Krishnamurthy Rao (Fat man)

10/2/69 – B. S. Krishna Rao in Bus

Trans. Amer. Math. Soc. Vols 36-C Bound 5 gr
1934-C

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11^2

$\frac{5032}{12} = 419.$

$\left\{\begin{array}{r} 4.20 \\ 3.10 \\ \hline 1:60 \end{array}\right\}$ $\left\{\begin{array}{r} 4.20 \\ 2.10 \\ \hline 2:60 \end{array}\right\}$

Spin 3/2 fields

✗ Federbush P – Phys. Rev. 120, ~~1960~~ 1926

✗ ✓ S. N. Gupta – Proc. Phys. Soc. A. 63, 681

✗ Moldauer – Phys. Rev. 102, 279

✗ Takahashi – Prog. Th. Phys, 9, 1

$\frac{G m_s}{r^2} \sim \frac{m_s R c^2}{M_s^2}$ $\frac{G M}{R c^2} \sim 1$

$\left\{\begin{array}{l} Ac' + bb' + cc' = k^2. \\ Ac'' + \quad = k'^2 \end{array}\right.$

Lecture Notes on Analytical Dynamics

given

during 1962–63. (Following Whittaker).

Syllabus on Analytical Dynamics

Lagrange's eqns for holonomic systems; The Lagrangian; Reduction of systems with ignorable coordinates; Integrals of momentum and angular momentum; the energy integral; systems of the Liouville type.

Theory of vibrations – Vibrations about equilibrium; Reduction to normal coordinates; Proof of the reality of the roots of the determinantal equation, and the condition for the roots being positive; Principle of superposition of vibrations; Stability of equilibrium; General nature of vibrations about steady motion.

Hamilton's Principle and Principle of Least Action for Conservative holonomic systems; Actual minima of the Action; Kinetic foci; Principle of least curvature and Appell's eqns.

Hamiltonian form of the equations of motion; Absolute and relative integral invariants; Integral invariants of order one; Theorem of Lie and Koenigs; The last multiplier and its applications to Hamiltonian systems.

Equations of a canonical transformation; The bilinear covariant; Lagrange and Poisson bracket expressions; Equations of motion in Poisson bracket notation; Infinitesimal contact transformations; Jacobi's identity; Poisson's theorem; Constancy of Lagrange bracket expressions; Involution systems.

Hamilton's principal and Characteristic functions; The Hamilton-Jacobi equation; Hamilton's integral as a solution of the Hamilton-Jacobi equation; Separation of variables in this equation; Action and angle variables.

Scope of Syllabus: (i) S. Goldstein – Classical Mechanics – Chaps VI to X

(ii) Whittaker. E.T – Analytical Dynamics – Chaps II, III, VII and Chaps IX to XII

1st Lecture — 16/7/62 — Dynamical system, and its specification by coordinates — Motion under forces — Newton's laws of motion — Change of coordinates by motion — Holonomic and non-holonomic coordinates and number of degrees of freedom of the system — Lagrangian equations of motion of dynamical system which is holonomic; preliminary.

2nd Lecture — 18/7/62 — Potential energy V — Lagrangian function L — Lagrangian eqns of motion in the form $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0$ — Lagrange's equations in explicit form — Natural systems with $T = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j$.

3rd Lecture — 20/7/62 — Case where $Q_r = -\frac{\partial V}{\partial q_r} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_r}\right)$ with $V = V(q, \dot{q})$ giving equations also of the Lagrangian form — Lagrange's equations for impulsive motion, not differential but algebraic equations — Case of initial motions, and Example on p. 46 to find the initial radius of curvature of the path of a point in a plane.

4th Lecture — 24/7/62 — Ex. 3, p. 51 of Whittaker to find ρ and τ at origin

[Solution: Proceeding as in the Ex. on p. 46 of Whittaker & generalising to the three-dimensional case by taking $\ddot{\xi} = X(x+\xi, y+\eta, z+\zeta)$, etc and assuming

$$\left.\begin{aligned} \xi &= a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6 + \cdots \\ \eta &= b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 + b_6 t^6 + \cdots \\ \zeta &= c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5 + c_6 t^6 + \end{aligned}\right\}$$

& since ξ, η, ζ and $\dot{\xi}, \dot{\eta}, \dot{\zeta}$ are initially zero (i.e. for $t = 0$), terms of order lower than t^2 ~~not~~ not being included, we have as on p. 46

$$\left.\begin{aligned} a_2 &= \tfrac{1}{2} X, \; b_2 = \tfrac{1}{2} Y, \; c_2 = \tfrac{1}{2} Z \\ a_3 &= 0, \; b_3 = 0, \; c_3 = 0 \\ a_4 &= \frac{1}{24}\left(X\frac{\partial X}{\partial x} + Y\frac{\partial X}{\partial y} + Z\frac{\partial X}{\partial z}\right), \; b_4 = \frac{1}{24}\left(X\frac{\partial Y}{\partial x} + Y\frac{\partial Y}{\partial y} + Z\frac{\partial Y}{\partial z}\right), \; c_4 = \frac{1}{24}\left(X\frac{\partial Z}{\partial x} + Y\frac{\partial Z}{\partial y} + Z\frac{\partial Z}{\partial z}\right) \end{aligned}\right\}$$

or using $u = \frac{1}{2} t^2$,

$$\left.\begin{aligned} \xi &= Xu + \frac{1}{6}\left(X\frac{\partial X}{\partial x} + \cdots\right) u^2 + \cdots \\ \eta &= Yu + \frac{1}{6}\left(X\frac{\partial Y}{\partial x} + \cdots\right) u^2 + \cdots \\ \zeta &= Zu + \frac{1}{6}\left(X\frac{\partial Z}{\partial x} + \cdots\right) u^2 + \cdots \end{aligned}\right\}$$

From Goursat, Vol I, pp. 470–75, we have the result that for the curve $x = f(t)$, $y = \phi(t)$, $z = \psi(t)$ the radius of ~~curve R etc~~ curvature R, and radius of torsion T are given by

$$R^2 = \frac{ds^6}{A^2 + B^2 + C^2}, \quad T = \pm \frac{A^2 + B^2 + C^2}{\Delta}$$

where $A = dy\, d^2z - dz\, d^2y$, $B = dz\, d^2x - dx\, d^2z$, $C = dx\, d^2y - dy\, d^2x$, and $\Delta = \begin{vmatrix} dx, & dy, & dz \\ d^2x, & d^2y, & d^2z \\ d^3x, & d^3y, & d^3z \end{vmatrix}$

From new edition of Kendall & Stuart

(1) (b) Derive the Gram-Charlier series of Type A for a ~~function~~ frequency function (in standard measure) in terms of Hermite polynomials. [Kendalls Stuart – Vol I, Ch. 6, § 6.17, p. 157)

1(a) Explain briefly the ~~laser~~ the method of fitting of Pearson ~~dist^n~~ curves by the method of moments.
[Ibid, p. 152 – 54.]

(c) Discuss the ~~basic~~ idea basic to the theory of confidence intervals in estimation, and give a simple example in which a graphical ~~rep^n~~ ~~of~~ can be ~~used~~ found for showing the confidence limits
[Ibid. Vol II, pp. 101–102.]

From old edition by Kendall only

1(a) to stand

(b) to be deleted - Instead,

Define general Pearson distributions, and show that the normal and rectangular distributions are both of the Pearson's types [pp. 137–42, Vol I

(c) to be deleted – Instead

Discuss the ~~notion~~ theory of "fiducial inference", and the relationship between this theory and the theory of confidence intervals [Vol II, pp. 86–87)

5th

ie $R^2 = \dfrac{\left\{\left(d\xi/du\right)^2 + \cdots\right\}^3}{\left(\frac{d\eta}{du}\frac{d^2\zeta}{du^2} - \frac{d\zeta}{du}\frac{d^2\eta}{du^2}\right)^2 + \left(\frac{d\zeta}{du}\frac{d^2\xi}{du^2} - \frac{d\xi}{du}\frac{d^2\zeta}{du^2}\right)^2 + \left(\frac{d\xi}{du}\frac{d^2\eta}{du^2} - \frac{d\eta}{du}\frac{d^2\xi}{du^2}\right)^2}$ expressing ξ, η, ζ in terms of parameter u.

we have to substitute here for the several derivatives their values at the initial time ie $t=0$ or $u=0$, and at the origin ie $x=0, y=0, z=0$ in the given exprns for X, Y and Z. Accordingly we have

$$\left.\begin{aligned}
\frac{d\xi}{du} &= X + \tfrac{1}{3}u\left(X\frac{\partial X}{\partial x} + Y\frac{\partial X}{\partial y} + Z\frac{\partial X}{\partial z}\right);\ \frac{d^2\xi}{du^2} = \tfrac{1}{3}\left(X\frac{\partial X}{\partial x} + \cdots\right)\\
\frac{d\eta}{du} &= Y + \tfrac{1}{3}u\left(X\frac{\partial Y}{\partial x} + Y\frac{\partial Y}{\partial y} + Z\frac{\partial Y}{\partial z}\right);\ \frac{d^2\eta}{du^2} = \tfrac{1}{3}\left(X\frac{\partial Y}{\partial x} + \cdots\right)\\
\frac{d\zeta}{du} &= Z + \tfrac{1}{3}u\left(X\frac{\partial Z}{\partial x} + Y\frac{\partial Z}{\partial y} + Z\frac{\partial Z}{\partial z}\right);\ \frac{d^2\zeta}{du^2} = \tfrac{1}{3}\left(X\frac{\partial Z}{\partial x} + \cdots\right)
\end{aligned}\right\}$$

From the given expressions for X, Y, Z, we have at the origin;

$$X_0 = a,\ Y_0 = 0,\ Z_0 = 0;\ \left(\frac{\partial X}{\partial x}\right)_0 = b;\ \left(\frac{\partial X}{\partial y}\right)_0 = \left(\frac{\partial X}{\partial z}\right)_0 = 0$$

$$\left(\frac{\partial Y}{\partial x}\right)_0 = c;\ \left(\frac{\partial Y}{\partial y}\right)_0 = \left(\frac{\partial Y}{\partial z}\right)_0 = 0 \text{ and } \left(\frac{\partial Z}{\partial x}\right)_0 = \left(\frac{\partial Z}{\partial y}\right)_0 = \left(\frac{\partial Z}{\partial z}\right)_0 = 0$$

Hence at the origin, at time $t = 0$, we have $\frac{d\xi}{du} = X_0 = a;\ \frac{d^2\xi}{du^2} = \frac{1}{3}X_0\left(\frac{\partial X}{\partial x}\right)_0 = \frac{1}{3}ab$

$$\frac{d\eta}{du} = Y_0 = 0;\ \frac{d^2\eta}{du^2} = \frac{1}{3}X_0\left(\frac{\partial Y}{\partial x}\right)_0 = \frac{1}{3}ac.$$

and $\frac{d\zeta}{du} = Z_0 = 0;\ \frac{d^2\zeta}{du^2} = 0.$

$\therefore$ $R^2 = a^6 \Big/ \frac{1}{9}a^4c^2$, the only non-vanishing term in the denominator being $\frac{d\xi}{du}\cdot\frac{d^2\eta}{du^2} = a\cdot\frac{1}{3}ac$

$= 9a^2/c^2$

ie $R = 3a/c$

To find T, we require to know the values of $d^3\xi/du^3$, etc. Proceeding as on p. 46 and finding the coeffts of higher powers of t in expansions of ξ, η and ζ, we have, $a_5 = b_5 = c_5 = 0$ and

$$a_6 = \frac{1}{720}\left[\left\{\frac{\partial X}{\partial x}\left(X\frac{\partial X}{\partial x} + Y\frac{\partial X}{\partial y} + Z\frac{\partial X}{\partial z}\right) + \frac{\partial X}{\partial y}\left(X\frac{\partial Y}{\partial x} + Y\frac{\partial Y}{\partial y} + Z\frac{\partial Y}{\partial z}\right) + \frac{\partial X}{\partial z}\left(X\frac{\partial Z}{\partial x} + Y\frac{\partial Z}{\partial y} + Z\frac{\partial Z}{\partial z}\right)\right\}\right]$$

$$+ \frac{1}{120}\left[X^2\frac{\partial^2 X}{\partial x^2} + Y^2\frac{\partial^2 X}{\partial y^2} + Z^2\frac{\partial^2 X}{\partial z^2} + 2YZ\frac{\partial^2 X}{\partial y\partial z} + 2ZX\frac{\partial^2 X}{\partial z\partial x} + 2XY\frac{\partial^2 X}{\partial x\partial y}\right]$$

with similar exprns for b_6 and c_6, and $\frac{d^3\xi}{du^3} = 48a_6$, etc. for $u = 0$ (ie $t = 0$)

and $\Delta = \begin{vmatrix} d\xi/du, & d\eta/du, & d\zeta/du \\ d^2\xi/du^2, & d^2\eta/du^2, & d^2\zeta/du^2 \\ d^3\xi/du^3, & d^3\eta/du^3, & d^3\zeta/du^3 \end{vmatrix} = \begin{vmatrix} a, & 0, & 0 \\ \frac{1}{3}ab, & \frac{1}{3}ac, & 0 \\ 48a_6, & 48b_6, & 48c_6 \end{vmatrix} = 16a^2c\cdot c_6$

Now $c_6 = \frac{1}{120}X_0^2\left(\frac{\partial^2 Z}{\partial x^2}\right)_0 = \frac{1}{120}\cdot a^2\cdot 2d = \frac{1}{60}a^2d$ & therefore $\Delta = \frac{4}{15}a^4cd$

$$\therefore\ T = \pm\frac{A^2+B^2+C^2}{\Delta} = \frac{\frac{1}{9}a^4c^2}{\frac{4}{15}a^4cd} = \frac{5c}{12d}$$

Did only R and asked the students to find T for themselves — Definition of similarity of Dynamical systems, and deduction of the relation $w = xyz^2$ (W. §33, p. 47).

5th Lecture — 25/7/62 — Particular cases of similarity in cases of gravity and mutual gravitation as

indicated in the Ex. at end of §33 — Motion with reversed forces as a special case of similarity (§34) — Example on p. 45 of Whittaker. done only the first two parts.

[Ex. p. 45 — Using the summation convention, and writing $T = \frac{1}{2} b_{ik} \dot{q}_i \dot{q}_k$, we can write the explicit form of Lagrange's equation as on p. 39 (§28) as, (also using Γ_i^{ik}),

$$\left.\begin{aligned} Q_i &= b_{ik}\ddot{q}_k + \Gamma_i^{km}\dot{q}_k\dot{q}_m \\ \text{and } Q_k &= b_{ik}\ddot{q}_i + \Gamma_k^{im}\dot{q}_i\dot{q}_m \end{aligned}\right\} \quad -- (1)$$

and get $\dfrac{\partial Q_i}{\partial \dot{q}_k} = 2\Gamma_i^{km}\dot{q}_m$ and $\dfrac{\partial Q_k}{\partial \dot{q}_i} = 2\Gamma_k^{im}\dot{q}_m$ -- (1')

From (i) the first part of the example follows viz $\dfrac{\partial Q_i}{\partial \ddot{q}_k} = \dfrac{\partial Q_k}{\partial \ddot{q}_i} = b_{ik}$. — (2)

Also $$\frac{\partial Q_i}{\partial \dot{q}_k} + \frac{\partial Q_k}{\partial \dot{q}_i} = 2\left(\Gamma_i^{km} + \Gamma_k^{im}\right)\dot{q}_m$$

$$= \dot{q}_m\left\{\left(\frac{\partial b_{ik}}{\partial q_m} + \frac{\partial b_{im}}{\partial q_k} - \frac{\partial b_{km}}{\partial q_i}\right) + \left(\frac{\partial b_{ik}}{\partial q_m} + \frac{\partial b_{km}}{\partial q_i} - \frac{\partial b_{im}}{\partial q_k}\right)\right\}$$

$$= 2\frac{\partial b_{ik}}{\partial q_m}\dot{q}_m = 2\frac{d}{dt}\left(b_{ik}\right) = \frac{d}{dt}\left(\frac{\partial Q_i}{\partial \ddot{q}_k} + \frac{\partial Q_k}{\partial \ddot{q}_i}\right) \quad ---(3)$$

As regards the last part of the example, write, using (1)

$$\frac{\partial Q_i}{\partial q_k} = \frac{\partial b_{ij}}{\partial q_k}\ddot{q}_j + \frac{\partial \Gamma_i^{lm}}{\partial q_k}\dot{q}_l\dot{q}_m$$

$$\text{and } \frac{\partial Q_k}{\partial q_i} = \frac{\partial b_{kj}}{\partial q_i}\ddot{q}_j + \frac{\partial \Gamma_k^{lm}}{\partial q_i}\dot{q}_l\dot{q}_m$$

$$\therefore \frac{\partial Q_i}{\partial q_k} - \frac{\partial Q_k}{\partial q_i} = \left(\frac{\partial b_{ij}}{\partial q_k} - \frac{\partial b_{kj}}{\partial q_i}\right)\ddot{q}_j + \left(\frac{\partial \Gamma_i^{lm}}{\partial q_k} - \frac{\partial \Gamma_k^{lm}}{\partial q_i}\right)\dot{q}_l\dot{q}_m \qquad (4)$$

using (1') $$\frac{1}{2}\frac{d}{dt}\left(\frac{\partial Q_i}{\partial \dot{q}_k} - \frac{\partial Q_k}{\partial \dot{q}_i}\right) = \left(\Gamma_i^{km} - \Gamma_k^{im}\right)\ddot{q}_m + \left(\frac{\partial \Gamma_i^{km}}{\partial q_l} - \frac{\partial \Gamma_k^{lm}}{\partial q_l}\right)\dot{q}_l\dot{q}_m \qquad (5)$$

The first terms on the R.H.S. of (4) and (5) if we note that

$$\Gamma_i^{km} - \Gamma_k^{im} = \frac{\partial b_{im}}{\partial q_k} - \frac{\partial b_{km}}{\partial q_i}$$

so that $$\left(\Gamma_i^{km} - \Gamma_k^{im}\right)\ddot{q}_m = \left(\frac{\partial b_{im}}{\partial q_k} - \frac{\partial b_{km}}{\partial q_i}\right)\ddot{q}_m = \left(\frac{\partial b_{ij}}{\partial q_k} - \frac{\partial b_{kj}}{\partial q_i}\right)\ddot{q}_j$$

so that to prove the last relation, we have only to show that

$$\left(\frac{\partial \Gamma_i^{lm}}{\partial q_k} - \frac{\partial \Gamma_k^{lm}}{\partial q_i}\right)\dot{q}_l\dot{q}_m = \left(\frac{\partial \Gamma_i^{km}}{\partial q_l} - \frac{\partial \Gamma_k^{im}}{\partial q_l}\right)\dot{q}_l\dot{q}_m \qquad (6)$$

$$\frac{\partial \Gamma_i^{lm}}{\partial q_k} - \frac{\partial \Gamma_k^{lm}}{\partial q_i} = \frac{1}{2}\left\{\frac{\partial^2 b_{il}}{\partial q_k \partial q_m} + \frac{\partial^2 b_{im}}{\partial q_k \partial q_l} - \frac{\partial^2 b_{kl}}{\partial q_i \partial q_m} - \frac{\partial^2 b_{km}}{\partial q_i \partial q_l}\right\}$$

and using $\dfrac{\partial^2 b_{il}}{\partial q_k \partial q_m}\dot{q}_l\dot{q}_m = \dfrac{\partial^2 b_{im}}{\partial q_k \partial q_l}\dot{q}_l\dot{q}_m$, and $\dfrac{\partial^2 b_{kl}}{\partial q_i \partial q_m}\dot{q}_l\dot{q}_m = \dfrac{\partial^2 b_{km}}{\partial q_i \partial q_l}\dot{q}_l\dot{q}_m$ $\left(l \rightleftarrows m\right)$

$$\therefore \text{L.H.S. of (6)} = \left(\frac{\partial^2 b_{im}}{\partial q_l \partial q_k} - \frac{\partial^2 b_{km}}{\partial q_l \partial q_i}\right)\dot{q}_l\dot{q}_m$$

* Gave references to my earlier papers:

(i) Über die ein mechanisches System betreffenden Theoreme von Liouville u. Stäckel
– Zs. f. Phys, 93, 1935, pp. 320–38.

(ii) On separable systems in classical mechanics and wave mechanics.
– Math. Annalen, 3, 1935, pp. 459–68.

and the R.H.S of (6) $\frac{1}{2}\sum_{\ell}\frac{\partial}{\partial q_\ell}\left\{\left(\frac{\partial b_{\ell i}}{\partial q_m}+\frac{\partial b_{\ell m}}{\partial q_i}-\frac{\partial b_{im}}{\partial q_\ell}\right)-\left(\frac{\partial b_{ki}}{\partial q_m}+\frac{\partial b_{km}}{\partial q_i}-\frac{\partial b_{im}}{\partial q_k}\right)\right\}\dot{q}_\ell\dot{q}_m$

$$=\left(\frac{\partial^2 b_{im}}{\partial q_\ell \partial q_k}-\frac{\partial^2 b_{km}}{\partial q_\ell \partial q_i}\right)\dot{q}_\ell\dot{q}_m = \text{L.H.S of (6)}$$

& hence the last part]

6th Lecture – 27/7/62 – Did the last part of Ex. on p. 45 – Integration of dynamical problems; problems soluble by quadratures – Set of $2n$ diffl. eqns of 1st order equivalent to a set of n diffl. eqns of a dynamical problem with n degrees of freedom – Systems with k ignorable coordinates and Process of ignoration using the Routhian function R.

7th Lecture – 31/7/62 – Worked Ex. on p. 57 of Whittaker, and Ex. 3, p. 69 of Whittaker (Solution on p. 1 of last year's notes) – Special cases of ignoration leading to Principles of conservation of linear momentum, and of angular momentum – General theorem of angular momentum – the energy equation; integral of energy, total energy constant for conservative systems.

8th Lecture – 3/8/62 – Reduction of a dynamical problem, using the integral of energy, to another dealing with a system with $(n-1)$ degrees of freedom.

9th Lecture – 7/8/62 – Case of $(n-1)$ cyclic coordinates; successive reduction by two methods: (i) using successively integrals of energy, and (ii) using the Routhian function & the integral of energy for the final system of one degree of freedom – Ex. on p. 67. W. completely worked out (Soln of on p. 7 of last year's notes) – Case of separation of variables; ordinary separable systems.

10th Lecture – 8/8/62 – Complete integration of dynamical systems of the Liouville type* – Problem of two centres of gravitation in particle dynamics as an example of the Liouville type (§53, p. 97 of W).

11th Lecture – 17/8/62 – First part of Ex. 5, p. 69 for the general case with r ignorable coordinates $q_1, \ldots q_r$ and $(n-r)$ non-ignorable coordinates $q_{r+1}, \ldots q_n$ (Proof for particular values of n and r and general case also worked on pp. 4, and pp. 5-6 of last year's notes)

12th Lecture – 21/8/62 – Second part of Ex. 5, p. 69 pointing out the error in Whittaker's form of the equations of motion and giving the correct form (This is done on pp. 4-5 of last year's notes) – Working of Ex. 7, p. 70 (Worked on p. 2 of last year's notes).

13th Lecture – 22/8/62 – Vibrations about an equilibrium configuration of a dynamical system which is conservative – Reduction of T and V to homogeneous quadratic forms in $\dot{q}_i$ and q_i respectly. with constant coefficients, and T positive definite – Mention of normal coordinates

14th Lecture – 28/8/62 – Introduction of normal coordinates by reducing T and V to the normal forms using initially a root of $\|\lambda b_{ij} - a_{ij}\| = 0$ for the first reduction – [In particular, the proof

that the terms in $\dot{q}_1'\dot{q}_2', \dot{q}_1'\dot{q}_3', \ldots \dot{q}_1'\dot{q}_n'$ vanish from the transformed T by the substitution $q_1 = q_1', q_2 = q_2' + \frac{A_{12}}{A_{11}} q_1', \ldots$ is shown by proving that the coefficients of $\dot{q}_1'\dot{q}_2'$ come from $\dot{q}_2^2, \dot{q}_1\dot{q}_2, \dot{q}_2\dot{q}_3, \dot{q}_2\dot{q}_4, \ldots$ ie collecting coeffts of $\dot{q}_1'\dot{q}_2'$ from $\frac{1}{2}\left\{ a_{22}\left(\dot{q}_2' + \frac{A_{12}}{A_{11}}\dot{q}_1'\right)^2 + 2a_{12}\dot{q}_1'\left(\dot{q}_2' + \frac{A_{12}}{A_{11}}\dot{q}_1'\right) + 2a_{23}\left(\dot{q}_2' + \frac{A_{12}}{A_{11}}\dot{q}_1'\right)\left(\dot{q}_3' + \frac{A_{13}}{A_{11}}\dot{q}_1'\right) + \ldots \right\}$

this coefficient $= a_{22}\cdot\frac{A_{12}}{A_{11}} + a_{12} + a_{23}\cdot\frac{A_{13}}{A_{11}} + \ldots + a_{2n}\frac{A_{1n}}{A_{11}}$

$= \frac{1}{A_{11}}\left\{ a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} + \ldots + a_{2n}A_{1n} \right\} = 0$

since expn within $\{\ \}$ is the sum of the products of the elements of the second row of the determinant $\|a_{ij}\|$ with the cofactors of the corresponding elements of the first row of the same determinant].

15th Lecture – 29/8/62. – Alternative proof of the simultaneous reduction of T and V to the normal form by using the method in Ah·13·4·3, p. 410 of Mirsky's Introduction to Linear Algebra – First reduction of the +ve definite form to $\frac{1}{2}(q_1^2 + \ldots + q_n^2)$ using Ah·13·2 (p. 400) & then the simultaneous reduction of both the forms – Also the proof that the coeffts in the reduced form of $V = \frac{1}{2}(\mu_1 q_1^2 + \ldots + \mu_n q_n^2)$ are the roots of the determinantal equation $\|\lambda a_{ij} - b_{ij}\| = 0$.

16th Lecture – 4/9/62 – Proof of the reality of the roots of $\|\lambda a_{ij} - b_{ij}\| = 0$ – Sylvester's proof – Proof as in Ferrar's algebra on the lines given in last year's notes – Given for homework, Ex. on p. 413 of Mirsky viz. simultaneous reduction of $y^2 + 2z^2 - 2yz + 2zx - 2xy$, and $2x^2 + 9y^2 + 2z^2 - 2yz + 2zx + 6xy$ to normal forms.

17th Lecture – 5/9/62 – Proof in Ferrar about the roots being real and +ve if V be also a +ve definite quadratic form – Proof that, since the roots of $\|\lambda A - B\| = 0$ in the given example be $-1, 1$ & 1, if T reduces to $\frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)$ or the second quad. form (which is +ve definite) reduces to $\xi^2 + \eta^2 + \zeta^2$, then V reduces to $\frac{1}{2}(-q_1^2 + q_2^2 + q_3^2)$ or the 1st quad form to $-\xi^2 + \eta^2 + \zeta^2$.

18th Lecture – 7/9/62 – Theory re. simultaneous reduction of two quadratic forms to diagonal form as given in Littlewood's University Algebra viz. Constructing right factors of zero X_r of the matrices $[\lambda_r a_{ij} - b_{ij}]$ corresp. to λ_r and taking $T = [X_1, \ldots X_n]$ and forming $\tilde{T}[a_{ij}]T$ & $\tilde{T}[b_{ij}]T$ – Worked the example given in Littlewood viz $2x^2 + 3y^2 + 2z^2 - 2zx$, and $3x^2 + 2y^2 + z^2 + 6yz - 4zx - 2xy$ with the first being +ve definite & the secular equation having distinct roots $1, 2, -1$.

19th Lecture – 11/9/62 – did two more examples of simultaneous reduction to sum of squares one from Littlewood viz $2x^2 + 5y^2 + 6z^2 + 2xy - 6zx + 2yz$, & $3y^2 + 4z^2 - 2xy - 2zx + 6yz$, and the other two being the one from Mirsky given for home work on 4/9/62.

20th Lecture – 14/9/62 – Proof in Whittaker about the condn for the roots of $\|\lambda a_{ij} - b_{ij}\| = 0$ being +ve viz. that $\frac{1}{2}\sum b_{ij}x_i x_j$ should also be +ve definite – Repetition of the essential points in Whittaker's proof

of the reality of the roots of $\|\lambda a_{ij} - b_{ij}\| = 0$ as follows :-

In the sequence $\Delta, \Delta_1, \Delta_2, \ldots \Delta_{n-1}, \Delta_n$, the last term is taken as $\Delta_n = 1$ and this can be justified by writing Δ as $\begin{vmatrix} \Delta & 0 \\ 0 & 1 \end{vmatrix}$ ie bordering Δ with an extra column and extra row with all elements zero except the last one taken $= 1$. Then we can think of Δ_n obtained by striking out the first n rows and first n cols. The known theorems to be used are (i) If $\Delta_r = 0$ for any value of λ, then for this value of λ, Δ_{r-1} and Δ_{r+1} have opp. signs

(ii) Since $\Sigma a_{ij} x_i x_j$ is $+^{ve}$ definite, the coefficients of the highest powers of λ in each of $\Delta, \Delta_1, \ldots \Delta_n$ are $+^{ve}$

(iii) From (ii) it follows that for $\lambda = \infty$, all the Δ's are $+^{ve}$, and for $\lambda = -\infty$, the Δ's are alternatively of positive & negative or negative & positive signs according as n is even or odd, and in either case Δ_n is $+^{ve}$ and there are n changes of sign in the sequence for $\lambda = -\infty$. Hence as λ varies from ~~$+\infty$~~ $-\infty$ to $+\infty$, there is a loss of n changes of sign.

We can now consider three possibilities :-

(a) λ passes through a root of $\Delta_{r-1} = 0$ — Since Δ_n is $+^{ve}$ it follows that Δ_{n-2} is $-^{ve}$ for this value of λ. Hence the signs of $\Delta_{n-2}, \Delta_{n-1}, \Delta_n$ before and after passing through this value are $- + +$ or $- - +$ and $- - +$ or $- + +$ resply and hence there is no loss of change of sign ; —. (A)

(b) λ passes through a root of $\Delta_r = 0$ ($r \neq n-1$ or $r \neq 0$, $\Delta_0 \equiv \Delta$). Here Δ_{r-1} & Δ_{r+1} have opp. signs when passing through this value of λ. Hence the possibilities are

$$\left.\begin{array}{l} \text{Before : } + + - \;;\; + - - \;;\; - + + \;;\; - - + \\ \text{After : } + - - \;;\; + + - \;;\; - - + \;;\; - + + \end{array}\right\} \quad (B)$$

ie again no loss of change of sign

From (a) and (b) alone we can conclude that a loss of change of sign can occur only in the last case (c) to be considered viz when λ passes through a root of $\Delta = 0$, and since there is a loss of n changes of sign, $\Delta = 0$ must have n real roots. But we can prove this also directly by considering case (c) explicitly ie (c) λ passes through a root of $\Delta = 0$ say $\lambda = \lambda_0$. Let $\Delta = f(\lambda) = a_0 \lambda^n + \ldots$, and $\Delta_1 = \phi(\lambda) = b_0 \lambda^{n-1} + \ldots$, where a_0 and b_0 are $+^{ve}$. $f(\lambda_0 + h) = h n a_0 \lambda_0^{n-1} + \ldots$, and $\phi(\lambda_0 + h) = \phi(\lambda_0) + h\phi'(\lambda_0) + \ldots = b_0 \lambda_0^{n-1} + \ldots + h\phi'(\lambda_0) + \ldots$ We notice that $a_0 \lambda_0^{n-1}$ and $b_0 \lambda_0^{n-1}$ have the same sign, and as h changes from a small $-^{ve}$ to a small $+^{ve}$ positive value, we have the possibilities for the signs in the sequence Δ, Δ_1 :

$$\left.\begin{array}{l} \text{Before : } - + \text{ or } + - \\ \text{and After : } + + \text{ or } - - \end{array}\right\} \quad (C)$$

ie there is a loss of change of sign. Since there is a loss of n changes of sign between $-\infty$ and $+\infty$ and in view of (a), (b) and (c), there are exactly n real roots of $\Delta = 0$ in $(-\infty, \infty)$.

Now regarding the condⁿ for all the roots being +ve, we notice from (C) that at consecutive roots of $\Delta = 0$, the change in sign of Δ while passing through a root will be reversed ie + - or - + becomes resply - + or + - so that Δ_1 changes sign between these roots from - to + or + to - ie there is a root of Δ_1 between two consecutive roots of $\Delta = 0$, and obviously $\Delta_1 = 0$ has $n-1$ real roots. Generalisation of this argument shows that the roots of $\Delta_{r-1} = 0$ separate those of $\Delta_r = 0$. Now $\Delta_n = 0$ has no root, and $\Delta_{n-1} = 0$ ie $\lambda a_{nn} - b_{nn} = 0$ has a +ve root if b_{nn} is +ve since a_{nn} is +ve. Next, the roots of Δ_{n-2} ~~sepa~~ are separated by this +ve root of $\Delta_{n-1} = 0$. If Δ_{n-2} has the same sign at $\lambda = 0$ and at $\lambda = -\infty$, it has either no negative root or two negative roots, and in the latter case, these roots cannot be separated by the +ve root of $\Delta_{n-1} = 0$. But $\Delta_{n-2} = 0$ has two real roots & therefore both these must be +ve. Generalising the conditions that none of the functions in the sequence $\Delta, \Delta_1, \cdots \Delta_n$ have a -ve root is that each of the fns has the same sign for $\lambda = -\infty$ and $\lambda = 0$ ie each fn has the sign $(-1)^{n-r}$ at $\lambda = 0$ ie each of the $\|b_{ij}\|$ and its cofactors is +ve which is the condⁿ for $\frac{1}{2} b_{ij} x_i x_j$ being +ve definite //

— Solution of the differential eqns; periods & stability — Example of particle moving on fixed smooth surface under gravity.

(21st) 18/9/62 — Soln of Ex. 12, p. 58 of Littlewood by students in the Class.

(22nd) 19/9/62 — Soln of Ex. 13 ibid, and part of Ex. 14. ibid by students in the Class

(25th) 21/9/62 — did the case where both quadratic forms are +ve definite viz

$$6x^2 + 6y^2 + 6z^2 + 8zx + 4xy$$

and $\quad 3x^2 + 3y^2 + 3z^2 - 2yz + 2zx + 2xy.$

and taking the second one as A and the first one as B, roots of $\|\lambda A - B\| = 0$ are $\lambda = 1, 2, 3$ and roots of $\|\lambda B - A\| = 0$ are $1, \frac{1}{2}, \frac{1}{3}$.

Dealt with the first case & got the columns of the T-matrix corresponding $\lambda = 1, 2, 3$ as $\begin{pmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$, $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}$ and taking $T = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & -1/2 \end{pmatrix}$

we get $2x = \eta + \zeta$, $2y = \xi - \eta$, $2z = \xi - \zeta$, and these transformations reduce the quadratics (the second as A & first as B) to $\xi^2 + \eta^2 + \zeta^2$ and $3\xi^2 + 2\eta^2 + \zeta^2$, ~~and take~~ or taking

$T = \Big($

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* This argument is wrong as can be seen by taking the case $n = 3$. In this case eliminating λ_1^2 from the second & third linear equations, we get the <u>same</u> equation as the first in μ_2^2 and μ_3^2. The correct argument is that from each linear equation for eg $\mu_2^2(\lambda_1^2 - \lambda_2^2) + \cdots + \mu_n^2(\lambda_1^2 - \lambda_n^2) = 0$, since $(\lambda_1^2 - \lambda_2^2), \cdots$ can be arbitrary, the conclusion to be drawn is $\mu_2^2 = \mu_3^2 = \cdots = \mu_n^2 = 0$. Similarly for the other equations.

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Condition for stability of eqm

28/9/62 – 24th Lecture – Effect of a new constraint on the periods of a vibrating system

25th Lecture – 3/10/62 – Stationary character of normal vibrations. Expression of $(n-1)$ constraints by the equations $q_1 = \mu_1 q$, $q_2 = \mu_2 q$, $\ldots$, $q_n = \mu_n q$ is explained by the fact that from $(n-1)$ linear eqns, expressing the constraints, of the form $A_1 q_1 + A_2 q_2 + \ldots + A_n q_n = 0$, we can solve for q_1/q_n, q_2/q_n, $\ldots$ and q_{n-1}/q_n and write $\frac{q_1}{q_n} = \frac{\alpha_1}{\alpha_n}, \ldots, \frac{q_{n-1}}{q_n} = \frac{\alpha_{n-1}}{\alpha_n}$ or

$$\frac{q_1}{\alpha_1} = \frac{q_2}{\alpha_2} = \ldots = \frac{q_n}{\alpha_n} = q, \text{ say.} \quad //$$

26th Lecture – 5/10/62 – Re. the stationary character of the constrained system, we have n eqns $\frac{\partial \lambda^2}{\partial \mu_1^2} = 0, \ldots \frac{\partial \lambda^2}{\partial \mu_n^2} = 0$.

The first is a linear eqn in $\mu_2^2, \mu_3^2, \ldots \mu_n^2$, the second one in $\mu_1^2, \mu_3^2, \ldots \mu_n^2$, $\ldots$, and the last one in $\mu_1^2, \mu_2^2, \ldots \mu_{n-1}^2$ [Eliminating a_1^2 from the last $(n-1)$ eqns we get $(n-2)$ linear eqns in $\mu_2^2, \mu_3^2, \ldots \mu_n^2$ and this, along with the first, gives $(n-1)$ linear eqns in these variables of which the solution is $\mu_2^2 = \mu_3^2 = \ldots = \mu_n^2 = 0$. Similarly we get the other $(n-1)$ sets of stationary values. For all these sets, the stationary value of λ^2 is a_1^2 or a_2^2, $\ldots$ or a_n^2.] *

– & Worked Example (ii) p. 187 of Whittaker.

27th Lecture – 14/11/62 – Vibrations about steady motion – Definition of steady motion and the two examples of top motion and $\odot^r$ central orbit (where non-ignorable coords are const. & ignorable velocities are constant) — stable vibratory motion, definition – Expn. for T – use of $\frac{\partial L}{\partial \dot{p}_r} = p_r$ in obtaining expression for R performing ignoration of coordinates

[We can simplify, or rather make clearer, the derivation of R in Whittaker (p. 194) by a proper choice of indices for denoting the ignorable coordinates $(p_1, p_2, \ldots p_k)$ and the non-ignorable coordinates $(q_1, q_2, \ldots q_n)$. Let us use $i, j (= 1, \ldots n)$ for the latter, and $r, s, t, u = (1, \ldots k)$ for the former, and also use the summation convention. Then

$$T = \tfrac{1}{2} a_{ij} \dot{q}_i \dot{q}_j + b_{ir} \dot{q}_i \dot{p}_r + \tfrac{1}{2} c_{rs} \dot{p}_r \dot{p}_s \quad \ldots (1)$$

so that the K-integrals $\frac{\partial L}{\partial \dot{p}_r} = p_r$ or $\frac{\partial T}{\partial \dot{p}_r} = p_r$ become

$$c_{rs} \dot{p}_s + b_{ir} \dot{q}_i = p_r$$

and multiplying by C_{rt} (co-factor of c_{rt} in $|c_{rs}|$)

$$C_{rt} c_{rs} \dot{p}_s + C_{rt} b_{ir} \dot{q}_i = C_{rt} p_r$$

$$\text{i.e.} \quad \delta_{st} \dot{p}_s + C_{rt} b_{ir} \dot{q}_i = C_{rt} p_r$$

$$\text{i.e.} \quad \dot{p}_t = C_{rt} (p_r - b_{ir} \dot{q}_i)$$

Changing $t \to r$, $r \to s$ and using $C_{sr} = C_{rs}$, we get

$$\dot{p}_r = C_{rs} (p_s - b_{is} \dot{q}_i) \quad \ldots (2)$$

Substituting (2) in the expression (1) for T

* (7) represents a non-natural system with terms of degree 2, 1 and 0 in the $\dot{q}_i$

$$T = \tfrac{1}{2} a_{ij}\dot{q}_i\dot{q}_j + b_{ir}\dot{q}_i C_{rs}(\beta_s - b_{js}\dot{q}_j) + \tfrac{1}{2} C_{rs}C_{rt}(\beta_t - b_{it}\dot{q}_i)C_{su}(\beta_u - b_{ju}\dot{q}_j)$$

using the convention that no summation index should appear more than twice in a term

ie $$T = \tfrac{1}{2} a_{ij}\dot{q}_i\dot{q}_j + b_{ir}C_{rs}\beta_s\dot{q}_i - C_{rs}b_{ir}b_{js}\dot{q}_i\dot{q}_j + \tfrac{1}{2}C_{tu}(\beta_t - b_{it}\dot{q}_i)(\beta_u - b_{ju}\dot{q}_j)$$

(using $C_{rs}C_{rt}C_{su} = \delta_{st}C_{su} = C_{tu}$)

$$= \left(\tfrac{1}{2} a_{ij}\dot{q}_i\dot{q}_j - C_{rs}b_{ir}b_{js}\dot{q}_i\dot{q}_j + \tfrac{1}{2}C_{tu}b_{it}b_{ju}\dot{q}_i\dot{q}_j\right)$$
$$+ \left(b_{ir}C_{rs}\beta_s\dot{q}_i - \tfrac{1}{2}C_{tu}b_{it}\beta_u\dot{q}_i - \tfrac{1}{2}C_{tu}b_{ju}\beta_t\dot{q}_j\right) + \tfrac{1}{2}C_{tu}\beta_t\beta_u$$
$$= \left(\tfrac{1}{2} a_{ij}\dot{q}_i\dot{q}_j - C_{rs}b_{ir}b_{js}\dot{q}_i\dot{q}_j + \tfrac{1}{2}C_{rs}b_{ir}b_{js}\dot{q}_i\dot{q}_j\right)$$
$$+ \left(C_{rs}b_{ir}\beta_s\dot{q}_i - \tfrac{1}{2}C_{rs}b_{ir}\beta_s\dot{q}_i - \tfrac{1}{2}C_{rs}\beta_s b_{ir}\dot{q}_i\right) + \tfrac{1}{2}C_{rs}\beta_r\beta_s$$
$$= \tfrac{1}{2}(a_{ij} - C_{rs}b_{ir}b_{js})\dot{q}_i\dot{q}_j + \tfrac{1}{2}C_{rs}\beta_r\beta_s \quad \text{, the second term vanishing } (C_{rs} = C_{sr}) \qquad \text{---(3)}$$

Performing the ignoration of co-ordinates using

$$R = L - \dot{p}_r\beta_r \qquad \text{---------(4)}$$
$$= T - \dot{p}_r\beta_r - V \qquad \text{---}$$

and using (2) and (3), $$= \tfrac{1}{2}(a_{ij} - C_{rs}\beta_{ir}\beta_{js})\dot{q}_i\dot{q}_j - C_{rs}\beta_r(\beta_s - b_{is}\dot{q}_i) + \tfrac{1}{2}C_{rs}\beta_r\beta_s - V$$
$$= \tfrac{1}{2}(a_{ij} - C_{rs}\beta_{ir}\beta_{js})\dot{q}_i\dot{q}_j + C_{rs}\beta_r b_{is}\dot{q}_i - \tfrac{1}{2}C_{rs}\beta_r\beta_s - V \qquad \text{---(5)}$$

In the vibration about steady motion, the non-ignorable coords $q_1, q_2, \ldots q_n$ are all constant & without loss of generality we can assume all these constants as zero. We can then expand the coefficients in R, which are all functions of $q_1, q_2 \ldots q_n$ only since $p_1 \ldots p_k$ are ignorable coordinates (also V is a fn of q_i's), by Taylor's theorem, and neglect all terms of degree higher than two in q_i & $\dot{q}_i$. The only possible terms that could be left over are of the types $\dot{q}_i\dot{q}_j$, $q_i\dot{q}_j$, q_iq_j, $\dot{q}_i$ and q_i. of these the last type does not appear since we are putting $q_i = 0$; also the terms linear in $\dot{q}_i$ (last but one) vanish from the equations

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0 \qquad \text{---(6)}$$

the $\dot{q}_i$

automatically, and so can be omitted. We have therefor R finally of the form

$$R = \tfrac{1}{2}\alpha_{ij}\dot{q}_i\dot{q}_j + \tfrac{1}{2}\beta_{ij}q_iq_j + r_{ij}q_i\dot{q}_j \qquad \text{---(6) (7)}^*$$

where $\alpha_{ij} = \alpha_{ji}$, $\beta_{ij} = \beta_{ji}$ but $r_{ij} \neq r_{ji}$ in general, and the last terms in (7) are called the gyroscopic terms. From (7), the eqns of motion (6) become

(since α_{ij}, β_{ij} & r_{ij} are constant)

$$\frac{d}{dt}\left(\alpha_{ij}\dot{q}_j + r_{ji}q_j\right) - \beta_{ij}q_j - r_{ij}\dot{q}_j = 0$$
$$\text{ie } \alpha_{ij}\ddot{q}_j - \beta_{ij}q_j + (r_{ji} - r_{ij})\dot{q}_j = 0 \qquad \text{-- (8)}$$

These are also linear eqns with constant coefficients as in the case of vibrations about eqm with $\ddot{q}_j$ & q_j

but the additional third term in (8) makes the reduction to normal coords not possible. It can be shown however that every vibration can be considered as the superposition of n normal modes of vibration. The proof is very complicated & is not in the Syllabus.]

Steady motion is given by $\partial R/\partial q_i = 0$ for as can be seen from the form of R in (5) and the eqns of motion (6), we have $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot q_i}\right) = 0$ since in steady motion $\dot q_i$ and $\ddot q_i$ are permanently zero.

Examples. (1) Corollary p. 204 of Whittaker: A particle of unit mass is describing a circular orbit of radius a in a plane about a centre of force at the centre of the circle, the potential energy being $\phi(r)$, where r is the distance from the centre.

Here $T = \frac{1}{2}(\dot r^2 + r^2\dot\theta^2)$

$V = \phi(r)$ ie θ is an ignorable coordinate leading to

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = 0$ ie $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = 0$ or $\frac{\partial T}{\partial \dot\theta} = K$ or $r^2\dot\theta = K$

and $R = T - V - K\dot\theta$

$= \frac{1}{2}(\dot r^2 + r^2\dot\theta^2) - \phi(r) - r^2\dot\theta^2 = \frac{1}{2}\dot r^2 - \frac{1}{2}r^2\dot\theta^2 - \phi(r) = \frac{1}{2}\dot r^2 - \phi(r) - \frac{K^2}{2r^2}$

Steady motion is given by $\left(\frac{\partial R}{\partial r}\right)_{r=a} = 0$ ie $\frac{K^2}{a^3} - \phi_a = 0$ or $K^2 = a^3\phi_a$

and $R = \frac{1}{2}\dot r^2 - \frac{a^3\phi_a}{2r^2} - \phi(r)$

To find vibrations about steady motion, put $r = a + \rho$ in R & retain terms up to 2nd order in ρ and $\dot\rho$, getting

$R = \frac{1}{2}\dot\rho^2 - \frac{a^3\phi_a}{2}\cdot\frac{1}{a^2}\left\{1 - \frac{2\rho}{a} + \frac{3\rho^2}{a^2}\right\} - \left\{\phi(a) + \rho\phi_a + \frac{\rho^2}{2}\phi_{aa}\right\}$

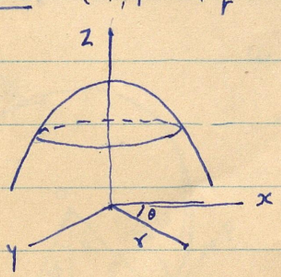
$= \frac{1}{2}\dot\rho^2 - \frac{1}{2}\rho^2\left(\phi_{aa} + \frac{3}{a}\phi_a\right) - \phi(a) - \frac{1}{2}a\phi_a$ (No terms like $\rho, \dot\rho$)

and $\frac{d}{dt}\left(\frac{\partial R}{\partial \dot\rho}\right) - \frac{\partial R}{\partial \rho} = 0$ gives $\ddot\rho + \rho\left(\phi_{aa} + \frac{3}{a}\phi_a\right) = 0$

Since there are no gyroscopic terms in R, this is a case of normal vibrations and

for stability $\phi_{aa} + \frac{3}{a}\phi_a > 0$ ie soln corresponds to $\ddot x + \mu x = 0$ $(\mu > 0)$ S.H.M

Ex. 2: Ex(ii), p. 204 of Whittaker



$L = \frac{1}{2}(\dot r^2 + r^2\dot\theta^2 + \dot z^2) - gz$ [$z = f(r)$ is the surface

θ is ignorable & corresp. integral is $r^2\dot\theta = K$, and

$R = \frac{1}{2}\dot r^2\{1 + f'^2(r)\} - gf(r) - K^2/2r^2$.

For steady motion, $\left(\frac{\partial R}{\partial r}\right)_{r=a} = 0$ gives $\left[\frac{1}{2}\dot r^2 \cdot 2f'(r)f''(r)\right]_{\dot r = a} - gf'(a) + \frac{2K^2}{2a^3} = 0$

Since $\dot r = 0$ for $r = a$, this gives $K^2 = ga^3 f'(a)$

ie $R = \frac{1}{2}\dot{r}^2\{1+f'^2(r)\} - qf(r) - qa^3 f'(a)/2r^2$

The remaining portion is as worked in Whittaker

For particular case of paraboloid of revolution, as given on p. 205 of Whittaker, $z = r^2/2l$

and the period $= \frac{2\pi}{\sqrt{q}}\left\{\frac{1+f''(a)}{f''(a)+3f'(a)/a}\right\}^{1/2} = \frac{2\pi}{\sqrt{q}}\left\{\frac{1+a^2/l^2}{1/l+3q/al}\right\}^{\frac{1}{2}} = \frac{2\pi}{\sqrt{q}}\left\{\frac{a^2+l^2}{4l}\right\}^{\frac{1}{2}}$

$$= \pi\left(\frac{l^2+a^2}{ql}\right)^{1/2} \text{ as required}$$

Whittaker p. 208 - Ex. 1. – Referring to §45, p. 74, we can reduce the rotating curve to rest and take the potential energy as $V(s) - \frac{1}{2}\omega^2 r^2$ [$r = f(s)$

$$T = \frac{1}{2}\dot{s}^2, \quad V' = V(s) - \frac{1}{2}\omega^2 r^2 \qquad L = T - V'$$

To find position of equilibrium of particle, $\frac{\partial L}{\partial s} = 0$ gives $\frac{dV}{ds} - \omega^2 r\frac{dr}{ds} = 0$

$$\text{or } \omega^2 = \frac{dV/ds}{r\,dr/ds}. \quad \text{---(1)}$$

We can take the value of s given by this equation as corresponding to $s = 0$.

Expanding V' by Taylor's theorem in powers of s, and neglecting the constant term

we have coefficient of $s = \frac{dV}{ds} - \omega^2 r\cdot\frac{dr}{ds} = 0$ in virtue of (1)

& coefficient of $\frac{s^2}{2} = \frac{d^2V}{ds^2} - \omega^2\frac{d}{ds}\left(r\frac{dr}{ds}\right)$

$$= \frac{d^2V}{ds^2} - \frac{1}{r}\frac{dV/ds}{dr/ds}\left\{\left(\frac{dr}{ds}\right)^2 + r\frac{d^2r}{ds^2}\right\}$$

$$= \frac{d^2V}{ds^2} - \frac{1}{r}\frac{dV}{ds}\cdot\frac{dr}{ds} - \frac{dV}{ds}\cdot\frac{d^2r/ds^2}{dr/ds} = \mu, \text{ say} \quad \text{---(2)}$$

Since $T = \frac{1}{2}\dot{s}^2$ and $V' = \frac{1}{2}\mu s^2$, the period of vibration $2\pi(\mu)^{-\frac{1}{2}}$.

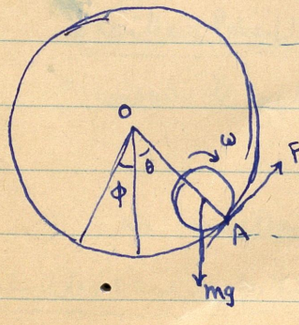
It is easy to see that $-\frac{dV}{ds}\cdot\frac{d}{ds}\log\left(\frac{-r\,dr/ds}{dV/ds}\right) = \mu$, for the L.H.S

$$= -\frac{dV}{ds}\cdot\frac{d}{ds}\left\{\log\left(-r\frac{dr}{ds}\right) - \log\frac{dV}{ds}\right\}$$

$$= -\frac{dV}{ds}\left[\frac{\left(\frac{dr}{ds}\right)^2 + r\frac{d^2r}{ds^2}}{r\,dr/ds} - \frac{d^2V/ds^2}{dV/ds}\right]$$

$$= \frac{d^2V}{ds^2} - \frac{1}{r}\frac{dV}{ds}\cdot\frac{dr}{ds} - \frac{dV}{ds}\cdot\frac{d^2r/ds^2}{dr/ds} = \mu, \text{ as from (2).}$$

Ex. 2.



Here the inner cylinder is assumed to roll without slipping on the inner surface of the outer cylinder. If direction of F is that acting on the outer cylinder, $Mk^2\ddot{\phi} = -F\cdot k$ or $Mk\ddot{\phi} = -F$ ---(1)

For motion of inner cylinder, $\frac{1}{2}ma^2\dot{\omega} = F\cdot a$ or $\frac{1}{2}ma\dot{\omega} = F$ --(2)

and for motion of centre of inner cylinder

$$m(k-a)\ddot{\theta} = -F - mg\sin\theta \quad \text{--(3)}$$

condⁿ for rolling ie that A of both cylinders has the same velocity, gives

$$(b-a)\dot\theta - a\omega = -b\dot\phi \quad \ldots (4)$$

From (2) $a\dot\omega = 2F/m$, using this in (4) after differentiation:

$$(b-a)\ddot\theta - \frac{2F}{m} = -b\ddot\phi = \frac{F}{M} \text{ using (1)}$$

ie $(b-a)\ddot\theta = F\left(\frac{2}{m} + \frac{1}{M}\right) = \frac{F(2M+m)}{Mm}$

finally using (3), $(b-a)\ddot\theta = \frac{2M+m}{Mm}\{-m(b-a)\ddot\theta - mg\sin\theta\}$

or $M(b-a)\ddot\theta = -(2M+m)(b-a)\ddot\theta - (2M+m)g\sin\theta$

ie $(3M+m)(b-a)\ddot\theta + (2M+m)g\sin\theta = 0$

& for small oscillations $\ddot\theta = \frac{(2M+m)}{(b-a)(3M+m)}\theta = 0$. Hence the result.

~~28th, 29th,~~

28th, 29th and 30th Lectures on 20/11/62, 21/11/62 and 23/11/62 – Finite vibrations about steady motion – did examples above except the last one – Gave terminal examination answer books working problems therein.

31st Lecture – 27/11/62 – Hamilton's Principle – Introduction to least Action Principle.

32nd Lecture – 28/11/62 – least Action Principle – In proving this Hamilton's Principle is used in view of the relation $\sum \dot q_r \frac{\partial L}{\partial \dot q_r} - L = h$ so that

$$\int_{CD} \sum \dot q_r \frac{\partial L}{\partial \dot q_r}\cdot dt - \int_{AB} \sum \dot q_r \frac{\partial L}{\partial \dot q_r}\cdot dt = \int_{CD}(h+\Delta h)\,dt - \int_{AB} h\,dt + \int_{CD} L\,dt - \int_{AB} L\,dt$$

$$= (h+\Delta h)(t_1+\Delta t_1 - t_0 - \Delta t_0) + \left[\sum \frac{\partial L}{\partial \dot q_r}\Delta q_r + \left(L - \sum \dot q_r \frac{\partial L}{\partial \dot q_r}\right)\Delta t\right]_A^B$$
$$- h(t_1 - t_0)$$

& neglecting $\Delta h\cdot\Delta t_1$ & $\Delta h\cdot\Delta t_0$,

$$= \Delta h(t_1 - t_0) + \left[\sum \frac{\partial L}{\partial \dot q_r}\Delta q_r - h\Delta t\right]_A^B$$
$$+ h(\Delta t_1 - \Delta t_0)$$

$$= \Delta h(t_1 - t_0) + \left[\sum \frac{\partial L}{\partial \dot q_r}\Delta q_r\right]_A^B$$

$$= \left[\sum \frac{\partial L}{\partial \dot q_r}\Delta q_r + t\Delta h\right]_A^B$$

If C coincides with A and D with B & both trajectories have same h, then $\Delta q_r = 0$, $\Delta h = 0$ which leads to the Action Principle – For natural problems Action Principle relates to $\int T\,dt$.

Hamilton's Principle for non-conservative systems:– Proved for the case where neither T nor Q_r contain t explicitly in the form $\int_{t_0}^{t_1}\left(\delta T + \sum Q_r \delta q_r\right)dt = 0$ in exactly the same way as Hamilton's Principle when L does not contain T explicitly.

Examples 1 and 2, p. 248 – Solved in last year's notes. and done in the Class.

* $mu = X$, $mv = Y$, $mw = Z$

$mu' = X + X'$, - - , with $X'u' + \cdots = 0$

i.e. $\Sigma m\{(u'-u)u' + - + -\} = 0$ i.e. $\Sigma m(u'^2 + \cdot) - \Sigma m(u'^2 + \cdot)$

$= \Sigma m\{(u-u')^2 + \cdot\}$ i.e. $T > T'$

† As in the above but with $X'u + \cdot = 0$ leads to $\Sigma m\{(u'-u)u + \cdot\} = 0$

leads $T - T' = -\Sigma m\{(u-u')^2 + \cdot\} < 0$.

(§) - The symmetrical form mentioned in Whittaker, p. 264 (bottom) is really equivalent to

$\delta(p_r dq_r - H dt) = d(p_r \delta q_r)$ since $\delta t = 0$ as is implied in the eqn $\delta H = \dot{q}_r \delta p_r - \dot{p}_r \delta q_r$ i.e. (only $\delta q_r, \delta p_r \neq 0$)

‡ This is further equivalent to $\delta p_r\, dq_r + p_r \delta(dq_r) - \delta H\, dt - H\, d(\delta t) = dp_r \delta q_r + p_r d(\delta q_r)$

or $\delta p_r\, dq_r - \delta H\, dt = dp_r \delta q_r$ since $\delta(dq_r) = d(\delta q_r)$ & $\delta(dt) = d(\delta t) = 0$

L.H.S $= \delta p_r dq_r - (\dot{q}_r \delta p_r - \dot{p}_r \delta q_r) dt = \delta p_r dq_r - dq_r \delta p_r + dp_r \delta q_r$

$= dp_r \delta q_r =$ R.H.S

If δt is taken $\neq 0$, the relⁿ $\delta(p_r dq_r - H dt) = d(p_r \delta q_r - H \delta t)$ means

$dH \delta t - \delta H\, dt = dp_r \delta q_r - \delta p_r dq_r$

‡ L.H.S $= (\dot{q}_r\, dp_r - \dot{p}_r dq_r)\delta t - (\dot{q}_r \delta p_r - \dot{p}_r \delta q_r) dt$

$= (\delta q_r dp_r - \delta p_r dq_r) - (dq_r \delta p_r - dp_r \delta q_r) = 2(dp_r \delta q_r - \delta p_r dq_r)$

$\neq$ R.H.S.

(*) Ex 1, p. 268, $\frac{d}{dt}(\delta x - t\delta u) = \frac{d}{dt}(t\delta a) - \delta a = \delta a - \delta a = 0$

Ex. 2, p. 268, $\frac{d}{dt}(u\delta x - x\delta u) = \frac{du}{dt}\delta x + u\frac{d}{dt}(\delta x) - \frac{dx}{dt}\delta u - x\frac{d}{dt}(\delta u)$, and with $\frac{dx}{dt} = u$, $du/dt = -\mu x$

$= -\mu x \delta x + u\delta u - u\delta u + x\mu\delta x = 0$.

33rd Lecture – 5/12/62 – Representation of the motion of dynamical systems by geodesics (W. § 104; p. 253) in the case of two degrees of freedom – did a little of theory of projectives – Ex. 1, p. 254 solved, but not strictly proved to be a geodesic as on p. 12 of last year's notes.

34th Lecture – 11/12/62 – Did the above rigorously – used this line element for the ruled surface $z = f(p)$ to solve Ex. 2, p. 254 also – did the least curvature principle of Gauss and Hertz up to the eqn

$$\Sigma (m_r \ddot{x}_{r0} - X_r)(\ddot{x}_r - \ddot{x}_{r0}) = 0$$

35th Lecture – 12/12/62 – Did the above completely – Also obtained the expression for the curvature of the path in generalised coordinates in the form $\frac{1}{D} A_{ij} S_i S_j$, using summation convention.

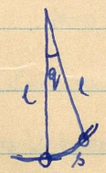
36th Lecture – 18/12/62 – Appell's eqns for the holonomic case as in last year's notes – Also non-holonomic case – Derivation of Euler's eqns of motion of rigid body with one point fixed from Appell's eqn (Ex. p. 259)

37th Lecture – 19/12/62 – Bertrand's theorem as in Whittaker* – Same in generalised coords as in Ramsey II – Kelvin's theorem as in Ramsey II – Kelvin's theorem as in Whittaker†

38th Lecture – 21/12/62 – Examples at end of Chapter IX p. 261 (worked in last year's notes).

39th Lecture – 9/1/63 (After Christmas vacation) – Chap. X – Hamiltonian form of the equations – Hamiltonian form of eqn of virtual work ie $\delta H = \dot{q}_r \delta p_r - \dot{p}_r \delta q_r$ [and its symmetrical form (§) viz $\delta(p_r dq_r - H dt) = d(p_r \delta q_r - H \delta t)$] – Hamilton's eqns of motion – Integral of energy ie $p_r \dot{q}_r - L = \text{const}$ as $H = h$, and in case of L & hence H not involving t explicitly, the expression $H = T + V$.

40th Lecture – 11/1/63 – Symmetrical form $\delta(p_r dq_r - H dt) = d(p_r \delta q_r - H \delta t)$ as equivalent to $\delta H = \dot{q}_r \delta p_r - \dot{p}_r \delta q_r$ – Hamiltonian for simple pendulum: Ex. on p. 265



$$T = \tfrac{1}{2} m \dot{s}^2 = \tfrac{1}{2} m l^2 \dot{q}^2, \text{ putting } s = lq$$

$$V = mg(l - l\cos q) - mgl = -mgl\cos q$$

Hence $H = \frac{1}{2}$. Here $L = \frac{1}{2} l^2 \dot{q}^2 + gl\cos q$ (with $m = 1$) $= T - V$

and $p = \frac{\partial L}{\partial \dot{q}} = l^2 \dot{q}$ or $\dot{q} = p/l^2$ and so $T = \frac{p^2}{2l^2}$.

The Lagrange eqn is $\frac{d}{dt}(l^2 \dot{q}) = -gl \sin q = -gl q$ (for small q)

ie $\ddot{q} = -\frac{g}{l} q$ giving the usual eqn of the pendulum

$$\therefore H = T + V = \frac{p^2}{2l^2} - gl\cos q$$

& the Hamiltonian eqns gives $\dot{q} = p/l^2$ and $\dot{p} = -\frac{\partial H}{\partial q} = -gl \sin q$

the second one giving $\ddot{q} = -\frac{g}{l} q$ (with q small).

– Integral invariants ie $\frac{d}{dt} \int = 0$ ie $\frac{d}{dt}(\text{Integrand}) = 0$ for arbitrary region – Ex ex. 1 & 2 on p. 268 – Extended eqns & variational equations: $\frac{dx_r}{dt} = X_r$ & $\frac{d}{dt}(\delta x_r) = \frac{\partial X_r}{\partial x_k} \delta x_k$ (together form extended system & if a particular soln (x_r) be known, the second set are variational eqns

* In $D = A_{ijk}\frac{\partial x_i}{\partial y_{jk}}$, there is no summation over i, but only over j

for eg. $D = A_{1jk}\frac{\partial x_1}{\partial y_{jk}}$; But $\frac{\partial D}{\partial y_j} = A_{ik}\frac{\partial^2 x_i}{\partial y_j \partial y_k}$ with summation over i.

~~From $D = A_{jk}\frac{\partial x_1}{\partial y_k}$, $D\frac{\partial y_k}{\partial x_1} = A_{1k}\frac{\partial x_1}{\partial x_1}$~~

From $D = A_{1j}\frac{\partial x_1}{\partial y_{jk}}$, $D\frac{\partial y_k}{\partial x_1} = A_{1j}\frac{\partial x_1}{\partial y_j}\cdot\frac{\partial y_k}{\partial x_1} = A_{1j}\,\delta_{jk} = A_{1k}$

~~Similarly $A_{ij}\frac{\partial x_i}{\partial y_{jk}}$~~ Similarly $D\frac{\partial y_k}{\partial x_i} = A_{ik}$ ~~or $A_{ik}\delta_{ik}$~~ or $\frac{\partial y_k}{\partial x_i} = \frac{1}{D}A_{ik}$

– To an integral invariant $\int F_r \delta x_r$ of order one of original system corresponds $\frac{d}{dt}(F_r \delta x_r) = 0$ ie $F_r \delta x_r = \text{const}$, an integral of extended system – Conds for $\int M_r \delta x_r$ being integral invariant of order one are given by

$$\frac{\partial M_r}{\partial t}\delta x_r + \frac{\partial M_r}{\partial x_k} X_k \delta x_r + M_r \frac{\partial X_r}{\partial x_k}\delta x_k = 0$$ & equating coeffts of each $\delta x_r = 0$ gives

$$\text{or} \quad \frac{\partial M_r}{\partial t} + \frac{\partial M_r}{\partial x_k} X_k + M_k \frac{\partial X_k}{\partial x_r} = 0 \quad -(A)$$

Cor 1. from (A): If $F = \text{const}$ be integral of the given system $dx_r/dt = X_r$

41st Lecture – 15/1/63. $\int \frac{\partial F}{\partial x_r}\delta x_r$ is an integral invariant

Cor 2 – Converse of Cor. 1 ie if $\int \frac{\partial U}{\partial x_r}\delta x_r$ is an integral invariant, then $\frac{dU}{dt} = \phi(t)$

or $U - \int \phi(t)\,dt = \text{const}$ is an integral.

– Relative integral invariants & absolute integral invariants, former of one of order p equivalent to latter one of order $(p+1)$. – $\int p_r \delta q_r$ being a relative integral invariant of all Hamiltonian systems – Converse of this theorem – Any system of diff. eqns possessing infinite number of absolute and relative integrals of 1st order.

42nd Lecture – 16/1/63 – Portions left over from previous lecture – Theorem of Lie and Koenigs :–

In proving this Pfaff's theorem may be assumed in the form that q_n, p_n fns of $(x_1 \cdots x_K)$ exist such that $\sum_k^i \delta x_k^i \to p_r \delta q_r \qquad (i = 1 \cdots k,\ r = 1 \cdots n)$

Infinite number of $\int \xi_i \delta x_i$ (init. invs.) exist.

Let q_n, p_n, u_j be independent fns of $(x_1 \cdots x_K)$ $\qquad (K = 2n+s),\ (j = 1 \cdots K-2n)$

such that $\frac{dx_r}{dt} = X_r \longrightarrow \frac{dq_r}{dt} = Q_r,\ \frac{dp_r}{dt} = P_r,\ \frac{du_j}{dt} = V_j \quad (r = 1 \cdots n,\ j = 1 \cdots K-2n)$

P_r, Q_r, V_j are fns of (q_r, p_r, u_j)] Now $\int p_r \delta q_r$ is an init. inv. of this system ie a H exists as $H(q_r, p_r, t)$

such that $Q_r = \partial H/\partial p_i$, $P_r = -\partial H/\partial q_i$ ie $Q_r + P_r$ also may be taken fns of q_r, p_r, t.

Thus original system → Hamiltonian system + $\frac{du_j}{dt} = V_j$.

Theorem of Last multiplier; definition – Lemma leading to the theorem ie $\frac{\partial X_r}{\partial x_r} = \frac{1}{D}\frac{\partial (D Y_r)}{\partial y_r}$. Proof without using summation signs ie using suitable convention, say with i, j, k

$$\frac{\partial X_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{dx_i}{dt}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial x_i}{\partial y_j}\frac{dy_j}{dt}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial x_i}{\partial y_j} Y_j\right)$$

$$= \frac{\partial}{\partial y_K}\left(\frac{\partial x_i}{\partial y_j} Y_j\right)\frac{\partial y_K}{\partial x_i} = \left(\frac{\partial^2 x_i}{\partial y_j \partial y_K} Y_j + \frac{\partial x_i}{\partial y_j}\cdot\frac{\partial Y_j}{\partial y_K}\right)\frac{\partial y_K}{\partial x_i}$$

$$= \frac{\partial y_K}{\partial x_i}\frac{\partial^2 x_i}{\partial y_j \partial y_K} Y_j + \delta_{jK}\frac{\partial Y_j}{\partial y_K} = \frac{\partial y_K}{\partial x_i}\frac{\partial^2 x_i}{\partial y_j \partial y_K} + \frac{\partial Y_j}{\partial y_j} \quad (*)$$

~~For $D = A_{ij}\frac{\partial x_i}{\partial y_j}$ $\therefore D\frac{\partial y_K}{\partial x_i} = A_{ij}\delta_{jK} = A_{iK}$~~ ie $\frac{\partial y_K}{\partial x_i} = \frac{1}{D}A_{iK}$

~~Also $\frac{\partial D}{\partial y_K} = A_{iK}\frac{\partial x_i}{\partial y_K}$~~ + $\frac{\partial D}{\partial y_j} = A_{iK}\frac{\partial^2 x_i}{\partial y_j \partial y_K}$

So the first term of $(*) = \frac{1}{D}\frac{\partial D}{\partial y_j} Y_j$. Hence the lemma.

(*) Condⁿ for $P\,dx + Q\,dy$ being a perfect differential is

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

i.e. $\frac{\partial}{\partial x}\left(\frac{M' X'}{\Delta'}\right) = -\frac{\partial}{\partial x_n}\left(\frac{M' X_n'}{\Delta'}\right)$ which is same as (4)

Also from $\frac{dx_n}{X_n'} = \frac{dx}{X'}$ giving $X' dx_n - X_n' dx = 0$, an integral may not be derivable since to make the expⁿ a perfect diffl, the integrating factor is $\frac{M'}{\Delta'}$.

43rd Lecture – 23/1/63 – Portions left over from previous lecture viz. (i) Proof of $\frac{\partial x_i}{\partial x_i} = \frac{1}{D}\frac{\partial (D Y_i)}{\partial y_i}$. (Lemma)

(ii) Proof that $D = 1/\Delta$ for the transformation $(x_1, x_2, \ldots x_n, x) \to (f_1, f_2, \ldots f_{n-1}, x_n, x)$

In this case $D = \dfrac{\partial (x_1, \ldots x_{n-1}, x_n, x)}{\partial (f_1, \ldots f_{n-1}, x_n, x)} =$

$$\begin{array}{|c|cc|} \hline \dfrac{\partial x_1}{\partial f_1} \ldots \dfrac{\partial x_1}{\partial f_{n-1}} & 0 & 0 \\ \vdots \quad \Delta \quad \vdots & \vdots & \vdots \\ \dfrac{\partial x_{n-1}}{\partial f_1} \ldots \dfrac{\partial x_{n-1}}{\partial f_{n-1}} & 0 & 0 \\ \hline \cdots\cdots & 1 & 0 \\ \cdots\cdots & 0 & 1 \\ \hline \end{array} \qquad \ldots (1)$$

and expanding in terms of the last two columns by Laplace's method of development of a determinant, we have $D = \dfrac{\partial (x_1, \ldots x_{n-1})}{\partial (f_1, \ldots f_{n-1})} = \dfrac{1}{\Delta}$

(iii) Use of Lemma for the transformation of eqns (unity equal to dt)

$$\frac{dx_1}{X_1} = \ldots = \frac{dx_n}{X_n} = \frac{dx}{X} = dt \quad \text{to} \quad \frac{dx_n}{X_n'} = \frac{dx}{X'} = dt$$

In this case $\dfrac{\partial X_i}{\partial x_i} + \dfrac{\partial x}{\partial x} = \dfrac{\Delta'}{\Delta'}\left\{\dfrac{\partial (X_n'/\Delta')}{\partial x_n} + \dfrac{\partial (X'/\Delta')}{\partial x}\right\} \qquad \ldots (2)$

(iv) Equation of last multiplier in terms of the transformed coordinates:

The eqn of L.M is $\dfrac{\partial}{\partial x_i}(MX_i) + \dfrac{\partial}{\partial x}(MX) = 0$.

i.e. $\left(X_i \dfrac{\partial M}{\partial x_i} + X\dfrac{\partial M}{\partial x}\right) + M\left(\dfrac{\partial X_i}{\partial x_i} + \dfrac{\partial x}{\partial x}\right) = 0$ or $\left(X_i \dfrac{\partial}{\partial x_i} + X\dfrac{\partial}{\partial x}\right)\log M + \left(\dfrac{\partial X_i}{\partial x_i} + \dfrac{\partial x}{\partial x}\right) = 0 \qquad \ldots (3')$

i.e. $\left(\dfrac{\partial M}{\partial x_i}\dfrac{dx_i}{dt} + \dfrac{\partial M}{\partial x}\dfrac{dx}{dt}\right) + M(\quad " \quad) = 0$

or $\dfrac{dM}{dt} + M(\quad " \quad) = 0$

or $\dfrac{1}{M}\dfrac{dM}{dt} + (\quad " \quad) = 0 \qquad \ldots (3)$

which on transformation becomes $\dfrac{1}{M'}\dfrac{dM'}{dt} + \Delta'\left\{\dfrac{\partial}{\partial x_n}\left(\dfrac{X_n'}{\Delta'}\right) + \dfrac{\partial}{\partial x}\left(\dfrac{X'}{\Delta'}\right)\right\} = 0$

or $\dfrac{1}{M'}\left(\dfrac{\partial M'}{\partial x_n}\dfrac{dx_n}{dt} + \dfrac{\partial M'}{\partial x}\dfrac{dx}{dt}\right) + \Delta'\{\quad " \quad\} = 0.$

or $\dfrac{1}{\Delta' M'}\left(X_n'\dfrac{\partial M'}{\partial x_n} + X\dfrac{\partial M}{\partial x}\right) + \{\quad\} = 0$

or $\dfrac{\partial}{\partial x_n}\left(\dfrac{M'X_n'}{\Delta'}\right) + \dfrac{\partial}{\partial x}\left(\dfrac{M'X'}{\Delta'}\right) = 0 \qquad \ldots (4)$

(v) Derivation of the integral of the transformed eqns:

The above shows that the expression

$\dfrac{M'}{\Delta'}(X'dx_n - X_n'dx)$ is a perfect differential (*)

and hence $\int \dfrac{M'}{\Delta'}(X'dx_n - X_n'dx) = \text{const.}$ is an integral $\ldots (5)$

(vi) Proof that the quotient of two last multipliers gives an integral of the eqn:

Eqn (3) of the last multiplier can be written as

* Done only up to the theorem of the L.M. giving the integral

$$\int \frac{\partial t_1}{\partial c} dq_1 + \frac{\partial t_2}{\partial c} dq_2 = \text{const.}$$

(**) Condn for $\iint \cdots \int M \delta x_1 \cdots \delta x_n$ being an integral invariant is $\frac{d}{dt} \iint \cdots \int = 0$

Putting $\delta x_1 \cdots \delta x_n = \frac{\partial(x_1 \cdots x_n)}{\partial(c_1 \cdots c_n)} \delta c_1 \cdots \delta c_n$ & since $\delta c_1 \cdots \delta c_n$ is a const ie region is constant

Condn for integral invariancy requires $\frac{d}{dt}$ (Integrand) $= 0$

ie $\frac{d}{dt} \left\{ M \frac{\partial(x_1 \cdots x_n)}{\partial(c_1 \cdots c_n)} \right\} = 0.$

$$\frac{d}{dt}(\log M) + \left(\frac{\partial X_i}{\partial x_i} + \frac{\partial X}{\partial x}\right) = 0$$

Since $\frac{d}{dt}(\log N) + (\quad '' \quad) = 0 \quad \therefore$ Subtracting $\frac{d}{dt}\left(\log \frac{M}{N}\right) = 0$

ie $\log(M/N) = \text{const}$ or $\frac{M}{N} = \text{const}$, is an integral. — — — . . . (6)

(vii) Application of L. M. to H-systems: —

For the Hamiltonian system $\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$, $\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$

$$\frac{\partial X_i}{\partial x_i} = \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = 0$$

ie $\log M = \text{Const}$ or $M = \text{Const}$ & may be taken equal to unity ie $M = 1$ — . . (7)

or directly from (3') $\left(X_i \frac{\partial}{\partial x_i} + X\frac{\partial}{\partial x}\right)\log M + \left(\frac{\partial X_i}{\partial x_i} + \frac{\partial X}{\partial x}\right) = 0$, we have $\log M = 0$ ie $M = 1$

~~(viii)~~

Remaining portions (vi) & (vii) of last lecture —

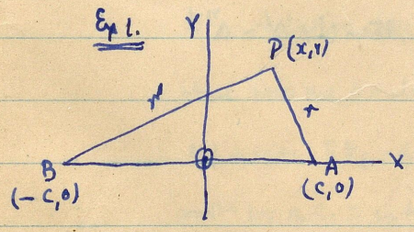
44th Lecture — 29/1/63 — (viii) Integration of any conservative holonomic system of two degrees of freedom when one integral beside the integral of energy is known.*

45th Lecture — 30/1/63 — Proof of $\frac{\partial f_2}{\partial q_1} = \frac{\partial f_1}{\partial q_2}$ and hence $f_1\,dq_1 + f_2\,dq_2 = d\theta(q_1, q_2, h, c)$ and $\frac{\partial \theta}{\partial c} = \text{const.}$ being an integral — Proof that the other integral is $t = \partial\theta/\partial h + \text{const}$ — condition for $\iint\cdots\int M\,\delta x_1 \ldots \delta x_n$ being an integral invariant of order n (**) viz. M a last multiplier — $\iint\cdots\int \delta q_1 \ldots \delta q_n\, \delta p_1 \ldots \delta p_n$ being an integral invariant for H-systems — Proof of $N = MD$ Reduction of system of diff. eqns to Lagrangian form; use of ~~last~~ last multiplier for the case one single equation —

46th Lecture — 1/2/63 — Remaining portions from previous lecture — § 124, p. 285 of Whittaker (usually not done for the classes any time) —

47th Lecture — 5/2/63 — Miscellaneous examples on p. 286. —

Ex 1.



under each centre of force with $V \propto -1/r$ orbit is a conic & hence under both centres of force together also orbits are conics (by Bonnet's theorem with foci A, B)

Put $x = c\cosh\xi\cos\eta$, $y = c\sinh\xi\sin\eta$

$\xi = \text{const}$ & $\eta = \text{const}$ are ellipses & hyperbolas with foci A & B.

$$V = -\frac{\mu}{r} - \frac{\mu'}{r'} = -\frac{\mu}{\sqrt{(x-c)^2+y^2}} - \frac{\mu'}{\sqrt{(x+c)^2+y^2}} = -\frac{\mu}{c(\cosh\xi - \cos\eta)} - \frac{\mu'}{c(\cosh\xi + \cos\eta)}$$

Putting $q_1 = c\cosh\xi$, $q_2 = c\cos\eta$, $V = -\frac{\mu}{q_1 - q_2} - \frac{\mu'}{q_1 + q_2}$

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}c^2(\cosh^2\xi - \cos^2\eta)(\dot{\xi}^2 + \dot{\eta}^2)$$

$$= \frac{1}{2}(q_1^2 - q_2^2)\left(\frac{\dot{q}_1^2}{q_1^2 - c^2} + \frac{\dot{q}_2^2}{c^2 - q_2^2}\right)$$

$$\therefore p_1 = \frac{\partial T}{\partial \dot{q}_1} = \frac{(q_1^2 - q_2^2)}{(q_1^2 - c^2)}\dot{q}_1 \; ; \; p_2 = -\frac{q_1^2 - q_2^2}{q_2^2 - c^2}\dot{q}_2$$

* Is this correct?

If $NX = f(x_1 \cdots x_n)$, then from the diff eqns $\frac{dx_1}{X_1} = \cdots = \frac{dx_n}{x_n} = \frac{dx}{x}$

we can write $x_1 \ldots x_n$ as fns of x. Hence $NX = 1$ is correct.

† Last years notes, p. 19.

§. If $\Delta = (a_1 b_2 \cdots k_n)$ & $\Delta' = (A_1 B_2 \cdots K_n)$, A_1 cofactor of a_1, then (i) $\Delta' = \Delta^{n-1}$

Proof for $n = 3$, $\Delta = (a_1 b_2 c_3)$

$$a_1 A_1 + b_1 B_1 + c_1 C_1 = \Delta$$

$$a_1 A_2 + b_1 B_2 + c_1 C_2 = 0$$

$$a_1 A_3 + b_1 B_3 + c_1 C_3 = 0.$$

$$\Delta\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ & & \\ & & \end{vmatrix} = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

$$\Delta' = \Delta^2$$

So if A_1, etc be replaced by A_1/Δ', we have $a_1 A_1 + b_1 B_1 + c_1 C_1 = 1$
$= 0$
$= 0$

This gives H in the req^d form & the eqns follow.

Ex 3. Last year's notes, p. 19.

Ex. 4. ibid, p. 18.

48th Lecture – 6/2/63

Ex. 5, Whittaker, p. 287 – Last year's notes, p. 18 – Here the L.M of the transformed eqn given by

$$\frac{\partial}{\partial x}(NX) = 0 \text{ is taken as } NX = \text{const} = 1 \text{ or } N = 1/X$$

More correctly,* $NX = f(x_1 \cdots x_n)$ ie $N = f/X$, so that $M = \frac{f}{X}\frac{\partial(\theta_1 \cdots \theta_n X)}{\partial(x_1 \cdots x_n, x)} = \frac{f}{X}\frac{\partial(\theta_1 \cdots \theta_n)}{\partial(x_1 \cdots x_n)}$

$= f M_0$ say.

To prove that $f M_0$ is a L.M, we write

$$\frac{\partial}{\partial x_1}(f M_0 X_1) + \cdots + \frac{\partial}{\partial x_n}(f M_0 X_n) + \frac{\partial}{\partial x}(f M_0 X) = 0$$

ie if $\frac{\partial}{\partial x_1}(M_0 X_1) + \cdots + f\frac{\partial}{\partial x_n}(M_0 X_n) + f\frac{\partial}{\partial x}(M_0 X)$ } } unnecessary See ⊛

$$+\left[M_0 X_1 \frac{\partial f}{\partial x_1} + \cdots + M_0 X_n \frac{\partial f}{\partial x_n}\right] = 0$$

the expn within square brackets $= M_0\left[\frac{\partial f}{\partial x_1}\cdot\frac{dx_1}{dx} + \cdots + \frac{\partial f}{\partial x_n}\cdot\frac{dx_n}{dx}\right] = M_0\frac{df}{dx} = 0$. }

– Defn of general contact transformations – extended point transformation – group property of contact transformations mentioning notion of product, associative law, identity and inverse – Ex 1, p. 293 & Exs 2, 3 on p. 294 – Ex. p. 296, and Ex. p. 307 (How are these related to each other?)

49th Lecture – 8/2/63 – Extended point transformations and point transformations – Explicit analytical expression for a contact transformation – Ex. $Q = (2q)^{1/2}K^{-1/2}\cos p$; $P = (2q)^{1/2}K^{1/2}\sin p$ where there is no relation like $\Omega = 0$. Here $W = \frac{1}{2}Q(2qK - K^2Q^2)^{1/2} - q\cos^{-1}\left(Q\sqrt{K/2q}\right)$ [W, Ex, p. 296]† – Case where there are K relations $\Omega_s = 0$ $(s = 1, \cdots K)$ – Bilinear covariant of a differential form (a Pfaffian expression) $\delta\theta_q - d\theta_s = a_{ij}\,dx_i\,\delta x_j$ $\left(a_{ij} = \frac{\partial x_i}{\partial x_j} - \frac{\partial x_j}{\partial x_i}\right)$. Invariance of bilinear covariant for $x \to y$ – conds for a C. T. in terms of bilinear by means of bilinear covariant ie $(\delta p_r\,dq_r - dp_r\,\delta q_r)$ invariant – conds in terms of L. B's

50th Lecture – 12/2/63. Invariance of $\delta p_r dq_r - dp_r \delta q_r$ directly from invariance of $\delta\theta_q - d\theta_s$ – Verification from $Q = (2q)^{1/2}K^{-1/2}\cos p$ & $P = (2q)^{1/2}K^{1/2}\sin p$ – conds in terms of L. B's – Connection between P. B's & L. B's, $(u_k, u_l)[u_t, u_s] = \delta_{rs}$ ie determinants reciprocal (§)

Exs. 1 & 2, p. 300 – Conds for C. T. in terms of P. B's –

51st Lecture – 13/2/63 – Repetition of obtaining conditions in terms of L. B's. Here we can write

$$dp_r\,\delta q_r - \delta p_r\,dq_r = \frac{\partial p_r}{\partial u_K}\frac{\partial q_r}{\partial u_L}du_K\,\delta u_L - \frac{\partial p_r}{\partial u_K}\frac{\partial q_r}{\partial u_L}\delta u_K\,du_L$$

Since there is summation over K & L, we can write coeff of $du_K\,\delta u_L = \frac{\partial p_r}{\partial u_K}\cdot\frac{\partial q_r}{\partial u_L} - \frac{\partial p_r}{\partial u_L}\cdot\frac{\partial q_r}{\partial u_K}$

Symbolic coeff of $\delta u_K\, du_L = \dfrac{\partial p_r}{\partial u_L}\dfrac{\partial q_r}{\partial u_K} - \dfrac{\partial p_r}{\partial u_K}\dfrac{\partial q_r}{\partial u_L}$

Hence coeff of $(du_L\,\delta u_K - \delta u_L\, du_K) = \dfrac{\partial q_r}{\partial u_K}\cdot\dfrac{\partial p_r}{\partial u_L} - \dfrac{\partial p_r}{\partial u_K}\cdot\dfrac{\partial q_r}{\partial u_L}$

& expn $= [u_K, u_L](du_L\,\delta u_K - \delta u_L\, du_K)$

– Connection between P.B's and L.B's as forming reciprocal determinants – Expression for C.T. in terms of P.B's – Exx 1, 2, p. 300 and Ex 1, p. 301.

[In a determinant, sum of products of elements of a row by their respective cofactors $= \Delta$, and sum of products of elements of a row by cofactors of elements of another row $= 0$. If $|a_{ij}|$ be the determinant & A_{ij} the cofactor of a_{ij}, this can be written as

$$a_{iK}\,A_{jK} = \Delta\cdot\delta_{ij} \quad [\text{or by cols } a_{Ki}\,A_{Kj} = \Delta\cdot\delta_{ij}]$$

For reciprocal determinants the elements corresponding to a_{ij} are A_{ij}/Δ & hence for such determinants $a_{iK}\,A_{jK} = \delta_{ij}$. In the case of L.B's & P.B's, when $[u_t, u_s] = a_{ts}$ and $(u_t, u_r) = A_{tr}$ we have $a_{ts}\,A_{tr} = \delta_{rs}$ ie we have multiplication of elements of a column by cofactors/Δ of elements of same column or different columns.

– In expns of C.T. in terms of P.B's, the relations follow from conditions for C.T in terms of L.B's from $(u_t, u_r)[u_t, u_s] = \delta_{rs}$. In fact $(u_t, u_r)[u_t, u_s] = 0$ $(r \neq s)$ lead to $(q_r, p_s) = 0$, $(q_r, q_s) = 0$ and $(p_r, p_s) = 0$, while $(u_t, u_r)[u_t, u_r] = 1$ lead to $(q_r, p_r) = 1$. ie forming conditions same for L.B's and P.B's.

– Exx 1, 2, p. 300 – Worked on pp. 20–21 of last year's notes.

Ex. 1 (Alternatively) – Replacing f, ϕ, ψ by f_1, f_2, f_3

$$((f_1, f_2), f_3) = \frac{\partial(f_1, f_2)}{\partial q_s}\cdot\frac{\partial f_3}{\partial p_s} - \frac{\partial(f_1, f_2)}{\partial p_s}\cdot\frac{\partial f_3}{\partial q_s}$$

$$= \frac{\partial^2 f_1}{\partial q_r^2}\cdot\frac{\partial f_2}{\partial p_r}\frac{\partial f_3}{\partial p_r} - \frac{\partial^2 f_1}{\partial q_r \partial p_r}\left(\frac{\partial f_2}{\partial q_r}\cdot\frac{\partial f_3}{\partial p_r} + \frac{\partial f_2}{\partial p_r}\cdot\frac{\partial f_3}{\partial q_r}\right) + \frac{\partial^2 f_1}{\partial p_r^2}\cdot\frac{\partial f_2}{\partial q_r}\cdot\frac{\partial f_3}{\partial q_r}\cdot$$
$$- \frac{\partial^2 f_2}{\partial q_r^2}\cdot\frac{\partial f_3}{\partial p_r}\cdot\frac{\partial f_1}{\partial p_r} + \frac{\partial^2 f_2}{\partial q_r \partial p_r}\left(\frac{\partial f_3}{\partial q_r}\cdot\frac{\partial f_1}{\partial p_r} + \frac{\partial f_3}{\partial p_r}\cdot\frac{\partial f_1}{\partial q_r}\right) - \frac{\partial^2 f_2}{\partial p_r^2}\cdot\frac{\partial f_3}{\partial q_r}\cdot\frac{\partial f_1}{\partial q_r}$$

(after simplification & arrangement)

In the cyclic interchange $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$, the 6 four terms in the first row after the interchange cancel with the four terms in the 2nd row & hence the result.

– Ex. 1, p. 301 – Last year's notes p. 21 – Start with the right hand side expn & use P.B. conditions for C.T.

52nd Lecture – 15/2/63 – Ex. 1, p. 301 given for H.W. last time – Subgroups of Mathieu transfns and extended point trfns –

53rd Lecture – 19/2/63 – Ex. p. 302 – 2nd part obvious from theorem on homogeneous functions, since P_r is homogeneous

and of first degree in p_k ie $p_k \frac{\partial P_r}{\partial p_k} = P_r$. As regards the first part, we see from $dQ_r = \frac{\partial Q_r}{\partial q_k} dq_k + \frac{\partial Q_r}{\partial p_k} dp_k$

that if $p_k \to \mu p_k$, $dQ_r \to dQ_r$ ie $Q_r \to Q_r$ ie Q_r is homogeneous of degree zero in p_k

ie $p_k \frac{\partial Q_r}{\partial p_k} = 0$.

– Infinitesimal C. T's – motion of a dynamical system as unfolding of an inf. C.T (a continuous group); from initial values (α_r, β_r) at $t = t_0$ to values (q_r, p_r) at $t = t$ (ie solns of the diff. eqns of motion express a C. T)

– Jacobi's theorem on transformation of one dynamical system into another

– Ex. p. 307 – This is proved a C. T. on p. 297 and only form of K is required

$$H = \tfrac{1}{2}(p^2 + k^2 q^2) = \tfrac{1}{2}\left(2QK \sin^2 P + K^2 \cdot 2Q \cdot K^{-1} \cos^2 P\right) = KQ = K$$

– Representation of a dynamical problem with a differential form – Pfaffian eqns – Pfaffian eqns of the diff. form $p_r dq_r - H dt$.

54th lecture – 20/2/63 – Writing $p_r dq_r - H dt = 0$ in the form $p_1 dq_1 + \cdots + p_n dq_n + 0 \cdot dp_1 + \cdots + 0 \cdot dp_n - H dt$

we have $X_r = p_r$, $X_{n+r} = 0$, $X_{2n+1} = -H$ $(r = 1, \cdots n)$

$a_{ij} dx_i = 0$ for $j = r$ gives $\left(\frac{\partial X_i}{\partial q_r} - \frac{\partial p_r}{\partial x_i}\right) dx_i = 0$ ie $-\frac{\partial H}{\partial q_r} dt - dp_r = 0$

$a_{ij} dx_i = 0$ for $j = n + r$ gives $\left(\frac{\partial X_i}{\partial p_r} - 0\right) dx_i = 0$ ie $dq_r - \frac{\partial H}{\partial p_r} dt = 0$.

$a_{ij} dx_i = 0$ for $j = 2n+1$ gives $\left(\frac{\partial X_i}{\partial t} + \frac{\partial H}{\partial x_i}\right) dx_i = 0$ ie $-\frac{\partial H}{\partial t} dt + \frac{\partial H}{\partial q_r} dq_r + \frac{\partial H}{\partial p_r} dp_r + \frac{\partial H}{\partial t} dt = 0$

ie $\frac{\partial H}{\partial q_r} dq_r + \frac{\partial H}{\partial p_r} dp_r = 0$

and this is a consequence of the first two eqns. Hence Pfaffian eqns of $p_r dq_r - H dt$ are the Hamiltonian eqns. – Eqns of motion in terms of variables $(x_1, \cdots x_{2n}, \tau)$ are the first Pfaffian eqns of $X_1 dx_1 + \cdots + X_{2n} dx_{2n} + T d\tau$ obtained from $p_r dq_r - H dt$ by a C. T. –

Another proof of the conservation of Hamiltonian form under all C.T's and finding the form of the transformed Hamiltonian K in $\frac{dQ_r}{dt} = \frac{\partial K}{\partial P_r}$; $\frac{dP_r}{dt} = -\frac{\partial K}{\partial p_r}$.

– Case where t is also changed to T – New formulation of the integration problem

55th lecture – 27/2/63 – Chapter XII of Whittaker to be started.

Hamilton – Jacobi partial diff. eqn; the Q_r's in W are constants in the eqns problem & hence W can be considered a function of $(q_1, \cdots q_n, t)$ only – Solution $W(q_1 \cdots q_n, \alpha_1, \cdots \alpha_n, t)$ of the H-J eqn lead to the soln $\beta_r = -\partial W/\partial \alpha_r$; $p_r = \partial W/\partial q_r$ enpressing q_r, p_r as fns of t & $2n$ arbitrary constants of the dynamical problem ie soln of system of n degrees of freedom depends on soln of one partial diff. eqn of first order in $(n+1)$ variables – Hamilton's integral as soln of the H-J eqn –

– Connection of integrals with infinitesimal contact transformations.

56th Lecture – 5/3/63 – Remaining portion of above previous lecture – Poisson's theorem of $(\phi, \psi) = \text{const}$ being an integral of $\phi = \text{const}$, $\psi = \text{const}$. be integrals – Constancy of L. B. expns – Involution systems – Solns when half the integrals in involution are known.

57th Lecture – 6/3/63 – Remaining part of last lecture – Examples to be worked. Chap II – Miscellaneous Examples p. 311,

Ex 1. Showing $(Q_1, Q_2) = (P_1, P_2) = (Q_1, P_2) = (Q_2, P_1) = 0$, and $(Q_1, P_1) = (Q_2, P_2) = 1$.
That transformed Hamiltonian is Q_2 is evident from the form of Q_2.

Ex. 2. p. 24 of last year's notes.

Ex. 3. p. 25 of last year's notes – First prove for a C.T. $\dfrac{\partial(Q_r, P_r)}{\partial(q_r, p_r)} = 1$

Ex. 4, "

Examples, p. 336

Ex. 2. Here $H = h$ is integral of energy & H does not contain t explicitly
$\phi = \text{const}$ being an integral $\rightarrow \dfrac{d\phi}{dt} = \dfrac{\partial \phi}{\partial t} + \dfrac{\partial \phi}{\partial q_r}\cdot\dfrac{\partial H}{\partial p_r} - \dfrac{\partial \phi}{\partial p_r}\cdot\dfrac{\partial H}{\partial q_r} = 0$
Differentiating partially w.r. to t, leads to the req^{d}. result.

Ex. 3. p. 27 of last year's notes.

(1) Uspensky - Introduction to mathematical probability

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(2)

14/3/69.

Questions on General Relativity (from Synge)

(A) Killing vector, Killing tensor and Killing's equation

If $\xi^i = dx^i/ds$

and $\xi_{i|j} + \xi_{j|i} = 0$ ——— (1)

for all i, j, ξ_i is called a Killing vector. $\xi_{i|j}$ = Killing tensor.

(1) consists of 10 equations

Using $T_{i|jk} - T_{i|kj} = R^a_{\cdot ijk} T_a$ we get $\xi_{i|jk} - \xi_{i|kj} = R_{aijk} \xi^a$. (2)

and applying two cyclic permutations to ijk, adding and using (1) and $R_{iabc} + R_{ibca} + R_{icab} = 0$, we get

$$\xi_{i|jk} + \xi_{j|ki} + \xi_{k|ij} = 0 \qquad (3)$$

ie the Killing tensor has a vanishing cyclic divergence

Applying (1) to the first term in (3)

$$-\xi_{j|ik} + \xi_{j|ki} + \xi_{k|ij} = 0 \qquad -- (4)$$

The first two terms may be combined as in (2) and hence the second order covariant derivative of the Killing vector can be expressed in terms of the Riemann tensor & the Killing vector itself as follows:

$$\xi_{k|ij} = R_{aijk} \xi^a = R_{kija} \xi^a \qquad (5)$$

Multiplying this by g^{ij}, we have (in terms of the generalised d'Alembertian)

$$\Box \xi_k = g^{ij} \xi_{k|ij} = R_{ka} \xi^a$$

(B) From FOCK

(a) Parallel transport. Increment of vector components due to ∥ˡ transport has the form

$$\delta A_\alpha = \Gamma_{\gamma,\alpha\beta} A^\gamma \delta x_\beta \qquad -- (1)$$

where $\Gamma_{\gamma,\alpha\beta}$ is the Christoffel Symbol $\frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x_\beta} + \frac{\partial g_{\beta\gamma}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_\gamma}\right)$

Also $\delta A^\gamma = -\Gamma^\gamma_{\alpha\beta} A^\alpha \delta x_\beta$ -- (2)

For scalar product of two vectors $\delta(A^\nu B_\nu) = 0$ ie scalar product remains unchanged in ∥ˡ transport; in particular magnitude of a vector is unaltered

(b) Along a curve $x_\beta = x_\beta(p)$, ∥ˡ transport gives $\frac{dA^\nu}{dp} = -\Gamma^\nu_{\alpha\beta} A^\alpha \frac{dx_\beta}{dp}$

Cf. with eqn of geodesic viz $\frac{d^2 x_\nu}{dp^2} + \Gamma^\nu_{\alpha\beta} \frac{dx_\alpha}{dp} \cdot \frac{dx_\beta}{dp} = 0$

The two sets are same if we put $A^\nu = dx_\nu/dp$ ie if geodesic be time-like (point vel $< c$), proper time τ can be used in place of p & vector A^ν = 4-dimnl velocity. Thus in this case, geodesic can be interpreted (in 4 dimnl terms) as the equation of

the parallel transport of ~~a body~~ the velocity vector in the origin given by the vector itself

Eqns for $\parallel^{l}$ transport of a tensor $T_{\mu\nu}$ given by

$$\delta T_{\mu\nu} = T_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}\,\delta x_\beta + T_{\alpha\nu}\Gamma^{\alpha}_{\mu\beta}\,\delta x_\beta$$

(v) covariant derivatives

~~Defined as~~ of a ~~vector~~ covariant vector A_α defined by $\nabla_\beta A_\alpha = \dfrac{\partial A_\alpha}{\partial x_\beta} - \Gamma^{\nu}_{\alpha\beta} A_\nu$

and contravariant A^ν $\quad \nabla_\beta A^\gamma = \dfrac{\partial A^\gamma}{\partial x_\beta} + \Gamma^{\nu}_{\alpha\beta} A^\alpha$

" tensor $T_{\mu\nu} \rightarrow \nabla_\beta T_{\mu\nu} = \dfrac{\partial T_{\mu\nu}}{\partial x_\beta} - \Gamma^{\rho}_{\mu\beta} T_{\rho\nu} - \Gamma^{\rho}_{\nu\beta} T_{\mu\rho}$

& $\quad \nabla_\beta T^{\mu\nu} = \dfrac{\partial T^{\mu\nu}}{\partial x_\beta} + \Gamma^{\mu}_{\rho\beta} T^{\rho\nu} + \Gamma^{\nu}_{\rho\beta} T^{\mu\rho}$

& $\quad \nabla_\beta T^{\mu}_{\nu} = \dfrac{\partial T^{\mu}_{\nu}}{\partial x_\beta} - \Gamma^{\rho}_{\beta\nu} T^{\mu}_{\rho} + \Gamma^{\mu}_{\rho\beta} T^{\rho}_{\nu}$

Ricci's lemma: If $T_{\mu\nu} = g_{\mu\nu}$, covariant derivatives vanish

(Detailed proof on p. 141 of Fock's new edition)

covariant derivative of product of two tensors follows same rule as that of ordinary derivatives

Examples. (i) $\mathrm{Div}\, A = \nabla_\nu A^\nu = \dfrac{\partial A^\nu}{\partial x^\nu} + \Gamma^{\nu}_{\sigma\nu} A^\sigma = \dfrac{1}{\sqrt{(-g)}}\dfrac{\partial}{\partial x_\nu}\left\{\sqrt{(-g)}\, A^\nu\right\}$

If $A_\mu = \dfrac{\partial \varphi}{\partial x_\mu}$ so that $A^\nu = g^{\mu\nu}\dfrac{\partial \varphi}{\partial x_\mu}$

divergence of this vector $= \Box\varphi = \dfrac{1}{\sqrt{(-g)}}\cdot\dfrac{\partial}{\partial x_\nu}\left\{\sqrt{(-g)}\cdot g^{\mu\nu}\dfrac{\partial \varphi}{\partial x_\mu}\right\}$

~~If A_μ~~ and in this case $\nabla_\mu A_\nu - \nabla_\nu A_\mu = 0$

More generally for any A_μ this expn $\neq 0$

(ii) & putting $(\mathrm{Rot}\, A)_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \dfrac{\partial A_\nu}{\partial x_\mu} - \dfrac{\partial A_\mu}{\partial x_\nu}$

(iii) If $A_{\mu\nu}$ = symmetric part of covariant derivative of A_μ

$$A_{\mu\nu} = \frac{1}{2}\left(\nabla_\nu A_\mu + \nabla_\mu A_\nu\right)$$

and hence $\quad = \dfrac{1}{2}\left(\dfrac{\partial A_\mu}{\partial x_\nu} + \dfrac{\partial A_\nu}{\partial x_\mu}\right) - \Gamma^{\alpha}_{\mu\nu} A_\alpha$

using this we define covariant second derivative of A_μ by

$$\nabla_{\mu\nu} A_\sigma = \nabla_\nu A_{\mu\sigma} + \nabla_\mu A_{\nu\sigma} - \nabla_\sigma A_{\mu\nu} = \text{etc}$$

Sum for $A^{\mu\nu}$

(iv) using $\nabla_\sigma F_{\mu\nu} = \frac{\partial F_{\mu\nu}}{\partial x_\sigma} - \Gamma^\rho_{\mu\sigma} F_{\rho\nu} - \Gamma^\rho_{\nu\rho} F_{\mu\rho}$, etc.

we get if $F_{\mu\nu}$ is antisymmetric ie $F_{\mu\nu} = -F_{\nu\mu}$ the result

$$\nabla_\sigma F_{\mu\nu} + \nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\sigma\mu} = \frac{\partial F_{\mu\nu}}{\partial x_\sigma} + \frac{\partial F_{\nu\rho}}{\partial x_\mu} + \frac{\partial F_{\sigma\mu}}{\partial x_\mu}$$

Calling R.H.S = $F_{\mu\nu\sigma}$, this is antisymmetric in all three suffixes & is hence a tensor of third rank & is called the cyclic derivative of the antisymmetric $F_{\mu\nu}$.

(17/3/69) – Theory of Relativity by Möller:

(1) The Christoffel symbols do not transform like a tensor – for linear (affine) transformations they transform like a tensor. for eg.

$$\Gamma'^{\,l}_{kl} = \alpha^i_r \frac{\partial \bar{x}^r_k}{\partial x'^l} + \alpha^i_r \bar{\alpha}^s_k \alpha^t_l \Gamma^r_{st}$$

and for affine transformations $\bar{\alpha}^i_k$, α^i_k are constant & the first term on R.H.S. vanishes. Hence Christoffel symbols are called affine tensors

If g_{ik} = const, these symbols vanish

(2) Covariant derivatives applied to tensor fields also. for eg

$$t^{ik}_{\ ;l} = \frac{\partial t^{ik}}{\partial x^l} + \Gamma^i_{lr} t^{rk} + \Gamma^k_{lr} t^{ir}$$

$$t_{ik;l} = \frac{\partial t_{ik}}{\partial x_l} - \Gamma^r_{il} t_{rk} - \Gamma^r_{kl} t_{ir}$$

$$t^i_{k;l} = \frac{\partial t^i_k}{\partial x^l} + \Gamma^i_{lr} t^r_k - \Gamma^r_{kl} t^i_r$$

(3) curvature tensor: Because of $R_{iklm} = -R_{ikml} = -R_{kilm} = R_{lmik}$

and $R_{iklm} + R_{ilmk} + R_{imkl} = 0$

the number of algebraically independent components of the curvature tensor is 20 in a 4-dimensional space.

Bianchi identities: – $R^i_{klm;n} + R^i_{kmn;l} + R^i_{knl;m} = 0$

Contracted curvature tensor: $R_{ik} = R^r_{rik} = -R^r_{irk} = -R^r_{kri}$

$\longrightarrow R_{ik} = R_{ki}$

$R = g^{ik} R_{ik} = R^i_i$

Covariant divergence of $R^{ik} - \frac{1}{2} g^{ik} R = 0$. (ie this symm. tensor has 10 comps)

Explicit expns for $R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^k} - \frac{\partial \Gamma^l_{lk}}{\partial x^l} + \Gamma^r_{ik} \Gamma^l_{kr} - \Gamma^r_{ik} \Gamma^l_{lr}$

$$= \frac{\partial^2 \log\sqrt{|g|}}{\partial x^i \partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \Gamma^r_{il}\,\Gamma^l_{kr} - \Gamma^r_{ik}\,\frac{\partial \log\sqrt{|g|}}{\partial x^r}.$$

(4) $R_{iklm} = 0$ is necessary and sufficient for a space being flat – Sufficient condⁿ proved at great length in Möller: Appendix, p. 377.

(i) to show that the resultant of a $\|^k$ displacement of a vector is unique & is independent of the path along which the displacement is performed

(ii) to show that $\Gamma^{\prime i}_{kl} = 0 \longrightarrow \partial g'_{ik}/\partial x'^l = 0$

(B) Synge's General Relativity

(1) $(AB)_{;i} = A_{;i}\cdot B + A\cdot B_{;i}$ for covariant derivative of product of two tensors A and B (suffixes suppressed) (i') $g_{ij;k} = g^{ij}_{\ ;k} = 0$

(2) Absolute derivatives for vector & tensor fields defined on a curve $x^i = x^i(u)$

21|3|69 – At Stat.

Let $G^{\beta\delta} = c_1 R^{\beta\delta} + c_2 R g^{\beta\delta} + c_3 g^{\beta\delta}$ (Adler, Bazin & Schaffer p. 158.)

Using the Bianchi identities in the form (with $\|$)

$$\{ R_{\alpha\eta\beta\gamma\|\delta} \}_{(\beta,\gamma,\delta)} = 0$$

& using the first two identities

$$\{ R^{\alpha\eta}_{\ \ \beta\gamma\|\delta} \}_{(\beta,\gamma,\delta)} = 0$$

[$R^{\alpha}_{\rho\alpha\delta} = R_{\rho\delta}$

$R^{\alpha\eta}_{\ \ \eta\delta} = R^{\alpha}_{\ \eta\delta}$

$\eta\delta\alpha\eta = g^{\eta\epsilon} R^{\alpha}_{\epsilon\eta\delta}$

$= g^{\alpha\epsilon} R^{\eta}_{\epsilon\alpha\delta}$

$= g^{\alpha\epsilon}$]

i.e. $R^{\alpha\eta}_{\ \ \beta\gamma\|\delta} + R^{\alpha\eta}_{\ \ \gamma\delta\|\beta} + R^{\alpha\eta}_{\ \ \delta\beta\|\gamma} = 0$

Contracting α with β and η with γ gives

$$R^{\alpha\eta}_{\ \ \alpha\eta\|\delta} + R^{\alpha\eta}_{\ \ \eta\delta\|\alpha} + R^{\alpha\eta}_{\ \ \delta\alpha\|\eta} = 0$$

By definition of the contracted Riemann tensor & by symmetry properties of the Riemann tensor, we find

$$R^{\eta}_{\eta\|\delta} - R^{\alpha}_{\delta\|\alpha} - R^{\eta}_{\delta\|\eta} = 0$$

Relabelling indices & rearranging terms, we obtain

$$R^{\eta}_{\eta\|\delta} = 2 R^{\beta}_{\delta\|\beta}$$

Now with $R^{\eta}_{\eta} = R$, we can write $\frac{1}{2} R_{\|\delta} = R^{\beta}_{\delta\|\beta}$

For with indices in contravariant position, $R^{\beta\delta}_{\ \ \|\beta} = \frac{1}{2}(g^{\beta\delta} R)_{\|\beta}$.

i.e. $G^{\beta\delta} = R^{\beta\delta} - \frac{1}{2} g^{\beta\delta} R$ has the property $G^{\beta\delta}_{\ \ \|\beta} = 0$.

Suppose $R^{\beta\delta} = 0$. Then $R = 0$ and $G^{\beta\delta} = \frac{1}{2} R^{\beta\delta} - \frac{1}{2} g^{\beta\delta} R = 0$.

Conversely if $G^{\beta\delta} = 0$, then $G^{\beta}_{\delta} = R^{\beta}_{\delta} - \frac{1}{2} g^{\beta}_{\delta} R = 0$

Contracting we see that $G^{\beta}_{\beta} = 0 = R^{\beta}_{\beta} - \frac{1}{2} g^{\beta}_{\beta} R = R - 2R$

$$\therefore R = 0$$

$$\therefore R^{\beta}_{\delta} = G^{\beta}_{\delta} + \frac{1}{2} g^{\beta}_{\delta} R = 0$$

$\therefore G^{\beta}_{\delta} = 0$ if and only if $R^{\beta}_{\delta} = 0$

24/3/69

Sci. Amer. ~~1960~~ July 1960 – p. 142 – Things that go faster than light by Milton A. Rothman

Example of a pair of scissors (cutting pt moving with $v > c$) – group velocity & phase vel (~~former~~ latter may be $> c$) – Einstein only said that messages & energy and mass cannot travel faster than c – Examples of plasma etc where $v > c$ appear

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Ibid, Sept 1960 – p. 171 – Studies of meteorites show that their elements are some 4.9 billion years old – Article on age of solar system by J. H. Reynolds

May, 1961, p. 148 – Mathematician as an explorer by Sherman K. Stein – Mentions about the Sanskrit word (1000 years old) used by Drummers viz

Good for a popular lecture

yumátárájabhánasalagám

(0111010001)

June, 1961 – p (?) – nice pictures of baboons, lion, zebras, elephants, Gnus etc in Kenyan forests

where $\frac{1}{2}\beta x^2 = \beta x^2$ and $\beta = m\omega/\hbar$

The square of the first expression may be written in simpler form;

$$|\varphi_n|^2 \simeq \frac{(2/\pi)}{\sqrt{a_n^2 - x^2}} \cos^2\left[\frac{1}{2}\beta x\sqrt{a_n^2 - x^2} + \frac{1}{2}\beta a_n^2 \sin\left(\frac{x}{a_n}\right) - \frac{1}{2}\pi n\right] ; \; x < a_n$$

where $a_n^2 = (2n+1)/\beta = 2E_n/m\omega^2 = 2E_n/k$ with $k = m\omega^2$. (The average value (of the classical $\cos^2$ term leaving $1/2$) of $|\varphi_n|^2$ for $n \gg 1$ is thus on the average equal to the classical P.

(4) When $\hbar\omega$ is not small, the difference between $|\varphi_n|^2$ & P is pronounced. $|\psi_n|^2$ in the quantum case, has a series of nodes a distance approximately $2\pi/\beta a_n$ apart near the origin for a state having just the energy E_n.

For large n, the average of $|\psi_n|^2$ over several wave lengths approaches P for $x < a_n$.

(Anything new in all this ?)

28/11/69 Read in some issue of New Scientist (England) a new theory of elementary particles by J. Schwinger based on dyons (instead of quarks) with +ve & -ve magnetic charges - Ref. to Science Vol. 165, p. 757 (not found in the Library)

1/12/69 - found the above and read it.

3/12/69 - Receiving some more articles in Science - Borrowed Pauli's & Wilson's book from Physics Lib

5/12/69 - Went to Physics Library & saw some books - also gave 2 reprints to Dr. K. Govindarajan who had asked for them on 3rd inst - Looked into Math. Zs. Vol. 39 (1935), pp. 634-68 and Vol. 43 (1938), p. 101 (both by Hahn)

- Also looked at J. Math & Phys. Vol. 12, p. 291. Giving a table of Hermite functions $\varphi_n(x)$ [referred to in the earlier pages] upto $\varphi_{11}(x)$ for values of $x = 0.00(0.04)8.00$

‖ This must be used to draw the $\varphi_n^2(x)$ graphs

10/12/69 Topics in recreational mathematics by J. H. Cadwell, C.U.P, 1966 also contains a Chapter on curves of constant breadth and rotors - Send reference to K.V.

29/12/69 Saw 2 new books in the Library

(i) An intro to field quantisation by Y. Takahashi* (Pergamon, 1969

(ii) Particles & fields by D. Lurie (Interscience, 1968)

(i) treats with Spin 3/2 & spin 2 fields but no references to me at all

(ii) " Spin 3/2 " (disgraceful)

* This is the fellow who K.S. Viswanathan said had great regard for me!

Notes from the Proc. Heidelberg. Int. Conf. on Elem. particles 1968 (a) Fund. Th. Qns by K. Hepp.

p. 1 – It is hard to be original at the present rate of international conferences. – In its own shortcomings it will be strictly original – Talk on fund. theoretical questions has much resemblance to a play of tennis with invisible, imaginary tennis balls [Ref. Blow up (Metro Goldwyn Mayer, 1960)

(b) p. 79 – Nucleon resonance & low energy scattering by C. Lovelace (Cern). – I shall discuss low energy scattering first, to let you recover from your lunches before facing the appalling spectacle of nucleon resonances. – Now, after today you may well wonder: if resonances continue to divide and multiply at their present rate, where are we going to end up? To finish, I shall ANSWER this question. It is not generally known that the CERN missing mass spectrometer was anticipated in 500 B.C., by an instrument which had even finer resolution. With it, a discovery was made so shattering that everything you have heard it pales in comparison (fig. 24) (?)

(c) Quarks & symmetries by H. J. Lipkin, p. 253 – Most interesting introduction ✓. (This ~~lecture~~ article might be carefully studied and represented at one of Alladis Conferences

29/10/69 = Eastern Theoretical Physics Conference Report (Univ. of Virginia), Oct 26-27, 1962 – After dinner speech by J. A. Wheeler, p. 441.

19/11/69 – Read all articles of 1969 Il Nuovo Cimento & found references to "Chiral group as group of trns of internal variables not conserving parity" – Also many other interesting things – Also cosmology.

– New Books Section – (i) old & new problems in elementary particle physics – Academic Press N.Y, 1968 – Also some articles of a general nature.

(ii) Also Enrico Fermi Series of special problems in elem. particle physics – Articles on current Algebra, Regge pole theory etc.

26/11/69 – Morse & Feshbach – Methods in Th. Phys. Part II., pp. 1640–44

(1) Classical $P = \begin{cases} \dfrac{1/\pi}{\sqrt{(2E/K) - x^2}}, & x^2 < 2E/K \\ 0, & x^2 > 2E/K. \end{cases}$

(2) $\displaystyle\int_{-\infty}^{\infty} x^2 \varphi_n^2(x) = \frac{n + 1/2}{\beta} = \frac{E_n}{K}$ $\left[\beta = \frac{M\omega}{\hbar},\ E_n = \hbar\omega\,(n + \tfrac{1}{2}\right.$

(3) Asymptotic value of ψ for n large as an application of the method of steepest descent and as a comparison with the classical limit, one has

(i) For $n \gg 1$,

$$\varphi_n \simeq \begin{cases} \dfrac{\beta^{1/4}\sqrt{2/\pi}}{(2n - z^2)^{1/4}} \cos\left[\tfrac{1}{2} z\sqrt{2n - z^2} - \tfrac{1}{2}\pi n + n \sin^{-1}(z/\sqrt{2n})\right]; & (z^2 < 2n \\ \sqrt{n!}\,(\beta/\pi)^{1/4} z^n e^{-\frac{1}{2}z^2} & (z^2 > 2n \end{cases}$$

pp. 145–146 – Complete factorisations of $2^{119}-1$, $2^{129}+1$, and $2^{141}+1$ are given

[but $2^{128}+1$ does not come under any of these heads]

pp. 166–73 – Review of a number of books on ALGOL (Compilers)

p. 517 – Review of a book on "Elementary inequalities" by D. S. Mitrinovic, P. Noordhoff Ltd, Groningen, Netherlands (Is this the fellow who asked for my reprints?) – Review says it is a good book for colleges.

p. 675 – An article of maximal gaps between successive primes.

Vol. 20 – Nos. 93–96 (1966).

p. 424 – Some integrals of Ramanujan and related contour integrals.

p. 634 – Review of a new book on "Calculus Vol I" by R. Courant & Fritz John – Laying emphasis on numerical methods.

Vol. 21. Nos 97–100 (1967).

p. 87 – Some factorisations of $2^n \pm 1$ and related results by Brillhart & Selfridge

p. 242 – Systematic computations on amicable numbers.

p. 245 – Primes of the form n^4+1 by M. Lal (working in Canada)

p. 258 – Review of M. Lal's "Expansion of $\sqrt{2}$ to 19600 D.

p. 705 – Lal's constant and generalisations by D. Shanks.

p. 731 – Review of Lal's "$\sqrt{3}$ to 19600 D"

p. 745 – Review of Jagjit Singh's book on "Great ideas in Information Theory, Language and Cybernetics" – Dover Publrs.

30/6/69 Vol 22 – Nos 101–104 – In the Library upstairs (just looked in, but looked into Loney & Whittaker & borrowed the former for Gauhati valn)

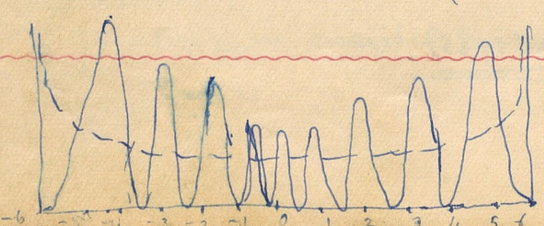
Found in M. R. Sept. 63. p. 511 – An article giving

$$|H_n(x)| \leq \frac{(2m+2)!}{(m+1)!}|x|e^{x^2/2}, \text{ if } n=2m+1$$

$$\leq 2\frac{(2m+1)!}{m!}e^{x^2/2}, \text{ if } n=2m$$

27/10/69 – Wilson & Pauling – Fig on p. 76, for $|\psi_{10}(\xi)|^2 \uparrow$

The probability distn function $|\psi_{10}(\xi)|^2$ for the state $n=10$ of the harmonic oscillator. Note how closely the function approximates in its average value the probability distn fn for the classical harmonic oscillator (with the same total energy) represented by the dashed curve.



(6) Sc. American - March 69.

p. 113 - In an advertisement by the Planning Research Corporation, at Los Angeles, Calif, is found a wood carving of S. Ramanujan by William Ransom and photographed by Max. Yavno and the following quotation from Hardy's book (C.U.P, 1940) on Ramanujan

" [Ramanujan] was not particularly interested in his own history or psychology; he was a mathematician anxious to get on with the job. And after all I too was a mathematician and a mathematician meeting Ramanujan had more interesting things to think about than historical research. It seemed ridiculous to worry him about how he had found this or that known theorem, when he was showing me half a dozen new ones almost every day"

Back cover - Advertisement by Union Carbide Co. about donation of blood - mention that the revolutionary advance in the new science of cryogenics (pioneered by Union Carbide Co) makes it possible for blood or its red cells can be stored for a century or more as fresh as the day it was taken from a donor (eg. of liquid nitrogen of -320° F).

p. 116 - Math. Games by Martin Gardner on " The multiple fascinations of the Fibonacci sequence"

(7) - Do - April 69

p. 15 - The dynamics of the arms race.

(8) - Do - Feb. 69.

p. 50 - The astrophysics of cosmic rays by V.L. Ginzburg.

p. 76 - Subsistence Herding in Uganda - Mention is made (with illustrations) as to how cattle blood is taken from an animal. This is done only once every two to three months. Blood is caught in a bowl held by a herd boy. From 4 to 8 pints will be taken and bleeding stops when the thong is loosened. A mixture of blood and milk is the standard food of the cattle camps, so that the herders literally live off their animals - Illustration of making butter for homestead use (just like what Sonya does) and then boiling it to make ghee; because they have no means of refrigeration, ghee is the only milk product they can preserve.

p. 110 - Math. Games - Boolean algebra, Venn diagrams and the propositional calculus.

(9) - Do - Jan. 69

p. 73 - The neurophysiology of remembering.

p. 130 - The dance of the solids - some verses by John Updike

p. 116 - Math. Games - Dr. Matrix gives his explanation of why Mr. Nixon was elected President.

27/6/69 - Maths of Computation Vol. 19, Nos: 89-92 (1965).

~~Pengard~~

Vol 19 - Nos. 89-92 (1965).

25/6/69 (1) The New Scientist 27/3/69, Vol. 47, No. 642 (Weekly) - upstairs in the Library

p. 669 - History in not bunk - Editorial about closing the history and philosophy of Science Deptt in the University College, London - Editorial criticises this step.

p. 675 - Article by Herbert Dingle on the above saying that such a closing would be "a cause of external reproach and internal shame" (For I.I.W.C. Lecture)

p. 672 - Living with Technology - Statement that "all of those in the audience carried DDT test residues in their tissues, and had been dusted with radioactive fall out from Weapon tests.

p. 679 - The moon's mysterious mascons

p. 682. Conservation of natural resources (animals and plant species, 384 & 124 species reps)

p. 705 - In a review a support for the present "Space race".

(2) The American Scientist - Winter, 1968

p. 303 - Science and the social imperatives. by F. C. Lindvall

p. 344 - on the effectiveness of research and development organisations by C. Argyris

|| p. 375 - Mathematics as a creative art by P. R. Halmos. - 3rd paragraph.

p. 414 - The computer game by J. D. Meng.

(3) The Advancement of Science, Vol. 25, No. 124, Dec. 1968

p. 129 - Computers and the future of Britain.

|| p. 135 - The modish Maths by D. B. Scott - About teaching of mathematics: gives illustration of proofs of $1 = 2$ (i) Let $x = 1$ $\therefore x = x^2$ $\therefore x - 1 = x^2 - 1 = (x-1)(x+1)$

$\therefore 1 = x + 1$ ie $1 = 2$ (ii) Suppose $1 = 2$,

Then $2 = 1$

Adding $= 3 = 3$ which is correct & so original is correct

p. 151 - Schools without class ~~rooms~~ rooms - wither primary education? by L. J. Burroughs.

p. 241 - Technological change in developing countries by Kenneth Little - Relates mostly to Africa, but interesting nevertheless.

(4) Science - Vol 162. A - 26/11/68 - Whole issue of 200 & odd pages devoted as a "Guide to scientific instruments" with index of firms and suppliers and names of types of instruments.

(5) - Do - 4/10/68

p. 73 - Computer-assisted Instruction by Atkinson & Wilson.

when $\lambda = 0$. for eg. $H_0^\lambda(\xi) = 1$, $H_1^\lambda(\xi) = 2\xi$

$$H_2^\lambda(\xi) = 4\xi^2 - 2(1+2\lambda)$$

$$H_3^\lambda(\xi) = 8\xi^3 - 4(3+2\lambda)\xi$$

$$H_4^\lambda(\xi) = 16\xi^4 - 16(3+2\lambda)\xi^2 + 4(1+2\lambda)(3+2\lambda), \text{ etc}$$

all of which → usual forms when $\lambda = 0$.

Method of computation of zeros of $H_K^\lambda(\xi)$: First evaluate zeros of $L_n^\alpha(x)$ which satisfies diff eqn

$$x L_n^{\alpha''}(x) + (\alpha+1-x) L_n^{\alpha'}(x) + n L_n^\alpha(x) = 0$$

use Hoffsommer's iteration method (MTAC. V. 12, 1958, pp. 58-60) to find zero. Using for a guess (to start from a x_{ni}^α) use Szegö's asymptotic formula for $L_n^\alpha(x)$ and obtain $x_{ni}^\alpha \cong \frac{\pi^2}{n}\left[(4i-1)/8 + \frac{\alpha}{4}\right]^2 + O(n^{-1/2})$. Next to find zeros of $H_K^\lambda(\xi)$ note that the

$$(\pm i)^{th} \text{ zero of } H_{2m}^\lambda(\xi) = \pm\left[i^{th} \text{ zero of } L_m^{\lambda-.5}(x)\right]^{1/2} \quad 1 \leq i \leq m$$

and $$(\pm i)^{th} \text{ zero of } H_{2m+1}^\lambda(\xi) = \pm\left[i^{th} \text{ zero of } L_m^{\lambda+.5}(x)\right]^{1/2} \quad 1 \leq i \leq m$$

$$= 0 \text{ for } i = 0.$$

Tables are given for Laguerre & Hermitian polynomials, and in particular for the latter with $\lambda = 0(1.0)4.0$, $K = 8, 16, 32, 64$ to 25 D.

We can verify that for $\lambda = 0$, $K = 8$ & 16, the largest zeros as found in the tables of Salzer et als (See earlier pages) agree to 15 D & 14 D of the latter

[All this to be taken up to prove $x_{0,n+1}'^2 - x_{0,n}'^2 < 2$ later on]

p. 640 – On maximal gaps between successive primes by D. Shanks – In a personal communication P. A. Carlson asks the author if he could give a rough "ball-park" estimate of where one could find a run of a million or more consecutive composite numbers.

Let $p(g)$ be the first prime that follows a gap of g or more consecutive primes. Thus $p(1) = 5$ (following 4), $p(2) = p(3) = 11$, $p(4) = p(5) = 29$, $p(6) = p(7) = 97$ etc. What is wanted here is an estimate of $p(10^6)$.

After lots of considerations, the author makes the conjecture (with considerable confidence) that $10^{300} < p(10^6) < 10^{400}$ ie almost surely $p(10^6)$ is greater than a googol (10^{100}). On the other hand the eight known Mersenne primes $M_{2203}, \ldots M_{11213}$ exceed 10^{600} & almost surely $p(10^6)$ is < any of these. In fact it is probable that somewhere below M_{11213} there is a gap of fifty million or more consecutive composites.

p. 664 – Paper on numerical soln of diff. eqns

p. 680 – Review of Courant-Hilbert's Methods of Math. Phys Vol II by Diaz with extracts from Courant's Preface to the book (very interesting)

factors. [I think the theorem used was that "if $n \neq 1$, then 2^n+1 is prime if and only if $3^{2^{n-1}} \equiv -1 \pmod{2^n+1}$". Using this theorem for the case $m = 10$ i.e. $(2^{1024}+1)$ the least +ve residue of $3^{2^{1023}} \pmod{2^{1024}+1}$ was found to be

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by Emma Lehmer to the base 16, using 0, 1, 2, 3, — 9, 4, v, w, x, y, z as digits & thus establishing $2^{2^{10}}$ is composite. Perhaps same method must have been used by in proving compositeness of for F_7]

p. 339 – Review of Proc. 7th Kharagpur Conference, 1961 (Seth as Ex. Editor) – only one article of computational interest is mentioned

pp. 340-43 – A huge review of ~~Hoffmann~~ Walter Hoffmann's book "Digital Information Processes" Wiley & Sons, 1962, $27.00 – Volume meant for those who are already ~~know~~ familiar with computers and for computing specialists (~~workers~~) – The review itself is very rewarding since it brings out the numerous technical terms (some very funny) and the uses of the computer techniques in various fields – Specially the last article by Hoffmann himself

~~p/pegs.~~ [Ps p. 509

23/6/69 – Reread review of book by Hoffmann.

p. 569 – Triangle formulas in the complex plane by Philip J. Davis – Simple understandable paper.

p. 594 – Tables of zeros and Gaussian weights of certain associated Laguerre polynomials and ~~assoc~~ the ~~later~~ related Hermite polynomials by Shao, Chen & Frank. –

Defines usual $L_n^{\alpha}(x)$ & $H_k(\xi)$ as in Szegö, taking

$$H_k(\xi) = k! \sum_{m=0}^{[k/2]} (-1)^m (2\xi)^{k-2m} / m!\,(k-2m)!$$

Mentions the relation between Hermite & Laguerre polynomials viz

$$H_{2m}(\xi) = (-1)^m 2^{2m} \cdot m!\, L_m^{-.5}(\xi^2)$$

$$H_{2m+1}(\xi) = (-1)^m \cdot 2^{2m+1} \cdot m!\, \xi L_m^{+.5}(\xi^2)$$

and obtains the generalised Hermitian polynomials $H_k^{\lambda}(\xi)$ as

$$H_{2m}^{\lambda}(\xi) = (-1)^m \cdot 2^{2m} \cdot m!\, L_m^{\lambda-.5}(\xi^2)$$

$$H_{2m+1}^{\lambda}(\xi) = (-1)^m \cdot 2^{2m+1} \cdot m!\, \xi L_m^{\lambda+.5}(\xi^2)$$

further obtains recursion relations, differential relations, integral relations and also the expns for the generalised Hermitian polynomials – Also mentions how the generalised Hermite polynomials enhance the systematisation of solutions for the case of the spatial harmonic oscillator in cartesian, cylindrical & sph. polar coordinates.

The above relations reduce to the usual relations for Hermitian polynomials

[Looked at the Chapter on Hermite polynomials in Rainville's book, but found nothing that would be of use in my work].

p. 146 – Fermat numbers and Mersenne numbers by Selfridge and Hurwitz – description of computer techniques, debugging aids and so on – Confirmation that $F_{14} = 2^{2^{14}}+1$ is composite. And that 2^p-1 is composite if $5000 < p < 6000$ – Complete testing of F_{17} would take 128 full weeks of machine time on the IBM 7090 & it seems more economical to search for smaller factors of F_{17} than to perform this test. – Also mentions how the machines can make errors.

p. 148 – Lucas' test showing that for $6000 < p < 7000$, M_p is prime

p. 151 – In some review of tables of primes, the reviewer criticises the author of his ignorance of the list of primes up to 21,000,000 and of punched tape lists (?) to 36,000,000 published in U.S.A. (V.K) & work of Lehmer.

p. 166 – In a review of Ore's book on Graphs & their uses (not available in the library) [The Collected Papers of Oystein Ore is available], the reviewer points out an interesting error viz the defn of the "Dodecahedron" as a polyhedron of twenty faces, and enquires "Is it typographical, or did he multiply when he should have added?" !

pp. 172–173 – Reviews of books on computing reveal how complicated the whole thing is.

p. 319 – Table of the first 100 zeroes of $J_0(x)$ accurate 19 significant figures

p. 324 – New factors of Fermat numbers. – Eleven new factors of F_m are listed in Table 1, A summary of the present status of the sequence F_m is presented in Table 2.

Table 2

m	Character of F_m
0, 1, 2, 3, 4	Prime
5, 6	composite and completely factored
10, 11, 12*, 19, 30, 38	Two or three* prime factors known
9, 15, 16, 18, 21, 23, 25, 26, 27, 32	
36, 39, 42, 52, 55, 58, 63, 73, 77	
81, 117, 125, 144, 150, 207, 226, 228	
258, 267, 268, 284, 316, 452, 1945	only one prime factor known
7, 8, 13, 14	Composite but no factor known
17, 20, 22, 24, 28, 29, 31, etc	Character unknown.

Saw in Math. Rev. Jan 61, p. 4, a reference to Brennan Math. Comput. 14 (1960), pp. 73-7 about complete factorisation of $2^{132}+1$

(a) Ref. to R. M. Robinson – Proc. Amer. Math. Soc. 1958, pp. 673-81

[Looked this up & found a reference to same author in same Journal, 1954, p. 842. But it was not clear how for $m = 7$, compositeness was found by the computer without finding the

[Remark in (a) p. 677 – Since for $m \geq 2$, every prime factor p of F_m satisfies $p \equiv 1 \pmod{2^{m+2}}$ we need to try the number N (prime) $= K \cdot 2^n + 1$ as a factor of F_m only for $m \leq n-2$]

The Fibonacci sequence is defined by:

$$F_1 = 1,\ F_2 = 1,\quad F_n = F_{n-2} + F_{n-1} \quad (n \geq 2)$$

Rollet has posed the problem: There are only three known Fibonacci numbers which are squares viz $F_1 = 1$, $F_2 = 1$, $F_{12} = 144$. Are there any others? It is shown by using computer technique that for $n \leq 1{,}000{,}000$ there are no other square F_n's except the three above.

p. 470 – Reference to some Pergamon Math. Table series by Berryland, Gavrilova and Prudnikov (Russian Academy Publications) translated by Prasenjit Basu quite many of the tables of $H_n^*(x)$ defined in terms of standard Hermite polynomials as below:

$$H_{2n}^*(x) = H_{2n}(x)/B_{2n} \text{ and } H_{2n-1}^*(x) = H_{2n-1}(x)/B_{2n} \text{ where}$$

$B_{2n} = (-1)^n (2n)!/n!$ so that $H_{2n}^*(0) = 1$

The table gives 6 S values of $H_n^*(x)$ for $n = 1(1)30$, $x = 0(0.01)10$.

The Referee points out some errors in B_{22} & says "It is regrettable that the accuracy of these extensive tables does not match with the very attractive appearance of the binding"! (Referee is J.W.W.?)

p. 479 – Very interesting review of a book by George S. Sebestyen on "Decision-Making Processes in Pattern Recognition" – Specially about pattern recognition in military uses.

Vol. 18. Nos 85–88 (1964).

p. 19. Article by A.K. Aziz & B.E. Hubbard supported by Naval Ordnance Laboratory through the Foundational Research program, and the Air Force Office of Scientific Research

[Some article too about the Goursat problem]

p. 87 – On factors of certain Mersenne numbers II by J. Brillhart

Completes the listing of the corresponding literature of all prime factors $q < 2^{35}$ for $103 \leq p \leq 257$, and $q < 2^{34}$ for $257 < p < 20{,}000$.

p. 93 – Three new Mersenne primes and a statistical theory by D.B. Gillies –

Description of finding the largest known primes M_{9689}, M_{9941} & M_{11213} (bringing to 23 the number of known primes viz for

($p = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281,$
$3217, 4253, 4423, 9689, 9941, 11213$)

$2^{11} - 1 = 2047 = 23 \times 89$ (20|6|69)

The large gap between 4423 & 9689 raises the question of what the distribution is of prime Mersenne primes & of factors of Mersenne composites. E.g. I.J. Good suggests that the number of prime $M_p < x$ is asymptotic to $2 \cdot 3 \log\log x$, and D. Shanks suggests approximately $\frac{5}{\log 10} \log\log x$. It is suggested here that $\frac{2}{\log 2} \log\log x$ is the correct asymptotic function. [Digital Computer Laboratory Univ. of Illinois, Urbana, Illinois]

[Ref. to Rainville's Special functions]

A table is given showing for the eight points the corresponding longitude & colatitude and also the rectangular coords X, Y, Z. It is found that the approximate volume of this polyhedron is 1.8157. Fig 1. shows the stick figure and also the convex hull of the ~~so~~ polyhedron.

It is known [Ref. to L. Fejes Tóth "Lagerungen in der Ebene auf der Kugel und im Raum, Springer-Verlag 1953 p. 127] (not in the Library) that a polyhedron of maximal volume among all polyhedra with n vertices ~~on the sphere~~ inscribed in a sphere must be bounded solely by triangular faces. By Euler's theorem, in any polyhedron with Δ^r faces and eight vertices, the total number of edges is $3(8-2)=18$, so the average number of edges at each vertex is $2 \cdot 18/8 = 4.5$. In the polyhedron tabulated above, the number of edges at each vertex is either four or five and is therefore as close as possible ~~as~~ to the average.

Referee's note :- The 8 points are located ~~as close as possible~~ close to the points of tangency of the eight faces of the largest known octahedron circumscribed about the sphere [Goldberg's article pp. 234, 236].

The dual of Goldberg's conjecture (suggested by Polya) is that the number of edges leading to any vertex is as nearly as possible to the average of such numbers.

p. 308 - A book of essays in honour of George Polya (Stanford U. Press, 1962, $10.00) by Gabor Szegö et al, called Studies in Mathematical Analysis and Related topics.

The volume consists of sixty new research papers by leading mathematicians published on 13/12/62 in honour of Prof. Polya's 75th birthday. There is included a list of Polya's 217 papers (up to 1961) and his six books. The book "How to solve it" has been translated into Arabic, Croatian, French, German, Hebrew, Hungarian, Japanese and Russian (so far). An imposing list of the names 60 mathematicians is given [it is disgraceful that not a single name of an Indian mathematician is found!]

[Saw the book in the Library with a fine portrait of Polya and read the short Preface]. Also looked at the Articles briefly and found on p. 392 an article by Szego on Turan's inequality to Jacobi polynomials (found no reference to work of myself & V.R.J) and another on p. 399 on an Eigenvalue problem occurring in quantum mechanics by E. C. Titchmarsh dealing with the H-problem.

p. 414 - A paper by R. Sankar and V. Malini of the National Aeronautical Laboratory, Bangalore, India on "A note on the relative merits of Padé and Machey's diagonal convergents in computing e^x [Authors ~~are~~ give thanks to Dr P. Nilakantan Director of the Laboratory, and to Referee for pointing out a flaw in the first draft]

p. 455 - On the non-existence of Fibonacci squares by M. Wunderlich

p. 478 – factors of Mersenne prime

p. 482 – Properties of hyperperfect and trimate numbers – References to Chowla's work

p. 507 – Review of a book on "Experimental music composition with an electronic computer" McGraw Hill, N.Y. 1959, $6.00 – Review refers to Computers & Composers

Vol. 17 – Nos. 81–84 (1963)

(18/6/69) p. 129 – A mention of Hermite polynomials in connection with approximate integration formulas. Gives an example of Hermite polynomials in 2 dimensions from which approximate integration formulas cannot be derived viz

$$H_{3,0} = 8x^3 - 12x, \quad H_{1,2} = 8xy^2 - 4x$$

$$H_{2,1} = 8x^2y - 4y, \quad H_{0,3} = 8y^3 - 12y.$$

Also ref. to Szegö as v. 23.

p. 170 – Calculation of Euler's constant γ to 3566 places by Dura W. Sweeney. Earlier one to 1271 D by Kmuth & a later one to by Uhler to ? D.

p. 183 – A fast method for computing number of primes less than a given limit ie $\pi(x)$. by David C. Mapes – Work done under auspices of the U.S. Atomic Energy Commission & use of an IBM 709 at the Lawrence Radiation Laboratory of the Univ. of California

p. 194 – on the congruences

(1) $(p-1)! \equiv -1 \pmod{p^2}$ – Wilson primes (for eg. 5, 13 and 563)

(2) $2^{p-1} \equiv 1 \pmod{p^2}$

References to earlier work – Author (Emma H. Pearson) tested primes $30,000 < p \leq 200,183$ in (1) and $10^5 < p \leq 200183$ in (2) & found no such primes

1093 1093 and 3511 are the only primes $< 50,000$ of this type (2) } up to limit
and the above 3 Wilson primes are the only known ones so far } tested by author.

p. 197 – Search for largest polyhedra by D. W. Grace.

Configuration of 8 points on the unit sphere which determines the convex polyhedron of maximum volume is not known. It is not the cube whose volume is $8\sqrt{3}/9$ or about 1.5396, since the volume of the double pyramid (one point at each of the poles and the other six distributed uniformly around the equator) is $\sqrt{3}$ or about 1.732 [Reference to Polya: "Mathematics as a subject for learning plausible reasoning" The Mathematics Teacher v. 52, 1959 (not available in library)] : the

The search in the present paper was carried out using gradient methods on the Burroughs 220 computer (?) and by use of Goldberg's conjecture (?) [Ref. to Goldberg "The isoperimetric problem for polyhedra" Johnson Math. Journal v. 40, April 35, pp. 226-36 (not in the library)] a polygon larger than the double pyramid was found.

Refers to von Neumann's interest in studying the frequency distribution of the various digits in expansions of e and π.

p. 249 – New Mersenne primes – If p is prime $M_p = 2^p - 1$ is called a Mersenne number. The primes M_{4253} and M_{4423} are the two largest primes known (using Lucas-Lehmer test for the IBM 7090)

p. 275 – Euler's constant to 1271 decimal places

i.e. $\gamma = \lim_{n \to \infty} (1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n)$

Previously known value was up to 328D. The evaluation of γ to many places is considerably more difficult than that of e or π – Done here by using the Burroughs 220 computers – Total time of computation was approximately 8 hours.

p. 282 – Calculation of lattice points in circle and sphere

If $A_K(x)$ be the number of points $(y_1, y_2, \cdots y_K)$ satisfying

$$y_1^2 + y_2^2 + \cdots + y_K^2 \leq x$$

where $y_1, y_2, \cdots y_K$ are integers ($+^{ve}$, $-^{ve}$ or zero), then $A_K(x)$ is the number of lattice points in a K-dimensional hypersphere of radius $x^{1/2}$. This paper describes the calculation of a table of $A_2(x)$ and $A_3(x)$ on an IBM 650 Computer.

p. 407 – Machine proof of a theorem on cubic residues by D.H. Lehmer et als.

If p is a prime of the form $6m+1$, the numbers

$$1^3, 2^3, \cdots (p-1)^3$$

when reduced modulo p consist of only $(p-1)/3 = 2m$ distinct numbers between 1 and $p-1$. These $2m$ numbers are known as the cubic residues of p. Thus the cubic residues of 13 are 1, 5, 8 and 12, while those of 97, when arranged monotonely begin

1, 8, 12, 18, 19, 20, 22, 27, 28, 30, · ·

Here we observe the triplet (18, 19, 20) consisting of three consecutive numbers among the cubic residues of 97, while no such phenomenon exists for $p = 13$. We will call any set of three consecutive $+^{ve}$ integers a triplet. A prime $p = 6m+1$ is called exceptional if does not have a triplet of cubic residues. Thus 13 is exceptional, while 97 is not. It has been known since 1928 [Brauer] that all "sufficiently large" primes have a triplet of cubic residues. Thus there are only a finite number of exceptional primes. By using machine methods, we have proved much more viz

(a) The only exceptional primes are

2, 3, 7, 13, 19, 31, 37, 43, 61, 67, 79, 127, 283

(b) Every non-exceptional prime has a triplet of cubic residues that does not exceed (23532, 23533, 23534) --- (1)

(c) There are ∞-ly many primes whose smallest triplet of cubic residues is (1). Hence result (b) is the best.

It is stated that all the results (a), (b), (c) are due to the machine !

No. 76

~~Vol~~ 76 : p. 420 – The compositeness of the 13th Fermat number ($2^{2^n}+1$ where $n = 13$) shown by computer IBM 7090 to be prime, computation taking 6 hrs & 17 mins of machine time

16/6/69

Vol. 16 (1962). ~~p. 76~~ – No. 77–80.

✓ || ~~Vol 77~~, p. 76 – Calculation of π to 100,000 decimals by Shanks and Wrench

Formula used was $\pi = 24 \tan^{-1} 1/8 + 8 \tan^{-1} 1/57 + 4 \tan^{-1} 1/239$ . . (1)

due to Störmer. Earlier computations used Machin's formula

$$\pi = 16 \tan^{-1} 1/5 - 4 \tan^{-1} 1/239$$

Computation using (1) to more than 100,000 D required 8 hours 43 mins on an IBM 7090 system. The computation time breaks down as follows:–

$8 \tan^{-1} 1/57$: 3 hrs 7 mins

$4 \tan^{-1} 1/239$: 2 hrs 20 mins

$24 \tan^{-1} 1/8$: 2 hrs 34 mins

Conversion (binary to decimal) : 42 mins

Check using Gauss's formula similar to (1) & discussion of time taken for checking

A million decimals? The program would ~~take~~ require time of the order of months. But since the memory of 7090 is too small, by a factor of ten, a modified program, which writes and reads partial results, would take longer still. One would really want a computer 100 times as fast, 100 times as reliable, and with a memory ten times as large. No such machine now exists.

Entirely different procedures are possible. For eg. we can compute $1/\pi$ and then takes its reciprocal. This sounds fantastic, but in fact, it can be faster than the use of eqn (1). One can compute $1/\pi$ by Ramanujan's formula [Coll. papers, Cambridge, pp. 23–39]

$$\frac{1}{\pi} = \frac{1}{4}\left(\frac{1123}{882} - \frac{22583}{888^2}\cdot\frac{1\cdot 3}{2\cdot 4^2} + \frac{44043}{882^5}\cdot\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2} - \cdots\right) \qquad (2)$$

Using (6) we can actually reduce the time required by (2) by an hour. But unfortunately we lose our overlapping check, and ~~the~~ in any case, this small gain is inadequate for the present question.

One could hope for a theoretical approach to this question of optimisation – a theory of the "depth" of numbers – but no such theory now exists. One can guess that e is not as "deep" as π, but try to prove it! (We have computed e on a 7000 to 100,265 D by the obvious program & this took 2.5 hours instead of the 8-hour run for π by (1).

p. 188 – A statistical study of randomness among the first 10,000 digits of π by R. K. Pathria of Deptt. of Physics, Univ. of Delhi, Delhi-6 – Refers to the above papers & several others

J. B. Russell in J. Math. & Phys. Vol (12), 1933, pp. 291-97 — Looked at this (in the Lib) & found the table might be of use in drawing figures up to $\phi_{10}(x)$ — 35 errata are shown on p. 71

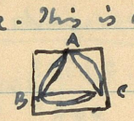
p. 226 — Review of von Neumann's book — The computer and the brain — Yale Univ. Press New Haven, Connecticut, Price $3.00. — Last contribution of Neumann — A review very pertinent to cases of Nalini & Shakuntala Devi. Neumann states (using notion of "neurons" in the brain that (i) the language used by the brain is probably statistical in nature in which correlation processes play a very important role, and (ii) Logics and mathematics in the central nervous system, when received as languages, must structurally be different essentially from those languages to which our common experience refers. (Consider both digital and analog computers)

Nos. 69-72 (1960)

p. 1. Article on mathematics of computation by H.P (?) — A beautiful summary article, specially para. 3, which describes the several applications of computer technique.

13/6/69 p. 199 — Numerical evaluation of the 18th perfect number by using the IBM 709 at the National aviation facilities Exp. Centre in Atlantic City. This computed the largest known perfect number [Every number of the form $2^{n-1}(2^n-1)$ is perfect if 2^n-1 is prime ie a Mersenne number] viz. $2^{3216}(2^{3217}-1)$. Running time was approximately 5 mins. and the number has 6433 proper divisors [$2^{3217}-1$ is the 18th Mersenne prime — Ref. MTAC. V. 12, 1958, p. 60.

p. 229. Rotors in polygons and polyhedra — Most interesting article by Michael Goldberg [For eg. a rotor in a square is a curve which may rotate while remain tangent to all sides of the square. This is a generalisation of the earlier well known curves of constant breadth.



[arcs having centres at A, B, C at vertices of the eqn. ...]

p. 262 — The permanent of an n matrix $A = [a_{ij}]$ with elements in a commutative ring is defined as $\text{per}\, A = \sum a_{1i_1} a_{2i_2} \cdots a_{ni_n}$, where the sum extends over all $n!$ permutations of $1, 2, \ldots n$

p. 302 — A scathing review by J. C. P. Miller of some tables published ~~by J. C. P. Miller~~.

p. x — Ref. to Cramer-Rao inequality

Ibid V. 15-16 (1961-62).

Nos. 73 & 74 — nil

No. 75 (July 61)

p. 250 — A computational approach to the 4-colour problem.

p. 311 — Mention of Airy integral, giving tables of solutions of the differential eqn

$y'' = xy$

$\beta_i^{(n)} = \alpha_i^{(n)} \exp\left[(x_i^{(n)})^2\right]$ ~~and this is Chris~~ Christoffel number corresponding to $\alpha_i^{(n)}$. Both $\alpha_i^{(n)}$ & $\beta_i^{(n)}$ are tabulated for $n = 1$ to 20.

(6/6/69) Ibid - Vols 9-11 (1955-57).

Vol 9. Nos. 49-52. p. 139 - Polynomial root solving on the electronic differential analyser by C. Atkinson, Univ. of Cal, Berkeley, Calif. - Gives Laplace's method giving general wiring diagram and so on

Gives a reference to F. W. J. Olver on "Evaluation of high degree polynomials" in Phil. Trans. Roy. Soc. Vol 244, 1952, pp. 385-415 - Looked at this paper describing in particular the "root-squaring process" and found that it would not be of much use.

p. 164 - Orthogonal polynomials arising in numerical solving inverse Laplace transforms by H. E. Salzer. [Number of references to Szegö] - no references to Hermite polynomials

p. 224 - Ref. to death of R. C. Archibald on 26/7/55.

Nos. 53-56 - Nothing in particular.

Nos. 57-60 (1957)

p. 53 - obituary notice of Sir Edmund Whittaker.

p. 127 - " of John von Neumann.

(9/6/69) - Same nos. & looking at Burnside and Panton Vol 1.

(11/6/69) Ibid Vol. 12-14 (1958-60) (Nos. 61-72).

Nos. 61-64 (1958).

p. 60 - Reference to the establishment of $2^{3217} - 1$ to be a Mersenne prime using the Swedish electronic digital computer BESK. Method used was a Lucas test and running time was $5^h 30^m$. The new prime has 969 decimal digits. Actual one found by BESK given [For $2^n - 1$ to be prime, n must be a prime but the converse not true. (3217 is prime?)]

p. 247 - Mechanical resolution of linguistic problems. Acad. Press. Inc. New York, 1958 by A. D. Booth etc - Q^n of translation from one language into another by using modern electronic computers.

Nos. 65-68 (1959)

p. 65 - Mention of a theorem of A. A. Markoff (one of the two brothers Markoff) viz

If $\Pi_n(x)$ be a polynomial of degree at most n such that

$$|\Pi_n(x)| \leq 1,$$

then we have $|\Pi_n'(x)| \leq n^2$, and the result is the best possible one

p. 71 - Table of errata by (Sexton)^3 on ~~Hubbs~~ a table of Hermite functions by

two zeros and about 10^{-3} for the last two.

Quite a complicated procedure. Anyway check (*).

[p. 30 – Ref. to a paper by V. R. T & B. S. Ramakrishna about about a 2D table ~~for~~ re. Bessel fns derived from heat flow, paper in Quarter App. Math. 10, 1952, pp. 255–262]

Vol. 8. Nos. 45–48 (1954).

p. 90 – Table of the zeros and ~~weg~~ weight factors of the first 20 Hermitian polynomials by Salzer et als. – Jn. Research NBS, 48, 1952, pp. 111–16.

Jn. Research NBS. ~~D~~, 48, 1952, pp. 111–16 (available in the library). The $|x_i^{(n)}|$ are the zeros in the 1st column. For the largest zeros, the table gives

$n=2$,	0.70710	67811	86548.	(ie 15D)
$n=3$,	1.22474	48713	91589	
$n=4$,	1.65068	01238	85785	
$n=5$,	2.02018	28704	56086	
$n=6$,	2.35060	49738	74492	
$n=7$,	2.65196	13568	35233	
$n=8$,	2.93063	74202	57244	
$n=9$,	3.19099	32017	81528	
$n=10$,	3.43615	91188	37738	
$n=11$,	3.66847	08465	59583.	
$n=12$,	3.88972	48978	69782	
$n=13$,	4.10133	75961	78640	
$n=14$,	4.30444	85704	7363.	(14D)
$n=15$,	4.49999	07073	0939	
$n=16$,	4.68873	89393	0582	
$n=17$,	4.87134	51936	744	(13D)
$n=18$,	5.04836	40088	745	
$n=19$,	5.22027	16905	375	
$n=20$,	5.38748	08900	112.	

Mentions as ~~an~~ most important consequence of $\int_{-\infty}^{\infty} e^{-x^2} H_m H_n \, dx = \pi^{\frac{1}{2}} 2^n n! \, \delta_n^m$ --- (10)

is $\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx = \sum_{i=1}^{n} \alpha_i^{(n)} f(x_i^{(n)}) + R_n$ --- (1)

where $\alpha_i^{(n)} = \dfrac{1}{H_n'(x_i^{(n)})} \displaystyle\int_{-\infty}^{\infty} \frac{e^{-x^2} H_n(x)}{x - x_i^{(n)}}\, dx$ --- (11)

The $\alpha_i^{(n)}$ are also expressible in the following form

$$\alpha_i^{(n)} = \frac{\pi^{1/2} \cdot 2^{n+1} \cdot n!}{n \{H_{n-1}(x_i^{(n)})\}^2} \quad (13) \longrightarrow \text{Ref. Szego, p. 344.}$$

more useful for computation.

the factor 226663 was discovered by the SWAC in 25 minutes of running time

D.H.L]

Nos. 37-40 (1952), p. 232 – Harvard ~~Laboratory~~ University, Computation Laboratory, Annals, 1952.

The fns tabulated are $\phi^{(n)}(x) = \frac{d^n}{dx^n}\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right)$, $n = 0, 1, \ldots$ up to 20. Guth has a chapter indicates the connection of these to Hermite polynomials. A 10D table of all the zeros of the all the functions tabulated in the volume is also given.

[p. 152 – Ref. to ménage ménage numbers ie the general number $U_{n,r}$ being the number of arrangements of n married couples seated at a circular table, gentlemen alternating with ladies in which precisely r husbands are seated the next to their wives. for eg. $U_{5,2} = 40$ – D.H.L]

[p. 256 – Defn of a large prime – A. Ferrier has established the primality of $(2^{148}+1)/17$ probably the "largest" prime to be identified by hand computing methods. (Primes like $2^{1279}-1$ would require more than a century of desk calculator work to establish by any known method]

p. 61. – D.H.L. remarks (i) ~~the~~ Ref. to Ferriers work alone (ii) using the EDSAC primality of $1+K(2^{127}-1)$ for $K = 114, 124, 388, 408, 498, 696, 738, 744, 780, 934, 978$ has been established (iii) Smith that of $1+180(2^{127}-1)^2$, a number of 79 decimal digits, (iv) using the Lucas test for Mersenne primes, the SWAC has discovered the primes $2^{521}-1$ and $2^{607}-1$ on 30/1/52. These lead to the 13th & 14th perfect numbers]

Nos. 41-44 (1953) (Vol. 7)

p. 260 – Asymptotic formulas for the Hermite polynomials by H. E. Salzer.

Use of a method based on integral eqns & using an ~~an~~ iterative process Refers to Szego, pp. 212-213 stating that Szegos gives only proof of existence of asymptotic expansion & mentions existence of first term only. Extension is here made beyond the first term & formulae obtained were used to calculate $H_{19}(x)$ and $H_{20}(x)$ for $x = 1(1)5$.

These formulae ~~to~~ were also used to calculate the earlier zeros of $H_n(x)$ using $A(x)$ & $B(x)$ [quite complicated expns] given below, we have

$$0 = A(x) + N^{-1/2} \tan\left(N^{\frac{1}{2}}x - \tfrac{1}{2}n\pi\right) B(x) \qquad [N = 2n+1$$

using iterative formula, $x_{i+1} = N^{-\frac{1}{2}}\left\{\left(\frac{1}{2}n+m\right)\pi - \arctan\left\{N^{\frac{1}{2}} A(x_i)/B(x_i)\right\}\right.$ for the rth zero, derives the initial approximation

$$x_0 = \begin{cases} N^{-1/2}(r-\frac{1}{2})\pi & n \text{ even} \\ N^{-1/2}(r-1)\pi & n \text{ odd} \end{cases} \quad (*)$$

Iterative Process applied to H_{19} & H_{20} for the gave a relative error of about 10^{-9} for the first

30/5/69 – Merely did some manipulations with $\psi_n(x)$ & $\psi_n'(x)$ & looked at several tables of functions.

2/6/69 – Ref. Math. Zeit. Vol. 43 (1938), p. 101

This article refers to $\varphi^r(x)$ being an orthogonal set only in 3 cases & mentions that the case of $r=1$ has been established in Math. Zs. Vol. 39, p. 634.

pp. 634-38 Math. Zs. Vol 39, p. 634 (1935) ⊛⊛ – Calls $\frac{1}{n} p_n'(x)$ as normalised derivatives – Nothing to do with zeros. (several nos from Vol I onwards)

while looking at above, saw the Journal of Math. Tables and other aids to computation and saw lots of references to Hermite polynomials computations.

Vol 3-5, 1948-51, pp. 167-169, formulae expressing $x^0, x^1, x^2, x^3, \ldots x^{30}$ in terms of the $H_n(x)$ are given, for eg.

$$32x^5 = H_5 + 20H_3 + 60H_1$$

$$64x^6 = H_6 + 30H_4 + 180H_2 + 120H_0, \text{ etc}$$

Ibid. p. 416 – Paper by Greenwood and Miller on zeroes of H-polynomials in Bull. Amer. Math. Soc. V. 54, Aug. 48, pp. 765-769.*

$H_n(x)$ defined by $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$

$$= (2x)^n - \frac{n(n-1)(2x)^{n-2}}{1!} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} \ldots$$

If $h_n(x) = e^{\frac{1}{2}x^2} d^n(e^{-\frac{1}{2}x^2})/dx^n$

$$H_n(x) = -(2^{\frac{1}{2}})^n h_n(2^{\frac{1}{2}}x)$$

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx \simeq \sum_{i=1}^{n} \lambda_{i,n} f(x_{i,n}) \quad \text{---} (1)$$

where the set $\{x_{i,n}\}$ is the set of roots of $H_n(x)=0$ & where the set $\{\lambda_{i,n}\}$ is given by $\lambda_{i,n} = \pi^{\frac{1}{2}} 2^{n+1} n! / [H_n'(x_{i,n})]^2$ [Ref. Szego. ~~Chap XV~~ p. 344]

If $f(x)$ is a polynomial of degree $(2n-1)$ or less (1) is exact. [Szego. Chap XV]

Tables for some $\{x_{i,n}\}$ and $\{\lambda_{i,n}\}$ are given in this article.

[* not available in the Library; I could have verified with some more $x_{i,n}$ since the tabulation is here is indicated as $n = 1(1)10$, $i = 1(1)5$?]

4/6/69 Nos. 33-36, 1951, p. 82

For 108514808571661 read 226663.478749547 correction in Kraitchik's article in Scripta Math. V. 14, 1948, p 24-26.

[The fact that this large "prime" factor of $18!-1$ is composite was proved by A. Ferrier who showed that (for $N = 108\ldots661$)

$$2^{N-1} \equiv 51070566362894 \pmod{N}$$

The problem of factoring N was proposed during a demonstration of the SWAC on 29/1/51.

⊛⊛ Gives the formula $H_n''(x) - xH_n'(x) + nH_n(x = 0$

while the usual one is $H_n''(x) - 2xH_n'(x) + 2nH_n(x)$

Christoffel's formula

$$\sum_{k=0}^{n} \frac{H_k(x)\,H_k(y)}{2^k k!} = \frac{H_{n+1}(x)\,H_n(y) - H_{n+1}(y)\,H_n(x)}{2^{n+1}\cdot n!\,(y-x)}.$$

Runge's formula.

$$2^{n/2} H_n\left(\frac{x+y}{\sqrt{2}}\right) = \sum_{r=0}^{n} \binom{n}{r} H_r(x)\,H_{n-r}(y).$$

Laguerre's theorem. If $F(x)$ is a polynomial of degree n with real coefficients, and (p. 312) with all its roots real and distinct, then if x_0 is a root of $F(x)$ we have the inequality

$$3(n-2)F''^2(x_0) - 4(n-1)F'(x_0)F'''(x_0) \geq 0$$

[Proof by Szegö given on p. 116].

12/5/69

Burnside & Panton. Vol I (i) If 1st negative term of $x^n + b_1 x^{n-1} + \dots + b_n = 0$ be $-b_r x^{n-r}$ and greatest $-$ve coeff be $-b_k$, then $\sqrt[r]{b_k} + 1$ is a superior limit of the $+$ve roots

(ii) If each $-$ve coeff be taken positively and divided by the sum of all the positive coefficients which precede it, the greatest quotient thus obtained increased by unity is a superior limit of the $+$ve roots.

Ex: $x^4 - 5x^3 + 40x^2 - 8x + 23 = 0$

(i) gives 9 as superior limit 8(ii) gives 6 as superior limit

(iii) Any number which renders $+$ve $f(x)$ & all its derivatives $f_1(x) \dots f_n(x)$ is a superior limit of the $+$ve roots of $f(x) = 0$ (Newton's theorem)

(iv) Limiting eqns: If all the real roots of $f'(x) = 0$ could be found, it would be possible to determine the no. of real roots of $f(x) = 0$

(v) Horner's method of finding $f(x)$ when $x = a$.

14 to 17th – working in particular

19/5/69 – Entering Calcutta marks lists!

21/5/69 – found that $x'^2_{0,10}$ lies between 16.6 and 16.7 thus again verifying $x'^2_{0,n+1} - x'^2_{0,n} < 2$.

23/5/69 – Handbuch der Physik. Bd III – Math. Hilfsmittel p. 336.

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - \dots$$

(stop the last term being of the 1st order or 0th order in x.

26/5/69 – Simply looked at Møller, Synge, Conference on Relativity & the articles in J. Math & Phys. & found something similar in Eddington's book.

The R.H.S. is a Fourier series in θ. The L.H.S is square-integrable over $0 \le \theta \le 2\pi$. Hence applying Parseval's formula

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|^2 d\theta = P_0^2 + \cdots + \frac{P_n^2(x)}{(n!)^2} + \cdots$$

Now $f(\theta) = e^{-\cos 2\theta + 2x\cos\theta}$ & the exponent $-\cos 2\theta + 2x\cos\theta$ is less than $1 + 2|x|$. Thus the exponential itself is less than $e^{1+2|x|}$ which is independent of θ & its mean value in $[0, 2\pi)$ is therefore less than that value. Each term of this series on the right, being less than the sum, has to be less than $e^{1+2|x|}$. Finally

$$\frac{|P_n(x)|}{n!} < e^{(1/2)+|x|}$$

which is a strict inequality.

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p. 350 - If Q_m be a polynomial of degree $m < n$, we can write

$$Q(x) = K_m P_m(x) + K_{m-1} P_{m-1}(x) + \cdots + K_1 P_1(x) + K_0 P_0.$$

Orthogonal functions by Sansone

pp. 303 onwards:

If $H_n(x) = \sum_{k=0}^{n} a_k x^{n-k}$, we have

$$a_k = -(n-k+2)(n-k+1)\, a_{k-2}/2k$$

with $a_0 = (-1)^n 2^n$ we get

$$a_2 = \frac{n(n-1)}{1!}(-1)^{n-1} 2^{n-2}$$

$$a_4 = \frac{n(n-1)(n-2)(n-3)}{2!}(-1)^{n-2} 2^{n-4}$$

$$a_6 = \frac{n(n-1)\cdots(n-5)}{3!}(-1)^{n-3} 2^{n-6}$$

$$\cdots\cdots\cdots\cdots$$

$$a_{2k} = \frac{n(n-1)\cdots(n-2k+1)}{k!}(-1)^{n-k} 2^{n-2k}$$

Hence

$$H_n(x) = (-1)^n \sum_{k=0}^{[n/2]} (-1)^k \frac{n(n-1)(n-2)\cdots(n-2k+1)}{k!}(2x)^{n-2k}$$

or

$$H_n(x) = (-1)^n \sum_{k=0}^{[n/2]} (-1)^k \frac{\Gamma(n+1)}{\Gamma(k+1)\,\Gamma(n-2k+1)}(2x)^{n-2k}$$

where $[n/2]$ is the largest integer $\le n/2$

and $H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$, $H_{2n+1}(0) = 0$

$H'_{2n}(0) = 0$, $H'_{2n+1}(0) = 2(-1)^{n+1}\frac{2n!}{n!}(2n+1)$

6/5/69 - Pauling & Wilson

p. 76. Fig. showing probability density function $\{\psi_{10}(x)\}^2$ for state $n=10$ cf. with the classical prob. distn curve.

$\xi = \sqrt{\alpha}\, x$

$H_6(x) = 64\xi^6 - 480x^4 + 720x^2 - 120$

$H_7(x) = 128x^7 - 1344x^5 + 3360x^3 - 1680x$

$H_8(x) = 256x^8 - 3584x^6 + 13440x^4 - 13440x^2 + 1680$

$H_9(x) = 512x^9 - 9216x^7 + 48384x^5 - 80640x^3 + 30240x$

$H_{10}(x) = 1024x^{10} - 23040x^8 + 161280x^6 - 403200x^4 + 302400x^2 - 30240.$

Rev. Mod. Phys: Vol 1, 1929, p 183 - Article by E. C. Kemble gives

$H_n(x) \doteq (2x)^n - \frac{n(n-1)}{1!}(2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!}(2x)^{n-4} - \ldots$

7/5/69 - References from Szegö.

(1) Zeros of H-polynomials - Math. Annalen 107 (1932), pp. 87-89 by Brauer

(2) Erdelyi, - Math. Zeis 44 (1938), pp. 201-11

(3) M. Fekete - " 17 (1923), pp. 228-249

(4) Annals of Math (2), 37 (1936), pp. 483-517

(5) Math. Zs. ~~Vol 39 (1936)~~ Vol. 43 (1937), p. 101 ✓

(6) Bull. Amer. Math. Soc. 42 (1936), pp. 423-48

(7) Math. Annalen 105 (1931), pp. 408-436

Smirnov's Analysis Vol. 3.

$H_{2n+1}(0) = 0, \quad H_{2n}(0) = (-1)^n \cdot 2^n \cdot 1 \cdot 3 \cdot 5 \cdot \ldots (2n-1).$

Exercises in Mathematics by G. Bass.

pp. 353 onwards: $\exp(-t^2 + 2tx) = \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n(x)$ $\quad [P_n = H_n$

$\exp\left[-(t-x)^2\right] = \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n(x) \exp(-x^2)$

$(-1)^n \cdot 2t \sum \frac{t^n}{n!} P_n(x) = \sum \frac{t^n}{n!} P_n'(x)$

$P_n' = 2n P_{n-1}$ equating coeffts of $t^n/n!$

$P_n = \frac{P_{n+1}'}{2(n+1)}; \quad P_{n-1} = \frac{P_{n+1}''}{4n(n+1)}.$

Choose for t a complex number whose absolute value is 1, $t = e^{i\theta}$, then

$\exp(-e^{2i\theta} + 2xi\theta) = \sum_{n=0}^{\infty} \frac{e^{ni\theta}}{n!} P_n(x)$

For the least +ve zero x_{1n}, we obtain ($x_{12} = 2^{-1/2}$; $x_{13} = (3/2)^{1/2}$)

$$\left.\begin{matrix} \dfrac{\pi/2}{(2n+1)^{1/2}} \\ \dfrac{\pi}{(2n+1)^{1/2}} \end{matrix}\right\} < x_{1n} \leqq \begin{cases} \left(\dfrac{5/2}{(2n+1)}\right)^{1/2}, \\ \left(\dfrac{21/2}{2n+1}\right)^{1/2} \end{cases} \qquad n \geqq 2.$$

From the last of these, we can derive bounds for the minimum distance d_n between consecutive zeros. The convex property mentioned above easily furnishes

$d_n = x_{1n} - x_{0n}$, that is $d_n = 2x_{1n}$ if n is even, and $d_n = x_{1n}$ if n is odd. It follows that

$$\frac{\pi}{(2n+1)^{1/2}} < d_n \leqq \begin{cases} \dfrac{10^{1/2}}{(2n+1)^{1/2}} \\ \left(\dfrac{21/2}{2n+1}\right)^{1/2} \end{cases} \qquad n \geqq 2$$

In every case we have

$$\frac{\pi}{(2n+1)^{1/2}} < d_n \leqq \frac{(21/2)^{1/2}}{(2n+1)^{1/2}}$$

Extensive literature on the subject (none available in the library)

Hille obtains better upper bounds for x_{1n} as

$$x_{1n} \begin{cases} \dfrac{\pi/2}{(2n+1)^{1/2}} \left\{ \dfrac{1}{2} + \dfrac{1}{2}\left[1 - \left(\dfrac{\pi}{2n+1}\right)^2\right]^{1/2} \right\}^{-1/2} \\ \dfrac{\pi}{(2n+1)^{1/2}} \left\{ \dfrac{1}{2} + \dfrac{1}{2}\left[1 - \left(\dfrac{2\pi}{2n+1}\right)^2\right]^{1/2} \right\}^{-1/2}. \end{cases}$$

(11) The largest zeros of Hermite polynomials.

Let $x_1 > x_2 > x_3 > \cdots$

be the enumeration of the zeros $x_\nu = x_{\nu n}$ of $H_n(x)$ in decreasing order. Then for $\{x_\nu\}$

$$x_\nu < (2n+1)^{1/2} - 6^{-1/2}(2n+1)^{-1/6} i_\nu$$

[$i_1 < i_2 < i_3 < \cdots$ are the real zeros of Airy's function $A(x) \simeq 2^{-1} 3^{-\frac{1}{4}} \pi^{\frac{1}{2}} |x|^{1/4} \exp\left\{-2\left(|x|/3\right)^{3/2}\right\}$ $x \to -\infty$

Further, we have for a fixed ν

$$x_\nu^{1/2} = (4n + 2\alpha + 2)^{1/2} - 6^{-1/2}(4n + 2\alpha + 2)^{-1/6}\{i_\nu + \epsilon_n\}$$

$$x_\nu = (2n+1)^{1/2} - 6^{-1/3}(2n+1)^{-1/6}\{i_\nu + \epsilon_n\}, \text{ where } \lim \epsilon_n \to 0.$$

We notice that $(2n+1)^{1/2} - 6^{-1/3}(2n+1)^{-1/6} i_1$ is an upper bound for the zeros of $H_n(x)$

[Here the const $6^{-1/3} i_1 = 1.85575\ldots$]

Alternative forms are $x_\nu^{1/2} = (4n)^{1/2} - 6^{-1/2} i_\nu (4n)^{-1/6} + O(n^{-1/6})$

& $x_\nu = (2n)^{1/2} - 6^{1/2} i_\nu (2n)^{-1/6} + O(n^{-1/6})$

(12) p. 136 – Electrostatic repns of the zeros of the classical polynomials.

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Retd.Lok.Tilak Professor of Applied
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For the least +ve zero x_{1n}, we obtain $\left(x_{12} = 2^{-1/2};\ x_{13} = (3/2)^{1/2}\right)$

$$\left.\begin{matrix} \dfrac{\pi/2}{(2n+1)^{1/2}} \\ \dfrac{\pi}{(2n+1)^{1/2}} \end{matrix}\right\} < x_{1n} \leqq \left\{\begin{matrix} \left(\dfrac{5/2}{(2n+1)}\right)^{1/2}, \\ \left(\dfrac{21/2}{2n+1}\right)^{1/2} \end{matrix}\right. \qquad n \geqq 2.$$

From the last of these, we can derive bounds for the minimum distance d_n between consecutive zeros. The convex property mentioned [illegible] furnishes

It follows that

$x_\nu \}$

$\left\{-2\left(|x|/3\right)^{3/2}\right\}$

$x \to -\infty$

$H_n(x)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = 0.$$

$$q_r = q$$

Alternative forms are $x_\nu^{1/2} = (4n)^{1/2} - 6^{-1/2} i_\nu (4n)^{-1/6} + O(n^{-1/6})$

& $x_\nu = (2n)^{1/2} - 6^{1/2} i_\nu (2n)^{-1/6} + O(n^{-1/6})$

(12) p. 136 – Electrostatic repns of the zeros of the classical polynomials.

(7) The zeros of $H_n(x)$ are simple

(8) For Hermite polynomials, from $y'' - 2xy' + 2ny = 0$, $y = H_n(x)$

or $z'' + (2n+1-x^2)z = 0$, $z = e^{-x^2/2} H_n(x)$

$$f''(x_0) = 2x_0 f'(x_0); \quad f'''(x_0) = 2(2x_0^2 - n + 1) f'(x_0)$$

$$3(n-2)x_0^2 - 2(n-1)(2x_0^2 - n + 1) \leqq 0$$

whence $|x_0| \leqq \dfrac{2^{\frac{1}{2}}(n-1)}{(n+2)^{1/2}} = (2n+1)^{1/2} - \dfrac{9}{2}(2n+1)^{-\frac{1}{2}} + \cdots$

(9) The zeros x_ν of $L_n^{(\alpha)}(x)$ arranged in increasing order have the bounds

$$\xi_\nu^2 \leqq x_\nu \leqq \eta_\nu^2$$

where ξ_ν & η_ν denote the positive zeros of the Hermite polynomials $H_{2n}(x)$ and $H_{2n+1}(x)$ respectly.

(10) Assuming $n \geqq 2$ we obtain the convexity of the sequence of the zeros

$$x_{0n} < x_{1n} < x_{2n} < \cdots$$

of $H_n(x)$ where $x_{0n} = 0$ if n is odd, and $x_{0n} = -x_{1n}$ if n is even. In all cases $x_{1n}, x_{2n}, \ldots$ denote the $+^{ve}$ zeros in increasing order.

On comparing the same eqn with $z'' + (2n+1)z = 0$ we find

$$x_{\nu n} > \begin{cases} \dfrac{\nu - 1/2}{(2n+1)^{1/2}}\pi & \text{(upper corresponds to } n \text{ even} \\ \dfrac{\nu}{(2n+1)^{1/2}}\pi & \text{lower } \text{"} \quad n \text{ odd)} \end{cases}$$

Now let ω be a fixed $+^{ve}$ number, $\omega < (2n+1)^{1/2}$, then

$$x_{\nu n} < \begin{cases} \dfrac{\nu - 1/2}{(2n+1-\omega^2)^{1/2}}\pi \\ \dfrac{\nu}{(2n+1-\omega^2)^{1/2}}\pi \end{cases}, \quad \nu = 1, 2, 3, \cdots [n/2]$$

provided the right hand members are not greater than ω. For a fixed ν, we see that the constants $(\nu - 1/2)\pi$ & $\nu\pi$ are the best possible

By introducing $x = (2n+1)^{-\frac{1}{2}}\xi$, the diff eqn mentioned transforms to

$$\frac{d^2z}{d\xi^2} + \left\{1 - (2n+1)^{-2}\xi^2\right\} z = 0$$

$$z = \exp\left\{-(2n+1)^{-1}\xi^2/2\right\} H_n\left\{(2n+1)^{-\frac{1}{2}}\xi\right\}$$

The coefficient of z increases with n; hence (for fixed ν) $(2n+1)^{1/2} x_{\nu n}$ decreases as n increases. Therefore we have $(2n+1)^{1/2} x_{\nu n} \leqq (4\nu+1)^{1/2} x_{\nu, 2\nu}$ or $(4\nu+3)^{\frac{1}{2}} x_{\nu, 2\nu+1}$ respectly. Thus

$$\left.\begin{array}{l} \dfrac{\nu - 1/2}{(2n+1)^{1/2}}\pi \\ \dfrac{\nu}{(2n+1)^{1/2}}\pi \end{array}\right\} < x_{\nu n} < \begin{cases} \dfrac{(4\nu+1)}{(2n+1)^{1/2}} \\ \dfrac{(4\nu+3)}{(2n+1)^{1/2}} \end{cases}, \quad \nu = 1, 2, \cdots [n/2]$$

$$w_{q.m} = \psi_0^2(x) = \frac{1}{x_0\sqrt{\pi}}\, e^{-x^2/x_0^2}$$ also shown in fig. 26

[More detailed consideration of zero point energy case]

Introductory quantum mechanics by R. Rojansky – p. 104 – figures as in Bushman for $\bar{\psi}_n \psi_n$ or ψ_n^2 (for $n = 0, 1, 2, 3, 4$ shown in the figure).

21/4/69 – Again from Szegö's orthogonal polynomials: – No notes taken down

23/4/69 Quantum Mechanics etc. by A. March, p. 61.

A broad indication of the classical oscillation is present, because according to classical mechanics for every ψ_i there is a definite relation between the position of the maximum and the amplitude of the oscillation · for example, the maximum of ψ_i

$$\left[\psi_i = \alpha_i\, e^{-(2\pi^2 m\nu_0 x^2/h)}\, H_i\left(2\pi\sqrt{\frac{m\nu_0}{h}}\,x\right),\ \alpha_i = \frac{1}{\sqrt{2^i i!}\,\pi^{\frac{1}{2}}}\sqrt[8]{\frac{ma}{\hbar^2}}\right]$$

lies at $x' = 1, 1$ $\left[x' = x\sqrt[4]{\frac{4\pi^2 ma}{h^2}} = 2\pi x\sqrt[4]{\frac{am^2}{4\pi^2\hbar^2 m}} = 2\pi x\sqrt{\frac{m\nu_0}{h}}\right]$

and therefore $x = 1, 1\sqrt{h/4\pi^2 m\nu_0}$. In classical mechanics, however, the amplitude A with the energy $3h\nu_0/2$ is assigned a value a $A^2/2 = 3h\nu_0/2$ & therefore $A = \sqrt{3}\sqrt{h/4\pi^2 m\nu_0}$. In general, it turns out that every eigenfunction gives a probability that differs from zero within a domain that approximately covers the range of the classical oscillation

25/4/69 – Reading issues of Physics Today jan 68 to April 68.

28/4/69 – Szegö's orthogonal polynomials:

(1) Formula: $H_n(x) = 2x H_{n-1}(x) - 2(n-1) H_{n-2}(x)$ for calculation starting with $H_0(x) = 1$, $H_1(x) = 2x$.

(2) The system $e^{-x^2} x^n$ is closed in $L(-\infty, +\infty)$, p. 106.

(3) For the largest zero x_0 of $H_n(x)$, $x_0 > \left\{\frac{1}{2}(n-1)\right\}^{1/2}$. (Equality for $n = 2$)

2/5/69 (4) – Continuation of Szegö.

(4) $\lim_{x\to\infty} x^{-n} H_n(x) = 2^n$.

(5) $H_0(x) + \frac{H_1(x)}{1!}w + \frac{H_2(x)}{2!}w^2 + \cdots + \frac{H_n(x)}{n!}w^n + \cdots = \exp(2xw - w^2)$

(6) Let $f(x)$ be a π_n and x_0 one of its zeroes. Then any circle through the points x_0 and $x_0' = x_0 - 2(n-1)\frac{f'(x_0)}{f''(x_0)}$ [Laguerre's theorem]

[π_n = arbitrary polynomial of degree n] contains some zeroes of $f(x)$ in both domains bounded by it, unless all the zeroes lie on the circle of this circle. The same is true if a st. line replaces the circle.

For Hermite polynomials $x_0' = x_0 - \frac{n-1}{x_0}$ so that ~~$\mathrm{Im}(x_0') > \mathrm{Im}(x_0)$~~

if $\mathrm{Im}(x_0') > \mathrm{Im}(x_0)$ if $\mathrm{Im}(x_0) > 0$.

[pp. 168–169 – $\psi_n(\xi) = e^{-\xi^2/2} H_n(\xi)$

where $H_n(\xi)$ is the Hermite polynomial of degree n

$$H_n(\xi) = \frac{(-1)^n}{\sqrt{2^n \cdot n! \sqrt{\pi}}} e^{\xi^2} \frac{d^n}{d\xi^n}(e^{-\xi^2})$$

the coeff^t of e^{ξ^2} is so chosen that $\int_{-\infty}^{\infty} \psi_n^2(\xi)\, d\xi = \int_{-\infty}^{\infty} e^{-\xi^2} H_n^2(\xi)\, d\xi = 1$

$$\psi_n(x) = \frac{e^{-\xi^2/2}}{\sqrt{x_0}} H_n(\xi), \quad \xi = x/x_0$$

ψ_n is even or odd according as the principal quantum number n is even or odd

$$\int_{-\infty}^{\infty} \psi_n^2(x)\, dx = 1$$

$$\psi_0(x) = \frac{1}{\sqrt{x_0 \sqrt{\pi}}} e^{-x^2/2x_0^2} \qquad n = 0$$

$$\psi_1(x) = \frac{1}{\sqrt{2 x_0 \sqrt{\pi}}} e^{-x^2/2x_0^2} \cdot 2\frac{x}{x_0} \qquad n = 1$$

$$\psi_2(x) = \frac{1}{\sqrt{2^2 \cdot 2 x_0 \sqrt{\pi}}} e^{-x^2/2x_0^2} \left(4\frac{x^2}{x_0^2} - 2\right) \qquad n = 2$$

Plotting of the first three functions (not well done in the book on p. 170 as many ψ_1 appears > than ψ_0.)] ✓

The probability $w_{q.m}$ is also plotted in Fig. 25. It can be seen that the quantum probability has maxima near the classical turning points (or, more precisely, at the analogues of the classical turning points, since, for $E_1 = \frac{3}{2}\hbar\omega_0$, $OA = OB = \sqrt{\frac{3\hbar}{\mu\omega_0}}$ and $OA' = OB' = \sqrt{\hbar/\mu\omega_0}$, but that, in contrast to the classical case, the probability of finding the particle does not go to zero beyond the turning points. This result is not self-contradictory because $E = T + U$ or $T = E - U$ does not hold in quantum mechanics. T and U are not simultaneously measurable.

The difference is particularly striking in the state of lowest energy (ground state).

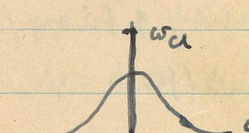
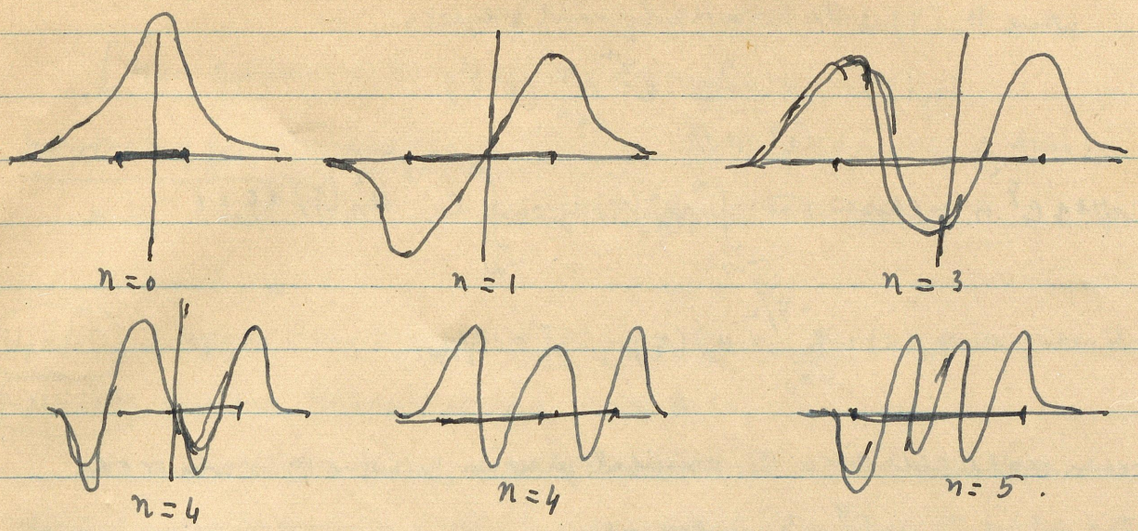


Fig. 26

In classical case minimum energy of oscillator is $E = 0$, the corresponding to a particle at rest at the eq^m position. The position probability $w_{cl}(x)$ associated with this state has the form shown in Fig. 26 being everywhere equal to zero except at the point $x = 0$. In quantum theory the minimum energy of the oscillator is $E_0 = \frac{\hbar\omega_0}{2}$ (zero-pt. energy) & the associated position probability with this state is

Figs on p. 24



14/4/69 – Ref. to Condon & Morse – Ruark & Urey – N.F. Mott – K.K. Darrow – Kemble & Hill

Pauling & Wilson – pp 67 – 83.

Fig 11.4, p. 76 giving probability distn for $|\psi_{10}(\xi)|^2$ for $n = 10$ of L.H.O. – ~~Remarks~~ Compares with the classical probability distn function – Remarks that the wave-mechanical amplitude lies in the neighbourhood of the classical value for the same energy (p. 77)

16/4/69 —

18/4/69 – Principles of Quantum mechanics by D. I. Blokhintsev.

p. 169. The probability of finding the particle, $w(x)\,dx$ (in classical mechanics) in the interval $(x, x+dx)$ is $\propto$ time dt ~~in~~ which the particle takes to travel across the segment ~~dx~~ dx. If period of oscillation be $T = 2\pi/\omega_0$, we may set

$$w_{cl}(x)\,dx = \frac{dt}{T} = \frac{\omega_0}{2\pi}\frac{dx}{v}$$

where v is the vel. of the particle. Since $x = a \sin \omega_0 t$, where a = amplitude of the oscillations given by $a = \sqrt{\frac{2E}{\mu\omega_0^2}}$, we have $v = a\omega_0 \cos\omega_0 t = a\omega_0\sqrt{1 - x^2/a^2}$.

& hence

$$w_{cl}(dx) = \frac{1}{2\pi a}\frac{dx}{(1 - x^2/a^2)^{1/2}} \qquad -a \le x \le a \quad \text{— (1)}$$

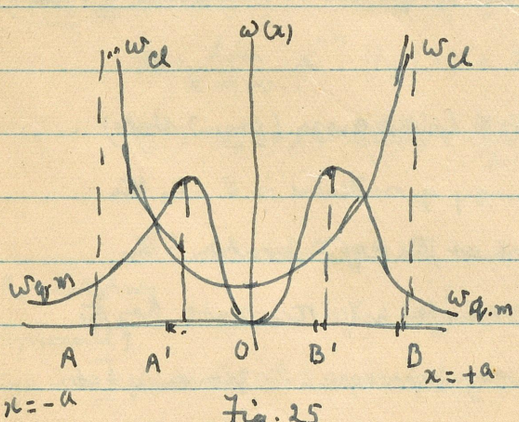


Fig. 25

This is plotted in Fig. 25. As expected, the probability is highest at the turning points A & B.

In quantum mechanics, the probability of finding the particle in the interval $(x, x+dx)$ is (for $n = 1$)

$$w_{q.m}(x)\,dx = \psi_1^2(x)\,dx$$

$$\text{where } \psi_1 = \frac{1}{\sqrt{2x_0\sqrt{\pi}}}\, e^{-x^2/2x_0^2} \cdot 2 \cdot \frac{x}{x_0} \quad \text{— (2)}$$

$$\left[\xi = x/x_0\,;\ x_0 = \sqrt{\frac{h}{\mu\omega_0}},\ \lambda = \frac{2E}{h\omega_0}\right]$$

$\lambda = 2n+1$ ($n = 0, 1, 2, \ldots$

ie $\sum_n \frac{s^{n-1}}{(n-1)!} H_n(x) + 2\sum_n \frac{s^{n+1}}{n!} H_n(x) - 2\sum_n \frac{s^n}{n!} x H_n(x) = 0$ . . (5)

equating to zero the coefficient of s^n, we get

$$H_{n+1}(x) - 2xH_n(x) + 2n H_{n-1}(x) = 0 \qquad 16]$$

[$H_{-1}(x)$ should be interpreted as 0 when used with (3) or (6)]

p.20 pp. 16 – 23 – Shows that the only eigen-values of the operator $-\frac{d^2}{dx^2} + x^2$ are the odd positive integers $\lambda_1, \lambda_2, \ldots$ where $\lambda_n = 2n+1$ and let the ~~Hermite~~ eigen-fns be $u_1, u_2, \ldots$

If $u = v e^{-x^2/2}$, substituting in

$$\left(-\frac{d^2}{dx^2} + x^2\right) u = \lambda u, \text{ we have for } v \quad \ldots (7)$$

$$v'' - 2xv' + (\lambda - 1)v = 0 \quad \ldots (8)$$

If we put $\lambda = \lambda_n = 2n+1$, this gives deff eqn for $H_n(x)$ viz

$$H_n''(x) - 2xH_n'(x) + 2n H_n(x) = 0$$

p. 26 – Examples.

(1) Verify that $H_{K+1}(x) e^{-x^2} = -\frac{d}{dx}\left[H_K(x) e^{-x^2}\right]$ and show by using Rolle's theorem and mathematical induction that the n nodes of $H_n(x)$ ie [roots of the equation $H_n(x) = 0$] are all real & distinct

(2) Show that every solution of eqn (7) above inflects at each of its nodes, at $x = -\lambda^{1/2}$ and $x = \lambda^{1/2}$; that every soln of (7) is concave towards the ~~origin~~ x-axis outside this region $-\lambda^{1/2} < x < \lambda^{1/2}$ and convex towards the x-axis outside this region; that no solution of (7) can have more than one node in the region $x \le -\lambda^{1/2}$ or more than one node in the region $x \ge \lambda^{1/2}$; and that if a soln of (7) has a node at a finite value of x in the region $x \le -\lambda^{1/2}$ or at a finite value of x outside in the region $x \ge \lambda^{1/2}$, then this soln is ill-behaved. Show further that $u_n = A_n H_n(x) e^{-\frac{1}{2}x^2}$ has n nodes in the region $-\sqrt{(2n+1)} < x < \sqrt{(2n+1)}$, one node at $x = \infty$ & another at $x = -\infty$, ie in all $(n+2)$ nodes.

(3) Show that the fns $u_n = A_n H_n(x) e^{-\frac{1}{2}x^2}$ satisfy the identities

$$xu_n = \frac{nA_n}{A_{n-1}} u_{n-1} + \frac{A_n}{2A_{n+1}} u_{n+1} \quad \ldots (9)$$

and $$\frac{d}{dx}(u_n) = \frac{nA_n}{A_{n-1}} u_{n-1} - \frac{A_n}{2A_{n+1}} u_{n+1} \quad \ldots (10)$$

& in general deduce expns for $x^2 u_n$ and $\frac{d^2}{dx^2} u_n$ & prove in general

$$\left(\frac{d}{dx} + x\right)^K u_n = \frac{2^K \cdot n!\, A_n}{(n-K)!\, A_{n-K}} u_{n-K} \quad \text{for } K \le n,$$

p. 385 – Problems (a) – Prove the identity

$$\sum_{\nu=0}^{n} \binom{n}{\nu} H_\nu(x)(2y)^{n-\nu} = H_n(x+y)$$

[Hint use (a)]

p. 386 · Problems – (1) Prove the following identities

$$y'' + 2xy' + 2(n+1)y, \quad y = e^{-x^2} H_n(x)$$

(2) $$\int_0^y e^{-t^2/2}\,dt = 6^{-1/2} \sum_{\nu=0}^{\infty} \left(-\frac{1}{12}\right)^\nu \frac{H_{2\nu+1}(x)}{(2\nu+1)\,\nu!}$$

(a) (3) Prove that $\lim_{n\to\infty} \left(\frac{y}{n}\right)^n H_n\left(\frac{n}{2y}\right) = e^{-y^2}$

[Hint – use (b)]

Re. Zeros of Hermite polynomials numberg results mixed up with general orthogonal polynomials in Ch. VI on pp. 116 ff, 122, 126 ff, 140 ff, p. 238 (ref. to p. 131) and p. 350 where one has for the Hermite case $\{x_1 > x_2 > -\infty,\ \lambda_1 < \lambda_2 < \cdots < \lambda_{[(n+1)/2]}\}$?

(B) Table of Hermite functions by J. B. Russell – J. Math. Phys. 12 (1933), p. 291 using which graphs of $\psi_{0,1,2,3}$ have been drawn in Pauling & Goudsmit's quantum mechanics

In Russell's article general form of $H_n(x)$ where $\Phi_n(x) = e^{-x^2/2} H_n(x)$ is the Hermitian polynomial function is given as

$$H_n(x) = 2^n \cdot x^n - 2^{n-1} C_2^n x^{n-2} + 1\cdot 3\cdot 2^{n-2} C_4^n x^{n-4} - \cdots$$

(C) Amer. Math. Monthly 43 (1936) not available

11/4/69 Rojansky's Introduction Quantum mechanics

p. 24. Six Hermite forms $u_n = A_n H_n(x) e^{-\frac{1}{2}x^2}$ with $A_n = (2^n n!\, \pi^{\frac{1}{2}})^{-\frac{1}{2}}$ are given. The thick horizontal lines extend along x-axis from $-\sqrt{2n+1}$ to $\sqrt{2n+1}$

(Drawn from figure in Pauling & Wilson

p. 25: Generating fn of Hermite polynomials is

$$z(x,s) = e^{x^2 - (s-x)^2} = e^{-s^2 + 2sx} = \sum_n \frac{s^n}{n!} H_n(x), \qquad n = 0, 1, 2, \cdots \quad (1)$$

p. 26 ~~Examples~~ Also from this we have $z(x,s)$ satisfying the equation

$$\frac{\partial}{\partial x} z(x,s) = 2sz(x,s)$$

and in view of (1), we have $\sum_n \frac{s^n}{n!} H_n'(x) = \sum_n \frac{2s^{n+1}}{n!} H_n(x)$ – (2)

which holds identically in s. Equating coeffts of s^n on both sides

$$H_n'(x) = 2n H_{n-1}(x) \quad (3)$$

$z(x,s)$ also satisfies $\frac{\partial}{\partial s} z(x,s) = -2sz(x,s) + 2xz(x,s)$ (4)

$z'' + (2n+1-x^2)z = 0$, $z = e^{-x^2/2} H_n(x)$

$e^{-x^2} H_n(x) = (-1)^n \left(\frac{d}{dx}\right)^n e^{-x^2}$

$\frac{H_n(x)}{n!} = \sum_{\nu=0}^{[n/2]} \frac{(-1)^\nu}{\nu!} \cdot \frac{(2x)^{n-2\nu}}{(n-2\nu)!}$ ——— (b)

$H_{2m}(0) = (-1)^m \frac{(2m)!}{m!}$; $H'_{2m+1}(0) = (-1)^m \frac{(2m+2)!}{(m+1)!}$

$\lim_{x\to\infty} x^{-n} H_n(x) = 2^n$.

$H_0(x) + \frac{H_1(x)}{1!}\omega + \frac{H_2(x)}{2!}\omega^2 + \cdots + \frac{H_n(x)}{n!}\omega^n + \cdots = \exp(2x\omega - \omega^2)$ .. (a)

$H_n(x) = 2x H_{n-1}(x) - 2(n-1) H_{n-2}(x)$

$(n = 2, 3, 4 \cdots)$; $H_0(x) = 1$, $H_1(x) = 2x$

$\sum_{\nu=0}^{n} (2^\nu \nu!)^{-1} H_\nu(x) H_\nu(y) = (2^{n+1} n!)^{-1} \frac{H_{n+1}(x) H_n(y) - H_n(x) H_{n+1}(y)}{x-y}$

∴ Individual properties viz

$H_n'(x) = 2n H_{n-1}(x)$, $H_n(x) = 2x H_{n-1}(x) - H'_{n-1}(x)$

From (a) we get

$\sum_{\nu=0}^{n} \binom{n}{\nu} H_\nu(x) H_{n-\nu}(y) = 2^{n/2} H_n\{2^{-\frac{1}{2}}(x+y)\}$

and by Cauchy's formula, $H_n(x) = \frac{1}{2\pi i} \int \omega^{-n-1} \exp(2x\omega - \omega^2)\, d\omega$

where the contour encloses the origin

[Ref. to Bateman Manuscript Project. Vol 2, Ch. 10, pp. 188-196 & also Exx. & problems, 67, 68, 72-80 for further formal properties]

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p. 174 : 1. The function $2n\{H_n(x)\}^2 + \{H_n'(x)\}^2$ is decreasing for $x < 0$ and increasing for $x > 0$. Method of proof not quite clear. Mentions that it follows from differentiating $y'' - 2xy' + 2ny = 0$ as in § 7.3 (1)

p. 175 – Th. 7.6.3 – The successive relative maxima of

$e^{-x^2/2} |H_n(x)|$ ——— → follows by using $z'' + (2n+1-x^2)z = 0$,

form an increasing sequence for $x \geq 0$.

and applying the Theorem of Sonin (p. 164 of Szegö) viz

If $y = y(x)$ satisfy the differential equation

$y'' + \phi(x) y = 0$

where $\phi(x)$ is a +ve function having a continuous derivative of a constant sign in $x_0 < x < X_0$. Then the successive relative maxima of $|y|$, as x increases from x_0 to X_0 form an increasing or decreasing sequence according as $\phi(x)$ decreases or increases

and hence mean value of $|\psi|^2$ is (since $2n \sim \lambda$) given by $P(x) = 1/\pi\sqrt{\lambda - x^2}$.

(9/4/69) Look up references:- (1) Szász - J. Ind. Math. Soc (NS) 15 (1951), 129-134

Proves Indritz's result in essentially the same way. [M.R. Nov. 62]

(2) E. Hill - Reciprocal functions, Ann. of Math. Vol 26-27 (1924-26), 427-64

Reference given by Indritz.

(3) S.K. Chatterjea (Math. Rev. April 62, p. 485) - In a Some Journal Rev. Mat. Hisp. Amer proves that $[H_n(x)]^2 - H_{n-1}(x) H_{n+1}(x)$ can be expressed as sum of squares so that $|\Delta_n(x)| \geq 0$ follows for all real x (Turán's inequality) (done by me and V.R.T. in 1949) [what about Q.M. interpretation?]

(4) Ibid in some journal [M.R. Sept. 61, p. 1381] proves

$H_{n+h}(x) H_{n+k}(x) - H_n(x) H_{n+k+h}(x) > 0$ (for n, h, k integers ≥ 1, x real) and gets Turán's inequality - method not given in M.R.

(5) M.R. Sept. 63, p. 511 - Reference to the inequalities

$$|H_n(x)| \leq \frac{(2m+2)!}{(m+1)!} |x| e^{x^2/2} \text{ if } n = 2m+1$$

$$\leq 2 \frac{(2m+1)!}{m!} e^{x^2/2} \text{ if } n = 2m$$

by J. Balázs - Derived from a more complicated type of inequality - To prove directly & to find Q.M. interpretation.

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(1) Not in the D.J.Sc. library - from M.R. Vol. 13, 1952, p. 842

Denote by $\mu_{0,n}, \mu_{1,n}, \ldots$ the successive relative maxima of $|\phi_n(x)|$ as x decreases from $+\infty$ to 0, where the $\phi_n(x)$ are the Hermite orthogonal functions. It is shown that $\mu_{r,n} > \mu_{r,n+1}$ for $n \geq r \geq 0$. Combining this result with the classical fact that $\mu_{r,n} > \mu_{r+1,n}$ $(r = 0, 1, \ldots)$, it follows that $|\phi_n(x)| < 1$ confirming a conjecture of G.E.H. Reuter [J. Lond. Math. Soc. 24, 159-60; these Rev. 11, 29]

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(2) not in the Library.

(A) Szegö - orthogonal polynomials - Amer. Math. Soc. Coll. Publns. Vol XXIII.

Properties of Hermite polynomials (p. 105)

(1) Defined by $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \pi^{1/2} 2^n \cdot n!\, \delta_{nm}$ $(n, m = 0, 1, 2$

The coefficient of x^n in the nth polynomial is $+2^n$. (Vide Hilbert-Courant pp. 77-79)

(2) $y'' - 2xy' + 2ny = 0$, $y = H_n(x)$

7/4/69 Quantum Mechanics by Powell & Grasemann [Schrödinger eqn is taken as $\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - \frac{1}{2}Kx^2)\psi = 0$

$H_n'(x)$ satisfies the diff eqn $\phi'' - 2x\phi' + 2(n-1)\phi' = 0$

$H_n' = 2n H_{n-1}$ [classical frequency of oscillator is $\omega = \sqrt{K/m}$

$H_{n+1} - 2x H_n + 2n H_{n-1} = 0$

$H_{n+1} = 2x H_n - H_n'$

$\psi_n = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-x^2/2} H_n(x)$ $\left[H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \right.$

$\sqrt{n+1}\,\psi_{n+1} - \sqrt{2}\,x\psi_n + \sqrt{n}\,\psi_{n-1} = 0$

$x\psi_n + \psi_n' = \sqrt{2n}\,\psi_{n-1}$

$x\psi_n - \psi_n' = \sqrt{2(n+1)}\,\psi_{n+1}$

[Graphs of $\psi_{0,1,2,3}$ from R. J. B. Russell's Tables of Hermite functions' J. Math. Phys 12, 291 (1933)]

n zeros of ψ_n are contained within the interval of length $2\sqrt{2E_n/K}$

[ψ_n oscillates very rapidly] $\left[e^{x^2} = \sum_{(K\,even)}^{0} x^K/(K/2)! \right]$

and the average interval between zeroes is therefore $2\sqrt{\frac{2E_n}{K}}/n \approx \frac{2}{\sqrt{n}}\sqrt{\frac{2\hbar\omega}{K}}$ (ergs) which $\to 0$ as $n \to \infty$

Ref. to E. R. Smith "Zeroes of the Hermitian polynomials", Am. Math. Monthly 43, 354 (1936).

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classical probability-density $P(x)\,dx = \frac{dx}{\pi\sqrt{\lambda - x^2}}$, taking $x = \pm\sqrt{\lambda}$ as limits and $x = \sqrt{\lambda}\sin\omega t$ & where $\omega\sqrt{\lambda}\cos\omega t$.

and $\frac{dT}{T} = \frac{1}{T}\frac{2dx}{\omega\sqrt{\lambda}\cos\omega t} = \frac{1}{T}\cdot\frac{2dx}{\omega\sqrt{\lambda - x^2}}$ in which $T = \frac{2\pi}{\omega}$ = period of oscillation & hence $P(x)\,dx = \frac{dx}{\pi\sqrt{\lambda - x^2}}$ as written above

It can be shown that if n is large, the Hermite Polynomial is approximated by

$$H_n(x) \sim \frac{2^{n+1}(n/2e)^{n/2}}{\sqrt{2\cos\alpha}} e^{n\alpha^2} \cos\left\{(2n+\tfrac{1}{2})\alpha - \frac{n\pi}{2}\right\}, \text{ where } \alpha \text{ is the} \quad (2)$$

smallest $+^{ve}$ angle whose sin is $x/\sqrt{2n}$ (A) [Ref. Szegö, orthogonal polynomials N.Y. Amer. Math. Soc. Coll. Publns, 1939, Vol XXIII]. The probability for for the harmonic oscillator is hence

$$|\psi_n|^2 = \frac{1}{2^n n!\sqrt{\pi}} e^{-x^2} H_n^2(x) \sim \frac{2}{\pi\sqrt{2n - x^2}} \cos^2\left[(2n+\tfrac{1}{2})\frac{x}{\sqrt{2n}} - \frac{n\pi}{2}\right]$$

using $n \gg 1$ to simplify (i). The average value of $\cos^2[\ ]$ over many periods is $\frac{1}{2}$

viz 3, 5, 17, 257 and 65537 ($= 2^{2^4}+1$) are prime & expressed both the belief that this is true for all n (if this were true, one would have had a general formula of a simple form yielding only primes for integral values 1, 2, 3, - - of n). Later Fermat himself began to doubt his conjecture and in 1732, Euler showed (1707-83) showed that the next Fermat no. $2^{2^5}+1$ is not prime but divisible by 641. Calculations have been made for n = 6, 7, 8, 9, 11, 12 all giving composite nos (using Calculating machines), and with large divisors. We still do not know whether among the fermat nos. there are infinitely many primes or only a finite number viz. the five corresponding to n = 0, 1, 2, 3, 4 given by fermat: one has no idea as to what how to approach this question with the mathematical methods at our disposal.

About all n's examined & found to lead to composite nos. are under 25. Besides these n = 36 and n = 38, and in 1905 Morehead showed that $2^{2^{73}}+1$ is not prime but is divisible by $5 \cdot 2^n + 1$ (n = ?). It is difficult to describe the stupendous magnitude of this number. To write it in the decimal notation we should have to use thousands of quintillions of digits (trillion is a million millions, a quintillion is a million trillions). Were we to write this number in numerals (Roman) 5 mm wide each, it would occupy a length exceeding five trillion Rm & in order to write it we should have to cover the equator more than a hundred billion times.

Geometrical significance of $2^n + 1$ viz. with aid of ruler & compasses alone it is possible to divide the circle into 17, 257 and 65, 537 parts.

Another problem viz. every prime number except 2 is of the form $4n+1$ or of the form $4n+3$ leading to viz. the theorem every prime of the form $4n+1$, but no prime of the form $4n+3$, can be expressed as the sum of two squares of $x^2 + y^2$. There is an interesting letter of the famous Eisenstein (1823-52) who died at age of 29 and whom Gauss (1777-1855) [called Prince of mathematicians & whose work Dirichlet.. compared at the age of 20 led the formulation of the theory of numbers, notwithstanding the important achievements of the ancient Greeks & Hindus, & is called the Magna Carta of the History of nos] regarded as the ablest mathematician of his generation. In this letter he relates how he attempted to explain this theorem in his family circle and adds:
I have an uncle who maintains that it suffices to take someone from an insane asylum, he will certainly be an excellent mathematician. I, on the other hand, say - and Dirichlet agrees with me - that mathematics may, indeed, be a form of insanity, but not every form of insanity need be of a mathematical nature.

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p. 2 - discovery of 0 by the Hindus a great achievement in which Archimedes & Apollonius (287-212 BC) (265?-170 BC) fails - Ref. to article by B. Datta - Bull. Cal. Math. Soc. Vol. 24 (1933).

(7) Goldbach's conjecture. Communicated to Euler (greatest mathematician of his time) in 1742 viz. ex. $8 = 3+5$, $16 = 13+3 = 11+5$. Not solved so far despite efforts of greatest mathematicians during many generations. Decisive progress about 1920 after a keen keen method invented by Ramanujan (1887-1920) and Hardy (1877-1947) & Littlewood.

Schnirelmann (1907-38) a great Russian Jew demonstrated the following theorem: There exists a natural number N such that one can express every natural number (except 1) as the sum of at most N primes. His proof is fairly elementary and "constructive" & leads to $N \sim \frac{1}{3}$ of a million. In 1936 it was proved that N need not be greater than 71 if the nos. in question are above a certain limit. A year later the Russian mathematician Vinogradoff was able to prove that every sufficiently large odd integer can be represented as the sum of at most three primes. This means that to represent a sufficient large even number there is no need to use more than 4 primes ie $N \leq 4$ when large integers are concerned; thus a mathematical dream of centuries is approaching reality (Vide Estermann's Intn to modern prime number theory, London, 1952, p. 33)

(8) Partition of the circle. - Theorems discovered by induction (ie by actual calculation or experiment) and later demonstrated mathematically. Even Gauss (founder of theory of nos. as a systematic discipline) employed calculations to a great extent (partly due to his proficiency in numerical calculations from which he derived great pleasure). However, conclusions formed inductively even when based on large no of expts cannot be considered as established; we may accept the conclusion only after it has been proved mathematically. For eg: Euler found by testing all n up to 2500 that it was possible to express every odd number as the sum of prime no. p and the double of a square ie $n = p + 2m^2$. Later calculations were extended to n up to 9000 & it was found that among these there are two viz. 5777 and 5993 which do not satisfy the Euler property; thus, among 4500 consecutive tests only two give negative results, the rest being +ve

Again for eg. the expn $n^2 - 79n + 1601$ yields primes for $0 \leq n \leq 79$, yet, alas, for $n = 80$ gives a composite number.

There is a much more famous example viz. prime numbers of the form $2^m + 1$. It is easy to see (that if m has an odd factor ($\neq 1$), $2^m + 1$ is not prime [Proof · bottom of p. 39] Hence one has only to consider nos. of the form $2^{2^n} + 1$ in which n assumes all values 0, 1, 2, .. Fermat (1601-65) one of the greatest mathematicians of the 17th century and perhaps the greatest French student of the Theory of nos. pointed out that the five nos. of this form

even perfect number is of the form $2^{n-1}.(2^n-1)$ (last factor a prime) – Re. odd perfect numbers we know nothing and we don't know if they exist & there is no proof that they don't exist; if such nos. exist they must be very large – Primes of form 2^n-1 are called Mersenne numbers; 2^n-1 is prime only if n is prime; but if n is prime, 2^n-1 need not necessarily be prime. The largest n for which 2^n-1 is prime is $n=2281$ (1952); in fact $2^{2281}-1$ is the largest prime number known.

p. 51 – 6 conditions to be satisfied by odd perfect numbers (if they exist at all)

(3) Amicable numbers – known to Greeks & before them to the Hindus, for eg. 220 & 284. Sum of all divisors of 220 except 220 itself $= 1+2+4+5+10+11+20+22+44+55+110 = 284$ & sum of divisors of $284 = 1+2+4+71+142 = 220$. ie m & n are amicable if sum of nos. of divisors of $m = n$ & sum of divisors of $n = m$ (not counting m or n always divisors as in the case of perfect numbers). About 70 such are known, but no proof as to whether number of such pairs is finite or not.

(4) Let property T be that all numbers less than m and prime to m are prime numbers for eg 3, 4, 6, 8, 12, 18, 24, 30. Not every number has the property T for eg. 10, 28 & 36 for 9 is prime to 10 and to 28 & 25 is prime to 36 and yet neither 9 or 25 is a prime. The remarkable property of 30 is that there is no number above 30 possessing the property T = Proof of 2 pages.

(5) Bertrand's postulate that between n and $2n$ there is always a prime when n is an integer (or a real no >1 was proved by Tchebycheff. It was later found that from $n=23$ onward there is always a prime number between n and $n+n/4$; from $n=47$ onward between n and $n+n/8$. For $n=10^8$, there are between n and $n/2$ about a million prime nos, not merely one or four. On the other hand we do not know whether between every two consecutive squares ie between n^2 and $(n+1)^2$ there is always a prime number.

(6) Density of prime numbers – decreases on the average ie if we consider sufficiently large intervals; local density means density in the neighbourhood of a given prime. Question whether there is a last pair of twin primes in view of decreasing density of primes in general or whether such pairs exist after any given number however large. This problem has not yet been solved (Brun discovered a partial answer by using infinite series, Bottom of pp. 36 & 3); at any rate there exist fairly large twins for eg. 10,006,427 and 10,006,429. Converse of local density problem viz whether in the sequence of prime nos. there may be arbitrarily great jumps or whether, for any n, we can find n consecutive integers none of which is prime is easily answered in the affirmative; for in the n successive numbers $(n+1)!+2, (n+1)!+3, \ldots (n+1)!+(n+1)$ there is no prime among them for since each of these nos is divisible by its second term ie $(2, 3, \ldots n+1)$.

(1910)

Attempts to get him a scholarship failed and he was himself unwilling to be a burden on anybody & took a job in Port Trust (1911) – In 1912 due to recommendation of G. T. Walker F. R. S (Head of the Meteorological Dept & formerly fellow and math. lecturer at Trinity College, Cambridge) who recognised the quality of his work, he got in 1912 "a research studentship of Rs 75 p month for two years"!

(Died 26/4/1920)

Nature 105 (1920), p. 494 – obituary by Hardy – It is perhaps useless to speculate as to his history had he been introduced to modern ideas and methods at sixteen instead of at twenty-six. It is not extravagant to suppose that he might have become the greatest mathematician of his time. What he did actually is wonderful enough. Twenty years hence when the researches which his work would have suggested have been completed, it will probably seem to a good deal more wonderful than it does today

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Nature. 123 (1929), pp. 631-33 – Littlewood's review of Collected Papers of Srinivasa Ramanujan edited by G. H. Hardy, P. V. Seshu Iyer & B. M. Wilson (C. U. P, 1927

– Most interesting remarks about the joint paper by Hardy & Ramanujan ~~about~~ re. the formula for how $p(n)$ [partitions]. – Littlewood's statement that his ~~most~~ Ramanujan's mathematical ~~work~~ ~~takes second place to~~ this interest to the romance of his life and mathematical career, his unusual psychology, etc. does not sound correct.

2/4/69 – Saw Sir C. V. – He did not even have the courtesy to ask what I was doing

4/4/69 A. A. Fraenkel's Integers & Theory of Numbers

(1) Checking whether $7744 \cdot 88 = 681472$ is correct

Sum of digits of 7744 = 22 & sum of digits of 22 = 4

" 88 = 16 & " 16 = 7

" 681472 = 28 ie $4 \cdot 7 = 28$ Correct

(This is true if the sums in question exceed 9 for eg. $22 \& 16 \& 28$.

(2) Numbers which are greater than the sum of their divisors are called "abundant" for eg. $10 > 1+2+5$ & if smaller number is deficient for eg. $12 < 1+2+3+4+6$

Perfect if equal, for eg. $6 = 1+2+3$, $28 = 1+2+4+7+14$. Greeks discovered two such viz 496 & 8128. The fifth perfect number = 33,550,336. – Every number of the form $2^{n-1}(2^n-1)$ is perfect if 2^n-1 is prime. Two such large ones correspond to $n = 2203, 2281$ & have 1335 & 1373 digits. They all end with 6 or 8 & it is not difficult to show that every even perfect number ends with 6 or 8. Euler's theorem that every

31/3/69 . Proc. Roy. Soc. A, 99 (1921) - xiii - xxix (Obituary notice of Ramanujan by Hardy)

Every +ve integer one of his personal friends - nothing abnormal in his mode of thought - He had no special secret. My belief is that all mathematicians think, at bottom, in the same way and that Ramanujan was no exception. He had of course an extraordinary memory. He could remember the idiosyncracies of numbers in an almost uncanny way. - Re. Taxicab No. 1729 (7.13.19) not being an ill omen & his saying that it is a very interesting number, being the smallest number expressible as a sum of two cubes in two different ways. I asked him whether he knew the answer to the corresponding problem of 4th powers & he replied after a moment's thought that he could see no obvious example & thought that the first such number must be very large. His memory and powers of calculation were very unusual, but they could not be reasonably called 'abnormal'. If he had to multiply two large numbers, he multiplied them in the ordinary way; but he would do it with unusual rapidity and accuracy but not more rapidly or more accurately that any mathematician who is naturally quick & has the habit of computation. The table of partitions in one of our joint papers was calculated, for the most part, calculated independently by Ramanujan and Major Mac Mahon and Major Mac Mahon was, in general, slightly the quicker and more accurate of the two.

In his insight into algebraical formulae, transformation of ∞-series & so forth, there was most amazing. On this side most certainly I have never met his equal, and I can compare him only with Euler or Jacobi. He ... With his memory, his patience and his power of calculation, he combined also a power of generalisation, a feeling for form, and a capacity for rapid modification of his hypotheses, that was often really startling, and made him, in his own peculiar field, without a rival in his day.

One gift his work has which no one can deny is his profound and invincible originality.

$$\frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2}$$ correct to 8 decimal places.

He was not a man who talked much about himself - At 7 extraordinary powers appear to have been recognised immediately. He used to entertain his friends with theorems and formulae even in those early days - Borrowing Carr's synopsis & verifying formulae - Had an extraordinary memory and could easily repeat the complete lists of Sanskrit roots (atmanepada and parasmepada); he could give the values of $\sqrt{2}$, π, e, .. to any number of decimal places - In manners he was simplicity itself - Passed matric in 1904 (at 17)

a vector field]

Note – The theorem is true not merely to a one-parameter group of trns but can be generalised to n-parameter group of trns. ~~Con~~ Condition becomes that the group should admit n Killing vector fields. Such a group of trns is often called a group of motions of the space ~~[illegible]~~ with the metric $g_{\mu\nu}$.

28/3/69 – Read Hardy's Lectures on Ramanujan (with biographical sketch & discussion of his work) – C.U.P., 1940

Ramanujan – Born, 1887

Even at 12 or 13 recognised to be a abnormal boy – proved $e^{i\theta} = \cos\theta + i\sin\theta$.

Passed matric in Dec 1903 (ie at 16) & got Subramanyam scholarship & joined Govt. College, Kumbakonam for 1st F.A. – failed ~~[illegible]~~ to secure promotion to Senior F.A. & hence scholarship discontinued – disappointed & went to Telugu country north, but retd. to Kumbakonam & rejoined the College – did not obtain term certificate in 1905, left the College & joined Pachaiyappa's College in 1906, fell ill & retd to Kumbakonam – Appeared for F.A. in private in 1907 & failed – got married in 1909 (ie at 22) & so wanted some employment – In 1910 some Indian friends helped him to get a job ~~as~~ a clerk in office of Port Trust, Madras in 1912 (at £30 per year) (ie nearly 37.5 Rs/month) – Through help of Englishmen (& his writing to Hardy in 1913) he went to Cambridge in 1914 – uninterrupted work till 1917 – Fell ill in 1917, ~~never~~ returned, never really recovered & died in 1920 (ie at 33) [F.R.S in 1918, first Indian].

[For mathematicians crucial years are between 18 & 25 & it was exactly during these years that his life was made most miserable]

[All this because of India being a slave country dominated by the English scoundrels who cared so little for Indian education!]

Ref: (1) Sesham Iyer – J.I.M.S. 12 (1920), 81–86

(2) G.H. Hardy – J.I.M.S. 9 (1917), 30–48

(3) " – Nature 105 (1920), 494 & J.I.M.S. 12 (1920), 90–91

(4) Littlewood – Collected Papers of Ramanujan – Math. Gaz. 14 (1929), 425–28 & Nature, 123 (1929), 631–33

(5) Hardy – on Ramanujan – PLMS (2), 19 (1921), xl–(viii) & Proc. Roy. Soc. (A), 99 (1921), xiii–xxix

(6) R. Ramachandra Rao – In Memoriam – J.I.M.S, 12 (1920), 87–90

[Papers by S. Narayana Iyer, S. Chowla, H. Gupta, S.S. Pillai]

$\frac{2.5 \times 15}{37.5}$ 1

symmetry group that a Riemannian space-time can admit or it is also $\simeq$ 5 Dimensional rotational rotation group with 10 parameters. A double valued 4×4 matrix repn for the de Sitter group can be found. It is the single-valued repn of a subgroup of the unimodular complex linear group in 4-Dimensions. The de Sitter group has two Casimir operators. In the limit $R \to \infty$, the Casimir operators take the form

$$I_1 \Rightarrow -P_\lambda P_\lambda = m^2; \quad I_2 \to -V_\lambda V_\lambda = m^2 s(s+1)$$

where m & s are respectively the rest mass & spin that characterise the repn so that I_1 & I_2 go over into the Casimir operators of the inhomogeneous Lorentz group (Poincaré group)

For a given cosmological background the group of motion that has more than local validity is no longer the L.T. In an expanding universe the de Sitter group is a better approxn to the group admitted by the universe as a whole than the P.G. is

p. 683. The Yang-Mills group in general is the ∞-Dimensional group which is obtained by taking the continuous direct product of any arbitrary Lie group with itself over space-time. - Corresponds dynamically to the Y-M field is non-linear. Combined gravitational, Yang-Mills & scalar fields

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Intn to gen. rel. by Bolla, Bazin & Schiffi McGraw Hill, 1965

p. 352. In deriving particular forms of Schwarzschild for the centrally symmetric field & in establishing the Robertson-Walker line element in Cosmology, one appealed to the symmetry of the physical and geometrical situation in order to simplify the ansatz. Now the qn arises: What would we say in an arbitrary coordinate system in which this symmetry is not clearly displayed? i.e. how could we discover hidden symmetries in an arbitrarily given Coord. system. Clearly the problem is natural in the context of General relativity, where we try to rid ourselves of the accidental peculiarities of the Coord. frame. Answer given by the following theorem:

A necessary & suff. condn for the metric $g_{\mu\nu}$ to have a hidden symmetry is that it admits a Killing vector field $\psi_\mu(x^\nu)$ *

[Defn of Killing vector. Covariant derivative of a vector is denoted by

$$\xi^i_{\|K} = \frac{\partial \xi^i}{\partial x^K} - \Gamma^i_{kl}\,\xi^l \qquad \text{(ord. partial derivative by a single |)}$$

Vector-fields ξ_λ on Riemannian manifold with the differential

Condtns $\xi_{\lambda\|\eta} + \xi_{\eta\|\lambda} = 0$ are called Killing vector fields]

[* we consider a metric tensor $g_{\alpha\beta}(x^\nu)$ ~~with~~ which admits a one-parameter group of continuous transformations $\bar{x}^\mu = \varphi^\mu(x^\nu; \lambda)$, and $\psi^\mu(x^\nu) = \dot{\varphi}^\mu(x^\nu; 0)$ where dot over φ denotes diffn with respect to λ. $\psi^\mu(x^\nu)$ can be shown to be

Group properties of associativity, existence of unit & inverse established — The infinitesimal Lie group (structure constants) — Commutation rules satisfied by the infinitesimal generators — Coord trns & trn of parameters — Canonical form — Reps'ns and global properties of Lie groups — Adjoint repn of a Lie ring — length & vol in group space — The Casimir operator — Compactness & connectivity & Covering group — Universal covering group]

The 4-dimil rotation group O_4: Putting $J_{12} = M_3$, $J_{23} = M_1$, $J_{31} = M_2$
$J_{43} = N_3$, $J_{41} = N_1$, $J_{42} = N_2$ } (5)

for the 6 generators of the Lie ring of O_4. The commutator relations (4) take the form

$$\left.\begin{aligned} [M_i, M_j] &= i\,\epsilon_{ijk}\,M_k \\ [N_i, M_j] &= i\,\epsilon_{ijk}\,N_k \\ [N_i, N_j] &= i\,\epsilon_{ijk}\,M_k \end{aligned}\right\} \qquad (6)$$

where now all the indices go from 1 to 3. The space with const. +ve curvature on which the group acts can be embedded in flat-space in 4 dimns where it becomes the 3-dimil surface of a hypersphere of radius R with eqn

$$\xi_1^2 + \cdots + \xi_4^2 = R^2 \qquad (7)$$

By suitably choosing a basis using (7) we can transform (6) to show that the group decomposes into the direct product of two rotation groups. [$\vec{K}$ & $\vec{L}$ being their rotation operators]

The Casimir operators for O_4 — upto a const. factor

$$C_1 = \tfrac{1}{4} J_{\mu\nu} J_{\mu\nu} = \vec{K}\cdot\vec{K} + \vec{L}\cdot\vec{L} \qquad \left[\vec{K} = \tfrac{1}{2}(\vec{M}+\vec{N}),\ \vec{L} = \tfrac{1}{2}(\vec{M}-\vec{N})\right.$$

Eigenvalues of $C_1 = K(K+1) + L(L+1)$. Also the operator $C_2 = \frac{1}{4} J_{\mu\nu} \tilde{J}_{\mu\nu} = \vec{K}\vec{L} - \vec{L}\vec{L}$ commutes with all the elements of the Lie ring of O_4, i.e. with eigenvalues $C_2 = K(K+1) + L(L+1)$. Because of Schur's lemma both C_1 & C_2 characterise an irreducible repn. It follows that this repn may be labelled by the pair (L,K) with $K = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$ $L = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots$. The unitary repns will have the form $D^{(KL)}(\vec{\alpha}, \vec{\beta}) = D^K(\vec{\alpha})\, D^L(\vec{\beta})$

where the D^K are the unitary repns of the rotation group defined previously ||. Hence it follows that the Universal covering group of O_4 is $SU_2 \otimes SU_2$.

(2) Hom. L.G. 1 — omit

(3) The de Sitter group. is the group of motion admitted by the cosmological space with line element

$$-ds^2 = \phi^2 dx_\mu\, dx_\mu \quad (x_4 = ix_0) \text{ where}$$

$$\phi = \left(1 + \frac{r^2 - x_0^2}{4R^2}\right) \quad (r^2 = (x^1)^2 + \cdots + (x^3)^2$$

Considering case of +ve curvature — the de Sitter is isomorphic to the 10-parameter group of motion of a space-time with constant curvature, the maximum continuous

in Hilbert space – Construction of Hilbert space for SU_2 – Reps^{ns} of the Lie rings for SU_2 –

SU_2 is the universal covering group of O_3 (rotation group in 3 dimns)]

[Examples of group spaces – (i) R the additive group of real nos (1-dimnl translations) the group space is an ∞ str: line & R is non-compact & simply connected

(ii) O_2 : its group space is a circle. A closed curve that goes k times around the circle cannot be deformed into one that goes l times round it. Thus O_2 is compact & ∞ly connected

(iii) O_3 : parametrised by vectors having the diren of the rotation axis & the length of the rotation angle : diametrical opp. points on surface of a sphere have to be identified. It can be shown that O_3 is doubly connected

(iv) SU_2 : the group space of SU_2 stands to O_3 the way a Riemann surface with two sheets stands to a cut plane. It can be shown that SU_2 is compact & simply connected

(v) group of n-dimnl translations : group-space is n-dimnl Euclidean space & is simply connected.

for $n \geq 2$, the n-dimensional sphere is also simply connected]

of the symmetry properties of a particular group manifold viz the the underlying space of SU_2

The group O_n^+ ie the part of the n-dimnl orthogonal group connected with the identity.

Regarded as a transformation group on a real n-dimnl vector space, O_n^+ consists of all $n \times n$ ~~matrices~~ orthogonal matrices with unit determinant

$$\Omega \Omega^T = \Omega^T \Omega = 1 \qquad (1)$$

& Ω can be expressed by means of an anti-sym. matrix in the following way

$$\Omega = e^A \quad (A = -A^T) \qquad (2)$$

Hence the group has $\frac{1}{2} n(n-1)$ parameters. For an infinitesimal transformation, the matrix $A = \delta A$ is small & we have $x + \delta x \simeq x(1 + \delta A)$

where x stands for a row vector in the n-dimnl rep^{n} space. We can rewrite the last relation in the form $dx_i = x_j \delta A_{ji}$, and the generators of the infinitesimal operators are

$$i J_{ij} = -i J_{ji} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \qquad (3)$$

It is sufficient to use this special rep^{n} to derive the following commutation relations which could also be derived directly from the root diagram of the Lie ring

$$\left[J_{ab}, J_{rs} \right] = i \left(\delta_{ar} J_{bs} + \delta_{bs} J_{ar} - \delta_{as} J_{br} - \delta_{br} J_{as} \right) \qquad (4)$$

Where the indices go from 1 to n & the J_{ab} are antisymmetric

[Lie groups – General properties – Topological groups. Lie groups. A topological group which is a manifold is a Lie group – Lie group as a transformation group –

(B) Gravitation as geometry II – by J. A. Wheeler.

(1) Allowable and an arbitrary geometry – Cf. Kinematics & dynamics

(2) Physics as eqns of motion + initial value data – will physics energize the latter & also the former or should one go back to observation to get the latter, no answer till now + true of all branches top. Newtonian mechanics, electrodynamics & geometrodynamics (or gen. rel)

(3) Initial value data for electrodynamics – Possible – two steps (i) Pick out a space-like hypersurface slicing through space-time ie analogous to selection of a specific initial time in mechanics (ii) on this hypersurface give explicitly the value of $\vec{H}$ (analogue of coordinate in mechanics), and of $\vec{E}$ (analogue of momentum in mechanics). Check that these data are acceptable ie test whether div $H = 0$ & div $E = 0$ in empty space. Then the initial-value data are said to satisfy initial value constr. They are enough to fix the entire future and past of the e.m.f

(4) Intrinsic 3-geometry & extrinsic curvature of a space-like hypersurface

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(1) T. J. Willmore – Intro to Diff. Geometry, Oxford X

(2) P. Alexandroff – Elementary concepts of topology, Dover, N.Y. 1961 X

(3) Re. Groups in G.R.

Relativity, Groups & Topology – Witt & Witt

26/2/69 Three famous tests of GR – gravitational redshift, gravitational deflection of light, Perihelion rotation of Mercury.

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p. 147 – Some semi-simple Groups associated with G.R

Deals with (1) Repn of the 4-dimnl rotation group O_4 ie the group of motion of a 3-dimnl space with constant +ve curvature (static Einstein space) ie O_4 describes symmetries of a static Einstein space in G.R; secondly O_4 is the group of motion of the group space of SU_2. Hence the study of U_4 permits the understanding

[SU_2 = unimodular unitary group in 2 dimns ie regarded as a transf group in 2-dimnl complex space linear space, SU_2 corresponds to the transformation

$$\psi' = U\psi \quad \text{where } \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \ \psi' = \begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix}$$

are complex vectors of the space and the 2×2 matrix U has the properties

$$\text{Det } U = 1, \quad U U^+ = 1$$

U can be put in the canonical form $U(\vec{n}, \theta) = e^{i(\theta/2)\vec{\sigma}\cdot\vec{n}} \quad (\vec{n}\cdot n) = 1$

depending on 3 parameters – Another form for U – ∞ reducible unitary repn $2SU_2$

(4) Additivity of distance - extremal or 'geodesic' route obtained by calculation - Simplification No. 1.

(5) Cataloguing of points simplified by four families of coordinate surfaces - " No. 2

(6) Continuity & independence the only requirements; no preferred coord. system

(7) Coords as cataloging numbers

(8) dimension 4 $\longrightarrow ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$. $(g_{\alpha\beta} = g_{\beta\alpha})$ (ie $\rightarrow$ 10 quantities $g_{\alpha\beta}$ (metric coefficients or gr. potentials)

Summary in shorthand of all the symmetry of space-time

(Whole thing childish! Good for a popular lecture only!)

Appendix ~~for establishing~~ criterion for establishing that an ensemble of $(n+1)$ points defines an n-dimensional euclidean simplex

Do the $\frac{1}{2}n(n+1)$ distances between $(n+1)$ points define an n-dimensional simplex in an Euclidean space? Then the volume V of this figure will be non-zero. Denote points by $0, 1, 2, \ldots n$ (let 0 be origin) & let coords of Kth point be K_s $(s = 1, 2, \ldots n)$. Area $\Delta = \frac{1}{2}\begin{vmatrix} 1_1 & 1_2 \\ 2_1 & 2_2 \end{vmatrix}$. Vol of tetrahedron $= (1/6)\begin{vmatrix} 1_1 & 1_2 & 1_3 \\ 2_1 & 2_2 & 2_3 \\ 3_1 & 3_2 & 3_3 \end{vmatrix}$

& vol of n-simplex is

$$V = \frac{1}{n!}\begin{vmatrix} 1_1 & \cdot & \cdot \\ \cdot & - & \\ \cdot & - & n_n \end{vmatrix}$$

Rewrite eqn transposing rows & columns & multiplying the two determinants. The element in i^{th} row & k^{th} col. of the product = scalar product of vector $(0i)$ by vector $(0K)$

$$(0i)\cdot(0K) = \sum_{s=1}^{n} i_s K_s$$

In flat space this scalar product is given by the familiar formula for a Δ^{le}

$$(0i)(0K) = \tfrac{1}{2}\left[(iK)^2 - (0i)^2 - (0K)^2\right]$$

Therefore only distances are involved in standard formula for vol

$$V = \left(\frac{1}{n!}\right)\begin{vmatrix} (01)^2 & (01)(02) & \cdot \cdot \\ (01)(02) & (02)^2 & - \; - \\ \cdot \; \cdot & - \; - & (0n)^2 \end{vmatrix}^{1/2}$$

Its vanishing for $n \geq 5$ or more provides (1) a way to check in the case of nearby events that dimensionality is actually four (2) a way to determine $ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta$ & (3) to show presence of curvature ie if $V_5 \neq 0$ for ~~four~~ 5 or more points not nearby (not that dimensionality is 5 but that curvature is non-zero.

Ref: S. Picard: Sur les ensembles de distances des ensembles de points d'un espace euclidien, Gauthier Villars, Paris, 1940 for an analysis of the conditions that an ensemble of real nos. shall be the ensemble of distances between an ensemble of points in a Euclidean space.

attacks the cosmological principle as an unwarrantable assumption incapable of being tested and incompatible with the comparitive method; cosmological principle is similar to the cardinal principle of exp. science viz the indefinite repeatability of expts; questioning validity of cosmological principle ~~by not going to other~~ because one cannot go to other regions of space is equivalent to asking "Is the H-atom same everywhere or e same everywhere in the universe?"; the CP is one of the oldest if not the oldest of all the principles now in use.

Ch. II. Early cosmological history – Utter rubbish – No mention of Hindu mythology.

Ch. III. The dark ages – on p. 21 only a reference to ~~Islamism~~ Koran & Islamism – obviously the author is ~~unaware~~ unaware of ancient Indian Civilisation or Chinese civilisation.

Ch. IV – Age of Newton. – Galileo's expt. of letting down the two balls of different weights roll down an inclined plane (same in principle as falling down vertically) since accurate clocks were not available at that time (17th century)

Ch. V – Scale of things in earlier times –

Ch VI – The paradox – Olber's paradox is "The sky is dark at night"

Ch. 13 – $1/p$ = average density of matter. $p \sim$ electrical forces/grav. forces (between electrons & protons) Dirac suggested that this ratio and p are fns of age of the universe & also $G \propto 1/p$. This implies new matter in the form of protons & electrons must be continually coming into the universe. Dirac was unwilling to accept & hence ~~evaded~~ evaded it by assuming a variation of m_p with time. Jordan accepted Dirac's original result of creation of new matter & brought out a theory of creation of lumps of matter or even galaxies

Ch. 14 – References (i) Sc. Amer. Dec 1963 – Article on quasars by Greenstein

(ii) Hoyle, Burbidge & Sargent – Nature, 209, 1966, ~~751 ff~~ pp. 751 ff

(iii) " " – Nature, 210, 1966, p. 1346.

No. references to work other than British – Stupid remarks at the end

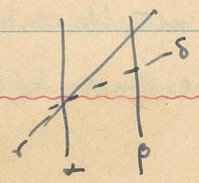
At end points "Olber's paradox" the article by Olbers in original (in German)! and of Cheseaux in French!

24/2/69 – Gravitation & Geometry – Benjamin N. Y, 1964

(A) Gravitation as geometry I, pp. 40–64 by Marzke & J. A. Wheeler

(1) Illustration of geometry of a curved manifold – Eg. of Airline distances between AZ, BE, BO, BA → Geometry of sphere.

(2) Points in space-time defined by events in a coord. ind. way – World lines and light rays



(3) distance between nearby points enough to define the geometry

eg. NY to S.F = NY to Chic + Chic + SF (add)

α, β, γ world lines of test particles
δ = light ray

years of relative calm, the red shift has again become the centre of furious activity.

(2) Scientific American - Sept. 1967 - Review of North's book by Sciama on p. 293 & calls the review as "Cosmology before and after quasars and the cosmic black body radiation" (Good for Annamalai lectures) ... Last sentence "Cosmology has at last become a science" (!) - Most cosmologists who are starved for basic observations have now been reduced to a state of bewildered euphoria - Geometrical period and the astrophysical period - Various "heretical" theories proposed by Eddington, Milne, Dirac & P. Jordan - Separate mention of steady state theory of Bondi, Gold & F. Hoyle, which of all the "heretical theories" is the one that has irritated and excited the most people, has provoked the most good astrophysics and has more or less survived to the present day. - Presence of cosmic radiation would go against SST since this proposes a ρ_0 independent of t. ~~other~~ - Also Ryle's interpretation that radio sources exhibit intrinsic evolution goes against SST, for such evolution could only be expected in a universe to which one can attach the conception of an age (no such conception in SST) - Hoyle's (& others) suggestion that quasars are in nearby galaxies goes against observation

Sci. America, October 68 - Article on 'pulsars' (White dwarfs or neutron stars).

Ibid, June 68 - p. 28 - The primeval fireball by Peebles & Wilkinson - About cosmic radio mentioned by Sciama above - Cosmological considerations (useful for Annamalai lectures?)

~~Pg~~ (21/2/69) - Popular book on Cosmology - "The Bowl of ~~my~~ night" by F. P. Dickson

Ch. I. Cosmology today may be defined as the study of the structure and history of the universe in terms of its large scale features. It is now a part of physics and no longer involves cosmogony which attempts to explain why & how the universe came into existence (& ?) - Wave length of green light is about $\frac{1}{2}$ a micron, this the millionth part of the length of a man's forearm - Always the possibility that some observation as yet unknown may go against a theory. ~~This problem is by no means unique to science~~; ~~there is~~ This is like the well known difficulty of proving a negative; one has to prove that there exists no other possibility. This problem is by no means unique to science: there is usual no legal difficulty in establishing who is the mother of a child, but he would be a rash man who would undertake to prove absolutely who is the father - Epistemology ~~is~~ whether we can gain genuine knowledge of reality - Cosmological principle; Whitrow's (Structure & evolution of universe, Hutchinson, 1961, p. 60) book gives the version "a sphere with its centre everywhere and its circumference nowhere"; H. Dingle in Sc. Amer. Sept. 1956, p. 234 -

disturbed by a doubling of all gravitational forces. Thus the acceleration is determined by the mass distribution in the universe, but is independent of the strength of the interactions. Designating the mass of the sun by m_s and its distance by r enables the acceleration to be expressed according to Newton as $a = Gm_s/r^2$ or, from dimensional arguments, in terms of the mass-distribution as $a \sim m_s Rc^2/Mr^2$. Combining the two expressions gives eqn (5) $\left[Gm_s/r^2 \sim m_s Rc^2/Mr^2 \right.$

$\left. \rightarrow GM/Rc^2 \sim 1 \right]$

Brans-Dicke theory of gravitation based on a scalar field in a Riemannian Geometry - Particular case of a static spherically symmetric field about a mass found mass given by

$$ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} \left[dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right] \qquad (6)$$

where α, β are fns of r only. The found soln of this shown to be in consonance with Mach's Principle. The cosmological model derived from this theory has

$$ds^2 = -dt^2 + a^2(t) \left[dr^2/(1-\lambda r^2) + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right] \quad - - (7)$$

with $\lambda = +1$ for a closed space, $\lambda = -1$ for open, & $\lambda = 0$ for a flat space. [Long & detailed discussion of this model on pp. 91-95 with two typos (i) a against t for $\lambda = +1, -1, 0$ -

(ii) ϕ against t also for the three cases. Quite long but interesting]

Appendix 9 - The Earth and Cosmology - The earth may be affected by the distant matter of the universe through a long-range interaction [Long discussion of 20 pages, but the final conclusion is all this is only conjecture, and that what is badly needed to raise the discussion above the level of conjecture is a good demonstration that the gravitational constant has indeed been slowly decreasing]

Dirac cosmology: that G decreases with time & that there exists a connection between the mass content of the universe, its size and G.

Appendix 10 - Implications for cosmology of stellar and galactic evolution rates - Summary of the current picture of stellar & galactic history and discussion of its implications for cosmology and gravitational theory - steady state & evolutionary theories; former inconsistent with general relativity as it violates energy-momentum conservation laws; evolution possible on basis of general relativity; two others considered here viz Dirac cosmology [Proc. Roy. Soc. A 165, 199, 1938] and the Brans-Dicke cosmology [Phys. Rev. 124, 925 (1961)]. This latter as well as SST were inspired by Mach's Principle

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17/2/69 at Jodrell - Science Journal - Nov. 1966, p. 52 - Article on redshift by D.V. Sciama.

(1) Questioning application of redshift to quasars - being too far away or even outside the whole galactic system - {or within our own galaxy - failure of Hubble's law? - After 40 or in nearby galaxy NGC 5128 (suggestion of Hoyle & Burbridge)}

$$\frac{1}{c^2}ds^2 = -\frac{a^2}{(1+kr^2)^2}\{(dx^1)^2 + \dots + (dx^3)^2] + dt^2 \qquad [R-W \text{ line element?} \quad \dots (1)$$

t is called cosmic time. If the tensor g_{ij} in (13) is substituted into Einstein's eqn (without λ-term)

$$R_{ij} - \frac{1}{2}g_{ij}R = -\frac{8\pi G}{c^4}T_{ij}$$, one gets the eqn

$$\left(\frac{da}{dt}\right)^2 + k = \frac{8\pi}{3}G\rho a^2 \quad \dots (2)$$

for the expansion parameter a. Here ρ refers to average matter density of the universe & k takes on the values $k = +1, 0$ and -1 for closed, flat & hyperbolic universes. The Hubble expansion age is given by

$$T_h = a\Big/\left(\frac{da}{dt}\right) \qquad (3)$$

Another parameter of importance, because it can be measured, albeit very imperfectly, is the acceleration parameter

$$q_0 = -a\frac{d^2a}{dt^2}\Big/\left(\frac{da}{dt}\right)^2 = \frac{4\pi}{3}G T_h^2(\rho + 3P/c^2) \quad \dots (4)$$

where P is the pressure. Assuming $T_h \sim 1.3 \times 10^{10}$ years (and P negligible) & q_0 is ~ 1.0, we get $\rho \sim 2 \times 10^{-29}$ gm/cm^3. Observed density of visible matter is $\sim 10^{-31}$ gm/cm^3. This vast discrepancy is interpreted by many cosmologists to mean that space contains matter outside the galaxies, perhaps ionised, perhaps neutral but at very low temperature. It has also been suggested that the required energy density may be provided by a vast swarm of almost unobservable neutrinos – Steady state cosmology: A particularly satisfying aspect of this type is that it provides a constant distribution of matter and permits Mach's Principle to be satisfied with a constant G.

Appendix 7. Mach's Principle and a relativistic theory of gravitation.

A principle as sweeping as that of Mach, having its origins in matters of philosophy, can be described in the absence of a theory in a qualitative way only. A model of a theory incorporating elements of Mach's Principle has been given by Sciama [M.N.R.A.S., 113, 34 (1953)]. From simple ~~and~~ dimensional arguments, as well as the discussion of Sciama, it has appeared that, with the assumption of the validity ~~of the~~ of Mach's Principle, the gravitational constant G is related to the mass distribution in a expanding universe in the following way

$$GM/Rc^2 \sim 1 \quad \dots (5)$$

Here M stands for the finite mass of the visible (ie causally related) universe, and R stands for the radius of the boundary of the visible universe. As follows:-

Mach's Principle according to Sciama means a gravitational force acting on a fixed laboratory due to the presence of distant accelerated matter. Consider a test body falling towards the sun. In a coordinate system so chosen that the object is not accelerating, the gravitational pull of the sun may be considered as balanced by another gravitational pull, the inertial reaction [Dicke, ~~Amer. Scientist~~ Am. Scientist, 47, 25 (1959)]. Note that the balance is not

there are more variables in the theories than can be determined from the observations, and you have to try to fit a plausible situation to all the evidence you have. The most promising methods are based on (i) the integrated sky brightness, (ii) measurement of the density of intergalactic hydrogen, (iii) Radio source counts, (iv) the angular diameter of radio sources

I think myself that the present results show so large a departure from those predicted by the SSM that we must recognise the plausibility of the EM. The alternative seems to regard the results as arising from a "local" irregularity as proposed by Hoyle, F and Narlikar J.V. (1961), M.N.R.A.S, 123, 133, but this is also difficult to accept since such an irregularity would have to extend to distances of about 350 Mpc, a distance which is not scarcely local. — Some discussion on Ryle's paper

(d) Angular diameters and luminosity function of extragalactic radio sources by H.P. Palmer

Gives curves in Fig 20/1 (p.231) & in Fig 20/2 and histograms in Fig 20/3 re. distn of absolute luminosities P inferred for sources of flux density $S \geq 12$ f.u and $b' \geq 12°$ (b' = galactic latitude) and also plotted for $Z \leq 1.25$ and $Z \geq 1.25$ corresponding to SSM and Einstein-de-Sitter model, and supports the latter. bf

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(14/2/69) — R.H. Dicke's Book on Relativity.

Appendix 5: Cosmology, Mach's Principle & Relativity.

In cosmology philosophical considerations still play a crucial if not a dominant role, because of very little observational basis — Conclusions about isotropy — Other more uncertain observations; Virtually nothing known about intergalactic matter; Hubble's expansion age of Universe is believed to be fairly reliable, but the fact that it has doubled twice in a decade shakes one's confidence in this number — Cosmological significance of Mach's Principle is due to the dominant role assigned by Mach to the distant matter in the universe. According to him, this influence of distant matter penetrates the shielded walls of the laboratory, affecting the physicists every experiment i.e. the idea of fixed frames of reference that the laboratory is tied to and influenced by matter in the remote parts of the universe. Thus the physicist could ill afford to ignore Cosmology — Cosmic fields: Long range tensor fields associated with chargeless zero-mass particles with spin 2, can be developed so as to provide, through the acceleration dependent force which it generates, the local inertial effects of distant matter in the universe — Cosmic fields and Cosmology — Long range scalar cosmic fields can also be constructed & the two long range tensor & scalar fields determine the gross evolutionary history of the Universe — If the assumption of isotropy & uniformity is made, Kinematic considerations give the metric tensor

of the Universe as in EC, (ii) SSC has the philosphically agreeable feature of yielding an effectively unique model instead of the infinitely many as in EC, (iii) Predictions of SSC have not been contradicted by observation (? Sandage ?), (iv) SSC has the disadvantage of not resting on any "field theory"

(i) other possibilities - (i) models using cosmical term λ (ii) Milnes Kinematic relativity offers a different approach, but the model to which it leads is included among those of relativistic cosmology. (iii) ~~Possibility of Cosmicills~~

(j) critical comments - There is at any rate one thing about which we are almost certain that no prediction can be made, and that is the future development of cosmological theory.

Ibid, p. 222 - Cosmology and radio astronomy. - A discussion

(a) Aspects of cosmology by G.C. McVittie - If indeed all this work of Hubble's which was the chief basis for our belief that the distribution of the galaxies is uniform, is unreliable (mentions in support the number/intensity counts in cosmology & observations on this by Prof. Ryle), I am puzzled to know why we continue to accept with such faith the notion of uniformity - In the discussion G.C.M. suggested new number/astr. may count of galaxies by telescope beyond Earth's atmosphere & said that greatest difficulty was in finding an enthusiastic band of young astronomers and a suitable telescope even on earth

(b) Observational tests in cosmology by W.H. McCrea - Smoothed out universe assumption a convenient simple one, but not an exact one - Tests for all theories are difficult and round-about - Difference between the predictions of spatial density on the EM and SSM theories is a factor of $(1+z)$ ($z = d\lambda/\lambda$ = measure of red shift), so that observations must be accurate to the order of z - The papers of Hoyle, A.W.K. & Morrison, P. in 1959 [M.N.R.A.S. 119, 657, (1959)] who applied information theory to cosmology and showed that if you take unity to represent the flow of information received from nearby parts of the universe, then the theory gives you z as the flow of information you get from, so to speak, the horizon. In fact, the flow of information does contain the factor $(1+z)^{-1}$. Thus we seem to be faced with the paradoxical situation that the information available from the universe falls off in exactly the ratio $(1+z)$ that represents the ratio of quantities predicted by the various cosmological theories. This may mean that a decisive test is not possible even in principle (?)

(c) Radio-astronomical tests of cosmological tests by M. Ryle. - At present it turns out that

$4\pi\rho q^3/T$. Hence new matter has to be created inside the sphere at the rate $3\rho/T$ per unit vol. per unit time. This must be the universal rate of creation in the model.

Let $f(t)\rho$ = density of material of age $\geq t$. Consider a sphere of radius q whose surface is moving with the medium. In time δt, the vol. changes from V to $V+\delta V$ say, and the material that was of age t becomes of age $t+\delta t$. So we have

$$f(t+\delta t)(V+\delta V)\rho = f(t)V\rho$$

giving $f'/f + V'/V = 0$

But $V = \frac{4}{3}\pi q^3$, $V' = 4\pi q^3/T$ and so, we obtain

$$f'/f + 3/T = 0 \qquad \text{---} \quad (24)$$

Hence $f(t) = e^{-3t/T}$ since we must have $f(0) = 1$. From this it follows that the mean age of the matter in any neighbourhood is

$$\frac{3}{T}\int_0^\infty t e^{-3t/T} dt = \frac{1}{3}T \qquad \text{---} \quad (25)$$

This is also the mean age of systems belonging to any general category in the model. These properties are all "local" so that they are the same whether we use classical or relativity theory. If we use (23) in (15) and put $K=0$, we have the space time

$$ds^2 = c^2dt^2 - e^{2t/T}(dx^2+dy^2+dz^2) \qquad \text{---} \quad (26)$$

Now changing t to $t+t_0$, this metric is unchanged if we also change x to $e^{-t_0/T}x$, etc. Thus the geometry is unchanged by a change in the origin of t. So we verify that (26) is appropriate for the description of a steady state universe. We should not have for this result with $K \neq 0$.

If we attempt Newtonian gravitational eqns or the field-relations of GRT, again taking $p=0$, we find that the motion is not according to these laws. In fact, the material is accelerated away from O with acceleration $\vec{q}/T^2$ and so it would be interpreted as showing gravitational repulsion rather than attraction. Many attempts have been made to modify this, but a satisfactory approach has not yet been discovered.

(h) Physical discussion (i) steady state model (SSM) avoids the objection to evolutionary (EM) models in which nothing at all is older than some definite age $< T$, (ii) no singularity in SSM as in EM (iii) observable differences between the two models exist for eg. in EM there is an age-distance effect but not in SSM. In practice no satisfactory observational tests exist (since we do not know how individual galaxies change in appearance with age) although strenuous efforts have been made.

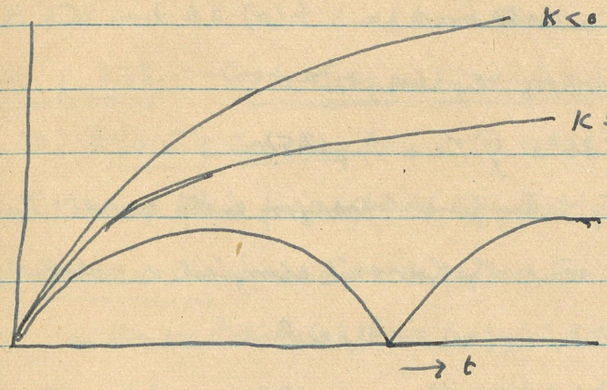
To sum up (i) SSC possesses all the satisfactory features of (previously noted) for Newtonian or relativistic cosmologies, while it avoids the unnecessary feature associated with a definite age

explains the phenomenon & gives the same eqns! But there are differences in the two treatments viz (i) Boundary conds (not disposed off in classical theory), (ii) geometry effected by matter in RT but not in CT, (iii) neglect of stress in the model in CT not satisfactory. In general Rel. Cosmology is a more satisfying treatment of the same phenomena physical phenomena as those treated in Newtonian cosmology.

(f) Evolutionary cosmology. We can write (19) in the form

$$R'^2 = \frac{C}{R} - Kc^2 \qquad (19')$$

where $C = \frac{8}{3}\pi G\rho_0 > 0$ if $\rho_0 > 0$. We are only interested in $R \geq 0$. Without solving (19') explicitly (in elliptic functions), we can see that the solutions are of the form shown in the adjoining figure. As already stated, we do not wish to consider the case $K > 0$. Then we are left with the evolutionary models & we see that these start in a singular state; this is the 'big bang' of such models. For definiteness take the case $K = 0$, then we find from (19') that $R(t) = \text{Const.}\, t^{2/3}$



giving for this case

$$\frac{R'}{R} = \frac{2}{3t} = \frac{1}{T} \quad \text{or} \quad t = \frac{2}{3}T$$

where the definition in (11) is used. This gives the time from the big bang to the epoch of observation or what we call the age of the Universe. Also we can show that for all these evolutionary models the age is between $\frac{2}{3}T$ and T. If we use $T = 10^{10}$ years, this gives the age to be between 7 and 10×10^9 years. This appears too short to accommodate what we know about stellar evolution. Also there is no known feature of the actual universe that gives an indication of its ever having been in a state of extreme congestion as required by the models.

(g) Steady-state Cosmology. For reasons such as those stated above, Bondi & Gold and Hoyle proposed in 1948 a system of steady-state cosmology. There were also more philosophical reasons. For instance, the concept of a changing universe behaving in accordance with unchanging laws is hard to accept because we naturally think of the Universe and its laws being interdependent. While we cannot say that the universe must be in a steady state, it is nevertheless of interest to examine the possibility.

Postulate: All characteristics of the model are independent of the epoch (PCP?)

Writing as before $R'/R = 1/T$, then T is a constant independent of t. Hence

$$R = e^{t/T} \quad \ldots \quad (23)$$ (nothing gained by introducing a const. of integn)

Again we must have a constant value of ρ. Considering a sphere about O of any given radius q, the matter flows across its surface with speed q/T and so total rate of outflow is

$$\frac{1}{2}\left(\frac{dq}{dt}\right)^2 = \frac{GM}{q} - \frac{1}{2}kc^2r^2 \quad \text{--- (12)}$$

$$G\rho T^2 = \frac{3}{8\pi}\left(1 + kc^2/R'^2\right) \quad \text{-- (13)}$$

(12) allows us to obtain from (13) [using $k \leq 0$, excluding $k > 0$ leading to periodicity]

$$G\rho T^2 \leq \frac{3}{8}\pi \quad \text{-- (14)}$$

From this it is easy to show that the actual universe conforms to the CP with $\rho \neq 0$. The empirical value of T got in this way is $\cong T = 10^{10}$ years & then (14) gives $\rho \leq 2 \times 10^{-29}$ gm/cm^3. Thus the model is satisfactory as not being static & explaining the expansion effect & also CP. — Eqn (14) can alternatively be considered as ~~giving~~ determining G given ρ and T (Dirac's cosmology?)

(e) Relativistic cosmology — We start with the same general notions about FOs. Using the basic notions of the ~~age~~ aggregate of events being points in a ~~Rem~~ R_4 & world lines of light signals are null-geodesics, we find only space-times conforming to these notions are of the form

$$ds^2 = c^2dt^2 - R^2(t)\left(dx^2 + dy^2 + dz^2\right)/\left(1 + \tfrac{1}{4}kr^2\right)^2 \quad \text{--- (15)}$$

where $r^2 = x^2 + y^2 + z^2$, these being the Robertson–Walker metrics.

The FOs have the world lines $x = \text{const}$, $y = \text{const}$, $z = \text{const}$. Let O be the one with $x = y = z = 0$. Then the 'ordinary' distance of any other from O is

$$q = r\,R(t) \quad \text{-- (16)}$$

where r is fixed for the other FO.

If we now appeal to Einstein's field eqns, they tell us that the density ρ and pressure p in the universe (15) are given by

$$8\pi G\rho = \frac{3\left(R'^2 + kc^2\right)}{R^2} \quad \text{-- (17)}$$

$$8\pi \frac{Gp}{c} = -\frac{2RR'' + R'^2 + kc^2}{R^2} \quad \text{-- (18)}$$

Now putting $p = 0$ ~~as in (d)~~

$$2RR'' + R'^2 + kc^2 = 0$$

which has the first integral $R\left(R'^2 + kc^2\right) = C$ -- (19) (C = int. const)

& from (17), $\frac{8}{3}\pi G\rho = C/R^3$ -- (20)

Writing $C = \frac{8}{3}\pi G\rho_0$, where ρ_0 is a constant, then (19), (20) become

$$R\left(R'^2 + kc^2\right) = \frac{8}{3}\pi G\rho_0 \quad \text{-- (21)}$$

$$\rho = \rho_0/R^3 \quad \text{-- (22)}$$

Eqns (16), (21) & (22) are seen to be the same as (1), (10) and (4)

~~Expansion of universe~~ ~~has~~ Rehearsing curious history of modern cosmology. Expansion of universe predicted by GRT. When Hubble discovered this as an empirical law property (the most extensive) of the actual universe, it was naturally hailed as a triumph for relativity theory. Later, as shown above, it was shown that classical theory also

(c) Cosmological theories : Newtonian Cosmology.

F a newtonian frame with origin O and t newtonian time reckoned from arbitrary zero. Matter in the form of an ideal fluid of negligible pressure & amount finite. $\vec{q}$ vector relative to O of any particle of the matter. Consider motion of matter such that

$$\vec{q} = \vec{r}\, R(t) \quad \text{---} \quad (1)$$

$R(t)$ same for all particles & $\vec{r}$ a fixed vector, characteristic of particle considered. Then

$$\frac{d\vec{q}}{dt} = \frac{R'}{R}\vec{q}, \quad \frac{d^2\vec{q}}{dt^2} = \frac{R''}{R}\vec{q}, \text{ etc.} \quad \text{---} \quad (2)$$

ie speed at any instant is $\propto$ to distance from O.

Let at some epoch $t = t_0$, let $\rho = \rho_0$

$$\left.\begin{aligned} \text{ie } \rho &= \rho_0 \quad (r \le r_0) \\ &= 0 \quad (r > r_0) \end{aligned}\right\} \quad (3)$$

where $r = |\vec{r}|$ and r_0 & ρ_0 are const. Conservation of matter is satisfied if & only if at epoch t [from (2) & (3)]

$$\left.\begin{aligned} \rho &= \rho_0 / R^3 \quad (q \le r_0 R) \\ &= 0 \quad (q > r_0 R) \end{aligned}\right\} \quad (4)$$

where $R_0 \equiv R(t_0) = 1$ for convenience $\quad \text{---} \quad (5)$

Mass $M(q)$ within distance q of O at epoch t is

$$M \equiv M(q) = \frac{4}{3}\pi\rho\, q^3 = \frac{4}{3}\pi\rho_0 r^3 \quad (q \le r_0 R) \quad (6)$$

From sph. symmetry, we have according to Newton's law of gravitation, the attraction at $\vec{q}$ is

$$-GM\vec{q}/q^3 = -\frac{4}{3}\pi G\rho\vec{q} \quad (G = \text{gr. const}) \quad \text{---} \quad (7)$$

Hence pressure being neglected, the eqn of motion of a particle of the fluid is

$$\frac{d^2\vec{q}}{dt^2} = -\frac{4}{3}\pi G\rho\vec{q} \quad \text{---} \quad (8)$$

Using (2) & (4), this becomes $R^2R'' = -\frac{4}{3}\pi G\rho_0 \quad \text{---} \quad (9)$

This has a first integral

$$R(R'^2 + Kc^2) = \frac{8}{3}\pi G\rho_0 \quad (Kc^2 = \text{const. of integration}) \quad \text{---} \quad (10)$$

It can be shown using (9) and (10) that the ~~Cosmological principle~~ spherical symmetry holds in Newtonian cosmology.

(d) Model universes & actual universes. Using the system as a model universe, (2) gives $R'/R = 1/T$ and T may be called the Hubble time for the model, which is in general a fn of t. The model is allowable if $R(t)$ satisfies (9). From this eqn we see that if $\rho \neq 0$, $R'' \neq 0$ and hence $R' \neq 0$; thus there is no static non-empty model & further no non-empty model for which T is independent of t. Using (2) and the eqn $\frac{R'}{R} = \frac{1}{T}$ ---(11), (10) may be written in two instructive forms

Gives also a reference to work of A. Raychaudhuri on inhomogeneous models & states an important theorem due to him & of his equation (Ref. 1955, Phys. Rev. 98, 1123).

10/2/69 – Proc. Eastern Th. Phys. Conference – Univ of Virginia, 1962 (edited by M. E. Rose)
– An after dinner talk by J. A. Wheeler, p. 441 – Quite interesting.

Radio Astronomy today – Jodrell Bank Summer School 1962 ~~symposium~~ (Summer School)
p. 206: Cosmological theories – A survey by W. H. McCrea

(a) Cosmology – study of the astronomical universe in the large. Physicist may be interested in properties say of a star, and ~~a~~ particle or interior of a star, but he is not interested in how much & how many of them exist. This is precisely what is of interest to the cosmologist. His concern is the universe as it actually exists. Purpose of cosmological theory is to infer the past state and to predict the future state of the universe in the large. I.e. are the various large-scale characteristics of the universe, the rate of apparent expansion, the mean density of material and so on, connected together by knowable relations? Or again, are these characteristics connected by knowable relations with the basic properties of matter, i.e. properties of the "elementary particles" of physics?

(b) Cosmological models – Construction of a theoretical model universe; characteristics common to all models (i) they relate to a smoothed-out universe i.e. no local irregularities are considered & does not deal with an individual galaxy, (ii) satisfying the cosmological principle (CP) i.e. postulation of the existence of a triple infinity of fundamental observers (FOs) such that no two ever meet & and experience of the rest of the universe got by one can be superimposed upon the experience of the rest of the universe got by any other; i.e. postulating universe being homogeneous (iii) the universe has spherical symmetry about some FO i.e. sph. symmetry about any FO as follows from (ii) i.e. postulating of universe being isotropic. From these postulates, it follows that a FO is moving with the material in his neighbourhood. A model in which two world-pictures as seen by any one FO are not all the same, and which are not periodic, will be called evolutionary. Periodic models do not appear to be of much interest (Mc Crea's personal opinion as in 1962) & we shall exclude them. If any two world pictures seen by some FO are ~~the same~~ identical, then all world pictures seen by all FOs. FOs are the same. This is the steady state model and is said to satisfy the perfect cosmological principle (PCP). Evolutionary models admit the existence of cosmic time while a steady-state model does not admit the existence of a unique cosmic time. Hence it is misleading to consider the steady-state model as a particular case of models in general. (d)

that the E-world is the direct topological product of a circle and a 3-dimensional space-form of a space with const. +ve curvature. Enumeration of all such possible space-time forms for the E-universe has not yet been given

(2) The de Sitter model - is an empty space-time with -ve const. curvature (λ is +ve & signature of the metric = -2) - 3 coordinate representations of the model, one of which is

$$ds^2 = dt^2 - e^{2\alpha t}(dx^2+dy^2+dz^2)$$

This has for each co-moving observer a stationary aspect since the group of trans

$$t' = t+\beta,\quad x' = e^{-\alpha\beta}x,\quad y' = e^{-\alpha\beta}y,\quad z' = e^{-\alpha\beta}z$$

transforms the world-model into itself. The model received a new interest in 1948 & 49 as the world model of the "steady state theory"

(3) Friedmann model - If the scalar of expansion vanishes → Einstein universe model; if density of matter vanishes → de Sitter model. The line element of Friedmann model must have (since the hypersurfaces $t = const$ are spaces of constant curvature) the form

$$\left.\begin{aligned} ds^2 &= dt^2 - R^2(t)\,d\sigma^2,\quad t = x^0,\quad T_{\mu\nu} = \rho(t)\,\delta^0_\mu\delta^0_\nu \\ \frac{4\pi}{3}\rho R^3 &= M,\quad M = const > 0, \text{ and} \end{aligned}\right\} \quad -(1)$$

$d\sigma^2$ = line element of a space of constant curvature $\epsilon = +1, 0, -1$ independent of time, ρ = density of matter. The world lines of matter are geodesics with constant space coordinates. The function $R(t)$ has to fulfil the Friedmann eq^n

$$\dot{R}^2 = \frac{\kappa M}{4\pi R} + \frac{1}{3}\lambda R^2 - \epsilon \qquad -\cdot- \qquad (2)$$

Each soln with a variable value of $R(t)$ gives a Friedmann model. For constant R one has besides (2), the condition $\lambda = \kappa\rho/2$. General soln of (2) can be obtained by means of elliptic functions. For vanishing λ & for cases $\epsilon = 0, -1, +1$, soln of the Friedmann eq^n take the forms:

$$\left.\begin{aligned} &\epsilon = 0,\quad R = \left(\frac{\rho\kappa M}{16\pi}\right)^{1/3} t^{2/3} \\ &\epsilon = -1,\quad R = \frac{\kappa M}{8\pi}(\cosh\tau - 1);\quad t = \pm\frac{\kappa M}{8\pi}(\sinh\tau - \tau) \\ &\epsilon = +1,\quad R = \frac{\kappa M}{8\pi}(1-\cos\tau);\quad t = \pm\frac{\kappa M}{8\pi}(\tau - \sin\tau) \\ &\tau = \int \frac{dt}{R(t)} \end{aligned}\right\} \quad (3)$$

All spaces $t = const.$ being of +ve curvature are closed. If $\epsilon = 0$, there exist 18 different types of space forms for 3-dimensional Euclidean space. Among them are closed space forms, for eg. the "hypertorus space" which can be constructed from the Euclidean space with Cartesian coordinates by means of the following group of motions:

$$x'^1 = x^1 + l\cdot A,\quad x'^2 = x^2 + m\cdot B,\quad x'^3 = x^3 + n\cdot C,\quad A, B, C = const \neq 0, \qquad (4)$$

and l, m, n are integers. If points are identified which are equivalent under (4), one obtains the hypertorus surface.

metric at the point at which surface B^r crosses - over".

pp. 438 - 453 - Article on relativistic cosmology by Heckmann and Schucking

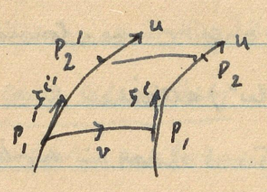
(1) The Einstein model. - Is a static space-time. This means that there exists a time-like hypersurface orthogonal Killing vector. The Einstein world contains uniquely defined

[From Synge p. 236 - The necessary and sufficient condition that ξ_i (any vector field) should define a group of motions is $\xi_{i|j} + \xi_{j|i} = 0$. These ten eqns are called Killing's equations, ξ_i the Killing vector, and $\xi_{i|j}$ the Killing tensor (skew-symmetric)

Also $\quad \xi_{i|jk} - \xi_{i|kj} = R_{aijk}\xi^a \quad -(i)$

$\xi_{i|jk} + \xi_{j|ki} + \xi_{k|ij} = 0 \quad -(ii)$

Ibid. p 235. Concept of a group of motions: If for two events P_1, P_1' in space-time with parameters u_1 and u_1', and for events P_2, P_2' with parameters $u_2 = u_1 + \Delta u_1$, $u_2' = u_1' + \Delta u$, the space admits of group of motions if the world function $\Omega(P_2 P_2') = \Omega(P_1 P_1')$



Ibid p. 47. If $P'(x')$ and $P(x)$ be two points of space-time, joined by a geodesic Γ with eqn $x^i = \xi^i(u)$, where u is a special parameter, then the integral

$$\Omega(P'P) = \Omega(x', x) = \frac{1}{2}(u_1 - u_0)\int_{u_0}^{u_1} g_{ij}\, v^i v^j\, du$$

taken along Γ with $v^i = d\xi^i/du$ has a value independent of the parameter chosen. $\Omega(P'P)$ or $\Omega(x'x)$ is called the world function of space-time.]

space-sections through each event of the manifold. These hypersurfaces are homogeneous and isotropic spaces of constant $+^{ve}$ curvature. They are filled with incoherent matter of constant density. The line element and the em-tensor is given by

$$ds^2 = dt^2 - R^2\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

$$R = \text{Const} > 0, \quad 0 \le \chi \le \pi, \quad 0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi, \quad t = x^0$$

$$T_{\mu\nu} = \rho\, \delta^0_\mu \delta^0_\nu, \quad \rho = \text{const} > 0, \qquad \Lambda\ (\lambda) = \frac{K\rho}{2} = \frac{1}{R^2}.$$

The whole space-time and its E-m tensor are invariant w.r.t a multiply transitive seven parameter group of motions. Since the space sections $t = \text{const}$ are closed, and the whole space-time may be regarded as the direct topological product of an open line with a 3-dimal sph. space, the Einstein universe has been called a 4-dimal cylindrical world. F. Klein drew attention to the fact that the space-sections of the E-universe might have the connectedness of an elliptical space. But the above line-element for $t = \text{const.}$ is also compatible with infinitely many topologically different space forms of a 3-dimal space of constant $+^{ve}$ curvature. There are still more possibilities for the topological structure of the E-universe. It may be assumed for eg

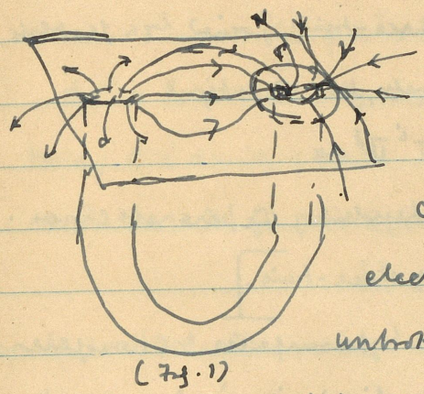
Sc. Amer. Dec. 66 – Quasars – Burbridge & Hoyle. – Discusses also "bing bang" & "Steady state theories" – Talks of uncertainty about quasars being at edge of universe or nearer to us – Also talks of quarks.

(5/2/69) At Institute

Gravitation – An introduction to current reseach by Louis Witten, 1962: –

pp. 412 – 437 – Chap. 10 – Article on geometrodynamics by J. G. Fletcher

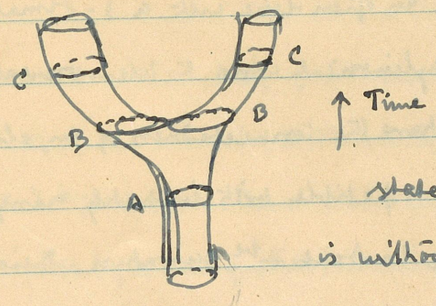
p. 416 – divergence of the electric field at some point within the regions enclosed by closed surfaces is in violation of Maxwell's free field equations. However in proving that the field diverges at some point within the ~~regions~~ surfaces, it is assumed that the topology within the surface is Euclidean. It is not necessary to make this assumption. There exist many topologies such that the surfaces, though closed, do not enclose any volume. Consider, for example, a handle (or in more picturesque language, a "wormhole") attached to a 3-space of ordinary Euclidean topology [A two-dimensional analogy to this is shown in ~~Fig. 1~~ the Fig. 1.



(Fig. 1)

The electric lines of force are represented by the solid lines with arrows. ~~Through~~ Through the closed line C (which is the analogy of the closed surface, in the actual 3-dimil case) there passes a non-vanishing electric flux. yet the electric field does not diverge; the lines of force continue unbroken down the wormhole, and emerge at its other end. Thus, the region of a wormhole mouth constitutes what is interpreted as electric charge. – In geometrodynamics, the topology of space-time is left unrestricted.

p. 428 – What is the nature of the singularity at the moment of topology change [for eg. the case of the wormhole breaking in the Schwarzschild soln for $x^0 = 4m$]. Strangely enough, there is no topological singularity. As an analogy, consider a universe with one space dimension and one time dimension. The universe is the two-dimensional surface of a Y-tube as shown in Fig 2, where the time is plotted vertically with the future towards the top of the page. The one-dimensional space-like surfaces change from a single circle at A to two circles at C, passing through a singular state at B. Yet the two-dimensional space-time continuum is without topological singularity. It is only the "slicing out" process of space-like surfaces which is singular. However there is a singularity in the



After a finite time, an observer stationed at the throat will reach the singularity $r = 0$. At this time the wormhole will pinch off.

By 2.6 for an example of multiply connected spaces i.e. those which contain two or more wormholes of this type.

Sc. American - April 67, p. 106 - Antimatter & Cosmology

If antimatter exists on a large scale, the current theories of cosmology viz "the big bang" theory and the "steady state" theory fall by the wayside. The former assumes that the universe started from an "ylem" or an extremely dense ball consisting only what we know as ordinary matter. If the original nucleus had contained anti-matter as well as matter, it would have annihilated itself, the big bang would have been a too big bang. As for the latter based on the continuous creation of matter, this idea likewise denies the creation of antimatter.

Oskar Klein's theory: the primordial "metagalaxy" was an attenuated plasma containing only protons & antiprotons (& perhaps some electrons & positrons). In this initial state the spherical metagalactic cloud had a radius of about a trillion light years, and the density was so low that particles & antiparticles practically never collided one another (1) Gravitation caused the cloud to contract very slowly until it reached a radius of a few billion light years, at which stage the p's and $\bar{p}$'s began to collide, annihilating each other and producing e.m radiation (2) When the radius of the cloud had shrunk to about a billion light years the radiation arising from proton - antiproton collision annihilation was so strong that the radiation pressure overcame the gravitational attraction (3) The cloud then began to expand completing its separation into regions of matter and anti-matter (4) this kind of "radiation explosion" is more consistent with known physical processes than the big bang is

Intense radio star 3 C 273 may derive its enormous energy from matter - antimatter annihilations in dense matter. Protruding jet suggests an explosion

It seems quite possible that some, if not all, of the star-like radio objects in the heavens including the quasars arise from matter - anti-matter collision.

3|2|69 - St Institute - Sc. American - Nov. 1967, p. 88 - Gravitational Collapse by K. S. Thorne - Contains reference to Penrose's theorem - Also general relativity and topology.

Also Dec. 67. p. 36 - X-ray stars by R. Giacconi.

" p. 76 - High energy scattering by Barger & Cline.

The boundary $S^n = B\sigma^{n+1}$ of a σ^{n+1} ~~(complex)~~ is called a closed simplicial n-complex or merely an n-sphere.

Exx: 0-sphere is just 2 points

1-sphere → perimeter of a rectangle

2-sphere - Hopf map.

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(25) T. Y. Thomas - Inadequacy of current cosmological theories for the explanation of the discretised galactic structure found by A. G. Wilson clearly necessitates a modification of our over-all view point of the universe. We give here an account of epochal cosmology. Essential features of this theory are as follows: - The universe has existed throughout the infinite past in a state devoid of ponderable matter, and that it would have continued to exist in such a state except for the accidental creation of material particles. The termination of the initial stage with the creation of material particles is based on the possibility of such creation within the framework of Einstein's field eqns ie $B_{AB} - \frac{1}{2} B h_{AB} = K T_{AB}$ $(A, B = 0, 1, 2, 3)$ in which h_{AB} & B_{AB} are the components of the metric & Ricci tensors resp'ly & B a scalar curvature & K an absolute or universal constant., T_{AB} are components of energy tensor. The explosion of an accumulation of material created towards the end of the initial stage produced the first cosmological epoch & also gave rise to the series of epochs of creations which continue as a creative force in the universe & are significantly related to the observed discretization in red shifts & ages of galaxies

Concludes that both the evolutionary & steady state theories do not afford basis for discretisation in galactic structure. [Evolutionary = Lemaitre & Gamow
Steady state = Gold & Hoyle ~~Ex~~

~~From~~ 27/1/69 From Brill re. figs re. Schwarzschild singularity

It has recently been shown by Kruskal that the singularity at $r = 2m$ can be removed by a suitable coordinate transformation. The result is an analytic continuation of the Schwarzschild soln which is regular for all values of $r > 0$. However the singularity at $r = 0$ remains. There exists no coordinate system in which the analytically continued soln is static. The newly recognised dynamic behaviour of the Schw-soln is shown in Fig 2.9. At some initial time the metric is everywhere regular and describes two regions in which space is asymptotically euclidean. These two sheets are connected by an inner region in which space is curved strongly, the funnel or "wormhole" shown in the figure. Near the throat of the wormhole, the coordinate r is time-like and decreases as time proceeds.

of the Taub soln & neither can describe the full history of all geodesic test particles

The physical interpretation of the late contracting phases of the model universes, and its relevance to the actual universe is far from clear. The Taub-NUT soln has invalidated the old view that at the moment of collapse ∞-curvatures always occur, so that Einstein's eqns are of no further use in analysing the situation, and that different laws govern the "beginning" and "end" of the Universe. One of the most important tasks in relativity today is to investigate more general models through the contraction and collapse phases, describing the whole universe, or a part of it contracting faster than the rest, like a collapsing star. Any clarification of these challenging issues should enlarge our understanding not only of general relativity, but also of various other fields of physics. (25)

Fig 2a - A "snapshot" of the Schwarzschild geometry at time $t=0$ continued analytically maximally. Both sheets at top & bottom should really be continued to ∞. In time proceeds we have the "pinch-off" after a finite proper time $t = 4m$ (for interpretation of m see Anderson)

Fig. 2b - Another interpretation S-geometry at time $t=0$. The two sheets have been "pasted together" at a very large distance from the "throats". The two $\odot$ edges which are left free are to be identified so that the space has the topology shown in Fig. 3.

Fig 3 - The topology of the Reissner-Nordström soln at $t=0$, after the two sheets have been pasted together.

Fig. 4 - A two-dimensional analog of the hypersurface of time symmetry of a manifold consisting of two "throats" is shown isometrically imbedded in flat three-space. The figure illustrates the curvature and topology for a system of two "particles" of equal mass m and separation large compared to m, described by the metric

$$ds^2 = (1 + m/2r_1 + m/2r_2)^4 ds^2_{flat}$$

Note: The ~~topology of the~~ great $\odot^3$ on the three-sphere which represent the preferred direction in the Taub ~~have~~ space have the interesting property that they all link each other. The topology of these great $\odot^3$, regarded as fibres in the fibering of the three-sphere over the two sphere (the Hopf map), has been investigated in detail by mathematicians.

Defns: A p-simplex σ^p is merely any set of $(p+1)$ objects $\{A_0 \ldots A_p\}$

A simplicial complex K is one whose elements are simplexes

$$X^2+Y^2+Z^2+W^2=1$$

Parametrize this surface via three coordinates x, y, z, such that

$$\left.\begin{aligned} X &= \sin x \cos(y-z) \\ Y &= \sin x \sin(y-z) \\ Z &= \cos x \cos(y+z) \\ W &= \cos x \sin(y+z) \end{aligned}\right\} \qquad (127)$$

The induced metric is easily computed & finds

$$ds^2 = dx^2+dy^2+2\cos x\, dy\, dz + dz^2 \qquad -(128)$$

The coord. grid (allowing only one parameter out of the three to vary) are great ⊙s in 3-sphere. Now generalise this metric so as to preserve this property & add a time-coord

$$ds^2 = r_1 dx^2 + (r_1 \sin^2 x + r_2 \cos^2 x)\, dy^2 + 2 r_3 \cos x\, dy\, dz + r_3 dz^2 - r\, dt^2 \qquad -(129)$$

where the r's are funcs of t only. This is Taub's metric. A time-symmetric soln for the free-space Einstein eqns is given by

$$r_1 = (1/4)\cosh t \operatorname{sech}^2(t/2),\quad r_2 = \operatorname{sech} t,\quad r = r_1^2 r_2 \qquad -(130)$$

This metric has the remarkable property that it is invariant under a 3-parameter group which is isomorphic to the group of 4-dimensional rotations without fixed points. Such a space is called homogeneous. Total space volume is finite

$$V = \int \sqrt{3g}\, d^3x = r^{1/2} \int \sin x\, dx\, dy\, dz = 8\pi^2 r^{1/2}$$

The volume shrinks to zero after a finite time

$$\tau = \int_0^\infty r^{1/2}\, dt \sim 0.31\pi \text{ for soln of (130)}$$

Comparing with other models:

Model — Time req'd for collapse from max. expansion / (measured in units of max. $R = (V/8\pi^2)^{1/2}$)

Model	Friedmann	Tolman	Taub
t/R_{max}	$\pi/2 \sim 1.6$	1	0.8

The fact that the volume of some surface t = const. vanishes does not necessarily imply a geometrical singularity. For Taub's metric, this surface is actually a null surface & the soln can be analytically continued across it. On the continuation (due to Newman, Unti & Tamburino & and often called NUT space), the preferred space direction & time exchange roles: time coord. becomes periodic, like the great ⊙s on the sphere, and one space co-ordinate extends to infinity where it behaves like a radial coordinate in asymptotically flat space — There are two distinct analytic continuations

for this case, found by R. Lindquist, is

$$ds^2 = e^{2\sqrt{3}\sin\eta}\left\{d\mu^2 + \left[\exp\left[\tfrac{2}{3}\cos^2\eta - 4\sqrt{3}\sin\eta\right]\right](d\eta^2 + \sin^2\eta\, d\varphi^2)\right\} \quad \ldots (120)$$

This eqn shows that solns exist for multiply connected closed spaces & since manifolds described by (120) has a high degree of symmetry we can make various identifications, cutting it up and pasting it together again, to obtain initial solns for other topologies. In particular, if we identify each point μ, η, φ with its antipode $\mu, -\eta, \varphi + \pi$ we obtain an initial soln with a nonorientable topology, the direct product of a one-sphere and a 2-dimnl projective space.

(3) Non-trivial solns excluded for certain topologies.

In genl, closed spaces will not admit the trivial flat space as a soln - for eg there is no way of defining a flat metric on a topological 3-sphere. The class of closed spaces that do admit a flat metric is well known as the Clifford-Klein spaces. Again the three-torus, a cube with opposite faces identified, is a typical example of a closed space admitting a flat metric.

(4) Dynamics of closed universes - Details of time-development of closed universes may be quite complex, universe as a whole may expand or contract. Investigate the time-development of the total three-space volume

$$V(t) = \int_{t=\text{const}} \sqrt{\det g_{ij}(x,t)}\, d^3x$$

and confining oneself to the time symmetric case, it can be shown that in the sourceless or the general case, the universe developing from the moment of time-symmetry will never expand; it will in general contract, thereafter.

The above investigation of time-development in powers of t is not, however, suited to give us the information about the long-range time development of the universe. To date, much information has been obtained almost exclusively for two particular models, such as the (dust-filled) Friedmann universe, or the (radiation filled) Tolman universe; but here development of singularities have not been investigated

Gravitational radiation can curve up space into a closed universe ie a universe filled uniformly with gravitational radiation will contract (in both terms of time) until its radius vanishes.

(5) Expanding and recontracting non-isotropic universes - An example of an exact soln of the free-space Einstein eqns for a closed universe which is not isotropic is given by Taub. Consider the unit 3-sphere imbedded in 4-space with cartesian coordinates X, Y, Z, W

planes interconnected by funnels or "wormholes" [Fig 4]. The neighbourhood of each pole is the asymptotically flat region at one of the bottom sheets in the figure. It can be verified that the metric near the pole describes a sheet of infinite extent.

Examples of other types of topology that have been constructed are two sheets connected by a number of wormholes. By a "cross-wise" identification of the two sheets, one can then obtain solutions for wormholes in single sheets as shown in Fig. 2.6. Asymptotically flat solutions with non orientable topology have not yet been constructed: however, there is no ~~reason~~ obvious reason why the initial value equations (and hence all of Einstein's equations) should not have solutions with all kinds of topological characteristics

Closed Universes

(1) Universes with 3-sphere topology → Restricting to the time-symmetric case (a) & using eqns (105) and (106) for a metric of type (89) we can construct solns for closed universes. To demand that $ds^2 \to 0$ at $r \to \infty$ is to demand regular solns of (105) satisfying

$$\psi \to 0 \quad \text{at} \quad r \to \infty \qquad -(117)$$

Out of the family of solns for q in (106), it can be shown that there is exactly one which satisfies the eigen-value condition (117). It has the well known radial dependence of a soln for a zero-energy eigen-function

$$\psi \to (\text{const}/r) + O(1/r^2) \qquad -(118)$$

The metric therefore has the asymptotic form

$$ds^2 \to (\text{Const}/r^2)(dr^2 + r^2 d\Omega^2) \quad \text{as } r \to \infty \qquad (119)$$

using $r' = 1/r$, the metric has the flat-space form

$$ds^2 \to (\text{const})(dr'^2 + r'^2 d\Omega^2) \quad \text{at } r' \to 0$$

This shows that by addition of the "point at infinity" $r' = 0$, the space described by the soln can be ~~completely~~ completed smoothly to a closed universe with the topology of a 3-sphere

(2) Universes with other topologies. - Analogously we can construct solns of the initial value eqn (105) representing closed universes with other topologies. We only need to choose a base metric $d\sigma^2$ which describes for eg. a multiply connected topology. Eqn (101) will then, in general, have solns only for a class of measure zero of base metrics, typically for one metric out of each one-parameter class. The correct value of the parameter for closure is a kind of eigen value of eq (101) just as the unique one in the above case.

This method can be illustrated by the case of a "wormhole" universe, the topological product of a two-sphere and a one-sphere, described by angle coordinates η, ϕ, and μ resply. An explicit soln of the time-symmetric initial value eqn for this case

automatically fulfilled, so that the initial value problem reduces to a single equation

$$^3R = -2T_0^0 \quad \text{--- (99)}$$

To solve this for the source-less case of pure gravitation consider a "bare" metric $d\sigma^2$ which is conformally related to the soln ds^2 by

$$ds^2 = \psi^4 d\sigma^2 \quad \text{--- (100)}$$

In terms of ψ and the quantities formed from $d\sigma^2$ (indicated by subscript σ), eqn (99) takes the form for $T_0^0 = 0$,

$$\Delta_\sigma^2 \psi + \tfrac{1}{8} R_\sigma \psi = 0 \quad \text{--- (101)}$$

Hence method of solving initial value eqn (99) consists of assuming an arbitrary metric $d\sigma^2$ & solving (101) for ψ & then (100) furnishes a soln of the initial value problem, which can then be continued in time to give a soln of Einstein's equations. We may think of the arbitrarily chosen function $d\sigma^2$ as the "wave part" of a gravitational field, and ψ as a generalisation of the static (Newtonian) potential

Positive definite energy of gravitational waves – To answer the question if the total mass-energy of the gravitational waves and is positive definite, which is equivalent to the statement that the solns of the Einstein eqns to be considered must asymptotically become identical with the space part of the Schwarzschild metric

$$\psi \to 1 + m^*/2r + \cdots \quad \text{(104)} \qquad (m^* = GM/c^2)$$

Let us evaluate the initial value equation for case (a) & time gravitation eq. (101) for an axially symmetric initial metric of type (89). The resulting eqn can be split up into two well-known eqns

$$\nabla^2 \psi + \Phi \psi = 0 \quad \text{(3-dimnl Schrödinger eqn)} \quad \text{--- (105)}$$

$$4\Phi = \frac{\partial^2 q}{\partial r^2} + \frac{\partial^2 q}{\partial z^2} \quad \text{(2-dimnl Poisson eqn)} \quad \text{--- (106)}$$

To solve the initial value eqn, choose a fn q, compute from it the Schrödinger potential Φ & solve the zero-energy S-eqn (105) with this potential for the "wave function" ψ. Using this method in solving, we can show that the mass-energy of any non-singular gravitational waves constructed in this way is $+^{ve}$ definite (i.e. even for case $T_{\alpha\beta} = 0$). This result can be generalised to case $T_{\alpha\beta} \neq 0$, provided T_{00} is $+^{ve}$ definite

Solutions with non-euclidean topologies – The initial metric corresponding to the Schwarzschild soln with $K_{ij} = 0$, $\psi \to \psi = 1 + m/2r$ (Corresp. to Sph.-symmetry case) may be interpreted as the description of an everywhere regular, two-sheeted manifold "wormhole" topology (Fig. 2a). In general any soln in which ψ has simple poles describes a multiple-sheeted manifold. In a 2-dimn analogy the manifold looks like many

The initial value eqn can now be written in terms of g_{ij} & its time derivative. Instead of $g_{ij,0}$ introduce the "extrinsic curvature" of the surface $t=0$

$$\left.\begin{aligned} K_{ij} &= -(1/2N)\, g_{ij,0} = -N\Gamma^0_{ij}, \\ K &= {}^3g_{ij}\, K_{ij} \end{aligned}\right\} \quad \text{-- (90)}$$

~~Dropping superscript 3 of~~ Assuming that ${}^3g^{ij}$ are the matrix inverse of the three-dimensional metric

ie ${}^3g^{ij}\, g_{jk} = \delta^i_k$ -- (91)

Dropping the superscript 3 for convenience, differentiation of (91) gives

$$g^{ij}{}_{,0}\, g_{jk} = -g^{ij} g_{jk,0} = 2N g^{ij} K_{ij}, \text{ hence}$$

$$g^{ij}{}_{,0} = 2NK^{ij}; \quad g_{ij,0} = -2NK_{ij} \quad \text{--- (92)}$$

In this notation the initial value ~~problem~~ eqns (61) take the form

$${}^3R + K^2 - K^{ij}K_{ij} = -2T^0_0, \quad \text{-- 93(a)}$$

$$(K_i{}^j - \delta_i{}^j K)_{;j} = -NT^0_i \quad \text{-- 93(b)}$$

The remaining Einstein field eqns for $\partial K_{ij}/\partial t$ are:

$$\partial K_{ij}/\partial t = -N_{;ij} + N\left[{}^3R_{ij} - T_{ij} + \tfrac{1}{2} g_{ij} T - 2K_{il}K^l{}_j + KK_{ij}\right] \quad \text{-- (94)}$$

where ${}^3R_{ij}$ & ${}^3R$ are the 3-dimnl Ricci tensor & curvature scalar of surface $t=0$, formed out of ${}^3g_{ij}$. Choosing initial surface as a "maximal" surface [~~for +ve definite metric~~ in the imbedding space, we have $K=0$ -- 95 [for +ve definite metric $K=0$ is minimal surface but for indefinite metric of gen. rel. it gives a maximal surface]. If (95) is also reqd in the neighbourhood of $t=0$, we can deduce (in case of pure gravitation ie $T_{\alpha\beta}=0$) that the condn $K=0$ specifies N as the soln of the elliptic eqn

$$\nabla^2 N = NR = NK^{ij}K_{ij} \quad \text{-- (96, b)}$$

(∇^2 = covariant Laplace operator in space-like 3 metric).

The ~~Sch~~ Schwarzschild exterior soln is an example of soln of (96, b). We can deduce from (96, b) that at most one surface $t=0$ can be a maximal surface in a closed space.

Solution of initial value equation. For the very special case of spherical symmetry ie assuming that there are other Killing vectors besides the ξ_α. It can be easily seen that the general spherically symmetric 3-dimnl metric can be brought into conformally flat form

$$g_{ij} = \psi^4 \delta_{ij} \quad \text{-- (97)}$$

and substituting in (93), we have $-8\psi^{-5}\nabla^2\psi + (8/9)K^2 = -2T^0_0$; for $T_{\alpha\beta}=0$, $K=0$, and the eqn becomes Laplace's eqn with boundary condn $\psi \to 1$ at $r \to \infty$

Besides sph. symmetric soln, there are two more general cases (a) the "time symmetric case", when $K_{ij}=0$. In this case the future behaviour of a soln exactly retraces its past history; (b) "the conformally flat case" when (97) holds. In case (a) eqns (93 b) are

(5) (Contd) (a) Initial value problem: For Einstein's eqns in which the highest time derivative appearing is the second, the appropriate initial data are the g_{ij} & $\partial g_{ij}/\partial t$ on an initial space-like surface $t=0$. Writing the contracted Bianchi identities in the form

$$\left(R^0_\alpha - \tfrac{1}{2}\delta^0_\alpha R\right)_{;0} \equiv -\left(R^i_\alpha - \tfrac{1}{2}\delta^i_\alpha R\right)_{;i} \quad \ldots (60)$$

The highest time-derivative on the R.H.S is the second (since $i \neq 0$). Since (60) is an identity the L.H.S. simly contains at most 2nd order t-differentiation. But the one x^0-derivative is written out explicitly, hence the expn $R^0_\alpha - \frac{1}{2}\delta^0_\alpha R$ can contain at most 1st order derivatives. Therefore the four Einstein eqns $R^0_\alpha - \frac{1}{2}\delta^0_\alpha R = T^0_\alpha$ (61) are the desired initial value condns. For pure gravitational case, R.H.S = 0. The eqns (61) are simpler than the full set of Einstein eqns mainly because they are purely elliptic eqns. Any regular soln of (61) can be continued for at least a finite time to give a regular soln of the full Einstein eqns; in particular, they can give information re. the possible topological structure of the spaces described by solns of Einstein's eqns.

(Axial symmetry case): - ie a geometry invariant under rotation about one axis. To make a covariant defn of this, we note that an infinitesimal rotation is an infinitesimal coord. trn described by a vector ξ^α, $x'^\alpha = x^\alpha + \epsilon\xi^\alpha$ --- (79)

New metric to 1st order in ϵ is $g'_{\alpha\beta} = g_{\alpha\beta} + \epsilon\xi_{(\alpha;\beta)}$ ... (80)

If the metric is invariant under an infinitesimal rotation, a ξ_α (a Killing vector) must exist satisfying $\xi_{(\alpha;\beta)} = 0$ .. (81) & which is orthogonal to some set of hypersurface, φ = const ie $\xi_\alpha \propto \varphi_{,\alpha}$ (φ = angle with range $0 \leq \varphi < 2\pi$ ie trajectories of the group are closed curves intersecting the hypersurface φ = const. The metric, using φ itself as a coord meta can be written as

$$ds^2 = g_{ab}(x^c)\,dx^a dx^b + \gamma(x^c)\,d\varphi^2 \qquad (a,b,c = 0,1,2) \quad \ldots (84)$$

and $\gamma(x^c) = \xi^\alpha\xi_\alpha\,(\theta^{ce}/2\pi)^2$ of the θ^{ce} x^c = const, $0 \leq \varphi < 2\pi$

Initial value problem (Contd) - Choose time coord along normal to initial surface ie

$$g_{0i} = g_{i0} = 0 \text{ at } t = 0 \quad \text{--- (85)}$$

Also let $g_{00} = -N^2$. The general initial metric becomes

$$ds^2 = -N^2 dt^2 + {}^3g_{ij}\,dx^i dx^j \quad \text{--- (86)}$$

In axially symmetric case {eqn (84)} we have

$$^3g_{ij}\,dx^i dx^j = \sum_{a,b=1,2} g_{ab}(x^c)\,dx^a dx^b + \gamma(x^c)\,d\varphi^2 \quad \ldots (87)$$

Introducing two new coordinates $\rho(x^a)$, $z(x^a)$ satisfying the condns

$$g_{\rho z} = 0, \quad g_{\rho\rho} = g_{zz} \quad \text{--- (88)},$$

the final form (with a choice motivated by future convenience) of line element is

$$^3g_{ij}\,dx^i dx^j = \psi^4\left[e^{2q}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\right]$$

$$= \psi^4\left[e^{2q}(dr^2 + r^2 d\varphi^2) + r^2\sin^2\theta\, d\theta^2\right] \quad \text{--- (89)}$$

$$r^2 = \rho^2 + z^2$$

Anderson's book to be borrowed:

Brandeis Inst. Lectures, 1964 – Article by Trautman

Epilogue: I want to finish this discussion of cosmology by saying that in my opinion it is not worthwhile to work in theoretical cosmology at the present time. I think it would be better to sit and wait for the astronomers to get more data of the motion and distribution of distant galaxies. I think it is the opinion of most people who work in that field that one can expect in the next few years to obtain much significant data, probably from radio-astronomy, and there are also hopes of putting a telescope into orbit.

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(1) Il nuovo Cimento – Suppl (I), 2, pp. 47–57 – D. R. Brill on closed Universes

According to Mach the universe as a whole is an essential ingredient in any physical solution. It may be replaced by boundary conditions such as asymptotic flatness, but only at a loss of understanding how inertial properties really come about.

We consider closed universes ie closed in space. This demand of closedness is the type of physical principle that is needed to select physically meaningful solns from the great variety of possible solns ie "to obtain boundary conditions without boundary conditions"; this restriction to closure is just a possible first step in a rigorous formulation of Mach's Principle

The demand of closure may be considered a demand that the $g_{\mu\nu}$ vanish at $r = \infty$, like the usual boundary conditions for static fields. This means that the distance between all points at $r = \infty$ vanishes ie that space has been pulled together and closed up at $r = \infty$. Of course we must ascertain that the resulting manifold is regular regular at the point

(1) → Universes with 3-sphere topology – To construct solns for closed universes we turn to the initial value eqns for eg. eqns $\nabla^2\psi + \Phi\psi = 0$ (3 dimnl Schrödinger eqn) —(1)

and $4\Phi = \frac{\partial^2 q}{\partial r^2} + \frac{\partial^2 q}{\partial z^2}$ (2 dimnl Poisson eqn)

for a metric of type $^3g_{ij}\,dx^i dx^j = \psi^4\{e^{2q}(d\rho^2 + dz^2) + \rho^2 d\varphi^2\}$

for the case "time-symmetric case" when $K_{ij} = 0$

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$K = {}^3g^{ij}K_{ij}$ $\quad [K_{ij} = \left(\frac{\partial}{\partial N}\right)4\psi^2\psi_0\,\delta_{ij} = (K/3)\,\delta_{ij}$ $\quad (g_{00} = -N^2)$

in this case $K_{ij} = 0$, the future behaviour of a soln exactly retraces its past history

Restricting attention to this case, to demand that $ds^2 \to 0$ at $r \to \infty$ is to demand regular solns of (1) satisfying $\psi \to 0$ at $r = \infty$. If an arbitrary q is chosen

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See p. 7

Notes on Cosmology

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