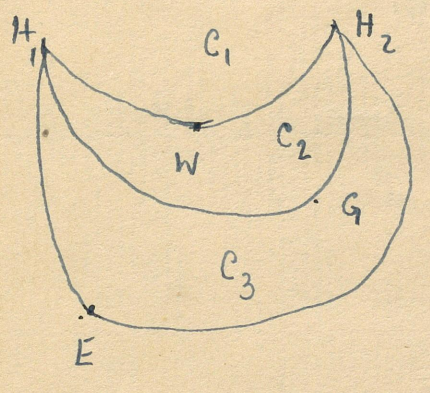
simple algebra

semi-simple algebra is an algebra which is not nil-potent and has no maximal nil-potent invariant sub-algebra. If, in addition, this semi-simple algebra has no invariant sub-algebra it is called a <u>simple</u> algebra. I can hardly think of a better example of the travesty of the word "simple" colours. The frontiers of the map form a connected graph, and the problem reduces to vertex-traversing. Any body in this audience who can show that a ~~cubic graph~~ connected cubic graph may be vertex-traversed with the end at the starting point will make a name for himself in history. Strangely enough the problem of colour mapping on more complicated surfaces like the torus is known (Seven on anchor-ring - Show model) while the problem for a sphere, the simplest of all surfaces, remains unsolved.

<u>Problem 4</u>: <u>Water-Gas-Electricity problem</u>: 3 houses, no two pipes to cross or come underneath a house. The solution is impossible and depends on the Jordan Curve Theorem ie a simple curve divides the plane into an outside and an outside (Aside re. simple & non-simple curves, and use of word simple in mathematics*). This theorem is not at all obvious, and is a characteristic topological theorem. For example it is not true for a torus (Show model). A simple closed curve is ~~the~~ a

homeomorph of a circle, and an arc of a closed linear interval (expt. with rubber band). Explanation of puzzle -



H_1WH_2, H_1GH_2, H_1EH_2 divide plane into three ~~[illegible]~~ regions C_1, C_2, C_3 and H_3 can lie in none of them.

<u>Model 5</u>: ~~When talking~~ <u>Möbius band</u> : When talking to you about the dodecahedron, I have mentioned to you the word manifold. Here you see such a curiously shaped surface known as a <u>Möbius strip</u>. It differs ~~from~~ in type from an ordinary belt-shaped surface ~~in~~ that its boundary consists of a single curve or it has only one side (no inside or outside) (Trace a path on model coming out from inside without crossing). Such a two dimensional manifold is said to be <u>non-orientable</u>. In general any two dimensional manifold is said to be <u>orientable</u> if no portion of it is homeomorphic

* mention that cutting or tearing changes ~~home~~ orientability

ie tearing is not a topological transformation.

with a Möbius strip, otherwise non-orientable. Thus a sphere and torus are both orientable while a Mobius strip is not. Cut this strip along the dotted line; the strip does not fall asunder but forms a bilateral surface* (Show fig) ie orientable, with two closed curves as margins, not knotted but interwoven. A similar band is given by the 360° twist Möbius band (Show fig). This surface (or previous one) cut along the middle gives rise to two interwoven strips (Fig). Another 180° twist ie a total of 540° again gives a unilateral surface the edge being a knotted curve (Show fig). The ordinary belt, 360° Mobius-band are both bilateral with edges unknotted, ~~Cunthera~~ while the 540°-band is unilateral with a knotted edge. Does there exist a bilateral surface with a knotted edge? A positive answer to this query is the model before you (Show fig). Here we have a bewildering variety of manifolds, ~~there~~ the sphere (Show model), the ~~two~~ torus, the 360°-band, and the last model all bilateral (orientable), then the ~~ordinary~~ and 540°-bands both unilateral (non-orientable). Does this mean that all the former ~~are top~~ ones and the latter ones topologically equivalent? In other words does orientability alone characterise homeomorphy. Certainly not, for as we have mentioned the sphere and torus are not homeomorphic. There is another characteristic for such two dimensional manifolds viz its connectivity or the maximum number of simple curves that can be traced on it without separating it into two or more pieces. Thus the sphere is of connectivity 0, and the ~~two~~ torus 2. The type of a manifold is completely fixed if we know its connectivity and whether or not it is ~~orientability~~ orientable. In other words connectivity and orientability are the complete set of invariants for 2 dimensional manifolds. For three dimensional manifolds much work has been done, but no complete set of topological invariants has ever been found, and most surprisingly some recent work in mathematical logic tends to show that it is quite possible that no such sets may ever be found!

<u>Model. 6</u> – <u>Knots</u>: Let me now go to another topological problem, the theory of knots. From their very nature knots cannot exist in 2 dimensions (Show with string) and it has been shown that the same is true of 2n

dimensions. In topological language a knot is ~~th~~ a homeomorph of a circle in Euclidean 3-space, just as a simple curve is its homeomorph in the plane. (Thread) I shall consider the problem of the knot in one particular aspect viz " the cutting of the Gordian knot" and replace it by " Gordian cutting of a knot" ie a change in the crossings from upper to lower or vice versa. Using the ordinary meaning of the phrase "Cutting the Gordian knot", it would appear that one Gordian cutting would be sufficient to untie the knot. But this is certainly not true (Show double clover leaf knot where 2 cuts are necessary). The simplicity of a knot (show single clover leaf knot) is no criterion for the smallness of this cutting number (Show 8_{20} and 9_{42} in Alexander's tables). Thus the least number of Gordian cuttings which will untie a knot is a very important number or a topological invariant for a knot. Given any knot could you straightaway say how many Gordian cuttings will untie it? If you were able to find a method of answering it, you will become a very famous mathematician tomorrow.

We can look at this question in another way. Consider a circular knot! Imagine a surface inscribed in it. The surface is not penetrated anywhere by the knot. Try with any other knot and you will find that such a surface is pierced through by the knot in one or more places. Of course, by changing the kind of surface you use for inscription you can vary the number of thrusts, but we can talk of the least possible number of such points. This is another characteristic of the knot called the "Pannwitz-invariant". Consider a very simple theorem like this: "The circle is the only knot for which the Pannwitz invariant zero". This looks obvious, but nobody has so far proved it, and it is assumed as an axiom & called Dehn's lemma.

In addition to the Gordian cutting number, and Pannwitz invariant a knot, being a manifold, has also its connectivity or genus invariant. What is the precise relation between these three invariants? No body has gone anywhere near a solution of this problem and to change the words of the poet slightly the problem here is

"The Knot to be, or the Knot not to be
Is the question"!

Point-Set topology:- So far we have confined ourselves to particular types of ~~manifolds~~ configurations or point sets viz manifolds, but we could generalise and consider any arbitrary set of points. This leads us on to the notion of abstract spaces ie any set of points or even elements characterised by some axioms including the notion of continuity. On such a space we can effect a homeomorphism or topological transformation (a 1-1 continuous correspondence). Topology concerns itself with properties of spaces which remain invariant for topological transformations (Explain here Klein's Erlanger programme). Different abstract spaces could be obtained by varying the axioms, and two such important types are connected (hanging in one piece) and closed spaces. A space both connected & closed is called a continuum the simplest examples of which are simple arcs & ~~cl~~ simple closed curves. Properties of continua play a great part in topology.

Among the numerous problems in point-set topology I shall choose only one viz the fascinating problem of "imbedding". Given a topological configuration C and a space S we say that C is imbedded or immersed or pictured in S if S contains a homeomorph of C. Thus ~~a knot can be imbedded in a Euclidean 2-space~~ a simple plane curve can be pictured topologically as a knot in Euclidean 3-space. This theorem has been generalised into the Menger-Nöbeling theorem that "Every n-dimensional compact metric space may be imbedded in a Euclidean space of 2n+1 dimensions". Another one is ~~that~~ the theorem of Urysohn that "the most general separable metric space may be imbedded in the fundamental parallelopiped of Hilbert space". I well realise that these theorems carry no meaning to most of you, but I hope they will serve to give you some idea of the type of problems studied in abstract topology.

Applications: Outside the range of pure mathematics where it plays a dominating role, topology had not so far had many applications in other fields of science. It is however interesting to note that in certain problems of dynamics, and recently in the theory of statistical mechanics topological methods are being used. A. Rosenthal (Ann. d. Phys. Vol 42 p. 796–886) has shown the impossibility of the existence of a certain kind of gas system. Bertrand Russell in his book on "The Analysis of matter" has applied topological methods to philosophic problems.

Conclusions – I hope that this rambling talk of mine has given you some idea of the scope and methods of a young and virile branch of modern mathematics. There is no doubt that, by virtue of its very general outlook, topology is bound to be used more and more in many branches of science. I shall be thoroughly satisfied if you have caught the spirit of the subject. If you have learnt to topologise I shall have no need to apologise for my very sketchy lecture.

Let me thank you, once again, for the very patient hearing you have given me.

B. S. Madhava Rao