

Temperature Distribution in an Electrically Heated Filament

THE distribution of temperature in a filament electrically heated *in vacuo* has been studied by several previous authors. The differential equation defining the steady state is:

$$\frac{d^2T}{dx^2} + \frac{5}{2} a (T_m^4 - T^4) = 0, \quad (1)$$

in which T is the temperature at a distance x from one of the ends, T_m is the value to which the temperature T_l at the centre tends as the length $2l$ of the filament is increased indefinitely, keeping the heating current the same, and a is a constant determined by the cross-section of the filament, its thermal conductivity and the emissivity of its surface. Using the boundary conditions $T = \Theta$ when $x = 0$, and $dT/dx = 0$ when $x = l$, one obtains¹

$$x = \int_{\Theta}^T [5aT_m^4 (T_l - T) - a (T_l^5 - T^5)]^{-\frac{1}{2}} dT, \quad (2)$$

the value of T_l occurring in (1) being determined by the condition that, when the upper limit of the integral is made equal to T_l , x should become l .

In a recent paper² it was shown that though the integral cannot be evaluated directly, it can be expanded as a convergent power series. The filament can be divided for this purpose into two distinct regions, region A comprising a certain length near the middle, and region B outside it, the power series referred to being different in the two regions.

Now, denoting $T_m - T$ by Δ , it can be shown that:

$$\Delta \exp f(\Delta) = \exp \pm x\sqrt{A} \quad (3)$$

are solutions of (1), where

$$A = 10 a T_m^3 \quad (4)$$

and

$$f(\Delta) = \frac{1}{2} \frac{\Delta}{T_m} + \frac{1}{16} \frac{\Delta^2}{T_m^2} - \frac{1}{240} \frac{\Delta^3}{T_m^3} - \dots \quad (5)$$

The effect of the finite temperature coefficient of the thermal conductivity, and of the other constants, can also be taken into account by replacing the factors $\frac{1}{2}$, $\frac{1}{16}$, $-\frac{1}{240}$, ... by A_1 , A_2 , A_3 , ... which can be calculated. For a tungsten filament with $T_m = 2,400^\circ \text{K}$., $A_1 = 0.75$, and the other A 's are much smaller.

In the region where $\exp f(\Delta)$ is close to unity, a general solution for the finite filament can be obtained by suitably combining the two particular solutions given by (3). The corresponding two terms in the general solution are obviously such that one of them increases rapidly and the other decreases rapidly as we move away from the centre of the filament. If $T_m - T_l$ is sufficiently small, the latter term may become negligible in comparison with the former, while Δ still remains small enough to permit our combining the two particular solutions. When this has happened, the need to keep $\exp f(\Delta)$ close to unity disappears. We thus obtain a *practical* general solution that is applicable over the whole length of the filament, namely:

$$\begin{aligned} & \Delta \exp f(\Delta) \\ &= \Delta_0 \exp f(\Delta_0) [\exp -x\sqrt{A} + \exp \{-(2l-x)\sqrt{A}\}] \\ &= \Delta_0 \exp f(\Delta_0) \exp -l\sqrt{A} (\exp q\sqrt{A} + \exp -q\sqrt{A}), \end{aligned} \quad (6)$$

where $\Delta_0 = T_m - \Theta$, and $q = l - x$ is the distance measured from the centre.

Now for a filament for which $T_m - T_l$ is small in comparison with T_m , which we shall refer to for convenience as a 'long' filament, the limit of the A -region is given by $\Delta_c \sim 3(T_m - T_l)$. At this point it is found that the ratio of the second term inside the brackets in (6) to the first term has decreased to about $\exp(-3.6)$, that is, below 3 per cent. Taking for convenience this fraction, namely 0.03, as small in comparison with unity, we can readily investigate how small $T_m - T_l$ should be in comparison with T_m in order that (6) may hold over the whole length of the filament to at least this degree of accuracy.

It may appear at first sight that the criterion for our being able to combine the two particular solutions, namely, that $\exp f(\Delta)$ should be close to unity, will become increasingly difficult to satisfy as Δ increases. But actually, owing to the disparity in the magnitudes of the two terms to be combined, which increases rapidly as one moves away from the centre, the working criterion becomes $f(T_m - T_l) < 0.03$, or $(T_m - T_l) < 0.04T_m$. When this is secured, (6) will fit with observation to a much closer accuracy than 3 per cent, the accuracy increasing rapidly as one moves away from the centre of the filament.

The results obtained earlier for such a filament, on the basis of the series-expansion of the integral—namely, the parabolic variation of temperature in the A -region, the closeness of the distribution outside the A -region to that of a similar infinitely long filament heated by the same current and the logarithmic distribution at the top of this range—all come out as special cases of (6).

By putting $x = l$ in (6), one obtains a useful relation for the temperature T_l at the centre of a long filament, as a function of its length, namely:

$$T_l = T_m - 2\Delta_0 \exp f(\Delta_0) \times \exp -l\sqrt{A}, \quad (7)$$

in view of which (6) may also be written in the convenient form,

$$\Delta \exp f(\Delta) = \frac{1}{2} (T_m - T_l) (\exp q\sqrt{A} + \exp -q\sqrt{A}), \quad (8)$$

which is applicable over the whole length of a long filament.

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Feb. 12.

¹ See Carslaw, H. S., and Jaeger, J. C., "Conduction of Heat in Solids", 135 (Oxford: Clarendon Press, 1947).

² Krishnan, Sir K. S., and Jain, S. C., *Nature*, 173, 166 (1954); *Proc. Roy. Soc., A* (in course of publication).

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We trust you will be interested in
the attached cutting from our issue
for 1 MAY 1954

3-28

Temperature distribution in an electrically heated filament

To
The Editor of 'Nature'

Dear Sir,

The distribution of temperature in a filament electrically heated in vacuo has been studied by several previous authors. The differential equation defining the steady state is

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in which T is the temperature at a distance x from one of the ends, T_m is the value to which the temperature T_c at the centre tends as the length $2l$ of the filament is increased indefinitely, keeping the heating current the same, and a is a certain constant determined by the cross-section of the filament, its thermal conductivity and the emissivity of its surface. Using the boundary conditions $T = \textcircled{H}$ when $\frac{dT}{dx} = 0$ at $x=0$ and $dT/dx = 0$ when $x=l$ one obtains

$$x = \int_{\textcircled{H}}^T \left[5a T_m^4 (T_c - T) - a (T_c^5 - T^5) \right]^{-1/2} dT, \quad \dots (2)$$

the value of T_c occurring in (2) being determined by the condition that when the upper limit of the integral is made equal to T_c , x should become l .

2)
 In a recent paper it was shown that though the integral cannot be evaluated directly, it can be expanded as a convergent power series. The filament can be divided for this purpose into two distinct regions, region A comprising a certain length near the middle, and region B outside it, the power series referred to being different in the two regions.

Now denoting $T_m - T$ by Δ it can be shown that

$$\Delta e^{f(\Delta)} = e^{\pm x \sqrt{A}} \quad (3)$$

are solutions of (1), where

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$$f(\Delta) = \frac{1}{2} \frac{\Delta}{T_m} + \frac{1}{16} \frac{\Delta^2}{T_m^2} - \frac{1}{240} \frac{\Delta^3}{T_m^3} - \dots \quad (5)$$

The effect of the temperature coefficient of the thermal conductivity, and of the other constants, can be taken into account by replacing the factors $1/2$, $1/16$, $-1/240$, ... by A_1 , A_2 , A_3 , ... which can be calculated. For tungsten filament with $T_m = 2400^\circ \text{K}$, $A_1 = 0.7$, and the other A_i 's are much smaller.

In the region where $\exp. f(\Delta)$ is close to unity, a general solution for the finite filament can be obtained by suitably combining the two particular solutions given by (2). The corresponding two terms

in the general solution are obviously such that one of them increases rapidly and the other decreases rapidly as we move away from the centre of the filament. If $T_m - T_c$ is sufficiently small, the latter term may become negligible in comparison with the former while yet Δ remains small enough to permit our combining the two particular solutions. When this has happened, the need to keep $\Delta^{(A)}$ close to unity disappears. We thus obtain a practical general solution that is applicable over the whole length of the filament, namely

$$\Delta e^{f(\Delta)} = \Delta_0 e^{f(\Delta_0)} \left[e^{-x\sqrt{A}} + e^{-(2l-x)\sqrt{A}} \right]$$

$$= \Delta_0 e^{f(\Delta_0) - l\sqrt{A}} \left[e^{q\sqrt{A}} + e^{-q\sqrt{A}} \right] \quad \text{--- (6)}$$

where $\Delta_0 = T_m - T_c$, and $q = l - x$ is the distance measured from the centre.

Now for a filament for which $T_m - T_c$ is small in comparison with T_m , which we shall refer to for convenience as a long filament the limit of the A-region is given by $\Delta_c = 3(T_m - T_c)$. At this point it is found that the ratio of the second term inside the square brackets in (6) to the first term has dropped down to about $e^{-3.6}$ i.e. below 3%. Taking for convenience this fraction, namely 0.03 as small in comparison with unity, we can readily investigate how small $T_m - T_c$ should be in comparison with T_m in order that (6) may hold over the whole length of the filament to at least this degree of accuracy.

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The results obtained earlier for such a filament, on the basis of the series expansion of the integral, namely the parabolic variation of temperature in the A-region, the closeness of the distribution outside the A-region to that of a similar infinitely long filament heated by the same current, the logarithmic distribution at the top of this range, all come out as special cases of (6).

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which is applicable over the whole length of a long filament.

Yours truly

see Carslaw, H.S., and Jaeger J.C.,
 Conduction of Heat in Solids p.135, (1947)
 Oxford: Clarendon Press.

Krishnan, Sir K.S., and Jain S.C., Nature, 173 (1954) K.S. Krishnan
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