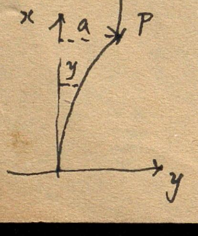
Introduction – I intend to speak this evening about some recent advances in Engineering mathematics. Many people have objected to this nomenclature on the score that mathematics cannot tolerate an adjective unless it be of an abstract nature like 'pure' or 'applied'. Such purists – purists who ~~gloat over~~ pray that their pure science, the purest mathematics should be unsullied by applications might however prefer "mathematical engineering". But this nomenclature is an insult to engineers suggesting that there is something in engineering which is inexact, ~~which is far from truth~~ which certainly is a travesty of truth. Taking all things into consideration, I personally prefer the first description especially in view of the fact that ~~the~~ investigations of recent years in many engineering problems have enriched pure mathematics itself just as in the case of physical investigations. Not only have men like Rankine, Prandtl, Perry, Osborne Reynolds, G. I. Taylor, Steinmetz others contributed to engineering problems, but their work has opened ~~new~~ out new methods of mathematical analysis like the iteration or the Stodola method, step-by-step integration, the Raleigh-Ritz method of finding maxima & minima and so on. ~~It is also not out of place to mention here that some of the greatest mathematicians~~ ~~mathematical minds~~ ~~the world has ever produced have been engineers. It is also remarkable how many of~~ This remarkably close connection between engineering & mathematics is next only to that between physics & mathematics. Archimedes who by common sense is one among the three greatest mathematicians of all time was ~~really~~ also a great engineer. Leonardo da Vinci the versatile prodigy was a genius in Engineering. Gauss went through a polytechnic, Euler was deeply interested in engineering, ~~Kelvin was great an Engineer as a physicist~~, Laplace & Lagrange ~~Lord Ray~~ have made notable contributions to engineering. Kelvin & Rayleigh were engineers as well as physicists. Coming to our own days it is not so well known that Prof. Dirac who is perhaps the greatest theoretical physicist of the day started as an engineer. Thus examples could be multiplied.

In the good old days Engineering mathematics connoted special methods of approximations, and graphical methods adapted for the solution of simple problems and attempts at bringing in the methods of advanced mathematical analysis were looked down contemptuously by most engineers. In fact this tendency persists even to this day in many circles. But in recent years thanks to the work of a specially well equipped group of workers in civil, mechanical, electrical, aeronautical, & chemical engineering, it has come to be realised that advanced research in engineering needs the help of many branches of mathematics. Thus tensors & matrices, ordinary and partial differential equations, operational calculus, boundary values, eigen value problems, expansion in orthogonal functions, functions of a complex variable, Conformal transformation, Integral equations, Calculus of variations are used in engineering problems like engine dynamics, Rotating electrical machinery, buckling & vibration of ~~beams~~ & stress distribution of beams, plates & shells, hydraulics of viscous & compressible fluids, aerofoils & ballistics. This does not of course mean that every engineer should be acquainted with all these branches of mathematics. What the engineer wants are numerical calculations based on these theories, and a ~~knowledge as to ho~~ an appeal to standard texts in mathematics ~~texts~~ will give the ~~data~~ method for these calculations. Even here one should notice a remarkable feature. The type of numerical methods required by engineers are exactly those based on the purely mathematical considerations of establishing ~~existing~~ existence theorems.

There is another recent tendency in the application of mathematics to engineering problems on which I would like to make a few remarks. Most of the branches of mathematics listed above from which the engineer draws his methods can, in a sense, be described "linear". Thus the type of differential eqns used have always been linear, or at least the Engineer makes them linear by suitable assumptions. In fact the engineer has a genius for 'linearising'. In this excellent text book on theory of structures by Prof. Morley, I have searched for a non-linear equation, and except for the catenary, I have failed to a single one. In recent problems however some equations have appeared which are non-linear, and which lose all sense by linearising. Such problems have presented a real challenge to the ~~mathemat~~ engineer since he can no longer rely on standard text-books but has to chalk out methods for himself. These problems also constitute a challenge to the mathematician since his methods of establishing existence theorems for the linear case cannot be blindly extended to the non-linear case also. Engineers & physicists have always looked upon the mathematicians' efforts at proving existence theorems with amusement & contempt since they actually amounted to proving the obvious. This was perhaps true in the linear case, but not at all so for the non-linear case where it is not at all obvious that a certain eq^n of this type has a solution. Attempts in this direction by Levi-Civita, Karman & others have already yielded ~~valu~~ results valuable both of engineering & mathematics. I want to present a few of them to you this evening.

(2) Consider first ~~the theory~~ problems in elasticity or applied mechanics if you like. The two guiding principles in ~~such~~ problems ~~specially~~ those relating to deflection are (i) that Hooke's law is valid (ii) that the deflections are small. These lead to linear differential equations for the stress distribution & deflection. Giving up one or the other or both of the assumptions leads to non-linear equations.

(a) Let us now retain (i) but give up (ii) i.e. in exp^n for strain we do not neglect squares & products of deflections & their derivatives. A classical example of this type is the problem of the "elastica" or better known to you as Euler's theory of 'long' pillars. I shall first take the classical solution as given in most text books (Morley p. 274) and take the case of both ends of the thin rod being on pivots or frictionless hinges. This can be immediately reduced to the case freely hinged at only one end. I shall deal with this at some length because it brings out clearly ~~the~~ view point of the 'pure' applied mathematicians



if one ~~might~~ might put it so. The equation of bending moment gives

$$\frac{M}{EI} = \frac{P(a-y)}{EI} = \frac{d^2y}{dx^2}$$

Calling EI the flexural rigidity as B ~~this gives~~ (or bending stiffness)

$$B\frac{d^2y}{dx^2} = P(a-y) \quad \text{, a linear equation} \qquad (1).$$

This is to be solved with the boundary or end conditions $x=0$, $y=0$, $\frac{dy}{dx}=0$ and when $x=l$, $y=a$. The solution is

$$y = a\left(1-\cos x\sqrt{P/B}\right)$$ using the first set of boundary conditions.

Using the second set we see that P cannot be arbitrary but

$$\cos l\sqrt{P/B} = 0$$

~~For those~~ ie $l\sqrt{P/B} = \pi/2,\ 3\pi/2,\ 5\pi/2,\ -$

For given l & B, the values of P are called the eigen-values (same notation even in atomic mechanics) and the lowest eigen-value is given by

$$l\sqrt{P/B} = \pi/2.$$ & the corresponding parameter P determines the buckling load or the "Euler load" For the case of both ends freely hinged this load is given by $P_E = \pi^2 B/l^2$.

~~The~~ value of y for this P_E the socalled "eigen-function" gives the shape of the buckling column but leaves the deflection undetermined since we initially assume this as 'a' & derive the solution. To determine the force-deflection reflection beyond the buckling load we have to now turn to ~~the~~ a non-linear equation. This eqn is obtained by ~~considering~~ considering the expn for the bending moment EI/R. Instead of putting $1/R = \frac{d^2y}{dx^2}$ we have to write if we do not neglect $\left(\frac{dy}{dx}\right)^2$

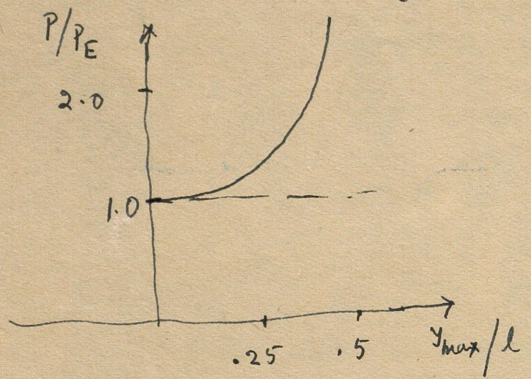
$$\frac{1}{R} = \frac{d^2y/dx^2}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}} \quad \text{or} \quad \frac{d^2y/ds^2}{\left\{1-\left(\frac{dy}{ds}\right)^2\right\}^{1/2}}$$

for $\frac{dy}{ds} = \sin\phi$, $\frac{d^2y}{ds^2} = \cos\phi.\frac{d\phi}{ds}$ ie $\frac{1}{R} = \frac{d^2y/ds^2}{\cos\phi} = \frac{d^2y/ds^2}{\sqrt{1-(dy/ds)^2}}$.

and the deflection equation is

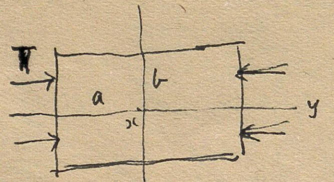
$$\frac{d^2y}{ds^2} + \frac{P}{B}\, y \left\{ 1 - \left(\frac{dy}{ds}\right)^2 \right\}^{1/2} = 0. \qquad (2)$$

(2) can be integrated by using elliptic functions & the adjoining figure shows this solution ie P/P_E as a fn of the ratio of deflection of midpt to the length.



The analogous ~~problem~~ but much more complicated problem of the finite deflection of plates loaded beyond their buckling limits ~~has~~ is important in the stressed-skin or "monocoque" method of constrn of aeroplane wings & fuselages. The idea is to load the thin metal skin beyond its buckling limit. In fact the skin in this buckled or "wave" state is able to carry stresses which are large multiples of the buckling stress. The max load is then determined by the ultimate strength of the buckled skin & the aim of theory is to find stress distribution & max. stress occurring in the wave state.

The correct mathematical description of the problem is rather complicated & to realise this one has only to look at Chap. 13 of Morley to realise the struggles of the Engineer to linearise the problem ~~for~~ of the flat plate (circular & rectangular). I shall mention only the important steps just to show the unsolved problems that arise here.



Consider the plate freely supported along $y = \pm b$ & subjected to compressive forces normal to $x = \pm a$. The deflection here is $w = w(x, y)$. Components of stresses per unit length are σ_x, σ_y & τ_{xy} derived from Airy's fn $F(x, y)$

$$\sigma_x = \partial^2 F/\partial y^2, \quad \sigma_y = \partial^2 F/\partial x^2, \quad \tau_{xy} = -\partial^2 F/\partial x \partial y$$

with $\Delta\Delta F = 0$ (biharmonic eqn).

For no normal external load $\Delta\Delta w = 0$ (cf $B\frac{d^4y}{dx^4}$ = loading) (linear eqn)

In the non linear case

$$\Delta\Delta F = E\left\{ \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \right\}$$

$$B\Delta\Delta w = t\left\{\frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right\}$$

given by Th. von. Karman

If in undeflected state $\sigma_x = -\Pi$, $\sigma_y = 0$, $\tau_{xy} = 0$ & we substitute in the above eqn we get the linearised eqn

$$B\Delta\Delta w + \Pi t \frac{\partial^2 w}{\partial x^2} = 0.$$

This can be solved & gives eigen values of $\Pi = \frac{B}{t} \cdot \frac{\pi^2 a^2}{4m^2}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$

(hinged support at $x = \pm a$, $y = \pm b$), where m & n are integers.

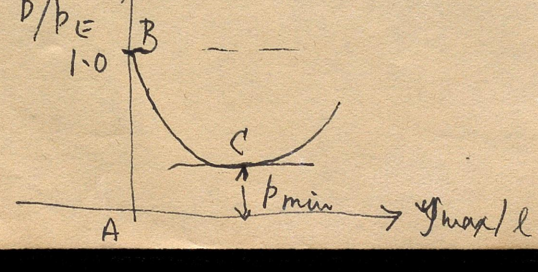
Lowest eigen value gives buckling or Euler load Π_E. If Π be slightly $> \Pi_E$ approximation method can be used but if Π be many times greater this cannot be used & no satisfactory has so far been possible.

Similar soln for circular plates for stressing beyond buckling have been more satisfactory & given by Timoshenko.

Another interesting case is that of a flat arch under lateral load. Let the two end points of a flat circular arch be pin-joined & let us consider large deflections of the order of the height of the arch. If p be uniform normal load per unit length & R rad. of curvature, the uniform load produces a compressive force $= pR$ & the first eigenvalue is given as in Euler's strut by

$$(pR)_E = \pi^2 B/l^2$$

The non-linear eqn taking large deflections into account is like that for the elastica but while in the elastica the thrust after the buckled stage is reached increases with deflection, here just the reverse is true



In the former case for $p < p_E$ there is only possible posn of eqm, here there several possible for every $p < p_E$. For a continuous transition

p first reaches p_E & then any other value on BC, but due to imperfect structure & want of uniformity in loading there may be a jump between eq^m states along AB & BC, so that in this case p_{min} is of greater interest than p_E. This explain systematic discrepancies found between break~~down~~ loads of curved shells found experimentally & on the linear theory, the experimental values being much less than the theoretical ones. Also the shape of the buckled shell is very different from the one given on the linear theory. The above non-linear theory explains these very satisfactorily.

Going from two dimensions to three dimensional isotropic elastic continua I shall only mention that in recent years it has been found that a correct theory for large deflections can only be obtained if one were to use the tensor calculus used in general relativity & Riemannian Geometry etc. For obvious reasons I cannot go into this theory.

(b) Now let us also give up the first assumption that Hooke's law which is also linear is valid. ~~but retain~~ ~~the~~ Two types of non-linear stress-strain relations are possible

(i) although there is a unique & reversible rel^n between stress & strain this rel^n cannot be expressed linearly

(ii) Beyond a certain limit the deformation is no longer purely elastic ie there is a permanent set after removing the load. Non reversible part of deformation is called plastic deformation or plastic flow & this is obtained after the yield point (stress limit). In general if plastic flow takes place yield point is raised ie additional plastic flow ~~takes~~ place only if further load is put. This is called cold-hardening.

A material capable of plastic deformation without cold hardening is called perfectly plastic & is amenable to mathematical treatment. For such a material the yield point is given by a constant difference between the largest & smallest principal stress (Law of plasticity) & this must be satisfied in the whole domain in which plastic flow takes place. Working for 2 dimensional plastic flow & with $\sigma_x = t$, $\sigma_y = r$, $\tau_{xy} = s$ (in terms of F) this condⁿ gives

$$(r+t)^2 - 4(rt - s^2) = K^2 \quad (K = \text{a characteristic of the material})$$

This is a non-linear partial diffl eqⁿ of the hyperbolic type amenable to what is called the method of characteristics. The soln of this eqⁿ can be exhibited geometrically & shows that plastic flow consists essentially of a gliding along the planes in which the maximum shearing stresses occur as observed by experimentally.

Remarkably enough the stress distribution in a mass of sand in the limiting state of equilibrium ie just about to collapse & neglecting the influence of its own weight is given by a very similar eqⁿ

$$(r+t)^2 - \frac{4}{1-f^2}(rt - s^2) = 0 \quad (f = \text{coeff of friction between sand particles})$$

This eqⁿ is identical with Rankine's fundamental eqⁿ of the theory of earth pressure & can be applied to various problems in soil mechanics. Taking gravity into account the eqⁿ becomes non-homogeneous containing linear terms in r, t & s. In important practical cases these can fortunately be reduced to ordinary diff. eqⁿˢ & solved.

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(3) Non-linear oscillations. – mechanical & electrical systems.

Usual linear eqⁿ with damping is

$$\ddot{x} + \omega^2 x + \frac{\beta}{m}\dot{x} = 0 \quad (\text{linear damping \& linear restoring force}$$

Generalising $\ddot{x} + \omega^2 x = f(x, \dot{x})$ includes non linear damping & non-linear restoring forces.

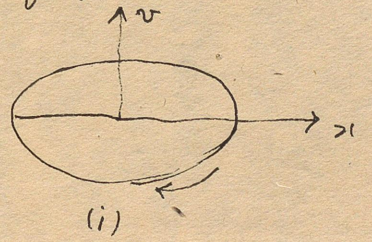
Case (i) $f(x, \dot{x}) = \mu\,\phi(x)\,\dot{x}$, $\dot{x} = v = f(x)$.

eqⁿ is $\dfrac{dv}{dx} = -\omega^2 \dfrac{x}{v} + \mu\,\phi(x)$. – Solution by method of isoclines (x, v diagram

~~$\mu = 0$~~,

(i) $\mu = 0$,

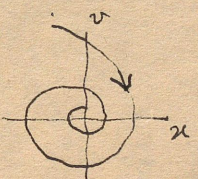
$v^2 + x^2\omega^2 = \text{const}$.



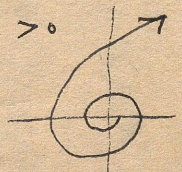
(i)

(ii) $\phi(x) = 1$, constant damping factor

$\mu < 0$



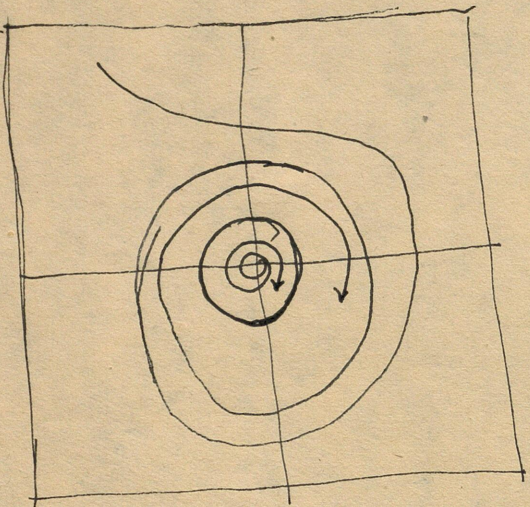
$\mu > 0$



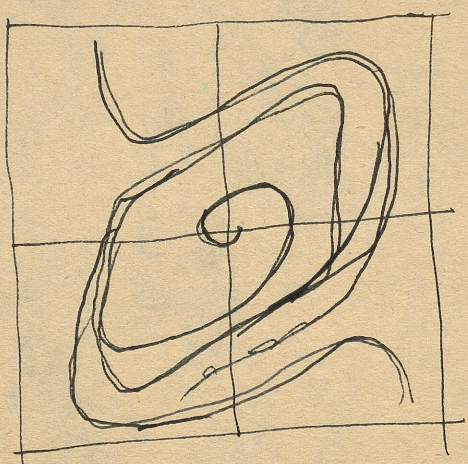
(iii) $\phi(x)$ variable : Van der Pol's case $\phi(x) = 1 - x^2$.

For $x < 1$, negative damping
$x > 1$ positive damping } $\phi(x)$ changes sign from $-^{ve}$ to $+^{ve}$.

Form of curve also depends on μ.



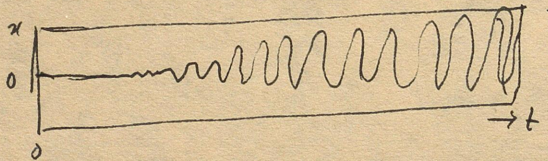
$\mu = .1$



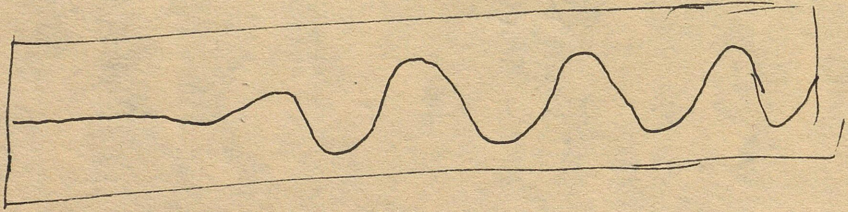
$\mu = 1$.

ie we get closed curves indicating periodic motion – self excited oscillations (constant amplitude). on $x - t$ diagram

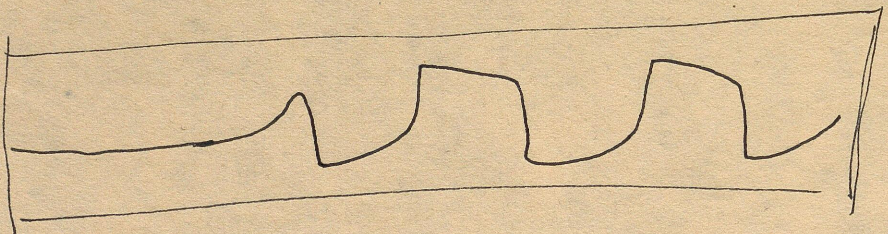
$\mu = .1$



$\mu = 1$



$\mu = 10$.



for large μ, large periods & hence name of relaxation oscillations – these appear in biological & economic phenomena and also in field of radio.

Case (ii): $f(x, \dot{x}) = -\mu \left\{ \dot{x} + \psi(x) \right\}$ linear damping & non linear restoring force

Of interesting case where external periodic force is present. In linear case for periodic external force "resonance" occurs i.e. when natural period of system = that of external force. In the non-linear case the external force excites oscillations whose frequency is a ~~frequency~~ fraction (& not a multiple) of its own frequency. This is subharmonic resonance (noticed by C.V. Raman also). In an airplane certain parts can be excited to violent oscillation by an engine running with a number of revolutions much larger than the natural frequency of the oscillating parts. This is explained by subharmonic resonance.

(4) Hydrodynamics – Many applications

'Heavy jet' – Priming in siphon spillway – Stability of jet – window round damsite being rattled within radius of several miles –