Examples from Ramsey's Hydrodynamics

Chap III onwards

(Chaps I & II in separate book)

(Second Edition of 1920)

Chapter III . p. 60.

(2) Very simple

(3) See Milne-Thomson set.

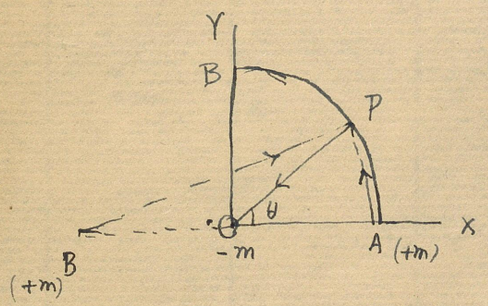
(5). Let sources be $(\pm a, 0)$ & sink at $(0,0)$. Then

$$w = -\log \frac{z^2-a^2}{z}.$$

giving $$\psi = \tan^{-1} \frac{a^2+r^2}{a^2-r^2} \tan\theta.$$

$$\phi = \log \frac{r}{(r^4 - 2a^2r^2\cos 2\theta + a^4)^{1/2}}$$

$$v = \left|\frac{dw}{dz}\right| = \left|\frac{a^2+z^2}{z(a^2-z^2)}\right| = \frac{1}{r}\left\{\frac{r^4+2a^2r^2\cos 2\theta + a^4}{r^4 - 2a^2r^2\cos 2\theta + a^4}\right\}^{1/2}$$



For $-m$ at O & $+m$ at A, the image consists of $+m$ at B. Obviously OX, OY are rigid boundaries

For a point P on the quadrant, normal vel at P

$$= -\left\{\frac{m}{BP}\cos OPB - \frac{m}{a} + \frac{m}{AP}\sin OPB\right\}$$

$$= -m\left\{\frac{\cos\frac{1}{2}\theta}{BP} - \frac{1}{a} + \frac{\sin\frac{1}{2}\theta}{AP}\right\}$$

$$= -m\left\{\frac{\cos\frac{1}{2}\theta}{2a\cos\frac{1}{2}\theta} - \frac{1}{a} + \frac{\sin\frac{1}{2}\theta}{2a\sin\frac{1}{2}\theta}\right\} = 0$$

Hence circular arc is also rigid boundary.

$\therefore w = -m\log\frac{z^2-a^2}{z}$ & ψ, ϕ are given as above

$\psi = c$ gives stream lines for different c. For the stream line leaving O at an angle α is given by $\theta = \alpha$ for $r = 0$ which gives $c = \alpha$ ie the stream line is

$$\frac{a^2+r^2}{a^2-r^2}\tan\theta = \tan\alpha$$

ie $$\frac{a^2}{r^2} = \frac{\sin(\alpha+\theta)}{\sin(\alpha-\theta)} \quad \text{ie } r^2\sin(\alpha+\theta) = a^2\sin(\alpha-\theta) \qquad (1)$$

This passes through $A(a,0)$ and at this point the tangent to the stream line makes with the radius vector an angle $= \left[\tan^{-1} r\frac{d\theta}{dr}\right]_{r=a,\theta=0}$.

From (1) $$2r\frac{dr}{d\theta}\sin(\alpha+\theta) + r^2\cos(\alpha+\theta) = -a^2\cos(\alpha-\theta)$$

$$\therefore \left(\frac{dr}{d\theta}\right)_{\substack{\theta=0\\ r=a}} = -a\cot\alpha$$

$$\therefore \left[\tan^{-1}\left(r\frac{d\theta}{dr}\right)\right]_{\substack{\theta=0\\ r=a}} = \pi-\alpha$$

ie the stream line also makes an angle α with the radius OA.

(6) $\phi + i\psi = \frac{1}{2}n(x+iy)^2 e^{2int}$ gives

$$\psi = \tfrac{1}{2}n\left\{2xy\cos 2nt + (x^2-y^2)\sin 2nt\right\}$$

$$\phi = \tfrac{1}{2}n\left\{(x^2-y^2)\cos 2nt - 2xy\sin 2nt\right\}$$

ψ = const gives the stream lines.

Also $\phi = \frac{1}{2}nr^2\cos(2\theta+2nt)$, so that for the paths of particles

$$\frac{dr}{dt} = -\frac{\partial\phi}{\partial r}, \text{ and } r\frac{d\theta}{dt} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} \text{ give}$$

$$\left.\begin{aligned}\frac{dr}{dt} &= -nr\cos(2\theta+2nt)\\ \frac{d\theta}{dt} &= n\sin(2\theta+2nt)\end{aligned}\right\}$$

The solution of this pair of simultaneous eqns gives the paths of particles. From the above eqns

$$\left.\begin{aligned}&\frac{dr}{dt}\cos(nt+\theta) - r\frac{d\theta}{dt}\sin(nt+\theta) = -nr\cos(nt+\theta)\\ \text{and}\quad &\frac{dr}{dt}\sin(nt+\theta) + r\frac{d\theta}{dt}\cos(nt+\theta) = nr\sin(nt+\theta)\end{aligned}\right\}$$

ie

$$\left.\begin{aligned}&\frac{dr}{dt}\cos(nt+\theta) - r\left(n+\frac{d\theta}{dt}\right)\sin(nt+\theta) = -nr\cos(nt+\theta)\\ &\qquad\qquad -nr\sin(nt+\theta)\\ \&\quad &\frac{dr}{dt}\sin(nt+\theta) + r\left(n+\frac{d\theta}{dt}\right)\cos(nt+\theta) = nr\sin(nt+\theta)+nr\cos(nt+\theta)\end{aligned}\right\}$$

ie

$$\left.\begin{aligned}\frac{d}{dt}\left\{r\cos(nt+\theta)\right\} &= -nr\left\{\cos(nt+\theta)+\sin(nt+\theta)\right\}\\ \frac{d}{dt}\left\{r\sin(nt+\theta)\right\} &= nr\left\{\cos(nt+\theta)+\sin(nt+\theta)\right\}\end{aligned}\right\}$$

Adding & Subtracting

$$\left.\begin{aligned}&\frac{d}{dt}\left\{r\cos(nt+\theta)+r\sin(nt+\theta)\right\} = 0\\ &\frac{d}{dt}\left\{r\sin(nt+\theta)\mp r\cos(nt+\theta)\right\} = 2nr\left\{\sin(nt+\theta)+\cos(nt+\theta)\right\}\end{aligned}\right\}$$

The first gives $r\cos(nt+\theta) + r\sin(nt+\theta) = K$, and the second then gives

$$\frac{d}{dt}\left\{r\sin(nt+\theta) - r\cos(nt+\theta)\right\} = 2nK$$

ie $r\sin(nt+\theta) - r\cos(nt+\theta) = 2Knt + K'$
$r\sin(nt+\theta) + r\cos(nt+\theta) = K$

$r\cos(nt+\theta) = \frac{1}{2}K(1-2nt) - \frac{1}{2}K'$
$r\sin(nt+\theta) = \frac{1}{2}K(1+2nt) + \frac{1}{2}K'$

ie $r\cos(nt+\theta) - \frac{1}{2}K = -Knt$
$r\sin(nt+\theta) - \frac{1}{2}K' = Knt$

ie $r\cos(nt+\theta) - \frac{1}{2}(K-K') = -Knt$
$r\sin(nt+\theta) - \frac{1}{2}(K+K') = Knt$

Putting $K = x_0 - y_0$ & $K' = -(x_0+y_0)$

$\frac{1}{2}(K-K') = x_0$; $\frac{1}{2}(K+K') = -y_0$

and $r\cos(nt+\theta) - x_0 = -nt(x_0-y_0)$

$r\sin(nt+\theta) + y_0 = nt(x_0-y_0)$

ie $nt(x_0-y_0) = x_0 - r\cos(nt+\theta) = y_0 + r\sin(nt+\theta)$.

The eqns as given in the problem are thus to be duly modified.

(7) If $f(x,y,\lambda) = 0$ be written as $\lambda = F(x,y)$ we have stream lines for different values of λ. Since the stream lines are also given by $\psi = \text{const}$, we have $\psi = G(\lambda)$. Now ψ satisfies Laplace's equation

$$\frac{\partial\psi}{\partial x} = G'(\lambda)\frac{\partial\lambda}{\partial x} \; ; \; \frac{\partial\psi}{\partial y} = G'(\lambda)\frac{\partial\lambda}{\partial y}$$

$$\frac{\partial^2\psi}{\partial x^2} = G''(\lambda)\left(\frac{\partial\lambda}{\partial x}\right)^2 + G'(\lambda)\frac{\partial^2\lambda}{\partial x^2} \; ; \; \frac{\partial^2\psi}{\partial y^2} = G''(\lambda)\left(\frac{\partial\lambda}{\partial y}\right)^2 + G'(\lambda)\frac{\partial^2\lambda}{\partial y^2}$$

$\therefore \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$ gives $\left(\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2}\right) \Big/ \left(\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right) = -\frac{G''(\lambda)}{G'(\lambda)}$

= a function of λ alone

which is the required condition

(9) See Milne-Thomson.

(10) " Bookwork.

(11) See Milne-Thomson.

(12) From (11), $w = -m\log\frac{(z-a)(z+a)}{(z-b)(z+b)}\frac{(z+ia)(z-ia)}{(z+ib)(z-ib)} = -m\log\frac{z^4-a^4}{z^4-b^4}$

and $v = \left|\frac{dw}{dz}\right|$. we find $\frac{dw}{dz} = -m\cdot\frac{4r^3e^{3i\theta}(a^4-b^4)}{\rho_1\rho_2}e^{-i(\phi_1+\phi_2)}$

where $r^4\cos 4\theta - a^4 = \rho_1\cos\phi_1$; $r^4\sin 4\theta = \rho_1\sin\phi_1$
$r^4\cos 4\theta - b^4 = \rho_2\cos\phi_2$; $r^4\sin 4\theta = \rho_2\sin\phi_2$

Thus $\frac{1}{m}\left|\frac{dw}{dz}\right| = \frac{4r^3(a^4-b^4)}{(r^8+a^8-2r^4a^4\cos 4\theta)^{1/2}(r^8+b^8-2r^4b^4\cos 4\theta)^{1/2}}$

(14) Not fully worked.

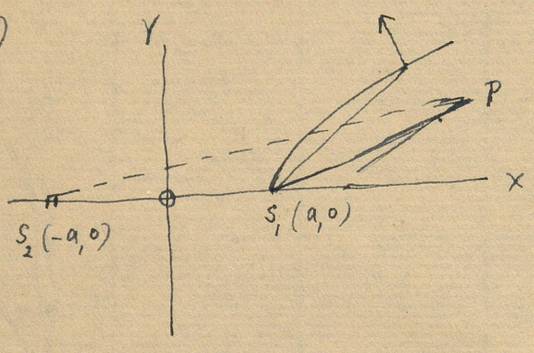
(15) See Ex. 5 – Here we have sinks at $(\pm a, 0)$ and a source at origin giving

$$w = \log (z^2-a^2)/z$$

giving $\psi = \tan^{-1}\frac{r^2+a^2}{r^2-a^2}\tan\theta$

Both $r = a$ and $\theta = \pi/2$ give the same value for ψ viz $\pm \pi/2$ ie two of the stream lines divide into the circle $r=a$ & the y-axis $\theta = \pi/2$.

(16)



If there be an equal source at $S_2(-a,0)$ the complex potential is given by

$$w = \log (z^2-a^2) \quad (-m = 1)$$

giving $\psi = \tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta - a^2}$

and for $\psi = \frac{\pi}{2}$, we get $r^2\cos 2\theta = a^2$ ie the hyperbola $x^2-y^2=a^2$ itself, showing that equal source at S_2 is the image of S_1 for the hyperbola being a boundary. Stream lines in general are given by

$$x^2-y^2-2cxy-a^2 = \text{const } 0$$

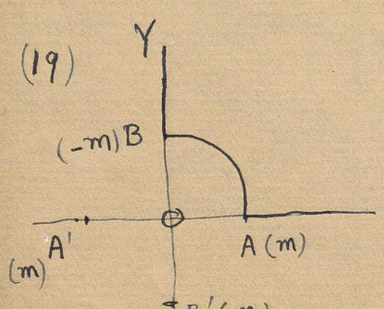
ie a system of hyperbolas. For $\psi = 0$, $r^2\sin 2\theta = 0$ ie $xy = 0$ ~~asymptotes~~

For $\psi = \pi/4$, $x^2-y^2-2xy = a^2$.

Also, $\frac{dw}{dz} = \frac{2z}{z^2-a^2} = \frac{2z}{(z-a)(z+a)}$

$$\therefore \left|\frac{dw}{dz}\right| = \frac{2OP}{PS_1 \cdot PS_2} = \frac{2OP}{P_1P_2}.$$

(19)



The image system consists of m at $A'(-a,0)$ and $-m$ at $B'(0,-a)$.

$$w = -m\log(z^2-a^2)/(z^2+a^2).$$

& $\psi = -m\tan^{-1}\left(\frac{2a^2r^2\sin 2\theta}{r^4-a^4}\right)$

$$\phi = \log\left(\frac{r^4 - 2a^2r^2\cos 2\theta + a^4}{r^4 + 2a^2r^2\cos 2\theta + a^4}\right)$$

The stream lines are given by

$$r^4 - a^4 = 2ca^2r^2\sin 2\theta \text{ for different values of } c.$$

from which $4r^3\frac{dr}{d\theta} = 4ca^2r\frac{dr}{d\theta}\sin 2\theta + 4ca^2r^2\cos 2\theta$

$$\therefore \left(\frac{dr}{d\theta}\right)_{a,0} = ca \qquad \therefore \tan\mu = \left(r\frac{d\theta}{dr}\right)_{a,0} = \frac{1}{c}$$

ie required stream line is

$$2a^2r^2\sin 2\theta = \tan\mu\,(r^4 - a^4)$$

giving $\frac{r^2}{a^2} = \left\{\cot\mu + \sqrt{\cot^2\mu + \operatorname{cosec}^2 2\theta}\right\}\sin 2\theta$

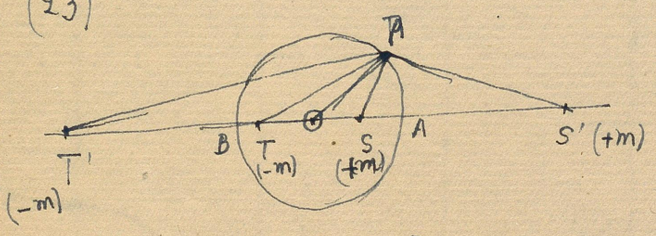
or $r = a\sin^{\frac{1}{2}}2\theta\left\{\cot\mu + \sqrt{\cot^2\mu + \operatorname{cosec}^2 2\theta}\right\}^{1/2}$.

(20) See Milne-Thomson.

(21) ibid. Ex. 34, p. 227.

(22) ibid, §8.51

(23)



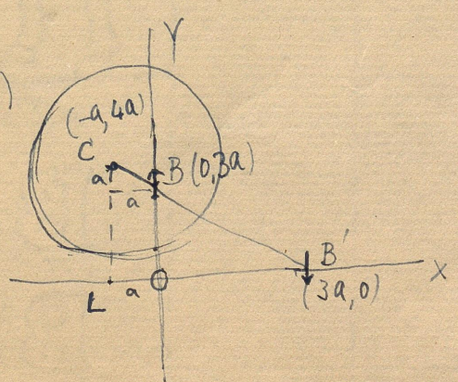
$[OS = OT = f]$

$$w = -m\log\frac{(z-f)(z-a^2/f)}{(z+f)(z+a^2/f)}$$

$$\frac{dw}{dz} = -\frac{2m(f^2+a^2)(z^2-a^2)}{f(z-f)(z+f)(z-a^2/f)(z+a^2/f)}$$

$$\therefore v = \left|\frac{dw}{dz}\right| = 2m\cdot\frac{f^2+a^2}{f}\cdot\frac{PA\cdot PB}{PS\cdot PS'\cdot PT\cdot PT'}$$

(24)



The image of doublet along OY at B(0,3a) is at B' such that $CB' = \frac{8a^2}{CB} = \frac{8a^2}{2a\sqrt{2}} = 4a\sqrt{2}$

If B' be the point at which CB meets the x-axis then from similar Δ^s $LB' = CL = 4a$ & $CB' = 4a\sqrt{2}$

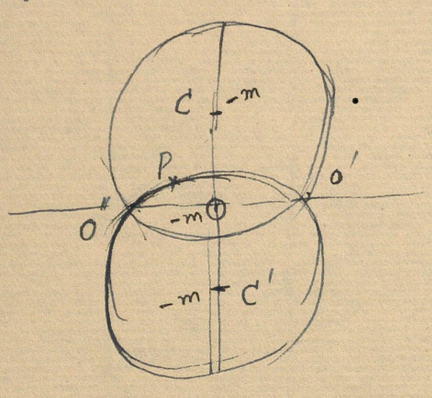
ie the image is the point where CB meets OX ie the point (3a, 0)

and the doublet at B' is of same strength but anti-parallel

$$\therefore \quad w = \frac{\mu e^{i\pi/2}}{z - 3ia} + \frac{\mu e^{-i\pi/2}}{z - 3a} = \frac{i\mu}{z - 3ia} - \frac{i\mu}{z - 3a}$$

which gives the required velocity potential.

(25)



Due to one circle only the image of $2m$ at O' is $-m$ at C. Also image of $-m$ at C in the other circle is $-m$ at O since ~~the~~ O is the inverse point the circles intersecting orthogonally. Thus with the arrangement $4m$ at O' and $-m, -m, -m$ at C, O & C' the two circles form a boundary. This could also be verified by consideration of the normal velocity at any point P due to the four sources & sinks. The complex potential is therefore given by

$$w = -\frac{m}{2\pi} \log \frac{(z-1)^4}{z(z^2+1)} = \frac{m}{2\pi} \log z(z^2+1)/(z-1)^4$$

$$-\frac{2\pi}{m}\frac{dw}{dz} = \frac{(z+1)^3}{z(z-1)(z^2+1)}$$

Hence $\left|\frac{dw}{dz}\right| = 0$ when $|z+1| = 0$ ie O'' is a double point, which is otherwise obvious since the ψs are stream lines and intersect at O''. Similarly for O'. The eqn for $\left|\frac{dw}{dz}\right|$ shows that there are no other double points.

(26) ~~Consider $z = c\cosh w$~~

~~$x = c\cosh\phi\cos\psi$~~

Consider

$$\left.\begin{aligned} x &= c\cosh\psi\cos\phi \\ y &= -c\sinh\psi\sin\phi \end{aligned}\right\} \qquad (1)$$

which is equivalent to $z = c\cos w$. ψ = const gives the ellipses $\frac{x^2}{c^2\cosh^2\psi} + \frac{y^2}{c^2\sinh^2\psi} = \text{const}$. with $\cosh\psi = \sqrt{a^2+\lambda}$, $\sinh\psi = \sqrt{b^2+\lambda}$

$$\tanh\psi = \sqrt{\frac{b^2+\lambda}{a^2+\lambda}} \quad \text{or} \quad \psi = \frac{1}{2}\log\frac{\sqrt{b^2+\lambda}+\sqrt{a^2+\lambda}}{\sqrt{a^2+\lambda}-\sqrt{b^2+\lambda}}$$

$$= A\log\left[\sqrt{a^2+\lambda}+\sqrt{b^2+\lambda}\right] + B$$

Also $\frac{dw}{dz} = -\frac{1}{c\sin w} = -\frac{1}{\sqrt{c^2-z^2}}$ $\therefore \quad q = \left|\frac{dw}{dz}\right| = \left|\frac{1}{\sqrt{(z+c)(z-c)}}\right| = \frac{1}{\sqrt{PS\cdot PS'}}$

Since $c = \sqrt{a^2 - b^2} = ae$.

(27) Vide Milne-Thomson.

(30) "

(33) Eqn of continuity is

$$r^2 \rho v = \text{const} = c$$

The eqn quoted is

$$v \frac{\partial v}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \quad ; \quad p = k\rho$$

$$\therefore \quad -\frac{c}{r^2} \frac{\partial v}{\partial r} = -k \frac{\partial \rho}{\partial r} \quad , \quad \rho = \frac{c}{r^2 v}$$

$$= -kc \left[-\frac{1}{r^2 v^2} \frac{\partial v}{\partial r} - \frac{2}{r^3 v} \right]$$

$$\therefore \quad \left(v - \frac{k}{v} \right) \frac{\partial v}{\partial r} = \frac{2k}{r} \quad \text{Integ.} \quad \frac{1}{2} v^2 - k \log v = 2k \log r$$

$$\text{or } \log r = \frac{v^2}{4k} - \frac{1}{2} \log v$$

$$r = \frac{1}{\sqrt{v}} e^{v^2/4k}$$

(36) $w = -\mu \log \frac{z - 2a}{z}$ if the sink be taken as the origin

$$\frac{dw}{dz} = -\frac{2a\mu}{z(z-2a)} \quad \therefore \quad \left| \frac{dw}{dz} \right| = \frac{2a\mu}{OP \cdot PS} \quad , \; S \text{ being the source}$$

$\therefore$ Velocity normal to the plane $= \dfrac{2a^2\mu}{(a^2+y^2+z^2)^{3/2}}$, (a, y, z) being the coordinates of any point on the plane. Hence the kinetic energy of liquid passing in unit time across the plane

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2} \rho q_n^2 \cdot q_n \, dy \, dz = 4 \int_0^{\infty} \int_0^{\infty} \frac{1}{2} \rho \cdot \frac{8a^6 \mu^3 \, dy \, dz}{(a^2+y^2+z^2)^{9/2}}$$

$$= 16 \rho a^6 \mu^3 \int_0^{\infty} \int_0^{\pi/2} \frac{dy \sqrt{a^2+y^2} \sin^2\theta \, d\theta}{(a^2+y^2)^{9/2} \sec^9 \theta}$$

$$= 16 \rho a^6 \mu^3 \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi/2} \frac{a \sec^2\phi \cdot d\phi}{a^8 \sec^8 \phi} = \frac{16 \rho \mu^3}{a} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \left(8\pi\rho\mu^3 / 7a \right).$$

Chapter IV – p. 90.

(3) The motion of pure strain is normal to

$$a\xi^2 + b\eta^2 + c\zeta^2 + 2f\eta\zeta + \cdots = \text{Const} \qquad (1)$$

where $a = \dfrac{\partial u}{\partial x}$, ..

$$\phi = \lambda xyz, \quad u = -\lambda yz, \text{ etc}$$

$$\therefore \frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z}$$

Hence (1) reduces to

$$2f\eta\zeta + - = \text{Const}$$

$$f = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \text{ etc}$$

$$= x$$

$\therefore$ The strain quadric has the eqn

$$x\eta\zeta + y\zeta\xi + z\xi\eta = 0 \text{ whose centre is } (x, y, z).$$

(4) From the eqns of motion, we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \cdots = X - \frac{1}{\rho}\frac{\partial p}{\partial x}, \text{ etc}$$

ie $$\lambda = X - \frac{1}{\rho}\frac{\partial p}{\partial x} - \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} + w\frac{\partial w}{\partial x}\right)$$

Similarly $$\mu = Y - \frac{1}{\rho}\frac{\partial p}{\partial y} - \left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial y}\right)$$

$$\nu = Z - \frac{1}{\rho}\frac{\partial p}{\partial z} - \left(u\frac{\partial u}{\partial z} + v\frac{\partial v}{\partial z} + w\frac{\partial w}{\partial z}\right)$$

$$\lambda\,dx + \mu\,dy + \nu\,dz = (X\,dx + Y\,dy + Z\,dz) - \frac{1}{\rho}\,dp - \frac{1}{2}\,d(u^2 + v^2 + w^2).$$

which is a perfect differential

(7) $q^2 = u^2 + v^2 + w^2$

$$\nabla^2 q^2 = (\text{grad}\,u)^2 + (\text{grad}\,v)^2 + (\text{grad}\,w)^2 + \left\{u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 v}{\partial x^2} + w\frac{\partial^2 w}{\partial x^2}\right\} + \cdots + \cdots$$

$$= \Sigma(\text{grad}\,u)^2 + u\nabla^2 u + v\nabla^2 v + w\nabla^2 w$$

From the eqn of continuity $\text{div}\,\vec{q} = 0$

& from the fact that motion is irrotational $\nabla^2 u = 0$, ..

Hence $\nabla^2 q^2 = +^{ve}$ ~~(This is no proof at all.)~~

For q^2 to be a max, $\nabla^2 q^2 = -^{ve}$ & hence q^2 cannot be a max at a point in liquid

If the motion be steady, from eqns of motion

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\therefore \frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} = -\frac{\partial^2 V}{\partial x^2} + (\text{grad } u)^2 + \left(u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 u}{\partial x^2} + w\frac{\partial^2 u}{\partial x^2}\right)$$

$$\therefore -\frac{1}{\rho}\nabla^2 p = \nabla^2 V + \Sigma(\text{grad } u)^2 + \Sigma u \nabla^2 u$$

$$= -\text{ve if } \nabla^2 V = 0$$

$\therefore$ p cannot be a minimum inside liquid.

(8) The motion is irrotational if & only if

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial \dot{x}}{\partial a} = \frac{\partial u}{\partial a} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial a}$$

$$\therefore \frac{\partial(\dot{x}, x)}{\partial(a, b)} = \frac{\partial x}{\partial b}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a}\right) - \frac{\partial x}{\partial a}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b}\right)$$

$$= \frac{\partial u}{\partial y}\left(\frac{\partial x}{\partial b}\frac{\partial y}{\partial a} - \frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\right)$$

Similarly $\dfrac{\partial(\dot{y}, y)}{\partial(a, b)} = \dfrac{\partial v}{\partial x}\left(\dfrac{\partial y}{\partial b}\dfrac{\partial x}{\partial a} - \dfrac{\partial y}{\partial a}\dfrac{\partial x}{\partial b}\right)$

$$\therefore \frac{\partial(\dot{x}, x)}{\partial(a, b)} + \frac{\partial(\dot{y}, y)}{\partial(a, b)} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\left(\frac{\partial y}{\partial b}\frac{\partial x}{\partial a} - \frac{\partial y}{\partial a}\frac{\partial x}{\partial b}\right)$$

$$= 0.$$

(9) $(u, v, w) = -\text{grad}\,\phi = (ax, by, cz)$.

Eqn of continuity viz $\text{div}\,\vec{q} = 0$ gives $a + b + c = 0$ which is satisfied showing that liquid is homogeneous.

The eqns of motion give

$$x\ddot{a} + a^2 x = -lx - \frac{1}{\rho}\frac{\partial p}{\partial x}, \text{ etc}$$

$$\therefore (l + a^2 + \dot{a})x^2 + \cdots + \cdots = -\frac{2}{\rho}p + C$$

Eqn of surface of equi-pressure is

$$(l + a^2 + \dot{a})x^2 + \cdots = \text{const}$$

If this be free surface (bounding surface)

we should have

$$x^2 \frac{\partial}{\partial t}\left(l+a^2+\dot{a}\right) + \cdots + \cdots$$

$$+ 2ax^2\left(l+a^2+\dot{a}\right) + \cdots \qquad = 0$$

leading to $\frac{\partial}{\partial t}\left(l+a^2+\dot{a}\right) + 2a\left(l+a^2+\dot{a}\right) = 0$, etc

Integrating $\left(l+a^2+\dot{a}\right)e^{2\int a\,dt} = \text{const}$, etc.

(10) Lines of flow are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$u = -\frac{\partial \phi}{\partial x} = A\left\{\frac{yz\left(x^2+y^2+z^2\right) + 3xz\left(x^2+y^2\right)\tan^{-1} y/x}{\left(x^2+y^2\right)\left(x^2+y^2+z^2\right)^{5/2}}\right\}$$

$$v = -A\left\{\frac{xz\left(x^2+y^2+z^2\right) - 3yz\left(x^2+y^2\right)\tan^{-1} y/x}{\text{"}}\right\}$$

$$w = -A\left\{\frac{\left(x^2+y^2-2z^2\right)\tan^{-1} y/x}{\left(x^2+y^2+z^2\right)^{5/2}}\right\}$$

Hence [From $\frac{\left(x^2+y^2+z^2\right)}{\left(x^2+y^2\right)^{4/3}} = K^{2/3}\left(x^2+y^2\right)^{2/3}$ (1)

we get by total differentiation,

$$x\left(x^2+y^2-2z^2\right)dx + y\left(x^2+y^2-2z^2\right)dy + 3z\left(x^2+y^2\right)dz = 0 \quad (2)\ ]$$

The lines of flow can be written as

$$\frac{x\left(x^2+y^2+z^2\right)dx}{-xyz\left(x^2+y^2+z^2\right)\left(x^2+y^2-2z^2\right) - 3x^2z\left(x^2+y^2\right)\left(x^2+y^2-2z^2\right)\tan^{-1} y/x}$$

$$= \frac{y\left(y^2+z^2-2z^2\right)dy}{xyz\left(x^2+y^2+z^2\right)\left(x^2+y^2-2z^2\right) - 3y^2z\left(x^2+y^2\right)\left(x^2+y^2-2z^2\right)\tan^{-1} y/x}$$

$$= \frac{3z\left(x^2+y^2\right)dz}{3z\left(x^2+y^2\right)\left(x^2+y^2-2z^2\right)\tan^{-1} y/x}$$

$$= \frac{x\left(x^2+y^2-2z^2\right)dx + - +}{0}$$

This gives (2) & hence by integration (1).

(11) Diff eqn of lines of flow : $\frac{d\theta}{u} = \frac{\sin\theta\, d\omega}{v}$

or $v\,d\theta - u\sin\theta\, d\omega = 0$.

Besant & Ramsey. Ex. 3, p. 16

Eqn of continuity in polar coordinates is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho q_r r^2\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho q_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\rho q_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\cdot\frac{\partial \rho q_\omega}{\partial \omega} = 0$$

Here $q_\theta = q_\omega = 0$ & eqn $\rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho q_r r^2\right) = 0$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} + \rho q_r \cdot 2r + \frac{1}{r^2}\cdot \rho r^2 \frac{\partial q_r}{\partial r} + \frac{1}{r^2}\cdot q_r \cdot r^2 \frac{\partial \rho}{\partial r} = 0$$

Putting $q_r = u$] $\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial r} + \frac{\rho}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2 u\right) = 0$

Eqn of continuity : $\frac{\partial}{\partial\theta}(u\sin\theta) + \frac{\partial v}{\partial\omega} = 0$

or $\frac{\partial}{\partial\theta}(-u\sin\theta) = \frac{\partial v}{\partial\omega}$

$\therefore$ $v\,d\theta - u\sin\theta\,d\omega$ = perfect differential $\equiv d\psi$, say

$v = \frac{\partial\psi}{\partial\theta}$, $u\sin\theta = -\frac{\partial\psi}{\partial\omega}$ or $u = -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\omega}$.

If ϕ be velocity potential

$$\frac{1}{a}\frac{\partial\phi}{\partial\theta} = -u, \quad \frac{1}{a\sin\theta}\frac{\partial\phi}{\partial\omega} = -v$$

$$\therefore \frac{\partial\phi}{\partial\theta} = -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\omega}\,; \quad \frac{1}{\sin\theta}\frac{\partial\phi}{\partial\omega} = \frac{\partial\psi}{\partial\theta}$$

Putting $x = \log\tan\frac{1}{2}\theta$

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\theta}\cdot\frac{\partial\theta}{\partial x} = \sin\theta\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial\omega}$$

Similarly $\frac{\partial\phi}{\partial\omega} = \frac{\partial\psi}{\partial x}$

$\therefore$ $\phi + i\psi = f(x + i\omega)$

$= F\left(e^{x+i\omega}\right) = F\left(e^{i\omega}\tan\frac{1}{2}\theta\right)$.

(12) From the previous example

$$\phi + i\psi = F\left(e^{i\omega}\tan\tfrac{1}{2}\theta\right)$$

$$\therefore \phi = F\left(e^{i\omega}\tan\tfrac{1}{2}\theta\right) + F\left(e^{-i\omega}\tan\tfrac{1}{2}\theta\right) \qquad \text{(imaginary parts cancelling)}$$

$$= F\left\{\frac{(\cos\omega + i\sin\omega)\,r\sin\theta}{r(1+\cos\theta)}\right\} + F\left\{\frac{(\cos\omega - i\sin\omega)\,r\sin\theta}{r(1+\cos\theta)}\right\}$$

$$= F\left(\frac{x+iy}{r+z}\right) + F\left(\frac{x-iy}{r+z}\right)$$

(16) See Milne-Thomson set

K.S.K. remarks for this example that result follows from the fact that Log q is a Laplacian.

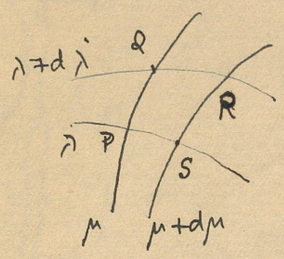
(17) Accelration along the normal $= q^2/\rho = lx'' + my'' + nz''$

$$x'' = \dot{u} = \frac{\partial u}{\partial x}\dot{x} + \cdots = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = (\vec{q}\cdot\operatorname{grad} u)$$

Similarly $y'' = (\vec{q}\cdot\operatorname{grad} v)$, $z'' = (\vec{q}\cdot\operatorname{grad} w)$

$$\frac{q^2}{\rho} = l\frac{\partial}{\partial x}\left(\tfrac{1}{2}q^2\right) + m\frac{\partial}{\partial y}\left(\tfrac{1}{2}q^2\right) + n\frac{\partial}{\partial x}\left(\tfrac{1}{2}q^2\right)$$

$$= \frac{\partial}{\partial \nu}\left(\tfrac{1}{2}q^2\right) = -\frac{\partial}{\partial \nu}\left(\frac{p}{\rho} + V\right)$$



Let $\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} - 1 = 0$ be system of confocal ellipses.

In confocal coordinates

$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)}{a^2-b^2}, \quad y^2 = \frac{(b^2+\lambda)(b^2+\mu)}{b^2-a^2}$$

Let P be (x,y), $Q\left(x+\frac{\partial x}{\partial \lambda}d\lambda, y+\frac{\partial y}{\partial \lambda}d\lambda\right)$

$$d\nu^2 = \left\{\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2\right\} d\lambda^2 = \frac{\lambda-\mu}{4(a^2+\lambda)(b^2+\lambda)}\,d\lambda^2$$

$$\frac{1}{\rho} = \frac{1}{q^2}\frac{\partial}{\partial \nu}\left(\tfrac{1}{2}q^2\right) = \frac{1}{q}\frac{\partial q}{\partial \nu} = \frac{\partial}{\partial \nu}(\log q) = \frac{\partial \lambda}{\partial \nu}\cdot\frac{\partial}{\partial \lambda}(\log q)$$

$$= \frac{2\sqrt{(a^2+\lambda)(b^2+\lambda)}}{\sqrt{\lambda-\mu}}\frac{\partial}{\partial \lambda}(\log q) \qquad (1)$$

$$\frac{1}{\rho} = \frac{p^3}{(a^2+\lambda)(b^2+\lambda)} \quad \& \quad p^2 = \frac{(a^2+\lambda)(b^2+\lambda)}{(\lambda-\mu)}$$

$$\therefore \frac{1}{\rho} = \frac{\sqrt{(a^2+\lambda)(b^2+\lambda)}}{(\lambda-\mu)^{3/2}} \qquad (2)$$

$\therefore$ From (1) & (2), $\quad \frac{\partial}{\partial \lambda}(\log q) = \frac{1}{2(\lambda-\mu)}$

$$\text{or } \log\left(\frac{q}{c}\right) = \tfrac{1}{2}\log(\lambda-\mu) \quad \text{ie } q = c\sqrt{\lambda-\mu}$$

Also $q = -\frac{\partial \phi}{\partial s}$ where δs is an element of a stream line

$$\delta s^2 = \left\{\left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2\right\} d\mu^2 = \frac{(\mu-\lambda)}{4(a^2+\mu)(b^2+\mu)}\,\delta\mu^2$$

$$\therefore \frac{\partial \phi}{\partial s} = \frac{\partial \phi}{\partial \mu}\cdot\frac{\partial \mu}{\partial s} = \frac{2\sqrt{(a^2+\mu)(b^2+\mu)}}{\sqrt{\lambda-\mu}}\cdot\frac{\partial \phi}{\partial \mu} = -c\sqrt{\lambda-\mu}$$

$$\therefore \frac{\partial \phi}{\partial \mu} = -\frac{c(\lambda-\mu)}{2\sqrt{-(a^2+\mu)(b^2+\mu)}}$$

$$\therefore \phi = \int \frac{c(\mu-\lambda)}{2\sqrt{-(a^2+\mu)(b^2+\mu)}}\,d\mu.$$

(19) Cf. p. 88, § 92. Example:

The velocity at a distance r from the axis is c^2/r at right angles to the radius-vector

$$T = \frac{l}{2}\cdot\rho\int_a^b \frac{c^4}{r^2}\cdot 2\pi r\,dr$$

$$= \pi\rho c^4 l \log(b/a)$$

velocity $= \overline{V}$ when $r = a$

$$\therefore\ c^2/a = \overline{V} \qquad \therefore c^2 = a\overline{V}$$

$$\therefore\ T = \pi\rho l a^2 \overline{V}^2 \log(b/a).$$

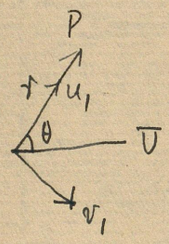
———o———

Chapter V, p. 113.

(1) The pressure is given by

$$\frac{p}{\rho} = \frac{\partial\phi}{\partial t} - \tfrac{1}{2}q^2 + F(t)$$

$$\phi = \frac{Ua^2}{r}\cos\theta$$



$$\frac{\partial\phi}{\partial t} = \frac{dU}{dt}\cdot\frac{a^2}{r}\cos\theta - \frac{Ua^2}{r^2}\cos\theta\cdot\frac{dr}{dt} - \sin\theta\cdot\frac{Ua^2}{r}\frac{d\theta}{dt}$$

$$\frac{dr}{dt} = -U\cos\theta\ ;\ \frac{d\theta}{dt} = \frac{U\sin\theta}{r}$$

$$\frac{\partial\phi}{\partial t} = \frac{dU}{dt}\cdot\frac{a^2}{r}\cos\theta + \frac{U^2a^2}{r^2}\cos 2\theta$$

$$\left.\begin{aligned}\tan\theta &= \frac{v_1}{u_1}\\ u_1^2 + v_1^2 &= U^2\\ f_1 &= \frac{dU}{dt}\cos\theta\end{aligned}\right\}$$

Also $$-\frac{\partial\phi}{\partial r} = \frac{Ua^2}{r^2}\cos\theta\ ;\ -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{Ua^2}{r^2}\sin\theta$$

$$\therefore\ q^2 = \frac{U^2a^4}{r^4}$$

$$\therefore\ \frac{p}{\rho} = -\frac{1}{2}\frac{U^2a^4}{r^4} + \frac{a^2}{r}\cos\theta\frac{dU}{dt} + \frac{U^2a^2}{r^2}(\cos^2\theta - \sin^2\theta) + F(t)$$

$$= -\frac{1}{2}(u_1^2+v_1^2)\frac{a^4}{r^4} + \frac{a^2}{r}f_1 + \frac{a^2}{r^2}(u_1^2 - v_1^2) + F(t) \qquad -(1)$$

The hydrostatic pressure is given by $q = 0$; or $\phi = 0$ & hence $= \rho F(t)$

The required difference is from (1) without the last term & can be re-arranged in the required form.

(2) $\phi = U(r + a^2/r)\cos\theta$, this gives

$$V^2 = \frac{U^2}{r^4}(a^4 + r^4 - 2a^2r^2\cos 2\theta)$$

$$= \frac{u^2}{r^4}(\quad\text{"}\quad) \text{ since } V = u \text{ for } r = \infty$$

The other parts are faulty. See Blasius theorem for case involving t in Milne-Thomson.

(3) In time t the centre of the cylinder traverses a distance Vt. Referred to the centre of the cylinder

$$\phi = \frac{Va^2}{r}\cos\theta = \frac{Va^2 x}{r^2}$$

Hence referred to the fixed origin

$$\phi = \frac{Va^2(x-Vt)}{(x-Vt)^2+y^2}$$

$$\therefore\ -\frac{\partial\phi}{\partial x} = -\frac{Va^2\{y^2-(x-Vt)^2\}}{\{(x-Vt)^2+y^2\}^2} = \frac{dx}{dt}$$

$$-\frac{\partial\phi}{\partial y} = \frac{dy}{dt} = \frac{2yVa^2(x-Vt)}{\{(x-Vt)^2+y^2\}^2}$$

$$\frac{dx}{dy} + i\frac{dy}{dt} = \frac{dz}{dt} = \frac{Va^2}{(z'-Vt)^2} \quad \text{on simplification}$$

In like manner, $\dfrac{dz'}{dt} = \dfrac{Va^2}{(z-Vt)^2}$.

(4) See p. 88. As in that article

$$\phi = -c^2\tan^{-1}\left(\frac{y}{x}\right) = -c^2\theta = A\theta.$$

$$\therefore\ T = \tfrac{1}{2}\rho\int_a^b \frac{c^4}{r^2}\cdot 2\pi r\,dr = \pi\rho A^2\log(b/a) \quad \text{etc.}$$

(not clear)

(5) $\phi = \dfrac{Ua^2}{r}\cos\theta$, $\psi = -\dfrac{Ua^2}{r}\sin\theta$

$$w = \phi + i\psi = \frac{Ua^2}{z}$$

with initial position of the centre as fixed origin

$$w = \frac{Ua^2}{z-Ut}.$$

Relative to the cylinder,

$$\phi = U\cos\theta\,(r + a^2/r)$$

$$-\frac{\partial\phi}{\partial r} = \frac{(a^2-r^2)U\cos\theta}{r^2}\ ;\ -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{a^2+r^2}{r^2}U\sin\theta$$

$$\therefore\ \frac{dr}{dt} = \left(\frac{a^2-r^2}{r^2}\right)U\cos\theta\ ;\ r\frac{d\theta}{dt} = \frac{a^2+r^2}{r^2}U\sin\theta$$

$$\frac{1}{r}\frac{dr}{dt} + i\frac{d\theta}{dt} = \frac{U}{r}\left\{\frac{a^2}{r^2}e^{i\theta} - e^{-i\theta}\right\}$$

Also the stream function being $\psi = (r - a^2/r)\sin\theta$

$\psi = b$ gives the other equation

For the second part see bookwork in Milne-Thomson.

(6) $\phi = \dfrac{a^2 V}{b^2 - a^2}\left(r + \dfrac{b^2}{r}\right)\cos\theta$ (p. 97)

The amount by which the pressure of the liquid increases the inertia of the cylinder is

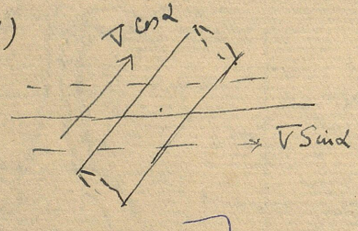
$$\frac{\pi\rho a^2 V}{b^2 - a^2}(b^2 + a^2).$$

Hence required impulse

$$= \frac{\pi\rho a^2 V}{b^2 - a^2}(b^2 + a^2) + \pi\sigma a^2 V \text{ , per unit length}$$

$$= \frac{\pi a^2}{b^2 - a^2}\left\{(\sigma + \rho)b^2 - (\sigma - \rho)a^2\right\} V.$$

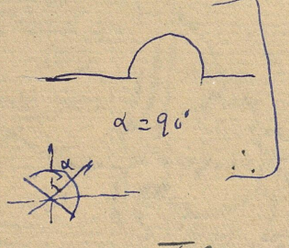
(7)



$$\phi = V\sin\alpha\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$\frac{\partial\phi}{\partial r} = V\sin\alpha\left(1 - a^2/r^2\right)\cos\theta$$

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = V\sin\alpha\left(1 + a^2/r^2\right)\sin\theta.$$



over the surface, $q^2 = 4V^2\sin^2\alpha\sin^2\theta$.

$$\therefore \frac{p}{\rho} = C - \frac{1}{2}q^2 = C - 2V^2\sin^2\alpha\sin^2\theta$$

$V\sin\alpha$ is also the velocity at infinity

$$\therefore \frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{1}{2}V^2\sin^2\alpha - 2V^2\sin^2\alpha\sin^2\theta \qquad \text{(a)}$$

Resultant pressure required

$$= \int_0^{\pi}\left(\Pi + \frac{1}{2}\rho V^2\sin^2\alpha - 2\rho V^2\sin^2\alpha\sin^2\theta\right)\sin\theta \cdot a\,d\theta$$

= required result (to be verified). [P. T. O]

(8) The forces that act on the cylinder are

(i) the impulse applied to generate a velocity U in it

(ii) the impulsive pressure ϖ.

Hence, if σ be the density of the inner cylinder,

P.T.O

$$\therefore \quad \pi a^2 \sigma V = \frac{\pi a^2}{b^2-a^2}\left\{(\sigma+\rho)b^2 - (\sigma-\rho)a^2\right\} V - \varpi \qquad (\text{Ex. } 6)$$

This gives $\varpi = \pi a^2 \rho V (b^2+a^2)/(b^2-a^2)$

This impulsive on length l of cylinder $= \varpi l =$ req^{d} answer.

(9) Vide Milne-Thomson.

(11)
$$\left.\begin{aligned} \phi &= Vb\sqrt{\frac{a+b}{a-b}}\, e^{-\xi}\cos\eta + V\sqrt{a^2-b^2}\cosh\xi\cos\eta \\ \psi &= -Vb\sqrt{\frac{a+b}{a-b}}\, e^{-\xi}\sin\eta + V\sqrt{a^2-b^2}\sinh\xi\sin\eta \end{aligned}\right\}$$

$$\phi + i\psi = Vb\sqrt{\frac{a+b}{a-b}}\, e^{-\xi}(\cos\eta - i\sin\eta) + Vz$$

since $\left.\begin{aligned} x &= c\cosh\xi\cos\eta \\ y &= c\sinh\xi\sin\eta \end{aligned}\right\}$ $(c^2 = a^2-b^2)$

and $e^{-\xi}(\cos\eta - i\sin\eta)$ can be shown equal to $\frac{1}{c}(z - \sqrt{z^2-c^2})$ with

$$z = c\cosh(\xi + i\eta)$$

$$\therefore \quad \phi + i\psi = \frac{Vb}{a-b}\left(z - \sqrt{z^2-c^2}\right) + Vz$$

$$= V\left(az - b\sqrt{z^2-c^2}\right)/(a-b).$$

(12)
$$\phi = Vb\sqrt{\frac{a+b}{a-b}}\, e^{-\xi}\cos\eta$$

$$\frac{\partial\phi}{\partial t} = Vb\sqrt{\frac{a+b}{a-b}}\, e^{-\xi}\left(-\frac{\partial\xi}{\partial t}\cos\eta - \sin\eta\,\frac{\partial\eta}{\partial t}\right)$$

$$\frac{\partial x}{\partial t} = -V, \quad \frac{\partial y}{\partial t} = 0$$

$$\left.\begin{aligned} -V = \frac{\partial x}{\partial t} &= \frac{\partial x}{\partial\xi}\frac{\partial\xi}{\partial t} + \frac{\partial x}{\partial\eta}\cdot\frac{\partial\eta}{\partial t} \\ 0 = \frac{\partial y}{\partial t} &= \frac{\partial y}{\partial\xi}\frac{\partial\xi}{\partial t} + \frac{\partial y}{\partial\eta}\frac{\partial\eta}{\partial t} \end{aligned}\right\}$$

& with $x = c\cosh\xi\cos\eta$, $y = c\sinh\xi\sin\eta$, this gives

$$\frac{\partial\xi}{\partial t} = -\frac{V\sinh\xi\cos\eta}{c(\sinh^2\xi + \sin^2\eta)}\,; \quad \frac{\partial\eta}{\partial t} = \frac{V\cosh\xi\sin\eta}{c(\sinh^2\xi + \sin^2\eta)}$$

and hence $\dfrac{\partial\phi}{\partial t} = V^2 b\sqrt{\dfrac{a+b}{a-b}}\cdot\dfrac{\cos 2\eta - e^{-2\xi}}{2c(\sinh^2\xi + \sin^2\eta)}$ (1)

Also $\dfrac{\partial\phi}{\partial\xi} = -Vb\sqrt{\dfrac{a+b}{a-b}}\, e^{-\xi}\cos\eta\,; \quad \dfrac{\partial\phi}{\partial\eta} = -Vb\sqrt{\dfrac{a+b}{a-b}}\, e^{-\xi}\sin\eta$

Ex. 6.7: For case $\alpha = \pi/2$, we have the formula

(case $\alpha = \pi/2$) $\quad p = p_\infty + \frac{1}{2}\rho V^2(1 - 4\sin^2\theta)$ $\quad$ [M. T., p. 157 (4th Edn.)

which is same as formula (a) overleaf

Resultant pressure required

$$= \int_0^{\pi} \left\{ p_\infty + \tfrac{1}{2}\rho V^2(1 - 4\sin^2\theta) \right\} \cdot a\sin\theta \cdot d\theta$$

$$= 2\pi a p_\infty + 2a p_\infty + \rho a V^2 - 2\rho a V^2 \int_0^{\pi} \sin^3\theta \cdot d\theta.$$

$$= 2a p_\infty + \rho a V^2 - \frac{8}{3}\rho a V^2.$$

$$= 2a p_\infty - \rho a V^2 - \frac{16}{3}\rho a V^2$$

$$= 2a p_\infty - \rho a V^2\left(\frac{8}{3} - 1\right) = 2a p_\infty - \frac{5}{3}\rho a V^2.$$

$$\int_0^{\pi} (1 - \cos^2\theta)\sin\theta \, d\theta$$

$$2 - \int \cos^2\theta \sin\theta \cdot d\theta$$

$$2 + \left[\frac{1}{3}\cos^3\theta\right]_0^{\pi} = 2 - \frac{2}{3} = \frac{4}{3}$$

$$2 - \frac{2}{3} - \frac{4}{3}.$$

$$\frac{5}{3} - 1 = \frac{4}{3}$$

$$q^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2$$

and $\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\phi}{\partial\eta}\cdot\frac{\partial\eta}{\partial x}$, etc

using $\frac{\partial\xi}{\partial x} = \frac{\partial\eta}{\partial y}$; $\frac{\partial\xi}{\partial y} = -\frac{\partial\eta}{\partial x}$ we find $q^2 = \dfrac{(\partial\phi/\partial\xi)^2 + \left(\frac{\partial\phi}{\partial\eta}\right)^2}{\left(\frac{\partial x}{\partial\xi}\right)^2 + \left(\frac{\partial x}{\partial\eta}\right)^2}$

ie $q^2 = \dfrac{U^2 b^2 \frac{a+b}{a-b} e^{-2\xi}}{c^2(\sinh^2\xi + \sin^2\eta)}$ (2) $\begin{cases} \frac{\partial x}{\partial\xi} = c\sinh\xi\cos\eta \\ \frac{\partial x}{\partial\eta} = -c\cosh\xi\sin\eta \end{cases}$

If the elliptic cylinder corresponds to $\xi = \alpha$, we find from 1

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{\partial\phi}{\partial t} - \tfrac{1}{2}q^2 = \frac{\Pi}{\rho} + \frac{U^2 b\left[(a-b)\cos 2\eta - a e^{-2\alpha}\right]}{2(a-b)^2(\sinh^2\alpha + \sin^2\eta)}$$

The maximum or minimum of the second term is ~~taken~~ given by $\eta = 0$ & $\eta = \pi/2$, the minimum ~~for~~ value for $\pi/2$ being negative.

Hence the least value of p is given by

$$\frac{p}{\rho} = \frac{\Pi}{\rho} - \frac{U^2 b}{2(a-b)^2}\cdot\frac{(a-b+ae^{-2\alpha})}{(\sinh^2\alpha + 1)} \quad \& \quad \left.\begin{matrix} a = c\cosh\alpha \\ b = c\sinh\alpha \end{matrix}\right\}$$

$$= \frac{\Pi}{\rho} - \frac{U^2 b}{2(a-b)^2}\cdot\frac{(a-b)+a(a-b)/(a+b)}{1 + b^2/(a^2-b^2)}$$

$$= \frac{\Pi}{\rho} - \frac{U^2 b(2a+b)}{2a^2}$$

If this be +ve, the pressure is everywhere positive ie if

$$\frac{\Pi}{\rho} > \frac{U^2 b(2a+b)}{2a^2}$$

ie $\rho U^2 < 2a^2\Pi/(2ab+b^2)$.

(13) Book work in Milne-Thomson.

(14). $\phi = U\sqrt{\frac{a+b}{a-b}}\, e^{-\xi}(b\cos\eta\cos\lambda + a\sin\eta\sin\lambda)$

$\psi = -U\sqrt{\frac{a+b}{a-b}}\, e^{-\xi}(b\sin\eta\cos\lambda - a\cos\eta\sin\lambda)$

$T = -\frac{1}{2}\rho\int\phi\, d\psi$

At the surface of the cylinder $\xi = \alpha$

$$\therefore T = \tfrac{1}{2}\rho U^2\frac{a+b}{a-b}e^{-2\alpha}\int_0^{2\pi}(b\cos\eta\cos\lambda + a\sin\eta\sin\lambda)^2 d\eta$$

which gives the required expression.

(15) By proper choice of units, we can write

$$x + iy = 2\cosh(\xi + i\eta) \qquad (1)$$
$$\phi + i\psi = e^{-(\xi + i\eta)} \qquad (2)$$

and putting $\zeta = \xi + i\eta$

$$z = x + iy = 2\cosh\zeta$$
$$w = \phi + i\psi = e^{-\zeta}$$

(2) gives $\psi = -e^{-\xi}\sin\eta$; $\phi = e^{-\xi}\cos\eta$

$$\tan\eta = -\frac{\psi}{\phi}$$
$$\psi^2 + \phi^2 = e^{-2\xi}$$

$$x = 2\cosh\xi\cos\eta\,;\quad y = 2\sinh\xi\sin\eta$$

$$\phi\left[1 + \frac{1}{\phi^2+\psi^2}\right] = e^{-\xi}\cos\eta\left[1 + \frac{1}{e^{-2\xi}}\right] = \cos\eta\,\frac{1+e^{2\xi}}{e^{\xi}} = 2\cos\eta\cosh\xi$$

$$\therefore\quad x = \phi\left[1 + \frac{1}{\phi^2+\psi^2}\right]$$

$$y = \psi\left[1 - \frac{1}{\phi^2+\psi^2}\right] \text{ similarly.}$$

(19) Vide Milne-Thomson.

(21) $\phi = \frac{1}{4}\omega(a+b)^2 e^{-2\alpha}\sin 2\eta$ for the surface of the cylinder. η does not change with t since there is no translation

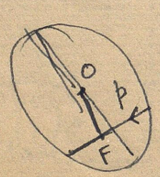
$$\frac{\partial\phi}{\partial t} = \frac{1}{4}\frac{d\omega}{dt}(a+b)^2\cdot\frac{(a-b)^2}{c^2}\sin 2\eta = \frac{1}{4}c^2\frac{d\omega}{dt}\sin 2\eta \qquad (1)$$

$$T = \frac{1}{16}\pi\rho\omega^2(a^2-b^2)^2 \text{ for the whole cylinder}$$

q^2 at a point on the cylinder ξ = const is $\dfrac{(\partial\phi/\partial\eta)^2}{(\partial x/\partial\eta)^2}$

$$= \frac{1}{16}\omega^2(a+b)^4 e^{-4\alpha}\cos^2 2\eta \Big/ c^2\cosh^2\alpha\sin^2\eta$$

$$\therefore\quad \frac{p}{\rho} = F(t) - \frac{1}{2}q^2 + \frac{\partial\phi}{\partial t}$$



Moment of fluid pressure round the centre

$$= \iint p\cdot OF\,dS\,.\quad \left[OF = \frac{dp}{d\psi} = -\frac{dp}{ds}\cdot\frac{ds}{d\psi} = -\frac{dp}{ds}\cdot r\frac{dr}{dp}\right.$$

$(p = \perp^r \text{ on tangt})$ $\qquad = -r\dfrac{dr}{ds}$

$\therefore$ Required moment $= \iint -p\, r \frac{dr}{ds}\, ds = \int p r\, dr$

$= -\frac{1}{2}\int p \frac{dr^2}{d\eta}\, d\eta$

$r^2 = x^2 + y^2 = c^2(\cosh^2\alpha \cos^2\eta + \sinh^2\alpha \sin^2\eta)$

$\frac{dr^2}{d\eta} = -c^2 \sin 2\eta \cosh 2\alpha$

and $p = F(t) - \frac{1}{8}\frac{\omega^2 (a+b)^4 e^{-4\alpha}}{c^2 \cosh^2\alpha} \cdot \frac{\cos^2 2\eta}{\sin^2\eta} + \frac{1}{4} c^2 \frac{d\omega}{dt} \sin 2\eta$

$\therefore$ Reqd moment $= \frac{1}{8}\rho\pi c^4 \frac{d\omega}{dt} - \frac{1}{16}\rho\omega^2 \frac{(a+b)^4}{\cosh^2\alpha} \int_0^{2\pi} \frac{\cos^2 2\eta}{\sin^2\eta} \sin 2\eta\, d\eta$

$= \frac{1}{8}\rho\pi c^4 \frac{d\omega}{dt}$, since second integral vanishes.

Alter: $\frac{d}{dt}\left(\frac{1}{2} M K^2 \omega^2 + \frac{1}{16}\pi\rho c^4 \omega^2\right)$ = rate at which forces do work

Also $MK^2 \frac{d\omega}{dt}$ = Moment of pressure about the axis. Hence result follows (?)

---

(22) Let $\psi = \sin 2\eta \,(A \cosh 2\xi + B \sinh 2\xi)$

$\psi = \frac{1}{4} c^2 \omega (x^2 + y^2) + C = \frac{1}{4} c^2 \omega (\cosh 2\alpha + \cos 2\eta) + C$

on surface of $\xi = \alpha$

and $= \frac{1}{4} c^2 \Omega (\cosh 2\beta + \cos 2\eta) + C$, on surface of $\xi = \beta$

$\therefore$ $\left.\begin{aligned} A\cosh 2\alpha + B \sinh 2\alpha - \tfrac{1}{4} c^2 \omega = 0 \\ A \cosh 2\beta + B \sinh 2\beta - \tfrac{1}{4} c^2 \Omega = 0 \end{aligned}\right\}$

from which A & B can be found, and

$\psi = \frac{1}{4} c^2 \frac{\cos 2\eta}{\sinh 2(\beta - \alpha)} \left[\omega \sinh 2(\beta - \xi) - \Omega \sinh 2(\alpha - \xi)\right]$

$\phi = \frac{1}{4} c^2 \frac{\sin 2\eta}{\sinh 2(\beta - \alpha)} \left[\omega \cosh 2(\beta - \xi) - \Omega \cosh 2(\alpha - \xi)\right]$

signs being adjusted so that $\frac{\partial\phi}{\partial\xi} = \frac{\partial\psi}{\partial\eta}$; $\frac{\partial\phi}{\partial\eta} = -\frac{\partial\psi}{\partial\xi}$

Regarding ξ as const, $d\psi = -\frac{1}{2} c^2 \frac{\sin 2\eta}{\sinh 2(\beta - \alpha)} [\quad "\quad]$

Hence for cylinder $\xi = \alpha$

$$-\tfrac{1}{2}\rho\int\phi\,d\psi = \frac{1}{16}\pi c^4\rho\omega\left[\int_0^{2\pi}\sin^2 2\eta\,d\eta\left\{-\Omega+\omega\cosh 2(\beta-\alpha)\right\}\right]\Big/\sinh 2(\beta-\alpha)$$

$$= \frac{\frac{1}{16}\pi c^4\rho\omega}{\sinh 2(\beta-\alpha)}\left\{-\Omega+\omega\cosh 2(\beta-\alpha)\right\}$$

and for $\xi = \beta$

$$-\tfrac{1}{2}\rho\int\phi\,d\psi = \frac{1}{16}\cdot\frac{\pi c^4\rho\Omega}{\sinh 2(\beta-\alpha)}\left\{\omega-\Omega\cosh 2(\beta-\alpha)\right\}$$

$$\therefore \text{ Total K.E} = \frac{1}{8}\frac{\pi c^4\rho}{\sinh 2(\beta-\alpha)}\left\{-\omega\Omega+\cosh 2(\beta-\alpha)(\omega^2+\Omega^2)\right\}$$

Putting $\Omega = \omega$, the K.E $= \frac{\rho}{8}\frac{\pi c^4}{\sinh 2(\beta-\alpha)}\omega^2\cdot 2\sinh^2(\beta-\alpha)$

$$= \frac{1}{8}\pi c^4\omega^2\rho\tanh(\beta-\alpha)$$

If cylinders be of unit length, $\pi(a_1b_1 - a_0b_0)\rho = M$

$$\therefore \text{ K.E} = \frac{1}{8}\frac{\pi c^4\omega^2\cdot M}{\pi(a_1b_1-a_0b_0)}\cdot\frac{(b_1a_0-b_0a_1)}{(a_1a_0-b_1b_0)}. \qquad \left[a_0 = c_0\cosh\alpha,\ b_0 = c_0\sinh\alpha,\ c_0^2 = a_0^2-b_0^2,\ \text{etc}\right]$$

(25) As in § 108, we have (since V is in the −ve diren of the x-axes)

$$\phi = -Vr\cos\theta + 2Va^{\frac{1}{2}}r^{\frac{1}{2}}\cos\tfrac{1}{2}\theta$$

Finding $\frac{\partial\phi}{\partial r}$ and $\frac{1}{r}\frac{\partial\phi}{\partial\theta}$, we find

$$q^2 = V^2\sin^2\tfrac{1}{2}\theta. \qquad (1)$$

$\frac{\partial\phi}{\partial t} = V^2\sin^2\frac{1}{2}\theta$, using $\frac{dr}{dt} = -V\cos\theta$, $r\frac{d\theta}{dt} = V\sin\theta$

$$\therefore\ p/\rho = \frac{\partial\phi}{\partial t} - \tfrac{1}{2}q^2 + F(t) = \tfrac{1}{2}V^2\sin^2\tfrac{1}{2}\theta + F(t).$$

For the parabola, $ds = a\sec^3\tfrac{1}{2}\theta\cdot d\theta$

and if ψ = angle of normal with the x-axis, $\psi = \frac{1}{2}\theta$

Resultant pressure, which by symmetry is along the axis, is

$$= -\rho\int_0^{2\pi}\cos\tfrac{1}{2}\theta\cdot a\sec^3\tfrac{1}{2}\theta\cdot d\theta\left[\tfrac{1}{2}V^2\sin^2\tfrac{1}{2}\theta + F(t)\right]$$

$$= -\tfrac{1}{2}\rho V^2 a\int_0^{2\pi}\tan^2\tfrac{1}{2}\theta\,d\theta + aF(t)\int_0^{2\pi}\sin^2\tfrac{1}{2}\theta\,d\theta$$

$$= \pi\rho aV^2, \text{ the second integral vanishing.}$$

(26) Over $r = a(1+\epsilon\cos n\theta)$,

$$\psi = -uy + C$$
$$= -ur\sin\theta + C = -ua\sin\theta(1+\epsilon\cos n\theta) + C$$
$$= -ua\sin\theta - \tfrac{1}{2}ua\epsilon\left[\sin(n+1)\theta - \sin(n-1)\theta\right] + C \qquad (1)$$

Therefore let

$$\psi = -\frac{ua^2}{r}\sin\theta + \frac{A}{r^{n+1}}\sin(n+1)\theta + \frac{B}{r^{n-1}}\sin(n-1)\theta$$

For $r = a(1+\epsilon\cos n\theta)$,

$$\psi = -\frac{ua^2}{a}\sin\theta(1-\epsilon\cos n\theta) + \frac{A}{a^{n+1}}\sin(n+1)\theta + \frac{B}{a^{n-1}}\sin(n-1)\theta$$

ie $+ua\epsilon\sin\theta\cos n\theta + \dfrac{A}{a^{n+1}}\sin(n+1)\theta + \dfrac{B}{a^{n-1}}\sin(n-1)\theta$

$$\equiv -\tfrac{1}{2}ua\epsilon\left[\sin(n+1)\theta - \sin(n-1)\theta\right]$$ from (1)

$$\left.\begin{aligned} \text{or } \tfrac{1}{2}ua\epsilon + \frac{A}{a^{n+1}} &= -\tfrac{1}{2}ua\epsilon \\ -\tfrac{1}{2}ua\epsilon + \frac{B}{a^{n-1}} &= \tfrac{1}{2}ua\epsilon \end{aligned}\right\} \text{ giving } A = -ua^{n+2}\epsilon,\ B = ua^n\epsilon$$

$$\therefore \psi = -\frac{ua^2}{r}\sin\theta - \frac{ua^{n+2}\epsilon}{r^{n+1}}\sin(n+1)\theta + \frac{ua^n\epsilon}{r^{n-1}}\sin(n-1)\theta$$

$$\therefore \phi = \frac{ua^2}{r}\cos\theta + \frac{ua^{n+2}\epsilon}{r^{n+1}}\cos(n+1)\theta - \frac{ua^n\epsilon}{r^{n-1}}\cos(n-1)\theta.$$

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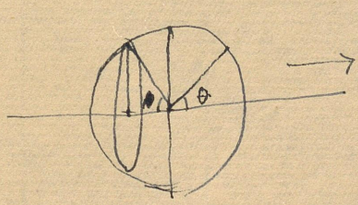
Chapter VI, p. 146. — no examples worked.

Chapter VII, p. 176.

(2)



$$\text{Pressure} = \int_{\pi/2}^{\pi} 2\pi a^2\sin\theta\,d\theta\cdot p\cos\theta$$

$$\phi = Ur\cos\theta + \tfrac{1}{2}V\frac{a^3}{r^2}\cos\theta$$

Finding $\dfrac{\partial\phi}{\partial r}$ & $\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}$, we get

$$q^2 = U^2\left\{\left(1-\frac{a^3}{r^3}\right)^2\cos^2\theta + \left(1+\frac{a^3}{2r^3}\right)^2\sin^2\theta\right\}$$

over the sphere, $q^2 = \dfrac{9}{4}U^2\sin^2\theta$.

$\dfrac{p}{\rho} = C - \tfrac{1}{2}q^2$ and $\dfrac{\Pi}{\rho} = C - \tfrac{1}{2}U^2$, since vel at $\infty = U$

$$\therefore \quad \frac{p}{\rho} = \frac{\Pi}{\rho} - \frac{9}{8} U^2 \sin^2\theta + \frac{1}{2} U^2 .$$

$$\therefore \quad \text{Thrust} = \int 2\pi a^2 \sin\theta \, d\theta \left(\frac{\Pi}{\cancel{\rho}} + \frac{1}{2}\rho U^2 - \frac{9}{8}\rho U^2 \sin^2\theta \right) \cos\theta$$

$$= \pi a^2 \left(\Pi - \frac{1}{16}\rho U^2 \right).$$

(3) $$\frac{\partial \phi}{\partial t} = \frac{1}{2} f \frac{a^3}{r^2} \cos\theta + \frac{1}{2} \frac{V^2 a^3}{r^3} (3\cos^2\theta - 1)$$

$$q^2 = \frac{1}{4} V^2 a^6 / r^6 \, (3\cos^2\theta + 1)$$

Over the sphere

$$\frac{p}{\rho} = F(t) + \frac{1}{2} f a \cos\theta + \frac{9}{8} V^2 \cos^2\theta - \frac{5}{8} V^2$$

(i) $$\int_0^{\pi/2} \left[\Pi + \frac{1}{2} f a \rho \cos\theta + \frac{9}{8}\rho V^2 \cos^2\theta - \frac{5}{8}\rho V^2 \right] 2\pi a^2 \sin\theta \, d\theta$$

$$= \pi a^2 \Pi - \frac{3}{64} \frac{V^2 M}{a} + \frac{1}{4} M f$$

(ii) $$\int_{\pi/2}^{\pi} (\text{same integral}) = -\left[\pi a^2 \Pi - \frac{1}{4} M f - \frac{3}{64} \frac{M V^2}{a} \right].$$

Hence the result, the second integral being taken with the negative sign

(4) $$q^2 = \frac{1}{4} V^2 \frac{a^6}{r^6} (3\cos^2\theta + 1)$$

ie $= \frac{1}{4} V^2 (3\cos^2\theta + 1)$ over surface of the sphere.

$$p = \rho F(t) - \frac{1}{8}\rho V^2 (5 - 9\cos^2\theta)$$

& $\rho F(t)$ is the pressure had the liquid been a rest

$$\therefore \quad \text{Reqd difference} = \frac{1}{8}\rho V^2 (9\cos^2\theta - 5) \qquad \frac{9}{2}(1+\cos^2\theta) - 5$$

$$= \frac{1}{16}\rho V^2 (9\cos 2\theta - 1)$$

(5) $$\text{Mean pressure} = \frac{1}{4\pi a^2} \cdot \int_0^{\pi} \left[\Pi - \frac{1}{8}\rho V^2 (5 - 9\cos^2\theta \right] 2\pi a^2 \sin\theta \, d\theta$$

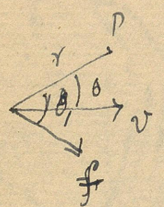
$$= \Pi - \frac{1}{4}\rho V^2 . \qquad \text{~~seco~~} \left(\text{cond}^n \ \Pi > \frac{5}{8}\rho V^2 \ ? \right)$$

(8) $$\frac{p}{\rho} = F(t) - \frac{1}{2} q^2 + \frac{\partial \phi}{\partial t}$$

$$p_0 = \rho F(t)$$

$$\frac{p-p_0}{\rho} = \frac{\partial \phi}{\partial t} - \frac{1}{2} q^2$$

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\frac{dv}{dt}\cdot\frac{a^3}{r^2}\cos\theta + \frac{1}{2}v^2\frac{a^3}{r^3}(3\cos^2\theta - 1)$$



$$q^2 = \frac{1}{4}\frac{v^2 a^6}{r^6}(3\cos^2\theta + 1)$$

$$\frac{dv}{dt}\cos\theta = f\cos\theta_1$$

$$\therefore \frac{p-p_0}{\rho} = \frac{1}{2}af\cos\theta_1 + \frac{1}{8}v^2(9\cos^2\theta - 5)$$

(9) $$\frac{p-p_0}{\rho} = \frac{1}{2}f\frac{a^3}{r^2} - \frac{1}{2}\frac{v^3a^3}{r^3} - \frac{1}{8}\frac{v^2a^6}{r^6} + \frac{3}{2}\frac{a^3}{r^3}q'^2 - \frac{3}{8}\frac{a^6}{r^6}q'^2$$

since $\frac{dv}{dt}\cos\theta = f$, $q\cos\theta = q'$

$$p - p_0 = \frac{1}{2}f\frac{a^3}{r^2} - \frac{a^3}{8r^6}q^2(4r^3 + a^3) + \frac{3}{8}\frac{a^3}{r^6}(4r^3 - a^3)q'^2$$

(10) $$p/\rho = \frac{\Pi}{\rho} + \frac{1}{2}f\cos\theta_1 + \frac{1}{2}v^2\frac{a^3}{r^3}(3\cos^2\theta - 1)$$

Neglecting $\frac{1}{2}f\cos\theta_1$, the pressure is given (Exc. 2) by $\pi a^2(\Pi - \frac{1}{16}\rho V^2)$

From Exc. 3, this is in a dirn opp. to that of v. The part contributed by $\frac{1}{2}f\cos\theta_1$ is $\frac{1}{3}\pi\rho a^3 f$ but in a dirn opp. to that of f.

(11) Let $$\phi = \left[Ar\cos\theta + \frac{B}{r^2}\cos\theta\right] + \left[Cr + \frac{D}{r^2}\right]\sin\theta$$

This ~~satisfies~~ the boundary conds are

$$\frac{\partial\phi}{\partial r} = -u\cos\theta, \quad r = a$$

$$\frac{\partial\phi}{\partial r} = -v\sin\theta, \quad r = b.$$

$$\therefore -u\cos\theta = \left(A - \frac{2B}{a^3}\right)\cos\theta + \left(C - \frac{2D}{a^3}\right)\sin\theta$$

$$-v\sin\theta = \left(A - \frac{2B}{b^3}\right)\cos\theta + \left(C - \frac{2D}{b^3}\right)\sin\theta$$

$A - \frac{2B}{a^3} = -u$, $C - \frac{2D}{a^3} = 0$, etc. These determine A, B, C, D & hence ϕ.

$$K\cdot E = -\tfrac{1}{2}\rho\int p\frac{\partial\phi}{\partial n}\,dS.$$

For K. E. see Milne-Thomson Book work.

(12) Milne-Thomson Book work. § 16·22. Ex. 11, p. 460.

(13) Ibid § 16·21.

(14) Ibid § 16·21.

(15) $-\frac{\partial \phi}{\partial r} = V\cos\theta + \dot{a}$, $r = a$

$\therefore \phi = \frac{1}{2} V \frac{a^3}{r^2}\cos\theta + \frac{\dot{a}}{r} a^2 + Vr\cos\theta$

$\frac{p}{\rho} + \frac{1}{2} q^2 - \frac{\partial \phi}{\partial t} = \text{const}$

$\frac{\partial \phi}{\partial t} = \frac{3}{2} V\dot{a}\cos\theta + 2\dot{a} + a\ddot{a}$

$\frac{\partial \phi}{\partial r} = -\dot{a}$, $\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{3}{2} V \sin\theta$

$q^2 = \dot{a}^2 + \frac{9}{4} V^2 \sin^2\theta$

$\int p\cos\theta \cdot 2\pi a^2 \sin\theta \, d\theta = 2\pi\rho a^2 V\dot{a}$, after simplification

$= \frac{1}{2} V \frac{d}{dt}\left(\frac{4}{3}\pi\rho a^3\right) = \frac{1}{2} V \frac{dM}{dt}$.

(16) See Milne-Thomson, Ex. 36, p. 436.

(19) The equation of continuity is

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial \phi}{\partial \mu}\right\} = 0 \qquad (1)$$

$r^2 \frac{\partial \phi}{\partial r} = -r^2 u = -\left\{\frac{2c}{r}\cos\theta + \frac{c^3}{4r^2}(1+3\cos 2\theta)\right\}$

$\frac{\partial \phi}{\partial \mu} = -\frac{1}{\sin\theta}\frac{\partial \phi}{\partial \theta} = \frac{ur}{\sin\theta}$

$= \frac{1}{\sin\theta}\left[\frac{c}{r^2}\sin\theta + \frac{c'}{r^3}\sin\theta\cos\theta\right]$

$= \frac{c}{r^2} + \frac{c'}{r^3}\cos\theta = \frac{c}{r^2} + \frac{c'}{r^3}\mu$.

Thus L.H.S of (1) $= -\left\{-\frac{2c}{r^2}\cos\theta - \frac{c'}{2r^3}(1+3\cos 2\theta)\right\}$

$+(1-\mu^2)\frac{c'}{r^3} + \left(\frac{c}{r^2} + \frac{c'\mu}{r^3}\right)(-2\mu)$ which $= 0$ on simplification

i.e. eqn of continuity is satisfied

If $U\,dr + Wr\,d\theta$ = perfect differential

$$\frac{\partial U}{\partial \theta} = \frac{\partial}{\partial r}(Wr)$$

i.e. $$\frac{\partial}{\partial \theta}\left[\frac{2C}{r^3}\cos\theta + \frac{C'}{4r^4}(1+3\cos 2\theta)\right] = \frac{\partial}{\partial r}\left[\frac{C}{r^2}\sin\theta + \frac{C'}{2r^3}\sin 2\theta\right]$$

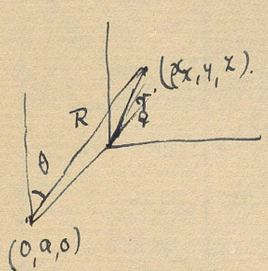
which can also be shown to be identically true.

If $C' = 0$, $U = \frac{2C}{r^3}\cos\theta$, $W = \frac{C}{r^3}\sin\theta$

from which $\phi = -\frac{C}{r^2}\cos\theta$

which is the velocity potential for sphere moving with its centre on a fixed line.

(20)



Velocity potential of doublet

$$= M\cos\theta/R^2 = Mz/R^3$$

$$R^2 = x^2 + (y-a)^2 + z^2$$

$$R^{-3} = (r^2 - 2ay + a^2)^{-3/2}$$

$$= \frac{1}{a^3}\left(1 - \frac{2y}{a} + \frac{r^2}{a^2}\right)^{-3/2}$$

$$= \frac{1}{a^3}\left(1 + \frac{3y}{a}\right), \text{ neglecting higher powers}$$

$$\phi = \frac{Mz}{a^3}\left(1 + \frac{3y}{a}\right) = \frac{Mz}{a} + \frac{3Myz}{a^4}.$$

Note: (i) z and yz both satisfy Laplace's eqn

(ii) Hence part of vel. potential due to introdn of sphere must also satisfy Laplace's eqn

(iii) If ϕ' be this part, the boundary condn at the sphere requires

$$\frac{\partial}{\partial r}(\phi + \phi') = 0.$$

$\therefore$ $\phi' = \frac{Az}{R^3} + \frac{Byz}{R^5}$, and the boundary condn (iii) gives

$$A = \frac{Mc^3}{2a^3}\ ;\ B = \frac{2Mc^5}{a^4}.$$

(22) Comparing $\psi = \frac{1}{2}V\left(a^4 r^{-2}\cos\theta - r^2\right)\sin^2\theta$ with (1)

$$\psi = -\frac{1}{2}Vr^2\sin^2\theta + C \qquad (2)$$

we see that the motion will be due to a solid of revolution moving along

if $C = \frac{1}{2} V \frac{a^4}{r^2} \cos\theta \sin^2\theta$

or $r^2 = \frac{1}{2} \frac{V a^4}{C} \cos\theta \sin^2\theta$ is the generating curve of a solid of revolution

If motion be irrotational ψ satisfies

$$r^2 \frac{\partial^2 \psi}{\partial r^2} + (1-\mu^2) \frac{\partial^2 \psi}{\partial \mu^2} = 0 .$$

This eqn has solutions of the type

$$\sin^2\theta \frac{r^{n+1}}{n+1} \frac{\partial P_n}{\partial \mu} , \text{ and } \frac{\sin^2\theta}{n r^n} \frac{\partial P_n}{\partial \mu}$$

Putting $n = 1$, $n = 2$ we have the solutions $r^2 \sin^2\theta$ and also (1) since $P_2 = \frac{1}{2}(3\mu^2 - 1)$, so that the motion is irrotational.

To find ϕ, we have

$$\frac{\partial \psi}{\partial r} = (1-\mu^2) \frac{\partial \phi}{\partial \mu}$$

ie $$-\frac{1}{2} V \sin^2\theta \left(\frac{2a^4 \cos\theta}{r^3} + 2r \right) = (1-\mu^2) \frac{\partial \phi}{\partial \mu}$$

or $$V \left(r + \frac{a^4 \mu}{r^3} \right) = \frac{\partial \phi}{\partial \mu}$$

or $$\phi = V \left(r\mu + \frac{a^4 \mu^2}{2r^3} \right)$$

$$= V \left(r \cos\theta + \frac{a^4 \cos^2\theta}{2r^3} \right)$$

~~with~~ the arbitrary constant being immaterial.

(23) Ex. 59, Milne-Thomson, p. 439.

(26) The velocity potential is given by

$$\phi = -\frac{b^2 - c^2}{b^2 + c^2} byz - \cdots - ux - vy - wz .$$

Hence we can find $U = -\frac{\partial \phi}{\partial x}$, etc, and

angular momentum about x-axis

$$= \rho \iiint (yW - zV) \, dx\,dy\,dz$$

$$= \rho \iiint \frac{b^2 - c^2}{b^2 + c^2} b (y^2 - z^2) \, dx\,dy\,dz$$

$$= \frac{4}{15} \pi \rho abc \cdot \frac{(b^2 - c^2)^2}{(b^2 + c^2)} b , \quad \text{evaluating the Dirichlet integrals}$$

See also Milne-Thomson, p. 465, Ex. 39.

(27) $\phi = -\dfrac{a^2-b^2}{a^2+b^2}\omega xy$

$$A \cdot M = \frac{4}{15}\pi\rho abc \cdot \frac{(a^2-b^2)^2}{a^2+b^2}\omega$$

Also $M = \frac{4}{3}\pi\rho abc$

$\therefore$ Required couple $= \frac{1}{5}M\omega(a^2-b^2)^2/(a^2+b^2)$.

(29) Let $\phi = x(A\alpha + B)$

$$\frac{\partial\phi}{\partial x} = (A\alpha+B) + Ax\frac{\partial\alpha}{\partial\lambda}\frac{\partial\lambda}{\partial x}$$

For the motion of the inner ellipsoid

$$\frac{\partial\phi}{\partial x} = (A\alpha_1+B) + Ax\left(-\frac{1}{a^2}\cdot\frac{2p_1^2x}{a^2}\right)$$

$$= A\alpha_1 + B - \frac{2p_1^2Ax^2}{a^4}.$$

$$\frac{\partial\phi}{\partial y} = -\frac{2p_1^2Ayx}{a^2b^2}\ ;\ \frac{\partial\phi}{\partial z} = -\frac{2p_1^2Azx}{a^2c^2}$$

For the boundary condition

$$-\frac{x}{a^2}\frac{\partial\phi}{\partial x} - \cdots - = \frac{Ux}{a^2}$$

$$\therefore\ -\frac{1}{a^2}(A\alpha_1+B) + \frac{2A}{a^2} = \frac{U}{a^2} \qquad (1)$$

For the outer ellipsoid

$$\frac{\partial\phi}{\partial x} = (A\alpha_2+B) + Ax\left\{\frac{-abc}{(a^2+\lambda)^{3/2}(b^2+\lambda)^{3/2}(c^2+\lambda)^{1/2}}\cdot\frac{2p^2x}{a^2+\lambda}\right\}$$

since in this case $\dfrac{\partial\alpha}{\partial\lambda}$ = first factor in $\{\ \}$ above.

$\therefore$ For the boundary condition

$$-\frac{1}{a^2+\lambda}(A\alpha_2+B) + \frac{2p_2^2Abc}{\Delta}\left\{\frac{x^2}{(a^2+\lambda)^2} + \cdots\right\} = \frac{U'}{a^2+\lambda}$$

$$\therefore\ \frac{2abcA}{\Delta(a^2+\lambda)} - \frac{A\alpha_2+B}{a^2+\lambda} = U'$$

or $2A\mu - (A\alpha_2+B) = U' \qquad (2)$

From (1) & (2), $A = (U-U')/(2-\alpha_1-2\mu-\alpha_2)$; $B = \dfrac{(2U-\alpha_2)(U-U')}{2-\alpha_1-2\mu-\alpha_2} - U'$

$$= \frac{\bar{v}'(\alpha_0 - 2) - \bar{v}(\alpha' - 2\mu)}{2 - \alpha_1 - 2\mu - \alpha_2}$$

$$\therefore \quad \phi = \frac{\left[(\bar{v} - \bar{v}')\alpha - \bar{v}(\alpha' - 2\mu) + \bar{v}'(\alpha_0 - 2)\right] x}{(2 - \alpha_1 - 2\mu - \alpha_2)}$$

(30) Vide Milne-Thomson, Ex. 32, p. 464.

(31) Ibid, Ex. 33, p. 464

(32) Ibid. Ex. 57, p. 439.

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Chapter VIII, p. 207.

(2) The boundary conditions are:

when $r = a$, $\frac{\partial \phi}{\partial r} = -v \cos\theta$

$r = b$, $\frac{\partial \phi}{\partial r} = 0$

To satisfy this assume

$$\phi = \left(Ar + \frac{B}{r}\right)\cos\theta$$

giving $A - \frac{B}{b^2} = 0$; $A - \frac{B}{a^2} = -v$

i.e. $A = \frac{a^2 v}{b^2 - a^2}$; $B = \frac{a^2 b^2 v}{b^2 - a^2}$

Impulsive force is given by $\rho\phi$. By symmetry the impulsive pressure on the inner shell is along the dirn of the velocity v, say along the x-axis

Total impulsive pressure

$$= -\int_0^{2\pi} \rho\phi \cos\theta \cdot b\, d\theta \text{ per unit length}$$

$$= \rho \int_0^{2\pi} \rho \cdot \frac{a^2 v}{a^2 - b^2} b\left(b + \frac{b^2}{b}\right)\cos^2\theta\, d\theta$$

$$= \rho \cdot \frac{2a^2 b^2 \pi v}{a^2 - b^2}$$

Δ this is the force per unit length which must be applied to keep the

cylinder at rest.

(3) If V be the resulting velocity, the effect of the liquid present is to increase its inertia by $\frac{1}{2}M'$, where M' is the mass of the liquid displaced.

Hence
$$(M + \tfrac{1}{2}M')V = P$$

But $M = M'$ since densities of the liquid & solid are equal

$$\therefore V = 2P/3M.$$

(7) Let $2T$ = K.E. of solid + K.E of the liquid inside

The K.E of the solid $= \frac{1}{2}MK^2\omega^2$ for ω round axis of suspension

The liquid, as a whole, has the same angular velocity round the axis

For rotation round the axis of figure, its
$$K.E = \frac{1}{8}m\frac{(a^2-b^2)^2}{(a^2+b^2)^2}\omega^2$$

$$\therefore \text{its } K.E = \left[\frac{1}{8}\frac{(a^2-b^2)^2}{(a^2+b^2)^2} + \frac{1}{2}d^2\right]m\omega^2$$

if d = distance between the axis, and axis of suspension

$$\therefore MK^2\ddot{\omega} + m\left[\frac{1}{4}\frac{(a^2-b^2)^2}{(a^2+b^2)^2} + d^2\right]\ddot{\omega} = -(M+m)gh\sin\theta$$

$$\therefore l = \frac{MK^2 + \left[\frac{1}{4} \quad \cdots \quad\right]m}{(M+m)h}$$

$$l' = \frac{MK^2 + \left[d^2 + \frac{1}{4}(a^2+b^2)\right]m}{(M+m)h}$$

$$\therefore l' - l = \frac{m}{(M+m)h}\cdot\frac{1}{4}\left[\frac{(a^2+b^2)^2 - (a^2-b^2)^2}{a^2+b^2}\right] = \frac{m}{M+m}\cdot\frac{a^2b^2}{a^2+b^2}\cdot\frac{1}{h}.$$

(8) As above

$$\left[MK^2 + m\left\{d^2 - \int\phi\frac{\partial\phi}{\partial n}\,ds\right\}\right]\ddot{\omega} = -(M+m)gl\sin\theta$$

where $\phi = -\frac{a^2-b^2}{a^2+b^2}xy$

$-\frac{\partial\phi}{\partial n}$ = velocity along normal to the surface

$= l(-y\omega) + m(x\omega) = \omega(mx - ly)$

$-\frac{1}{2}\int \rho \frac{\partial\phi}{\partial n}\, dS = -\frac{1}{2}\,\frac{a^2-b^2}{a^2+b^2}\int \rho xy(mx-ly)\,\omega\, dS$

integrated over the surface of the ellipsoid. This by Green's theorem is equal to $-\frac{1}{2}\,\frac{a^2-b^2}{a^2+b^2}\iiint (x^2-y^2)\,dx\,dy\,dz$

$$= -\frac{1}{2}\,\frac{a^2-b^2}{a^2+b^2}\cdot\frac{8abc}{8}\cdot\frac{\Gamma(3/2)\,\Gamma(1/2)\,\Gamma(1/2)}{\Gamma(3/2+\frac{1}{2}+\frac{1}{2}+1)}\cdot(a^2-b^2)$$

$$= -\frac{1}{2}\,abc\cdot\frac{(a^2-b^2)^2}{a^2+b^2}\cdot\frac{4}{15}\pi$$

$$= -\frac{1}{2}\cdot\frac{1}{5}\cdot\frac{4}{3}\pi abc\cdot\frac{(a^2-b^2)^2}{a^2+b^2}$$

$\therefore$ length of simple equivalent pendulum

$$= \frac{MK^2 + m\left[d^2 + \frac{1}{5}\cdot\frac{(a^2-b^2)^2}{a^2+b^2}\right]}{(M+m)\,l}$$

(10) For an ellipsoid moving along the x-axis

$$K.E = \frac{1}{2}\cdot\frac{\alpha_0}{\alpha-\alpha_0}\cdot\frac{4}{3}\pi abc\cdot\sigma u^2$$

For an oblate spheroid, $\alpha_0 = \frac{(1-e^2)^{1/2}}{3}\left[\cot^{-1}\frac{(1-e^2)^{\frac{1}{2}}}{e} - e(1-e^2)^{1/2}\right]$

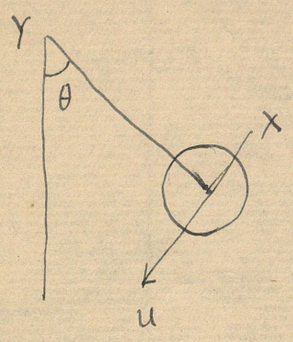
$\therefore$ K.E. for an oblate spheroid

$$= \frac{1}{2}\cdot\frac{(1-e^2)^{1/2}}{e^3}\left\{\sin^{-1}e - e(1-e^2)^{1/2}\right\}\frac{4}{3}\pi a^2c\sigma u^2 \Big/ \left\{2 - \frac{(1-e^2)^{\frac{1}{2}}}{e^3}\left[\sin^{-1}e - e(1-e^2)^{\frac{1}{2}}\right]\right\}$$

$$= \frac{1}{2}\cdot\frac{4}{3}\pi a^2c\sigma\cdot\frac{(1-e^2)^{1/2}\left\{\sin^{-1}e - e(1-e^2)^{1/2}\right\}}{e(1+e^2) - (1-e^2)^{1/2}\sin^{-1}e}\,u^2$$

K.E of the spheroid itself $= \frac{1}{2}\cdot\frac{4}{3}\pi a^2c\rho u^2$

$$\therefore P = \frac{\sigma}{\rho}\cdot\frac{(1-e^2)^{\frac{1}{2}}\{\quad\}}{e(1+e^2) - \cdots}$$



The velocity is always along the x-axis fixed in the spheroid

$$2T = Au^2 + K\omega^2$$
$$= (M + A_1)u^2 + K\omega^2$$

where $A_1 = -\int \phi \frac{\partial \phi}{\partial n} dS$, $K^2 = \frac{2}{5}a^2$,

there being no rotation about the axis the contribution to K^2 by the pressure of the liquid is zero.

$$u = -c\dot{\theta}$$
$$2T = (M + A_1)c^2\dot{\theta}^2 + \frac{2}{5}a^2\dot{\theta}^2$$

The eqn of motion is therefore

$$\left(M + A_1 + \frac{2}{5}\frac{a^2}{c^2}\right)c^2\dot{\theta}^2 = -(M - M')g c \sin\theta$$

where M' = mass of liquid displaced

$$\text{Length of S.E.P} = \frac{(M + A_1)c^2 + \frac{2}{5}a^2}{c(M - M')}$$

but $P = A_1/M$

$$\therefore \quad l = \frac{(1+P)c + \frac{2}{5}a^2/c}{1 - M'/M}$$
$$= \frac{(1+P)c + \frac{2}{5}a^2/c}{1 - \sigma/\rho}$$

(11) If $2T = Au^2 + Bv^2 + K\omega^2$

we can show easily that, $A = B = \pi\rho ab$

and $K = \frac{1}{4}\pi\rho ab \cdot \frac{(a^2 - b^2)^2}{(a^2 + b^2)}$ ⊛

For translation the liquid moves as a whole with the solid, and hence $A = B = \pi\rho ab$. For rotation K.E can be calculated.

In the latter case, when the plane is rough, since the point of contact is instantaneously at rest

$$u - y\omega = 0, \quad v + x\omega = 0$$

In the position of stable equilibrium

$$x = 0, \quad y = b$$

Hence at this motion, $u = b\omega$, $v = 0$

$$2T \text{ (at the instant)} = \pi\rho ab \cdot b^2\omega^2 + \tfrac{1}{4}\pi\rho ab \cdot \frac{\omega^2(a^2-b^2)^2}{a^2+b^2}$$

The eqⁿ of energy gives

$$\omega^2 = \frac{\pi ab\rho\,(a-b)g}{\pi ab\rho \cdot \frac{b^2}{2} + \frac{1}{8}\pi\rho ab \cdot (a^2-b^2)^2/(a^2+b^2)}$$

$$= \frac{8(a-b)(a^2+b^2)g}{4b^2(a^2+b^2)+(a^2-b^2)^2} \qquad (1)$$

In the first case, since the linear momentum in the horizontal direction is const, and equal to zero, we have

$$Au\sin\theta + Bv\cos\theta = 0 \qquad (2)$$

$$2T = Au^2 + Bv^2 + K\omega^2$$

$$= Au^2 + B \cdot \frac{A^2u^2\sin^2\theta}{B^2\cos^2\theta} + K\omega^2$$

$$= Au^2\left(1 + \frac{A}{B}\tan^2\theta\right) + K\omega^2$$

and (2) show that when $\theta = \pi/2$, $u = 0$ ie when the cylinder comes to its position of stable equilibrium, the horizontal velocity is zero

$$\therefore v = -\frac{Au}{B\cot\theta} = -\frac{A}{B}u\tan\theta$$

which is indeterminate in the position of equilibrium. Just before the coming to the posⁿ of stable eqᵐ $\tan\theta$ is finite while u is very small being zero at the eqᵐ posⁿ & cannot alter in magnitude abruptly. Hence $v = 0$ in the posⁿ of eqᵐ

ie $2T$ (at the moment) $= \frac{1}{4}\pi\rho ab\,(a^2-b^2)^2/(a^2+b^2)$

The eqⁿ of energy gives

$$\frac{1}{8}\pi\rho ab\,\frac{(a^2-b^2)^2}{(a^2+b^2)}\omega^2 = Mg(a-b)$$

$$\text{or } \omega^2 = \frac{8\pi\rho ab(a-b)}{\pi\rho ab(a^2-b^2)^2/(a^2+b^2)}$$

$$= \frac{8(a^2+b^2)}{(a+b)(a-b)^2}$$

∴ Required ratio is

$$(a^2-b^2)^2 + 4b^2(a^2+b^2) : (a^2-b^2)^2.$$

(13) If u and v are its velocities along the two symmetrical axes, and ω the angular velocity about its axis, the K.E is given by

$$2T = Au^2 + Bv^2 + K\omega^2$$

In general, $2T = Au^2 + 2Huv + Bv^2 + K\omega^2$

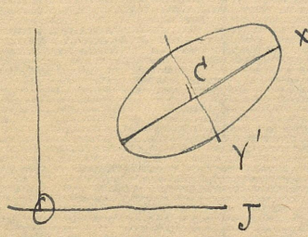
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega}\right) = v\frac{\partial T}{\partial u} - u\frac{\partial T}{\partial v}$$

$$\text{ie } \frac{d}{dt}(K\omega) = v(Au+Hv) - u(Bv+Hu)$$

$$= uv(A-B) + H(v^2-u^2)$$

$$J^2 = \left(\frac{\partial T}{\partial u}\right)^2 + \left(\frac{\partial T}{\partial v}\right)^2 = (Au+Hv)^2 + (Bv+Hu)^2$$

$$\tan\theta = \frac{Bv+Hu}{Au+Hv}$$



θ can be taken as the angle which any line fixed in the body makes with OJ, since OJ is has invariant direction

$$\therefore K\ddot{\theta} - J\cos\theta\cdot v + J\sin\theta\cdot u = 0$$

$$\left.\begin{aligned} Au+Hv &= J\cos\theta \\ Hu+Bv &= J\sin\theta \end{aligned}\right\} \quad \begin{aligned} u &= J(B\cos\theta - H\sin\theta)/(AB-H^2) \\ v &= J(A\sin\theta - H\cos\theta)/(AB-H^2) \end{aligned}$$

$$\therefore K\ddot{\theta} - J^2\{\cos\theta(A\sin\theta - H\cos\theta) - \sin\theta(B\cos\theta - H\sin\theta)\}/AB-H^2$$

Hence the equation required

Since there are no forces acting on the body

$$\xi = Au + Hv = \text{Const}$$

$$\eta = Hu + Bv = \text{Const}$$

A soln of ξ is given by $\tan\theta = \frac{\xi}{\mu} = \text{Const.}$

(14) $2T = Au^2 + Bv^2 + K\omega^2$

$A = (\pi\rho b^2 + \pi\sigma ab)$ $B =$ –

$2T = \pi(\rho b^2 + \sigma ab)u^2 + \pi(\rho a^2 + \sigma ab)v^2 + \pi\left(\frac{\rho}{\sigma}c^4 + \sigma ab \cdot \frac{a^2+b^2}{4}\right)\omega^2$

The last of the eqns of motion gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega}\right) = v\frac{\partial T}{\partial u} - u\frac{\partial T}{\partial v} \quad \text{if there be no external forces}$$

$$\text{ie } \pi\left(\frac{\rho c^4}{\sigma} + \frac{\sigma ab(a^2+b^2)}{4}\right)\dot{\omega} = \pi uv\rho(b^2-a^2)$$

$$\text{ie } \ddot{\theta}\left\{\frac{c^2}{g} + \frac{\sigma}{\rho}\cdot\frac{ab}{4}\,\frac{a^2+b^2}{a^2-b^2}\right\} + uv = 0, \quad (c^2 = a^2-b^2).$$

(15) Vide §171, p. 197

(17) Vide §144, p. 168.

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Chapter IX, p. 247.

(2) Spiral vortex – source & vortex – See Milne-Thomson § 13.33.

(3) $\psi = \frac{K}{4\pi}\log\frac{x^2+(y-b)^2}{x^2+(y+b)^2}$

For relative stream lines we reduce the vortices to rest by imposing a velocity such that

$$\left(-\frac{\partial\psi}{\partial y}\right) = -\frac{K}{4\pi b} \qquad \text{ie } \psi = \frac{Ky}{4\pi b}$$

$$\therefore \psi = \frac{K}{4\pi}\log\frac{x^2+(y-b)^2}{x^2+(y+b)^2} + \frac{Ky}{4\pi b}$$

(4) See Milne-Thomson, Ex. 4, p. 348.

(5) The image of (a,b) is $(a,-b)$, and

$$\psi = \frac{K}{4\pi}\log\frac{(x-a)^2+(y-b)^2}{(x-a)^2+(y+b)^2} = \frac{K}{2\pi}\log\frac{r}{r_1}$$

The pressure eqn is $\frac{p}{\rho} = f(t) - \frac{1}{2}q^2 + \frac{\partial\phi}{\partial t}$

$$w = \frac{iK}{2\pi} \log \frac{z-(a-ib)}{z-(a+ib)}$$

and $$\phi = \frac{K}{2\pi} \tan^{-1}\left\{\frac{2b}{(x-a)^2+y^2-b^2}\right\}$$

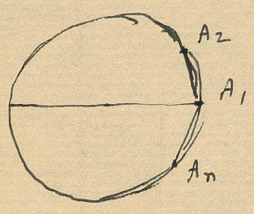
$$\therefore \frac{\partial\phi}{\partial t} = -\frac{K^2}{4\pi^2}\left\{\frac{y^2-b^2-(x-a)^2}{\{(x-a)^2+(y-b)^2\}\{(x-a)^2+(y+b)^2\}}\right.$$ using $\dot{a} = \frac{K}{4\pi b}$, $\dot{b} = 0$.

From $q = \left|\frac{dw}{dz}\right|$ we have

$$q^2 = \frac{b^2K^2}{\pi^2} \cdot \frac{1}{\{(x-a)^2+(y-b)^2\}\{(x-a)^2+(y+b)^2\}}$$

Hence the surfaces of equal pressure are obtained in the required form.

(6)



The velocity of A_1 due to $A_2 = \frac{K}{4\pi a \sin \pi/n}$ perpendicular to A_1A_2

" A_1 " A_n = " A_1A_n

and the resultant is along the tangent at A_1 and of magnitude $K/2\pi a$. Similarly we consider velocity of A_1 due to (A_3, A_{n-1}) etc. Also velocity of A_1 due to the diametrically opposite vortex is $K/4\pi a$ also along tangent at A_1. Hence resultant velocity is along the tangent and of magnitude $= \frac{n-2}{2}\cdot\frac{K}{2\pi a} + \frac{K}{4\pi a} = \frac{K(n-1)}{4\pi a}$

$\therefore$ Req^d angular velocity $= K(n-1)/4\pi a^2$

and time $= 2\pi \times 4\pi a^2/K(n-1) = 8\pi^2 a^2/K(n-1)$

To find the velocity we have

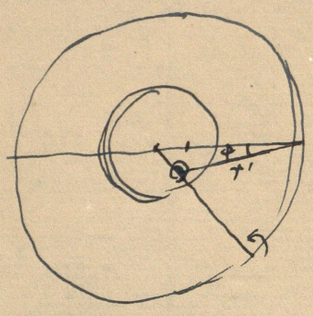
$$w = \frac{iK}{2\pi}\log(z^n - a^n)$$

$$\therefore q = \left|\frac{dw}{dz}\right| = \frac{Knr^{n-1}}{2\pi(r^{2n} - 2r^n a^n \cos n\theta + a^{2n})^{1/2}}.$$

(7) See Milne-Thomson, § 13·60.

(9) " § 13·31

(10) Due to the vortices on the outer cylinder the velocity at A_1 is, as in Ex 6,



given by $\frac{K(n-1)}{4\pi c}$ along the tangent at A_1,

For the images we have

$$\Sigma \frac{K}{\pi r'} \cos\phi = \Sigma \frac{K}{\pi r'^2}\left(c - \frac{a^2}{c}\cos\frac{2\pi}{n}\right)$$

$$= \Sigma \frac{Kc\left(c^2 - a^2\cos^2\pi/n\right)}{\pi\left(a^4 + c^4 - 2a^2c^2\cos^2\pi/n\right)}$$

but in the opposite direction. Velocity due to the two vortices situated on the diameter through A_1 is

$$\frac{K}{2\pi(c - a^2/c)} \quad \text{and} \quad \frac{K}{2\pi(c + a^2/c)}$$

The velocity is thus wholly along the tangent, and equal to

$$\frac{Kc}{2\pi(c^2-a^2)} + \Sigma \frac{Kc\left(c^2 - a^2\cos^2\pi/n\right)}{2\pi\left(c^4 + a^4 - 2c^2a^2\cos^2\pi/n\right)} - \frac{K(n-1)}{4\pi c}$$

$$= \frac{Kc\left(c^2 - a^2\cos^2\theta\right)}{2\pi\left(c^4 + a^4 - 2c^2a^2\cos\theta\right)} + \Sigma \frac{Kc\left\{c^2 - a^2\cos\left(\theta + \frac{2\pi}{n}\right)\right\}}{2\pi\left(c^4 + a^4 - 2c^2a^2\cos\frac{2\pi}{n}\right)} - \frac{K(n-1)}{4\pi c}$$

where $\theta = 0$.

$$= \frac{Kc}{4\pi} \cdot \frac{2nc^{2n-2}\left(c^{2n} - a^{2n}\right)}{c^{4n} - 2c^{2n}a^{2n} + a^{4n}} - \frac{K(n-1)}{4\pi c}$$

$$= \frac{K\left\{(n+1)c^{2n} + (n-1)a^{2n}\right\}}{4\pi c^2\left(c^{2n} - a^{2n}\right)}$$

To find the velocity of the point we use

$$w = \frac{ik}{2\pi} \log \frac{z^n - a^n}{z^n - b^n}$$

$$\left|\frac{dw}{dz}\right| = \left|\frac{nik\, z^{n-1}\left(c^n - b^n\right)}{2\pi\left(z^n - c^n\right)\left(z^n - b^n\right)}\right| = \text{required result.}$$

(11) See Milne-Thomson, Ex. 15, p. 350.

(12)

$$\left.\begin{aligned} \xi &= \cosh\frac{\pi y}{a}\cos\frac{\pi x}{a} \\ \eta &= -\sinh\frac{\pi y}{a}\sin\frac{\pi x}{a} \end{aligned}\right\} \qquad (1)$$

$x=0$, $x=a$, $y=0$ are all satisfied by $\eta=0$. Thus the boundary is transferred to $\eta=0$ in the ξ-plane.

$$\zeta = \xi + i\eta = \cos\frac{\pi z}{a}$$

If (ξ_1, η_1) corresponds to (x_1', y_1')

$$\psi = \frac{k}{4\pi}\log\frac{(\xi-\xi_1)^2+(\eta-\eta_1)^2}{(\xi-\xi_1)^2+(\eta+\eta_1)^2}$$

$$\frac{d\zeta}{dz} = -\frac{\pi}{a}\sin\frac{\pi z}{a}, \text{ and}$$

$$\left|\frac{d\zeta}{dz}\right| = \frac{\pi}{a}\sqrt{\cosh^2\frac{\pi y}{a}\left(1-\cos^2\frac{\pi x}{a}\right)+\cos^2\frac{\pi x}{a}\left(\cosh^2\frac{\pi y}{a}-1\right)}$$

For the path of P

$$\chi'(x_1', y_1') = \chi(\xi_1, \eta_1) + \frac{k}{4\pi}\log\left|\frac{d\zeta}{dz}\right|$$

$$\chi(\xi_1, \eta_1) = -\frac{k}{4\pi}\log\eta_1$$

$$\therefore\ \chi'(x_1', y_1') = -\frac{k}{4\pi}\log\left(\sinh\frac{\pi y}{a}\sin\frac{\pi x}{a}\right) + \frac{k}{4\pi}\log\frac{\pi}{a}\sqrt{\cosh^2\frac{\pi y}{a}-\cos^2\frac{\pi x}{a}}$$

Hence the path is

$$-\frac{k}{4\pi}\log\sinh^2\frac{\pi y}{a}\sin^2\frac{\pi x}{a} + \frac{k}{4\pi}\log\left\{\sin^2\frac{\pi x}{a}\cosh^2\frac{\pi y}{a}+\cos^2\frac{\pi x}{a}\sinh^2\frac{\pi y}{a}\right\} = \text{Const}$$

or $\cot^2\frac{\pi x}{a} + \coth^2\frac{\pi y}{a} = \text{Const.}$

Also $\xi^2+\eta^2 = \cosh^2\frac{\pi y}{a} - \sin^2\frac{\pi x}{a}$

$$\phi = -\frac{k}{2\pi}\left[\tan^{-1}\frac{-2\eta_1(\xi-\xi_1)}{(\xi-\xi_1)^2+(\eta^2-\eta_1^2)}\right] \quad \left[\text{ie } \tan^{-1}\frac{\eta-\eta_1}{\xi-\xi_1} - \tan^{-1}\frac{\eta+\eta_1}{\xi-\xi_1}\right.$$

$$-\psi+i\phi = -\frac{k}{2\pi}\log\frac{\cos\frac{\pi z}{a}-\cos\frac{\pi z'}{a}}{\cos\frac{\pi z}{a}-\text{Lee}\,\frac{\pi x'}{a}} = \omega_1$$

and $\left|\frac{d\omega_1}{dz}\right|$ = required velocity

(13) Lee Milne-Thomson, Exc. 6, p. 502.

(16) $\xi\frac{\partial u}{\partial x}+\ldots = \frac{\partial u}{\partial x}\left(\frac{\partial \omega}{\partial y}-\frac{\partial v}{\partial x}\right)+\ldots$

Consider $\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}$ ... (1)

For steady motion

$$u = \frac{dx}{dt}, \quad \therefore \; \frac{\partial u}{\partial x} = \frac{1}{u}\frac{d^2x}{dt^2}, \quad \frac{\partial v}{\partial x} = \frac{1}{u}\frac{d^2y}{dt^2}; \quad \frac{\partial u}{\partial x} = \frac{1}{u}\cdot\frac{d^2x}{dt^2}, \quad \frac{\partial v}{\partial x} = \frac{1}{u}\cdot\frac{d^2y}{dt^2}$$

Hence (1) = 0. By symmetry therefore

$$\xi\frac{\partial u}{\partial x} + \cdots = 0$$

with two more similar expressions

$$\xi\frac{\partial u}{\partial x} + \eta\left(\frac{\partial v}{\partial x} - 2\zeta\right) + \zeta\left(\frac{\partial w}{\partial x} - 2\eta\right) = 0$$

$$\therefore \; \xi\frac{\partial u}{\partial x} + \eta\frac{\partial v}{\partial x} + \zeta\frac{\partial w}{\partial x} = 0$$

∆ since ξ, η, ζ are constants when u, v, w are linear functions of x, y, z, we have

$$\xi u + \eta v + \zeta w = \text{const} \qquad (ii)$$

(ii) vanishes when $\xi = \eta = \zeta = 0$ in which case the motion is irrotational, the velocity potential being a solid harmonic of degree two in x, y, z.

If $u = ax + (h-\zeta)y$, $v = (h+\zeta)x - ay$, $w = 0$

the stream lines are given by

$$\frac{dx}{ax+(h-\zeta)y} = \frac{dy}{(h+\zeta)x - ay}$$

the soln of the differential eqn being

$$\left\{(h+\zeta) - 2a\frac{y}{x} - (h-\zeta)\frac{y^2}{x^2}\right\}x^2 = \text{Const}$$

ie $x^2(h+\zeta) - 2axy - (h-\zeta)y^2 = \text{Const}$ (iii)

$$\dot{x} = ax + (h-\zeta)y$$

$$\ddot{x} = a\dot{x} + (h-\zeta)\dot{y}$$

$$\therefore \; \ddot{x} + x\left\{(\zeta^2-h^2) - a^2\right\} = 0$$

Similarly $\ddot{y} + y\left\{(\zeta^2-h^2) - a^2\right\} = 0$

The motion is periodic if $\zeta^2 - (h^2+a^2)$ is +ve

In this case the path is

$$x = A\cos\left\{\sqrt{\zeta^2-(h^2+a^2)}\,t + B\right\}$$

$$y = A\cos\left\{\sqrt{\zeta^2-(h^2+a^2)}\,t + B_1\right\}$$

(17) $$T = \rho \iiint \left[u(y\zeta - z\eta) + v(z\xi - x\zeta) + w(x\eta - y\xi) \right] dx\,dy\,dz$$

$$\text{also} = \rho \iiint u \left\{ y\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial y}\right) - z\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \right\} dx\,dy\,dz \qquad (i)$$

omitting the surface integrals we have by the usual transformation

$$(i) = -\rho \iiint \left\{ (vy + wz)\frac{\partial u}{\partial x} - u^2 \right\} dx\,dy\,dz$$

$$\text{Thus } T = \rho \iiint \left[(u^2+v^2+w^2) + xu\frac{\partial u}{\partial x} + yv\frac{\partial v}{\partial y} + zw\frac{\partial w}{\partial z} \right] dx\,dy\,dz \qquad (iii)$$

$$\iiint xu\frac{\partial u}{\partial x}\,d\tau = \iint \tfrac{1}{2} xu^2\,dy\,dz - \tfrac{1}{2}\iiint u^2\,d\tau$$

$$= \iint \tfrac{1}{2} xu^2 l\,dS - \text{"}$$

the first integral vanishing since u, v, w vanish on the boundary

Hence Φ (ii) gives $T = \frac{1}{2}\rho \iiint (u^2+v^2+w^2)\,dx\,dy\,dz$

(19) The vortex lines are given by

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} = \frac{u\,dx + v\,dy + w\,dz}{\xi u + \eta v + \zeta w} = \frac{d\theta + \lambda\,dx}{\xi u + \eta v + \zeta w}$$

$$d\theta = \frac{\partial \theta}{\partial t}dt + \frac{\partial \theta}{\partial x}dx + \cdots ; \quad d\chi = \frac{\partial \chi}{\partial t}dt + \frac{\partial \chi}{\partial x}dx + \cdots$$

$$\therefore u = \frac{\partial \theta}{\partial x} + \lambda\frac{\partial \chi}{\partial x}, \quad v = \cdots, \quad w = \cdots$$

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial x} = \frac{D(\lambda, \chi)}{D(y, z)} ; \text{ similarly } \eta = \frac{D(\lambda, \chi)}{D(z, x)}, \quad \zeta = \frac{D(\lambda, \chi)}{D(x, y)}$$

$$\therefore \frac{dx}{D(\lambda,\chi)/D(y,z)} = \cdots = \cdots = \frac{\frac{\partial \lambda}{\partial x}dx + \cdots}{0} = \frac{\frac{\partial \chi}{\partial x}dx + \cdots}{0}$$

Hence $d\lambda = 0$, $d\chi = 0$

and the vortex lines are given by $\lambda = \text{const}$, $\chi = \text{const}$.

(20) If the vortex lines be $\perp^r$ to the stream lines

$$\xi u + \eta v + \zeta w = 0$$

ie $$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \cdots = 0$$

which is the condition that

$u\,dx + v\,dy + w\,dz + \chi\,dt = 0$ may be solved.

Let the soln be

$$\phi(x, y, z, t) = \text{const}$$

ie $\frac{\partial \phi}{\partial x} dx + \cdot + \cdot + \frac{\partial \phi}{\partial t} dt = 0$

$\therefore \quad u \Big/ \frac{\partial \phi}{\partial x} = v \Big/ \frac{\partial \phi}{\partial y} = w \Big/ \frac{\partial \phi}{\partial z} = \mu$.

(25) $\frac{D}{Dt}\left[\frac{\xi}{\rho}\frac{\partial \lambda}{\partial x} + \cdots\right] = 0$

$$\frac{\xi}{\rho}\frac{D}{Dt}\left(\frac{\partial \lambda}{\partial x}\right) + \frac{\partial \lambda}{\partial x}\frac{D}{Dt}\left(\frac{\xi}{\rho}\right) + \cdots = 0 \qquad (A)$$

The eqns of motion are given by

$$\frac{\partial u}{\partial t} - v\zeta + w\eta = -\frac{\partial x'}{\partial x}, \text{ etc}$$

where $x' = \int \frac{dp}{\rho} + \frac{1}{2}q^2 + \Omega$

From the second and third of the eqns of motion we have eqn stating $\frac{\partial^2 x'}{\partial y \partial z}$

and using $\zeta = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$,

$$\frac{\partial \zeta}{\partial t} + v\frac{\partial \zeta}{\partial y} + w\frac{\partial \zeta}{\partial x} - u\left(\frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z}\right) = \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} - \zeta\left(\frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}\right)$$

but $\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = 0$

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$\therefore \quad \frac{D}{Dt}\left(\frac{\xi}{\rho}\right) = \frac{\xi}{\rho}\frac{\partial u}{\partial x} + \cdots$, etc

Hence (A) reduces to

$$\frac{\xi}{\rho}\frac{\partial}{\partial x}\frac{D\lambda}{Dt} + \cdots = 0$$

or $\left(\frac{\xi}{\rho}\frac{\partial}{\partial x} + \cdots + \cdot\right)\frac{D\lambda}{Dt} = 0$.

since $\frac{\partial}{\partial x}\left(\frac{D\lambda}{Dt}\right) = \frac{D}{Dt}\left(\frac{\partial \lambda}{\partial x}\right) + \frac{\partial \lambda}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \lambda}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial \lambda}{\partial z}\frac{\partial w}{\partial x}$, etc.

(26)

$$\iiint (u^2 + v^2 + w^2)\, dx\, dy\, dz$$

$$= \iiint \left[(-\phi_x + H_y - G_z)^2 + \cdots + \cdot\right] dx\, dy\, dz.$$

$$= \iiint \left[\Sigma \phi_x^2 + \Sigma F_x^2 + \Sigma G_x^2 + \Sigma H_x^2 - J^2 \right] dx\,dy\,dz$$

$$-2 \iiint \left[\Sigma \phi_x (H_y - G_z) \right] dx\,dy\,dz - 2 \iiint \left[\Sigma (H_y G_z - H_z G_y) \right] dx\,dy\,dz$$

= req^{d} expression.

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Chapter X, p. 289

(5) The kinematical condition is $-\frac{\partial \psi}{\partial y} = V$ or $\psi + Vy = \text{const}$

The value of V is given by

$$V^2 = \frac{g\lambda}{2\pi} \tanh\left(\frac{2\pi h}{\lambda}\right) \qquad (p.265)$$

(6) $\phi = \frac{na}{m} e^{my} \cos(mx - nt) = \frac{ga}{n} e^{my} \cos(mx - nt)$

$\frac{p}{\rho} = F(t) - gy_1$ in the eq^{n} position

$\frac{p}{\rho} = F(t) - g(y_1 + y_0) - \frac{1}{2} q^2 + \frac{\partial \phi}{\partial t}$

where $y = a \sin(mx - nt)$

$\frac{\partial \phi}{\partial t} = ga e^{my} \sin(mx - nt)$

$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = g^2 a^2 \frac{m^2}{n^2} e^{2my}$

Hence $\frac{p}{\rho} = F(t) - g\left[y_1 + a \sin(mx - nt)\right] + ga \sin(mx - nt)$ to a^{1} approx^{n}

$= F(t) - gy_1$. Hence the result

(9) $V^2 = \frac{g}{m} \tanh(mh) = \frac{g}{m} \cdot \frac{2(mh + m^3h^3/3! + \cdots)}{2(1 + m^2h^2/2! + \cdots)}$

ie $V^2 = gh(1 + \epsilon)$ where ϵ vanishes with m

For $m = 0$, $V^2 = gh$

~~For~~ $m \to \infty$, $\frac{2\pi}{m} \to 0$ This proves the req^{d} results. (?)

(10) $\phi = \frac{ga}{n} e^{+my} \cos mx \cos nt$

$\frac{\partial \phi}{\partial x} = 0$, for $x = 0$; $\frac{\partial \phi}{\partial x} = 0$ for $x = l$, giving $\sin ml = 0$ ie $m = \frac{K\pi}{l}$

In this case $n^2 = gm = gK\pi/l$

$$\frac{2\pi}{n} = 2\pi \sqrt{\frac{l}{\pi K g}} = 2\sqrt{\frac{\pi l}{Kg}}$$ where K is an integer.

~~my is the~~ The amplitude $\propto e^{my} \propto e^{-S\pi z/l}$ where $z = -y$ is the depth.

(11) $$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - gy - \tfrac{1}{2} q^2 + F(t) \qquad (1)$$

At the free surface, p = const & hence

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} = 0$$

or $$\frac{\partial p}{\partial t} - \frac{\partial p}{\partial x}\frac{\partial \phi}{\partial x} - \frac{\partial p}{\partial y}\frac{\partial \phi}{\partial y} = 0 \qquad (2)$$

we can ~~take~~ omit $F(t)$ in (1) by assuming it contained in $\partial\phi/\partial t$. Substituting this value of p in (2)

$$\frac{\partial}{\partial t}\left\{\frac{\partial \phi}{\partial t} - gy - \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial y}\right)^2\right\} - \ldots = 0.$$

which on simplification gives the required result.

For $\phi = b \cos \frac{\pi x}{a} \cosh \pi (y+h)/a \sin pt$, the general eqn reduces to

$$\tan \frac{\pi(y+h)}{a} = ap^2/g\pi$$

1. $\frac{\partial \phi}{\partial y} = 0$ at $y = -h$
2. $\frac{\partial \phi}{\partial x} = 0$ at $x = 0$, & $x = a$
3. These are stationary waves
4. The surface of equal pressure is the plane

$$y = -h + \frac{a}{\pi} \tanh^{-1}\left(\frac{ap^2}{g\pi}\right).$$

(12) $$U^2 = \frac{g}{m} \frac{\rho - \rho'}{\rho \coth(mh) + \rho' \coth(mh')}$$ & for sufficiently small values of h & h' we have

$$v^2 = \frac{g}{m} \cdot \frac{\rho - \rho'}{\rho \cdot \frac{2}{2mh} + \rho' \cdot \frac{2}{2mh'}}$$

$$= \frac{g(\rho - \rho')hh'}{\rho h' + \rho' h}$$

(13) See Example 19.

(14) Let $x = 0$ at the sea front

Let $\phi = \frac{na}{m} \cdot \frac{\cosh m(y+h)}{\sinh mk} \cos mx \sin nt$

$$\left(\frac{\partial \phi}{\partial t}\right)_{x=0} = \frac{n^2 a}{m} \frac{\cosh m(y+h)}{\sinh mh} \cos nt$$

Hence $\frac{\partial \phi}{\partial t}$ for unit length of the sea front

$$= \frac{n^2 a}{m} \frac{\cos nt}{\sinh mh} \int_0^{-h} \cosh m(y+h)\, dy$$

$$= \frac{n^2 a}{m} \cdot \frac{\cos nt}{\sinh mh} \cdot \frac{1}{m} \left[\sinh m(y+h)\right]_0^{-h}$$

$$= \frac{n^2 a}{m^2} \cos nt,$$

$$= \frac{ag}{m} \cos nt \tanh mh, \text{ since } n^2 = gm \tanh mh.$$

The extra pressure is due (i) to the raised position of the particles above the mean position, (ii) to the term $\frac{\partial \phi}{\partial t}$.

For the free surface, $y = a \cos mx \cos nt$

Pressure at the sea front $= g\rho(h + y_0) \cdot \frac{1}{2}(h + y_0)$

since $\frac{1}{2}(h+y_0)$ is the distance of the C.G of the area from the free surface

Excess due $= g\rho h y_0$ where y_0 is the value of y at $x = 0 = g\rho h a \cos nt$

Hence ~~extra~~ excess pressure

$$= \frac{ag\rho}{m} \cos nt \tanh mh + g\rho h a \cos nt$$

$$= g\rho a \left(a + \frac{1}{m} \tanh mh\right) a \cos nt$$

but $m = 2\pi/\lambda$ and $n = 2\pi V/\lambda$. Hence the result.

(15) Let the barriers be at $x = 0$ and $x = l$.

Let $\phi = \frac{ga}{n} \frac{\cosh m(y+h)}{\cosh mh} \cos mx \cos nt$

$\frac{\partial \phi}{\partial x} = 0$ when $x = 0$

$\frac{\partial \phi}{\partial x} = 0$ when $x = l$, gives $\sin ml = 0$ or $m = R\pi/l$ where

R is an integer.

A period is given by $2\pi/n$, and

$n^2 = gm \tanh mh$

$= \frac{gR\pi}{l} \tanh \frac{R\pi h}{l}$

Hence the periods of the waves are found by giving R positive integral values

in $2\pi \left[\frac{\pi l}{gk} \coth \frac{R\pi h}{l}\right]^{1/2}$.

(16) Let $\psi_1 = c_1 e^{my} \cos(mx - nt)$ for the lower, and

$\psi_2 = c_2 e^{-my} \cos(mx - nt)$ " upper. Take the origin in the

common surface.

$y = -\infty$ and $y = +\infty$ give $\psi_1 = 0$, $\psi_2 = 0$ & hence are streem lines.

$$-\frac{\partial \psi_1}{\partial y_0} = c_1 m e^{my} \cos(mx - nt) = \dot{x}$$

$$\frac{\partial \psi_1}{\partial x} = -c_1 m e^{my} \sin(mx - nt) = \dot{y}$$

$$\left.\begin{aligned} \text{i.e. } x &= -\frac{c_1 m}{n} e^{my} \sin(mx - nt) \\ y &= -\frac{c_1 m}{n} e^{my} \cos(mx - nt) \end{aligned}\right\}$$

Hence the paths of the particles are circles

$$x^2 + y^2 = c_1^2 \frac{m^2}{n^2} e^{2my} \text{ and}$$

$$x^2 + y^2 = \frac{c_2^2 m^2}{n^2} e^{-2my}$$

about their mean positions.

Let $\phi_1 = A_1 e^{my} \sin(mx - nt)$ for the lower

$\phi_2 = A_2 e^{-my} \sin(mx - nt)$ "

$\dot{y} = -\frac{\partial \phi}{\partial y}$ at $y = 0$ when $y = a \cos(mx - nt)$ gives

$an = -A_1 m$ or $an = A_2 m$

$\therefore \phi_1 = -\frac{an}{m} e^{my} \sin(mx - nt)$

$\phi_2 = \frac{an}{m} e^{-my} \sin(mx - nt)$

$q_1^2 = \left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial y}\right)^2 = \frac{a^2 n^2}{m^2} e^{2my}$; $q_2^2 = \frac{a^2 n^2}{m^2} e^{-2my}$

when $y = \eta$ where η is small

$q_1 = \frac{an}{m}(1 + m\eta)$, $q_2 = \frac{an}{m}(1 - m\eta)$

giving the relative velocity $= 2an\eta$

$= 2a\eta \cdot \frac{2\pi V}{\lambda}$ $\left[\frac{2\pi}{n} = \frac{\lambda}{V}\right.$

$= \frac{4\pi V a \eta}{\lambda}$.

(17) Take the origin in the common surface

$\phi = C \cosh m(y + h) \cos(mx - nt)$ (i)

for the lower, and

$\phi' = (Ae^{my} + Be^{-my}) \cos(mx - nt)$ (ii)

for the upper.

At the free surface

$\frac{\partial^2 \phi'}{\partial t^2} + g \frac{\partial \phi'}{\partial y} = 0$

$\therefore$ from (ii) we have

$gm(Ae^{mh} - Be^{-mh}) - n^2(Ae^{mh} + Be^{-mh}) = 0$

i.e. $Ae^{mh}(gm - n^2) - Be^{-mh}(gm + n^2) = 0$ (iii)

Let $y = a \sin(mx - nt)$ at the common surface, then

$C = \frac{na}{m \sinh mk} = \frac{ga}{n \cosh nk}$

For (ii) we have $\dot{y} = -\frac{\partial \phi'}{\partial y}$ at $y = 0$, i.e.

$-an = -m(A - B)$ (iv)

$\therefore$ From (iii) & (iv) we have

$$A = \frac{an}{m} \cdot \frac{gm + n^2}{gm + n^2 - e^{2mh}(gm - n^2)}$$

$$B = \frac{an}{m} \cdot \frac{e^{2mh}(gm - n^2)}{gm + n^2 - e^{2mh}(gm - n^2)}$$

Also the pressure eqn gives

$$-g(\sigma - \rho)y = -\sigma \frac{\partial \phi}{\partial t} + \rho \frac{\partial \phi'}{\partial t} \qquad \text{at } y = 0$$

$$= \rho(A + B)n \sin(mx - nt)$$

$$- \sigma C n \cosh mh \sin(mx - nt)$$

which on substitution of values of A, B and C gives on some tedious simplification

$$\frac{\rho}{\sigma}(1 - u^2) = (u \coth mh - 1)(u \coth mk - 1).$$

(18) $$\phi_1 = \frac{a U}{\sinh mh} \cosh m(y + h) \cos(mx - nt) \qquad \text{(Lower)}$$

$$\phi_2 = \frac{-a U}{\sinh mh'} \cosh m(y - h') \cos(mx - nt) \qquad \text{(upper)}$$

Total ~~Total~~ Energy (See p. 276) for the lower liquid

$$= \tfrac{1}{4} g \rho a^2 \lambda + \tfrac{1}{4} \rho a^2 \lambda n^2/m \coth mh$$

$$\left(\frac{nq}{m} = \frac{ga}{n} \tanh mh \; ; \; g = \frac{n^2}{m} \coth mh \right)$$

$$P = \tfrac{1}{4} \frac{\rho a^2 \lambda}{g} \left\{ \frac{U^2 m}{g} \coth mh + 1 \right\}$$

$$= \tfrac{1}{4} \frac{\rho a^2 \lambda}{g} \left\{ \frac{(\rho - \rho') \coth mh}{\rho \coth mh + \rho' \coth mh'} + 1 \right\}$$

$$= \tfrac{1}{4} \frac{a^2 \lambda}{g} \rho \left\{ (2\rho - \rho') \coth mh + \rho' \coth mh' \right\} \Big/ \left(\rho \coth mh + \rho' \coth mh' \right)$$

$\therefore$ Required ratio is

$$= \frac{\rho \left\{ (2\rho - \rho') \coth mh + \rho' \coth mh' \right\}}{\rho' \left\{ (\rho - 2\rho') \coth mh' - \rho \coth mh \right\}}$$

(19) From the general result of Exc. 17, we have for the case of long waves m small. Hence

$$\frac{\rho}{\sigma}(1-u^2) = \left(u\cdot\frac{2}{2mk}-1\right)\left(u\cdot\frac{2}{2mh}-1\right)$$

$$u = \frac{mV^2}{g}$$

$$\therefore \quad \frac{\rho}{\sigma}\left(1-\frac{m^2V^4}{g^2}\right) = \left(\frac{V^2}{gk}-1\right)\left(\frac{V^2}{gh}-1\right)$$

and neglecting m^2, this gives

$$\frac{\rho}{\sigma} = \left(\frac{V^2}{gk}-1\right)\left(\frac{V^2}{gh}-1\right).$$

Expanding the above

$$\frac{V^4}{g^2kh} - \frac{V^2}{g}\left(\frac{1}{h}+\frac{1}{k}\right) - \frac{\rho}{\sigma} + 1 = 0$$

ie $$\frac{V^2}{g} = \frac{1}{2}\left[(h+k) \pm \sqrt{\{(h+k)^2 - (h-k)^2 + 4hk\rho/\sigma\}}^{1/2}\right]$$

which is the result required in Example. 13.

(20) $$\psi_1 = V\left[-y + \sin kc \operatorname{cosech} kc \left\{a\sinh k(y+c) - b\sinh ky\right\}\right]$$

$$\psi_2 = V\left[-y + b\sin kc\, e^{k(y+c)}\right]$$

Let $y = \alpha \sin kc$ be the free surface, then it must make ψ_1 a stream line so that $\alpha = a$ which gives $\psi_1 = 0$ at the free surface. Also the free surface is one of constant pressure & hence

$$\sin kc \cdot g\cdot\alpha + \tfrac{1}{2}V^2\left[1 - k\sin kc \operatorname{cosec} kc\left\{a\cosh kc - b\right\}\right]^2$$

must be constant ie

$$g\alpha - \tfrac{1}{2}V^2 \cdot 2k \operatorname{cosech} kc\,(a\cosh kc - b) = 0$$

$$\therefore \quad -\frac{g\alpha\sin kc}{V^2k} + \alpha\cosh kc = b \qquad (i)$$

Let $y = -c + \beta\sin kc$ be the free surface, then it must make both ψ_1 & ψ_2 the same stream line. This is $\psi_2 = Vc$ if

$$\left.\begin{aligned} -\beta + b &= 0 \\ -\beta + b &= 0 \end{aligned}\right\} \quad \text{ie} \quad b = \beta$$

The eq^{n} of the equality of pressure at the free surface for the two liquids gives

$$\rho g(-c+\beta\sin kx) - \tfrac{1}{2}\rho V^2 \cdot 2k\sin kx\,\mathrm{cosech}\,kc\,(a-b\cosh kc)$$

$$= \rho' g(-c+\beta\sin kx) - \tfrac{1}{2}\rho' V^2 \cdot 2kb$$

ie $\rho g\beta - \rho V^2 k\,\mathrm{cosech}\,kc\,(a-b\cosh kc) = \rho' g\beta - \rho' V^2 kb$

(i) gives $\alpha = \dfrac{V^2 k\beta}{V^2 k\cosh kc - g\sinh kc}$

$\therefore$

$b=\beta,\ a=\alpha$

$$\therefore \rho g\beta - \rho V^2 k\,\mathrm{cosech}\,kc\left[\frac{V^2 k\beta}{V^2 k\cosh kc - g\sinh kc} - \beta\cosh kc\right] = \rho' g\beta - \rho' V^2 k$$

which on simplification gives

$$V^4(\rho+\rho'\coth kc) - \frac{\rho' V^2 g}{k}(1+\coth kc) + \frac{g^2}{k^2}(\rho'-\rho) = 0$$

Solving this quadratic, we have

$$\frac{kV^2}{g} = 1 \text{ or } \frac{\rho'-\rho}{\rho+\rho'\coth kc}.$$

(21) Consider the ϕ $\phi_1 = A\cosh(y+h)\cos(mx-nt)$ (down)

$$\phi_2 = A(B_1 e^{my} + B_2 e^{-my})\cos mx\cos nt \quad \text{(up)}$$

$\dfrac{\partial\phi_2}{\partial x} = 0$ when $x=0$, obviously

$\dfrac{\partial\phi_1}{\partial x} = 0$ when $x=0$ if $\sin am = 0$ ie $m = \dfrac{k\pi}{a}$, k an integer

Also $\dfrac{\partial\phi_1}{\partial y} = 0$ when $y=-h$, obviously

Let $y = a_1\cos mx\sinh nt$ at the common surface

$$\therefore -an = -Am\sinh mh = -m(B_1 - B_2) \quad (i)$$

$$-n^2(B_1 e^{mh'} + B_2 e^{-mh'}) + gm\left[B_1 e^{mh'} - B_2 e^{-mh'}\right) \quad (ii)$$

is the cond^n to be satisfied at the free surface. Jan (i) & (ii)

$$B_1\left[1 - e^{2mh'}\frac{gm-n^2}{gm+n^2}\right] = \frac{a_1 n}{m}$$

$$B_2\left[e^{-2mh'}\frac{gm+n^2}{gm-n^2} - 1\right] = \frac{a_1 n}{m}$$

Also $g(\rho-\rho')a_1 = \rho An\cosh m(y+h) - \rho' n\left[B_1 e^{my} + B_2 e^{-my}\right]$

$\therefore\ g(\rho-\rho')a_1 = \rho n A \cosh mh - \rho' n (B_1+B_2)$

which gives on substituting for $A = \frac{an}{m}\operatorname{cosech} mh$, and also B_1, B_2 and after some simplification

$$\frac{\rho'}{\rho}(g^2m^2-n^4) = (n^2\coth mh - gm)(n^2\coth mh' - gm)$$

i.e. $$\left(n^2\coth\frac{k\pi h}{a} - \frac{gk\pi}{a}\right)\left(n^2\coth\frac{k\pi h'}{a} - \frac{gk\pi}{a}\right) + \frac{\rho'}{\rho}\left(n^4 - \frac{g^2k^2\pi^2}{a^2}\right) = 0.$$

(22) Let us first consider the following more general problem (from Basset):-

A horizontal rectangular box is filled completely with three liquids which do not mix, whose densities reckoned downwards are $\delta_1, \delta_2, \delta_3$ and whose depths when in equilibrium are l_1, l_2, l_3 respectively. Show that if long waves are propagated at their common surfaces, the velocity of propagation must satisfy the equation

$$\left[g(\delta_2-\delta_1) - V^2\left(\frac{\delta_1}{l_1}+\frac{\delta_2}{l_2}\right)\right]\left[g(\delta_3-\delta_2) - V^2\left(\frac{\delta_2}{l_2}+\frac{\delta_3}{l_3}\right)\right] = \frac{V^4\delta_2^2}{l_2^2}.$$

Let $\phi_3 = A\cosh m(y+l_3)\cos(mx-nt)$

$$\phi_2 = \left[B_1 e^{m(y+l_2)} + B_2 e^{-m(y+l_2)}\right]\cos(mx-nt)$$

$$\phi_1 = C\cosh m(y-l_1)\cos(mx-nt)$$

Let $y = a\sin(mx-nt)$ at the common surface of ϕ_3 & ϕ_2

$$\therefore\ -na = -m(B_1-B_2) = -Am\sinh ml_3$$

$$\therefore\ A = na/m\sinh ml_3$$

$$B_1 - B_2 = na/m$$

The pressure eqn gives $\left(\frac{\delta p}{\rho} + gy - \frac{\partial\phi}{\partial t} = 0;\ \frac{\delta p'}{\rho} + gy - \frac{\partial\phi'}{\partial t} = 0\right)$

$$-g(\delta_3-\delta_2)a = -\delta_3\left[\frac{n^2a\cosh ml_3}{m\sinh ml_3}\right] + \delta^2\left[n(B_1+B_2)\right]$$

Let $y' = a'\sin(mx-nt)$ at the common surface of ϕ_2 & ϕ_1.

$$\therefore\ -na' = -m\left[B_1 e^{ml_2} - B_2 e^{-ml_2}\right] = -Cm\sinh ml_1$$

The pressure eqn gives

$$-g(\delta_2-\delta_1)a' = \delta_1\left[\frac{n^2a'}{m}\coth ml_1\right] - \delta_2 n\left[B_1 e^{ml_2} + B_2 e^{-ml_2}\right]$$

$$\therefore\quad B_1 e^{ml_2} - B_2 e^{-ml_2} = na'/m$$

$$B_1 - B_2 = na/m$$

$$2B_1 \sinh ml_2 = \frac{n}{m}\left(a' - a e^{-ml_2}\right)$$

$$2B_2 \sinh ml_2 = \frac{n}{m}\left(a' - a e^{ml_2}\right)$$

$$\therefore\quad -g(\delta_3-\delta_2)a = -\delta_3\frac{n^2}{m}\cdot a\coth ml_3 + \frac{\delta_2 n}{\sinh ml_2}\cdot\frac{n}{m}\left(a' - a\cosh ml_2\right)$$

$$-g(\delta_2-\delta_1)a' = \frac{n^2a'}{m}\delta_1\coth ml_1 - \delta_2 n\left[\frac{na'}{m} + \frac{n}{m}\,\frac{(a'e^{-ml_2} - a)}{m\sinh ml_2}\right]$$

These give by multiplication & simplification

$$\left[-g(\delta_3-\delta_2) + \frac{n^2}{m}\left(\delta_3\coth ml_3 + \delta_2\coth ml_2\right)\right]\left[-g(\delta_2-\delta_1) + \frac{n^2}{m}\left(\delta_1\coth ml_1 + \delta_2\coth ml_2\right)\right] \qquad \text{---(A)}$$

$$= \frac{n^4}{m^2}\cdot\frac{\delta_2^{\,2}}{\sinh^2 ml_2}$$

This is the general result. If the waves are long

$$\left[-g(\delta_3-\delta_2) + \frac{n^2}{m}\left(\frac{\delta_3\cdot 2}{2ml_3} + \frac{\delta_2\cdot 2}{2ml_2}\right)\right]\left[\qquad\qquad\right] = \frac{n^4}{m^2}\cdot\frac{\delta_2^{\,2}}{m^2 l_2^{\,2}}$$

which gives, using $V^2 = n^2/m^2$, the result in the form stated.

Putting $l_1 = l_3 = \infty$, we have

$$\left[g(\delta_2-\delta_1) - V^2\delta_2/l_2\right]\left[g(\delta_3-\delta_2) - V^2\delta_2/l_2\right] = \frac{V^4\delta_2^{\,2}}{l_2^{\,2}}$$

giving $$V^2 = gl_2\,\frac{(\delta_2-\delta_1)(\delta_3-\delta_2)}{\delta_2(\delta_3-\delta_1)}$$

If l_2 be so small that the presence of the middle liquid can be neglected, we can put $\delta_2 = 0$ and $\coth ml_3$, $\coth ml_1 = 1$ in (A). We have

$$V^2 = \frac{g\lambda}{2\pi}\cdot\frac{\rho_3-\rho_1}{\rho_3+\rho_1}$$

$$\text{or}\quad V = \left[\frac{g\lambda}{2\pi}\cdot\frac{\rho_3-\rho_1}{\rho_3+\rho_1}\right]^{1/2}$$

(23) Continuity of the waves requires that

$$n = n'$$

$$m \tanh mh = m' \tanh m'h' \qquad (1)$$

Continuity also requires that the value of transmission of energy is the same in the two sections.

Rate of transmission of energy

$$= \tfrac{1}{2} g\rho a^2 b \times \text{group vel.}$$

$$= \tfrac{1}{2} g\rho a^2 b \cdot \frac{n}{m} \cdot \frac{1}{\sinh mh} \left(\sinh 2mh + 2mh\right) \qquad (2)$$

$$= \tfrac{1}{2} g\rho a'^2 b' \cdot \frac{n}{m'} \frac{1}{\sinh m'h'} \left(\sinh 2m'h' + 2m'h'\right)$$

Using (1) and (2) we have the required result.

(24) The relative importance of gravity and surface tension as dependent on the value of λ may be traced to the form of the expression for the potential energy of a deformation of the type

$$y = \alpha \cos mx \qquad (1)$$

The energy of extension of the surface of separation is

$$V = T_1 \int_0^\lambda \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx - T_1 \lambda$$

$$= \tfrac{1}{2} T_1 \int_0^\lambda \left(\frac{dy}{dx}\right)^2 dx = \tfrac{1}{4} T_1 m^2 \alpha^2 \cdot \lambda$$

Thus the part of the energy due to the extension of the bounding surface is, per unit area

$$= \tfrac{1}{4} T_1 \alpha^2 \Big/ \left(\frac{\lambda}{2\pi}\right)^2 = \frac{T_1 \pi^2 \alpha^2}{\lambda^2} \qquad (2)$$

Similarly the part due to gravity per unit area $= \tfrac{1}{4} g\rho\alpha^2 \qquad (3)$

For $\lambda = 2\pi\sqrt{T_1/\rho g}$, (2) reduces to $\tfrac{1}{4} g\rho\alpha^2 = (3)$

As λ increases from this value (2) goes on becoming smaller & smaller & gravity becomes more and more important as compared with surface tension. As λ decreases from the value (2) the reverse is the case.

(25) $$U = \frac{\sigma' V'}{\rho+\sigma'} + \left[V^2 - \frac{\rho\sigma' V'^2}{(\rho+\sigma')^2} \right]^{1/2}$$

If $V' = U$,

$$\left(U - \frac{\sigma' U}{\rho+\sigma'} \right)^2 = V^2 - \frac{\rho\sigma' U^2}{(\rho+\sigma')^2}$$

$$\therefore \frac{\rho^2 U^2}{(\rho+\sigma')^2} + \frac{\rho\sigma' U^2}{(\rho+\sigma')^2} = V^2$$

$$\therefore V^2 = \frac{\rho U^2}{\rho+\sigma'}$$

If $U = 0$, ~~fwies~~ we have

$$\frac{\sigma'^2 V'^2}{(\rho+\sigma')^2} + \frac{\rho\sigma' V'^2}{(\rho+\sigma')^2} = V^2$$

$$\therefore \frac{\sigma' V'^2}{\rho+\sigma'} = V^2$$

This gives the velocity of the wind

$$U = \frac{2\sigma' V'}{\rho+\sigma'} = \frac{2\sigma'^{1/2}}{(\rho+\sigma')^{1/2}} V = \frac{2\left(\frac{\sigma'}{\rho}\right)^{1/2}}{\left(1+\frac{\sigma'}{\rho}\right)^{1/2}} V$$

but $\sigma = \frac{\sigma'}{\rho}$, hence $U = \frac{2\sigma^{1/2}}{(1+\sigma)^{1/2}} V$

(26) $$V^2 = \left(\frac{g}{m} + \frac{mT}{\rho} \right) \tanh mh \quad (\text{p. } 280)$$

Minimum value of λ is $\lambda_0 = 2\pi\sqrt{T/g\rho}$ (§ 240)

$$U = \frac{d}{dm}(mV) \quad (\S\ 231)$$

$$m^2 V^2 = \left(gm + \frac{m^3 T}{\rho} \right) \tanh mh$$

$$\therefore 2mV \frac{d}{dm}(mV) = \left(g + \frac{3m^2 T}{\rho} \right) \tanh mh + m^2 \left(g + \frac{m^2 T}{\rho} \right) \operatorname{Sech}^2 mh$$

$\therefore \frac{U}{V}$ can be found, and we have $U >$ or $< V$ according as

$$\left[\left(g + \frac{3m^2 T}{\rho} \right) \tanh mh + m^2 \left(g + \frac{9m^2 T}{\rho} \right) \operatorname{Sech}^2 mh \right] > \text{or} < 2\left(g + \frac{m^2 T}{\rho} \right) \tanh mh$$

ie $$m^2 \left(g + \frac{m^2 T}{\rho} \right) \operatorname{Sech}^2 mh > \text{or} < \left(g - \frac{m^2 T}{\rho} \right) \tanh mh$$

For deep water, we put

$$\cancel{\operatorname{sech}^2} \operatorname{sech}^2 mh = 0, \quad \tanh mh = 1$$

$\therefore \quad U \gtrless V$ according as

$$\left(g - \frac{m^2 T}{\rho}\right) \lessgtr 0$$

ie $\lambda \lessgtr \lambda_0$.

(28) In this case the motion does not have a potential, but a stream fn ψ exists. Imposing a velocity

$$\frac{\partial \psi}{\partial y} = U$$ on the whole system we reduce the problem to a case of steady motion. A velocity μy at a height y above the bottom area is given by

$$-\frac{\partial \psi}{\partial y} = \lambda y$$

where $V = \lambda h$ ie $\lambda = V/h$

Hence $\psi = Uy - \frac{1}{2}\lambda y^2 - A \sinh my \sin mx$.

obviously $\psi = 0$ given $y = 0$ which shows the bottom to be a stream line. Let the eqn of the free surface be

$$y = h + a \sin mx.$$

At this surface

$$\psi = U(h + a \sin mx) - \tfrac{1}{2}\lambda(h + a \sin mx)^2 - A \sinh m(h + a \sin mx) \sin mx$$

which for $\psi = 0$, gives neglecting higher powers of a

$$Ua - \lambda h a - A \sinh mh = 0$$

The pressure eqn is

$$\frac{p}{\rho} + gy + \frac{1}{2}\left\{\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2\right\} = \text{Const}$$

where $y = h + a \sin mx$

Constancy at free surface then gives (coeffl of $\sin mx$ etc $= 0$)

$$ga - (U - \lambda h)(\lambda a + Am \cosh mh) = 0, \quad \lambda = V/h$$

$$a(U - V) = A \operatorname{sech} mh$$

giving $g = (U - V)(\lambda + \overline{U - V}\, m \coth mh)$

$$\therefore gh = (U-V)V + (U-V)^2 h \coth mh$$

$$m \coth mh = g/W^2$$

$$W^2 = (U-V)V\frac{W^2}{gh} + (U-V)^2.$$

(29) $$y_2 = \frac{K}{\cosh mh - \frac{g \sinh mh}{mU^2}} \sin mx$$

if $y_1 = K \sin mx$ (p. 286)

Putting $\cosh mh = 1$, $\sinh mh = mh$, for small values concerned

$$\frac{y_2}{y_1} = \frac{1}{1 - gh/U^2}$$

If y_2 be the depression,

$$\frac{y_2}{y_1} = \frac{1}{\frac{gh}{U^2} - 1}.$$

ie $$y_2 = y_1\left(\frac{gh}{U^2} - 1\right)^{-1}.$$

As $U^2 = gh$, the soln fails, and account has to be taken of dissipative forces.

(35) Let us impose a velocity v on the whole system to reduce it to the case of steady motion.

Taking the origin in the bed, let $y = h + a\cos mx$ be the free surface. The bed is given by $y = K\cos mx$. The conditions of the problem will be satisfied by

$$\phi = Ux + (A\cosh my + B\sinh my)\sin mx$$

$$\psi = Vy + (A\sinh my + B\cosh my)\cos mx$$

provided they make the bed a stream line, and the free surface a stream line as well as one of constant pressure

These conditions determine a, etc, neglecting squares of small quantities K & h.

(36) Reducing to a case of steady motion

$$\phi = (v-V)x + (A\cosh my + B\sinh my)\sin mx$$
$$\psi = (v-V)y + (A\sinh my + B\cosh my)\cos mx$$

The bed is $y = a\cos mx$. If free surface be $y = h + a'\cos mx$ the conds of the problem require

$$B = -(v-V)a$$
$$(v-V)a' = -[A\sinh mh + B\cosh mh]$$

The variable part of the pressure is given by

$$\frac{\delta p}{\rho} + g(h + a'\cos mx) + (v-V)\cdot m\cdot[A] = \text{const}$$

$$\delta p = -T\frac{\partial^2 \eta}{\partial x^2} = Tam^2\cos mx$$

$$\therefore \quad \frac{Tam^2}{\rho} + ga' + m(v-V)[A] = 0$$

$$a'\sinh mh\left(g + \frac{Tm^2}{\rho}\right) - ma(v-V)^2\cosh mh$$
$$= (v-V)^2 ma\left[\sinh^2 mh - \cosh^2 mh\right]$$
$$= -(v-V)^2 ma$$
$$\therefore \quad a = a'\left\{\cosh mh - \frac{g + m^2T/\rho}{m(v-V)^2}\sinh mh\right\}.$$

(37) Let $\frac{2\pi}{\lambda} = m$ & the free surface be $y = h + k\sin mx$. The conds of the problem will be satisfied by

$$\phi = -Vx + (A\cosh my + B\sinh my)\cos mx$$
$$\psi = -Vy + (A\sinh my + B\cosh my)\sin mx$$

provided conds of previous two examples are satisfied. If the bed be $y = \alpha\sin mx$, these give $k = \dfrac{V^2\alpha m}{mV^2\cosh mh - g\sinh mh}$

$$V^2 = \frac{g}{m}\tanh mH$$

$$\therefore \quad k = \frac{\alpha\sinh mH}{\sinh m(H-h)}$$

Hence referred to moving axes the surface is

$$y = h + \alpha \sinh m H \operatorname{cosech} m(H-h) \sin mx$$

and referred to fixed axes is

$$y = h + \alpha \sinh \frac{2\pi H}{\lambda} \operatorname{cosech} \frac{2\pi(H-h)}{\lambda} \sin \frac{2\pi(x-vt)}{\lambda}.$$

Chapter XI., p. 320

(1) $y = \chi(x)$; $\dot{\eta} = \theta(x)$.

Taking the general solution

$$\eta = f(ct-x) + F(ct+x)$$

we have when $t = 0$,

$$\chi(x) = f(-x) + F(x)$$

$$\theta(x) = cf'(-x) + cF'(x)$$

Integrating the last

$$\int^x \theta(x)\,dx = -cf(-x) + cF(x)$$

and therefore

$$f(-x) = \tfrac{1}{2}\chi(x) - \frac{1}{2c}\int^x \theta(x)\,dx$$

$$F(x) = \tfrac{1}{2}\chi(x) + \frac{1}{2c}\int^x \theta(x)\,dx$$

$$\text{or } \eta = \tfrac{1}{2}\left[\chi(x-ct) + \chi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \theta(x)\,dx.$$

The disturbance will travel uniformly in the two directions with velocity c. After time $t > a/c$ the disturbance will have travelled a distance ct in either direction, and hence a portion $2ct - 2\alpha$ will be straight. In this case

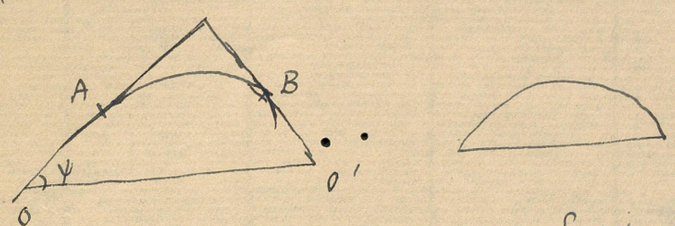
$$y = \frac{1}{2c}\int_{-\alpha}^{+\alpha} \theta(x)\,dx.$$

(3) $$y = \tfrac{1}{2}\left[k(x-ct)^2 + k(x+ct)^2\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \lambda x\,dx$$

$$= K(x^2 + c^2t^2) + \frac{\lambda}{4c}\left\{(x+ct)^2 - (x-ct)^2\right\}$$

$$= Kx^2 + \frac{Kt^2P}{\rho} + \lambda nt \quad , \text{ since } P = c^2\rho$$

(4)



At any time t, $y = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right]$

$$= \frac{\epsilon}{4}\left[(x-at)(l-x+at) + (x+at)(l-x-at)\right]$$

$$= \frac{1}{2}\epsilon\left[x(l-x) - 2a^2t^2\right]$$

up to time t, the disturbance has traversed a distance $\underline{a.t}$ from either end since the motion is symmetrical with respect to both the ends.

For the portion AB at time t, we have

$$\frac{dy}{dx} = \tan\psi = \frac{\epsilon}{2}(l-2x)$$

the value of which at $x = at$ is $\frac{\epsilon}{2}(l-2at)$

Since the disturbance now no longer exists in the portion OA and O'B, they must now be supposed to be straight (since the action of gravity is to be taken into account)

For the point A we have

$$y = \frac{\epsilon}{2}.at(l-2at) = at\tan\psi$$

which shows that OA itself is a tangent to the curved portion AB. Similarly it is obvious that O'B is a tangent to BA.

(5) Let displacement at time t be given by

$$y = \left(A\cos\frac{nx}{a} + B\sin\frac{nx}{a}\right)\cos(nt+\epsilon)$$

Since the string starts from rest at $t=0$, $\dot{y} = 0$ when $t=0$ for all x.

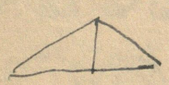
$$\therefore \quad \dot{y} = -n\sin(nt+\epsilon) = 0$$

Hence all the ϵ's are zero, and

$$y = \left(A\cos\frac{nx}{a} + B\sin\frac{nx}{a}\right)\cos nt$$

The conditions at the fixed ends are

$$\left.\begin{aligned} y &= 0, \; x = -l \\ y &= 0, \; x = +l \end{aligned}\right\} \text{ for all } t$$



$$\therefore \quad A\cos\frac{nl}{a} - B\sin\frac{nl}{a} = 0$$

$$A\cos\frac{nl}{a} + B\sin\frac{nl}{a} = 0$$

Also $y = b$, when $t = 0$ for $x = 0$

$$\therefore \; b = A$$

and since both A & $B \neq 0$, and $A = b$, we have $B = 0$ and $\cos\frac{nl}{a} = 0$

$$\text{ie} \quad n = \frac{(2r+1)\pi a}{2l}.$$

Hence the general solution is

$$y = \sum_{r=0}^{\infty} A_r \cos\frac{nx}{a}\cos nt$$

The initial conditions are

$$\left.\begin{aligned} y &= \frac{b(l+x)}{l} \quad \text{for } -l < x < 0 \\ y &= \frac{b(l-x)}{l} \quad \text{for } 0 < x < l \end{aligned}\right\} \text{ at } t = 0$$

$$\therefore \int_{-l}^{0} \frac{b(l+x)}{l}\cos\frac{nx}{a}\,dx + \int_{0}^{l} \frac{b(l-x)}{l}\cos\frac{nx}{a}\,dx = \int_{-l}^{+l} A_r \cos^2\frac{nx}{a}\,dx$$

which gives $A_r = \dfrac{2a^2b}{l^2n^2}$

$$\therefore y = \frac{8b}{\pi^2}\sum_{r=0}^{\infty}\frac{1}{(2r+1)^2}\cos\frac{(2r+1)\pi x}{2l}\cos\frac{(2r+1)\pi at}{2l}.$$

(6) Let us take the joint as the origin & let the displacements of the two portions be given by

$$y_1 = \left(A_1\cos\frac{nx}{a} + B_1\sin\frac{nx}{a}\right)\cos(nt+\epsilon)$$

$$y_2 = \left(A_2\cos\frac{nx}{a'} + B_2\sin\frac{nx}{a'}\right)\cos(nt+\epsilon)$$

$y_1 = 0$ for $x = -l$ for all values of t,

$$\therefore \; A_1\cos\frac{nl}{a} - B_1\sin\frac{nl}{a} = 0 \qquad (1)$$

$y_2 = 0$ for $x = l'$ for all t,

$$\therefore \; A_2\cos\frac{nl'}{a'} + B_2\sin\frac{nl'}{a'} = 0. \qquad (2)$$

At the joint $y_1 = y_2$ for $x = 0$ for all values of t

$$\therefore A_1 = A_2 \qquad (3)$$

Also here $\left(\frac{\partial y_1}{\partial x}\right)_{x=0} = \left(\frac{\partial y_2}{\partial x}\right)_{x=0}$ for all t

$$\therefore B_1 \frac{n}{a} = B_2 \frac{n}{a'} \quad \text{ie} \quad \frac{B_1}{B_2} = \frac{a}{a'} \qquad (4)$$

From (1) – (4) we therefore have

$$\frac{\tan nl/a}{\tan nl'/a'} + \frac{a'}{a} = 0$$

$\frac{2\pi}{n} = t$, $P = a^2 m = a'^2 m'$

$$\therefore n = \frac{2\pi}{t}, \quad a = \sqrt{\frac{P}{m}}, \quad a' = \sqrt{\frac{P}{m'}}$$

Hence the periods are given by

$$\frac{\tan\left(\frac{2\pi l}{t}\sqrt{\frac{m}{P}}\right)}{\tan\left(\frac{2\pi l'}{t}\sqrt{\frac{m'}{P}}\right)} + \sqrt{\frac{m}{m'}} = 0.$$

(7) (See p. 308).

$$y = \frac{2\beta l^2}{\pi^2 b(l-b)} \sum \frac{1}{s^2} \sin\frac{s\pi b}{l} \sin\frac{s\pi x}{l} \cos\frac{s\pi ct}{l}$$

$b = $ [illegible], $\beta = b$, $l = \frac{b}{5}$

Hence expn for y is obtained in the required form.

(8) Let the displacement be given by

$$y = \left(A\cos\frac{nx}{a} + B\sin\frac{nx}{a}\right)\cos(nt + \epsilon)$$

since $\dot{y} = 0$ at $t = 0$ for all x, all the ϵ's are zero

Also $y = 0$ for $x = 0$ for all t, which gives $A = 0$

$$\therefore \quad y = B\sin\frac{nx}{a}\cos nt$$

At the other end, $\frac{\partial y}{\partial x} = 0$ at $x = l$ for all t

$$\therefore \cos\frac{nl}{a} = 0 \quad \text{ie} \quad n = \frac{(2s+1)a\pi}{2l}$$

The general soln is $y = \sum B_s \sin\frac{nx}{a}\cos nt$.

Initially when $t=0$, $y = \frac{bx}{l}$ for all values of x

$$\therefore \int_0^l \frac{bx}{l} \sin\frac{nx}{a}\,dx = \int_0^l B_s \sin^2\frac{nx}{a}\,dx$$

$$\therefore \frac{b}{l}\left[-\frac{a}{n} l\cos\frac{nl}{a} + \frac{a^2}{n^2}\sin\frac{nl}{a}\right] = B_s \cdot \frac{l}{2}$$

$\cos\frac{nl}{a} = \cos(2s+1)\pi/2 = 0$, $\sin\frac{nl}{a} = \sin(2s+1)\pi/2 = (-1)^s$

$$\therefore y = \frac{8b}{\pi^2}\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s+1)^2}\sin\frac{(2s+1)\pi x}{2l}\cos\frac{(2s+1)\pi at}{2l}.$$

If $t < l/a$ it means the disturbance has not traversed the whole of the string, and hence the shape of the string for the undisturbed portion is $y = \frac{bx}{l}$ for $x = 0$ to $x = l - at$, and $y = \frac{b(l-at)}{l}$ beyond

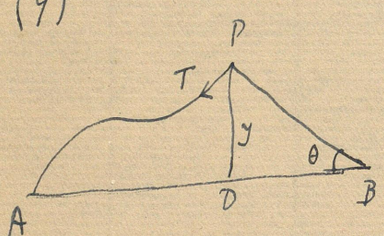
for $y = \frac{1}{2}\left[\phi(-at+x) + \phi(at+x)\right]$

$= \frac{1}{2}\left[\frac{b}{l}(x-at) + \frac{b}{l}(x+at)\right]$.

but $x = l - at$ at such points, since $\underline{at}$ is the distance which the ~~distance traversed~~ disturbance traverses from the other end

$\therefore y = b(l-at)/l$.

(9)



Let the displacement be given by

$$y = \left(A\cos\frac{nx}{a} + B\sin\frac{nx}{a}\right)\cos(nt+\epsilon)$$

At the fixed end

$A\cos\frac{nl}{a} - B\sin\frac{nl}{a} = 0$ if O is taken as the origin.

The components of the tension at P acting on the rod along the axes are

$-T$ and $-T\left(\frac{\partial y}{\partial x}\right)_x = 0$ ie $-T$, $-T\frac{Bn}{c}\cos(nt+\epsilon)$

$y = b\sin\theta$, $\therefore \ddot{y} = b\ddot{\theta}$ to a first approximation

Hence the eqn of the rod is

$$\frac{Mb^2}{3}\ddot{\theta} = \frac{Mb}{3}\ddot{y} = -\frac{TBn}{c}\cos(nt+\epsilon)b - T\cdot A\cos(nt+\epsilon)$$

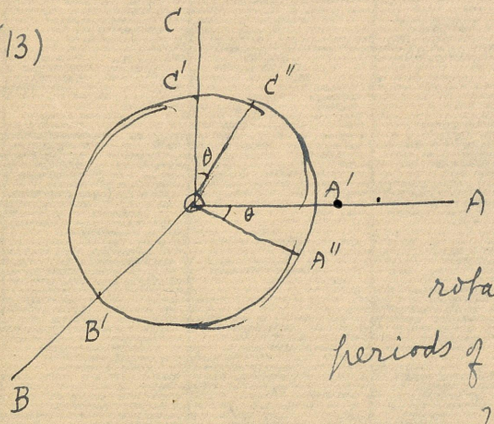
$$\therefore -n^2\frac{Mb}{3}A = -TA - \frac{Tbn}{c}\cdot B$$

$$n^2\frac{Mb}{3} = T + \frac{Tbn}{c}\cot\frac{nl}{c}$$

$$\frac{Mn^2}{3T} = \frac{1}{b} + \frac{n}{c}\cot\frac{nl}{c} \quad , \text{ putting } \frac{n}{c} = \xi$$

$$\frac{M}{3}\cdot\frac{\xi^2}{\rho} - \frac{1}{b} = \xi\cot l\xi \quad , \text{ since } T = c^2\rho$$

(13)



If the sphere be given a rotation round any one of the axes OA, OB, OC only two of the strings will be deformed. Thus giving the sphere three successive rotations about the axes we get the corresponding three periods of oscillation.

For a small rotation θ round OB let C', A' occupy the positions C'', A'' & the displacements of OA, OB referred to A' & C' as origins be

$$y_1 = \left(A_1\frac{\cos nx}{c} + B_1\frac{\sin nx}{c}\right)\cos(nt+\epsilon)$$

$$y_3 = \left(A_3\frac{\cos nx}{c} + B_3\sin\frac{nx}{c}\right)\cos(nt+\epsilon).$$

$$\left.\begin{aligned} y_1 &= 0 \text{ at } x = l_1 \\ y_3 &= 0 \text{ at } x = l_3 \end{aligned}\right\} \text{ for all } t$$

$$\therefore \left.\begin{aligned} A_1\cos\frac{nl_1}{c} + B_1\sin\frac{nl_1}{c} &= 0 \\ A_3\cos\frac{nl_3}{3} + B_3\sin\frac{nl_3}{c} &= 0 \end{aligned}\right\}$$

The components of the tensions of the strings acting on the sphere along the respective axes are

$$\left\{T_1 \,,\; T_1\left(\frac{\partial y_1}{\partial x}\right)_{x=0}\right\} \,,\; \left\{T_3 \,,\; T_3\left(\frac{\partial y_3}{\partial x}\right)_{x=0}\right\}$$

Their moment about the axis OB is

$$-T_1(y_1)_{x=0} - T_3(y_3)_{x=0} + T_1\left(\frac{\partial y_1}{\partial x}\right)_{x=0}a + T_3\left(\frac{\partial y_3}{\partial x}\right)_{x=0}a$$

Hence the eqn of motion of the sphere is

$$\frac{2}{5}Ma^2\ddot{\theta} = \text{this moment}$$

Using $y = a\sin\theta = a\theta$ (approx)

$$(\ddot{y})_{x=0} = a\ddot{\theta} = \text{same for both strings at } x = 0.$$

$\therefore A_1 = A_3$

$$\frac{2}{5} Ma\left(-\frac{n^2}{c^2}\right)\cos(nt+\epsilon)\cdot A_1$$

$$= \left(-T_1 A_1 - T_3 A_3\right)\cos(nt+\epsilon) + \left[T_1 a\cdot\frac{B_1 n}{c} + T_3\cdot a\,\frac{B_3 n}{c}\right]\cos(nt+\epsilon)$$

$$\frac{2}{5} Ma\frac{n^2}{c^2} = \left(T_1 + T_3\right) + T_1 a\frac{n}{c}\cot\frac{nl_1}{c} + T_3 a\frac{n}{c}\cot\frac{nl_3}{c}$$

and using $T_1 = c^2 P_1 = c^2 m_1/l_1$

$$T_3 = c^2 P_3 = c^2 m_3/l_3$$

we get the result in the required form.

(14) The eqn of motion in this case becomes

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} - 2K\frac{\partial y}{\partial t}$$

If we put $y = z e^{-Kt}$,

$$\frac{\partial^2 z}{\partial t^2} - K^2 z = e^2\frac{\partial^2 z}{\partial x^2}$$

Assuming $z \propto e^{im'x}$ where $m' = m\pi/l$

$$\frac{\partial^2 z}{\partial t^2} + p^2 z = 0, \text{ where } p^2 = c^2 m'^2 - K^2$$

and the solution is

$$y = Ae^{-Kt + im'x}\cos(pt+\alpha) = A_1 e^{-Kt}\sin m'x\cos(pt+\alpha)$$

$y = A\sin m'x$ when $t = 0$

$\therefore A_1\cos\alpha = A$ ... (1)

$$\dot{y} = -A_1 p e^{-Kt}\sin m'x\sin(pt+\alpha) - A_1 e^{-Kt} K\sin m'x\cos(pt+\alpha)$$

$\dot{y} = 0$ when $t = 0$ for all x

$\therefore p\sin\alpha + K\cos\alpha = 0$ (2)

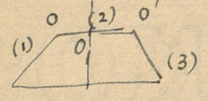
From (1) & (2) $A_1\cos\alpha = A$

$A_1\sin\alpha = -KA/p$

and $y = e^{-Kt}\sin m'x\left[A\cos pt + \frac{KA}{p}\sin pt\right]$ which reduces to the req'd form.

$= Ae^{-Kt}\left(\cos m't + \frac{K}{m'}\sin m't\right)\sin\frac{m\pi x}{l}$, where $m'^2 = \frac{m^2\pi^2 a^2}{l^2} - K^2$

(15) one of the notes which can be sounded is by plucking one of the points of junction and this is discussed in Ex. 6.



The second that can be sounded is that when both points of junction undergo a displacement. Let the displacements of the strings be

$$y_1 = \left(A_1 \frac{\cos nx}{a} + B_1 \frac{\sin nx}{a}\right) \cos(nt + \epsilon), \text{ etc, etc}$$

O being the origin for (1), O_1 for (2) and O' for (3). The several conditions are

$$\left.\begin{aligned} A_1 \cos\frac{nl'}{a} - B_1 \sin\frac{nl'}{a} &= 0 \\ A_3 \cos\frac{nl'}{a} + B_3 \sin\frac{nl'}{a} &= 0 \end{aligned}\right\} \quad (A)$$

$$\left.\begin{aligned} A_1 &= A_2 \cos\frac{nl'}{a'} - B_2 \sin\frac{nl'}{a'} \\ A_3 &= A_2 \cos\frac{nl'}{a'} + B_2 \sin\frac{nl'}{a'} \end{aligned}\right\} \quad (B)$$

$$\frac{B_1}{a} = \frac{1}{a'}\left[A_2 \sin\frac{nl'}{a'} + B_2 \cos\frac{nl'}{a'}\right]; \quad \frac{B_3}{a} = \frac{1}{a'}\left[-A_2 \sin\frac{nl'}{a'} + B_2 \cos\frac{nl'}{a'}\right]$$

$$\frac{A_1}{A_3} = \frac{1 - \frac{B_2}{A_2}\tan\frac{nl'}{a'}}{1 + \frac{B_2}{A_2}\tan\frac{nl'}{a'}}; \quad \frac{B_1}{B_3} = \frac{\frac{A_2}{B_2}\tan\frac{nl'}{a'} + 1}{1 - \frac{A_2}{B_2}\tan\frac{nl'}{a'}}$$

$$a\frac{A_1}{B_1} = a\tan\frac{nl'}{a} = a'\left\{\frac{\frac{A_2}{B_2} - \tan\frac{nl'}{a'}}{1 + \frac{A_2}{B_2}\tan\frac{nl'}{a'}}\right\}$$

$$a\frac{A_3}{B_3} = -a\tan\frac{nl'}{a} = a'\left\{\frac{\frac{B_2}{A_2}\tan\frac{nl'}{a'} + 1}{B_2/A_2 - \tan\frac{nl'}{a'}}\right\}$$

$$\therefore \frac{a\tan\frac{nl'}{a}\tan\frac{nl'}{a'} - a'}{a\tan\frac{nl'}{a} + a'\tan\frac{nl'}{a'}} = \frac{B_2}{A_2}$$

$$\frac{a\tan\frac{nl'}{a} + a'\tan\frac{nl'}{a'}}{a' - a\tan\frac{nl'}{a}\tan\frac{nl'}{a'}} = \frac{A_2}{B_2}$$

giving $\tan\frac{nl'}{a}\tan\frac{nl}{a} = a'/a$.

This gives the period for the second mode of vibration

(16) $$y = \sum B_s \sin\frac{s\pi x}{l}\cos\frac{s\pi ct}{l}$$

The initial values of y are

$$\left.\begin{aligned} y &= \frac{4bx}{l}, && 0 < x < \frac{l}{4} \\ y &= \frac{4b\left(\frac{l}{2}-x\right)}{l}, && l/4 < x < l/2 \\ y &= 0, && \frac{l}{2} < x < l \end{aligned}\right\}$$

$$\therefore \int_0^{l/4} \frac{4bx}{l} \sin\frac{s\pi x}{l}\,dx + \int_{l/4}^{l/2} \frac{4b\left(\frac{l}{2}-x\right)}{l} \sin\frac{s\pi x}{l}\,dx = \int_0^{l} B_s \sin^2\frac{s\pi x}{l}\,dx$$

which gives $\frac{B_s l}{2} = \frac{bl^2}{s^2\pi^2}\left[\sin\frac{s\pi}{2} - 2\sin\frac{s\pi}{4}\right]$.

$$= \frac{K}{s^2}\sin\frac{s\pi}{4}\sin\frac{s\pi}{8}$$

$$\text{Energy} \propto B_s^2 \propto \frac{1}{s^4}\sin^2\frac{s\pi}{4}\sin^2\frac{s\pi}{8}.$$

(17) The eqn to the catenary is

$$y = c\cosh\frac{x}{c}$$

$$\Delta \text{sag} = y - c = x^2/2c, \text{ approximately}$$

Horizontal tension $= T = c_0^2\rho = c\rho$ i.e. $c = c_0^2$

, sag is also $= x^2/2c_0^2 = \frac{l^2}{8c_0^2}$

$$\therefore T = 2\pi\sqrt{l^2/8c_0^2} = \frac{2l}{c_0}\cdot\frac{\pi}{2\sqrt{2}}$$

Swing $\frac{9}{10}T = \frac{2l}{c}$ approximately.

(18) The other part is same as Ex. 6.

(18) Let $p = na$,

$$y = \left(A\cos\frac{nx}{a} + B\sin\frac{nx}{a}\right)\sin pt$$

$$\left.\begin{aligned} y_1 &= K\sin pt, && \text{when } x = -l/2 \\ y_2 &= r\sin pt, && \text{when } x = 0 \end{aligned}\right\}$$

$$\therefore A = r, \quad K = A\cos\frac{bl}{2a} - B\sin\frac{bl}{2a}$$

$$\therefore B = \frac{r\cos\frac{bl}{2a} - K}{\sin\frac{bl}{2a}}.$$

$y_2 = k \sin pt$, when $x = l/2$

$y_2 = r \sin pt$, when $x = 0$

$$\therefore y_2 = \left(r \cos \frac{px}{a} + \frac{k - r \cos \frac{pl}{2a}}{\sin \frac{pl}{2a}} \sin \frac{px}{a} \right) \sin pt$$

$$M\ddot{y} = -P \frac{\partial y_1}{\partial x} + P \frac{\partial y_2}{\partial x}$$

$$\therefore M - p^2 r = -2P \cdot \frac{p}{a} \left(r \cos \frac{pl}{2a} - k \right) \Big/ \sin \frac{pl}{2a}$$

The last eqn gives r, which gives $y = r \sin pt$ for the motion of the mass.

(19) $$y = \left(A \cos \frac{nx}{c} + B \sin \frac{nx}{c} \right) \sin nt$$

$\frac{\partial y}{\partial x} = 0$, at $x = 0$ & $x = l$ for all t

$$\therefore -A \sin \frac{nx}{c} + B \cos \frac{nx}{c} = 0$$

$B = 0$

$A \sin \frac{nl}{c} = 0$ & $A \neq 0$ gives $\frac{nl}{c} = k\pi$

$$y = A \cos \frac{nx}{c} \sin nt$$

$y = r \sin nt$ where $x = l/2$

$$\therefore r = A \cos \frac{nl}{2c} \; ; \; A = \frac{r}{\cos \frac{nl}{2c}}$$

$\frac{\partial y}{\partial x}$ at $x = l/2$ is $-\frac{r}{\cos \frac{nl}{2c}} \cdot \frac{n}{c} \sin \frac{nl}{2c} \sin nt$

$$\therefore F \sin nt = -2P \frac{n}{c} \cdot \frac{r}{\cos \frac{nl}{2c}} \sin \frac{nl}{2c} \sin nt$$

$$\therefore r = -\frac{Fc}{2Pn} \cot \frac{nl}{2c}$$

$$\therefore y = -\frac{Fc}{2Pn} \operatorname{cosec} \frac{nl}{2c} \cos \frac{nx}{c} \sin nt$$

$$= -\frac{Fc}{2Pp} \operatorname{cosec} \frac{pl}{2c} \cos \frac{px}{c} \sin pt \qquad (\text{putting } p = n)$$

(20) Let $$y_1 = \left(A \cos \frac{nx}{a} + B \sin \frac{nx}{a} \right) \cos (nt + \epsilon) \qquad (1)$$

$$y_2 = \left(A_1 \cos \frac{nx}{a'} + B_1 \sin \frac{nx}{a'} \right) \cos (nt + \epsilon) \qquad (2)$$

Take the point of junction as the origin

$y = 0$ for $x = -l$ for (1)

$$\therefore A \cos\frac{nl}{a} - B \sin\frac{nl}{a} = 0$$

Similarly $$A_1 \cos\frac{nl'}{a'} + B \sin\frac{nl'}{a'} = 0$$

Let the motion of M be given by

$$y = r\cos(nt + \epsilon)$$

This gives $A = r = A_1$

$$B = r\cot\frac{nl}{a}\,; \quad B' = -r\cot\frac{nl'}{a'}$$

Hence (1) & (2) reduce to

$$y_1 = r\left[\cos\frac{nx}{a} + \frac{\sin nx}{a}\cot\frac{nl}{a}\right]\cos(nt+\epsilon)$$

$$y_2 = r\left[\cos\frac{nx}{a'} - \frac{\sin nx}{a'}\cot\frac{nl'}{a'}\right]\cos(nt+\epsilon)$$

$$M\ddot{y} = -P\frac{\partial y_1}{\partial x} + P\frac{\partial y_2}{\partial x} \qquad (at\ x = 0)$$

$$\therefore -Mrn^2\cos(nt+\epsilon) = -Pr\cos(nt+\epsilon)\left(\frac{n}{a}\cot\frac{nl}{a}\right) - P'\frac{nr}{a'}\cot\frac{nl'}{a'}$$

$$\therefore Mn = \frac{P}{a}\cot\frac{nl}{a} + \frac{P'}{a'}\cot\frac{nl'}{a'}$$

$$= a\rho\cot\frac{nl}{a} + a'\rho'\cot\frac{nl'}{a'}.$$

(22) Let the motion be given by

$$y_1 = \left(A\cos\frac{px}{a_1} + B\sin\frac{px}{a_1}\right)\sin(pt+\epsilon)$$

$$y_2 = \left(A_1\cos\frac{px}{a_2} + B_1\sin\frac{px}{a_2}\right)\sin(pt+\epsilon)$$

Take the point of junction as the origin & the several conditions give

$$\left.\begin{aligned} A\cos\frac{pb}{a_1} - B\sin\frac{pb}{a_1} &= 0 \\ A_1\cos\frac{p(l-b)}{a_2} + B_1\sin\frac{p(l-b)}{a_2} &= 0 \end{aligned}\right\}$$

Let the motion of the junction be given by, $y = r_1 \sin pt$

This gives $A = A_1 = r_1$, $t = 0$

$$y_1 = r_1 \left[\cos \frac{px}{a_1} + \cot \frac{pb}{a_1} \sin \frac{px}{a_1} \right] \sin pt$$

$$y_2 = r_1 \left[\cos \frac{px}{a_2} - \cot \frac{p(l-b)}{a_2} \sin \frac{px}{a_2} \right] \sin pt$$

$$r \sin pt = P \frac{\partial y_1}{\partial x} - P \frac{\partial y_2}{\partial x} \qquad \text{at } x = 0$$

$$\therefore \quad r = P \cdot \frac{p}{a_1} \cot \frac{pb}{a_1} \cdot r_1 + \frac{Pp}{a_2} \cot \frac{p(l-b)}{a_2} r_1$$

$$\therefore \quad r_1 = \frac{r}{\frac{Pp}{a_1} \cot \frac{pb}{a_1} + \frac{Pp}{a_2} \cot \frac{p(l-b)}{a_2}}$$

(23) See Arts 263, 266

(24) The eqn of motion is

$$\delta n \cdot \frac{m}{n^2} \ddot{y} = -P \sin \psi + P \sin \psi + \delta (P \sin \psi)$$

$$\frac{m}{n^2} \ddot{y} = \frac{P}{\rho} \frac{\partial^2 y}{\partial x^2}$$

$$\ddot{y} = a^2 n^2 \frac{\partial^2 y}{\partial x^2}$$

Let $y = f(x) \cos (nt + \epsilon)$ be its solution

$$-n^2 f(x) \cos (nt + \epsilon) = a^2 x^2 \frac{d^2}{dx^2} f(x) \cos (nt + \epsilon)$$

ie $\quad a^2 x^2 \frac{d^2 Y}{dx^2} + n^2 Y = 0 \quad , \qquad Y = f(x)$

The general solution of this is

$$y = x^{1/2} \left[C_1 \cos \left(\log x \sqrt{4n^2 - a^2}/2a \right) + C_2 \sin \left(\log x \sqrt{4n^2 - a^2}/2a \right) \right]$$

$$C_1 \cos \left(\log l_1 \sqrt{4n^2 - a^2}/2a \right) + C_2 \sin (\quad) = 0$$

$$C_1 \cos \left(\log l_2 \sqrt{4n^2 - a^2}/2a \right) + C_2 \sin (\quad) = 0$$

ie $\quad \tan \left[\log l_1 \sqrt{4n^2 - a^2}/2a \right] = \tan \left[\log l_2 \sqrt{4n^2 - a^2}/2a \right]$

$$\frac{\sqrt{4n^2 - a^2}}{2a} \log l_2 = K\pi + \frac{\sqrt{4n^2 - a^2}}{2a} \log l_1 \qquad [K \text{ an integer}$$

This gives $\quad \frac{4n^2}{a^2} = 1 + \left\{ 2K\pi / \log (l_2/l_1) \right\}^2$.

(25) The eqⁿ of motion is

$$\ddot{y} = a^2(1+kx)^2 \frac{\partial^2 y}{\partial x^2}$$

Putting $1+kx = x$, we have

$$\frac{\partial y}{\partial x} = k \frac{\partial y}{\partial x}$$

$$\therefore \frac{\partial^2 y}{\partial x^2} = k^2 \frac{\partial^2 y}{\partial x^2}$$

$$\therefore \ddot{y} = a^2 k^2 \frac{\partial^2 y}{\partial x^2}$$

whose solution is

$$y = x^{\frac{1}{2}}\left[C_1 \cos\left\{ \frac{\sqrt{4n^2 - a^2k^2}}{2ak} \log(1+kx) \right\} + C_2 \sin\{\quad\} \right]$$

$y = 0$ when $x = 0$ & $x = l$ for all t

This gives $C_1 = 0$, and

$$\sin\{\quad\} = 0$$

i.e. $\dfrac{\sqrt{4n^2 - a^2k^2}}{2ak} \log(1+kl) = i\pi$ where i is an integer

$$\sqrt{4n^2 - a^2k^2} \log(1+kl) = 2aik\pi$$

$i = 0$ gives $4n^2 = a^2k^2$ or $n = \dfrac{ak}{2}$

(28) Since the waves are harmonic, let us suppose that the incident wave is

$$\xi = A\cos(mx - nt) \qquad (1)$$

Since the density is unchanged

$$\frac{\partial^2 \xi}{\partial t^2} = a^2 \frac{\partial^2 \xi}{\partial x^2}$$

holds throughout the length of the string. For facility in working, we shall take (1) as the real part $Ae^{i(mx-nt)}$ and assume similarly expressions $A'e^{i(m'x-n't)}$ and $A_1 e^{i(m_1 x - n_1 t)}$ for the reflected & transmitted waves.

For continuity we should have $n_1 = n' = n$, and as the velocity of propagation has the same numerical value for all three waves, $m' = -m$, $m_1 = m$

Thus λ is unaltered by reflection & transmission. Hence taking the origin at the particle

$$\xi = Ae^{i(mx-nt)} + A'e^{-i(mx+nt)}$$

for one portion of the string, and

$$\xi_1 = A_1 e^{i(mx-nt)}$$ for the other portion

For the motion of the particle we have

$$M\ddot{\xi} = T\frac{\partial \xi_1}{\partial x} - T\frac{\partial \xi}{\partial x}$$

$$\therefore \; -M(A+A')n^2 = Tmi(A_1 - A + A')$$

Also $n^2 = a^2m^2$

$$\therefore \; \frac{Ma^2}{T}(A+A')n = A_1 - A + A'$$

Again $\xi = \xi_1$ at $x = 0$

$$\therefore A + A' = A_1$$

$$\therefore A_1 = \frac{2A}{2 - \frac{imMa^2}{T}}$$

$$\therefore \xi_1 = \frac{2A\left(2 + \frac{imMa^2}{T}\right)}{4 + \frac{m^2M^2a^4}{T^2}} e^{i(mx-nt)}$$

$$= \frac{2A}{\sqrt{4+k^2m^2}} e^{i(mx-nt+\epsilon)} \quad \text{where } k = \frac{Ma^2}{T}, \; \tan\epsilon = \tfrac{1}{2}km$$

$$A_1 = A + A' = \frac{2A}{2 - ikm}$$

$$\therefore A' = A_1 - A = \frac{ikmA}{2-ikm} = \frac{(2+ikm)ikm}{4+k^2m^2}A$$

$\therefore$ Reflected wave is given by

$$\xi' = A\frac{km(2i-km)}{4+k^2m^2}e^{-i(mx+nt)}$$

$$= -\frac{km}{\sqrt{4+k^2m^2}} A e^{-i(mx+nt+\epsilon_1)} \quad \text{where } \tan\epsilon_1 = 2/km$$

Hence the phases of the reflected & transmitted waves are different. The ~~ratio~~ ratio of the amplitudes is

$$\therefore \frac{km}{2} = \frac{Mma^2}{2T} = \frac{Mm \cdot a \cdot \frac{n}{m}}{2T} = \frac{Man}{2T}.$$

(30) Reflection takes place twice. Let M be taken as the origin

$$\xi_1 = A_1 e^{i(mx-nt)} + B_1 e^{-i(mx+nt)} \quad \text{from } x=-\infty \text{ to } x=0$$

$$\xi_2 = A_2 e^{i(mx-nt)} + B_2 e^{-i(mx+nt)} \quad \text{from } x=0 \text{ to } x=a$$

$\xi_1 = \xi_2$ at $x=0$

$$\therefore A_1 + B_1 = A_2 + B_2 \qquad (1)$$

$\xi_2 = 0$ at $x=a$

$$\therefore A_2 e^{ima} + B_2 e^{-ima} = 0 \qquad (2)$$

For the motion of the particle of mass M

$$M\ddot{\xi}_1 = c^2\rho\left[\frac{\partial \xi_2}{\partial x} - \frac{\partial \xi_1}{\partial x}\right]_{x=0}$$

$$-n^2 M\left[A_1 + B_1\right] = c^2\rho\left[(A_1 - B_1) + (A_2 - B_2)\right] mi$$

$$n^2 = c^2 m^2$$

$$Mmi\,(A_1+B_1) - \rho(A_1-B_1) = \rho(A_2-B_2)$$

$\therefore$ ~~Using~~ using (1) & (2)

$$B_1 = A_1 \frac{\left[\rho + i(\rho \cot ma - Mm)\right]}{\left[\rho - i(\rho \cot ma - Mm)\right]} = A_1 e^{2i\phi}$$

if $R\cos\phi = \rho$, $R\sin\phi = \rho\cot ma - Mm$

so that $\tan\phi = \cot ma - \frac{Mm}{\rho}$

Hence the change of phase $= 2\phi = 2\tan^{-1}\left[\cot\frac{pa}{c} - \frac{Mc\rho}{T}\right]$ ~~multiple~~.

(31) $$\xi_1 = A_1 e^{i(m_1 x - nt)} + B_1 e^{-i(m_1 x - nt)}$$

$$\xi_2 = A_2 e^{i(m_2 x - nt)}$$

$$m_1(A_1 - B_1) = m_2 A_2$$

Since $T\frac{\partial \xi_1}{\partial x} = \frac{T}{2}\frac{\partial \xi_2}{\partial x} + \frac{T}{2}\frac{\partial \xi_2}{\partial x}$, ξ_2 being the same for both the parallel strings

$$\xi_1 = \xi_2 \quad \text{at } x=0$$

$$\therefore A_1 + B_1 = A_2$$

$$\therefore \frac{m_1}{m_2}(A_1 - B_1) = A_1 + B_1$$

$$B_1 = A_1\left(\frac{m_1}{m_2} - 1\right)\Big/\left(\frac{m_1}{m_2} + 1\right)$$

$$\eta^2 = c_1^2 m_1^2 = c_2^2 m_2^2$$

$$\therefore B_1 = A_1 \frac{(c_2/c_1 - 1)}{(c_2/c_1 + 1)}$$

$$T = c_1^2 \rho \; ; \; T/2 = c_2^2 \rho$$

$$\therefore B_1 = A_1 \cdot \frac{\frac{1}{\sqrt{2}} - 1}{\frac{1}{\sqrt{2}} + 1} = -A_1 (\sqrt{2}-1)^2$$

$$A_1 = K$$

Hence the amplitude of the reflected waves is $K(\sqrt{2}-1)^2$

$$B_1 = -A_1 (\sqrt{2}-1)^2$$

$$A_2 = A_1 - A_1 (\sqrt{2}-1)^2 = 2A_1 (\sqrt{2}-1) = 2k(\sqrt{2}-1)$$

giving amplitude of the transmitted wave.

(32) Let $\xi_1 = A_1 e^{i(m_1 x - nt)} + B_1 e^{-i(m_1 x + nt)}$ for $x = 0$ to $x = +l$

$\xi = A e^{i(mx - nt)} + B e^{-i(mx + nt)}$ for $x = -\infty$ to $x = 0$

the point of junction being taken as the origin

$\xi_1 = 0$ at $x = l$ for all t

$$\therefore A_1 e^{im_1 l} + B_1 e^{-im_1 l} = 0$$

$\frac{\partial \xi_1}{\partial t} = 0$ at $x = -l$ gives the same condition

$\xi = \xi_1$ at $x = 0$

$$\therefore B + A = A_1 + B_1$$

$T = T_1$ at $x = 0$

$$\therefore a_1^2 \rho_1 (A_1 m_1 - B_1 m_1) = a^2 \rho (Am - Bm)$$

$$\therefore A - B = \frac{K_1}{K}(A_1 - B_1)$$

$$\therefore A = \frac{1}{2}\left[A_1 + B_1 + \frac{K_1}{K}(A_1 - B_1)\right]$$

$$B = \frac{1}{2}\left[A_1 + B_1 - \frac{K_1}{K}(A_1 - B_1)\right]$$

$$\therefore B = -A \frac{\frac{K_1}{K}\cos ml - i \sin ml}{\frac{K_1}{K}\cos ml + i \sin ml} = -A \times \text{exponential part since the}$$

moduli of both numerator & denominator are equal.

Hence the amplitudes are equal though the phase undergoes a change & hence the first result.

$$B_1 = -A \frac{\frac{ma}{m_1 a_1}\cos ml - i \sin ml}{\frac{ma}{m_1 a_1}\cos ml + i \sin ml}$$

$$\therefore \tan\alpha = \frac{ma}{m'a'} \tan\frac{pl}{a'} = \frac{ma}{m'a'} \tan\frac{ml}{\alpha} \qquad (?)$$

$$\therefore B = -A e^{-2i\alpha}, \quad A+B = A(1-e^{-2i\alpha})$$
$$A-B = A(1+e^{-2i\alpha})$$

$$A_1 + B_1 = A(1-e^{-2i\alpha})$$
$$A_1 - B_1 = \frac{k}{k_1} A(1+e^{-2i\alpha})$$

$$2A_1 = 2A\left[\frac{R_1^2}{R_1^2}\cos^2\alpha + \sin^2\alpha\right]^{1/2} \times \text{Exp. part}$$
$$= A \cdot \frac{2ap}{a'p'} \cos\alpha \left(1 + \frac{a'^2 p'^2}{a^2 p^2} \tan^2\alpha\right)^{\frac{1}{2}}$$

$$\therefore \frac{\sin\alpha}{\sin pl/a'} = \frac{\sin\alpha}{\frac{a'p'}{ap}\tan\alpha} \quad \left(\because\right)^{1/2} = \frac{ap}{a'p'}\cos\alpha\left(1+\frac{a'^2p'^2}{ap}\right)^{1/2}$$

---

(33) Let the incident wave be the real part of $A_0 e^{i(m_0 x - nt)}$ & the reflected wave $B_0 e^{-i(m_0 x + nt)}$ & the transmitted wave $Ae^{i(mx-nt)}$ in the string 'l' so that the reflected wave is $Be^{-i(mx+nt)}$. Let the transmitted wave be $A_1 e^{i(m_1 x - nt)}$

Thus $\xi_0 = A_0 e^{i(m_0 x - nt)} + B_0 e^{-i(m_0 x + nt)}$ for $-\infty < x < 0$

$\xi = Ae^{i(mx-nt)} + Be^{-i(mx+nt)}$ for $0 < x < l$

$\xi_1 = A_1 e^{i(m_1 x - nt)}$ for $l < x < \infty$

$\xi = \xi_0$ at $x = 0$

$$\therefore A + B = A_0 + B_0 \qquad (1)$$

$\xi = \xi_1$ at $x = l$

$$\therefore Ae^{iml} + Be^{-iml} = A_1 e^{im_1 l} \qquad (2)$$

At the joint, the tensions for the two portions are equal

$$T = E\frac{\partial \xi}{\partial x} = c^2 \rho \frac{\partial \xi}{\partial x}$$

At $x = 0$, we have

$$m_0 c_0^2 \rho_0 (A_0 - B_0) = mc^2 \rho (A - B)$$

At $x = l$,

$$mc^2\rho (Ae^{iml} - Be^{-iml}) = c_1^2 \rho_1 m_1 A_1 e^{im_1 l_1}$$

Let the typical quantity $mc^2\rho$ be denoted by K for brevity. To find A_1 in terms of A_0, we have

$$A + B = A_0 + B_0$$

$$A - B = (A_0 - B_0) K_0/K$$

$$A = \tfrac{1}{2}\left[(A_0 + B_0) + (A_0 - B_0) K_0/K\right], \text{ etc}$$

$$\therefore A_1 e^{iml} = \tfrac{1}{2} e^{iml}\left[(A_0 + B_0) + (A_0 - B_0) K_0/K\right] + \tfrac{1}{2} e^{-iml}\left[(A_0 + B_0) - (A_0 - B_0) K_0/K\right]$$

$$= A_0\left\{\cos ml + \frac{iK_0}{K} \sin ml\right\} + B_0\left\{\cos ml - \frac{iK_0}{K} \sin ml\right\}$$

Eliminating B_0, we have finally

$$A_1 = \frac{2A_0 \frac{K_0}{K} e^{-im_1 l}}{\cos ml \left(\frac{K_1}{K} + \frac{K_0}{K}\right) - i \sin ml \left(1 + \frac{K_1 K_0}{K^2}\right)}$$

Let $\cos ml \left(\frac{K_1}{K} + \frac{K_0}{K}\right) = R \cos\phi$

$$\sin ml \left(1 + \frac{K_1 K_0}{K^2}\right) = R \sin\phi$$

Hence the amplitude of the transmitted wave

$$= \frac{2A_0}{\sqrt{\left[\left(1 + \frac{K_1}{K_0}\right)^2 \cos^2 ml + \left(\frac{K}{K_0} + \frac{K_1}{K}\right)^2 \sin^2 ml\right]}}$$

$$= \frac{2A_0}{\sqrt{\left(1 + \frac{c_1\rho_1}{c_0\rho_0}\right)^2 \cos^2 \frac{ml}{c} + \left(\frac{\rho c}{\rho_0 c_0} + \frac{\rho_1 c_1}{\rho c}\right)^2 \sin^2 \frac{ml}{c}}}$$

Hence the ratio A_0/A_1 is easily derived.

34. (See Ex. 28)

Incidental wave : $\xi = A\cos(mx - nt)$

Reflected " : $\xi = \frac{-km}{\sqrt{4 + k^2 m^2}} \cos(mx + nt + \epsilon_1)$

Transmitted " : $\xi = \frac{2A}{\sqrt{4 + k^2 m^2}} \cos(mx + nt + \epsilon_1)$

K.E of ξ (incidental) for all wave lengths $\odot$ to λ

$$T = \tfrac{1}{2}\int_0^\lambda \dot{\xi}^2 dx = \tfrac{1}{4} A^2 \rho n^2 \lambda$$

$$V = \text{potential energy} = \tfrac{1}{2} T \Big/ \int_0^\lambda \left(\frac{\partial y}{\partial x}\right)^2 dx$$

$$= \tfrac{1}{4} T A^2 n^2 \lambda$$

$$\therefore \text{Energy} = \tfrac{1}{2} A^2 \lambda \rho^2 c^2 m^2$$

Hence the energies are in the ratios of

$$1 : \frac{m^2/\rho^2 \left(\frac{2\pi}{\lambda}\right)^2}{4 + \frac{m^2}{\rho^2}\left(\frac{2\pi}{\lambda}\right)^2} : \frac{4}{4 + \frac{m^2}{\rho^2}\left(\frac{2\pi}{\lambda}\right)^2}$$

ie $m^2n^2 + 4\rho^2 a^2 : m^2 n^2 : 4\rho^2 a^2$, since $n^2 = a^2 m^2$

Change of phase $= \tan^{-1}\left(\frac{\pi}{\lambda} \cdot \frac{m}{\rho}\right)$ (§ 269)

$$= \tan^{-1}\left(\frac{mn}{2\rho a}\right).$$

(36) Equation of motion is

$$\rho_1 \delta x \ddot{y} = \delta (P \sin \psi)$$

$$\rho_1 \ddot{y} = \frac{d}{dx}\left[(T_1 + E)\frac{\partial \xi}{\partial x} + T_1 \right] \frac{\partial y}{\partial x}$$

Since T_1 is the initial tension

$$T = E' \frac{\partial \xi}{\partial x} + T_1 \text{, where } E' = E l_1/l \text{, } T_1 = E\,\frac{l_1 - l}{l}$$

$$E' = T_1 + E.$$

$$\left.\begin{aligned} X &= x + \xi \text{ ; } \frac{\partial y}{\partial X} = \frac{\partial y}{\partial x} \cdot \frac{\partial x}{\partial X} \\ \frac{\partial X}{\partial x} &= 1 + \frac{\partial \xi}{\partial x} \end{aligned}\right\} \quad \frac{\partial y}{\partial x} = \frac{\partial y/\partial x}{1 + \partial \xi/\partial x}$$

$$\rho_1 \ddot{y} = \frac{d}{dx}\left[\left\{ T_1 \left(1 + \frac{\partial \xi}{\partial x}\right) + E \right\} \frac{\partial y/\partial x}{1 + \partial \xi/\partial x} \right]$$

$$= T_1 \frac{\partial^2 y}{\partial x^2} + E \frac{\partial}{\partial x}\left[\frac{\frac{\partial \xi}{\partial x}\frac{\partial y}{\partial x}}{1 + \partial \xi/\partial x} \right]$$

This gives the second result.

For longitudinal vibrations, $\rho_1 \frac{\partial^2 \xi}{\partial t^2} = E' \frac{\partial^2 \xi}{\partial x^2}$

$$\partial^2 \xi/\partial t^2 = \frac{T_1 + E}{\rho_1} \frac{\partial^2 \xi}{\partial x^2}$$

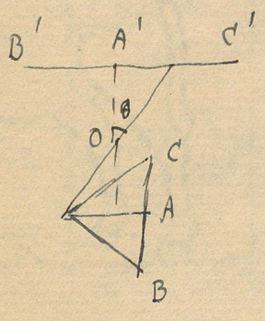
which gives the first result.

(37) Let the angular motion of the rod be given by

$$\theta = \frac{r}{a}\cos(nt+\epsilon)$$

so that the linear displacement of the midpt of the string is

$$r\cos(nt+\epsilon)$$



Taking the respective midpoints as origins, the arrangement is such that the extensible strings only perform longitudinal vibrations while the inextensible ones only the transverse ones. BAC is the inextensible string & B'A'C' the extensible. Further it is assumed that the initial tension T of B'A'C' is zero. The eqns of motion for them are

$$\frac{\partial^2\xi}{\partial t^2} = \beta^2\frac{\partial^2\xi}{\partial x^2}$$

$$\frac{\partial^2\xi}{\partial t^2} = \alpha^2\frac{\partial^2\xi}{\partial x^2}$$

Let $\xi_1 = \left(A\cos\frac{nx}{\beta} + B\sin\frac{nx}{\beta}\right)\cos(nt+\epsilon)$ $\quad -l'<x<0$

$\xi_1 = 0$ at $x = -l'$ for all t

$$\therefore A\cos\frac{nl'}{\beta} - B\sin\frac{nl'}{\beta} = 0$$

$\xi_1 = r\cos(nt+\epsilon)$ at $x=0$

$$\therefore A = r$$

$$\therefore \xi_1 = r\left[\cos\frac{nx}{\beta} + \cot\frac{nl'}{\beta}\sin\frac{nx}{\beta}\right]\cos(nt+\epsilon)$$

Similarly $\xi_2 = r\left[\cos\frac{nx}{\beta} - \cot\frac{nl'}{\beta}\sin\frac{nx}{\beta}\right]\cos(nt+\epsilon)$ $\quad$ for $0<x<l'$

For transverse vibrations we have similar expressions with l & α in place of l' & β (say η_1, η_2 instead of ξ_1 & ξ_2)

For the motion of the rod

$$M\frac{a^2}{3}\ddot{\theta} = M\frac{a}{3}\ddot{\xi} = M\frac{a}{3}\ddot{\eta} = (T_2 - T_1)a - 2T\left(\frac{\partial\eta_1}{\partial x}\right)_{x=0}$$

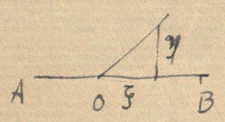
where $2a$ = length of the rod, and T_2, T_1 the tensions in ξ_2 & ξ_1 and T is the tension of BAC. $T_2 = E\frac{\partial\xi_2}{\partial x}$, $T_1 = E\frac{\partial\xi_1}{\partial x}$

$$\therefore M\frac{a}{3} - n^2 r\cos(nt+\epsilon) = -r\cos(nt+\epsilon)\cdot E\left[\frac{2n}{\beta}\cot\frac{nl'}{\beta}\right]$$

$$- 2T\frac{nr}{\alpha}\cos(nt+\epsilon)\cdot\cot\frac{nl}{\alpha}.$$

$$\therefore \frac{1}{b} Mn = \frac{E}{\beta} \cot \frac{nl'}{\beta} + \frac{I}{\alpha} \cot \frac{nl}{\alpha}.$$

(38)



Let the point where the ring is attached be suffered displaced a distance ξ by longitudinal vibrations. The eqn of motion is

$$\frac{\partial^2 \xi}{\partial t^2} = b^2 \frac{\partial^2 \xi}{\partial x^2} \qquad (1)$$

Let the motion of the ring along the wire be given by $\eta = r \cos(nt + \epsilon)$, so that the components of its displacement along axes through O are

$$x_0 = r \cos(nt + \epsilon) \cos\alpha$$
$$y_0 = r \cos(nt + \epsilon) \sin\alpha$$

The soln of (1) for OA is

$$\xi_1 = \left(A \cos \frac{nx}{b} + B \sin \frac{nx}{b} \right) \cos(nt + \epsilon)$$

$\xi_1 = 0$ at $x = -l_1$ for all t

$$\therefore A \cos \frac{nl_1}{b} - B \sin \frac{nl_1}{b} = 0$$

$\xi_1 = r \cos(nt + \epsilon) \cos\alpha$ at $x = 0$

$$\therefore A = r \cos\alpha$$

$$\therefore \xi_1 = r \cos\alpha \left[\cos \frac{nx}{b} + \cot \frac{nl_1}{b} \sin \frac{nx}{b} \right] \cos(nt + \epsilon)$$

Similarly for the portion OB

$$\xi_2 = r \cos\alpha \left[\cos \frac{nx}{b} - \cot \frac{nl_2}{b} \sin \frac{nx}{b} \right] \cos(nt + \epsilon)$$

In this motion let the tension of the string be $T = a^2\rho = b^2\rho \frac{\partial \xi}{\partial x} + T_0$

The eqn for transverse vibrations is

$$\eta_1 = \left(A \cos \frac{nX}{a} + B \sin \frac{nX}{a} \right) \cos(nt + \epsilon)$$

where $X = x + \xi$

$\eta_1 = 0$ when $X = -l_1$, $\eta_2 = r \cos(nt + \epsilon)$ at $X = 0$

$$\therefore \eta_1 = r \sin\alpha \left[\cos \frac{nX}{a} - \cot \frac{nl_1}{a} \sin \frac{nX}{a} \right] \cos(nt + \epsilon)$$

$$\eta_2 = r \sin\alpha \left[\cos \frac{nX}{a} + \cot \frac{nl_2}{a} \sin \frac{nX}{a} \right] \cos(nt + \epsilon)$$

For the motion of the ring

$$M\ddot{\eta} = -T_1 \cos(\alpha - \phi_1) + T_2 \cos(\phi_2 - \alpha)$$

where $\tan\phi_1 = \left(\frac{\partial \eta_1}{\partial x}\right)_{x=0}$; $\tan\phi_2 = \left(\frac{\partial \eta_2}{\partial x}\right)_{x=0}$

$$\therefore M\ddot{\eta} = -T_1\left[\cos\alpha + \sin\alpha\left(\frac{\partial \eta_1}{\partial x}\right)_{x=0}\right] + T_2\left[\cos\alpha + \sin\alpha\left(\frac{\partial \eta_2}{\partial x}\right)_{x=0}\right]$$

$$T_2 - T_1 = b^2\rho\left(\frac{\partial \xi_2}{\partial x} - \frac{\partial \xi_1}{\partial x}\right)$$

$$= -b^2\rho\,\frac{nr}{b}\cos\alpha\left(\cos\frac{nl_2}{b} + \cos\frac{nl_1}{b}\right)\cos(nt+\epsilon)$$

For transverse vibrations we can take

$$T_1 = T_2 = a^2\rho$$

$$\therefore M\ddot{\eta} = -b\rho\, nr\cos^2\alpha\left(\cos\frac{nl_2}{b} + \cos\frac{nl_1}{b}\right)\cos(nt+\epsilon)$$

$$- a^2\rho\,\frac{nr}{a}\sin^2\alpha\left(\cos\frac{nl_1}{a} + \cos\frac{nl_2}{a}\right)\cos(nt+\epsilon)$$

$\ddot{\eta} = -n^2 \cdot \cos(nt+\epsilon)\, r$ = same expn

$$\therefore Mn = b\rho\cos^2\alpha\left(\cos\frac{nl_2}{b} + \cos\frac{nl_1}{b}\right) + a\rho\sin^2\alpha\left(\cos\frac{nl_1}{a} + \cos\frac{nl_2}{a}\right).$$

— 0 —

Chapter XII – p. 348

(1) Assuming that there are no external, and including $F(t)$ in $\partial\phi/\partial t$

$$\int\frac{dp}{\rho} = \frac{\partial\phi}{\partial t} - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2$$

$$p = k\rho$$

$$\therefore k\log\rho = \frac{\partial\phi}{\partial t} - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2$$

Let $\rho = \rho_0(1+s)$

The eqn of continuity is

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u}{\partial x} = 0$$

$$\frac{\partial\rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial\rho}{\partial x} = 0$$

$$\rho_0\frac{\partial s}{\partial t} - \rho_0(1+s)\frac{\partial^2\phi}{\partial x^2} - \frac{\partial\phi}{\partial x}\rho_0\frac{\partial s}{\partial x} = 0$$

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\right)^2 + k \log \rho$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial t} + \frac{k}{\rho}\frac{\partial \rho}{\partial t}$$

$$= \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial t} + \frac{k\rho_0}{\rho_0(1+s)}\left\{(1+s)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial x}\frac{\partial s}{\partial x}\right\}$$

$$= \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial t} + k\frac{\partial^2 \phi}{\partial x^2} + \frac{k}{1+s}\frac{\partial \phi}{\partial x}\frac{\partial s}{\partial x}$$

Also $$\frac{\partial^2 \phi}{\partial x \partial t} = \frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} + \frac{k}{1+s}\frac{\partial s}{\partial x}$$

$$\therefore \frac{\partial^2 \phi}{\partial t^2} = 2\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial t} + k\frac{\partial^2 \phi}{\partial x^2} - \left(\frac{\partial \phi}{\partial x}\right)^2 \frac{\partial^2 \phi}{\partial x^2}$$

$$\frac{\partial^2 \phi}{\partial t^2} - 2\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial t} = k\frac{\partial^2 \phi}{\partial x^2} - \left(\frac{\partial \phi}{\partial x}\right)^2 \frac{\partial^2 \phi}{\partial x^2}$$

$$\text{i.e. } \frac{\partial}{\partial t}\left[\frac{\partial \phi}{\partial t} - \left(\frac{\partial \phi}{\partial x}\right)^2\right] = \frac{\partial}{\partial x}\left[k\frac{\partial \phi}{\partial x} - \frac{1}{3}\left(\frac{\partial \phi}{\partial x}\right)^3\right]$$

(3) Let $\phi = \left(A \cos\frac{nx}{a} + B \sin\frac{nx}{a}\right)\cos nt$

$\frac{\partial \phi}{\partial x} = 0$ at $x = 0$, $\therefore B = 0$

$$\text{i.e. } \phi = A \cos\frac{nx}{a}\cos nt$$

$$\delta p = \rho_0 \frac{\partial \phi}{\partial t} = -\rho_0 A \cos\frac{nx}{a} \cdot n \sin nt$$

and at $x = l$

$$\delta p = -\rho_0 A \cos\frac{nl}{a} \cdot n \sin nt = p_0 \sin nt$$

$$\therefore A = -\frac{p_0}{n\rho_0} \cdot \frac{1}{\cos\frac{nl}{a}}$$

$$\therefore \phi = -\frac{p_0}{n\rho_0}\frac{\cos nx/a}{\cos nl/a}\cos nt.$$

(5) The eqⁿ of statical equilibrium is

$$\frac{\partial p_0}{\partial x} = -g\rho \qquad (1)$$

The eqⁿ of motion is

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\frac{\partial p}{\partial x} - g\rho = -\frac{\partial}{\partial x}(p - p_0)$$

The eqⁿ of continuity is

$$\rho\left(1+\frac{\partial\xi}{\partial x}\right)=\rho_0$$

Also $p=k\rho^{\gamma}$; $p_0=k\rho_0^{\gamma}$, $p=p_0\left(1+\frac{\partial\xi}{\partial x}\right)^{-\gamma}$

$\therefore$ The eqn of motion is

$$\rho_0\ddot{\xi}=-\frac{\partial}{\partial x}\left[p_0\left(1+\frac{\partial\xi}{\partial x}\right)^{-\gamma}-p_0\right]$$

$$\cdots=-\frac{\partial}{\partial x}\left[-p_0\gamma\frac{\partial\xi}{\partial x}\right] \text{ to a first approximation}$$

$$=\gamma\left[\frac{\partial p_0}{\partial x}\frac{\partial\xi}{\partial x}+p_0\frac{\partial^2\xi}{\partial x^2}\right]$$

$$=\left[-g\rho_0\gamma\frac{\partial\xi}{\partial x}+\gamma p_0\frac{\partial^2\xi}{\partial x^2}\right]$$

$$\therefore \ddot{\xi}=-g\gamma\frac{\partial\xi}{\partial x}+\gamma\frac{p_0}{\rho_0}\frac{\partial^2\xi}{\partial x^2}$$

p_0, ρ_0 are the eqm pressure & density at a height x. We shall now express them in terms of the pressure p_0' and density ρ_0' at $x=0$

$$\frac{\partial p_0}{\partial x}=-g\rho_0$$

$$\left(\frac{p_0}{p_0'}\right)=\left(\frac{\rho_0}{\rho_0'}\right)^{\gamma}$$

$$\therefore \frac{\partial}{\partial x}\left(\frac{p_0'}{\rho_0'^{\gamma}}\rho_0^{\gamma}\right)=-g\rho_0$$

i.e. $$\frac{p_0'}{\rho_0'^{\gamma}}\cdot\gamma\cdot\rho_0^{\gamma-2}\,d\rho_0=-g\,dx$$

giving $$\frac{p_0'}{\rho_0'^{\gamma}}\cdot\gamma\cdot\frac{\rho_0^{\gamma-1}}{\gamma-1}=-gx+C$$

$\rho_0=\rho_0'$ at $x=0$

$$\therefore \frac{p_0}{\rho_0}=\frac{p_0'}{\rho_0'}-\frac{\gamma-1}{\gamma}gx$$

but $$\frac{p_0'}{\rho_0'}=\frac{g\lambda(\gamma-1)}{\gamma} \qquad \therefore \frac{p_0}{\rho_0}=\frac{g(\gamma-1)(\lambda-x)}{\gamma}$$

$$\therefore \ddot{\xi}=-g\gamma\frac{\partial\xi}{\partial x}+g(\gamma-1)(\lambda-x)\frac{\partial^2\xi}{\partial x^2}$$

$$=g\left[(\gamma-1)(\lambda-x)\frac{\partial^2\xi}{\partial x^2}-\gamma\frac{\partial\xi}{\partial x}\right]$$

(6) Take the fixed end as the origin

$\therefore$ let $\phi = \left(A\cos\frac{nx}{a} + B\sin\frac{nx}{a}\right)\cos(nt+\epsilon)$

$\frac{\partial\phi}{\partial x} = 0$ at $x = 0$ for all t, $\therefore B = 0$

$\therefore \phi = A\cos\frac{nx}{a}\cos(nt+\epsilon)$

If K is the area of the piston its motion is given by

$$M\ddot{\xi} = -K\delta p = -K\rho_0\frac{\partial\phi}{\partial t}$$

$$= K\rho_0 \cdot nA\cos\frac{nl}{a}\sin(nt+\epsilon)$$

$$\therefore M\dot{\xi} = -\frac{K}{n}A\rho_0\cos\frac{nl}{a}\sin(nt+\epsilon)$$

At $x = l$, $\dot{\xi} = \frac{\partial\phi}{\partial x}$ since the motion takes place inside the tube in the direction opposite to which x is measured

$$-\frac{1}{M}\cdot\frac{KA}{n}\rho_0\cos\frac{nl}{a}\cdot n\cos(nt+\epsilon)\cdot$$

$$= -\frac{n}{a}\left[A\sin\frac{nl}{a}\cos(nt+\epsilon)\right]$$

$$\therefore K\rho_0 = \frac{Mn}{a}\tan\frac{nl}{a}$$

$$\therefore K\rho_0 l = \frac{Mnl}{a}\tan\frac{nl}{a}$$

But $K\rho_0 l = M'$

$$\therefore \frac{nl}{a}\tan\frac{nl}{a} = \frac{M'}{M}.$$

(7) Let the motion of the piston be given by

$$\xi = r\sin(nt+\epsilon)$$

For the portion with a closed end let

$$\phi_1 = \left(A\cos\frac{nx}{a} + B\sin\frac{nx}{a}\right)\cos(nt+\epsilon)$$

Taking the origin at the piston itself

$$\frac{\partial\phi_1}{\partial x} = 0 \text{ at } x = -l$$

$$\therefore A\sin\frac{nl}{a} + B\cos\frac{nl}{a} = 0$$

$$\dot{\xi} = -\frac{\partial\phi_1}{\partial x} \text{ at } x = 0$$

$\therefore rn = -Bn/a \qquad \therefore B = -ar$

$\therefore A = ar \cot \frac{nl}{a}$

$\therefore \phi_1 = ar\left[\cot\frac{nl}{a}\cos\frac{nx}{a} - \sin\frac{nx}{a}\right]\cos(nt+\epsilon)$

For the other portion $\frac{\partial \phi_2}{\partial x} = 0$ at $x = l$, and we find

$\phi_2 = ar\left[\tan\frac{nl}{a}\cos\frac{nx}{a} - \sin\frac{nx}{a}\right]\cos(nt+\epsilon)$

For the motion of the piston

$M\ddot{\xi} = (p_1 - p_2)A$ if p_1 and p_2 be the pressures on the sides

ϕ_1 and ϕ_2 respectively

$M - rn^2 \sin(nt+\epsilon) = A(\delta p_1 - \delta p_2)$

$= A\rho_0\left(\frac{\partial \phi_1}{\partial t} - \frac{\partial \phi_2}{\partial t}\right)$

$= -A\rho_0\left[arn\cot\frac{nl}{a} - arn\tan\frac{nl}{a}\right]\sin(nt+\epsilon)$

$\therefore Mn = A\rho_0 a \cdot \frac{1 - \tan^2\frac{nl}{a}}{2\tan\frac{nl}{a}} \cdot 2 = 2A\rho_0 a \cot\frac{2nl}{a}.$

(8) Take the eqm position of the piston as the origin. Let the motion of the piston be given by

$\xi = A_1 \sin(nt+\epsilon)$

Let $\phi_1 = \left(A\cos\frac{nx}{a} + B\sin\frac{nx}{a}\right)\cos(nt+\epsilon)$

For one portion, $\frac{\partial \phi_1}{\partial t} = 0$ when $x = -l$

$\therefore A\cos\frac{nl}{a} - B\sin\frac{nl}{a} = 0 \qquad (1)$

Also $\dot{\xi} = -\frac{\partial \phi_1}{\partial x}$ at $x = 0$

$\therefore A_1 n\cos(nt+\epsilon) = -\frac{Bn}{a}\cos(nt+\epsilon)$

$\therefore A_1 a = -B$

$\therefore A = -A_1 a \tan\frac{nl}{a}.$

ie $\phi_1 = -A_1 a\left[+\tan\frac{nl}{a}\cos\frac{nx}{a} + \sin\frac{nx}{a}\right]\cos(nt+\epsilon)$

Also for the other portion

$\phi_2 = -A_1 a\left[-\tan\frac{nl}{a}\cos\frac{nx}{a} + \sin\frac{nx}{a}\right]\cos(nt+\epsilon)$

For the motion of the piston

$$M\ddot{\xi} + Mm^2\xi = \delta p_1 - \delta p_2$$

if the piston has unit cross section

$$\frac{M\ddot{\xi} + Mm^2\xi}{\sin(nt+\epsilon)} = \rho\left[\frac{\partial\phi_1}{\partial t} - \frac{\partial\phi_2}{\partial t}\right] \Big/ \sin(nt+\epsilon)$$

$$\therefore \quad = \rho\left[\tan\frac{nl}{a} + \tan\frac{nl'}{a}\right] A_1\, an$$

ie $-MA_1n^2 + MM^2A_1 =$ "

$$\therefore\ M(m^2-n^2) = \rho an\left[\tan\frac{nl}{a} + \tan\frac{nl'}{a}\right].$$

(9) At the piston, waves are reflected as well as transmitted. Let the incident wave be real part of $\phi = A_1 e^{i(mx-nt)}$, the reflected wave will be of the form $\phi = B_1 e^{-i(mx+nt)}$. Let $\underline{m}$ for the reflected wave change to $\underline{m_1}$, then the transmitted wave will be $\phi = A_2 e^{i(m_1x-nt)}$

Taking the origin at the piston, then

$\phi_1 = A_1 e^{i(mx-nt)} + B_1 e^{-i(mx+nt)}$ for one part of the tube, and

$\phi_2 = A_2 e^{i(m_1x-nt)}$ for the other.

The eqn of motion of the piston is

$$M\ddot{\xi} + Mp^2\xi = -(\delta p_2 - \delta p_1)$$

$$= \sin = -\sin e^{-int}\left[-A_2\rho_2 + (A_1+B_1)\rho_1\right]$$

whose particular soln is

$$\xi = \frac{\sin e^{-int}\left[-A_2\rho_2 + (A_1+B_1)\rho_1\right]}{M(n^2-p^2)}$$

At the piston ie at $x=0$

$$\dot{\xi} = \frac{Sn^2 e^{-int}}{M(n^2-p^2)}\left[-A_2\rho_2 + (A_1+B_1)\rho_1\right] = -\frac{\partial\phi_1}{\partial x} = -im(A_1-B_1)e^{-int}$$

& $\dot{\xi} = -\dfrac{\partial\phi_2}{\partial x} = -\dfrac{Sn^2e^{-int}}{M(n^2-p^2)}\left[-A_2\rho_2 + (A_1+B_1)\rho_1\right] = -A_2 im' e^{-int}$

$$\therefore\ \text{if}\ \frac{Sn^2}{M(n^2-p^2)} = K$$

we get the ratios of $A_1 : B_1 : A_2$ from which we deduce

$$A_1 + B_1 = -A_2 \, m'/m$$

$$\therefore \quad M\ddot{\xi} + Mp^2\xi + \frac{S\eta}{m'}\left(\rho_2 + \frac{m'}{m}\rho_1\right)\dot{\xi} = 0$$

$$\text{or} \quad M\ddot{\xi} + Mp^2\xi + S\dot{\xi}\left(a_2\rho_2 + a_1\rho_1\right) = 0.$$

(10) Let $\phi = (A\cos nx + B\sin nx)\cos nkt$

Take the piston without the spring as origin

$$\text{At } x = 0, \quad -\frac{\partial\phi}{\partial x} = a\cos nkt$$

$$\therefore \quad -Bn = a \quad \text{or} \quad B = -a/n$$

For the motion of the piston

$$M\ddot{\xi} + \mu\xi = -\delta p = -\rho\frac{\partial\phi}{\partial t} \quad \text{at } x = l$$

$$= \rho nk\left(A\cos nl + B\sin nl\right)\sin nkt$$

$$\therefore \quad \xi = \frac{\rho nk\left(A\cos nl + B\sin nl\right)}{\mu - Mn^2k^2}\sin nkt$$

But at $x = l$, $\dot{\xi} = \frac{\partial\phi}{\partial x}$

$$\therefore \quad \frac{\rho n^2k^2\left(A\cos nl + B\sin nl\right)}{\mu - Mn^2k^2} = -n\left[A\sin nl - B\cos nl\right]$$

Using the value of B, this gives

$$\text{i.e.} \quad \tan nt\left(A\cos nl + B\sin nl\right) = -\left(A\sin nl - B\cos nl\right)$$

This gives $A = -\frac{a}{n}\cot(nl + n\epsilon)$ using the value of $B = -a/n$

$$\therefore \quad \phi = -\frac{a}{n}\left[\cos nx\cot(nl + n\epsilon) + \sin nx\right]\cos nkt$$

$$= -\frac{a}{n}\,\frac{\cos n(l - x + \epsilon)}{\sin n(l + \epsilon)}\cos nkt.$$

(11) Take the junction as the origin

Let $\phi_1 = \left(A\cos\frac{nx}{a} + B\sin\frac{nx}{a}\right)\cos(nt + \epsilon)$ for air

$$\phi_2 = \left(A_1\cos\frac{nx}{a_1} + B_1\sin\frac{nx}{a_1}\right)\cos(nt + \epsilon) \quad \text{for the gas}$$

$$\frac{\partial\phi_1}{\partial t} = 0 \quad \text{at } x = -l/2$$

$$\therefore \quad A\cos\frac{nl}{2a} - B\sin\frac{nl}{2a} = 0$$

$\frac{\partial \phi_2}{\partial x} = 0$ at $x = -l/2$

$\therefore \; -A_1 \sin \frac{n l_1}{2a_1} + B_1 \cos \frac{n l_1}{2a_1} = 0$

At the junction $\delta p = \delta p_1$

$\therefore \; \rho A = \rho_1 A_1$

Also the velocity of the point of junction is the same for ϕ_1 & ϕ_2

$\therefore \; \frac{B}{a} = \frac{B_1}{a_1}$

$\tan \frac{nl}{2a} = A/B \; ; \; \tan \frac{n l_1}{2a_1} = \frac{B_1}{A_1}$

$\therefore \tan \frac{nl}{2a} \tan \frac{n l_1}{2a_1} = \frac{A}{A_1} \cdot \frac{B_1}{B} = \frac{a_1}{a} \cdot \frac{\rho_1}{\rho}$

& since $n = 2\pi/p$

$\tan \frac{\pi p l}{a} \tan \frac{\pi p l_1}{a_1} = \frac{a_1 \rho_1}{a \rho}$

(12) Let $\phi = \left(A \cos \frac{nx}{a} + B \sin \frac{nx}{a} \right) \cos (nt + \epsilon)$

$\frac{\partial \phi}{\partial x} = 0$ at $x = 0$, i.e. $B = 0$

$\therefore \; \phi = A \cos \frac{nx}{a} \cos (nt + \epsilon)$

$\frac{\partial \phi}{\partial t} = 0$ at $x = l \quad \therefore \cos \frac{nl}{a} = 0$, i.e. $n = \frac{(2k+1) a \pi}{2l}$.

$\int \frac{dp}{\rho} = \frac{\partial \phi}{\partial t}$, neglecting q^2 etc

$p = K_1 \rho^{\gamma}, \; dp = K_1 \gamma \rho^{\gamma - 1} d\rho$

$\therefore \; \frac{K_1 \gamma}{\gamma - 1} \rho^{\gamma - 1} = \frac{\partial \phi}{\partial t} = -A n \sin (nt + \epsilon)$

For the gravest mode, $n = \frac{a\pi}{2l}$

Let $\xi = \left(A \cos \frac{nx}{a} + B \sin \frac{nx}{a} \right) \cos (nt + \epsilon)$

Let the tube be open at $x = l$ and closed at $x = 0$

then $A = 0$, $\cos \frac{nl}{a} = 0$, whence $n = \frac{a\pi}{2l}$ for the fundamental mode

$\therefore \; \xi = B \sin \frac{nx}{a} \cos (nt + \epsilon)$

$\delta p = -c^2 \rho_0 \frac{\partial \xi}{\partial x} = -c^2 \rho_0 B \frac{n}{a} \cos (nt + \epsilon)$

$c^2 = \frac{dp}{d\rho} = \frac{\gamma p_0}{\rho_0} \quad \therefore \; p_0 = \frac{c^2 \rho_0}{\gamma}$

$$\therefore \text{ Variation} = \frac{c^2\rho_0}{\eta_1 r} = -c^2\rho_0 B\frac{n}{a} = -c^2\rho_0 B\pi/2l$$

$$\therefore B = \frac{2l}{n\pi r}$$

Hence since at the open end

$$\xi = B\sin\frac{nl}{a}\cos(nt+\epsilon)$$

the amplitude of vibration is

$$\frac{2l}{n\pi r}\sin\frac{nl}{a}$$

but $\frac{nl}{a} = \pi/2$ & hence this amp $= 2l/n\pi r$.

(14) Let $\phi_1 = \left(A_1\cos\frac{nx}{a} + \cdot\right)\cos(nt+\epsilon)$ for the closed end

$\phi_2 = \left(A_2\cos\frac{nx}{a'} + \cdot\right)$ " open

$$\frac{\partial\phi_1}{\partial x} = 0 \text{ at } x = l\ ;\ \frac{\partial\phi_2}{\partial x} = 0 \text{ at } x = l$$

$$\therefore A_1\sin\frac{nl}{a} + B_1\cos\frac{nl}{a} = 0 \quad (1)\ ;\ A_2\cos\frac{nl}{a'} + B_2\sin\frac{nl}{a'} = 0 \quad (2)$$

Let $\xi = r\cos(nt+\epsilon)$ give the motion of the piston

$$\frac{\dot{\xi}}{\sin(nt+\epsilon)} = -rn = -\frac{\partial\phi_1}{\partial x}\Big/\sin(nt+\epsilon) = -\frac{B_1\eta}{a} = -\frac{B_2\eta}{a'}$$

$$\therefore B_1 = ar,\quad B_2 = a'r$$

For the motion of the piston

$$Mr - n^2 = \omega\left(-\delta p_1 + \delta p_2\right) = \left[-\delta\frac{\partial\phi_1}{\partial t} + \rho\frac{\partial\phi_2}{\partial t}\right]_{x=0}^{\omega}$$

$$= -\omega\left[-A_1\delta n + A_2\rho n\right]$$

$$Mrn = -\left[\sigma A_1 - \rho A_2\right]\omega$$

$$A_1 = -ar\cot\frac{nl}{a}$$

$$A_2 = -a'r\tan\frac{nl}{a'}$$

$$\therefore \frac{Mn}{\omega} = \left[a\sigma\cot\frac{nl}{a} - a'\rho\tan\frac{nl}{a'}\right].$$

(15) Let ϕ_1, ϕ_2, ϕ_3 be the incident, reflected & transmitted waves with the usual expns

For the motion of the piston

$$zN\ddot{\xi} = -\left(\frac{\partial\phi_1}{\partial t} - \frac{\partial\phi_2}{\partial t}\right)$$

$$= in\left[A_1 e^{-imz/2} + A_2 e^{imz/2} - A_3 e^{imz/2}\right] e^{-int}$$

$$\therefore \xi = -\frac{i}{n}\cdot\frac{1}{Nz}\left[(A_2 - A_3)e^{\frac{1}{2}imz} + A_1 e^{-\frac{1}{2}imz}\right] e^{-int}$$

At the piston

$$-\frac{1}{Nz}\left[(A_2 - A_3)e^{\frac{1}{2}imz} + A_1 e^{-\frac{1}{2}imz}\right] = mi\left[A_1 e^{-\frac{1}{2}imz} - A_2 e^{\frac{1}{2}imz}\right]$$

$$-\frac{1}{Nz}\left[(A_2 - A_3)e^{\frac{1}{2}imz} + A_1 e^{-\frac{1}{2}imz}\right] = mi A_3 e^{\frac{1}{2}imz}$$

giving $A_3 = A_1 \cdot \dfrac{2e^{-imz}}{2 - imNz}$

Putting $\rho\cos\phi = 2$, $\rho\sin\phi = mNz$, we have

$$A_3 = \frac{2A_1 e^{\left(\tan^{-1}\frac{1}{2}mNz - mz\right)}}{\sqrt{4 + m^2N^2z^2}} = \frac{A_1}{\sqrt{1 + \frac{\pi^2N^2z^2}{\lambda^2}}} e^{\tan^{-1}\left(\frac{\pi Nz}{\lambda}\right) - \frac{2\pi z}{\lambda}}$$

Hence the intensity is diminished by $1 : 1 + \dfrac{\pi^2N^2z^2}{\lambda^2}$ and the phase is retarded by $\left\{\tan^{-1}\dfrac{\pi Nz}{\lambda} - \dfrac{2\pi z}{\lambda}\right\}$.

(16) Let $\phi = \left(A\cos\dfrac{nx}{a} + B\sin\dfrac{nx}{a}\right)\sin nt$

After the discs are drawn aside

$\dfrac{\partial\phi}{\partial t} = 0$ both at $x = l$ & $x = -l$

$\therefore B = 0$, & $n = \dfrac{(2s+1)\pi a}{2l}$

& $\phi = \sum A_{2s+1}\cos\dfrac{(2s+1)\pi x}{2l}\sin\dfrac{(2s+1)\pi at}{2l}$

Initially the increment of pressure $= \rho\dfrac{\partial\phi}{\partial t} = kp_0 t$ if p_0 be the density of the outside air and $a^2 = k$.

$$\left(\frac{\partial\phi}{\partial t}\right)_{t=0} = \sum \frac{(2s+1)\pi a}{2l} A_{2s+1}\cos\frac{(2s+1)\pi x}{2l}$$

$$\therefore a^2p_0 t = p_0\sum\frac{(2s+1)\pi a}{2l}A_{2s+1}\cos\frac{(2s+1)\pi x}{2l}$$

$$\therefore \int_{-l}^{0} a^2t\cos\frac{(2s+1)\pi x}{2l}dx + \int_0^l a^2t\cos\frac{(2s+1)\pi x}{2l}dx = \frac{(2s+1)\pi a}{2l}A_{2s+1}\int_0^l \cos^2\frac{(2s+1)\pi x}{2l}dx$$

which gives $A_{2s+1} = \dfrac{8atl(-1)^s}{(2s+1)^2\pi^2}$. Hence the result.

(17) Let $\xi = \left(A_1 \cos\frac{nx}{a} + B_1 \sin\frac{nx}{a}\right)\sin nt$

$\frac{\partial \xi}{\partial x} = 0$ at $x = 0$, $\therefore B_1 = 0$

$\xi = A_1 \cos\frac{nx}{a} \sin nt$,

$\xi = A \sin nt$, at $x = l$

$\therefore A = A_1 \cos\frac{nl}{a}$

$\therefore \xi = A_1 \cos\frac{nx}{a} \sin nt \Big/ \cos\frac{nl}{a}$.

Let the tube be of unit cross section. Then the

$$K.E = \frac{1}{2}\frac{M}{l}\int_0^l \dot{\xi}^2 dx$$

$$= \frac{1}{2}\frac{M}{l}\cdot\frac{A^2}{\cos^2\frac{nl}{a}}\cdot n^2\cos^2 nt\int_0^l \cos^2\frac{nx}{a}\,dx$$

$$= \frac{1}{4}\frac{M}{l}\cdot\frac{A^2}{\cos^2\frac{nl}{a}}\, n^2\cos^2 nt\left[l + \frac{a}{2n}\sin\frac{2nl}{a}\right]$$

$$= \frac{1}{4} MA^2\cos^2 pt\left[p^2\sec^2\frac{pl}{a} + \frac{ap}{l}\tan\frac{pl}{a}\right]$$ replacing n by p.

(18) $\phi_1 = Ae^{i(mx-nt)} + Be^{-i(mx-nt)}$

$\phi_2 = Ce^{i(mx+nt)}$

Take the rigid screen as origin

$\frac{\partial \phi_1}{\partial x} = 0$ at $x = 0$, $\therefore A - B = 0$

$\frac{\partial \phi_2}{\partial x} = 0$ at $x = 0$, $\therefore C = 0$

ie no sound waves are transmitted & the amplitudes of incident & reflected waves are equal.

$\therefore \phi_1 = A\{\cos(mx + nt) + \cos(nt - mx)\}$

$\left(\frac{\partial \phi_1}{\partial t}\right)_{x=0} = -2An\sin nt$; $\frac{n}{m} = a$

$= -2Aam\sin nt$

$\therefore \rho_0\frac{\partial \phi_1}{\partial t}$ = increment of pressure $= -2A\rho_0 am\sin nt$

ie the increment lies between $\pm 2Am a\rho_0$.

(19) Let $\phi_1 = Ae^{i(mx-nt)} + Be^{-i(mx+nt)}$

$\phi_2 = Ce^{i(mx-nt)}$

& let $\xi = De^{i(mx-nt)}$ give the displacement of the obstacle

$-iDn = -im(A-B) = -iCm$

$M\ddot{\xi} = -(\delta p_1 - \delta p_2) = i\rho_0 n(A+B-C)e^{-int}$

$-MDn^2 = i\rho_0 n(A+B-C)$

$\therefore -M(A-B) = i\rho_0[(A+B)-(A-B)]$

$\therefore -MAm = B(2i\rho_0 - Mm)$

$\therefore B = -Mam/(2i\rho_0 - Mm)$ (?)

(20) See §289.

$\phi_1 = A_1 e^{i(nt-m_1x)} + B_1 e^{i(nt+m_1x)}$

$\phi_2 = A_2 e^{i(nt-m_2x)}$

At the common surface

$\rho_1 \frac{\partial \phi_1}{\partial t} = \rho_2 \frac{\partial \phi_2}{\partial t}$ at $x = 0$

$\therefore \rho_1\{in(A_1+B_1)\} = \rho_2\{A_2 in\}$

$\therefore \rho_1(A_1+B_1) = \rho_2 A_2$ (i)

$\frac{\partial \phi_1}{\partial x} = \frac{\partial \phi_2}{\partial x}$ for the continuity of the normal velocity

$m_1(B_1 - A_1) = -m_2 A_2$ (2)

$\therefore \frac{\rho_1}{\rho_2}(A_1+B_1) = -\frac{m_1}{m_2}(B_1-A_1)$

$B_1\left(\frac{\rho_1}{\rho_2} + \frac{m_1}{m_2}\right) = A_1\left(\frac{m_1}{m_2} - \frac{\rho_1}{\rho_2}\right)$

$\frac{m_1}{m_2} = \frac{a_2}{a_1}$

$\therefore B_1 = \frac{a_2\rho_2 - a_1\rho_1}{a_2\rho_2 + a_1\rho_1} A_1$

$\rho_1\left\{A_1 + A_1\frac{a_2\rho_2 - a_1\rho_1}{a_2\rho_2 + a_1\rho_1}\right\} = \rho_2 A_2$

$\therefore A_2 = 2a_2\rho_1/(a_1\rho_1 + a_2\rho_2)$

(22)



Take M_1 as the origin

Let $\phi_1 = \left[A_1 \cos \frac{nx}{a_1} + B_1 \sin \frac{nx}{a_1} \right] \sin nt$ for 'l_1'

$\phi_2 = \left[A_2 \cos \frac{nx}{a_2} + B_2 \sin \frac{nx}{a_2} \right] \sin nt$ for 'l_2'

Let $\xi_1 = X \cos nt$, $\xi_2 = Y \cos nt$ give the motion of the two discs M_1, M_2

$$\therefore X n = \frac{B_1 n}{a_1}, \quad X n = \frac{B_2 n}{a_2} \qquad (1)$$

$$\left.\begin{aligned} Y n &= \frac{n}{a_1}\left[A_1 \sin \frac{nl_1}{a_1} + B_1 \cos \frac{nl_1}{a_1} \right] \\ Y n &= \frac{n}{a_2}\left[-A_2 \sin \frac{nl_2}{a_2} + B_2 \cos \frac{nl_2}{a_2} \right] \end{aligned}\right\} \qquad (2)$$

$$M\ddot{\xi} = -(\delta p_2 - \delta p_1)$$

$$\therefore -M_1 X n^2 = n(\rho_1 A_1 - \rho_2 A_2)$$

$$-M_2 Y n^2 = n\left[\rho_2 \left(A_2 \cos \frac{nl_2}{a_2} + B_2 \sin \frac{nl_2}{a_2} \right) - \rho_1 \left(A_1 \cos \frac{nl_1}{a_1} - B_1 \sin \frac{nl_1}{a} \right) \right] \qquad (3)$$

$$A_1 = a_1 Y \operatorname{cosec} \frac{nl_1}{a_1} - a_1 X \cot \frac{nl_1}{a_1}$$

$$A_2 = a_2 X \cot \frac{nl_2}{a_2} - a_2 Y \operatorname{cosec} \frac{nl_2}{a_2}$$

Hence (3) gives

$$X\left[M_1 n - \rho_1 a_1 \cot \frac{nl_1}{a_1} - \rho_2 a_2 \cot \frac{nl_2}{a_2} \right] = Y\left[\rho_1 a_1 \operatorname{cosec} \frac{nl_1}{a_1} + \rho_2 a_2 \operatorname{cosec} \frac{nl_2}{a_2} \right]$$

and the second gives

$$-M_2 n + \rho_1 \left[-a_1 \frac{\cos^2 \frac{nl_1}{a_1}}{\sin nl_1/a_1} - a_1 \sin \frac{nl_1}{a_1} \right] - \rho_2 \left[a_2 \frac{\cos^2 \frac{nl_2}{a_2}}{\sin \frac{nl_2}{a_2}} + a_2 \sin \frac{nl_2}{a_2} \right]$$

$$= -Y\left[\rho_1 a_1 \cot \frac{nl_1}{a_1} + \rho_2 a_2 \cot \frac{nl_2}{a_2} \right]$$

$$Y\left[M_2 n - \rho_1 a_1 \cot \frac{nl_1}{a_1} - \rho_2 a_2 \cot \frac{nl_2}{a_2} \right] = X\left[\rho_1 a_1 \operatorname{cosec} \frac{nl_1}{a_1} + \rho_2 a_2 \operatorname{cosec} \frac{nl_2}{a_2} \right]$$

$\therefore$ The period equation is

$$(M_1 n - \alpha)(M_2 n - \alpha) = \beta^2$$

and hence the result.

(23)

$$\phi_1 = \left[A_1 \cos \frac{px}{a_1} + B_1 \sin \frac{px}{a_1} \right] \sin pt$$

$$\phi_2 = \left[A_2 \cos \frac{px}{a_2} + B_2 \sin \frac{px}{a_2} \right] \sin pt$$

$\xi_1 = X\cos pt$, $\xi_2 = Y\cos pt$

give the motion of the other

$$X = \frac{pB_1}{a_1}, \quad X = \frac{pB_2}{a_2}$$

$$Yp = \frac{p}{a}\left[A_1 \sin\frac{pl_1}{a_1} + B_1\cos\frac{pl_1}{a_1}\right] = \frac{p}{a}\left[-A_2\sin\frac{pl_2}{a_2} + B_2\cos\frac{pl_2}{a_2}\right]$$

$$\frac{M\ddot{\xi}}{\sin pt} = pP(A_1 - A_2)$$

$$-XMp^2 = pP(A_1 - A_2) \qquad \therefore XMp = P(A_2 - A_1)$$

$$= -P\left[\frac{aY - aX\cos\frac{pl_1}{a}}{\sin pl_1/a} + \frac{Ya - Xa\cos\frac{pl_2}{a}}{\sin pl_2/a}\right]$$

$M/p = m$.

$$\therefore -mXp = aY\left(\operatorname{cosec}\frac{pl_1}{a} + \operatorname{cosec}\frac{pl_2}{a}\right) - aX\left(\cot\frac{pl_1}{a} + \cot\frac{pl_2}{a}\right)$$

$$\therefore X\left[a\left(\cot\frac{pl_1}{a} + \cot\frac{pl_2}{a}\right) - mp\right] = aY\left(\operatorname{cosec}\frac{pl_1}{a} + \operatorname{cosec}\frac{pl_2}{a}\right)$$

$$\therefore X = \frac{aY\left(\operatorname{cosec}\frac{pl_1}{a} + \operatorname{cosec}\frac{pl_2}{a}\right)}{a\left(\cot\frac{pl_1}{a} + \cot\frac{pl_2}{a}\right) - mp}\cos pt.$$

(24) Let $T_0 = 4l/c$ give the fundamental period with one end closed. When the end has a piston the period equation is

$$\cot\frac{nl}{c} + \frac{\rho nc}{\mu - Mn^2} = 0$$

Let $T = 2\pi/n$

$$\therefore \cot\frac{2\pi}{T}\cdot\frac{T_0}{4} + \frac{\rho l\cdot\frac{2\pi}{T_0}\cdot\frac{c}{l}}{\mu l - M\cdot 4\pi^2/T^2} = 0$$

$$\text{i.e. } \cot\frac{\pi T_0}{2T} + \frac{8\pi m T/T_0}{\mu T^2 - 4\pi^2 M} = 0$$

Since T differs from T_0 by an infinitesimal we can replace T^2 in the denominator of the second term by T_0^2 and put $\pi T_0/2T = x$. Hence

$$x\cot x + \frac{4\pi^2 m}{\mu T_0^2 - 4\pi^2 M} = 0$$

whose approximate value is $x = \pi/2$. If the real value be $\frac{\pi}{2} + h$, then

$$h = -\frac{4\pi^2 m/\mu T_0^2 - 4\pi^2 m}{(\cot x - x\operatorname{cosec} x)_{x=\pi/2}} = \frac{8\pi^2 m}{\mu T_0^2 - 4\pi^2 m}$$

$$\therefore \frac{\pi T_0}{2T} = \frac{\pi}{2} + \frac{8\pi^2 m}{\mu T_0^2 - 4\pi^2 m}$$

$$\therefore \quad \frac{T_0}{T} = 1 + \frac{16\pi m}{\mu T_0^2 - 4\pi^2 m}$$

$$\therefore \quad T = T_0 \left[1 - \frac{16\pi m}{\mu T_0^2 - 4\pi^2 m} \right]$$

(25) Let $\phi_1 = \left(A_1 \cos \frac{nx}{a_1} + B_1 \sin \frac{nx}{a_1} \right) \cos nt$ for 'x_1'

$\phi_2 = \left(A_2 \cos \frac{nx}{a_2} + B_2 \sin \frac{nx}{a_2} \right) \cos nt$ " 'x_2'

Let $\xi = D \sin nt$ give the motion of the piston

$$Dn = -\frac{B_1 n}{a_1} = -\frac{B_2 n}{a_2}$$

$$-mAn^2 D = -An(\rho_1 A_2 - \rho_2 A_2)$$

$$\therefore mnD = -(\rho_2 A_2 - \rho_1 A_1)$$

Also since the tube is closed at both ends

$$A_1 \sin \frac{nx_1}{a_1} + B_1 \cos \frac{nx_1}{a_1} = 0$$

$$-A_2 \sin \frac{nx_2}{a_2} + B_2 \cos \frac{nx_2}{a_2} = 0$$

$$\therefore mnD = \left[\rho_1 D a_1 \cot \frac{nx_1}{a_1} + \rho_2 D a_2 \cot \frac{nx_2}{a_2} \right]$$

$$\therefore \frac{2\pi m}{T} = a_1 \rho_1 \cot \frac{2\pi x_1}{a_1 T} + a_2 \rho_2 \cot \frac{2\pi x_2}{a_2 T}$$

(26) $\phi_1 = A_1 e^{i(nt - mx)} + B_1 e^{i(nt + mx)}$

$\phi_2 = A_2 e^{i(nt - mx)}$

Let the motion of the piston be given by

$$\xi = r e^{i(nt - mx)}$$

$$rin = -im(B_1 - A_1) = imA_2$$

$$-Mrn^2 = -\rho(A_1 + B_1 - A_2) in$$

$$\therefore -Mrn = -\rho\left(A_1 + A_1 - \frac{rn}{m} A_2\right) i$$

$$\therefore A_2(2\rho i + Mm) = 2\rho i A_1$$

$$A_2 = \frac{2\rho i A_1}{2\rho i + 2\pi M/\lambda}$$

$$\lambda\rho = m$$

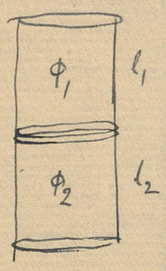
$$\therefore A_2 = \frac{2mA_1 i}{2mi + 2\pi M} = \frac{mA_1(m + i\pi m)}{m^2 + \pi^2 M^2}$$

Put $\left.\begin{aligned} \rho\cos\frac{2\pi b}{\lambda} &= m \\ \rho\sin\frac{2\pi b}{\lambda} &= \pi M \end{aligned}\right\}$ e

$$\therefore A_2 = A_1 \cdot \frac{m}{\sqrt{m^2 + \pi^2 M^2}}\, e^{i\frac{2\pi b}{\lambda}}$$

$$\therefore \text{The transmitted wave is } A_1 \cos\frac{2\pi b}{\lambda}\, e^{i(nt - mx + mt)}.$$

(27)



Let $\phi_1 = \left(A_1\cos\frac{nx}{a} + B_1\sin\frac{nx}{a}\right)\cos nt$

At the outset we have to will determine l_1, l_2 the two portions into which the piston divides the tube.

Let p_1 be the pressure in the compressed gas in l_2 in the equilibrium state, then since pv = const

$$p_1 \cdot l_2 = p \cdot l, \quad \text{but } p = g\sigma h, \text{ the atmospheric pressure}$$

$$\therefore p_1 l_2 = g\sigma h l$$

For the equilibrium of the piston

$$p_1 = \sigma k g + g\sigma h = g\sigma(k + h)$$

$$\therefore g\sigma(h+k) l_2 = g\sigma h l$$

$$\therefore l_2 = \frac{hl}{h+k}\,; \quad l_1 = \frac{kl}{h+k}$$

$$\frac{\partial\phi_1}{\partial t} = 0 \quad \text{at } x = -\frac{hl}{h+k}$$

$$\therefore A\cos\frac{nhl}{a(h+K)} - B\sin\frac{nhl}{a(h+K)} = 0.$$

Let $\xi = X\sin nt$ give the motion of the piston

$$Xn\cos nt = -\frac{Bn}{a}\cos nt$$

$$-aX = B$$

$$\phi_1 = -aX\left[\tan\frac{nhl}{a(h+k)}\cos\frac{nx}{a} + \sin\frac{nx}{a}\right]\cos nt$$

ϕ_2 Similarly we find

$$\phi_2 = -aX\left[\cot\frac{nkl}{a(h+K)}\cos\frac{nx}{a} + \sin\frac{nx}{a}\right]\cos nt$$

$$-Mn^2 = naX\left[\rho_1 \tan\frac{nhl}{a(h+K)} - \rho_2 \cot\frac{nkl}{a(h+K)}\right]$$

$$\therefore \frac{\gamma p}{\rho} = a^2 \quad , \quad \frac{\gamma g \sigma h}{\rho} = a^2$$

$$\therefore \sigma k n = a\left[\rho_2 \cot\frac{nkl}{a(h+K)} - \rho_1 \tan\frac{nhl}{a(h+K)}\right]$$

$$\frac{\gamma g \sigma h}{\rho_1} = a^2, \quad \frac{\gamma g \sigma (h+K)}{\rho_2} = a^2$$

$$\therefore \frac{2\pi a}{g\gamma T} = \frac{h+K}{K}\cot\frac{nkl}{a(h+K)} - \frac{h}{K}\tan\frac{nhl}{a(h+K)}.$$

(28) $$\phi_1 = A_1 e^{i(m_1 x - nt)} + B_1 e^{-i(m_1 x + nt)}$$

$$\phi_2 = A_2 e^{i(m_2 x - nt)}$$

Let $\zeta = \gamma e^{i(mx-nt)}$ give the motion of the piston

$$-im_1(A_1 - B_1) = -ni\gamma = -A_2 i m_2$$

$$-M\gamma n^2 = -i\omega n\left[A_2\rho' - (A_1 + B_1)\rho\right]$$

$$\therefore Mn\cdot\frac{m_1}{n}(A_1 - B_1) = i\omega\left[\frac{m_1(A_1 - B_1)}{m_2}\rho' - (A_1 + B_1)\rho\right]$$

$$\therefore A_1\left[Mm_1 - \frac{im\omega\rho'}{m_2} + i\rho\omega\right] = B_1\left[Mm_1 - i\omega\left(\frac{m_1\rho'}{m_2} + \rho\right)\right]$$

$$\therefore B_1 = A_1 \frac{\sqrt{M^2 m_1^2 + \omega^2\left(\frac{m_1\rho'}{m_2} - \rho\right)^2}}{\sqrt{M^2 m_1^2 + \omega^2\left(\frac{m_1\rho'}{m_2} + \rho\right)^2}} \times \text{Exponential part}.$$

Hence the ratio of intensities $= \dfrac{\left(\frac{m_1\rho'}{m_2} - \rho\right)^2 + \frac{M^2 m_1^2}{\omega^2}}{\left(\frac{m_1\rho'}{m_2} + \rho\right)^2 + \frac{M^2 m_1^2}{\omega^2}}$

$$\frac{n}{m_1} = a_1, \quad \frac{n}{m_2} = a_2; \quad m_1 = 2\pi/\lambda, \quad \frac{m_1}{m_2} = \frac{a_2}{a_1}$$

∴ the ratio of intensities is then given in the required form.

(29) $a^2 = \frac{\gamma p}{\rho}$, $a'^2 = \frac{\gamma p}{\rho'}$

$\phi_1 = Ae^{i(mx-nt)} + A_1 e^{-i(mx+nt)}$

$\phi_2 = A' e^{i(m'x-nt)}$

and $\xi = Be^{i(mx-nt)}$

$-Bin = -mi(A - A_1) = -m'iA'$

$M\ddot{\xi} + \mu\xi = \delta p_1 - \delta p_2$

$\therefore -Mn^2 B + \mu B = \left[ip'nA' - ip n (A+A_1) \right]$

$ni\left\{ A'\rho' - \rho(A+A_1) \right\} = B(\mu - Mn^2)$

$= \frac{m'A'}{n}(\mu - Mn^2)$

$ni\left[A'\rho' - \rho\left(2A - \frac{m'A'}{m}\right)\right] = \frac{m'A'}{n}(\mu - Mn^2)$

$A'\left[ni\left(\rho' + \frac{\rho m'}{m}\right) - \frac{m'}{m}(\mu - Mn^2)\right] = 2A\rho ni$

i.e. $A' = \frac{2A\rho n}{\sqrt{\left\{\frac{m'^2}{m^2}(\mu - Mn^2)^2 + n^2\left(\rho' + \frac{\rho m'}{m}\right)^2\right\}}}$ × Exponential part.

Also $ni\left[\frac{m\rho'}{m'}(A - A_1) - \rho(A + A_1)\right] = \frac{m}{n}(A - A_1)(\mu - Mn^2)$

Thus $A_1 = A\sqrt{\left\{\frac{\frac{m^2}{n^2}(\mu - Mn^2)^2 + n^2\left(\frac{m\rho'}{m'} - \rho\right)^2}{\frac{m^2}{n^2}(\mu - Mn^2)^2 + n^2\left(\frac{m\rho'}{m} + \rho\right)^2}\right\}}$ × exp. part

$\rho' + \frac{\rho m'}{m} = \frac{1}{a'^2}\left(a'^2\rho' + a^2\rho m/m'\right)$

$= \frac{\gamma p}{m'a'^2}(m + m')$, and using $a^2 = \frac{\gamma p}{\rho}$, $a'^2 = \frac{\gamma p}{\rho'}$

$A' = \frac{2A\rho n^2/m'}{\left\{(\mu - Mn^2)^2 + \gamma^2 p^2 (m+m')^2\right\}}$

and $A_1 = A\sqrt{\frac{(\mu - Mn^2)^2 + \gamma^2 p^2 (m'-m)^2}{(\mu - Mn^2)^2 + \gamma^2 p^2 (m'+m)^2}}$.

(30) Let $\phi_1 = A_1 e^{i(m_1 x - nt)} + B_1 e^{-i(m_1 x + nt)}$

$\phi_2 = A_2 e^{i(m_2 x - nt)} + B_2 e^{-i(m_2 x + nt)}$

$\phi_3 = A_3 e^{i(m_1 x - \omega t)}$

since the wavelengths of incident & transverse waves are the same.

$$\left.\begin{aligned} \delta p_1 &= \delta p_2 \quad \text{at } x = 0 \\ \delta p_2 &= \delta p_3 \quad \text{at } x = h \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} \frac{\partial \phi_1}{\partial x} &= \frac{\partial \phi_2}{\partial x} \quad \text{at } x = 0 \\ \frac{\partial \phi_2}{\partial x} &= \frac{\partial \phi_3}{\partial x} \quad \text{at } x = h \end{aligned}\right\} \qquad (2)$$

These four boundary conditions determine the various amplitudes.

$$\left.\begin{aligned} \rho_1 (A_1 + B_1) &= \rho_2 (A_2 + B_2) \\ \rho_2 \left\{ A_2 e^{i m_2 h} + B_2 e^{-i m_2 h} \right\} &= \rho A_3 e^{i m_1 h} \end{aligned}\right\} \qquad (A)$$

$$\left.\begin{aligned} m_1 (A_1 - B_1) &= m_2 (A_2 - B_2) \\ m_2 \left\{ A_2 e^{i m_2 h} - B_2 e^{-i m_2 h} \right\} &= A_3 m_1 e^{i m_1 h} \end{aligned}\right\} \qquad (B)$$

The eqns (A) & (B) determine B_1, A_2, A_3, B_2 in terms of A_1.

(32) $\phi_1 = A_1 e^{i(a_1 x + by + \omega t)} + B_1 e^{-i(a_1 x - by - \omega t)}$

$\phi_2 = A_2 e^{i(a_2 x + by + \omega t)} + B_2 e^{-i(a_2 x - by - \omega t)}$

$$\left.\begin{aligned} \frac{c_1}{\sin \alpha_1} &= \frac{c_2}{\sin \alpha_2} \\ \frac{\rho_1}{\rho_2} &= \frac{a_2^2}{a_1^2} = \frac{\sin^2 \alpha_2}{\sin^2 \alpha_1} \end{aligned}\right\} \qquad (A)$$

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$a_1 \tan \alpha_1 = a_2 \tan \alpha_2$

$\frac{\partial \phi_2}{\partial x} = 0$ at $x = l$

$\therefore A_2 e^{i a_2 l} - B_2 e^{-i a_2 l} = 0$

The boundary conditions give

$\rho_1 (A_1 + B_1) = \rho_2 (A_2 + B_2)$

$a_1 (A_1 - B_1) = a_2 (A_2 - B_2)$

To calculate B_1;

$A_2 + B_2 = \frac{\rho_1}{\rho_2} (A_1 + B_1)$; $A_2 - B_2 = \frac{a_1}{a_2} (A_1 - B_1)$

$$e^{ia_2 l}\left[\frac{\rho_1}{\rho_2}(A_1+B_1)+\frac{a_1}{a_2}(A_1-B_1)\right]$$

$$= e^{-ia_2 l}\left[\frac{\rho_1}{\rho_2}(A_1+B_1)-\frac{a_1}{a_2}(A_1-B_1)\right]$$

& finally obtain

$$B_1 = -A_1 \frac{\frac{a_1}{a_2}\cos a_2 l + \frac{\rho_1}{\rho_2} i \sin a_2 l}{\frac{a_1}{a_2}\cos a_2 l - i\frac{\rho_1}{\rho_2}\sin a_2 l}$$

$$2\tan^{-1}\left[\frac{\rho_1}{\rho_2}\cdot\frac{a_2}{a_1}\tan(a_2 l)\right]$$

$$= A_1 e$$

Hence the amplitudes of the incident & emergent ray are equal & the difference of phase is

$$2\tan^{-1}\left[\frac{\sin^2\alpha_1}{\sin^2\alpha_2}\cdot\frac{\tan\alpha_1}{\tan\alpha_2}\tan(a_2 l)\right]$$

$$= 2\tan^{-1}\left[\frac{\sin\alpha_2}{\sin\alpha_1}\tan\left(\frac{2\pi}{\lambda'}\cos\alpha_2 l\right)\right], \text{ since } a_2 = \frac{2\pi}{\lambda'}\cos\alpha_2 .$$

(33) In this case in which the second medium extends to infinity on one side of the plane interface

$$\frac{A'}{A} = \beta = \frac{a_1\rho_2 - a_2\rho_1}{a_1\rho_2 + a_2\rho_1} \qquad [p.343]$$

If θ be the angle of incidence

$$\left.\begin{aligned} &\frac{V_1}{\sin\theta_1} = \frac{V_2}{\sin\theta_2} \;;\; a_1\tan\theta_1 = a_2\tan\theta_2 \\ &\frac{\rho_1}{\rho_2} = \frac{V_2^2}{V_1^2} = \frac{\sin^2\theta_2}{\sin^2\theta_1} \end{aligned}\right\} \qquad (A)$$

$$\phi_1 = A_1 e^{i(a_1 x + by + \omega t)} + B_1 e^{i(-a_1 x + by + \omega t)}$$

$$\phi_2 = A_2 e^{i(a_2 x + \cdots)} + B_2 e^{i(-a_2 x + \cdots)}$$

$$\phi_3 = A_3 e^{i(a_3 x + \cdots)}$$

(θ_1, θ_2) are the angles of incidence & refraction at $x=0$, and (θ_2, θ_1) are the angles of incidence & refraction at $x = l$. The boundary conditions give

$$\left.\begin{aligned} \rho_1(A_1+B_1) &= \rho_2(A_2+B_2) \\ a_1(A_1-B_1) &= a_2(A_2-B_2) \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} \rho_2\left(A_2 e^{ia_2 l} + B_2 e^{-ia_2 l}\right) &= \rho_1 A_3 e^{ia_1 l} \\ a_2\left(A_2 e^{ia_2 l} - B_2 e^{-ia_2 l}\right) &= a_1 A_3 e^{ia_1 l} \end{aligned}\right\} \quad (2)$$

From these, we have

$$\frac{a_2\rho_1 - a_1\rho_2}{a_2\rho_1 + a_1\rho_2} = \frac{B_2}{A_2} e^{-2ia_2 l}$$

$$e^{2a_2 l}\,\frac{a_2\rho_1 - a_1\rho_2}{a_2\rho_1 + a_1\rho_2} = \frac{(a_2\rho_1 - a_1\rho_2) + \frac{B_1}{A_1}(a_2\rho_1 + a_1\rho_2)}{(a_2\rho_1 + a_1\rho_2) + \frac{B_1}{A_1}(a_2\rho_1 - a_1\rho_2)}$$

Putting $\dfrac{a_2\rho_1 - a_1\rho_2}{a_2\rho_1 + a_1\rho_2} = K$

we have $\dfrac{B_1}{A_1} = K\,\dfrac{e^{ia_2 l} - e^{-a_2 li}}{e^{ia_2 l} - K^2 e^{ia_2 l}}$

$$\frac{V_1}{\sin\theta_1} = \frac{V_2}{\sin\theta_2} \;;\; a_1 \tan\theta_1 = a_2 \tan\theta_2$$

$$\frac{V_1 a_1}{\cos\theta_1} = \frac{V_2 a_2}{\cos\theta_2} \;;\; \frac{2\pi}{\lambda}\cos\theta_1 = a_1$$

$$a_2 = \frac{V_1\cos\theta_2}{V_2}\cdot\frac{1}{\cos\theta_1}\cdot\frac{2\pi}{\lambda}\cos\theta_1$$

$$= \frac{2\pi V_1 \cos\theta_2}{\lambda V_2}$$

$a_2 l = x$, and $K = \beta$

$$\frac{B_1}{A_1} = -\frac{2i\beta\sin x}{e^{-ia_2 l} - \beta^2 e^{ia_2 l}}$$

$$e^{ia_2 l} + e^{-ia_2 l} = 2\cos x$$

$$e^{ia_2 l} - e^{-ia_2 l} = 2i\sin x$$

$$= -\frac{2i\beta\sin x}{(\cos x - i\sin x) - \beta^2(\cos x + i\sin x)}$$

$$= \frac{2\beta\sin x}{\sqrt{\sin^2 x\,(1+\beta^2)^2 + \cos^2 x\,(1-\beta^2)^2}} \times \text{Exponential part}$$

$$= \frac{2\beta\sin x}{\sqrt{1 - 2\beta^2\cos 2x + \beta^4}} \times \text{Exponential part}$$

giving the req'd ratio of amplitudes.

(35) $\phi = Ae^{i(ax+by+\omega t)} + Be^{i(-ax+by+\omega t)}$

$\phi_1 = A_1 e^{i(a_1 x+by+\omega t)} + B_1 e^{i(-a_1 x+by+\omega t)}$

$\phi_2 = A_2 e^{i(a_2 x+by+\omega t)}$

The various conditions at the boundaries are

$$\rho(A+B) = \rho_1(A_1+B_1)$$

$$a(A-B) = a_1(A_1-B_1)$$

$$\rho_1(A_1 e^{ia_1 l} + B_1 e^{-ia_1 l}) = \rho_2 A_2 e^{ia_2 l}$$

$$a_1(A_1 e^{ia_1 l} - B_1 e^{-ia_1 l}) = a_2 A_2 e^{ia_2 l}$$

$$V/\sin\theta = V_1/\sin\theta_1 = V_2/\sin\theta_2$$

$$a\tan\theta = a_1\tan\theta_1 = a_2\tan\theta_2$$

$$\frac{\rho_1}{\rho} = \frac{V^2}{V_1^2} = \frac{\sin^2\theta}{\sin^2\theta_1}$$

$$a\tan\theta\cdot\frac{V}{\sin\theta} = a_1\tan\theta_1\cdot\frac{V_1}{\sin\theta_1}$$

$$\frac{aV}{\cos\theta} = \frac{a_1 V_1}{\cos\theta_1}$$

$$\frac{2\pi}{\lambda}\cos\theta = a, \qquad E = \frac{al}{\rho}$$

$$\left(\frac{V^2}{V_1^2} - \sin^2\theta\right) = \sin^2\theta/\tan^2\theta_1$$

$$= \sin^2\theta \Big/ \frac{a^2}{a_1^2}\tan^2\theta = \frac{a_1^2}{a^2}\cos^2\theta = a_1^2\lambda^2/4\pi^2$$

$$\left.\begin{aligned} a_1 &= \frac{2\pi}{\lambda}\left(\frac{V^2}{V_1^2} - \sin^2\theta\right)^{1/2} \\ a_2 &= \frac{2\pi}{\lambda}\left(\frac{V^2}{V_2^2} - \sin^2\theta\right)^{1/2} \end{aligned}\right\} \quad \therefore \begin{aligned} E_1\rho_1 &= a_1 l \\ E_2\rho_2 &= a_2 l \\ E\rho &= al \end{aligned}$$

$$\frac{\rho(A+B)}{a(A-B)} = \frac{\rho_1}{a_1}\frac{(A_1+B_1)}{(A_1-B_1)}$$

$$A_1+B_1 = \frac{\rho}{\rho_1}(A+B)$$

$$A_1-B_1 = \frac{a}{a_1}(A-B)$$

$$\tfrac{1}{2}\rho_1\left[e^{ia_1 l}\left\{\frac{\rho}{\rho_1}(A+B) + \frac{a}{a_1}(A-B)\right\} + e^{-ia_1 l}\left\{\frac{\rho}{\rho_1}(A+B) - \frac{a}{a_1}(A-B)\right\}\right]$$

$$= \rho_2 A_2 e^{i a_2 l}$$

& finally

$$A_2 = A \cdot \frac{2 e^{-i a_2 l}}{\left(\frac{a_2}{a_1}\frac{\rho}{\rho_1} + \frac{\rho_2 a}{\rho_1 a}\right)\cos a_1 l - i\left(\frac{\rho\rho_2}{\rho_1^2} + \frac{a_2 a}{a_1 a}\right)\sin a_1 l}$$

If the change of phase be zero,

$$\frac{\tan^{-1}\left(\frac{\rho\rho_2}{\rho_1^2} + \frac{a_2 a}{a_1 a}\right)\tan a_1 l}{\frac{\rho_2 a}{\rho_1 a} + \frac{a_2 \rho}{a_1 \rho_1}} = a_2 l$$

$$\frac{\frac{a l}{\rho}\cdot\frac{a_2 l}{\rho_2} + \frac{a_1^2 l^2}{\rho_1^2}}{\frac{a_1 l_1}{\rho_1}\left(\frac{a l}{\rho} + \frac{a_2 l}{\rho_2}\right)} = \frac{E E_2 + E_1^2}{E_1(E+E_2)}$$

$$= \frac{\frac{a a_2}{\rho\rho_2} + \frac{a_1^2}{\rho_1^2}}{\frac{a_1}{\rho_1}\left(\frac{a}{\rho} + \frac{a_2}{\rho_2}\right)}$$

$$\frac{\tan a_2 l}{\tan a_1 l} = \frac{\frac{\rho\rho_2}{\rho_1^2} + \frac{a_2 a}{a_1 a}}{\frac{\rho_2 V}{\rho_1 V_1} + \frac{a_2 \rho}{a_1 \rho_1}} = \frac{E E_2 + E_1^2}{E_1(E+E_2)}$$

$$\therefore \text{if } \frac{\tan a_2 l}{\tan a_1 l} = \frac{\tan E_2 \rho_2}{\tan E_1 \rho_1} = \frac{E E_2 + E_1^2}{E_1(E+E_2)}$$

the phase of the transmitted wave will be unaltered.

(36) If A, A', A_1 be the amplitudes of the incident, reflected & transmitted waves

$$\frac{A}{a\rho_1 + a_1\rho} = \frac{A'}{a\rho_1 - a_1\rho} = \frac{A_1}{2a\rho} = K \text{ (say)}$$

$$A = K(a\rho_1 + a_1\rho) = ka\left(\rho_1 + \rho\frac{\tan\delta}{\tan\alpha_1}\right) = \frac{ka}{\cot\alpha\cot\alpha_1}\left(\rho_1\cot\alpha + \rho\cot\alpha_1\right)$$

$$A_1 = 2a\rho K$$

$$\text{Energy transmitted} = \tfrac{1}{2}\rho_1 c_1 \left(\frac{2\pi}{\lambda_1}\right)^2 \cdot A_1^2$$

$$= \frac{1}{2}\frac{\rho_1}{c_1}\left(\frac{2\pi}{T}\right)^2 A^2$$

$$\text{Required ratio} = \frac{\rho_1/c_1 \cdot A_1^2 \cos\alpha_1}{\frac{\rho}{c} \cdot A^2 \cos\alpha}$$

$$= \frac{\sin\alpha}{\sin\alpha_1} \cdot \frac{\cos\alpha_1}{\cos\alpha} \cdot \frac{\rho_1}{\rho} \cdot \frac{4a^2\rho^2 k^2 \cot^2\alpha \cot^2\alpha_1}{k^2a^2\left[\rho_1 \cot\alpha + \rho \cot\alpha_1\right]^2}$$

$$= \frac{\tan\alpha \tan\alpha_1 \cdot 4\rho\rho_1 \cdot \cot^2\alpha \cot^2\alpha_1}{\left(\rho_1 \cot\alpha + \rho \cot\alpha_1\right)^2}$$

$$= \frac{4\rho\rho_1 \cot\alpha \cot\alpha_1}{\left(\rho_1 \cot\alpha + \rho \cot\alpha_1\right)^2}.$$

(37) If $\phi = Ae^{i(bx+ay+\omega t)}$

$\phi' = A'e^{i(bx-ay+\omega t)}$

$\phi_1 = A_1 e^{i(bx+a_1y+\omega t)}$

and $\zeta = Be^{i(bx+\omega t)}$

gives the displacement of the membrane

$$-\omega B = a(A_0 - A') = a_1 A_1$$

$$\frac{A_0}{a\rho_1 + a_1\rho - iaa_1(Tn^2-\sigma)} = \frac{A'}{a\rho_1 - a_1\rho - iaa_1(Tn^2-\sigma)} = \frac{A_1}{2a\rho}$$

Here $\zeta = A\cos mx \sin pt$

$\therefore b = m, \ \omega = p$

& $A = Bi \qquad \therefore A_1 = -\frac{pB}{a_1}$

$$\therefore A_0 = \frac{Ap}{2aa_1\rho}\left\{aa_1\left(\frac{Tm^2}{p^2}-\sigma\right) + i\left(a\rho_1 + a_1\rho\right)\right\}$$

$p^2 = v^2(a^2+m^2), \quad p^2 = v_1^2(a_1^2+m^2)$

$\frac{p^2}{v^2} = m^2 + a^2, \qquad a^2 = -n^2$

$\frac{p^2}{v_1^2} = m^2 + a_1^2, \qquad a_1^2 = -n'^2$

$$A_0 = \frac{Ap}{2\rho}\left[\frac{Tm^2}{p^2} - \sigma + i\left(\frac{\rho_1}{a_1} + \frac{\rho}{a}\right)\right]$$

$$A' = \frac{Ap}{2\rho}\left[\frac{Tm^2}{p^2} - \sigma + i\left(\frac{\rho_1}{a_1} - \frac{\rho}{a}\right)\right]$$

$$\phi = A_0 e^{iay}\left[i\sin mx \cos pt\right]$$

The form $\zeta = A \cos mx \sin pt$ shows that the velocity potential must be the real part of $Ae^{imx} \sin pt$

$$\phi = Be^{ky} \sin mx \cos pt$$

$$\phi' = Ce^{k'y} \sin mx \cos pt$$

Since $\frac{\partial^2 \phi}{\partial t^2} = V^2 \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)$

we have $-p^2 = V^2 (-m^2 + k^2)$

$$\therefore \frac{p^2}{V^2} = m^2 - k^2 = m^2 - n^2 \qquad \therefore k = \pm n$$

$$\text{Simly } \frac{p^2}{V'^2} = m^2 - k'^2 = m^2 - n'^2 \qquad \therefore k' = \pm n'$$

From the continuity of the normal velocity

$$-\dot{\zeta} = \frac{\partial \phi}{\partial y} = \frac{\partial \phi'}{\partial y} \text{ at } y = 0$$

$$\therefore -Ap = B_1 k = C_1 k'$$

The eqn of motion of the membrane is

$$\sigma \ddot{\zeta} = T \frac{\partial^2 \zeta}{\partial x^2} + \delta p' - \delta p \qquad \text{when } y = 0$$

$$\sigma p' = \rho' \frac{\partial \phi'}{\partial t}, \quad \delta p = \rho \frac{\partial \phi}{\partial t}$$

$$\therefore \sigma p^2 A = T(-Am^2) + \rho'(-C_1 p) - \rho(-B_1 p)$$

$$\therefore A \left(\frac{Tm^2}{p^2} - \sigma \right) = \frac{1}{p} (B_1 \rho - C_1 \rho')$$

$$A \left(\frac{\rho}{n} + \frac{\rho'}{n'} \right) = \frac{1}{p} (B_1 \rho - C_1 \rho')$$

$$= \left(-\frac{A}{k} \rho + \frac{A}{k'} \rho' \right)$$

$$= A \left(\frac{\rho'}{k'} - \frac{\rho}{k} \right) = A \left(\frac{\rho'}{n} + \frac{\rho}{n} \right)$$

$$\therefore k' = n', \quad k = -n$$

Again $A \left(\frac{\rho}{n} + \frac{\rho'}{n'} \right) = \left(\frac{B_1 \rho}{p} + \frac{A \rho'}{n'} \right) \qquad \therefore B_1 = \frac{Ap}{n}$

$$\therefore \phi = -B_1 e^{-ny} \sin mx \cos pt = -Ap n^{-1} e^{-ny} \sin mx \cos pt$$

$$C_1' = B_1 p / k' = -B_1 n / n' = -\frac{Ap}{n} \cdot \frac{n}{n'} = -Ap / n'$$

$\therefore \phi' = A p n^{-1} e^{ny} \sin mx \cos pt.$

(38) Take the rigid wall as the origin. Let

$$\xi_1 = A_1 e^{i(m_1 x + nt)} + B_1 e^{i(-mx + nt)}$$

$$\xi_2 = A_2 e^{i(m_2 x + nt)} + B_2 e^{i(-m_2 x + nt)}.$$

$\xi_1 = \xi_2$ at $x = c$

$$\therefore A_1 e^{im_1 c} + B_1 e^{-im_1 c} = A_2 e^{im_2 c} + B_2 e^{-im_2 c}$$

$\frac{\partial \xi_1}{\partial t} = 0$ at $x = 0$ $\quad \therefore A_1 + B_1 = 0$

Since $\delta p_1 = \delta p_2$

$$\rho_1 c_1^2 m_1 \left(A_1 e^{im_1 c} - B_1 e^{-im_1 c} \right) = \rho_2 c_2^2 m_2 \left(A_2 e^{im_2 c} - B_2 e^{-im_2 c} \right)$$

but $\rho_1 c_1^2 = \rho_2 c_2^2$

$$m_2 \left(A_2 e^{im_2 c} - B_2 e^{-im_2 c} \right) = m_1 \left(A_1 e^{im_1 c} - B_1 e^{-im_1 c} \right)$$

$$\therefore A_2 e^{im_2 c} - B_2 e^{-im_2 c} = 2A \frac{m_1}{m_2} \cos m_1 c$$

$$A_2 e^{im_2 c} + B_2 e^{-im_2 c} = 2i A_1 \sin m_1 c$$

$$\therefore A_2 e^{im_2 c} = 2A_1 \left[\frac{m_1}{m_2} \cos m_1 c + i \sin m_1 c \right]$$

$$B_2 e^{-im_2 c} = -2A_1 \left[\frac{m_1}{m_2} \cos m_1 c - i \sin m_1 c \right]$$

$$\therefore \xi_1 = \frac{A_2 e^{im_2 c}}{2\left(\frac{m_1}{m_2} \cos m_1 c + i \sin m_1 c \right)} \left[e^{i(m_1 x + nt)} - e^{i(-m_1 x + nt)} \right]$$

Let $\frac{m_1}{m_2} \cos m_1 c = \rho \cos \phi$; $\sin m_1 c = \rho \sin \phi$

$$\therefore \xi_1 = \frac{A_2}{\rho} \sin m_1 x \sin (nt + m_2 c - \phi)$$

Let $m_2 c - \phi = -(\pi/2 - \alpha)$ & $\frac{A_2}{\rho} = A \operatorname{cosec}(m_1 c)$

Then $\xi_1 = A \sin m_1 x \cos (pt + \alpha) \operatorname{cosec} m_1 c.$

Again, $B_2 = -A_2 e^{2im_2 c} \left[\frac{\frac{m_1}{m_2} \cos m_1 c - i \sin m_1 c}{\text{"} \quad + \quad \text{"}} \right]$

$$\therefore \xi_2 = 2A_2 \cos(pt+\alpha)\cos(m_2 x - \alpha)$$

$$= 2Ap \operatorname{cosec} m_1 c \cos(pt+\alpha)\cos(m_2 x-\alpha)$$

$$\cos(m_2 c - \alpha) = \operatorname{cosec} cm_2 \left[\frac{\sin m_2 c}{\rho} \cos cm_2 - \sin cm_2 \frac{m_1}{m_2} \cos m_1 c \right]$$

$$+ \sec m_2 c \left[\frac{m_1}{m_2} \cdot \frac{\cos m_1 c}{\rho} \cos cm_2 + \frac{\sec cm_1}{\rho} \sec cm_2 \right]$$

$$= \frac{\sin m_1 c}{\rho}\left(\cos^2 cm_2 + \sin^2 cm_2\right) = \frac{\sin m_2 c}{\rho}$$

Hence $\xi_2 = 2A\cos(pt+\alpha)\cos(m_2 c - \alpha)\sec(cm_2 - \alpha)$

But $m_1 = \frac{n}{a_1}$, $m_2 = n/a_2$

$$\therefore \xi_1 = A\cos(nt+\alpha)\sin\frac{nx}{a}\operatorname{cosec}\frac{nc}{a_1}$$

$$\xi_2 = A\cos(nt+\alpha)\cos\left(\frac{nx}{a_2}-\alpha\right)\sec\left(\frac{nc}{a_2}-\alpha\right).$$

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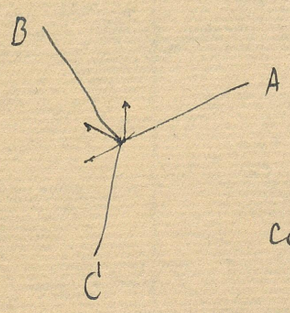
(39) See similar example in Basset, p. 143, Ex. 4



$$y_1 = \left(A_1 \cos\frac{nx}{a_1} + B_1 \sin\frac{nx}{a_1}\right) e^{int}$$

$$y_1 = 0,\ x = l_1,\quad A_1\cos\frac{nl_1}{a_1} + B_1\sin\frac{nl_1}{a_1} = 0$$

$$\cot\frac{nl_1}{a_1} = -\frac{B_1}{A_1};\quad \cot\frac{nl_2}{a_2} = -\frac{B_2}{A_2};\quad \cot\frac{nl_3}{a_3} = -\frac{B_3}{A_3}$$

$$\frac{2\pi}{n} = T,\quad n = \frac{2\pi}{T}$$

When O & A are fixed

$$y = \left(C\cos\frac{nx}{a_1} + D\sin\frac{nx}{a_1}\right)e^{int}$$

$$\sin\frac{nl_1}{a_1} = 0$$

$$\frac{nl_1}{a_1} = s\pi = \pi \text{ for the gravest mode.}$$

$$T_1 = \frac{2\pi}{n} = \frac{2\pi l_1}{\pi a_1} = \frac{2l_1}{a_1}$$

$$\cot\frac{nl_1}{a_1} = \cot\frac{2\pi l_1}{Ta_1} = \cot\frac{\pi T_1}{T}$$

$$\cot\frac{\pi T_1}{T} = -\frac{B_1}{A_1};\quad \cot\frac{\pi T_2}{T} = -\frac{B_2}{A_2};\quad \cot\frac{\pi T_3}{T} = -\frac{B_3}{A_3}$$

Displacements of the three strings at O regarded as vector quantities

should be such that their resultant is zero.

$$\frac{A_1}{\sin\alpha} = \frac{A_2}{\sin\beta} = \frac{A_3}{\sin\gamma} = \lambda$$

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{Bn}{a} e^{int}$$

$$\frac{B_1}{a_1} + \frac{B_2}{a_2} + \frac{B_3}{a_3} = 0$$

$$\lambda \sin\alpha \cos\frac{\pi T_1}{T} = -B_1, \text{ etc}$$

If P is the tension of a string

$$P_1 = \rho a^2, \text{ but } \frac{P_1}{\sin\alpha} = \frac{P_2}{\sin\beta} = \frac{P_3}{\sin\gamma}$$

$$\therefore \frac{a_1^2}{\sin\alpha} = \frac{a_2^2}{\sin\beta} = \frac{a_3^2}{\sin\gamma} \quad \text{since } \rho \text{ is constant}$$

$$\therefore \sqrt{\sin\alpha} \cos\frac{\pi T_1}{T} + \sqrt{\sin\beta} \cos\frac{\pi T_2}{T} + \sqrt{\sin\gamma} \cos\frac{\pi T_3}{T} = 0$$

——— 0 ———