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Sept. 5, 1975

IN REPLY Group T-8
REFER TO:
MAIL STOP: 453

Dr. B. S. Madhavarao
Center for Theoretical Studies
Indian Institute of Science
Bangalore 560012
INDIA

Dear Dr. Madhavarao,

Thank you for your letter of 28 July, which I am finally getting around to answering. Under separate cover I am sending to you I, II, and III, of our series.

As to the question of proper recognition of your work, that is a question which I always try to be sensitive to, and these "Bhabha" fields turn out to be a problem.

First let me state that what we were trying to say about the special cases was the following: In your 1942 paper you indeed worked out the special 3/2 and 2 cases. However, as we read the literature, (and thank you for the J. Mysore Univ. preprints which we had not seen), it was only after Bhabha's Current Sci Science 1945 paper that you printed the result the so(5) equations imply a multi-mass solution for the case max spin greater than 3/2. That was why we have been giving credit to Bhabha, and things are further complicated still. In 1942 Lubánski wrote two papers on the so(5) system in which he did the algebra and and found the spin and multi-mass states. In that sense, Lubánski was before everyone. 3/2 and 2

To add to the confusion, once having started the series on "Bhabha" fields, I, II, III, etc., I it is my impression that P.R. wants to give the same titles.

Your letter reached me just as IV was being completed. What we have done in IV and in V, which is being written, is to write at the very beginning of the articles references and names and a discussion of the history as it appears in print, stating that credit should be given to all three of you.

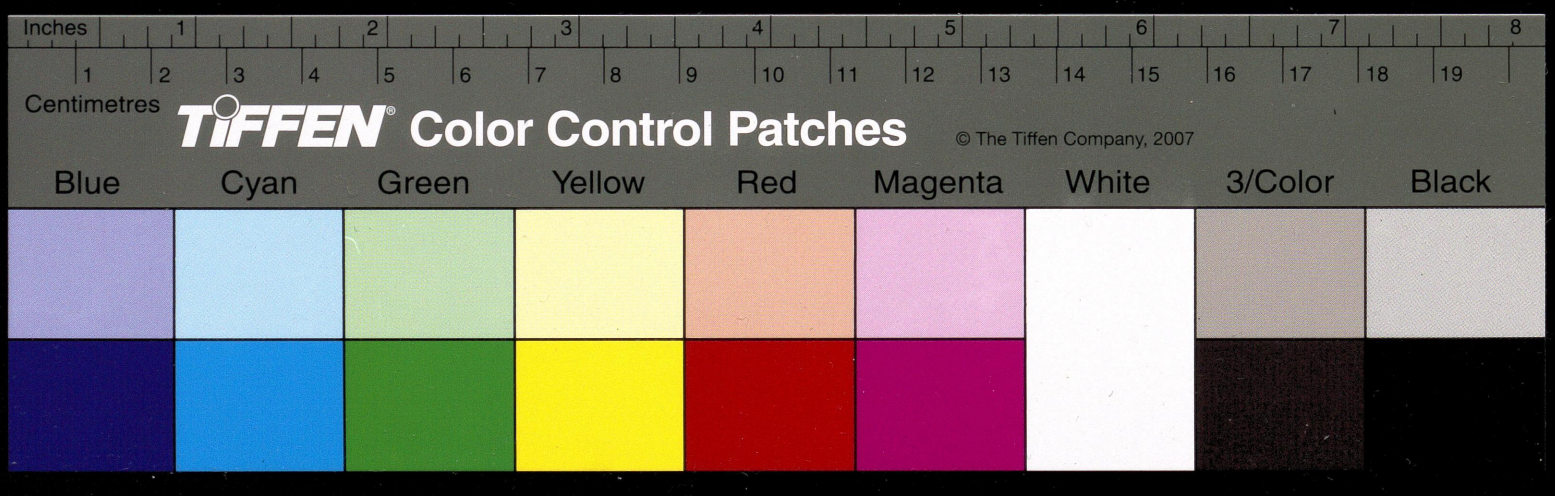
I hope that this helps to give you proper credit for your pioneering work.

Sincerely yours,

Michael Martin Nieto

cc: R. A. Krajcik

AN EQUAL OPPORTUNITY EMPLOYER



With thanks for your helpful comments

# Historical development of the Bhabha first-order relativistic wave equations for arbitrary spin

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From letters, personal reminiscences, and the published works of the authors involved, we describe the historical development of the Bhabha system of first-order relativistic wave equations for arbitrary spin. The simplest special cases of these equations reduce to the Dirac and DKP (Duffin–Kemmer–Petiau) equations for spin-1/2 and spins 0 and 1, respectively.

## I. INTRODUCTION

Elsewhere, in a detailed series of seven papers,[1–7] we have reinvestigated, from a modern point of view, the Bhabha system of arbitrary-spin, relativistic wave equations. These equations are of the form

$$(ip_\mu\alpha_\mu + \chi)\psi = 0, \tag{1.1}$$

where the $\alpha_\mu$ are given by representations of the SO(5) algebra

$$\alpha_\mu = J_{\mu 5}, \quad J_{\mu\nu} = -i[\alpha_\mu,\alpha_\nu], \quad J_{55} = 0, \quad (\mu,\nu) = 1,2,3,4, \tag{1.2a}$$

$$[J_{ab},J_{cd}] = i(\delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{bc}J_{ad} - \delta_{ad}J_{bc}), \quad (a,b) = 1,2,3,4,5. \tag{1.2b}$$

The representations are labeled by $(\mathcal{S},S)$, where $\mathcal{S} \geq S$ are both integers or half integers. These representations are of dimension

$$d_5(\mathcal{S},S) = (1/6)(2\mathcal{S}+3)(2S+1) \times [(\mathcal{S}+1)(S+2) - S(S+1)]. \tag{1.3}$$

Among themselves the $\alpha_\mu$ satisfy an algebra of order $(2\mathcal{S}+1)$.

The simplest nontrivial SO(5) representation of Eq. (1.2) is the $(1/2, 1/2)$ representation and yields the four-dimensional relativistic Dirac equation for spin-1/2 particles. For the five- and ten-dimensional SO(5) representations (1,0) and (1,1), Eq. (1.1) is the Duffin–Kemmer–Petiau (DKP) equation for particles of spin-0 and spin-1, respectively. For $\mathcal{S} > 1$ the equations represent multispin and multimass states of maximum spin $\mathcal{S}$. $d_5$ is then a linear sum of quantities $2(2s+1)$ for each mass state of spin $s$, with additional built-in subsidiary components for integer spin.

Subsidiary components ensue for integer spin because the generators $J_{ab}$ have representations which are (or can be rotated into) angular momentum matrices. Thus, they have eigenvalues from $-\mathcal{S}$ to $+\mathcal{S}$. In particular, this means there are "zero" eigenvalues for integer-spin representations. Since the mass states are given by $\chi/j$, where $j$ is a particular eigenvalue, for $j = 0$ one has "$\infty$-mass" states, or "subsidiary components." For example, the spin-0 and spin-1 DKP matrices mentioned above have three and four subsidiary components, respectively. However, as discussed in Sec. II and Ref. 2, these "subsidiary components" can be projected out. The remaining components contain all that is necessary to describe the integer-spin particles. This is discussed in detail in Ref. 2.

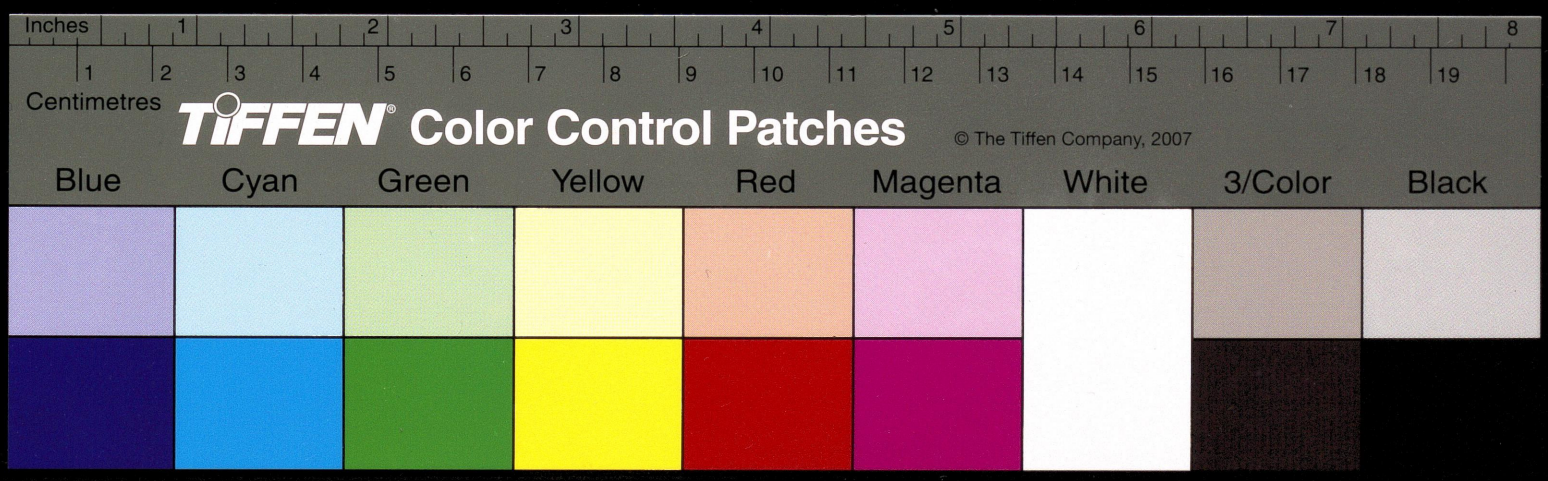
Equations (1.1) and (1.2) were first investigated by Lubański, Madhavarao, and Bhabha, and have come to be known as "Bhabha" wave equations.[8] The structure and properties of these equations can be obtained from our series.[1–7] In particular, for those who wish to have a quick review of the results drawn from our study and other work, paper VII gives a summary and set of conclusions.[7]

As is well known, for spin $s > 1$, no totally consistent, relativistic, massive, *interacting* wave equations have yet been discovered. For example, although the Bhabha system is causal with minimal electromagnetic interaction,[4] except for spin-1/2 (and spin-0 and spin-1 with the interpretative ansatz of a charge probability density), the system has an indefinite metric.[4,5] Using the usual quantum interpretation, this implies negative probabilities. (We have also investigated aspects of indefinite-metric spaces in another series.[9,10]) Nonetheless, since one does see high-spin particles in nature, the thus far unsuccessful search for a consistent high-spin quantum theory is of at least historical interest.

The reader may wonder, however, why one does not avoid the indefinite metric by using the Rarita–Schwinger or Bargmann–Wigner forms.[11] The advantage of these equations is that one has single-mass, single-spin, positive-definite metric states. But the price one pays is profound. To limit the number of components in these theories to $2(2s+1)$, one must introduce *external* (versus "built-in") subsidiary conditions, which amounts to introducing extra derivatives into the theory.[11] By the very nature of the partial differential equations which ensue, even for *c-number* minimal electromagnetic coupling, these extra derivatives imply that the solutions are *noncausal*. This result, known as the Velo–Zwanziger phenomena,[12] is discussed in more detail in Ref. 4.

To summarize, then, this paper is devoted to a historical résumé of the discovery and development of the Bhabha first-order wave equations; that development involves many of the well-known physicists of the first quarter century of quantum mechanics.[13] As we have noted,[8] by no means are


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these the only first-order wave equations which have been studied. However, as the correspondence, conversations, and publications of many of the authors involved shows, the interactions of the people working with these equations crucially affected the road actually taken in reaching our current understanding of relativistic wave equations.

## II. HISTORICAL RESUME

By the mid 1930's the success of the Dirac equation[14] in relativistically describing particles of spin-1/2 gave certain people the desire to find a similar first-order wave equation for spin-0 particles (then not yet discovered) and spin-1 particles. They felt such a formalism might be preferable to the already accepted "relativistic second-order Schrödinger equation," or Klein–Gordon (KG) equation,[15] as it is now known, for spin-0 particles and the second-order Proca equation[16] for the spin-1 particles.

The first major effort in this direction was initiated by de Broglie,[17] who for 40 years now[17,18] has hoped that the photon has a nonzero rest mass. De Broglie based his original investigations of first-order wave equations on $16 \times 16$ matrices that were products of different Dirac $\gamma$ matrix spaces. He hoped he could combine two leptons to yield massive photons. One of his students, Petiau, looked at a modification of de Broglie's algebra, becoming the first to obtain the $16 \times 16$ DKP algebra,[19]

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \beta_\mu\delta_{\nu\lambda} + \beta_\lambda\delta_{\mu\nu}. \tag{2.1}$$

(Géhéniau later decomposed the algebra into the five, ten, and trivial one-dimensional representations.[20] Also, elsewhere, in work with its own interesting history,[21] Géhéniau anticipated some of the specific mathematical results of our Ref. 4.) However, Petiau's 1936 work remained unknown to most workers[22]; and in particular was not heard of by Kemmer until after World War II.[23]

It was the work of Proca which inspired Kemmer and Duffin. Kemmer realized[23] that the Proca equations could be described as a set of coupled first-order equations, and he wrote them down along with the analogous spin-0 case.[24] He also realized, therefore, that the above could be written in explicit $5 \times 5$ and $10 \times 10$ matrix forms for spin-0 and spin-1, but at first he[23] did not have "any knowledge of the commutation rules satisfied by them (the matrices)."

Soon after, Duffin, a mathematician, attended[25] a physics seminar at Purdue where the aforementioned Ref. 24 by Kemmer on the Proca equations was discussed. When Duffin asked how one knew the theory was spin-1, a complicated answer was given which was esthetically unpleasing to Duffin. Since Duffin too preferred[25] a first-order Dirac approach, he looked into the matter, with the result that he was quickly able to put the spin-0 and spin-1 equations in a first-order $\beta$-matrix formulation, with three of the four constituent commutation relations represented in Eq. (2.1) (not for $\lambda = \mu \neq \nu$).

When Kemmer saw the note that Duffin published on his results,[26] he wrote to Duffin[25] saying he knew how to extend the theory but would first wait for Duffin to publish anything further. However, Duffin was by then involved in a collaboration with A. C. Schaeffer on function theory, and so wrote[25] to Kemmer to go ahead. Kemmer[23] then "quickly put together" all he had been doing, and produced his classic 1939 paper.[27] He[23] "sent it (a preprint) to Pauli, who typically, answered by telling me how to proceed." (See the footnotes on p. 102 and p. 104 of Ref. 27.) The fact that at this time it was already becoming clear that the primary components of the low-energy nuclear force are due to pseudoscalar and vector interactions,[28] excited many people about the DKP formalism which contained them both.

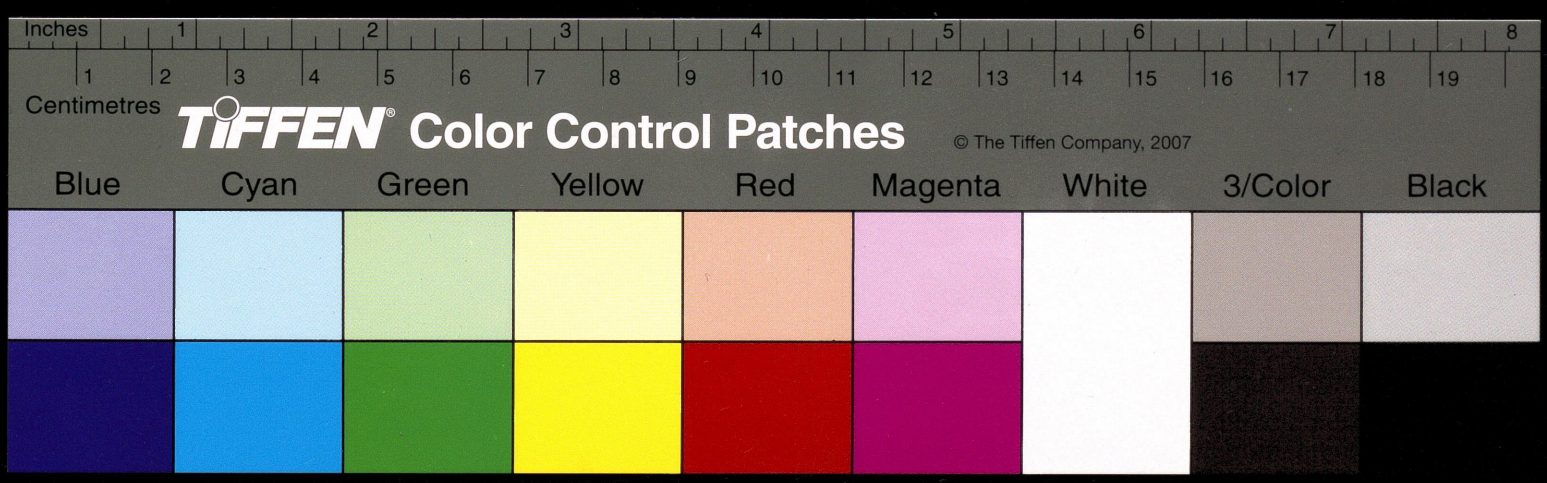
Two other important contributions were made in this period. The first was another version of the coupled equation results above, the work of Belinfante[29,30] on "undors." Undors are products (so as to include spatial reflections) of Dirac wave functions, and are generalizations "of Dirac wave functions in the same sense as tensors form a generalization of vectors." [29] In addition, a separate investigation into the first-order Dirac-like DKP equation had been undertaken by Sakata and Taketani (ST).[31,32] However, their investigation was delayed because Taketani was imprisoned.[33] By the time that Taketani was released, they had seen the work of Duffin and Kemmer, and so they concentrated on what we have called the ST decomposition[2]; removing the built-in "subsidiary components" from the physical "particle components" for integer-spin fields. In our series[1–7] we found that the generalization of this procedure is always a necessary extra step to properly handle integer-spin Bhabha fields.

Note that with the removing of the "subsidiary components," the ST "particle components" equations for spin-0 and spin-1 have two and six components, respectively; i.e., $2(2s + 1)$ components. [From another viewpoint, by looking at linear combinations of a second-order field and its time derivative, Tamm, and later Feshbach and Villars (FV), obtained $2(2s + 1)$ component matrix equations of the same form.[34] The FV work for spin-0 is the most widely known of all these works, and at the end of Ref. 34 we refer to a discussion of the relation of the ST and FV formulations.]

With the above basis, the work of Lubański, Madhavarao, and Bhabha was produced, all three of them deserving credit for their individual efforts. The first of them to publish was Lubański. His earliest work[35–37] began[38] in 1940 as a collaboration which involved Belinfante,[36] Kramers,[36] and Rosenfeld.[37] In 1942 his main work appeared in two papers[39] where he proposed in detail[40] the SO(5) algebra for the first-order wave equations and recognized the type of spin and mass spectrum which would be obtained.[41] However, his papers were published in the occupied Netherlands. This, and his untimely death due to carbon monoxide poisoning from a not properly ventilated room,[38] led to Lubański's papers not receiving the recognition they deserved.

Also in 1942, Madhavarao[42] began looking at first-order wave equations, having been inspired by Duffin's and Kemmer's work. At first he concentrated on what turned out to be the specific commutation relations of the $\alpha_\mu$ for the algebras $\mathcal{S} = 3/2$ and $\mathcal{S} = 2$. He then tried to obtain restrictions on the general algebra which would yield a single-mass KG equation,[43] a quest which was the starting point for the later work of Harish–Chandra.[44–48] [However, if one remains in the SO(5) algebra, such a single-mass wave equation is only possible for the Dirac and DKP cases, as we now know.] Madhavarao communicated with Bhabha on this question,[49] and later observed[50] that Bhabha, in a 1945 preliminary version[51] of his later major work,[52] had resolved the physical question by accepting that for high spin the SO(5) equations satisfy a multimass KG equation of the form





$$\prod_{j=1/2}^{s} (\Box - \chi^2/j^2)\psi = 0, \quad \text{half-integer spin,} \tag{2.2a}$$

$$(\partial \cdot \alpha) \prod_{j=1}^{s} (\Box - \chi^2/j^2)\psi = 0, \quad \text{integer spin,} \tag{2.2b}$$

$\pm\chi/j$ being the particle and antiparticle mass states. Madhavarao[53,54] and coworkers[53] went on to investigate further the algebraic properties of the SO(5) algebra. As the reader who wishes to follow our series[1–7] closely can observe, he made major contributions there.

The milieu in which Bhabha found himself makes part of the origin of his interest in first-order wave equations clear. He was a friend of Kemmer. In fact: "The decision to use the word meson (for the Yukawa particles) was made in a certain private house in Cambridge (that of Egon Bretscher[55]) by Maurice Price, Homi Bhabha, and myself (Kemmer)." [56] In the years 1945–1949, Bhabha published a number of papers in which the properties of these SO(5) equations were investigated.[51,52,57–59]

Over the succeeding years a great number of papers dealing with the DKP and general Bhabha wave equations were published.[13] However, after the early 1950's physical interest in them waned because[60] it was ultimately believed that the KG and the DKP equations are equivalent. The reason was that in the five-dimensional spin-0 representation of the DKP equation, the fifth component ($\psi_5$) behaves like the KG field and the first four components behave like the four-derivatives of $\psi_5$. So the two equations yielded the same energy levels for the $\pi$ mesic atom, and the same results for processes like $\pi - e$ scattering, $e^+e^- \to \pi^+\pi^-$, and, in fact, everywhere one checked,[44,61–67] especially for quantum electrodynamics.[62,63,65,67] Thus, the "complicated" five-component spinor equation was abandoned in favor of the simple one-component (albeit second-order) equation which was "easier" to handle and yielded the same results.

The above viewpoint was reached during the time when people only conceived of one meson ($\pi$), and for a long time no one ever examined if in symmetry-breaking situations physical quantities would remain identical when calculated with the KG and DKP equations. That is to say, the "equivalence" of the KG and DKP systems was proven with the mass of the incoming meson equal to the mass of the outgoing mesons (such as in Q.E.D.). When mesons of different mass were discovered (like the $K$), it was necessary to see if the equivalence proofs done for equal initial and final masses could be generalized to the case of unequal masses, such as for $K^0 \to \pi^+ e^- \bar{\nu}$.

In 1970, a collaboration began investigating this question[8,60,68–78] and found that the "equivalence" of the DKP and KG equations in general only holds[79] in the symmetry case. (Of course there have also been other investigations into hadronic processes.[80,81]) The basic point is, no matter what formalism one uses, using it amounts to an assumption as to how much of a vertex function is to be taken in the symmetry limit [say SU(3)] and how much is to be taken in the physical limit (on mass shell wave functions). Physically, one can reason that a first-order field with dimensions $(\text{mass})^{3/2}$, will extrapolate to a new mass differently than a KG field with $(\text{mass})^1$ dimensions.

This, plus an interest in the relativistic properties of wave equations,[82–85] eventually led to our series[1–7] on the Bhabha equations.[86] Since, as discussed in part IV,[4] as of yet there is no high-spin field equation which is devoid of all problems, and since the DKP work tells us that the Bhabha first-order system could in some cases yield different results than a second-order formalism, we undertook our series[1–7] to see what the Bhabha first-order wave equations could reveal about the physics of of high-spin fundamental fields and particles.

## ACKNOWLEDGMENT

Work supported by the United States Energy Research and Development Administration.

[1] R. A. Krajcik and M. M. Nieto, Phys. Rev. D **10,** 4049 (1974). I of our Bhabha series on $C$, $P$, and $T$.

[2] R. A. Krajcik and M. M. Nieto, Phys. Rev. D **11,** 1442 (1975). II of our Bhabha series on the mass and spin composition, the Hamiltonians, and the general Sakata–Taketani reductions of Bhabha fields. Also see Tables I and II of Ref. 7 on the mass–spin spectrum.

[3] R. A. Krajcik and M. M. Nieto, Phys. Rev. D **11,** 1459 (1975). III of our Bhabha series on the Poincaré generators.

[4] R. A. Krajcik and M. M. Nieto, Phys. Rev. D **13,** 924 (1976). IV of our Bhabha series, on causality with minimal electromagnetic interaction.

[5] R. A. Krajcik and M. M. Nieto, Phys. Rev. D **14,** 418 (1976). V of our Bhabha series, on the indefinite metric and Foldy–Wouthuysen transformations.

[6] R. J. Krajcik and M. M. Nieto, Phys. Rev. D **15,** 433 (1977). VI of our Bhabha series, on exact, closed-form Foldy–Wouthuysen transformations and solutions.

[7] R. A. Krajcik and M. M. Nieto, Phys. Rev. D **15,** 455 (1977). VII of our Bhabha series, giving a summary and our conclusions.

[8] By "Bhabha" first-order wave equations we specifically mean when the $\alpha_\mu$ satisfy Eq. (1.2). The terminology sometimes is used to designate all first-order wave equations of the type (1.1). Also, E. Fischbach, J. D. Louck, M. M. Nieto, and C. K. Scott, J. Math. Phys. **15,** 60 (1974), give a discussion of SO($n$) algebras, with specific emphasis on the DKP ring.

[9] R. A. Krajcik and M. M. Nieto, Phys. Rev. D **13,** 2245 (1976). I of a series on Foldy–Wouthuysen (FW) transformations in an indefinite-metric space.

[10] R. A. Krajcik and M. M. Nieto, Phys. Rev. D **13,** 2250 (1976). II of our series on FW transformations in an indefinite-metric space. R. A. Krajcik and M. M. Nieto, Phys. Rev. D **15,** 416 (1977). III of our series on FW transformations in an indefinite-metric space. R. A. Krajcik and M. M. Nieto, Phys. Rev. D **15,** 426 (1977). IV and conclusion of our series on FW transformations in an indefinite-metric space.

[11] W. Rarita and J. Schwinger, Phys. Rev. **60,** 61 (1941); V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U.S.A. **34,** 211 (1948); also, see C. L. Critchfield, J. Math. Phys. **8,** 1037 (1967). It has been shown that the free RS or BW formulations are equivalent to taking the Bhabha representations $(n + 1/2, 1/2)$ or $(n/2, n/2)$, respectively, and adding particular subsidiary conditions. See M. Kõiv, K. Loide, and I. Meitre, Tallinn (Estonia) Polytechnic Inst. Trans., Ser. A, No. 289 (1970); R. K. Loide, Inst. Phys. Astron., Estonian SSR Acad. Sci., Tartu, Report FAI-8 (1971); M. Kõiv and R. Saar, Inst. Phys. Estonian SSR Acad. Sci., Tartu, Report FI-33 (1974). Also compare to Tables I and II of Ref. 7 and the diagrams in Ref. 39 below.

[12] G. Velo and D. Zwanziger, Phys. Rev. **186,** 1337 (1969); *ibid.* **188,** 2218 (1969).

[13] References 1–7, especially 4 and 5, list pertinent works on the Bhabha equations, References 9 and 10, especially 9, list pertinent works on general indefinite-metric spaces. References 8 and 60 and 68–78 below list pertinent works on the Duffin–Kemmer–Petiau spin-0 and spin-1 first-order wave equations, they being special cases of the Bhabha system.



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[14]P. A. M. Dirac, Proc. R. Soc. A **117,** 610 (1928); A **118,** 341 (1928); *The Principles of Quantum Mechanics,* 4th ed. (Claredon, Oxford, 1958).
[15]O. Klein, Z. Phys. **37,** 895 (1926); V. Fock. Z. Phys. **38,** 242 (1926); **39,** 226 (1926); E. Schrödinger, Ann. Phys. (Leipz.) **81,** 109 (1926); J. Kudar, *ibid.* **81,** 109 (1926); W. Gordon, Z. Phys. **40,** 117 (1926); Th. de Donder, Compt. Rend. **182,** 1380 (1926); Th. de Donder and H. F. van den Dugen, *ibid.* **183,** 24 (1926).
[16]A. Proca, Compt. Rend. **202,** 1366 (1936); **202,** 1490 (1936); **203,** 709 (1936); J. Phys. Radium Ser. VII **7,** 347 (1936); **8,** 23 (1936).
[17]L. de Broglie, Compt. Rend. **199,** 445 (1934).
[18]L. de Broglie and J. P. Vigier, Phys. Rev. Lett. **28,** 1001 (1972). Also consult the alphabetical references under de Broglie in A. S. Goldhaber and M. M. Nieto, Rev. Mod. Phys. **40,** 296 (1971) for some of de Broglie's many works on the subject, and also Library of Congress listings of de Broglie's works.
[19]G. Petiau, Acad. R. Belg. Cl. Sci. Mém. Collect 8 **16,** No. 2 (1936).
[20]J. Géhéniau, Acad. R. Belg. Cl. Sci. Mém. Collect 8 **18,** No. 1 (1938).
[21]To show causality of the *q-number* Bhabha fields interacting with minimal electromagnetic coupling, a preliminary step is to derive the "KG-Divisor," which is the object $D$ that satisfies

$$(\partial \cdot \alpha + \chi)D = \prod_{j=1/2 \text{ or } 1}^{S} (\Box - \chi^2/j^2).$$

In a letter to M.M.N. dated November 1976, Géhéniau observed that in his study in Acad. R. Belg. Cl. Sci. Mém. Collect 8 **16,** No. 5 (1948), he discussed the general existence of what we call the KG-Divisor, calculating specific cases in particular. See his Eqs. (149) and (153) and p. 45. Géhéniau mentioned that the work was originally done in 1942 but (due to World War II, we assume) was not published until 1948. What Géhéniau did not inform us of, but which we discovered by finding Bull. Acad. R. Belg. Cl. Sci. **28,** 616, 737, 968, 973 (1942), is that his study was actually done as a prize-winning essay submitted to the Belgium Academy of Sciences as a response to the question, "*On demande une contribution importante à la théorie mathématique de la seconde quantification de la mécanique ondulatoire.*"
[22]Letter from G. Petiau to M.M.N. dated 6 May 1971.
[23]Letter from N. Kemmer to M.M.N. dated 16 July 1971.
[24]N. Kemmer, Proc. R. Soc. **166,** 127 (1938).
[25]Letter from R. J. Duffin to M.M.N. dated 17 May 1971.
[26]R. J. Duffin, Phys. Rev. **54,** 1114 (1938).
[27]N. Kemmer, Proc. R. Soc. A **173,** 91 (1939).
[28]C. Møller and L. Rosenfeld, Nature **143,** 241 (1939); K. Danske Vidensk. Selsk. Math. Fys. Medd. **17,** No. 8 (1940).
[29]F. J. Belinfante, Physica **6,** 849, 870, 887 (1939).
[30]F. J. Belinfante, *Theory of Heavy Quanta* (Nijhoff, Gravenhage, 1939).
[31]M. Taketani and S. Sakata, Proc. Phys. Math. Soc. Jpn. **22,** 757 (1940); S. Sakata and M. Taketani, Sci. Pap. Inst. Phys. Chem. Res. **38,** 1 (1940).
[32]The articles of Ref. 31 have been reprinted in Prog. Theor. Phys. Suppl. No. 1, 84, 98 (1955).
[33]The first page of the first work in Ref. 31 has an allusion to the incident. "The authors themselves . . . tried to write the meson equation in the form analogous to Dirac's electron equation. But before they could finish this task, certain circumstances prevented one of the authors (M.T.) from working in this problem and in the meantime Duffin's paper appeared." In a conversation on 19 July 1971, Taketani explained his side of the story to M.M.N. Taketani had authored an article in an internationalist journal under a pseudonym in which he had declared that scientists and science should be an international community. He related this did not set well with the then Japanese military government, who discovered who he was, and for his statements placed him in prison.
[34]Ig. Tamm, Compt. Rend. (Doklady) Acad. Sci. URSS **29,** 551 (1940); H. Feshbach and F. Villars, Rev. Mod. Phys. **30,** 24 (1958). Ref. 60, below, gives a discussion of the relation of the ST and FV formulations, and the connection to "symmetry breaking," a concept we discuss near the end of this section.
[35]J. K. Lubański, Physica **8,** 44 (1941).
[36]H. A. Kramers, F. J. Belinfante, and J. K. Lubański, Physica **8,** 597 (1941).
[37]J. K. Lubański and L. Rosenfeld, Physica **9,** 117 (1941).
[38]Letter from F. J. Belinfante to M.M.N. dated 9 May 1976.
[39]J. K. Lubański, Physica **9,** 310 (1942); *ibid.* **9,** 325 (1942).
[40]On pp. 613–614 of Ref. 36 there is a report on the work of Lubański which Belinfante, in his letter of Ref. 38, attributes to Kramers.
[41]A later work by J. K. Lubański appeared in Ark. Mat. Astron. Fys. **30B,** No. 7 (1944).
[42]B. S. Madhavarao (Madhava Rao), Proc. Indian Acad. Sci. A **15,** 139 (1942).
[43]B.S. Madhavarao (Madhava Rao), J. Mysore Univ. **B 3,** 59 (1942).
[44]Harish-Chandra, Proc. R. Soc. A **186,** 502 (1946).
[45]Harish-Chandra, Proc. Cambridge Philos. Soc. **43,** 414 (1947).
[46]Harish-Chandra, Phys. Rev. **71,** 793 (1947).
[47]Harish-Chandra, in *Report of an International Conference on Fundamental Particles and Low Temperatures,* Cavendish Laboratory, Cambridge, 22–27 July 1946 (Phys. Soc., London, 1947), Vol. I, p. 185.
[48]Harish-Chandra, Proc. R. Soc. A **192,** 195 (1948).
[49]B. S. Madhavarao, in a letter to M.M.N. dated 29 April 1976, recalls that during this period he "pointed out to Bhabha the possibility of the existence of polynomial equations for $\chi^2$, and he readily accepted the idea . . ." For a time Madhavarao preferred to continue the search for a single-mass solution, since it appealed to him on mathematical grounds.
[50]B. S. Madhavarao (Madhava Rao), J. Mysore Univ. B **6,** 57 (1945).
[51]H. J. Bhabha, Curr. Sci. **14,** 89 (1945).
[52]H. J. Bhabha, Rev. Mod. Phys. **17,** 200 (1945).
[53]B. S. Madhavarao, V. R. Thiruvenkatachar, and K. Venkatachaliengar, Proc. R. Soc. A **187,** 385 (1946).
[54]B. S. Madhavarao (Madhava Rao), Proc. Indian Acad. Sci. A **26,** 221 (1947).
[55]Letter from N. Kemmer to M.M.N. dated 27 April 1976.
[56]N. Kemmer, in *Hadronic Interactions of Electrons and Photons, Proceedings of the 11th Scottish University Summer School in Physics, 1970,* edited by J. Cumming and H. Osborn (Academic, New York, 1971), p. 1. The mentioned quote is on p. 14. This article, entitled "Some Recollections from the Early Days of Particle Physics," provides a great deal of information about the period we are discussing. Also see the article by N. Kemmer, Prog. Theor. Phys. Suppl. Extra Number, 602 (1965), appropriately entitled, "The Impact of Yukawa's Meson Theory on Workers in Europe—A Reminiscence—."
[57]H. J. Bhabha, Proc. Indian Acad.Sci. A **21,** 241 (1945).
[58]H. J. Bhabha, in *Report of an International Conference on Fundamental Particles and Low Temperatures,* Cavendish Laboratory, Cambridge, 22–27 July 1946 (Phys. Soc., London, 1947), Vol. I, p. 22.
[59]H. J. Bhabha, Rev. Mod. Phys. **21,** 451 (1949).
[60]E. Fischbach, M. M. Nieto, and C. K. Scott, Prog. Theor. Phys. **48,** 574 (1972).
[61]F. Booth and A. H. Wilson, Proc. R. Soc. A **175,** 483 (1940); A. H. Wilson, Proc. Cambridge Philos. Soc. **36,** 363 (1940).
[62]M. Neuman and W. H. Furry, Phys. Rev. **76,** 1677 (1949); R. G. Moorhouse, Phys. Rev. **76,** 1691 (1949).
[63]T. Kinoshita, Prog. Theor. Phys. **5,** 473 (1950); T. Kinoshita and Y. Nambu, *ibid.* **5,** 749 (1950).
[64]S. N. Gupta, Proc. Cambridge Philos. Soc. **46,** 649 (1950).
[65]D. C. Peaslee, Phys. Rev. **81,** 94, 107 (1951).
[66]A. Klein, Phys. Rev. **82,** 639 (1951).
[67]A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics,* 2nd ed. (Interscience, New York, 1965), Chap. IX, p. 819, discuss DKP Q.E.D. in great detail.
[68]E.Fischbach, F. Iachello, A. Lande, M. M. Nieto, and C. K. Scott, Phys. Rev. Lett. **26,** 1200 (1971).
[69]E.Fischbach, M. M. Nieto, H. Primakoff, C. K. Scott, and J. Smith, Phys. Rev. Lett. **27,** 1403 (1971).
[70]E. Fischbach, M. M.Nieto, and C. K. Scott, Phys. Rev. D **6,** 726 (1972).
[71]E. Fischbach, M. M. Nieto, H. Primakoff, and C. K. Scott, Phys. Rev. Lett. **29,** 1046 (1972).

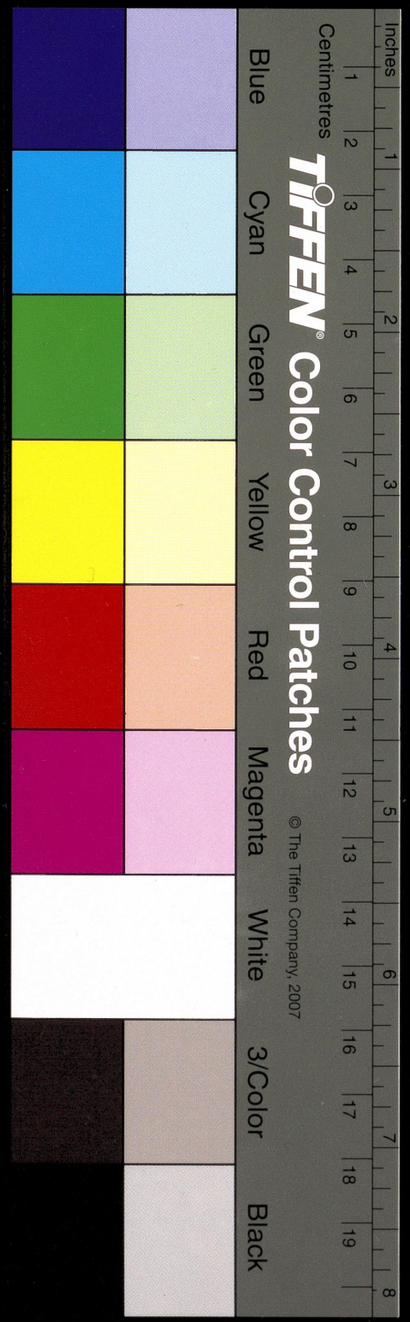


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[72]E. Fischbach, M. M. Nieto, and C. K. Scott, Phys. Rev. D **7,** 207 (1973).
[73]E. Fischbach, M. M. Nieto, and C. K. Scott, Phys. Rev. D **7,** 3544 (1973).
[74]E. Fischbach, M. M. Nieto, and C. K. Scott, J. Math. Phys. **14,** 1560 (1973).
[75]F. T. Meiere, E. Fischbach, A. McDonald, M. M. Nieto, and C. K. Scott, Phys. Rev. D **8,** 4209 (1973).
[76]E. Fischbach, M. M.Nieto, H. Primakoff, and C. K. Scott, Phys. Rev. D **9,** 2183 (1974).
[77]E. Fischbach, M. M. Nieto, and C. K. Scott, Prog. Theor. Phys. **51,** 1585 (1974).
[78]B. G. Kenny, D. C. Peaslee, and M. M. Nieto, Phys. Rev. D **13,** 757 (1976).
[79]A. S. Wightman has repeatedly emphasized that the Wigner theorem on the unitary equivalence of same mass-spin representations of ISL(2,*C*) "is true for free, i.e., noninteracting particles, but false for interacting particles." See p. 444 of his article, in *Partial Differential Equations, Proceedings of the 23rd Symposium in Pure Mathematics, Berkeley, California, 1971*, edited by D. C. Spencer (Am. Math. Soc., Providence, RI, 1973), p. 441.
[80]B. Sakita and K. C. Wali, Phys. Rev. **139,** B1355 (1965).
[81]E. Golowich and V. Kapila, Phys. Rev. D **8,** 2180 (1973).
[82]R. A. Krajcik and L. L. Foldy, Phys. Rev. Lett. **24,** 545 (1970).
[83]L. L. Foldy and R. A. Krajcik, Phys. Rev. Lett. **32,** 1025 (1974).
[84]R. A. Krajcik and L. L. Foldy, Phys. Rev. D **10,** 1777 (1974).
[85]L. L. Foldy and R. A. Krajcik, Phys. Rev. D **12,** 1700 (1975).
[86]Specifically, from the results of Refs. 8, 60, and 68–77, it was realized that many of the formal aspects of the DKP system, and hence of the Bhabha system, had never been investigated.



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**LOS ALAMOS SCIENTIFIC LABORATORY**
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*To:*
Dr. B. S. Madhavarao
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Bangalore 560-012
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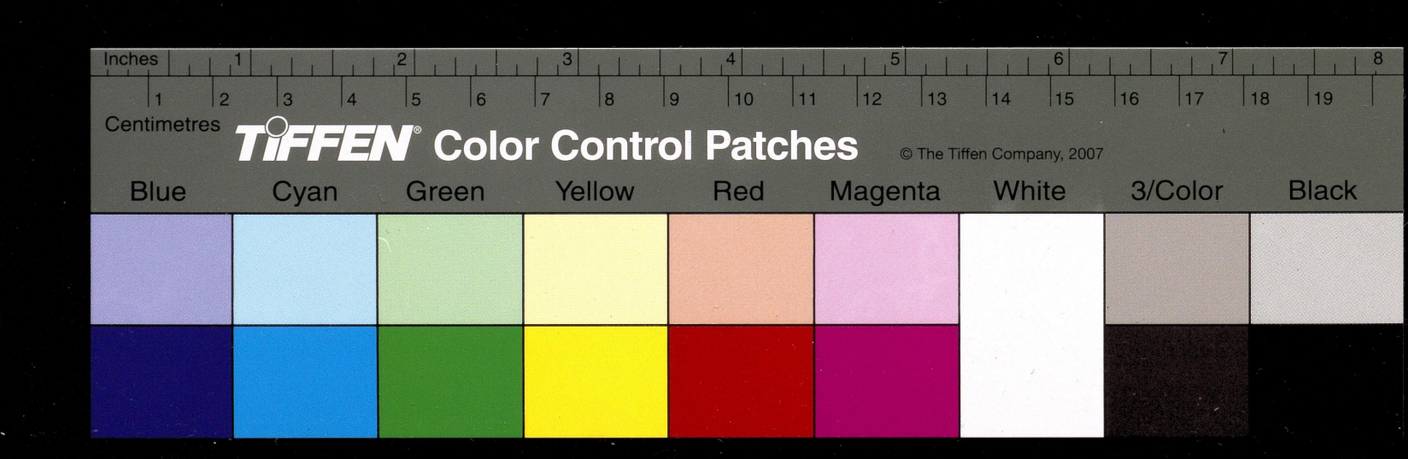
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15 April 1976

IN REPLY
REFER TO: T-8
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Dr. B. S. Madhavarao
Center for Theoretical Studies
Indian Institute of Science
Bangalore 560012
INDIA

Dear Dr. Madhavarao:

As I mentioned in an earlier letter, R. A. Krajcik and myself hope to include a historical section in the final paper of our series on the so(5) Bhabha equations.

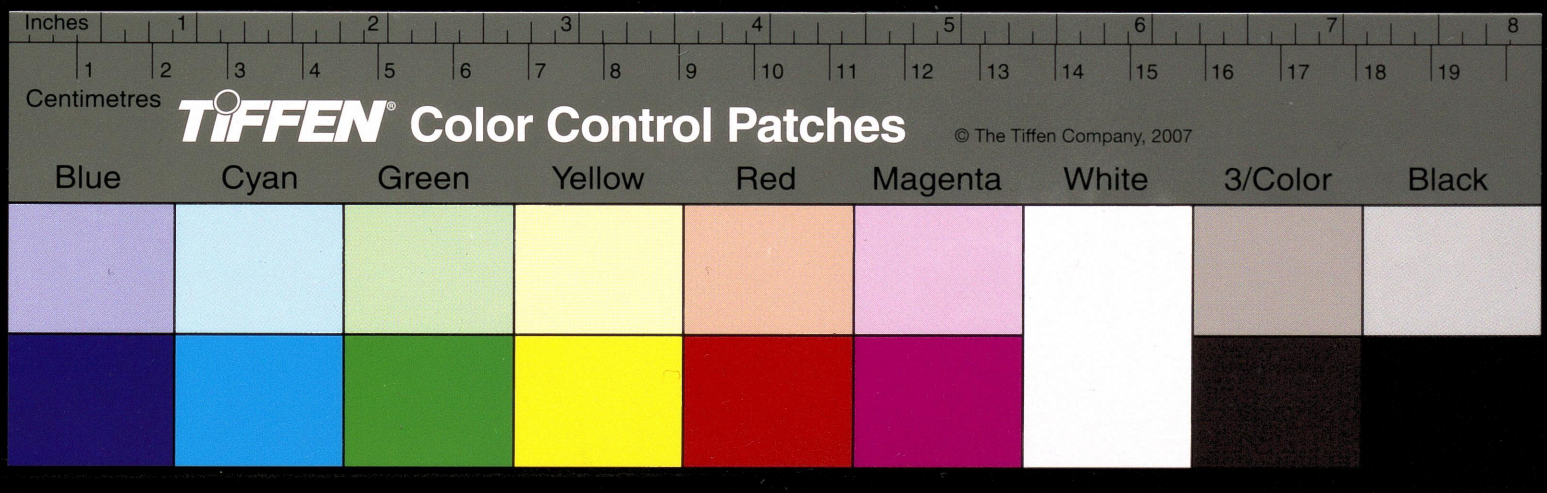
Enclosed you will find a copy of that section with appropriate references. We are sending this preliminary version to you now so that you will have the opportunity to suggest any changes before the manuscript is submitted for publication.

Sincerely yours,

Michael Martin Nieto

MMN:ce
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cy: R. A. Krajcik

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# BHABHA FIRST-ORDER WAVE EQUATIONS. VII. SUMMARY AND CONCLUSIONS

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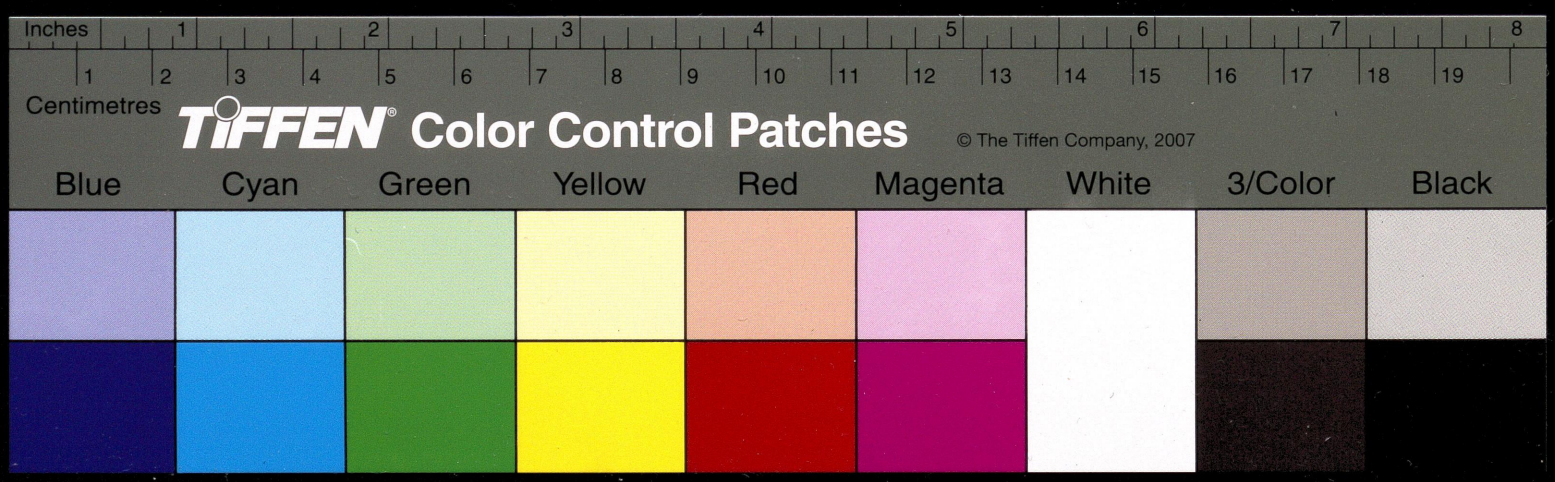
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ABSTRACT

We give a summary of the results we have obtained in this series on the Bhabha first-order wave equations for arbitrary spin. On the basis of these and other calculations we present a set of conclusions about their physical implications, especially with respect to the possibility of finding and properly interpreting a mathematically consistent field theory of particles with arbitrary spin. In an appendix we present a short historical review on the development of the Bhabha equations.

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*Work supported by the Univted States Energy Research and Development Administration.



## APPENDIX A: HISTORICAL RESUME

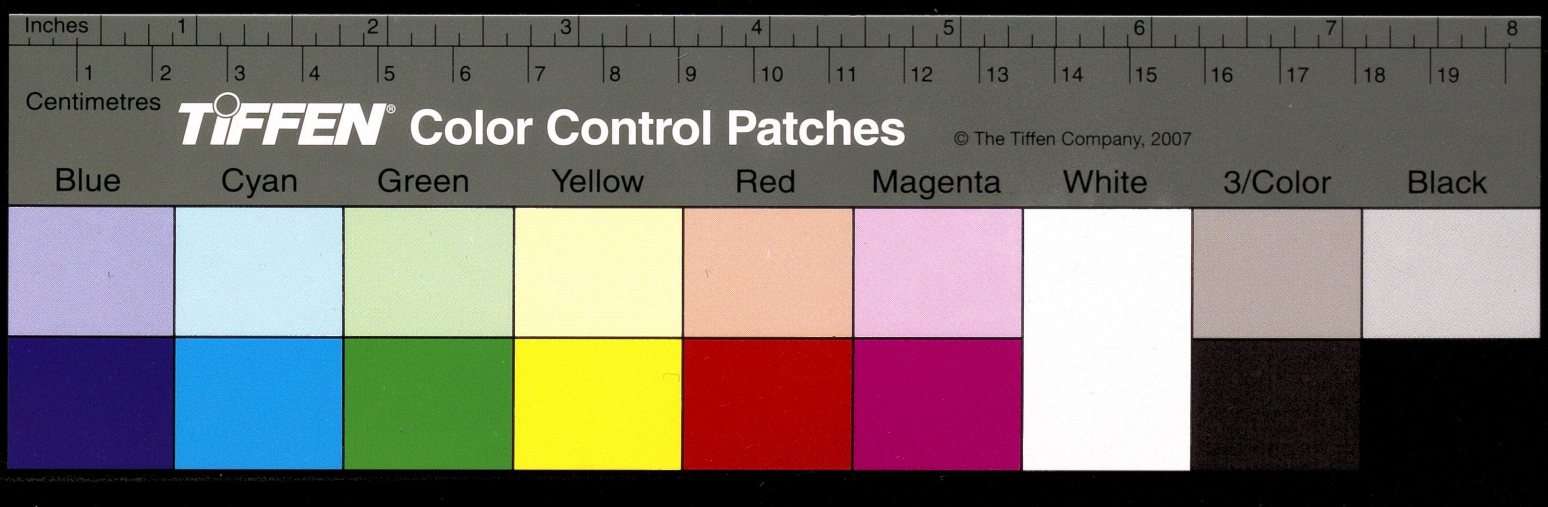
By the mid 1930's the success of the Dirac equation[A1] in relativistically describing particles of spin-1/2 gave certain people the desire to find a similar first-order wave equation for spin-0 particles (then not yet discovered) and spin-1 particles. They felt such a formalism might be preferable to the already accepted "relativistic second-order Schrödinger equation," or Klein-Gordon (KG) equation[A2] as it is now known, for spin-0 particles and the second-order Proca equation[A3] for the spin-1 particles.

The first major effort in this direction was initiated by de Broglie,[A4] who for 40 years now[A4-A6] has hoped that the photon has a nonzero rest mass. De Broglie based his original investigations of first-order wave equations on $16 \times 16$ matrices that were products of different Dirac $\gamma$ matrix spaces. He hoped he could combine two leptons to yield massive photons. One of his students, Petiau, looked at a modification of de Broglie's algebra, becoming the first to obtain the $16 \times 16$ DKP algebra,[A7]

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \beta_\mu\delta_{\nu\lambda} + \beta_\lambda\delta_{\mu\nu} \quad . \tag{A1}$$

(Géhéniau later decomposed the algebra into the 5, 10, and trivial 1-dimensional representations.[A8]) However, Petiau's 1936 work remained unknown to most workers;[A9] and in particular was not heard of by Kemmer until after World War II.[A10]

It was the work of Proca which inspired Kemmer and Duffin. Kemmer realized[A10] that the Proca equations could be described as a set of coupled first-order equations, and he wrote them down along with the analogous spin-0 case.[A11] He also realized, therefore, that the above could be written in

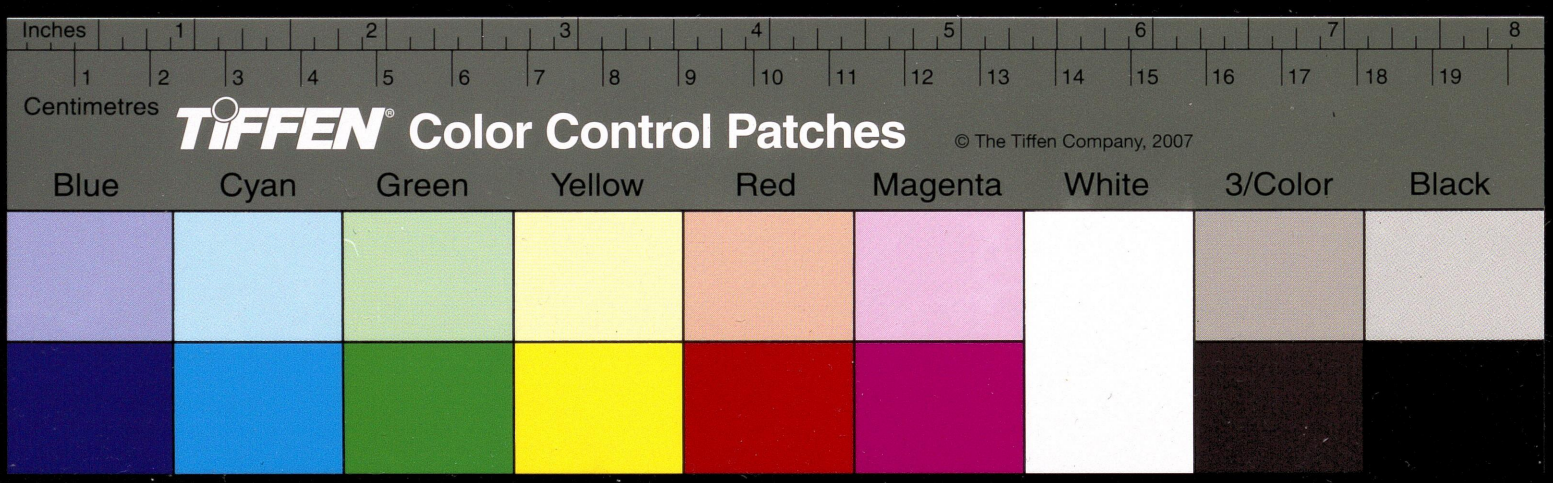


explicit 5 × 5 and 10 × 10 matrix forms for spin-0 and spin-1, but at first he[A10] did not have "any knowledge of the commutation rules satisfied by them (the matrices)."

Soon after, Duffin, a mathematician, attended[A12] a physics seminar at Purdue where the aforementioned Ref. A11 by Kemmer on the Proca equations was discussed. When Duffin asked how one knew the theory was spin-1, a complicated answer was given which was esthetically umpleasing to Duffin. Since Duffin too preferred[A12] a first-order Dirac approach, he looked into the matter, with the result that he was quickly able to put the spin-0 and spin-1 equations in a first-order $\beta$-matrix formulation, with three of the four constituent commutation relations represented in Eq.(A1) (not for $\lambda = \mu \neq \nu$).

When Kemmer saw the note that Duffin published on his results,[A13] he wrote to Duffin[A12] saying he knew how to extend the theory but would first wait for Duffin to publish anything further. However, Duffin was by then involved in a collaboration with A. C. Schaeffer on function theory, and so wrote[A12] to Kemmer to go ahead. Kemmer[A10] then "quickly put together" all he had been doing, and produced his classic 1939 paper.[A14] He[A10] "sent it (a preprint) to Pauli, who typically, answered by telling me how to proceed." (See the footnotes on p. 102 and p. 104 of Ref. A14). The fact that at this time it was already becoming clear that the primary components of the low energy nuclear force are due to pseudoscalar and vector interactions[A15] excited many people about the DKP formalism, which contained them both.

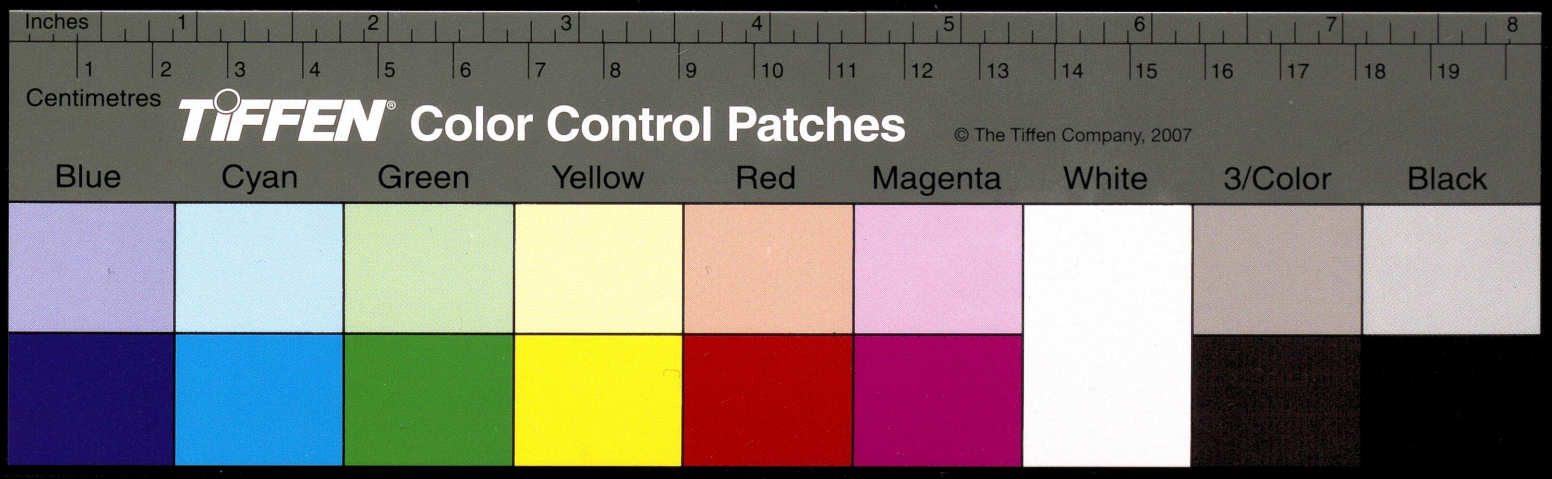
Two other important contributions were made in this period. The first was the work of Belanfante[A16,A17] on "undors," which was another version of the coupled equation results above. In addition a separate investigation into the first-order Dirac-like "DKP" equation had been undertaken by Sakata and Taketani (ST).[A18-A20] However, their investigation was delayed because



Taketani was imprisoned.[A21] By the time that Taketani was released, they had seen the work of Duffin and Kemmer, and so they concentrated on what we call the ST-decomposition, removing the built-in subsidiary components from the physical "particle components." In this series we have seen that the generalization of this procedure is always a necessary extra step to properly handle integer-spin Bhabha fields.

With that basis, the work of Lubánski, Madhavarao, and Bhabha was produced, all three of them deserving credit for their individual efforts. It was Lubánski[A22] who, in 1942, first proposed the so(5) algebra for the first-order wave equations and recognized the type of spin and mass spectrum which would be obtained. However, his papers were published in occupied Holland and so did not receive the recognition they deserved.

Also in 1942, Madhavarao[A23] began looking at first-order wave equations, having been inspired by Duffin's and Kemmer's work. At first he concentrated on what turned out to be the specific commutation relations of the $\alpha_\mu$ for the algebras $S = \frac{3}{2}$ and $S = 2$. He then tried to obtain restrictions on the general algebra which whould yield a single mass KG equation,[A24] a quest which was the starting point for the later work of Harish-Chandra.[A25-A27] However, if one remains in the so(5) algebra, such a single mass wave equation is only possible for the Dirac and DKP cases, as we now know. Madhadvarao[A30] later pointed out that Bhabha, in a 1945 preliminary version[A31] of his later major work,[A32] had resolved the question by discovering that for high spin the so(5) equations satisfy a multimass KG equation. Mahavaro[A33,A34] and coworkers[A33] went on to investigate further the algebraic properties of the so(5) algebra. As the reader who has been following our series closely has observed, he made major contributions there.

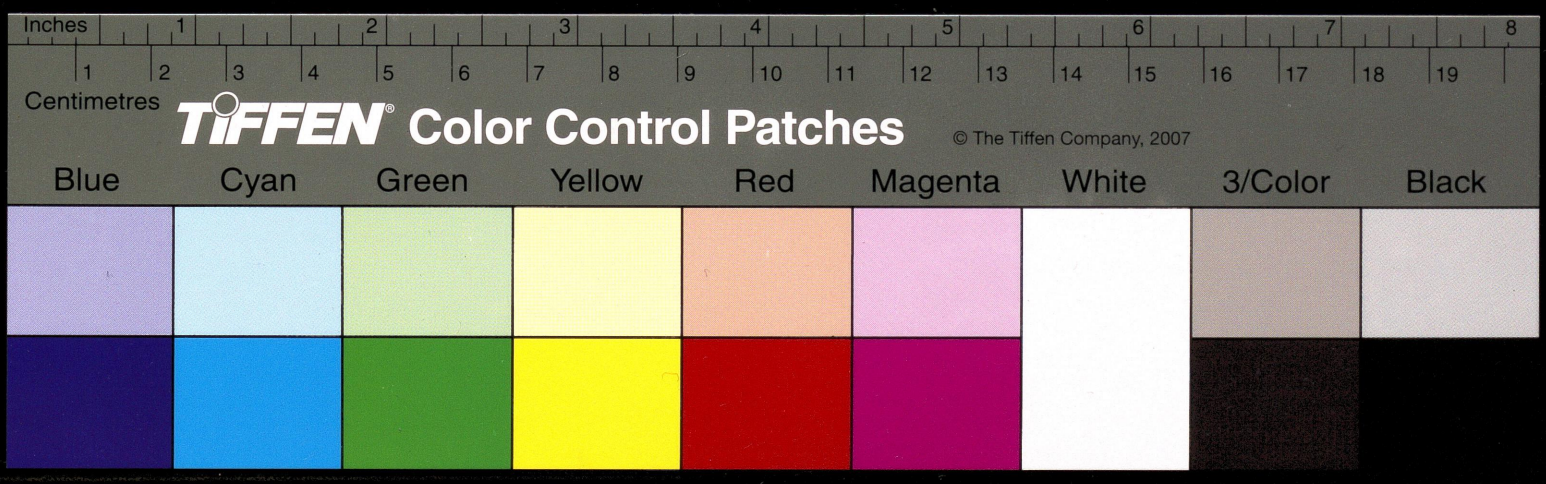


The milieu in which Bhabha found himself makes part of the origin of his interest in first-order wave equations clear. He was a colleague of Kemmer at Cambridge. In fact, "The decision to use the word meson (for the Yukawa particles) was made in a certain private house in Cambridge by Maurice Price, Homi Bhabha, and myself (Kemmer)."[A35] In the years 1945-1949 Bhabha published a number of papers in which the properties of these so(5) equations were investigated.[A31,A32,A36-A38]

Over the succeeding years a great number of papers dealing with the DKP and general Bhabha wave equations were published.[12] However, after the early 1950's physical interest in them waned because[A39] it was ultimately believed that the KG and the DKP equations are equivalent. The reason was that in the 5-dimensional spin-0 representation of the DKP equation, the fifth component ($\psi_5$) behaves like the KG field and the first four components behave like the 4-derivatives of $\psi_5$. So, the two equations yielded the same energy levels for the $\pi$ mesic atom, and the same results for processes like $\pi$-e scattering, $e^+e^- \rightarrow \pi^+\pi^-$, and, in fact, everywhere one checked.[A25,A40-A46] Thus, the "complicated" 5-component spinor equation was abandoned in favor of the simple 1-component (albeit second-order) equation which was "easier" to handle and yielded the same results.

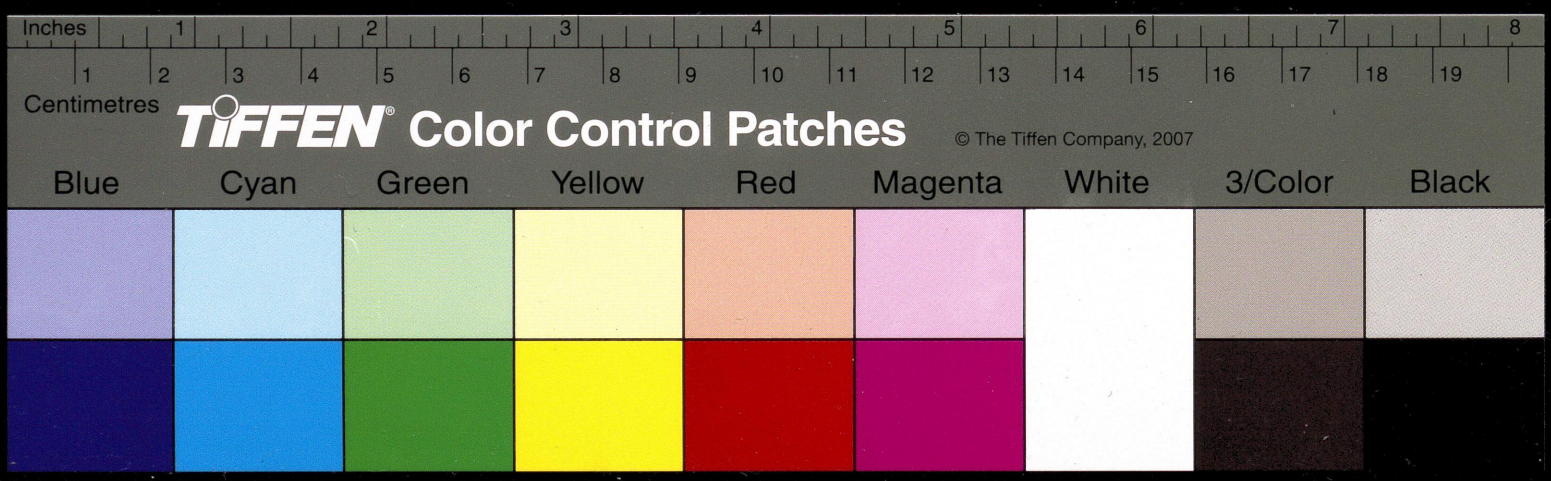
The above viewpoint was reached during the time when people only conceived of one meson ($\pi$), and for a long time no one ever reexamined if symmetry-breaking effects, such as mass splittings, would change the result that the KG and DKP equations are equivalent. But in 1970 a collaboration began reinvestigating the question[7,A39,A47-A57] and found that the "equivalence" of the DKP and KG equations in general holds only[A58] in the symmetry case. (Physically one can reason that a first-order field with dimensions (mass)$^{3/2}$

will extrapolate to a new mass differently that a KG field with $(\text{mass})^1$ dimensions.) This, plus an interest in the relativistic properties of wave equations,[A59-A62] led us[A63] to consider the general arbitrary-spin Bhabha wave equations. Since, as discussed in IV,[4] there as of yet is no high-spin field equation which is devoid of all problems, and since the DKP work tells us that the Bhabha first-order system is physically different than a second-order formalism, we undertook this series to try to discover what the Bhabha-first-order wave equations could reveal about the physics of high-spin fundamental fields and particles.



REFERENCES


1. R. A. Krajcik and M. M. Nieto, Phys. Rev. D 10, 4049 (1974). I of this series on C, P, and T.
2. R. A. Krajcik and M. M. Nieto, Phys. Rev. D 11, 1442 (1975). II of this series on the mass and spin composition, the Hamiltonians, and the general Sakata-Taketani reductions of Bhabha fields.
3. R. A. Krajcik and M. M. Nieto, Phys. Rev. D 11, 1459 (1975). III of this series on the Poincaré generators.
4. R. A. Krajcik and M. M. Nieto, Phys. Rev. D 13, 924 (1976). IV of this series on causality with minimal electromagnetic interaction.
5. R. A. Krajcik and M. M. Nieto, Phys. Rev. D XX, XXXX (197X). V of this series, on the indefinite metric and Foldy-Wouthuysen transformations.
6. R. A. Krajcik and M. M. Nieto, Phys. Rev. D XX, XXXX (197X). VI of this series, on solutions and exact, closed form Foldy-Wouthuysen transformations.
7. E. Fischbach, J. D. Louck, M.M.Nieto, and C. K. Scott, J. Math. Phys. 15, 60 (1974).
8. J. N. O. Hietarinta, Phys. Rev. D XX, XXXX (1976).
9. R. A. Krajcik and M. M. Nieto, Phys. Rev. D 13, XXXX (1976). I of a series on Foldy-Wouthuysen (FW) transformations in indefinite metric spaces.
10. R. A. Krajcik and M. M. Nieto, Phys. Rev. D 13, XXXX (1976). II of a series on FW transformations in indefinite metric spaces.
11. R. A. Krajcik and M. M. Nieto, Phys. Rev. D XX, XXXX (197X). III and conclusion of a series on FW transformations in indefinite metric spaces.
12. Refs. 1-7, especially 4 and 5, list pertinent works on the Bhabha equations. Refs. 9-11, especially 9, list pertinent works on general indefinite metric spaces. Refs. 7 and A39 and A47-A57 below list pertinent works on the

Duffin-Kemmer-Petiau spin-0 and spin-1 first order wave equations, they being special cases of the Bhabha system.

A1. P. A. M. Dirac, Proc. R. Soc. A117, 610 (1928); A118, 341 (1928); The Principles of Quantum Mechanics, 4th ed. (Claredon Press, Oxford, 1958).

A2. O. Klein, Z. Phys. 37, 895 (1926); V. Fock. Z. Phys. 38, 242 (1926); 39, 226 (1926); E. Schrödinger, Ann. Phys. (Leipz.) 81, 109 (1926); J. Kudar, ibid. 81, 109 (1926); W. Gordon, Z. Phys. 40, 117 (1926); Th. de Donder, Compt. Rend.182, 1380 (1926); Th. de Donder and H. F. van den Dugen, ibid. 183, 24 (1926).

A3. A. Proca, Compt. Rend. 202, 1366 (1936); 202, 1490 (1936); 203, 709 (1936); J. Phys. Radium, Ser. VII 7, 347 (1936); 8, 23 (1936).

A4. L. de Broglie, Compt. Rend. 199, 445 (1934).

A5. Consult the alphabetical references under de Broglie in A. S. Goldhaber and M. M. Nieto, Rev. Mod. Phys. 40, 296 (1971) for some of de Broglie'e many works on the subject, and also Library of Congress listings of de Broglie's works.

A6. L. de Broglie and J. P. Vigier, Phys. Rev. Lett. 28, 1001 (1972).

A7. G. Petiau, Acad. R. Belg. Cl. Sci. Mem. Collect 8 16, No. 2 (1936).

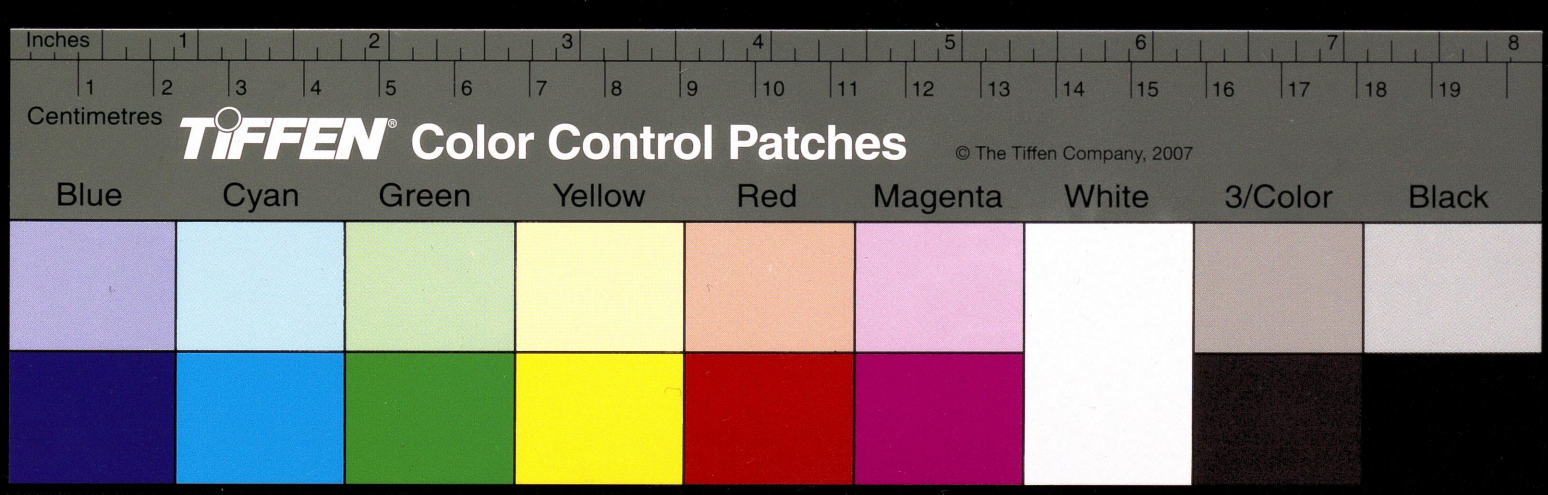
A8. J. Géhéniau, Acad. R. Belg. Cl. Sci. Mem Collect 8 18, No. 1 (1938).

A9. Letter from G. Petiau to M.M.N. dated 6 May 1971.

A10. Letter from N. Kemmer to M.M.N. dated 16 July 1971.

A11. N. Kemmer, Proc. R. Soc. 166, 127 (1938).

A12. Letter from R. J. Duffin to M.M.N. dated 17 May 1971.



A13. R. J. Duffin, Phys. Rev. 54, 1114 (1938).

A14. N. Kemmer, Proc. R. Soc. A173, 91 (1939).

A15. C. Møller and L. Rosenfeld, Nature 143, 241 (1939); Kgl. Danske Videnskab. Selskab. Math.-fys. Medd. 17, No. 8 (1940).

A16. F. J. Belinfante, Physica 6, 849, 870, 887 (1939).

A17. F. J. Belinfante, Theory of Heavy Quanta (M. Nijhoff, Gravenhage, 1939).

A18. M. Taketani and S. Sakata, Proc. Phys. Math. Soc. Jpn. 22, 757 (1940).

A19. S. Sakata and M. Taketani, Sci. Pap. Inst. Phys. Chem. Res. 38, 1 (1940).

A20. Refs. A18 and A19 have been reprinted in Prog. Theor. Phys. Suppl. No. 1, 84, 98 (1955).

A21. The first page of Ref. A18 had an allusion to the incident. "The authors themselves...tried to write the meson equation in the form analogous to Dirac's electron equation. But before they could finish this task, certain circumstances prevented one of the authors (M.T.) from working in this problem and in the meantime Duffin's paper appeared." In a converstation on 19 July 1971, Taketani explained his side of the story to M.M.N. Taketani had authored an article in an internationalist journal under a pseudonym in which he had declared the scientists and science should be an international community. This did not set well with the then Japanese military govenment. They discovered who he was, and for his statements placed him in prison.

A22. J. K. Lubánski, Physica 9, 310, 325 (1942).

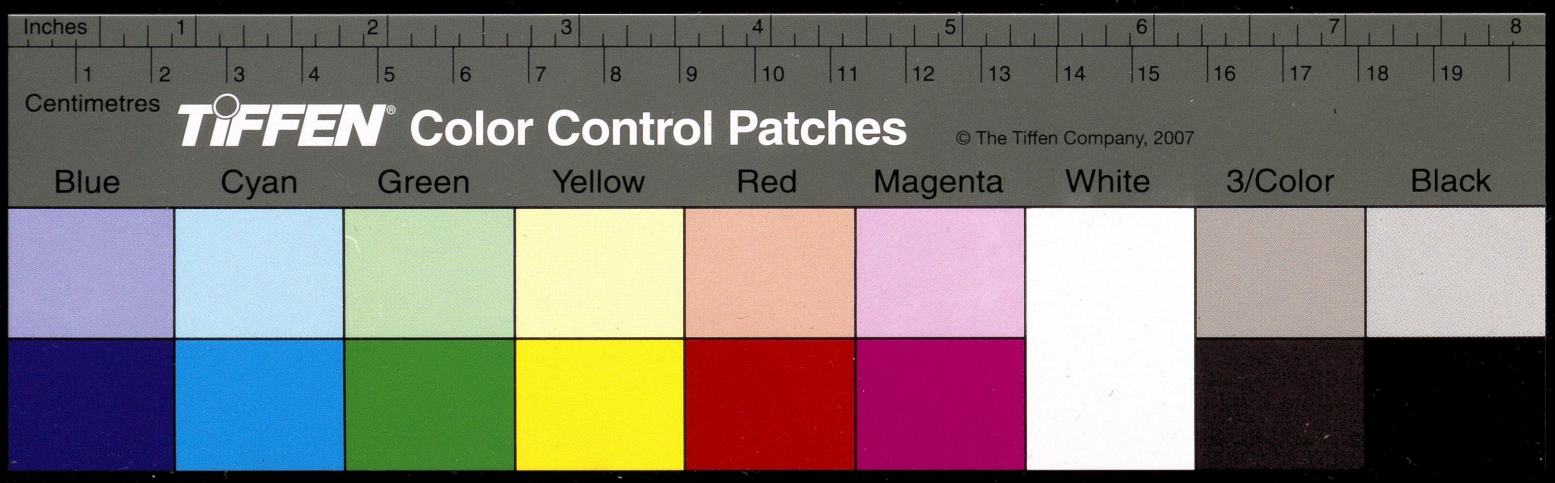
A23. B. S. Madhavarao (Madhava Rao), Proc. Indian Acad. Sci. A15, 139 (1942).

A24. B. S. Madhavarao (Madhava Rao), J. Mysore Univ. B3, 59 (1942).

A25. Harish-Chandra, Proc. Roy. Soc. A186, 502 (1946).

A26. Harish-Chandra, Proc. Camb. Phil. Soc. 43, 414 (1947).

A27. Harish-Chandra, Phys. Rev. 71, 793 (1947).



A28. Harish-Chandra, in Report of an International Conference on Fundamental Particles and Low Temperatures held at the Cavendish Laboratory, Cambridge on 22-27 July 1946, Vol. I (Physical Society, London, 1947), p. 185.

A29. Harish-Chandra, Proc. Roy. Soc. A192, 195 (1948).

A30. B. S. Madhavarao (Madhava Rao), J. Mysore Univ. B6, 57 (1945).

A31. H. J. Bhabha, Curr. Sci. 14, 89 (1945).

A32. H. J. Bhabha, Rev. Mod. Phys. 17, 200 (1945).

A33. B. S. Madhavarao, V. R. Thiruvenkatachar, and K. Venkatachaliengar, Proc. R. Soc. A187, 385 (1946).

A34. B. S. Madhavarao (Madhava Rao), Proc. Indian Acad. Sci. A26, 221 (1947).

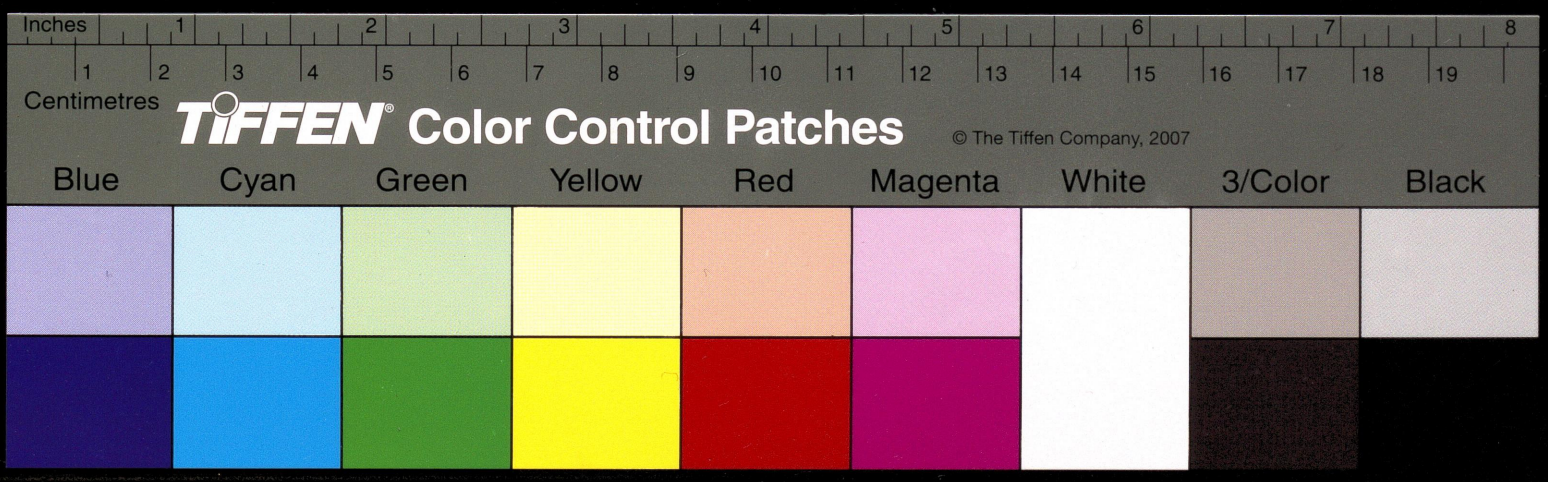
A35. N. Kemmer, in Hadronic Interactions of Electrons and Photons, Proc. of 11th Scottish Univ. Summer School in Physics, 1970, edited by J. Cumming and H. Osborn (Academic Press, New York, 1971), p. 1. The mentioned quote is on p. 14. This article, entitled "Some Recollections from the Early Days of Particle Physics," provides a great deal of information about the period we are discussing. Also see the article by N. Kemmer, Prog. Theor. Phys. Suppl, Extra Number, 602 (1965), appropriately entitled, "The Impact of Yukawa's Meson Theory on Workers in Europe - A Reminiscence - ."

A36. H. J. Bhabha, Proc. Indian Acad. Sci. A21, 241 (1945).

A37. H. J. Bhabha, in Report of an International Conference on Fundamental Particles and Low Temperatures held at the Cavendish Laboratory, Cambridge on 22-27 July 1946, Vol I (Physical Society, London, 1947), p.22.

A38. H. J. Bhabha, Rev. Mod. Phys. 21, 451 (1949).

A39. E. Fischbach, M. M. Nieto, and C. K. Scott, Prog. Theor. Phys. 48, 574 (1972).



A40. F. Booth and A. H. Wilson, Proc. R. Soc. A175, 483 (1940); A. H. Wilson, Proc. Camb. Philos. Soc. 36, 363 (1940).

A41. M. Neuman and W. H. Furry, Phys. Rev. 76, 1677 (1949).

A42. T. Kinoshita, Prog. Theor. Phys. 5, 473 (1950); T. Kinoshita and Y. Nambu, ibid. 5, 749 (1950.

A43. S. N. Gupta, Proc. Camb. Philos. Soc. 46, 649 (1950).

A44. D. C. Peaslee, Phys. Rev. 81, 94 (1951).

A45. A. Klein, Phys. Rev. 82, 639 (1951).

A46. A. I. Akhiezer and V. B. Berestetskii, Quantum Electrodynamics, 2nd ed. (Interscience, New York, 1965), Chap IX, p. 819, discuss DKP QED in great detail.

A47. E. Fischbach, F. Iachello, A. Lande, M. M. Nieto, and C. K. Scott, Phys. Rev. Lett. 26, 1200 (1971).

A48. E. Fischbach, M. M. Nieto, H. Primakoff, C. K. Scott, and J. Smith, Phys. Rev. Lett. 27, 1403 (1971).

A49. E. Fischbach, M. M. Nieto, and C. K. Scott, Phys. Rev. D 6, 726 (1972).

A50. E. Fischbach, M. M. Nieto, H. Primakoff, and C. K. Scott, Phys. Rev. Lett. 29, 1046 (1972).

A51. E. Fischbach, M. M. Nieto, and C. K. Scott, Phys. Rev. D 7, 207 (1973).

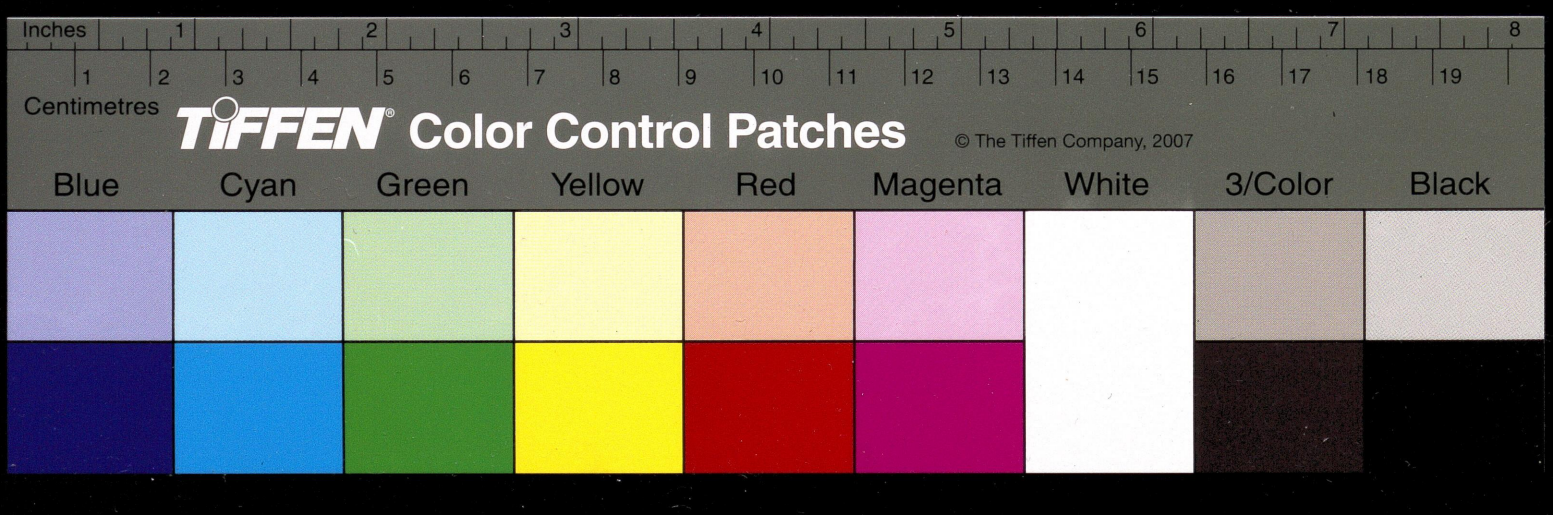
A52. E. Fischbach, M. M. Nieto, and C. K. Scott, Phys. Rev. D 7, 3544 (1973).

A53. E. Fischbach, M. M. Nieto, and C. K. Scott, J. Math. Phys. 14, 1560 (1973).

A54. F. T. Meiere, E. Fischbach, A. McDonald, M. M. Nieto, and C. K. Scott, Phys. Rev. D 8, 4209 (1973).

A55. E. Fischbach, M. M. Nieto, H. Primakoff, and C. K. Scott, Phys. Rev. D 9, 2183 (1974).

A56. E. Fischbach, M. M. Nieto, and C. K. Scott, Progr. Theor. Phys. 51, 1585 (1974).



A57. B. G. Kenny, D. C. Peaslee, and M. M. Nieto, Phys. Rev. D 13, 757 (1976).

A58. A. S. Wightman has repeatedly emphasized that the Wigner theorem on the unitary equivalence of same mass-spin representations of ISL(2,C) "is true for free-i.e., noninteracting-particles but false for interacting particles." See p. 444 of his article in Partial Differential Equations, proceedings of the 23rd Symposium in Pure Mathematics, Berkeley, California, 1971, edited by D. C. Spencer (Amer. Math. Soc., Providence, Rhode Island, 1973), p. 441.

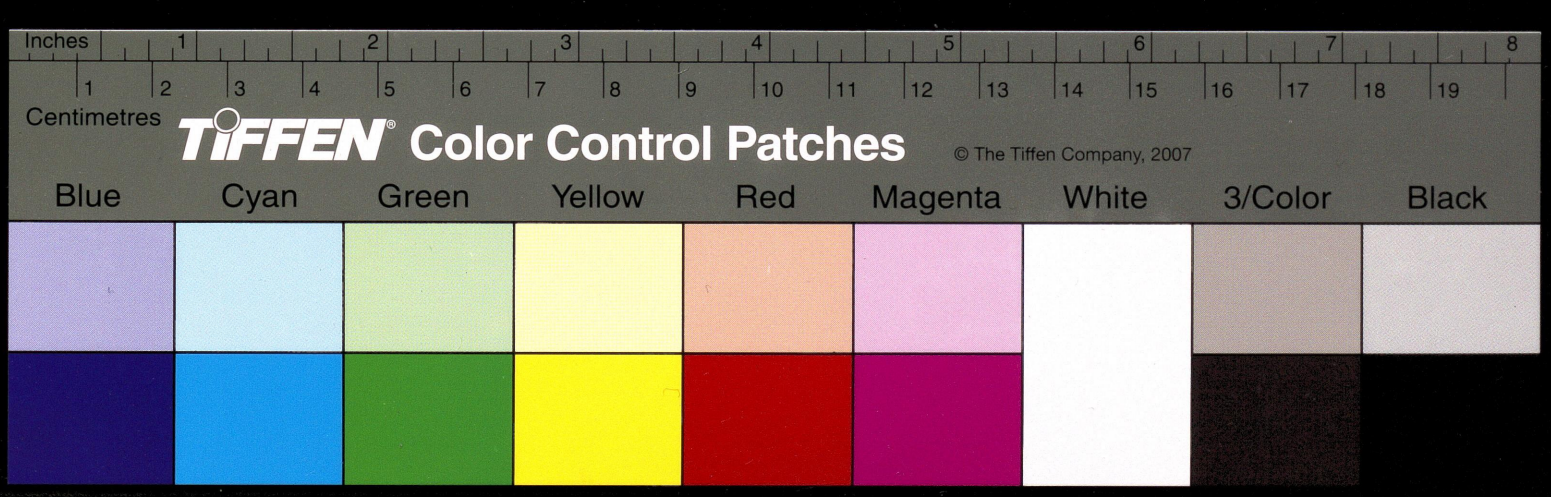
A59. R. A. Krajcik and L. L. Foldy, Phys. Rev. Lett. 24, 545 (1970).

A60. L. L. Foldy and R. A. Krajcik, Phys. Rev. Lett. 32, 1025 (1974).

A61. R. A. Krajcik and L. L. Foldy, Phys. Rev. D 10, 1777 (1974).

A62. L. L. Foldy and R. A. Krajcik, Phys. Rev. D 12, 1700 (1975).

A63. Specifically, from the results of Refs. 7, A39, and A47-A57, it was realized that many of the formal aspects of the DKP system, and hence of the Bhabha system, had never been investigated.



# I. INTRODUCTION

In this paper we close our series[1-6] on the Bhabha first-order wave equations for arbitrary spin. These equations are defined by

$$(\partial\cdot\alpha + \chi)\psi = 0, \tag{1.1}$$

where the $\alpha_\mu$ represent the $J_{\mu 5}$ generators of the so(5) algebra[7]

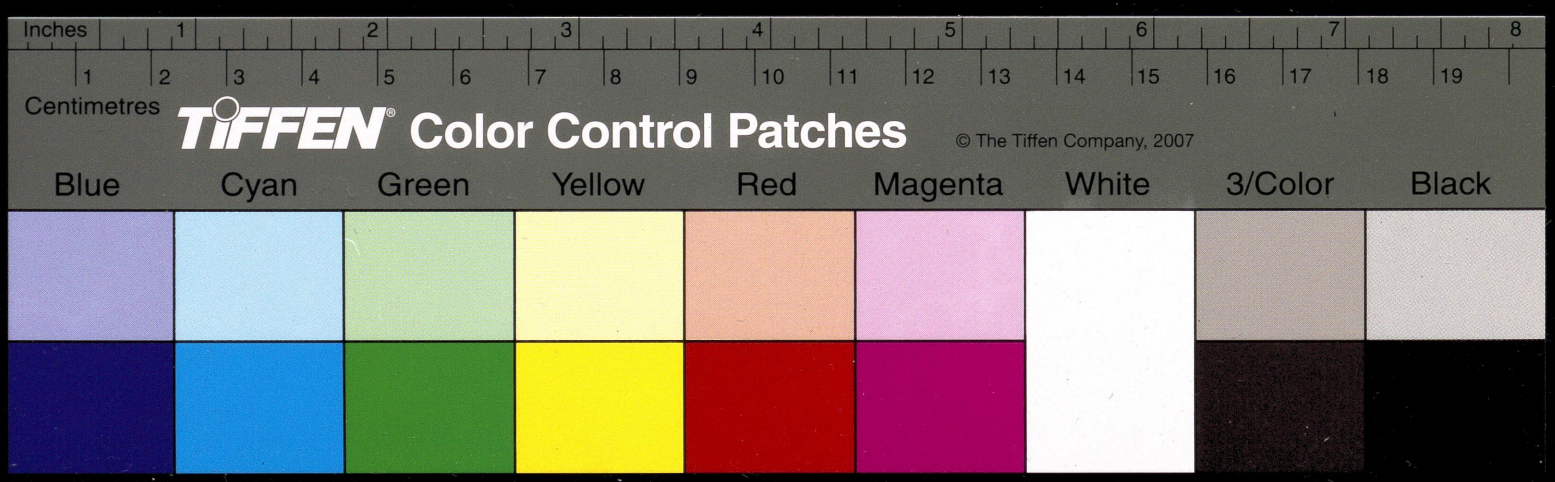
$$\alpha_\mu = J_{\mu 5} = - J_{5\mu}, \quad J_{\mu\nu} = - i[\alpha_\mu,\alpha_\nu], \quad J_{55} = 0 \ , \tag{1.2}$$

$$[J_{ab},J_{cd}] = i(\delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{bc}J_{ad} - \delta_{ad}J_{bc}) \ , \tag{1.3}$$

$$J_{ab} = - J_{ba}, \quad (a,b) = 1,2,3,4,5. \tag{1.4}$$

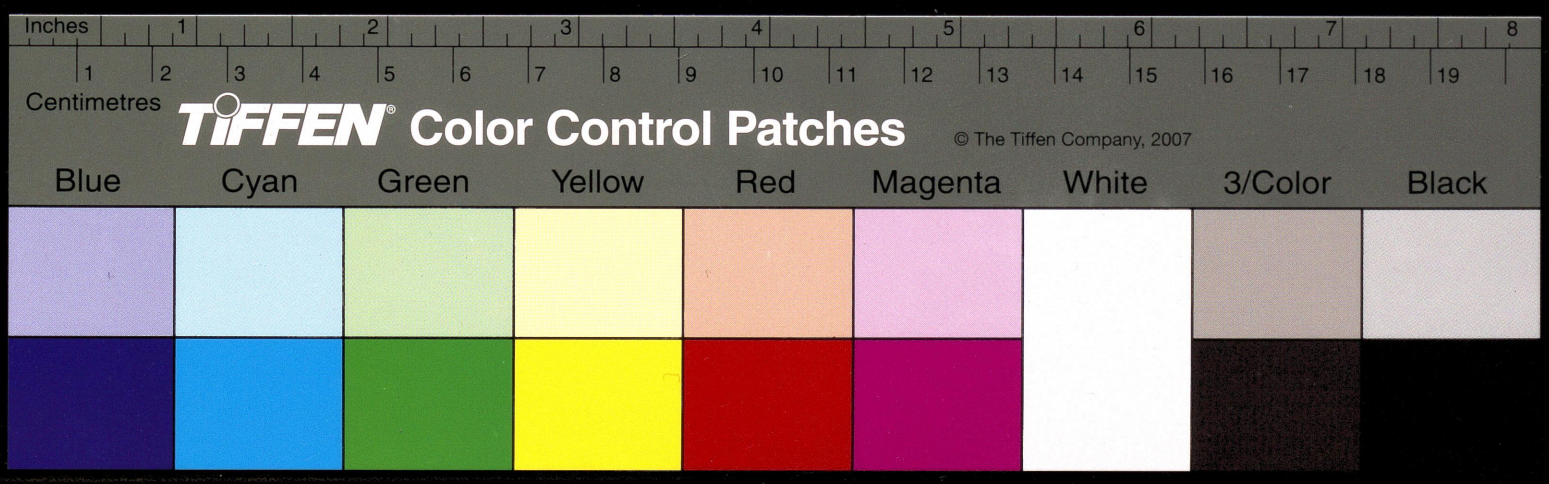
For a particular equation (1.1), the matrices $\alpha_\mu$ are taken to be a particular representation (S,S) of the so(5) algebra. For example, the (1/2,1/2) representation yields the Dirac equation and the (1,0) and (1,1) representations yield the Duffin-Kemmer-Petiau (DKP) spin-0 and spin-1 equations. Also, we define "Bhabha equations" to mean the $\alpha_\mu$ are of necessity so(5) matrices, not any first-order wave equation matrices, a definition that is sometimes used.

In the following section we will present a summary of the results we have obtained in this series. From this and the information available from other calculations[8-11] we will present in Sec. III a set of physical and mathematical conclusions about the Bhabha first-order wave equations. We have special interest in what can be said about the possiblility of finding a high-spin field theory which is both mathematically consistent and has an



acceptable physical interpretation.

As this is the last of our series, we take this occasion to add two appendices. The first, Appendix A, gives a short historical resume of the development of the Bhabha first-order wave equations. The second lists the misprints we have noted in the previously published articles of this series. Detailed lists of pertinent references can be found in our previous work.[12]



# Bhabha first-order wave equations. IV. Causality with minimal electromagnetic coupling*

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Theoretical Division, Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87544

We demonstrate that the arbitrary-spin Bhabha fields with minimal electromagnetic coupling are causal in both the c-number and q-number theories. We first obtain the Klein-Gordon (KG) divisors in closed form in terms of the elementary symmetric functions. c-number causality is easily demonstrated for half-integer spin with the Velo-Zwanziger method, and for integer spin by using Wightman's suggestion involving the KG divisors. For the q-number demonstration we set up an indefinite metric second-quantized formalism, and use the above KG divisors to show causality in closed form for arbitrary spin. In both the c-number and q-number theories a special handling of the integer spin subsidiary components is necessary. Our discussion focuses on the Bhabha indefinite metric and on the connection between the number of derivatives in a theory and the occurence or nonoccurence of causality.

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## I. INTRODUCTION

In this series of papers[1-3] we have been investigating the properties of the Bhabha first-order wave equations for arbitrary spin[4-6]

$$(\partial\cdot\alpha + \chi)\psi = 0 \ , \tag{1.1}$$

where the Bhabha matrices $\alpha_\mu$ for representations up to maximum spin $S$ are defined by the equations

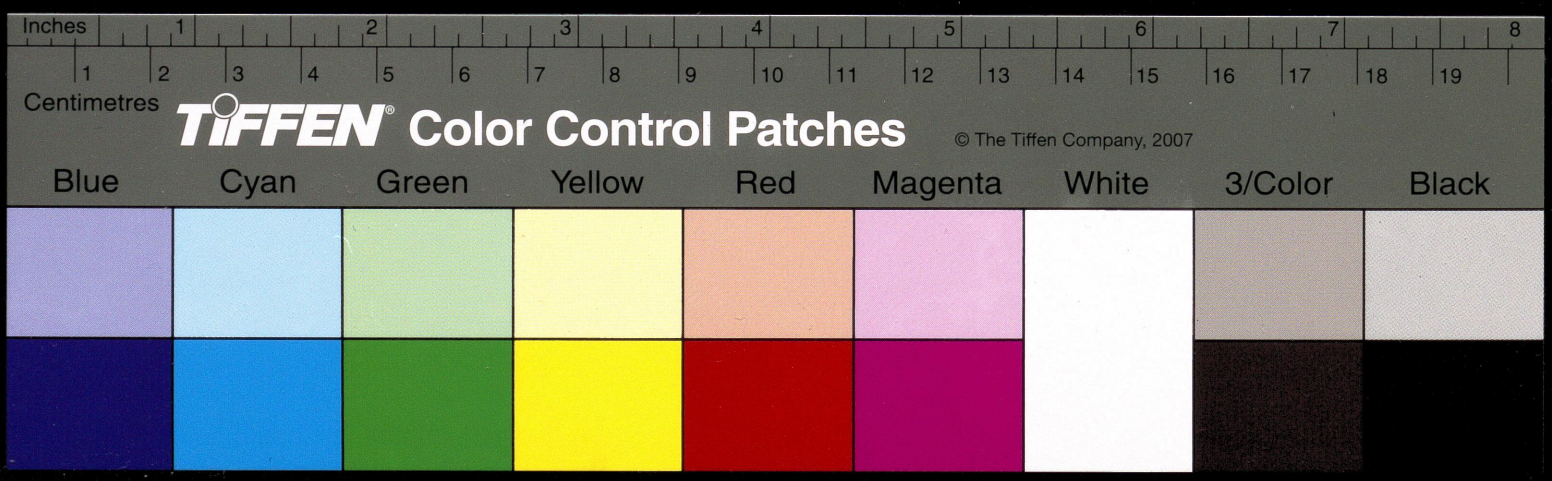
$$\left[[\alpha_\mu, \alpha_\nu], \alpha_\lambda\right] = \alpha_\mu \ \delta_{\nu\lambda} - \alpha_\nu \ \delta_{\mu\lambda} \ , \tag{1.2}$$

$$\prod_{n=-S}^{S}(\alpha_\mu - nI) = 0 \ , \tag{1.3}$$

with unity I added by hand for integer-spin representations.[7] That is, the $\alpha_\mu$ correspond to the $J_{\mu 5}$ generators of the algebra so(5).[2,4-9] It should be observed that in two papers obscured by World War II, Lubánski actually preceded Bhabha in investigating the so(5) Equations (1.1) - (1.3). Also, work by Madhavarao at least partially did the same. (See Ref. 9 for details.)

As before, our $\alpha_\mu$ matrices will be self-adjoint, we will use the metric $\delta_{\mu\nu}$ relating four-vector quantities $x_\mu = (\vec{x}, ix_0)$, $\mu = 1,2,3,4$, $\partial\cdot\alpha = \partial_\lambda\alpha_\lambda$, and $\alpha_4$ will be diagonal with eigenvalues $+S$ to $-S$. For the volume element integration in Sec. IV we will use $d^4x = dx_1dx_2dx_3dx_0$ to conform to standard usage.

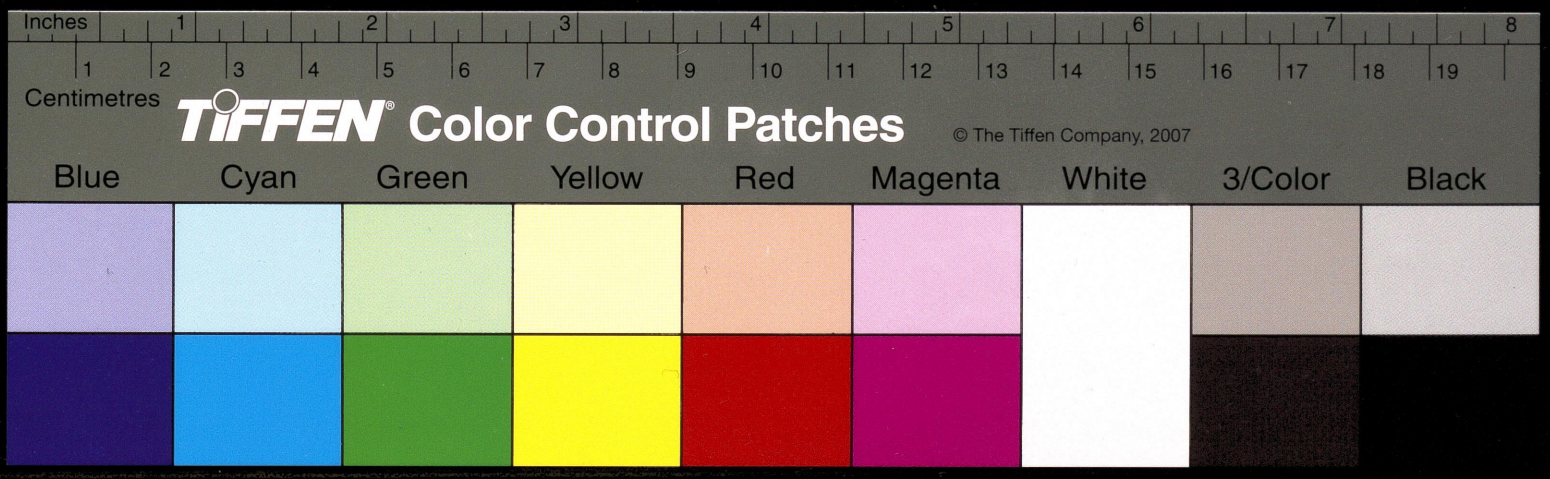
Paper I (Ref. 1) discussed the C, P, and T transformations of these equations, paper II (Ref. 2) discussed the mass and spin compositions, the Hamiltonians, and the general Sakata-Taketani reductions of the equations, and in paper III (Ref. 3) the Poincaré generators were investigated.



At the end of III[3] we re-emphasized a point made by Jauch and Rohrlich[10] for the Dirac field Poincaré generators. Given that the uncoupled generators satisfy the associated Lie algebra, they observed that, since the interaction Lagrangian $ie\bar{\psi}\gamma_\lambda A_\lambda \psi$ contains no derivatives, simply by showing that the commutation relations of the second-quantized minimally-coupled fields are preserved, one has also shown that the interacting field Poincaré generators satisfy the associated Lie algebra.

In principal this observation should go through for minimally coupled high-spin fields, such as the Bhabha field where we have already shown[3] that the free Bhabha generators in Eqs. (IIIl.5)-(IIIl.9) satisfy the associated Lie algebra. However, in practice deep problems have arisen, essentially from the constraint derivatives inherent in most high-spin equations. Johnson and Sudarshan[11] focused on the problem by showing that the spin-3/2 Rarita-Schwinger (RS) field[12] no longer obeys the proper equal-time field anti-commutation relation (zero at space-like points) when it is minimally coupled to an external electromagnetic field using the standard canonically conjugate formalism. Johnson and Sudarshan also demonstrated the same effect for a mixed spin-1/2 and -3/2 field invented by Bhabha.[13] However, as has been emphasized,[3,14-16] this field is not one of the fields we are talking about in this paper, it is another field. In fact, for the spin-3/2 piece the Lorentz group content of this special field is that of the RS field.

The results of Johnson and Sudarshan were placed on a new foundation by the important discovery of Velo and Zwanziger[17,18] that the problem could be traced even to the c-number theory. By using the method of characteristics to study the form of the hyperbolic differential equations involved, they were able to determine the propagation properties of a number of



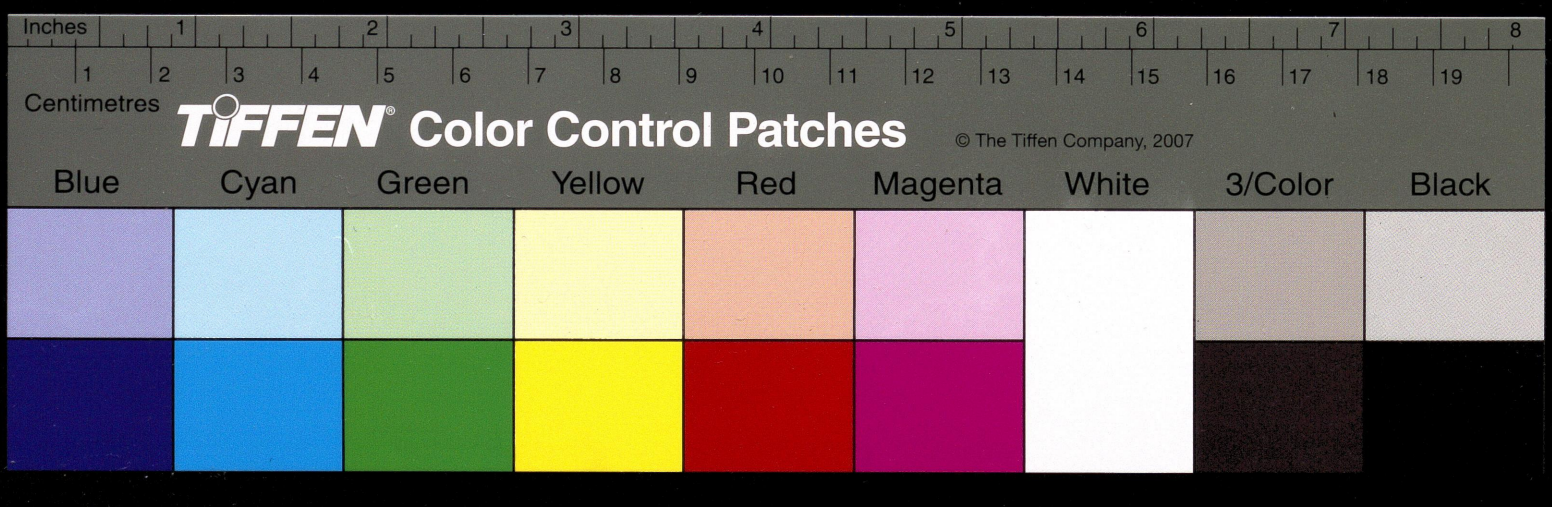
interacting field theories. In particular, they found that when minimal electromagnetic coupling is introduced into the RS spin-3/2 and the tensor spin-2 theories, they become noncausal. That is, the propagation goes outside the forward light-cone, no matter how small the interaction. The same was found to be the case for the spin-1 Proca field with electric quadrupole coupling.

Given this background a large literature has grown on the causality problem, and we site a number of examples.[15-51] These references contain discussions on arbitrary or multiple-spin fields,[15-34] as well as on specific spin-1 fields,[35-39] spin-3/2 fields,[40-47] and spin-2 fields,[48-49] among which there are a few disagreements.[38,40-41] We also specifically refer the reader to the general discussions of Wightman on causality and other problems in constructing interacting field theories.[23,24,50,51]

As of yet, no universally accepted demonstration has been given that there exists an interacting high-spin ($\geq 3/2$) field theory devoid of all problems. In fact one of the motivations of this series of papers has been to see how devoid of problems the Bhabha system is. It is the specific purpose of this paper to demonstrate that noncausality from minimal electromagnetic interaction is not one of them.

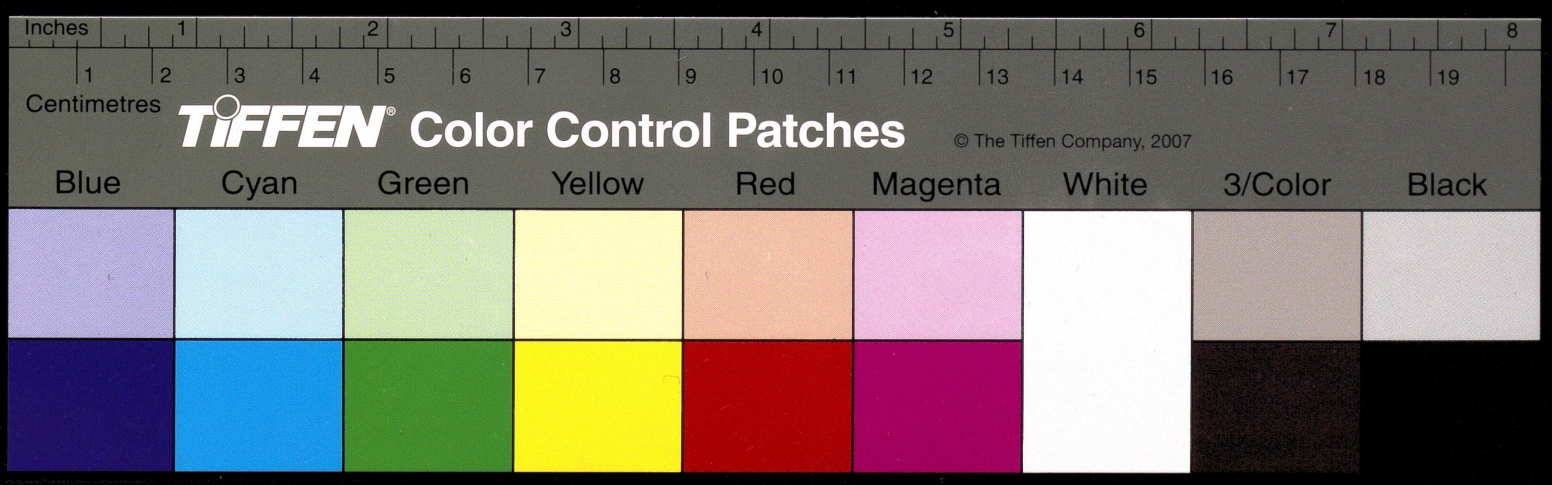
We begin in Sec. II by using the theory of symmetric functions,[52-54] discussed in the Appendix, to obtain general closed expressions for the Klein-Gordon (KG) divisors of arbitrary-spin Bhabha fields. We will use these closed expressions for the divisors to help us in both our c-number and q-number demonstrations of causality.

Section III gives the easier c-number demonstration. We first review the prescriptions for determining causality or noncausality given for nonsingular field equation matrices by Velo and Zwanziger [17,18,20] and by



Wightman,[23] and then for singular field equation matrices as discussed by Wightman [50] for DKP, based on the work of Hörmander.[55,56] We then simplify the calculations of Nagpal[16,30] and others[32] on causality for the Bhabha c-number theory with minimal electromagnetic interaction. Nagpal's first calculation[16] actually only showed causality for the half-integer-spin case since he was assuming a nonsingular $\alpha_4$. To show causality for the integer-spin case, where $\alpha_4$ is singular, involves what is a special handling of the subsidiary components, something that will also be necessary in the q-number theory. Recently Nagpal[30] realized, as we did independently,[30] that his previous c-number calculation did not apply to the integer-spin case because $\alpha_4$ is singular. He then used manipulation of the multi-mass Klein-Gordon (KG) equation to explain how the integer-spin c-number fields are causal.[30] This method is essentially the method we will use, i.e., that discussed by Wightman.[50,56] However, our demonstration will be simpler, being based on the closed form for the KG divisors given in Sec. II, and the properties of the $\alpha_\mu$ matrices we have previously discussed in this series. We will conclude this section with calculations on non-minimal coupling, renormalization, and a method of determining if there is causality in the case of constant coefficients.[23]

The causality of the minimally coupled spin-3/2 Bhabha q-number field was first shown by Baisya.[15,22] Nagpal later extended these results by an iteration method to show how the calculation goes through for arbitrary half-integer-spin Bhabha fields, but he also came to the conclusion that the integer-spin q-number fields were noncausal.[16] However, recently Nagpal[30] changed his previous conclusions[16] about the integer-spin fields being non-causal and, by case by case analyses of the systems of q-number equations, came to the causality conclusions we have independently[30] arrived at. The



advantage of our method is that we will use a proper indefinite metric quantization and will be able to do the general calculations in simple closed form for arbitrary spin by using the theory of symmetric functions[52-54] reviewed in the Appendix.

In Secs. IV-V we will present our demonstration that arbitrary spin Bhabha fields with minimal electromagnetic coupling are causal in the q-number theory. Section IV presents the formalism. In this discussion we will emphasize a physically important variation of our method from that of Baisya and Nagpal. We will perform an indefinite metric second-quantization with an interaction Hamiltonian that is metric-(pseudo) Hermitian (which, as discussed in Sec. II of III,[3] it properly should be). We also do this because one should come squarely face to face with the fact that there is a built in indefinite metric[57-61] in the Bhabha system.[62-64]

Having presented the formalism, we then demonstrate in Sec. V that the coupled Heisenberg fields have the same form as the free auxiliary fields for the half-integer spin case, and hence show that the field anti-commutation relations are preserved. For the integer-spin case we will show that although the coupled Heisenberg fields contain a new piece, these new pieces are multiplied by the projection operators onto the subsidiary components $I_0(S)$ defined in Eq. (II 3.8). Thus, the physical particle components fields have their commutation relations preserved and so are also causal. (This extra piece is well known to exist in the DKP case.)

In Sec. VI we will review our results and comment on a number of points. These include the connection of the extra subsidiary components terms in the integer-spin q-number theory both to the c-number integer-spin subtlety and also to the effects of extra derivatives in couplings. We will also discuss the significance of the indefinite metric to the physical interpretation and mathematical consistency of the Bhabha theory. In V of this series[65] we will discuss the indefinite metric in more detail, and calculate the generalized Foldy-Wouthuysen transformations for Bhabha fields.

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## II. KLEIN-GORDON DIVISORS

The Klein-Gordon divisors for general Bhabha fields were discussed in principle by Umezawa and Visconti.[66] (But see Glass[67] concerning errors with the Harish-Chandra[68-72] algebra.) Special cases were calculated by Umezawa and Visconti,[66] Baisya[15,22] and Nagpal,[16] who also gave a complicated expression for general integer spin.[30] In this section we will derive general, closed form expressions for the KG divisors in terms of the elementary symmetric functions. These explicit expressions have, to our knowledge, never been given before. We will discuss the half-integer and integer-spin cases separately, and then proceed in the following two sections to perform the causality calculations.

### A. Half-integer spin

For the half-integer-spin case the Klein-Gordon divisors $\mathcal{D}(\partial, S=n+\frac{1}{2})$ are the solutions to the equations

$$\mathcal{D}(\partial, S = n + \tfrac{1}{2})\Lambda \equiv \mathcal{D}(\partial, S = n + \tfrac{1}{2})(\partial\cdot\alpha + \chi)$$

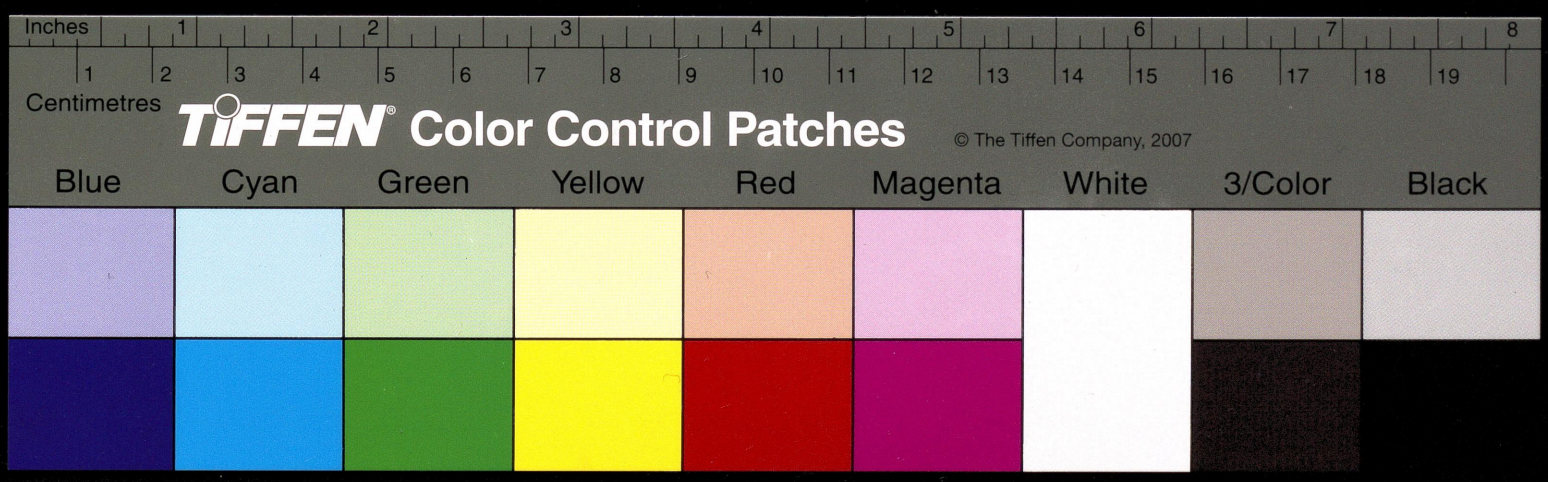
$$= \prod_{j=1}^{S+\frac{1}{2}} \left( \Box - \chi^2/(j - \tfrac{1}{2})^2 \right)$$

$$= \sum_{k=0}^{S+\frac{1}{2}} (-1)^{S+\frac{1}{2}-k} \Box^k (\chi^2)^{S+\frac{1}{2}-k} \phi_{S+\frac{1}{2}-k}(\ell_{\{S+\frac{1}{2}\}}) , \qquad (2.1)$$

where the elementary symmetric functions $\phi_{S+\frac{1}{2}-k}$ of the $(S+\frac{1}{2})$ quantities

$$\ell_k = \frac{1}{(k - \frac{1}{2})^2} \qquad (2.2)$$

are described in the Appendix. For the reasons given in the next paragraph, we can write the KG divisors in the form



$$\mathcal{D}(\partial, S = n + \tfrac{1}{2}) = \sum_{i=0}^{S-\frac{1}{2}} \chi^{2S-2i} \left[1 - (\partial\cdot\alpha)/\chi\right] \sum_{j=0}^{i} \Box^{i-j} (\partial\cdot\alpha)^{2j} c_{ij} , \qquad (2.3)$$

where the $c_{ij}$, $i \geqslant j$, remain to be determined.

The factor $(1 - (\partial\cdot\alpha)/\chi)$ in Eq. (2.3) comes about because, since

$$\chi \left(1 - (\partial\cdot\alpha)/\chi\right)\Lambda = \left(\chi^2 - (\partial\cdot\alpha)^2\right) \qquad (2.4)$$

and the right hand side of Eq. (2.4) is a factor in the characteristic KG equation for the $\alpha_\mu$, only then will one have the proper even powers of $\chi$ on the left hand side of Eq. (2.1). In particular, rewriting (2.1) as

$$\sum_{i=0}^{S-\frac{1}{2}} \chi^{2S-1-2i} \left[\chi^2 - (\partial\cdot\alpha)^2\right] \sum_{j=0}^{i} \Box^{i-j} (\partial\cdot\alpha)^{2j} c_{ij}$$
$$= \sum_{k=0}^{S+\frac{1}{2}} (-1)^{S+\frac{1}{2}-k} \Box^k (\chi^2)^{S+\frac{1}{2}-k} \phi_{S+\frac{1}{2}-k} \left(\ell_{\{S+\frac{1}{2}\}}\right), \qquad (2.5)$$

one has just the powers $\chi^0, \chi^2, \ldots \chi^{2S+1}$.

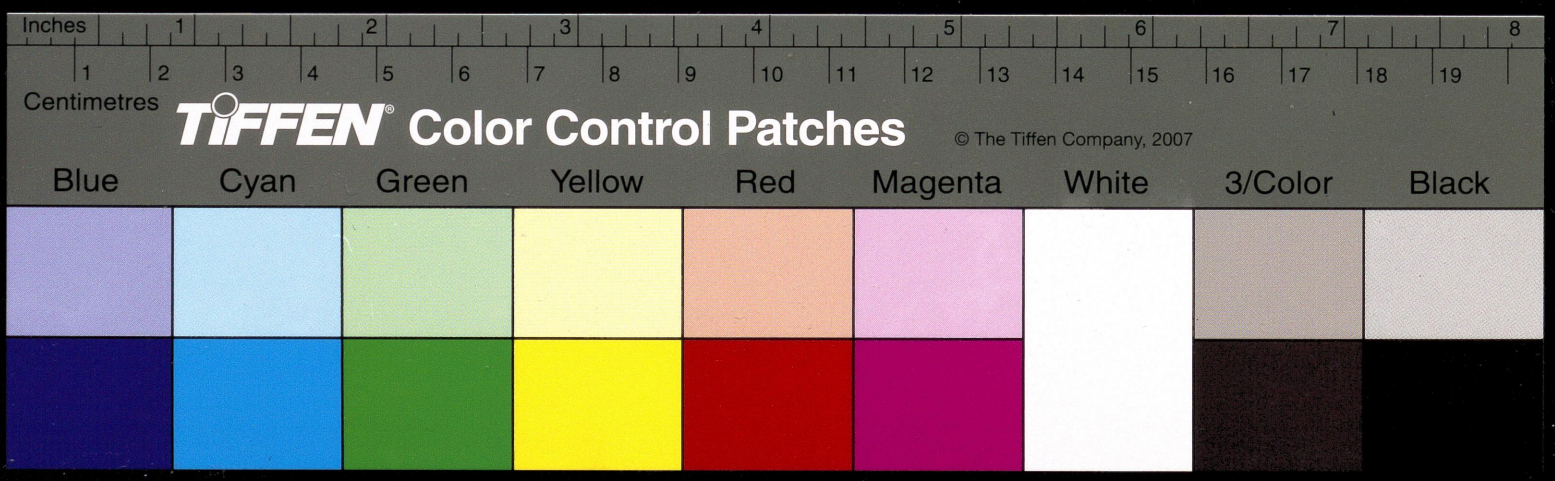
Also, the second sum on the left of (2.5) is the sum of all the possible products $\Box^{i-j}(\partial\cdot\alpha)^{2j}$ such that each term of the outside sum has the proper dimension $(\text{mass})^{2S+1}$. Our problem is now reduced to finding the quantities $c_{ij}$.

Starting with the $\chi^{2S+1}$ term in Eq. (2.5) one sees by inspection that

$$c_{00} = (-1)^{S+\frac{1}{2}} \phi_{S+\frac{1}{2}} (\ell) . \qquad (2.6)$$

Next taking the $\chi^{2S-1}$ term in Eq. (2.5), one has the equation

$$\Box c_{10} + (\partial\cdot\alpha)^2 c_{11} - (\partial\cdot\alpha)^2 c_{00} = (-1)^{S-\frac{1}{2}} \Box \phi_{S+\frac{1}{2}-1}(\ell) , \qquad (2.7)$$



which yields the solution

$$c_{11} = c_{00},$$

$$c_{10} = (-1)^{S-\frac{1}{2}} \phi_{S-\frac{1}{2}}(\ell) . \tag{2.8}$$

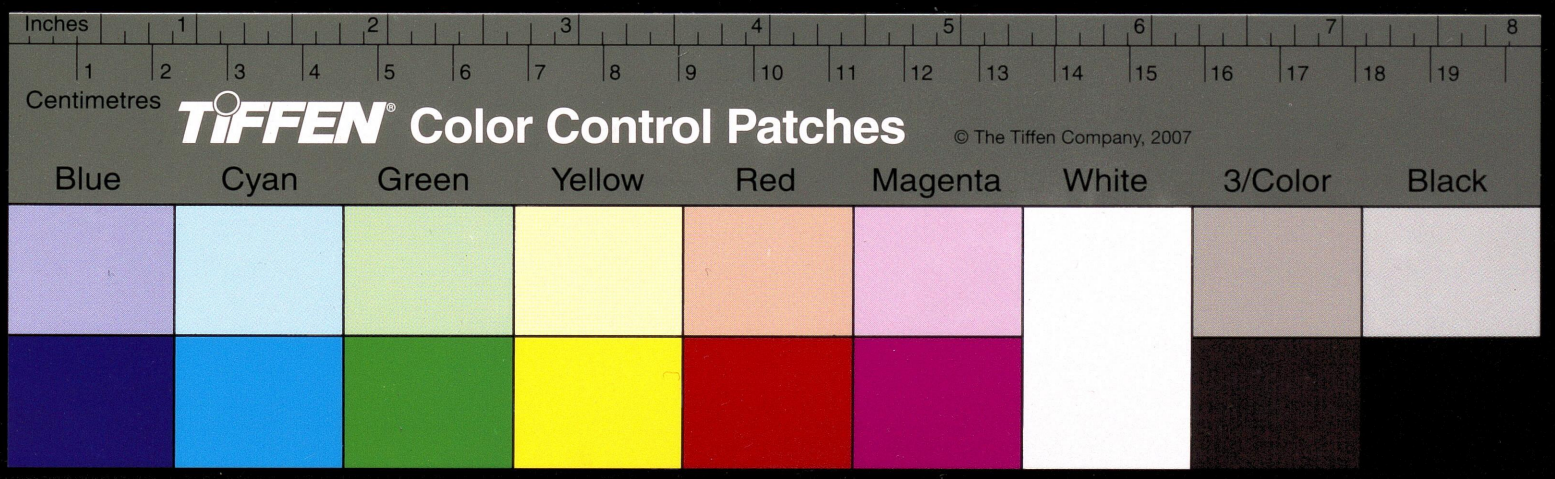
Similarly proceeding to the $\chi^{2S-3}$ and $\chi^{2S-5}$ terms yields the solutions

$$\begin{aligned} c_{33} = c_{22} = c_{11} = c_{00} &= (-1)^{S+\frac{1}{2}} \phi_{S+\frac{1}{2}}(\ell) , \\ c_{32} = c_{21} = c_{10} &= (-1)^{S+\frac{1}{2}-1} \phi_{S+\frac{1}{2}-1}(\ell) , \\ c_{31} = c_{20} &= (-1)^{S+\frac{1}{2}-2} \phi_{S+\frac{1}{2}-2}(\ell) , \\ c_{30} &= (-1)^{S+\frac{1}{2}-3} \phi_{S+\frac{1}{2}-3}(\ell) , \end{aligned} \tag{2.9}$$

suggesting that the general solution for the $c_{ij}$ is

$$c_{ij} = (-1)^{S+\frac{1}{2}-i+j} \phi_{S+\frac{1}{2}-i+j}\left(\ell_{\{S+\frac{1}{2}\}}\right) . \tag{2.10}$$

The solution (2.10) can be demonstrated by induction. From Eq. (2.9), Eq. (2.10) is true for $(i,j) = 0,1,2,3$. Now assume that Eq. (2.10) is true for $n = i \geqslant j$. Then by considering the case $(n+1) = i \geqslant j$ one finds that Eq. (2.10) still holds for $(S-\frac{1}{2}) \geqslant (n+1)$. Finally, by inspection the solution holds for $\chi^0$ term.



Thus, the KG divisors for half-integer spin Bhabha fields are

$$\mathcal{D}(\partial, S = n + \tfrac{1}{2}) = \sum_{i=0}^{S-\frac{1}{2}} \chi^{2S-2i} \left[1 - \frac{(\partial\cdot\alpha)}{\chi}\right] \sum_{j=0}^{i} \Box^{i-j} (\partial\cdot\alpha)^{2j} (-1)^{S+\frac{1}{2}-i+j} \, \phi_{S+\frac{1}{2}-i-j}\left(\ell_{\{S+\frac{1}{2}\}}\right) \tag{2.11}$$

$$\phi_0 = 1 \,, \tag{2.12a}$$

$$\phi_{j\neq 0}\left(\ell_{\{S+\frac{1}{2}\}}\right) \equiv \left\{ \sum_{i(1) > i(2) > \ldots > i(j) = 1}^{S+\frac{1}{2}} {}^{[\ell_{i(1)} \ell_{i(2)} \ldots \ell_{i(j)}]} \right\} \,, \tag{2.12b}$$

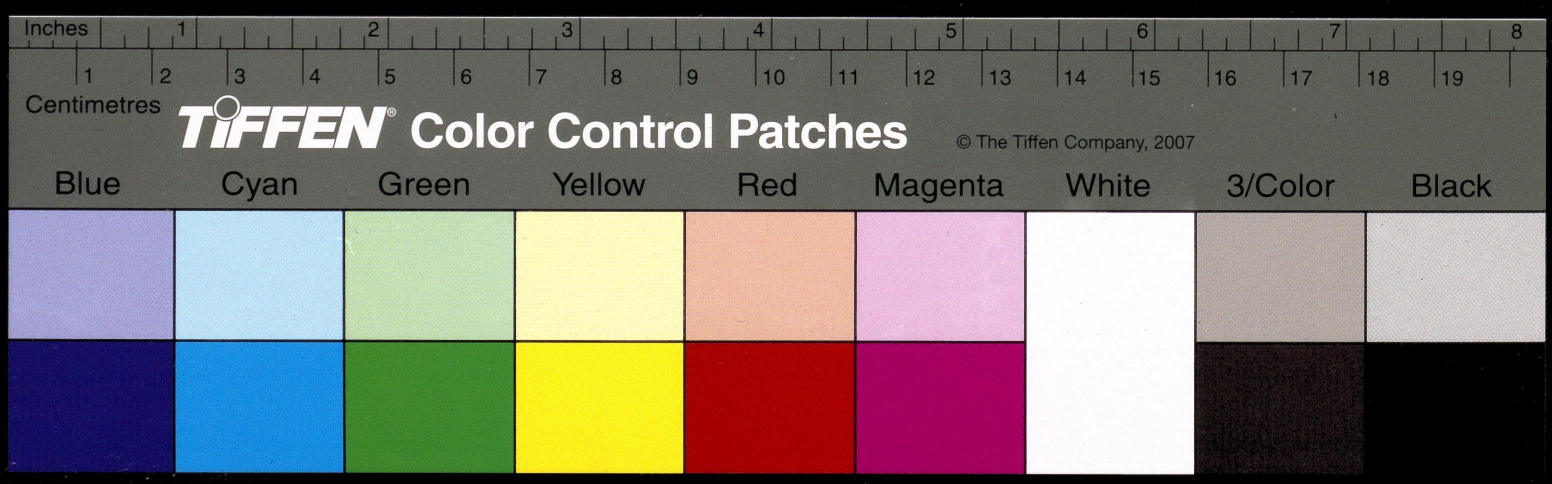
$$\ell_k \equiv \frac{1}{(k - \frac{1}{2})^2} \,. \tag{2.12c}$$

Special cases are

$$\mathcal{D}(\partial, S = \tfrac{1}{2}) = -4\chi \left[1 - \frac{(\partial\cdot\alpha)}{\chi}\right] \,, \tag{2.13}$$

$$\mathcal{D}(\partial, S = \frac{3}{2}) = \frac{16}{9} \chi^3 \left[1 - \frac{(\partial\cdot\alpha)}{\chi}\right] + \frac{16}{9} \chi \left[1 - \frac{(\partial\cdot\alpha)}{\chi}\right] \left[(\partial\cdot\alpha)^2 - \frac{40}{16} \Box\right] \,, \tag{2.14}$$

$$\mathcal{D}(\partial, S = \frac{5}{2}) = -\frac{64}{225} \chi^5 \left[1 - \frac{(\partial\cdot\alpha)}{\chi}\right] \left[1 + \chi^{-2} \left((\partial\cdot\alpha)^2 - \frac{35}{4} \Box\right) + \chi^{-4} \left((\partial\cdot\alpha)^4 - \frac{35}{4} (\partial\cdot\alpha)^2 \Box + \frac{259}{16} \Box^2\right)\right] \,. \tag{2.15}$$



$$\mathcal{D}(\partial, S = n) = \sum_{i=0}^{S} \left(1 - \frac{\partial\cdot\alpha}{\chi}\right) \chi^{2(S-i)-1} \sum_{j=0}^{i} A_{ij} \tag{2.18a}$$

$$= \left[\sum_{i=0}^{S-1} \left(1 - \frac{\partial\cdot\alpha}{\chi}\right) \chi^{2(S-i)-1} \sum_{j=0}^{i} A_{ij}\right] + \chi^{-1} \sum_{j=0}^{S} A_{Sj} \quad , \tag{2.18b}$$

$$A_{ij} = \Box^{i-j} (\partial\cdot\alpha)^{2j} (-1)^{S-i+j} \phi_{S-i+j}(L_{\{S\}}) \quad , \tag{2.19}$$

$$\phi_0 = 1 \ , \tag{2.20a}$$

$$\phi_{j\neq 0}(L_{\{S\}}) = \left\{ \sum_{i(1) > i(2) > \cdots > i(j) = 1}^{S} \left[L_{i(1)} L_{i(2)} \cdots L_{i(0)}\right] \right\}, \tag{2.20b}$$

$$L_k = \frac{1}{k^2} \quad . \tag{2.20c}$$

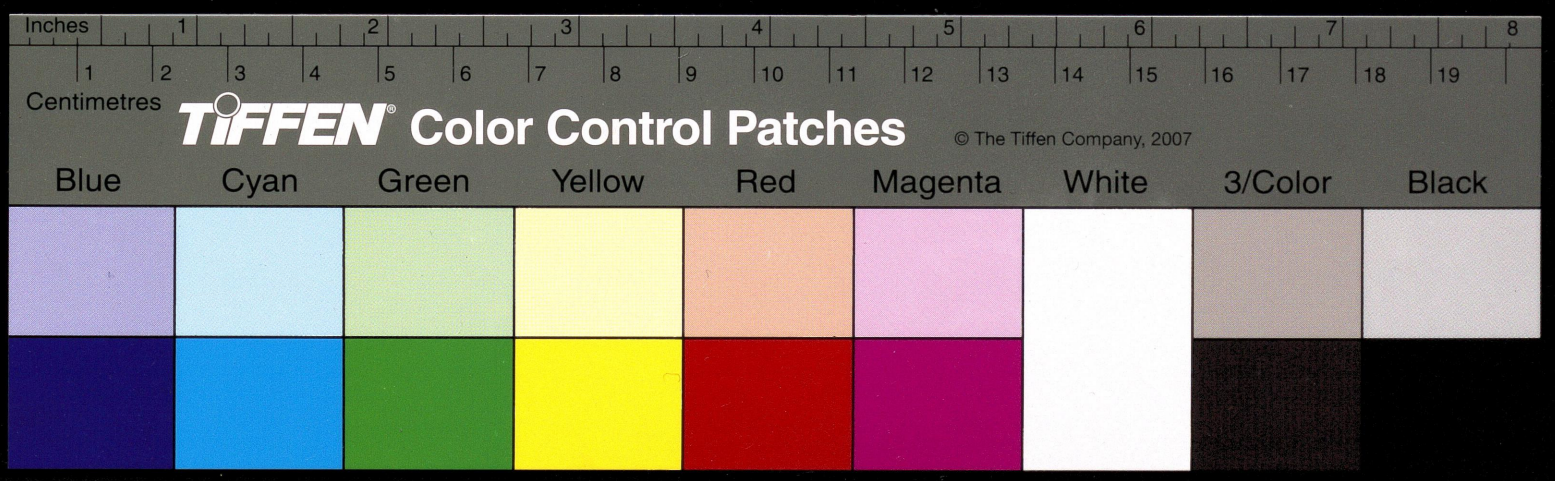
The second form in Eq. (2.18) comes by taking the characteristic equation (1.3) for $\alpha_4$, multiplying it by $(\partial_4)^{2S+1}$ and then transforming it to an arbitrary system, yielding

$$(\partial\cdot\alpha) \sum_{j=0}^{S} A_{Sj} = 0 \tag{2.21}$$

The extra factor $(\partial\cdot\alpha)$, of course, comes from the subsidiary components (the 0 eigenvalues of $\alpha_4$), and is not accounted for in the KG operator of Eq. (2.16). This will cause $\mathcal{D}(\partial, S = n)$ to yield extra pieces in the integer-spin Heisenberg interacting fields, as we will see in Sec. V.

Special cases of Eq. (2.18) are

$$\mathcal{D}(\partial, S = 1) = \chi^{-1}[(\partial\cdot\alpha)(\chi - \partial\cdot\alpha) + (\Box - \chi^2)] \ , \tag{2.22}$$



Equation (2.13) is the Dirac equation KG divisor, only in the Bhabha normalization instead of the usual $\gamma$-matrix normalization. Equation (2.14) for $S = 3/2$ agrees with the Baisya[15,22] and Nagpal[16] calculations, and Eq. (2.15) agrees[73] with Nagpal's[16] calculation. Also, one can easily demonstrate that the KG divisors in Eqs. (2.13)-(2.15) satisfy the original defining Eq. (2.1).

## B. Integer spin

For the integer-spin case the Klein-Gordon divisors $\mathcal{D}(\partial, S = n)$ are the solutions to the equations

$$\mathcal{D}(\partial, S = n)\Lambda \equiv \mathcal{D}(\partial, S = n)\ (\partial\cdot\alpha + \chi)$$

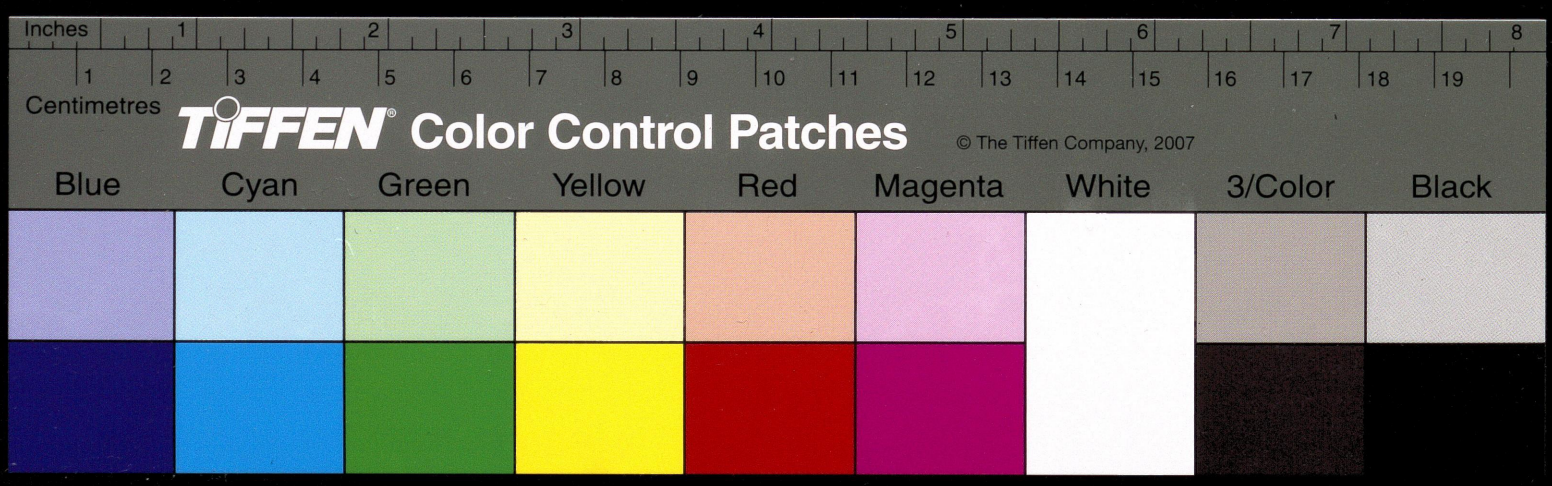
$$= \prod_{j=1}^{S} (\Box - \chi^2/j^2)$$

$$= \sum_{k=0}^{S} \Box^k\ (\chi^2)^{S-k}\ (-1)^{S-k}\ \phi_{S-k}\left(L_{\{S\}}\right), \qquad (2.16)$$

where now we are dealing with the elementary functions $\phi_{S-k}$ of the $S$ quantities

$$L_k = \frac{1}{k^2} \ . \qquad (2.17)$$

The solution proceeds in the same manner as for the half-integer spin case, and one finds that



$$\mathcal{D}(\partial, S=2) = \frac{1}{4}\chi\left(1-(\partial\cdot\alpha)/\chi\right)\left[\chi^2+(-5\Box+(\partial\cdot\alpha)^2\right]$$

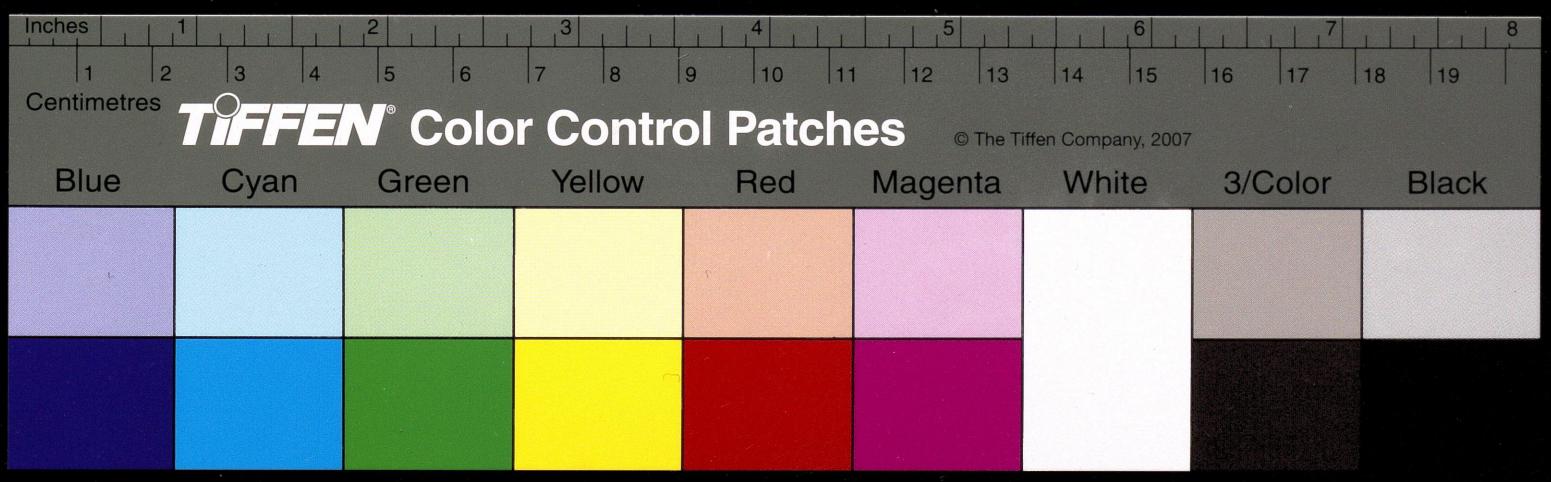
$$+\frac{1}{4}\chi^{-1}\left[4\Box^2-5\Box(\partial\cdot\alpha)^2+(\partial\cdot\alpha)^4\right] , \qquad (2.23)$$

$$\mathcal{D}(\partial, S=3) = \frac{-1}{36}\chi^5\left(1-\frac{\partial\cdot\alpha}{\chi}\right)+\frac{\chi^3}{36}\left(1-\frac{\partial\cdot\alpha}{\chi}\right)\left[14\Box-(\partial\cdot\alpha)^2\right]$$

$$+\frac{\chi}{36}\left(1-\frac{\partial\cdot\alpha}{\chi}\right)\left[-49\Box^2+14\Box(\partial\cdot\alpha)^2-(\partial\cdot\alpha)^4\right]$$

$$+\frac{\chi^{-1}}{36}\left[36\Box^3-49\Box^2(\partial\cdot\alpha)^2+14\Box(\partial\cdot\alpha)^4-(\partial\cdot\alpha)^6\right] . \qquad (2.24)$$

Equation (2.22) is the DKP Klein-Gordon Divisor. The special cases (2.23) and (2.24) agree with the calculations of Nagpal,[16] as well as the $q=0$, $n=2,3$ special cases of Wightman's[23] Eq.(2.35) generalized Harish-Chandra[68-72] KG divisors. Again one can easily demonstrate that the special cases (2.22) - (2.24) satisfy the defining Eq. (2.16).



## III. CAUSALITY OF THE c-NUMBER THEORY

The method used by Velo and Zwanziger[17,18] to discuss causality in the c-number theory consists of first writing the wave equation in the form

$$\Lambda(\partial,\chi,C)\psi = (\partial\cdot\alpha + \chi + C)\psi = 0 \ , \tag{3.1}$$

where $C$ is a matrix describing the interaction, and then solving the determinant equation

$$\text{Det}\ \hat{\Lambda}\left(\partial_\lambda \to in_\lambda, \chi, C(\partial_\lambda \to in_\lambda)\right) = 0 \tag{3.2}$$

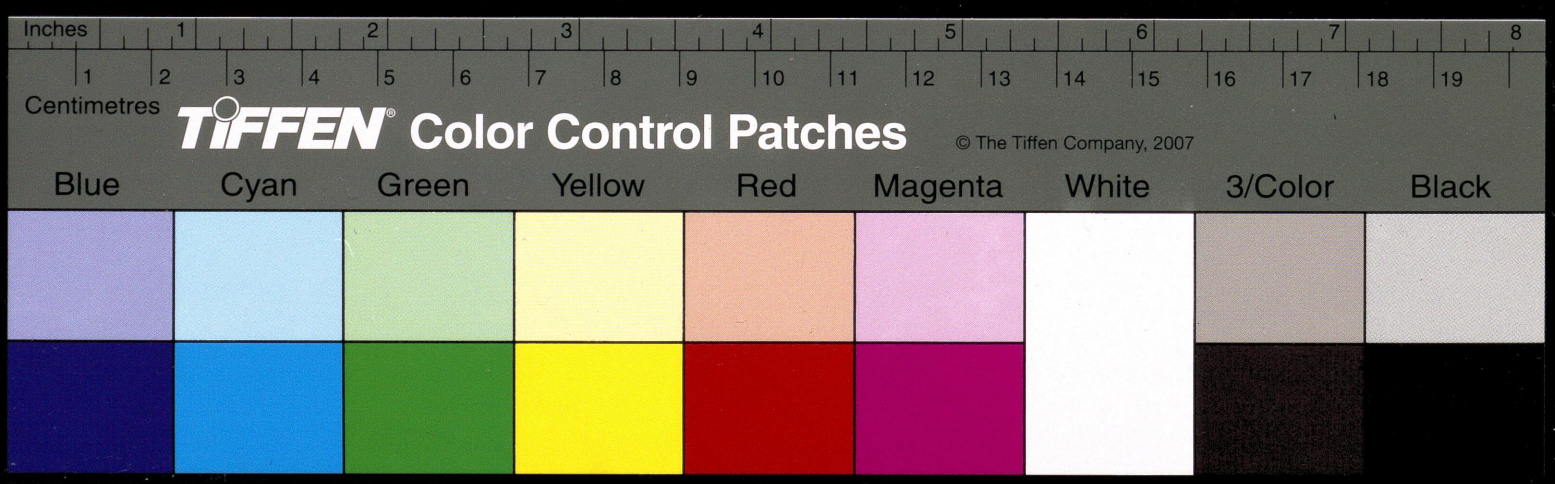
for $n_\lambda$, where $\hat{\Lambda}$ is that matrix piece of $\Lambda$ which will yield the highest powers of $\partial_\lambda$ in the determinant. The $n_\lambda$ are the normals to the characteristic surfaces, so for the propagation to the causal one wants solutions of $n_\lambda$ from (3.2) which are never timelike. This method is applicable if the matrix $\hat{\Lambda}$ is nonsingular.

For Bhabha fields, with the minimal substitution

$$\partial_\lambda \to \partial_\lambda^- \equiv \partial_\lambda - ieA_\lambda ; \qquad p_\lambda^- = p_\lambda - eA_\lambda \ , \tag{3.3}$$

the interaction $-ieA_\lambda$ enters into the matrix elements of $\Lambda$ only as $\partial_\lambda^-$. Thus, the determinant of the highest derivatives will have only the highest products of $\partial_\lambda$ without any $A_\lambda$ involved, so that

$$\text{Det}\ \hat{\Lambda}(in_\lambda) = \text{Det}\ (in\cdot\alpha) = 0 \ . \tag{3.4}$$



To rule out timelike $n_\lambda$ one can show a contradiction. (At this stage it is an assumption that such solutions do or do not exist.) Suppose that a timelike $n_\lambda$ existed as a solution to Eq. (3.4). Then one could perform a Lorentz transformation to a frame where

$$0 = \text{Det}\ \hat{\Lambda} = \text{Det}\ (in_4\ \alpha_4) \quad . \tag{3.5}$$

But then taking the diagonal representation for $\alpha_4$ in the notation of II,

$$\alpha_4 = \text{block diagonal}\ [SI_S^+,\ (S\text{-}1)I_{S-1}^+,\ \ldots\ ,\ -(S\text{-}1)I_{S-1}^-,\ -SI_S^-] \quad , \tag{3.6}$$

one would have

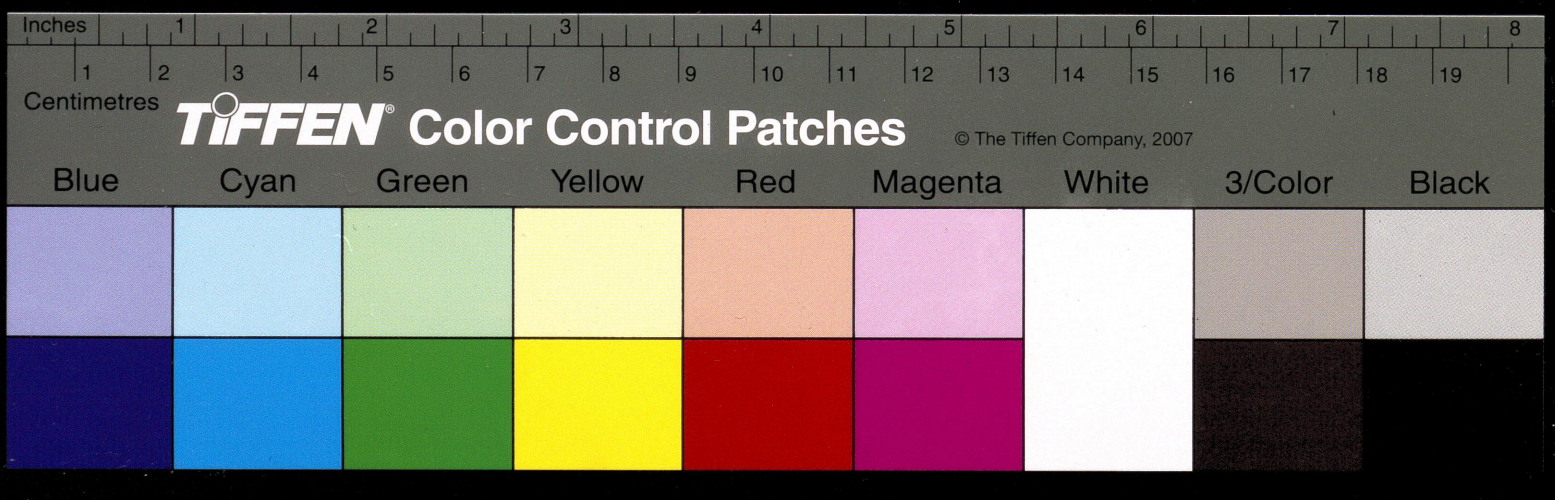
$$0 = \text{Det}\ \hat{\Lambda} = (in_0)^{[d_5(S,S)-a(0)]}\ 0^{a(0)} \prod_{j=(\frac{1}{2},1)}^{S} (S\text{-}j)^{2a(j)} \quad , \tag{3.7}$$

where the a(j) are the dimensionalities of the j eigenvalue block of $\alpha_4$ and the dimensionality of the matrices of an arbitrary Bhabha algebra (S,S) is

$$d_5(S,S) = \tfrac{1}{6}(2S + 3)(2S + 1)[(S + 1)(S + 2) - S(S + 1)] \tag{3.8}$$

$$\equiv a(0) + \sum_{j=(\frac{1}{2},1)}^{S} 2a(j) \quad . \tag{3.9}$$

For half-integer spin representations, there is no a(0), so (3.7) implies the contradiction $n_0 = 0$, ruling out timelike solutions. (The same argument can be used to rule out spacelike solutions for the $n_\lambda$ . In fact, the solutions are light-like; i.e., the propagation is bounded by the light cone.)



Because of the factor $0^{a(0)}$ in Eq. (3.7) for integer-spin Bhabha fields (the $\alpha_\lambda$ are singular), the above method is not applicable there. Then one has to use the method Wightman[50,56,74] attributed to Svensson[74] and discussed in Ref. 50 for Dirac and DKP fields. This method states that given the KG divisor, if one can show that

$$\mathcal{D}(\partial^-,S)\Lambda(\partial^-) = \prod_{j=(\frac{1}{2},1)}^{S} (\partial^-\cdot\partial^- - \chi^2/j^2) + K \ , \qquad (3.10)$$

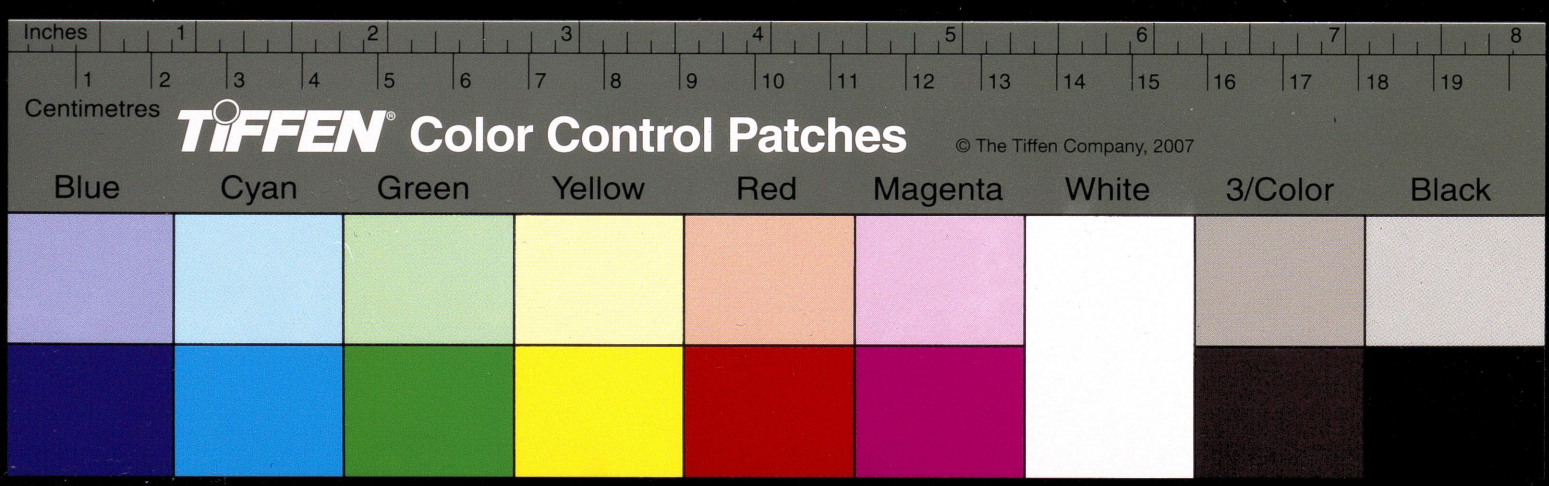
where K is a matrix whose principle part involves derivatives of order less than those in the KG multimass operator on the right of (3.10); i.e. less than order $(2S + 1)$ for half-integer spin and less than order $(2S)$ for integer spin, then the fields are causal whether or not the $\alpha_\lambda$ matrices are singular.

From the general closed forms for the KG divisors given in Sec. II, this can easily be shown.

First one realizes that the term K is non-zero only because the $\partial_\lambda^-$ do not commute,

$$[\partial_\mu^-,\partial_\nu^-] = - ie\, F_{\mu\nu} \ . \qquad (3.11)$$

So, the maximum power of $\partial_\lambda$ in K comes from that piece involving a single commutation relation of the type (3.11). Since the maximum power of $\partial_\lambda$ in the half-integer spin KG divisor (2.11) is $(2S)$, the product of $\mathcal{D}(\partial^-)$ with $\Lambda(\partial^-)$ will yield a maximum power of $\partial_\lambda$ of $(2S + 1)$. Therefore, the maximum derivative in K will be of order $(2S - 1)$, two orders less than that in the KG operator on the right of (3.11), implying causality.



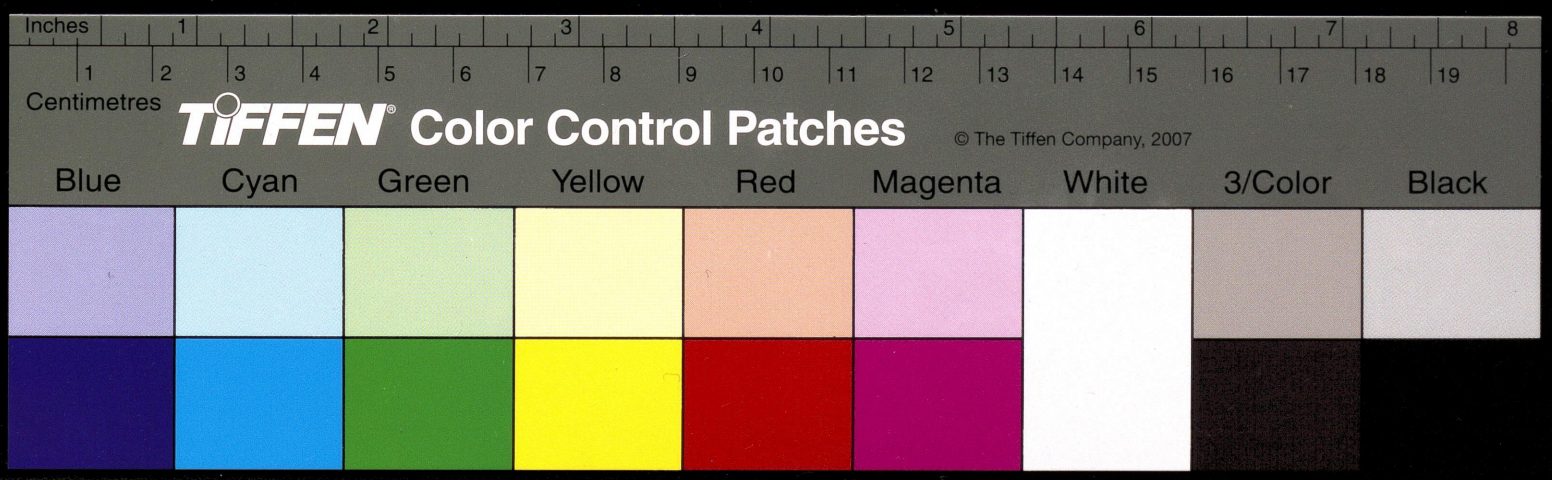
For integer-spin fields, one should start with the form of the KG divisor given in Eq. (2.18b). Then, in a similar manner; the KG divisor is of order (2S), meaning the product $\mathcal{D}\Lambda$ will have deriviatives of order (2S + 1), so that the maximum power derivative in the matrix K will be of order (2S - 1). This is one order less than the (2S) maximum power in the KG operator on the right of (3.10), so one has shown causality for Bhabha minimally coupled integer-spin fields.

For reference, using the standard notation instead of the Bhabha notation, the Dirac and DKP explicit results are

$$\mathcal{D}(\partial^-, S = \tfrac{1}{2})\Lambda(\partial^-) = (\partial^-\cdot\partial^- - m^2) + e\, F_{\mu\nu}\,[\partial_\mu,\partial_\nu]/(4i)\ , \tag{3.12}$$

$$\begin{aligned}\mathcal{D}(\partial^-, S = 1)\Lambda(\partial^-) = (\partial^-\cdot\partial^- - m^2) + \frac{ie}{2m}\Big[&(\partial^-\cdot\beta)\, F_{\mu\nu}\beta_\mu\beta_\nu \\ &+ F_{\mu\nu}\beta_\mu(\partial^-\cdot\beta)\beta_\nu + \partial^-_\lambda\, F_{\mu\nu}\beta_\mu\beta_\nu\beta_\lambda \\ &- \partial^-_\mu\, F_{\mu\nu}\beta_\nu - F_{\mu\nu}\,\partial^-_\mu\,\beta_\nu\Big]\ .\end{aligned} \tag{3.13}$$

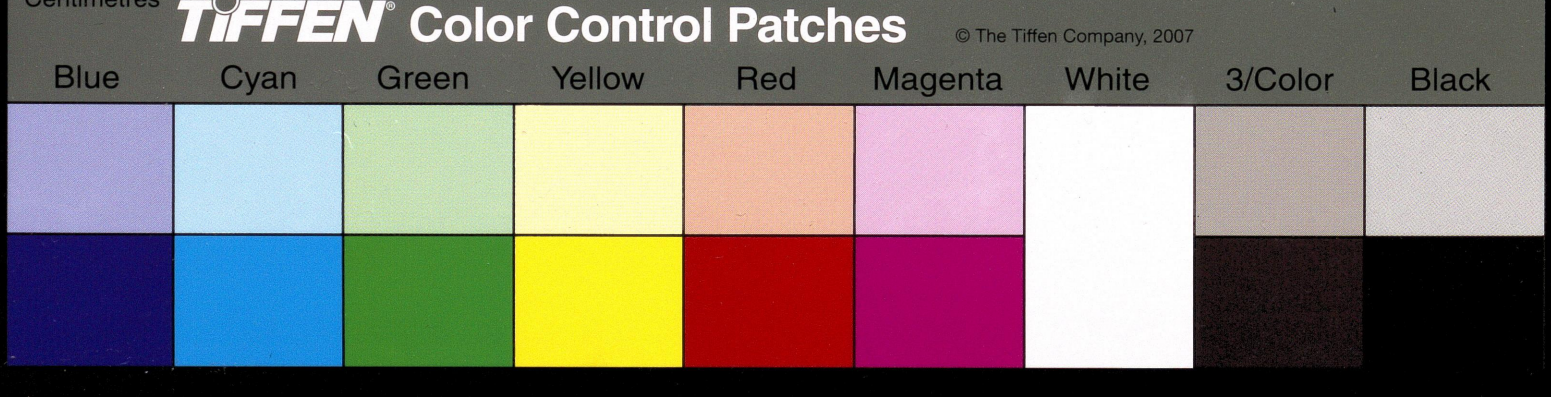
Finally we mention a method for investigating causality which is potentially extremely powerful, but for now is limited to interaction matrices with constant coefficients. Wightman[23] used this method to show noncausality for certain higher multipole interactions in the DKP spin-0 and spin-1 so(5) representations (1,0) and (1,1). Specifically, Wightman found noncausality for the couplings $F_{\mu\nu}\,[\beta_\mu,\beta_\nu]$ and $G_{\mu\nu}\,[\{\beta_\mu,\beta_\nu\} + \frac{1}{2}\,\delta_{\mu\nu}\beta\cdot\beta]$ in the spin-0 case $\left[F_{\mu\nu}\ (G_{\mu\nu})\right.$ is antisymmetric (symmetric) in $\left.\mu\nu\right]$, and noncausality for the electric quadrapole coupling $\partial_\lambda F_{\mu\nu}\,(5 + 2\,\beta\cdot\beta)\beta_\mu\beta_\nu\beta_\lambda$ in the spin-1 case. This last agrees with the Velo-Zwanziger[18] Proca field result, and lends further



interest to the calculation of Peaslee[75] that(due to the higher order of the algebra and the associated derivatives) the spin-1 DKP current has new dipole and quadrapole moment divergences in second order beyond the usual first-order charge and mass renormalizations.[75-79] (These divergences do not exist in the DKP spin-0 case, and perhaps emphasize the fact that, since $[\beta_\mu,\beta_\nu]$ is a spin operator, it would be curious to have such a coupling for a spin-0 particle, even if it were causal.)

This method[23,80] states that the principle part of the determinant of the entire matrix $\Lambda$ will test causality, vs. the determinant of the principle part $\hat{\Lambda}$, as in Eq. (3.2). Thus, it is only necessary that $\Lambda$ be non-singular, even if $\hat{\Lambda}$ is not. The problem is that this method has so far only been shown to be a theorem for constant coefficients[23,80] ($C$ in Eq. (3.1) is a matrix of constant coefficients). However, it still has two useful applications for us. The first comes from observing that if we were to assume this method is valid for nonconstant coefficients, we would, exactly as in Eqs. (3.2)-(3.9), come to the same conclusion of the causality of Bhabha fields with minimal interaction. This is a further[23,80] indication that this method may be able to be shown to hold for nonconstant coefficients, which would be a very important result, indeed.

The second application is in demonstrating the difference between the Dirac and DKP cases for $F_{\mu\nu}$ coupling. Assume constant $F_{\mu\nu}$ and $A_\nu$ in $p_\nu^-$. Further note that since both $F_{\mu\nu}$ and $A_\nu$ are constant: (i) they should not be considered to be mathematically related as in Eq. (3.11) but rather both taken as fundamental fields; (ii) in taking the determinant below, all matrix elements commute. Then we have for Dirac and spin-0 DKP



$$\text{Det } \Lambda^{D} = \text{Det } (ip^{-}\cdot\gamma + m - \kappa F_{\mu\nu}\,[\gamma_{\mu},\gamma_{\nu}]/4i) \tag{3.14a}$$

$$= \left[(p_0^{-})^2 - (\vec{p}^{-})^2 - m^2\right]^2 + \kappa^2 \Big\{ 8p_0^{-}\,\vec{p}^{-}\cdot(\vec{E}\times\vec{B}) + 4\left[(\vec{E}\cdot\vec{p}^{-})^2 + (\vec{B}\cdot\vec{p}^{-})^2\right] - 2\left[(p_0^{-})^2 + (\vec{p}^{-})^2\right](\vec{E}^2 + \vec{B}^2) + 2m^2\,(\vec{E}^2 - \vec{B}^2) + (\vec{E}^2 - \vec{B}^2)^2 + 4(\vec{E}\cdot\vec{B})^2 \Big\}\,, \tag{3.14b}$$

$$\text{Det } \Lambda^{DKP} = \text{Det } (ip^{-}\cdot\beta + m - \kappa F_{\mu\nu}\,[\beta_{\mu},\beta_{\nu}]/i) \tag{3.15a}$$

$$= (-m)\Big\{ m^2\left[(p_0^{-})^2 - (\vec{p}^{-})^2 - m^2 + \kappa^2\,(\vec{E}^2 - \vec{B}^2)\right] + \kappa^2\Big[2\,p_0^{-}\,\vec{p}^{-}\cdot(\vec{E}\times\vec{B}) + (\vec{E}\cdot\vec{p}^{-})^2 + (\vec{B}\cdot\vec{p}^{-})^2 + (\vec{E}\cdot\vec{B})^2 - (p_0^{-})^2\,\vec{B}^2 - (\vec{p}^{-})^2\,\vec{E}^2\Big]\Big\}\,. \tag{3.15b}$$

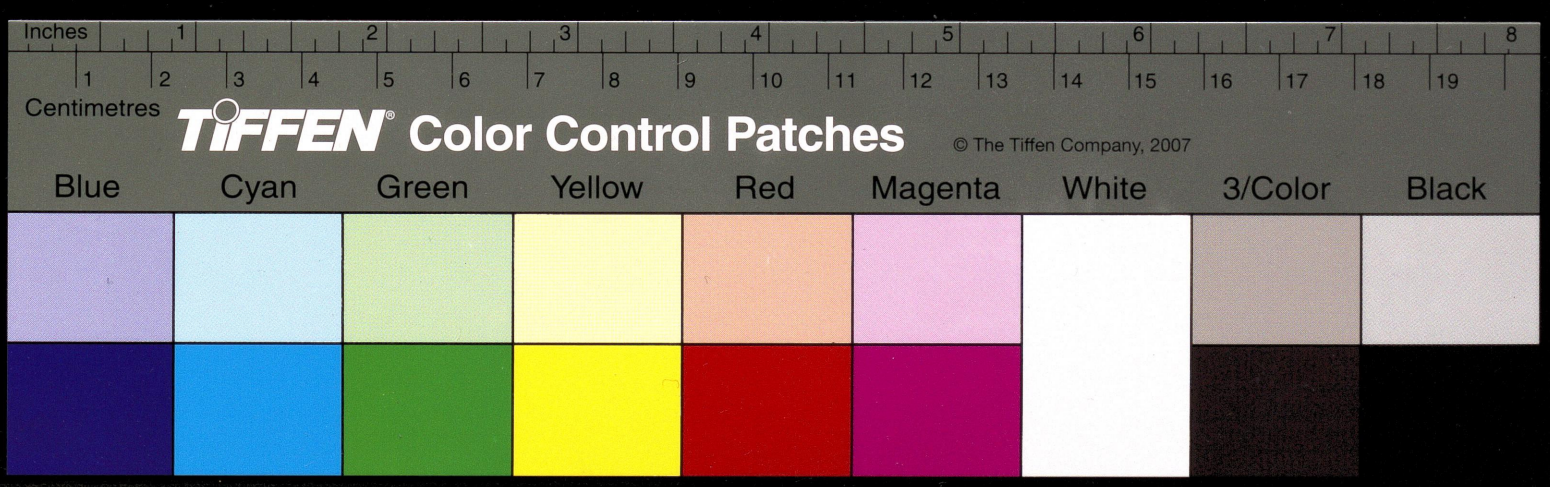
For Dirac the principle part being zero means

$$0 = \text{P.P.}(\text{Det } \Lambda^{D}) = \left[p_0^{\,2} - (\vec{p})^2\right]^2\,, \tag{3.16}$$

which implies causal solutions for these constant interaction fields. However, for DKP

$$0 = \text{P.P.}(\text{Det } \Lambda^{DKP}) = -m\Big\{ m^2\left[p_0^{\,2} - \vec{p}^{\,2}\right] + \kappa^2\left[2\,p_0\,\vec{p}\cdot\vec{E}\times\vec{B} + (\vec{E}\cdot\vec{p})^2 + (\vec{B}\cdot\vec{p})^2 - p_0^{\,2}\,\vec{B}^2 - \vec{p}^{\,2}\,\vec{E}^2\right]\Big\}\,, \tag{3.17}$$

which implies noncausal propagation for non-zero spin-0 DKP coupling of the form $F_{\mu\nu}[\beta_{\mu},\beta_{\nu}]$. For example, take $\vec{B}$ parallel to $\vec{p}$. Then Eq. (3.15) means

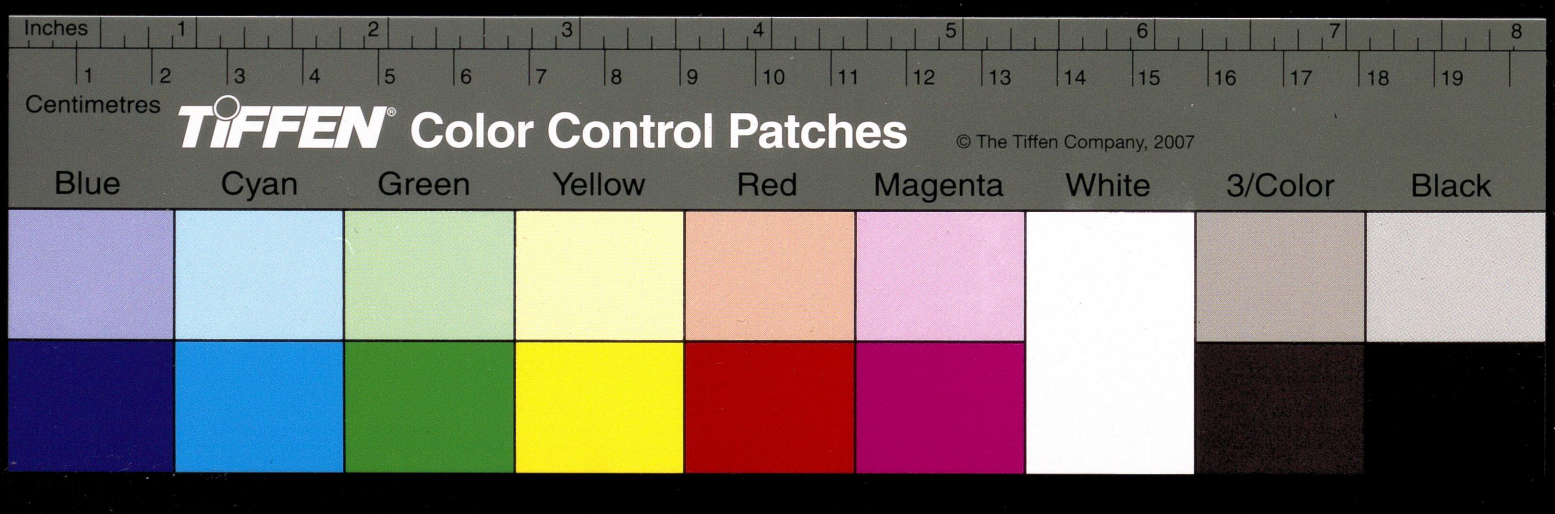


$$0 = (p_0^2 - \vec{p}^2)\,(1 - \kappa^2\,\vec{B}^2/m^2) \quad , \tag{3.18}$$

which allows noncausal propagation in a frame where

$$1 = \kappa^2\,\vec{B}^2/m^2 \quad . \tag{3.19}$$

This is the same type of mathematical solution that Velo and Zwanziger[17] found in their Eq. (2.17) for minimally coupled RS fields.



## IV. q-NUMBER FORMALISM

In this section we will perform an indefinite metric second-quantization of the Bhabha fields. (For reference we list a few[10,81-90] of the standard works which describe second-quantized formalisms.) Our method is similar in procedure to the positive norm multi-mass method of Baisya[15,22] and Nagpal,[16] which in turn is a generalization of the single mass techniques of Katayama,[83] Takahashi,[85-88] and Umezawa.[84-86] (The clear second-quantized calculations of Akhiezer and Berestetskii[90] for minimally coupled Dirac and DKP fields are also very useful to consult.) Besides properly taking account of the indefinite metric, we will also discuss the renormalizations involved in interacting second-quantized field theories. Ultimately this section is devoted to deriving the critical Eq. (4.53), which will be evaluated in Sec. V to show causality.

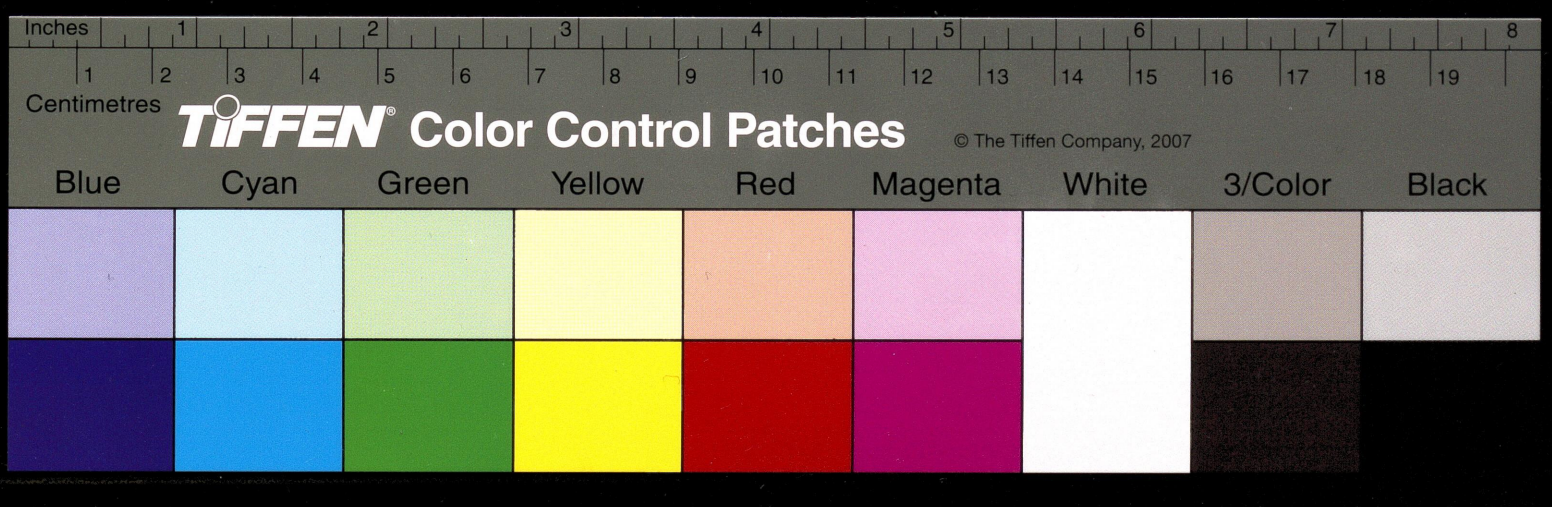
### A. Free fields

The second-quantized Bhabha free fields and adjoint fields can be written as

$$\psi(x) = \sum_{j,s} \int \frac{d^3p}{(2\pi)^{3/2}} \Big\{ a(\vec{p},j,s)\, u(\vec{p},j,s)\, e^{ip\cdot x} + b^{\dagger}(\vec{p},j,s)\, v(\vec{p},j,s)\, e^{-ip\cdot x} \Big\} , \tag{4.1}$$

$$\bar{\psi}(x) = \psi^{\dagger}(x)\,\eta_4 = \sum_{j,s} \int \frac{d^3p}{(2\pi)^{3/2}} \Big\{ a^{\dagger}(\vec{p},j,s)\, \bar{u}(\vec{p},j,s)\, e^{-ip\cdot x} + b(\vec{p},j,s)\, \bar{v}(\vec{p},j,s)\, e^{ip\cdot x} \Big\} , \tag{4.2}$$

where $\eta_4$ is the adjoint operator defined in Eq. (I3.47), and $a(\vec{p},j,s)$



and $a^{\dagger}(\vec{p},j,s)$ $b(\vec{p},j,s)$ and $b^{\dagger}(\vec{p},j,s)$ are the annihilation and creation operators for particles (antiparticles) of momentum $\vec{p}$, mass state $\chi/j$, and spin state s.

The annihilation and creation operators satisfy the relations

$$[a(\vec{p},j,s),a^{\dagger}(\vec{p}',j',s')]_{\pm} = [b(\vec{p},j,s),b^{\dagger}(\vec{p}',j',s')]_{\pm}$$

$$= \tau_{jj}\ \delta_{jj'}\ \delta_{ss'}\ \delta^{3}(\vec{p} - \vec{p}')\ , \qquad (4.3a)$$

$$[a,a]_{\pm} = [b,b]_{\pm} = [a,b^{\dagger}]_{\pm}\ldots \text{ (all other relations)}\ldots = 0\ , \qquad (4.3b)$$

where anticommutation relations

$$[a,a^{\dagger}]_{+} \equiv \{a,a^{\dagger}\} \equiv aa^{\dagger} + a^{\dagger}a \qquad (4.4a)$$

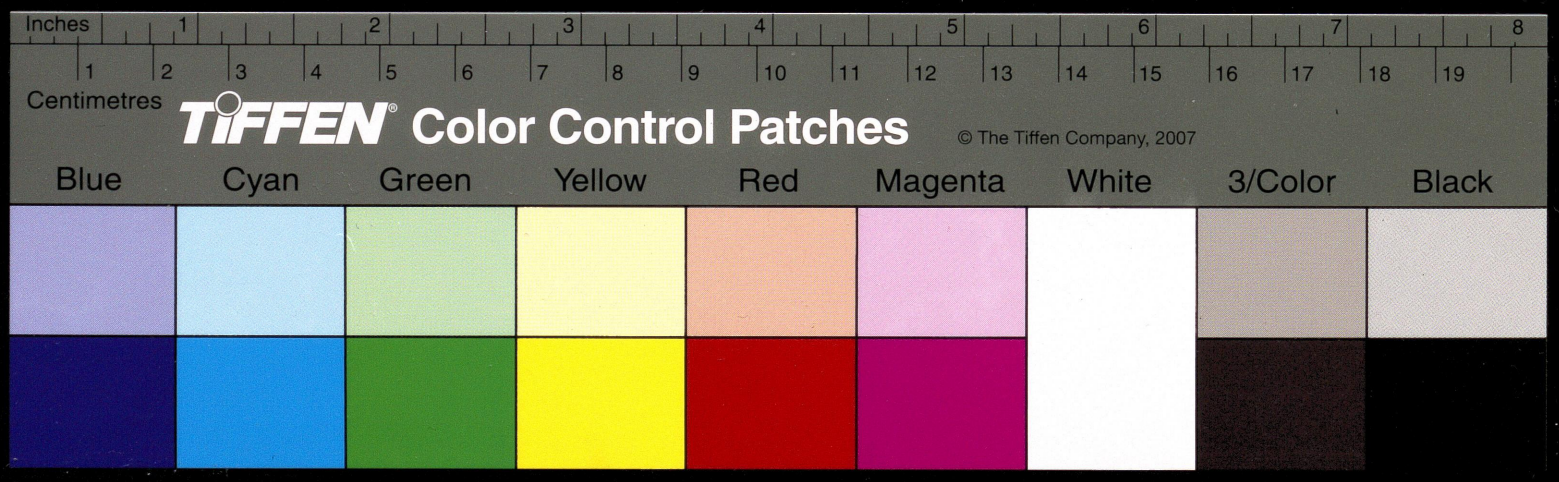
are to be used for half-integer spin fields, and commutation relations

$$[a,a^{-}]_{-} \equiv [a,a^{\dagger}] \equiv aa^{\dagger} - a^{\dagger}a \qquad (4.4b)$$

are to be used for integer spin fields. (We will always take the upper sign for half-integer spin fields, and the lower sign for integer spin fields). If one accepts the standard arguments about the connection of spin and statistics,[91-95] along with the requirement of a positive definite metric and energy density, then one desires

$$\tau_{jj'} \rightarrow \hat{\tau}_{jj'} \equiv \delta_{jj'}\ , \quad \text{(standard requirements)}. \qquad (4.5)$$

However, because of their indefinite metric,[62-64] Eq. (4.5) will not hold for the general Bhabha fields, as we will explicitly show.



Proceeding with the formalism, Eqs. (4.1)-(4.3) imply that

$$\left[\psi_\alpha(x),\bar{\psi}_\beta(x')\right]_\pm = \sum_j \int \frac{d^3p}{(2\pi)^3} \left[F^+_{\alpha\beta}(x,x',\vec{p},j) \pm F^-_{\alpha\beta}(x,x',\vec{p},j)\right], \quad (4.6)$$

$$F^+_{\alpha\beta}(x,x',\vec{p},j) \equiv \tau_{jj} \sum_s u_\alpha(\vec{p},j,s)\bar{u}_\beta(\vec{p},j,s)e^{ip\cdot(x-x')} \quad (4.7)$$

$$F^-_{\alpha\beta}(x,x',\vec{p},j) \equiv \tau_{jj} \sum_s v_\alpha(\vec{p},j,s)\bar{v}_\beta(\vec{p},j,s)e^{-ip\cdot(x-x')} \quad . \quad (4.8)$$

For the well-known Dirac case ($2\chi = m$, $2\alpha_\mu = \gamma_\mu$),

$$j \equiv \tfrac{1}{2}, \quad \tau_{\frac{1}{2}\frac{1}{2}} = 1, \quad (4.9)$$

$$F^+_{\alpha\beta}(x,x',\vec{p},\tfrac{1}{2}) = -\frac{1}{2E}\mathcal{D}_{\alpha\beta}(ip,S=\tfrac{1}{2})\, e^{ip\cdot(x-x')}, \quad (4.10)$$

$$F^-_{\alpha\beta}(x,x',\vec{p},\tfrac{1}{2}) = +\frac{1}{2E}\mathcal{D}_{\alpha\beta}(-ip,S=\tfrac{1}{2})\, e^{-ip(x-x')}, \quad (4.11)$$

with $\mathcal{D}_{\alpha\beta}(\partial,S=\tfrac{1}{2})$ given by Eq. (2.13), so that

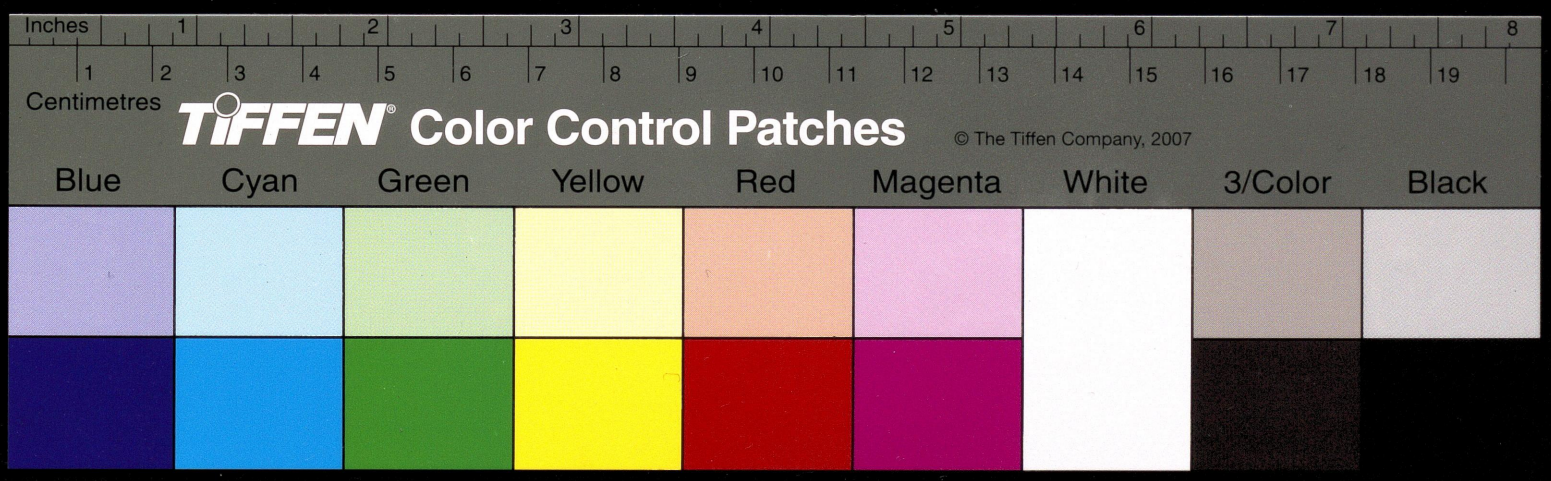
$$\left\{\psi_\alpha(x),\bar{\psi}_\beta(x')\right\}^D = -i\mathcal{D}_{\alpha\beta}(\partial,S=\tfrac{1}{2})\left[\Delta^+(x-x',2\chi) + \Delta^-(x-x',2\chi)\right]$$

$$= -i\mathcal{D}_{\alpha\beta}(\partial,S=\tfrac{1}{2})\ \Delta(x-x',2\chi), \quad (4.12)$$

where $\Delta(x-x',\chi/j)$ is the single particle invariant $\Delta$-function of mass $\chi/j$.

For the DKP case,[90] one has ($\chi \equiv m$, $\alpha_\mu \equiv \beta_\mu$)

$$j \equiv 1, \quad \tau_{11} = 1, \quad (4.13)$$



$$F^{+}_{\alpha\beta}(x,x',\vec{p},1) = \frac{ip\cdot\beta}{2mE}(ip\cdot\beta - m)\, e^{ip\cdot(x-x')} \quad , \tag{4.14}$$

$$F^{-}_{\alpha\beta}(x,x',\vec{p},1) = \frac{ip\cdot\beta}{2mE}(ip\cdot\beta + m)\, e^{-ip(x-x')} \quad , \tag{4.15}$$

so that

$$\left[\psi_\alpha(x),\bar{\psi}_\beta(x')\right]^{DKP} = -\, i\left[\frac{(\partial\cdot\beta)[m-(\partial\cdot\beta)]}{m}\right]\left[\Delta^{+}(x-x',m) + \Delta^{-}(x-x',m)\right] . \tag{4.16}$$

Thus, since

$$\left(\Box - \frac{\chi^2}{j^2}\right)\Delta(x-x',\chi/j) = 0 \quad , \tag{4.17}$$

where

$$\Delta(x-x',\chi/j) = \Delta^{+}(x-x',\chi/j) + \Delta^{-}(x-x',\chi/j) \quad , \tag{4.18}$$

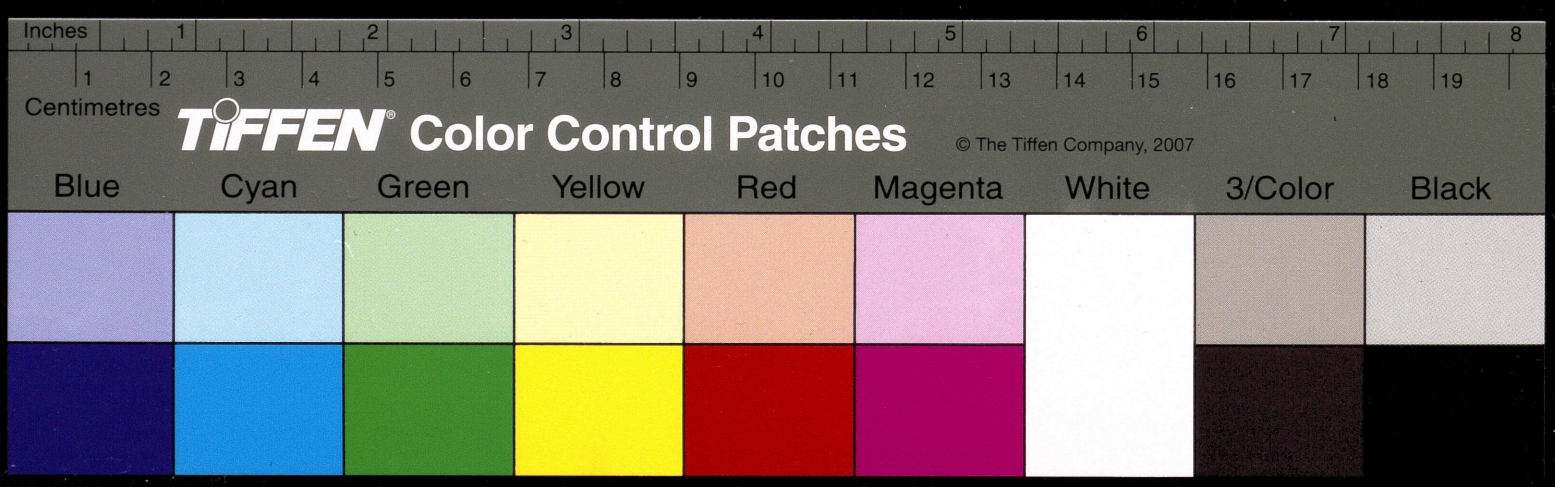
then from the form of $\mathcal{D}(\partial,S=1)$ given in Eq. (2.22) combined with Eq. (4.16) we obtain

$$\left[\psi_\alpha(x),\bar{\psi}_\beta(x')\right]^{DKP} = -\, i\mathcal{D}_{\alpha\beta}(\partial,S=1)\Delta(x-x',m) \quad . \tag{4.19}$$

In obtaining the results (4.12) and (4.19) for the Dirac and DKP cases, the crucial signs come out correctly because of the normalizations for the quantities

$$\sum_s \bar{u}(\vec{p},s)\gamma_4\, u(\vec{p},s) = \sum_s u^{\dagger}(\vec{p},s)\, u(\vec{p},s) = +1 \quad , \tag{4.20}$$

$$\sum_s \bar{v}(\vec{p},s)\gamma_4\, v(\vec{p},s) = \sum_s v^{\dagger}(\vec{p},s)\, v(\vec{p},s) = +1 \quad , \tag{4.21}$$



for Dirac, and

$$\bar{u}(\vec{p})\ \beta_4\ u(\vec{p}) = u^{\dagger}(\vec{p})\ \eta_4\ \beta_4\ u(\vec{p}) = +\ 1\ , \tag{4.22}$$

$$\bar{v}(\vec{p})\ \beta_4\ v(\vec{p}) = v^{\dagger}(\vec{p})\ \eta_4\ \beta_4\ v(\vec{p}) = -\ 1\ , \tag{4.23}$$

for DKP. These signs are due to the general Bhabha metric

$$M = \eta_4\alpha_4 \tag{4.24}$$

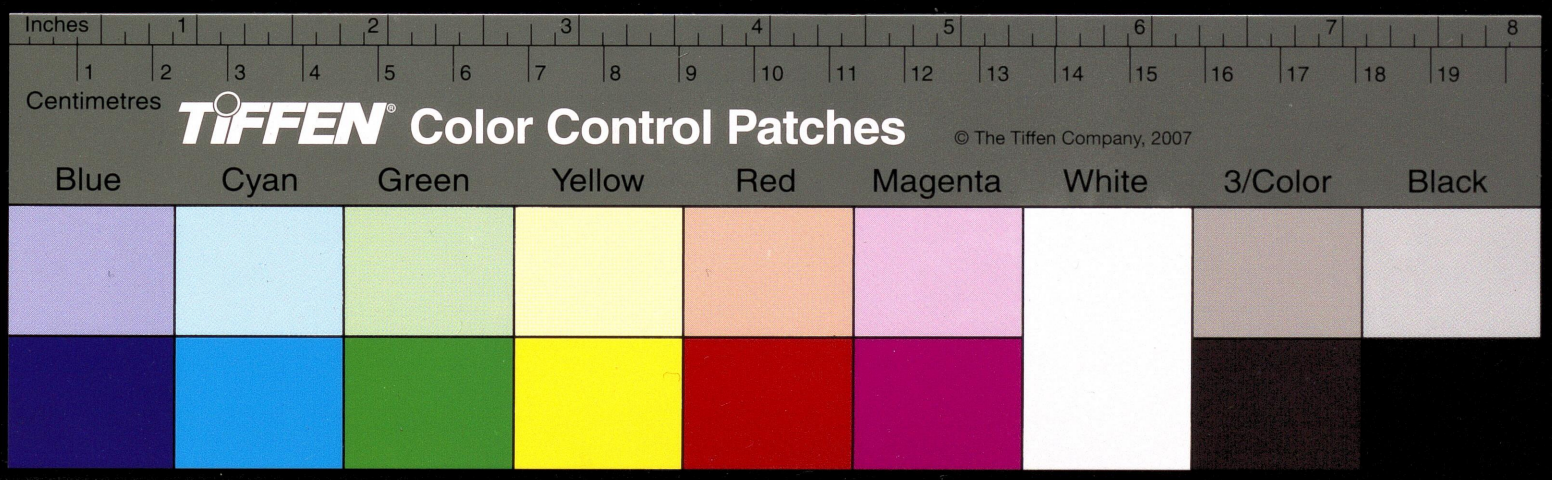
which, from the defining equations (IA18) for $\eta_4$ and the characteristic equation (I2.31) for $\alpha_4$ with $\alpha_4$ diagonal, has the form

$$M(S=n+\tfrac{1}{2}) = \text{block diagonal}\ \Big[+SI^{+}_{S},\ -(S-1)I^{+}_{S-1},\ +(S-2)I_{S-2},\ \ldots,$$
$$(-1)^{S-\frac{1}{2}}\tfrac{1}{2}I^{+}_{\frac{1}{2}},(-1)^{S-\frac{1}{2}}\tfrac{1}{2}I^{-}_{\frac{1}{2}},\ \ldots,\ -(S-1)I^{-}_{S-1},+\ SI^{-}_{S}\ \Big]\ , \tag{4.25}$$

$$M(S=n) = \text{block diagonal}\ \Big[+SI^{+}_{S}\ ,\ -(S-1)I^{+}_{S-1},+(S-2)I_{S-2}\ ,\ \ldots\ ,$$
$$(-1)^{S+1}I^{+}_{1}\ ,\ 0I_{0}\ ,\ -(-1)^{S+1}I^{-}_{1}\ ,\ \ldots\ ,$$
$$+\ (S-1)I^{-}_{S-1},\ -\ SI^{-}_{S}\ \Big]\ . \tag{4.26}$$

The $I^{\pm}_{j}(S)$ are the mass state $\pm\ \chi/j$ projection operators and $I_0(S)$ is the subsidiary component projection operator discussed in Secs. III-B and III-C of paper II[2] and also in Eq. (5.10)-(5.22) below. (The matrix M in (4.24) is also the $\lambda = 4$ case of the matrices $\xi_\lambda = \eta_\lambda\alpha_\lambda$ (no sum) defined by Madhavarao, Thiruvenkatachar, and Venkatachaliegar.[96]

For half-integer spin, Eq. (4.25) shows that the particle and anti-particle ground states $\pm\ \chi/S$ have positive norms, the first excited states $\pm\ \chi/(S-1)$ have negative norms, and so on. The simplest case ($S = \frac{1}{2}$) is the



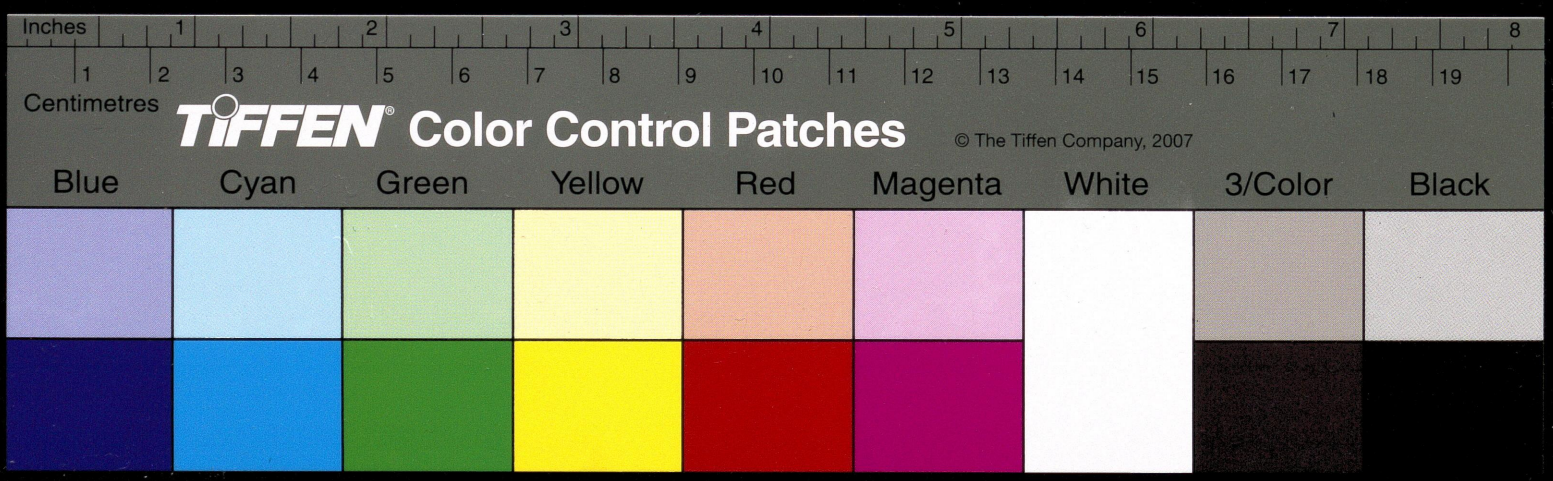
Dirac equation which has only positive normal states. For integer spin, Eq. (4.26), there is an extra minus sign. From Eq. (IA18) this is due to the extra minus sign in $\eta_4$ caused by the extra subsidiary components block projected by $I_0$ .

Thus, for DKP where $S = 1$, although the particle state has a positive norm, the antiparticle state has a negative norm (this being the origin of the sign difference between Eqs. (4.21) and (4.23) ). The standard resolution of this most simple negative norm is the Pauli-Weisskoff[97] device of saying this is a charge probability density instead of the particle probability density for the Dirac case. But for $S > 1$, such a resolution is no longer possible for either integer or half-integer spin, since the first excited state of a particle necessarily has the opposite norm, oscillating back and forth with each further excitation, with an added minus sign for the antiparticle normalization of integer spin particles.

Going back to Eq. (4.3a), we can now see that $\tau_{jj'}$ is just the sign of the metric (except for zero in the integer spin subsidiary components block), or, as a matrix,

$$\tau = \text{sign } (M)$$
$$= \text{block diagonal } \left[ +I_S^+, -I_{S-1}^+, +I_{S-2}^+, \ldots, \pm I_{S-2}^-, \mp I_{S-1}^-, \pm I_S^- \right] . \tag{4.27}$$

What Eqs. (4.3a) and (4.27) tell us is that we are quantizing the Bhabha fields with a metric similar to the Dirac[57] indefinite metric used by Gupta,[98,99] Bleuler,[100,101] and Heitler[101] for the photon field in quantum electrodynamics, but without the fortunate happenstance of quantum electrodynamics that the negative normed states are eliminated by a certain subset of the positive normed states. At this stage the negative normed states remain in the Bhabha theory, but we can proceed with the general quantization.



Returning to Eqs. (4-7)-(4.8), the normalization methods of Takahashi[88] and Baisya[15,22] now straightforwardly lead to

$$\sum_s u_\alpha(\vec{p},j,s)\bar{u}_\beta(\vec{p},j,s) = \frac{-\tau_{jj}\mathcal{D}_{\alpha\beta}(ip)}{2E_j\, X(S,j)} \quad , \tag{4.28}$$

$$\sum_s v_\alpha(\vec{p},j,s)\bar{v}_\beta(\vec{p},j,s) = \frac{\pm\tau_{jj}\mathcal{D}_{\alpha\beta}(-ip)}{2E_j\, X(S,j)} \quad , \tag{4.29}$$

$$X(S,j) \equiv \prod_{\substack{k=(\frac{1}{2},1)\\ k\neq j}}^{S} \left(\frac{\chi_j^2}{} - \frac{\chi_k^2}{}\right) \quad . \tag{4.30}$$

Putting these into Eqs. (4.7)-(4.8) we see that the $\tau_{jj}$ quantities are all squared, so that Eq. (4.6) will read

$$\left[\psi_\alpha(x),\bar{\psi}_\beta(x')\right]_\pm = -i\,\mathcal{D}_{\alpha\beta}(\partial)\sum_j^S \int \frac{d^3p}{(2\pi)^3 2E_1 X(S,j)}\left[-ie^{ip\cdot x} + ie^{-ip\cdot x}\right]$$

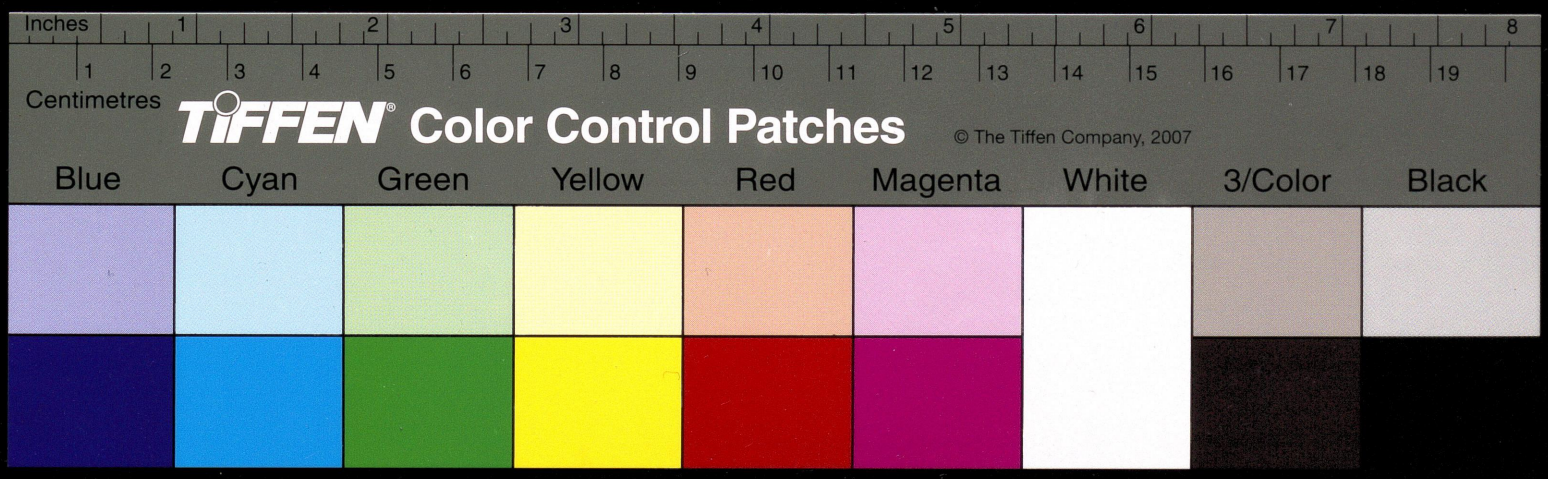
$$= -i\,\mathcal{D}_{\alpha\beta}(\partial)\sum_j^S \frac{\Delta(x-x',\chi/j)}{\prod_{\substack{k=(\frac{1}{2},1)\\ k\neq j}}^{S}\left(\frac{\chi_j^2}{} - \frac{\chi_k^2}{}\right)}$$

$$\equiv -i\,\mathcal{D}_{\alpha\beta}(\partial,S)\,\Delta_S(x-x') \quad , \tag{4.31}$$

which is to be combined with the other free field commutation relations

$$\left[\psi_\alpha(x),\psi_\beta(x')\right]_\pm = \left[\bar{\psi}_\alpha(x),\bar{\psi}_\beta(x')\right]_\pm = 0 \quad . \tag{4.32}$$

A crucial point to be emphasized is that if we had not used the indefinite metric quantization procedure in Eq. (4.3), then the second equality in Eq. (4.30) would have had an alternating minus sign within the sum, i.e.,



a $\tau_{jj}$ . Such a propagator,

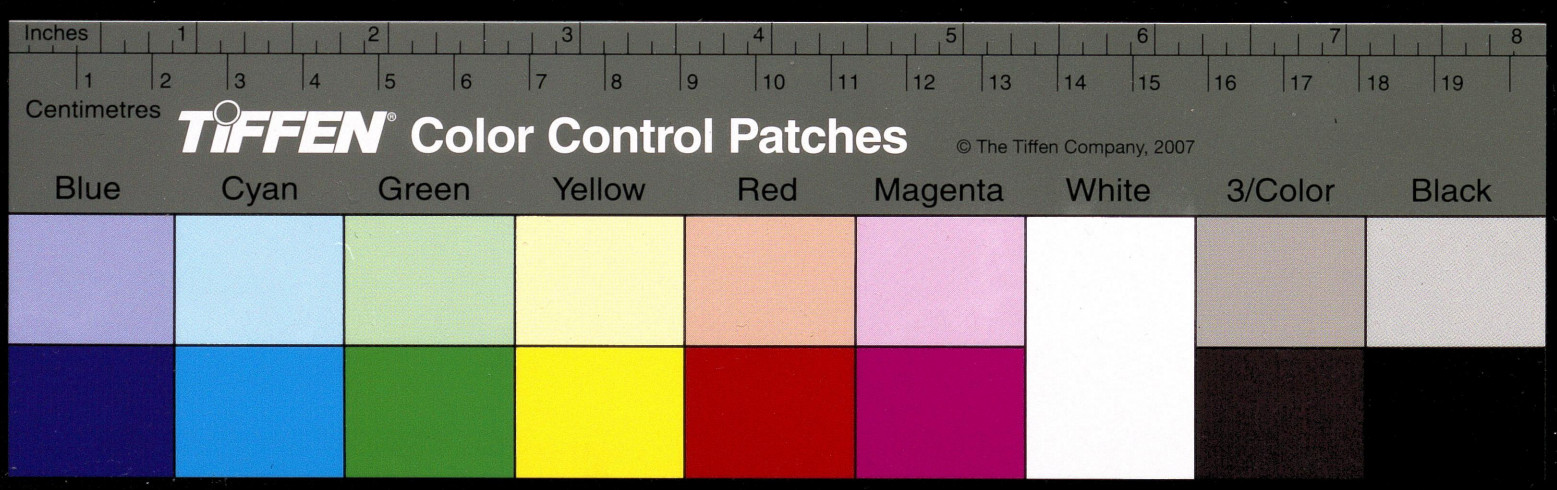
$$\hat{\Delta}_S(x-x') = \sum_j^S \frac{\tau_{jj}\,\Delta(x-x',\chi/j)}{\prod_{\substack{k=(\frac{1}{2},1)\\ k\neq j}}^{S}\left(\frac{\chi^2}{j^2}-\frac{\chi^2}{k^2}\right)} \quad , \tag{4.33}$$

as we shall see in the next section, would not yield causal fields with minimal coupling. The extra sign is important.

We note that Baisya[15,22] and Nagpal (at least at first[16]) avoided $\hat{\Delta}_S$ in another way. They kept the ordinary positive normed quantization, but then considered not the fields $\psi$ and $\bar{\psi}$ , but rather $\psi$ and $\hat{\psi}$, $\hat{\psi}$ being

$$\hat{\psi} \equiv \bar{\psi}\varepsilon(j) \quad , \tag{4.34}$$

where $\varepsilon(j)$ was an undefined signed quantity, which turns out to be our $\tau_{jj}$. However, with this method the Hamiltonian is not metric-(pseudo-) Hermitian. We prefer to have the Hamiltonian metric-Hermitian and use the indefinite metric quantization procedure. We feel this is the correct method. Observe, however, that in his recent paper[30] Nagpal considered the commutators $[\psi(x),\bar{\psi}(x')]_{\pm}$ without stating that he had used an indefinite metric second quantization instead of the positive normed quantization of his[16] and Baisya's[15,22] previous work. Strictly speaking, Nagpal's[30] use of Eq. (4.31) is only proper if, unstated, he were using our indefinite metric quantization. (Of course he could still use the fields $\psi$ and $\hat{\psi}$ with positive normed quantization to obtain the invariant function $\Lambda_S(x-x')$ and not $\hat{\Lambda}_S(x-x')$. ) Once $\Lambda_S(x-x')$ is obtained, though, the calculation proceeds as follows.



B. Interacting fields

Having the free field commutation relations (4.31) and (4.32), we can now proceed to the interacting field case. Here we are dealing with the coupled field equations

$$\Box A_\mu(x) = + ie\bar{\Psi}(\chi)\alpha_\mu\Psi(x) \equiv - j_\mu(x) \ , \tag{4.35}$$

$$\Lambda(\partial,\chi_{bare})\Psi(x) = (\partial\cdot\alpha + \chi_{bare})\psi(x)$$

$$= ieA_\mu(x)\alpha_\mu\Psi(x) \equiv j(x) \ , \tag{4.36}$$

where $A_\mu(x)$ and $\Psi(x)$ denote the interacting Heisenberg photon and Bhabha fields.

Following, as we do in this subsection, the discussion of Chap. 8 of Ref. 88, $\chi_{bare}$ refers to the fact that with the standard mass renormalization, Eq. (4.36) with its bare mass should be changed to

$$\Lambda(\partial,\chi_{bare} + \delta\chi)\Psi(x) = \Lambda(\partial,\chi_{obs})\Psi(x)$$

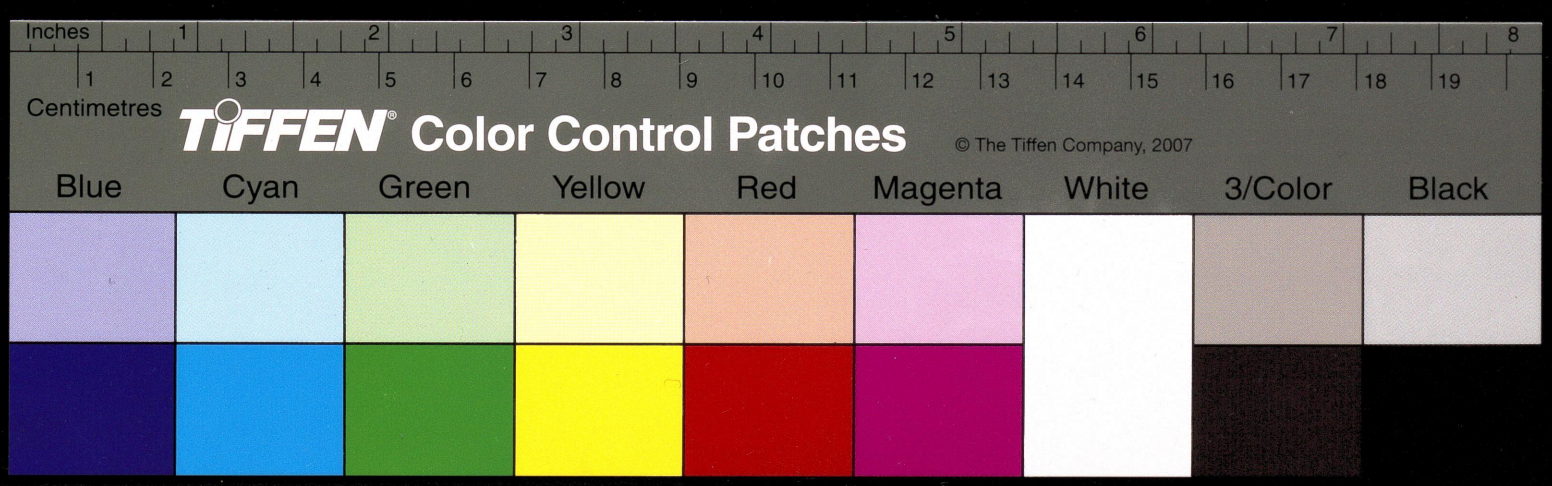
$$= \left[(\Delta\chi)\Psi(x) + ieA_\mu\alpha_\mu\Psi(x)\right] \equiv \hat{j}(x) \ , \tag{4.37}$$

where with our simple first order equation, $\delta\chi = \Delta\chi$ . Equation (4.37) (hopefully) allows the relation

$$\langle 0|\hat{j}(x)|\text{one particle state}\rangle = 0 \tag{4.38}$$

to be satisfied so that in turn one can have

$$\lim_{t\to\pm\infty} \langle a|\hat{j}(x)|b\rangle = 0 \ , \tag{4.39}$$



allowing necessary integrals to converge in the Källen[102] and Yang and Feldman[103] discussion of the Heisenberg field operators which will follow. In our case, since both $j(x)$ and $\hat{j}(x)$ involve no derivatives and are linear functionals of $\Psi(x)$, it will turn out that the existence of a $\delta\chi \neq 0$ will have no bearing on the resulting final form of the Heisenberg field commutation relations for $\Psi(x)$. Therefore we will take

$$\begin{aligned} \delta\chi \equiv \Delta\chi &= 0 \ , \\ \chi_{bare} &= \chi_{obs} = \chi \ , \\ \hat{j}(x) &= j(x) \ , \end{aligned} \tag{4.40}$$

with the knowledge that in our final results Eqs. (5.12) and (5.20) below, one could resubstitute $\hat{j}(x)$ for $j(x)$, etc., and still obtain the same causal physics.

Continuing with the Källen-Yang-Feldman procedure, one defines the Heisenberg photon and Bhabha fields as

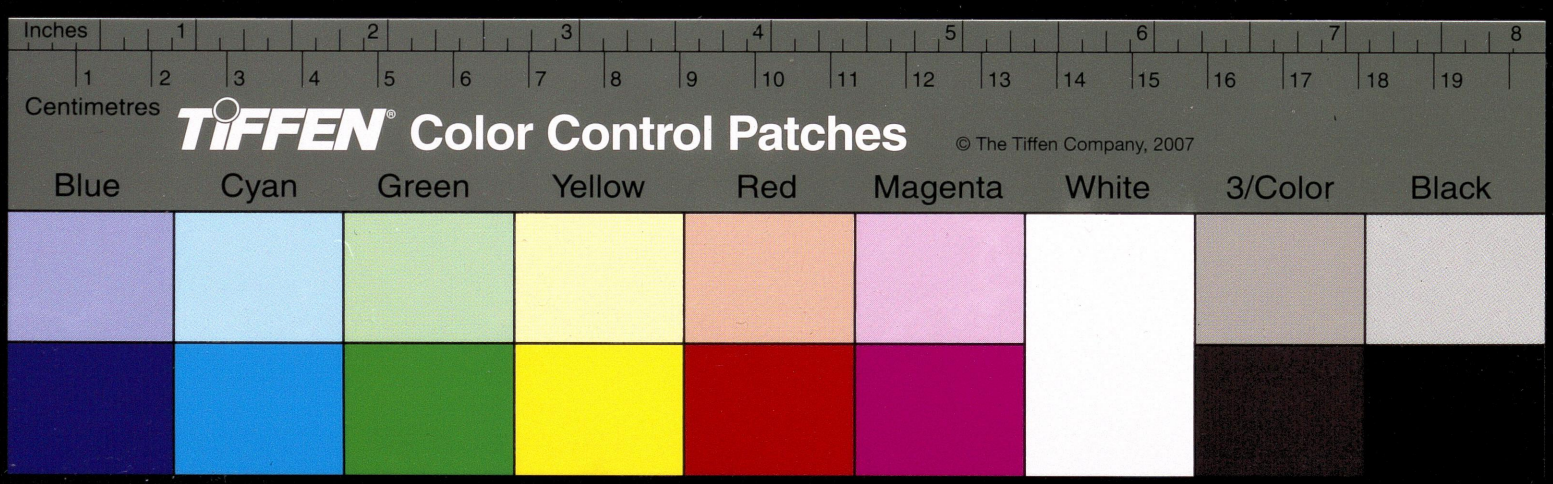
$$A_\mu(x) = z_3^{\frac{1}{2}} A_\mu(x)_{IN} + \int_{-\infty}^{\infty} d^4x' \ D^{(ret)}(x - x') j_\mu(x') \ , \tag{4.41}$$

$$\Psi(x) = z_2^{\frac{1}{2}} \psi(x)_{IN} - \int_{-\infty}^{\infty} d^4x' \ S^{(ret)}(x - x') j(x') \ . \tag{4.42}$$

One also defines associated auxiliary fields as a function of a space-like surface $\sigma$

$$A_\mu(x,\sigma) = z_3^{\frac{1}{2}} A_\mu(x)_{IN} + \int_{-\infty}^{\sigma} d^4x' \ D(x - x') j_\mu(x') \ , \tag{4.43}$$

$$\psi(x,\sigma) = z_2^{\frac{1}{2}} \psi(x)_{IN} - \int_{-\infty}^{\sigma} d^4x' \ S(x - x') j(x') \ . \tag{4.44}$$



In the above, $z_3$ and $z_2$ are the usual charge and wave function renormalization constants. The auxiliary fields $A_\mu(x,\sigma)$ and $\psi(x,\sigma)$ are photon and Bhabha fields which satisfy the free wave equations and commutation relations, and which become the renormalized IN fields as $\sigma \to -\infty$. The propagator Green's functions are

$$D(x - x') = \Delta(x - x', m = 0) \ , \tag{4.45}$$

$$D^{(ret)}(x - x') = \Theta(x_0 - x_0')D(x - x') \ , \tag{4.46}$$

$$S(x - x') = \mathcal{D}(\partial,S)\Delta_S(x - x') \ , \tag{4.47}$$

$$S^{(ret)}(x - x') = \mathcal{D}(\partial,S)\Theta(x_0 - x_0')\Delta_S(x - x') \ , \tag{4.48}$$

where $\Theta(x)$ is the usual unit step function.

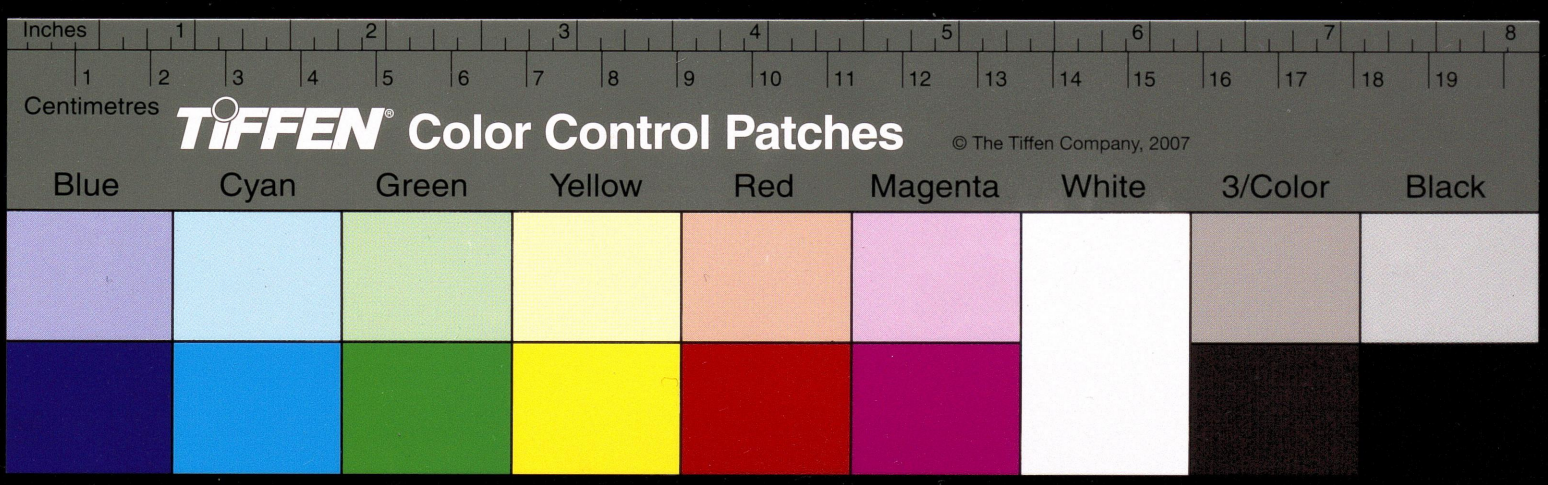
Denoting by $(x/\sigma)$ when x is on the surface $\sigma$, Eqs. (4.43) and (4.44) yield

$$A_\mu(x/\sigma) = z_3^{\frac{1}{2}} A_\mu(x)_{IN} + \int_{-\infty}^{\infty} d^4x' \ \Theta^\sigma(x_0 - x_0')D^\sigma(x - x')j_\mu(x') \ , \tag{4.49}$$

$$\psi(x/\sigma) = z_2^{\frac{1}{2}} \psi(x)_{IN} - \int_{-\infty}^{\infty} d^4x' \ \Theta^\sigma(x_0 - x_0')S^\sigma(x - x')j(x') \ , \tag{4.50}$$

where the superscripts $\sigma$'s mean that x is understood to be on $\sigma$ in the argument. (We will now drop these superscripts since the restriction $(x/\sigma)$ in the integrands will not affect the rest of the calculation.) Equations (4.41) and (4.49) trivially imply, as in ordinary Dirac quantum electrodynamics, that

$$A_\mu(x) = A_\mu(x/\sigma) \ , \tag{4.51}$$



so that the photon commutation relations

$$\left[A_\mu(x),\, A_\nu(x')\right] = i\,\delta_{\mu\nu}\,D(x - x') \tag{4.52}$$

are preserved. Equations (4.42) and (4.50) yield

$$\Psi(x) = \psi(x/\sigma) + \int_{-\infty}^{\infty} d^4x' \left[\Theta(x_0 - x_0'),\, \mathcal{D}(\partial,S)\right] \Delta_S(x - x')j(x')$$

$$= \psi(x/\sigma) + \frac{1}{2}\int_{-\infty}^{\infty} d^4x' \left[\varepsilon(x_0 - x_0'),\, \mathcal{D}(\partial,S)\right] \Delta_S(x - x')j(x')\ , \tag{4.53}$$

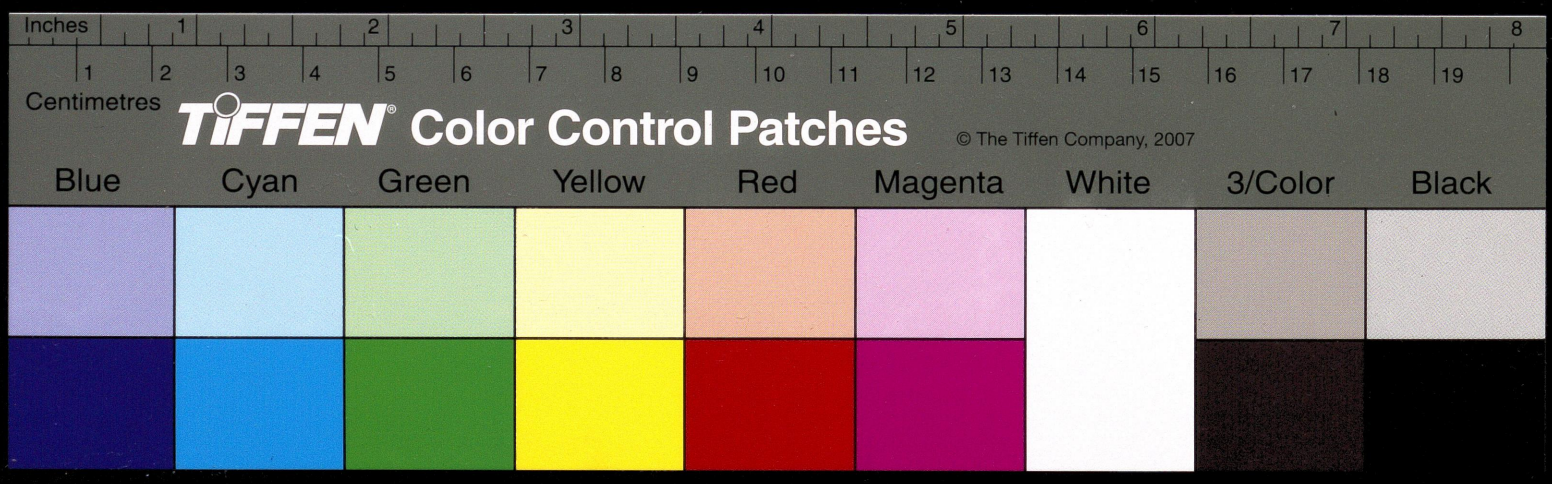
where

$$\varepsilon(x) = \Theta(x) - \Theta(-x)\ . \tag{4.54}$$

Equation (4.53) is the critical one. We want to show that

$$\Psi(x) \to \psi(x/\sigma) \tag{4.55}$$

so that the Bhabha free field commutation relations (4.31) - (4.32) are preserved, meaning the Bhabha minimally coupled fields are causal.



## V. CAUSALITY OF THE q-NUMBER THEORY

### A. Half-integer spin

From Eq. (4.53), the quantity we need to calculate is

$$Y \equiv \frac{1}{2}\left[\varepsilon(x_0 - x_0'),\ \mathcal{D}(\partial,S)\right]\Delta_S(x - x') \tag{5.1}$$

$$= \sum_{i=0}^{S-\frac{1}{2}} (\chi^{2S-2i}) \sum_{j=0}^{i} (-1)^{S+\frac{1}{2}-i+j}\ \phi_{S+\frac{1}{2}-i+j}\ (\ell_{\{S+\frac{1}{2}\}})$$

$$\times \frac{1}{2}\left[\varepsilon(x_0 - x_0'),\ (1 - \frac{\partial\cdot\alpha}{\chi})\,\Box^{i-j}(\partial\cdot\alpha)^{2j}\right]\sum_{k=\frac{1}{2}}^{S} \frac{\Delta(x-x',\chi/k)}{X(S,k)}\ , \tag{5.2}$$

where in (5.2) we have used Eqs. (2.11), (4.30), and (4.31). To evaluate the commutator in (5.2) we recall the Katayama[83,104] result

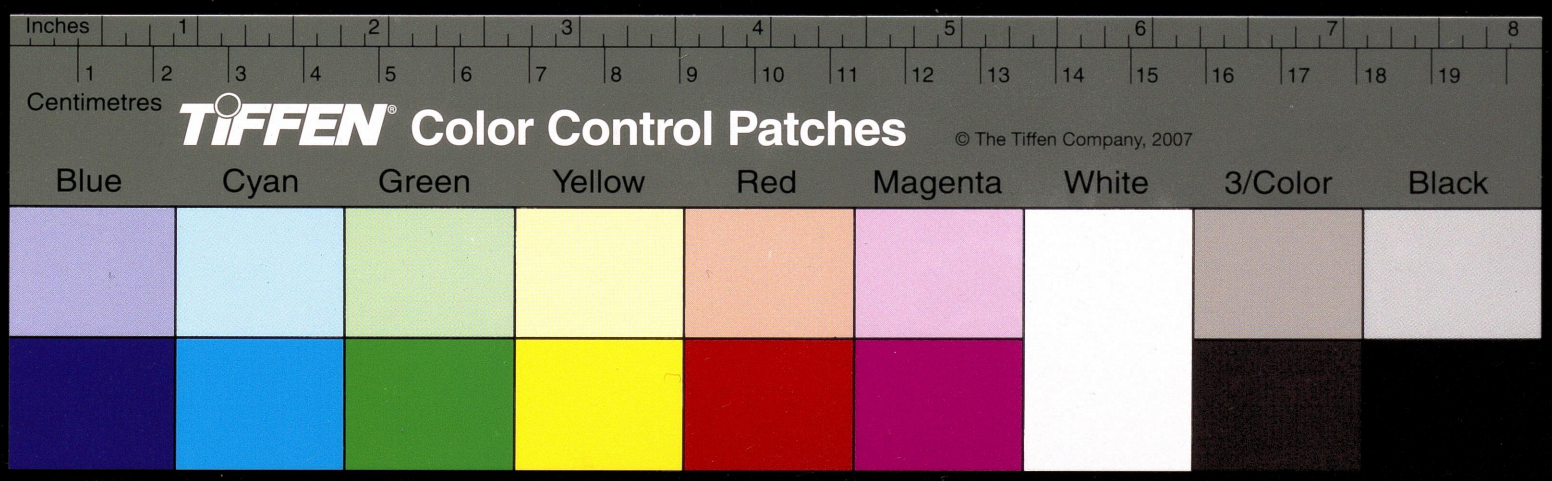
$$\frac{1}{2}\left[\varepsilon(x_0 - x_0'),\ \partial_{\mu(1)}\partial_{\mu(2)} \cdots \partial_{\mu(r)}\right]\Delta(x - x', m)$$

$$= \left(\frac{-1}{\Box - m^2}\right)\sum_{\substack{q=0 \\ \text{cyclic}}}^{r} (-1)^q\ \hat{n}_{\mu(1)} \cdots \hat{n}_{\mu(q)}\ \partial^s_{\mu(q+1)} \cdots \partial^s_{\mu(r)}$$

$$\times \left[(\hat{n}\cdot\partial)^q - \frac{(\Delta - m^2)^{q/2}}{2}\left\{(1 + (-1)^q) + (1 - (-1)^q)\,\frac{(\hat{n}\cdot\partial)}{(\Delta - m^2)^{\frac{1}{2}}}\right\}\right]\delta^4(x - x')\ , \tag{5.3}$$

where $\hat{n}$ is a time-like unit vector and the superscript s signifies the space part, i.e.,

$$\hat{n}_\mu = (0,\ 0,\ 0,\ i)\ , \tag{5.4}$$



$$\partial_\mu^s = \partial_\mu + \hat{n}_\mu(\hat{n}\cdot\partial) \ , \tag{5.5}$$

and cyclic means all interchanges of the type $\hat{n}_\nu\partial_\lambda \to \hat{n}_\lambda\partial_\nu$ .

Since in the commutator of (5.2) $\Box^{i-j}(\partial\cdot\alpha)^{2j}$ is a sum of products of an even number of derivatives, take r to be even in (5.3). The first two terms in the sum of (5.3) (q = 0 and q = 1) are zero. The square bracket of the q = 2 term is just $[-(\Box - m^2)]$ , which cancels the round bracket in front of the sum. The square bracket of the q = 3 term is $[-(\partial\cdot n)(\Box - m^2)]$ . In general the q = 2n = (even) terms in the square bracket are of the form

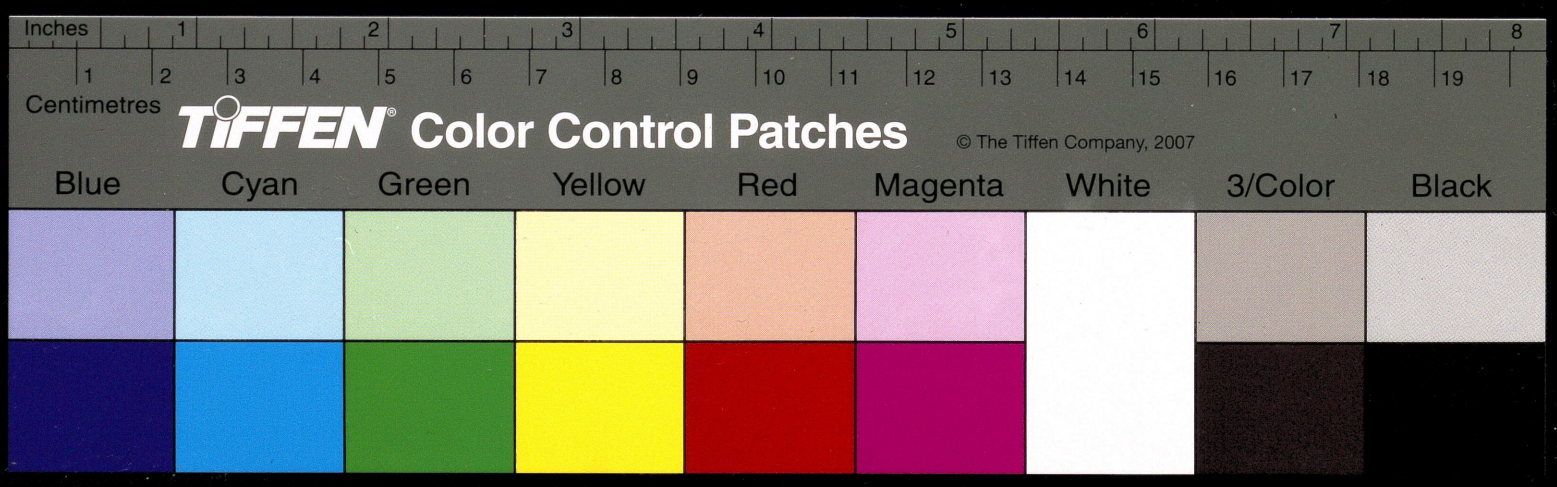
$$\begin{aligned}[(\hat{n}\cdot\partial)^{2n} - (\Delta - m^2)^n] &= [(-\partial_4^2)^n - (\Delta - m^2)^n] \\ &= -(\Box - m^2)\sum_{u=0}^{n-1}(-\partial_4^2)^{n-1-u}(\Delta - m^2)^u \\ &= -(\Box - m^2)\sum_{u=0}^{n-1}(-\partial_4^2)^{n-1-u}\sum_{v=0}^{u}(-1)^v\binom{u}{v}\Delta^{u-v}(m^2)^v \ ,\end{aligned} \tag{5.6}$$

so that the $-1/(\Box - m^2)$ is cancelled, the power series in $(m^2)$ goes to $(m^2)^{m+1}$, and the highest power occurs when 2n = r. The next term, q = 2n + 1 = (odd) is of the form

$$(\hat{n}\cdot\partial)\,[(\hat{n}\cdot\partial)^{2n} - (\Delta - m^2)^n] \ , \tag{5.7}$$

so it has the same powers of $(m^2)$ as Eq. (5.6).

Now consider the second term in the commutator of (5.2), the one which contains the $[-(\partial\cdot\alpha)/\chi]$ in front of the $\Box^{i-j}(\partial\cdot\alpha)^{2j}$. In the relation (5.3) this would add to any particular q = 2n = (even) square bracket term, like



(5.6), a term of the form (5.7), and to any $q = 2n + 1 = (\text{odd})$ square bracket term, like (5.7), a term of the form (5.6) with $n \to (n + 1)$. Further, the highest power term would be when $q = r + 1 = (\text{odd})$, so that one would have the same highest power of $(m^2)$, i.e., a term like

$$- \frac{(\hat{n}\cdot\partial)}{\chi} \left[(\hat{n}\cdot\partial)^r - (\Delta - m^2)^{r/2}\right] . \tag{5.8}$$

Putting all these factors together means that in (5.2) the commutator times an individual particle propagator $\Delta(x,-x', \chi/k)$ can be written as (remember, $\Box$ in (5.2) is a derivative to the second power)

$$\frac{1}{2}\left[\varepsilon(x_0 - x_0'), \left(1 - \frac{\partial\cdot\alpha}{\chi}\right)\Box^{i-j}(\partial\cdot\alpha)^{2j}\right]\Delta(x - x', \chi/k)$$

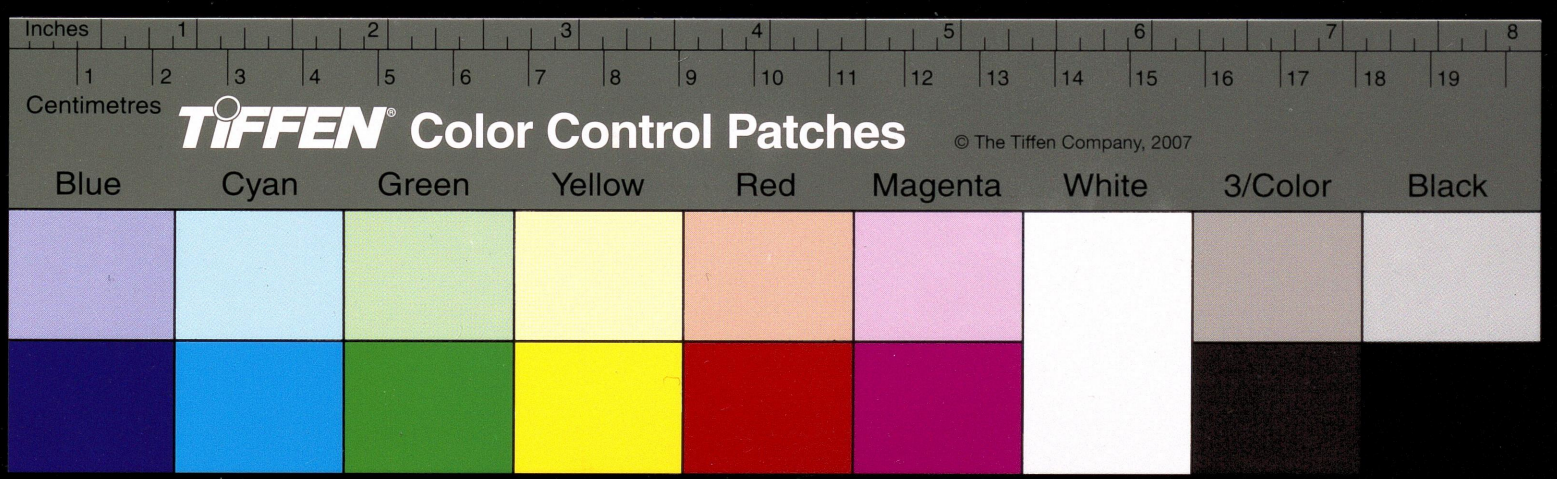
$$= \sum_{n=0}^{i} M(\hat{n},\alpha,\partial,i,j,n) \sum_{u=0}^{n-1} \left(-\partial_4^2\right)^{n-1-u} \sum_{v=0}^{u} (-1)^v \binom{u}{v} \Delta^{u-v} \left(\frac{\chi^2}{k^2}\right)^v . \tag{5.9}$$

The matrices $M(\hat{n},\alpha,\partial,i,j,n)$ can be explicitly determined from (5.2), (5.3), and (5.9). However, except for one case in the next subsection, it will not be necessary for us to calculate the M's. All that is necessary for us is to know that they are <u>not</u> functions of the mass $(\chi/k)$. Putting (5.9) into (5.1) gives

$$Y = \sum_{i=0}^{S-\frac{1}{2}} (\chi^{2S-2i}) \sum_{j=0}^{i} (-1)^{S+\frac{1}{2}-i+j} \phi_{S+\frac{1}{2}-i+j}\left(\ell_{\{S+\frac{1}{2}\}}\right)$$

$$\times \sum_{n=0}^{i} M(\hat{n},\alpha,\partial,i,j,n) \sum_{u=0}^{n-1} \left(-\partial_4^2\right)^{n-1-u}$$

$$\times \sum_{v=0}^{u} (-1)^v \binom{u}{v} \Delta^{u-v} \left[\sum_{k=\frac{1}{2}}^{S} \frac{(\chi^2/k^2)^v}{\prod_{\substack{\ell=\frac{1}{2} \\ \ell\neq k}}^{S} \left(\frac{\chi^2}{k^2} - \frac{\chi^2}{\ell^2}\right)}\right] \delta^4(x - x') . \tag{5.10}$$



But the square bracket terms in (5.10) are just the homogeneous symmetric functions $h_a$ of Eqs. (A3) and (A4), which here reduce to

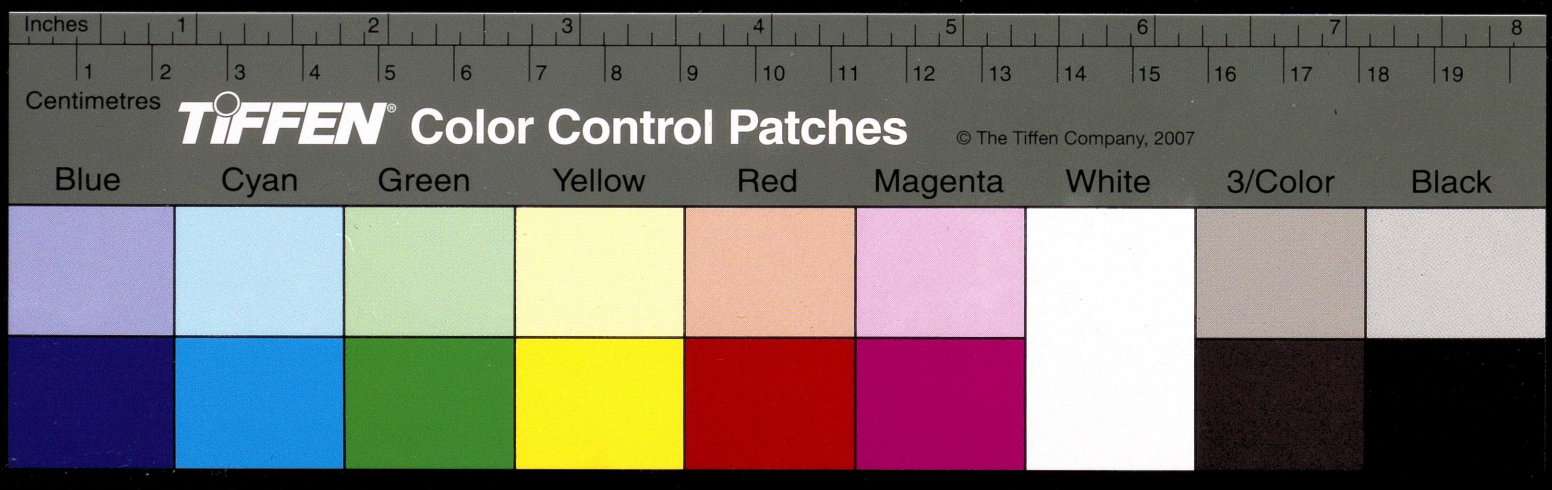
$$h_{v-S+\frac{1}{2}}\left(\left(\frac{\chi^2}{k^2}\right)_{\{S\}}\right) = 0 \text{ , since } 0 \leqslant v \leqslant u \leqslant n-1 \leqslant i-1 \leqslant S-\frac{3}{2} . \qquad (5.11)$$

Therefore, Eq. (5.1) for Y is zero, meaning the second term on the right in (4.53) is zero. Thus,

$$\Psi(x) = \psi(x/\sigma) \text{ , } S \text{ a half integer , } \qquad (5.12)$$

which in turn yields the desired result that the field anticommutation relations are preserved for the minimally coupled half-integer spin Bhabha fields, and so they are causal. (The well known Dirac field is, of course, a special case of our result.)

Two final points. If we had not used the indefinite metric quantization, meaning instead of the propagator $\Delta_S(x - x')$ we would have had the propagator $\hat{\Delta}_S(x - x')$ of (4.33), the square bracket in (5.10) would have had the extra factor $\tau_{kk}$ . Then the quantities in the bracket would not have been the homogeneous symmetric functions, so (5.12) would not have held, and noncausality would have ensued. The indefinite metric is necessary for causality. Also, because (5.12) does hold, our result would have been the same if we had used the renormalized masses in the KG divisors and the propagators and also had used the renormalized current $\hat{j}$ of (4.37) in the second term on the right of (4.53).



B. Integer spin

For integer spin Y is of the form

$$Y \equiv \frac{1}{2}\left[\varepsilon(x_0 - x_0'),\ \mathcal{D}(\partial,S)\right]\Delta_S(x - x') \tag{5.13}$$

$$= \sum_{i=0}^{S} \chi^{2(S-i)-1} \sum_{j=0}^{i} (-1)^{S-i+j}\, \phi_{S-i+j}\left(L_{\{S\}}\right)$$

$$\times\ \frac{1}{2}\left[\varepsilon(x_0 - x_0'),\ \left(1 - \frac{\partial\cdot\alpha}{\chi}\right)\Box^{i-j}(\partial\cdot\alpha)^{2j}\right]\sum_{k=1}^{S}\frac{\Delta(x-x',\chi/k)}{x(S,k)}\ , \tag{5.14}$$

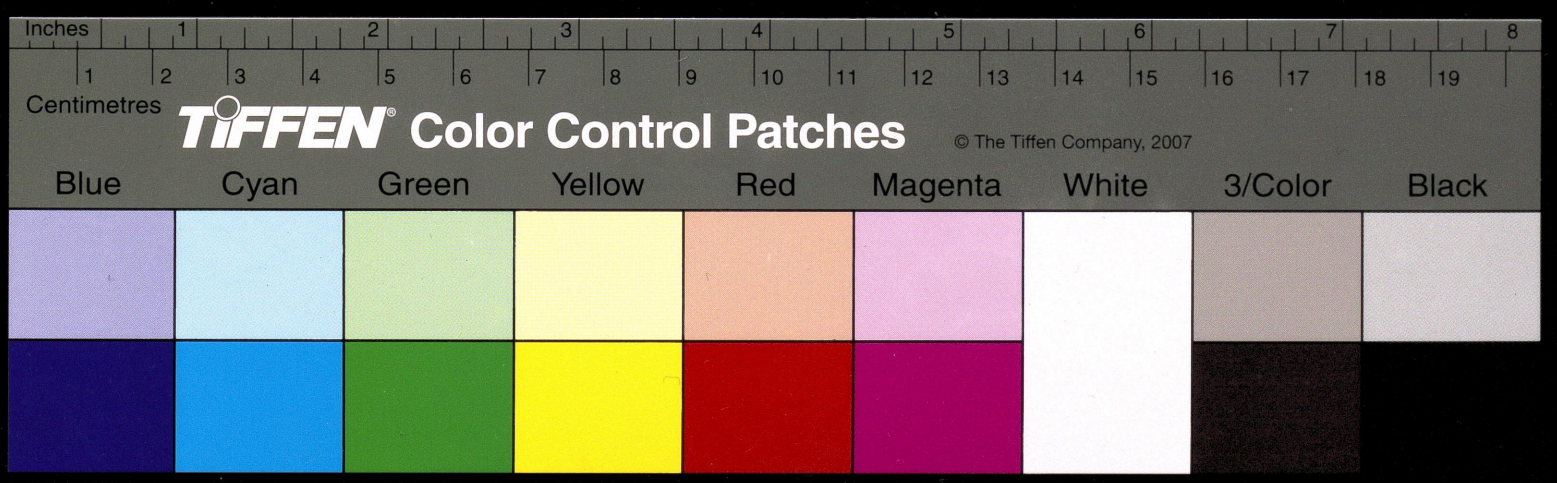
where this time Eqs. (2.18), (4.30), and (4.31) have been used. The discussion proceeds exactly as for half-integer spin, up to the point where one has obtained

$$Y = \sum_{i=0}^{S} \chi^{2(S-i)-1} \sum_{j=0}^{i} (-1)^{S-i+j}\, \phi_{S-i+j}\left(L_{\{S\}}\right)$$
$$\times \sum_{n=0}^{i} M(\hat{n},\alpha,\partial,i,j,n) \sum_{u=0}^{n-1} (-\partial_4^2)^{n-1-u}$$

$$\times \sum_{v=0}^{u} (-1)^v \binom{u}{v} \Delta^{u-v}\left[\sum_{k=1}^{S}\frac{(\chi^2/k^2)^v}{\prod_{\substack{\ell=1\\ \ell\neq k}}^{S}\left(\frac{\chi^2}{k^2} - \frac{\chi^2}{\ell^2}\right)}\right]\delta^4(x - x')\ . \tag{5.15}$$

But now the square brackets are the homogeneous symmetric functions

$$h_{v-S+1}\left(\left(\frac{\chi^2}{k^2}\right)_{\{S\}}\right) = \begin{cases} 0 & \text{for all } 0 \leqslant v \leqslant u \leqslant n-1 \leqslant i-1 < S-1 \\ 1 & \text{for } v = u = n-1 = i-1 = S-1\ , \end{cases} \tag{5.16}$$



so that we have one piece remaining in Y,

$$Y = \chi^{-1} \sum_{j=0}^{S} (-1)^j \phi_j \left(L_{\{S\}}\right) M(\hat{n},\alpha,\partial,S,j,S)(-1)^{S-1} \delta^4(x - x') . \quad (5.17)$$

Now we need to evaluate $M(\hat{n},\alpha,\partial,S,j,S)$. Observing that $M(\hat{n},\alpha,\partial,S,j,S)$ comes from the $A_{Sj}$ term in the KG divisor of Eq. (2.18), we use the second form, Eq. (2.18b), which eliminates the term in the commutator (5.14) containing the $(-\frac{\partial\cdot\alpha}{\chi})$ in front of $\Box^{S-j}(\partial\cdot\alpha)^j$ . Then using Eq. (5.3), one can see that

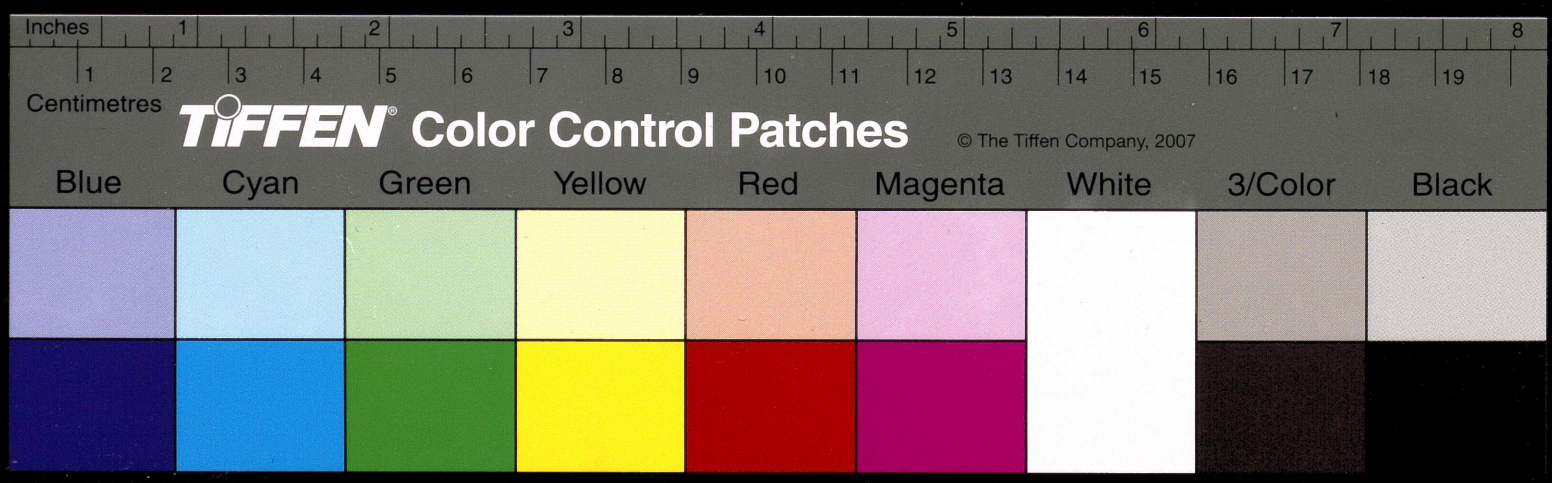
$$\begin{aligned} M(\hat{n},\alpha,\partial,S,j,S) &= (-1)^{2S} (\hat{n}\cdot\hat{n})^{S-j} (\hat{n}\cdot\alpha)^{2j} \\ &= (-1)^{S-j} (-1)^j (\alpha_4)^{2j} \\ &= (-1)^S (\alpha_4^2)^j \quad . \end{aligned} \quad (5.18)$$

This result obtains because, since (in the notation of (5.3) ) q=r=2S, only $(\hat{n}_\mu)$'s and no $(\partial_\mu^S)$'s are obtained on the right in (5.3). Of these $(\hat{n}_\mu)$'s, (S-j) pairs of them are dotted into one another because of the (S-j) powers of $\Box$ in the commutator. The 2j remaining $(\hat{n}_\mu)$'s are dotted into $\alpha$'s because the remaining original derivatives in the commutator were.

Thus, putting (5.18) into (5.17) we have

$$Y = -\chi^{-1} \left[ \sum_{j=0}^{S} (-1)^j (\alpha_4^2)^j \phi_j \left(L_{\{S\}}\right) \right] \delta^4(x - x') . \quad (5.19)$$

But from (A1) or (2.16) the quantity in the square brackets of (5.19) can be written as



$$\left[\prod_{j=1}^{S}\left(1 - \frac{\alpha_4{}^2}{j^2}\right)\right] = \frac{\prod_{j=1}^{S}(\alpha_4{}^2 - j^2)}{\prod_{k=1}^{S}(-k^2)} \equiv I_0(S) \ . \qquad (5.20)$$

The $I_0(S)$ are exactly the projection operators onto the subsidiary components for arbitrary integer spin defined in (II3.8);
i.e., they are the generalizations of the special DKP case operator

$$I_0(1) \equiv (1 - I), \qquad (5.21)$$

which Sakata and Taketani[105-109] first used for the $S = 1$ DKP system to decouple the particle component fields and Hamiltonian from the subsidiary components quantities. They have the idempotent properties necessary for projection operators

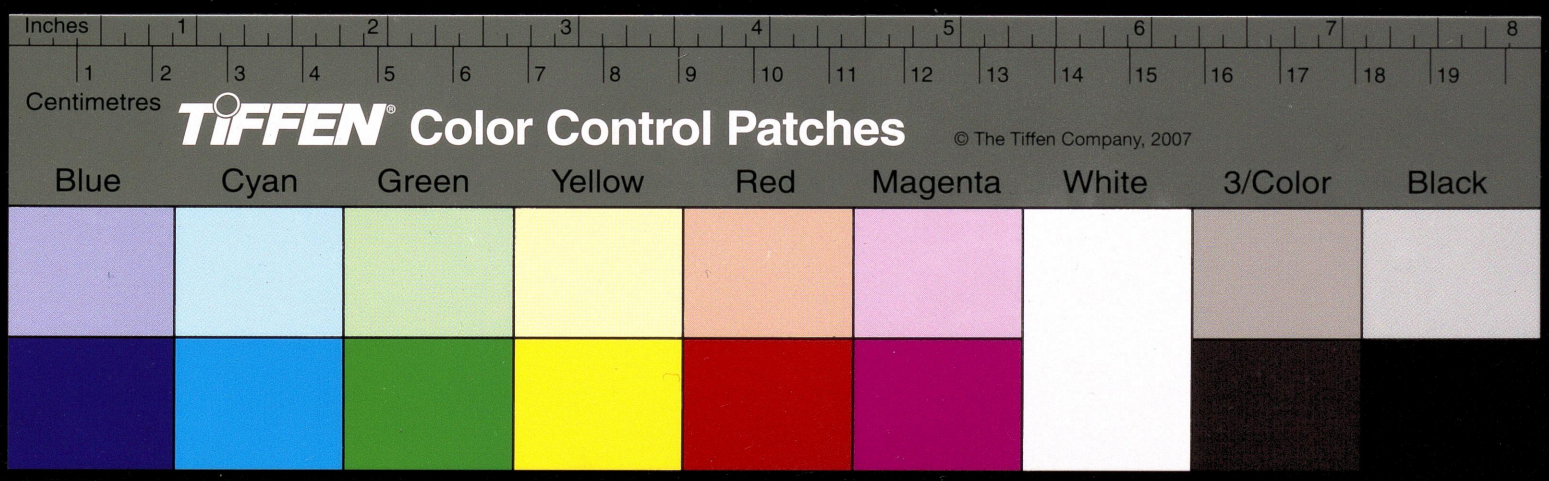
$$\left[I_0(S)\right]^2 = I_0(S) \ , \qquad (5.22)$$
$$I_0(S)\left[1 - I_0(S)\right] = 0 \ .$$

Thus, (5.19) and (5.20) mean that

$$Y = -\chi^{-1} I_0(S)\, \delta^4(x - x') \ , \qquad (5.23)$$

which from (4.53) means that

$$\Psi(x) = \psi(x/\sigma) - I_0(S)\chi^{-1} j(x) \qquad (5.24)$$



$$= \psi(x/\sigma) - I_0(S)\chi^{-1}\left[ie\, A_\mu(x)\alpha_\mu\, \Psi(x)\right] . \qquad (5.25)$$

It is an informative and simple exercise to explicitly verify Eq. (5.25) for the cases $S = 1$ and $S = 2$. One combines the KG divisors for $S = 1$ and $S = 2$ given in Eqs. (3.22) and (3.23) with the $S = 1$ and $S = 2$ invariant functions $\Delta_S(x - x')$ of Eq. (4.31), and then uses the corrected[104,110] special case commutation formulas for up to four derivatives of Katayama[83] and Takahashi[88] to end up with the result (5.25) with the correct projection operators of Eq. (II3.11),

$$I_0(1) = 1 - \alpha_4{}^2 \ ,$$

$$I_0(2) = \frac{1}{4}(\alpha_4{}^2 - 4)(\alpha_4{}^2 - 1) . \qquad (5.26)$$

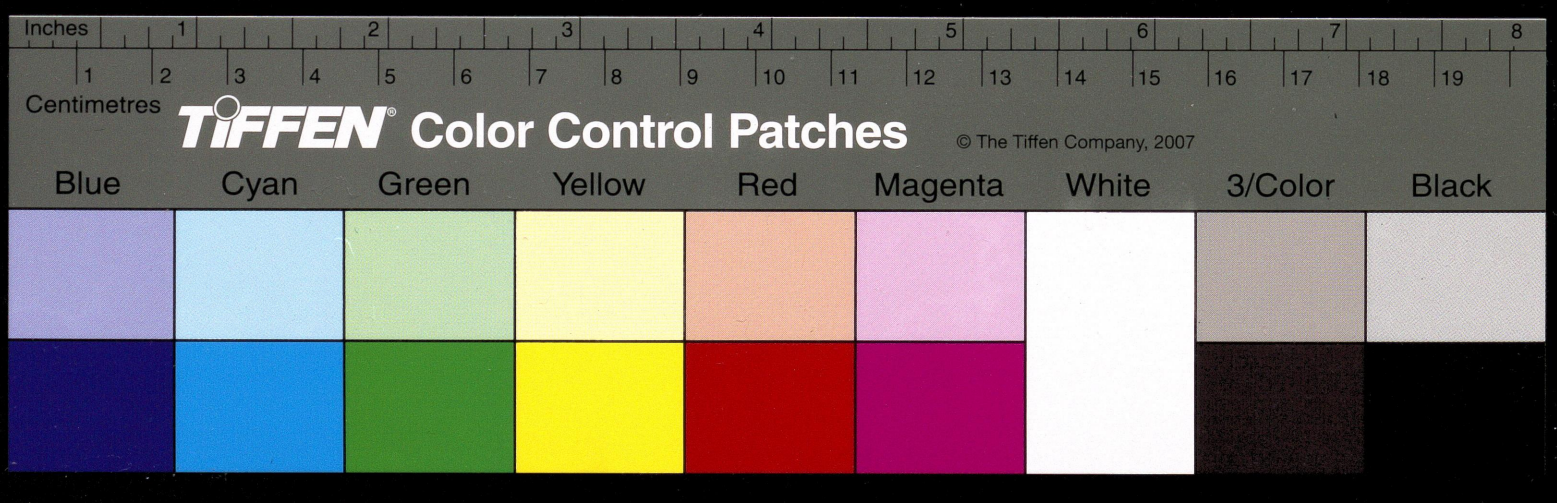
What (5.25) tells us is that the commutation relations of the physical particle components of the fields,

$$(1 - I_0)\Psi(x) = (1 - I_0)\psi(x/\sigma) , \qquad (5.27)$$

are preserved. That is, the physical fields are causal. The extra piece in the subsidiary components equation,

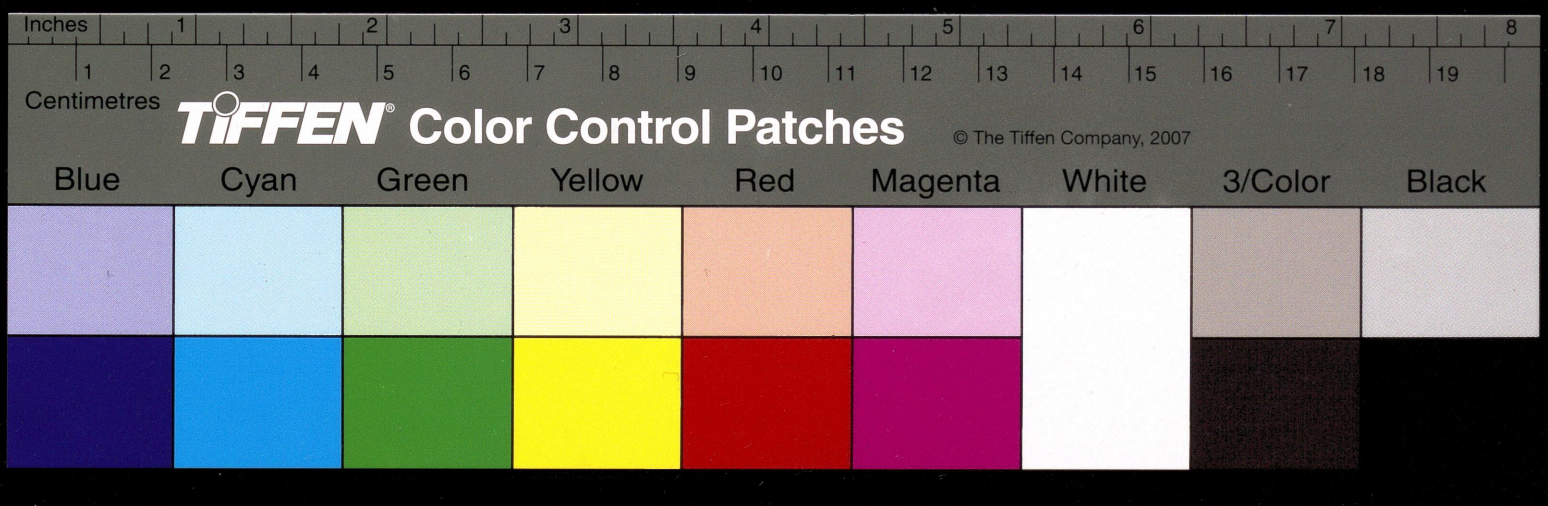
$$I_0(S)\Psi(x) = I_0(S)\psi(x/\sigma) - I_0(S)\chi^{-1}\, j(x) , \qquad (5.28)$$

is the q-number analogue of the $0^{a(0)}$ factor obtained in Eq. (3.7) if one tries to do the c-number problem for integer spin using the Velo-Zwanziger method.



These two apparent noncausality problems are resolved when one realizes it is only the nonphysical subsidiary components that are involved, and then handles the calculation accordingly.

Finally, the comments made at the end of the previous subsection pertain here too. The calculation needs the indefinite metric quantization of Eq. (4.3a) to succeed. Also, if the renormalized mass $\chi_{Obs}$ had been used, Eq. (5.24) would still have been obtained, only with $j(x) \to \hat{j}(x)$, meaning the same causal physics for the particle components fields would have resulted.



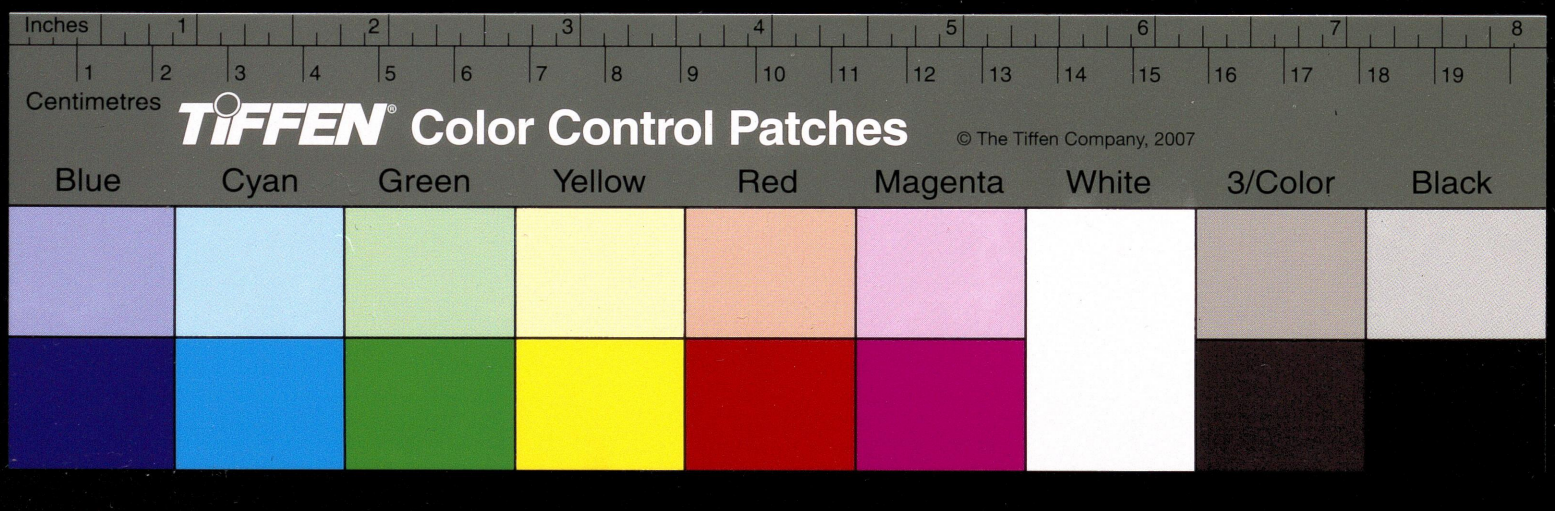
## VI. DISCUSSION

In this paper we have demonstrated, in a simple closed form, the causality of both the c-number and the q-number arbitrary-spin Bhabha fields with minimal electromagnetic coupling. The c-number demonstration was done with the Wightman[23] modification of the Velo-Zwanziger[17,18] method, as this modification allows the integer-spin subsidiary components to be handled without further complications in the calculations.

The q-number demonstration was done starting with an indefinite metric second quantization technique, necessary because the entire Bhabha system has negative normed states built into it. Then, with the aid of the symmetric functions to keep track of the many terms, causality could be demonstrated in closed form for arbitrary-spin Bhabha fields. For integer spin there is an extra piece in the Heisenberg fields which at first glance can appear to imply noncausality. But this extra piece turns out to be entirely composed of subsidiary components, so that the physical particle components fields are causal.

In both the c-number and the q-number causality demonstrations, the subsidiary components required a slightly special handling. (This, of course, has been true in the calculations throughout this series of papers.) Here the need for special handling can be technically traced to the extra derivative in the integer-spin defining algebra.

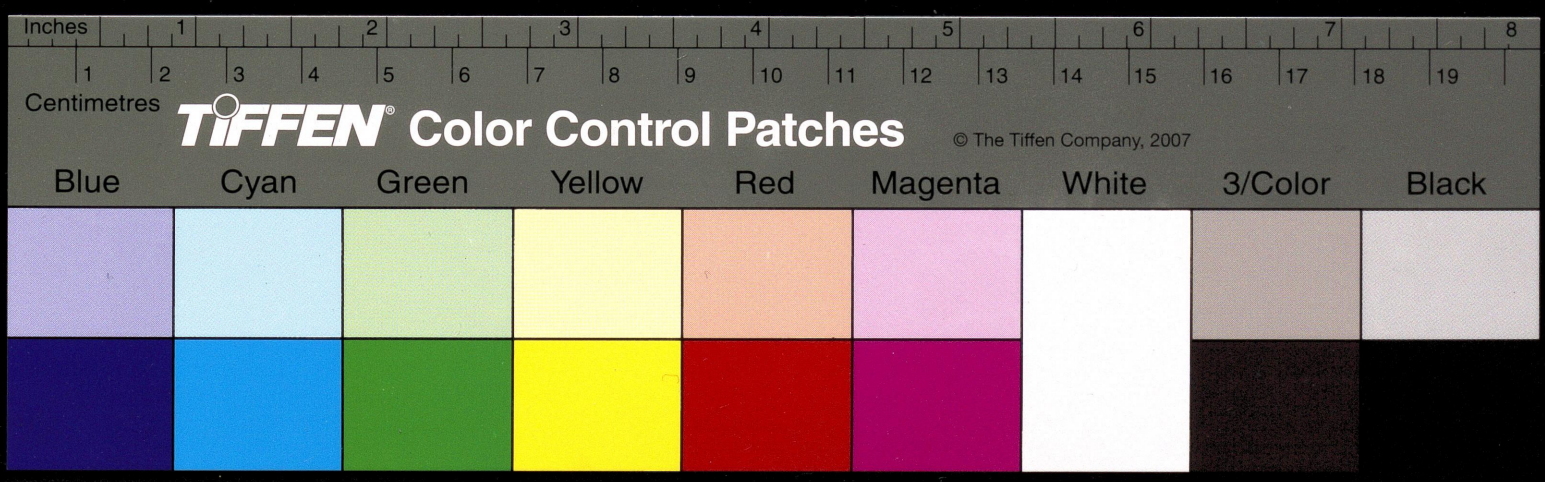
For half-integer spin both the KG operator of (3.1) and the defining Bhabha algebra (see the characteristic equation (1.3) and its relation to the entire "half-integer-spin KG equation" (I2.35) ) are of order $(2S + 1)$. This means that the KG divisor (3.11) has terms with derivatives up to a maximum power of $(2S)$ = (odd). This number is one less than the number of physical mass states.



For integer spin, the KG operator of (2.16) used to obtain the KG divisor is only of order $2S$, even though the defining Bhabha algebra (see (1.3) and now the entire "integer-spin KG equation" (I2.34) ) is still of order $(2S + 1)$. Thus in the KG divisor (2.18b) there appear extra non-zero terms with derivatives of maximum order not of $(2S - 1)$, but of order $(2S)$ = (even). This number is now equal to the number of physical mass states. These terms are the $\Box^{S-j}\ (\partial\cdot\alpha)^{2j}$ in the $A_{Sj}$, and are the terms which lead to the additional pieces in the subsidiary components of the Hiesenberg fields. The effects of the relative extra derivative were limited here to the subsidiary components Heisenberg fields because those were the ones where the extra derivative came from to begin with (the extra derivative in (I2.34) vs. the KG equation (2.16) ).

However, one can thus technically, as well as physically, see what higher derivatives will do; they will eventually lead to noncausality. For example, one could find in place of Eqs. (5.11) and (5.16), homogeneous symmetric functions of the type $h_{a>0} \neq (0,1)$. Another example is in the standard discussion[11] of the RS[12] spin-3/2 field, where noncausality is ultimately due to the imposition of external constraint conditions which involve derivatives.[17] A related example, as we mentioned in Sec. III, is Wightman's [23] demonstration for the DKP $S = 1$ case that $[\beta_\mu,\beta_\nu]F_{\mu\nu}$ coupling introduces noncausality in the spin-0 representation. Here the problem is involved with the order and eigenvalues of the algebra.

Given all this, it is the minimal derivative nature (first-order wave equation with no external derivative constraints) and the particular algebra of the Bhabha system which leads to causality with minimal electromagnetic interactions.

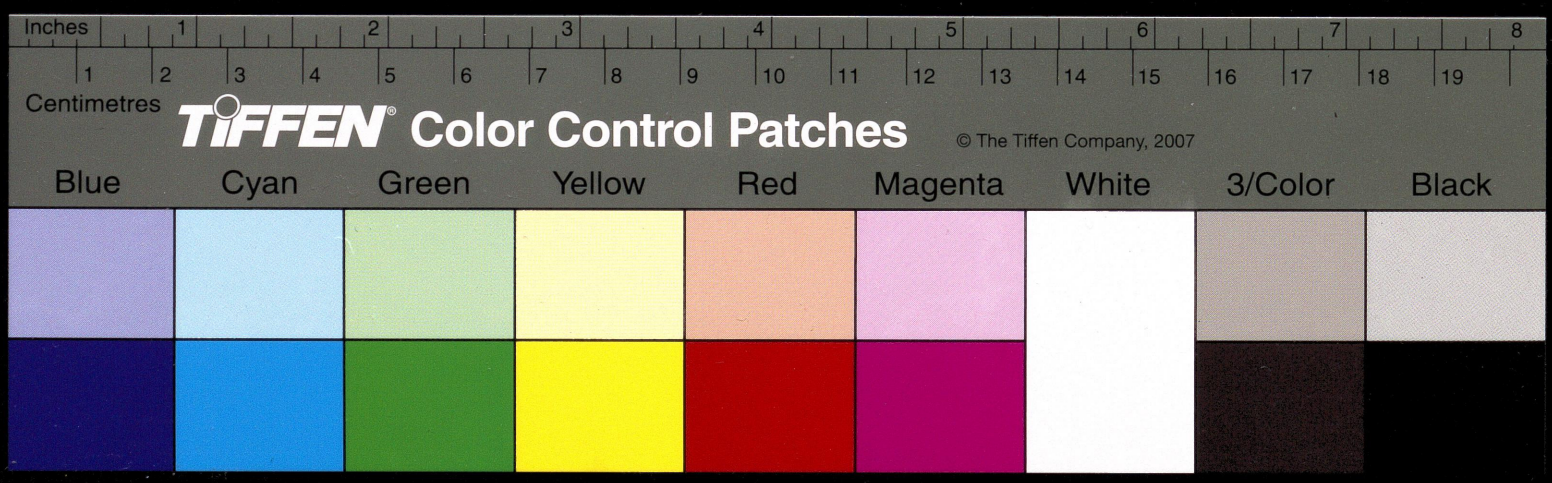


Thus, so far we have seen that the Bhabha system is CPT and Lorentz invariant with a well defined multi-mass and multi-spin spectrum, and is causal with minimal electromagnetic interaction. However, at this point, as was discussed around Eq. (4.24), one has an indefinite metric. Since (for convenience in our representation of the $\alpha_\lambda$ matrices) the different normed states are coupled by $\vec{\alpha}$ matrices, this metric implies that the standard quantum mechanical probablity (or charge) density interpretation is in doubt, for in principle unitarity can be violated by a nonconversation of probabiltiy (or charge). This is despite the conserved current condition

$$\partial_\lambda j_\lambda = \partial_\lambda \left[ ie\bar{\psi}\alpha_\lambda\psi \right] = 0 \ , \tag{6.1}$$

which trivially follows from the free field and adjoint field equations (I2.24) and (I2.45).

We will discuss this further in V, where we will concentrate on the indefinite metric and its meaning, as well as its relation to generalized Foldy-Wouthuysen transformations[111-113] for various-spin fields,[114-123] and the Bhabha system in particular.



ACKNOWLEDGMENTS

We wish to thank numerous colleagues for their helpful comments on this work. Of them, there are five people whom we would like to especially acknowledge:

K. J. LeCouteur made observations on the Bhabha indefinite metric.

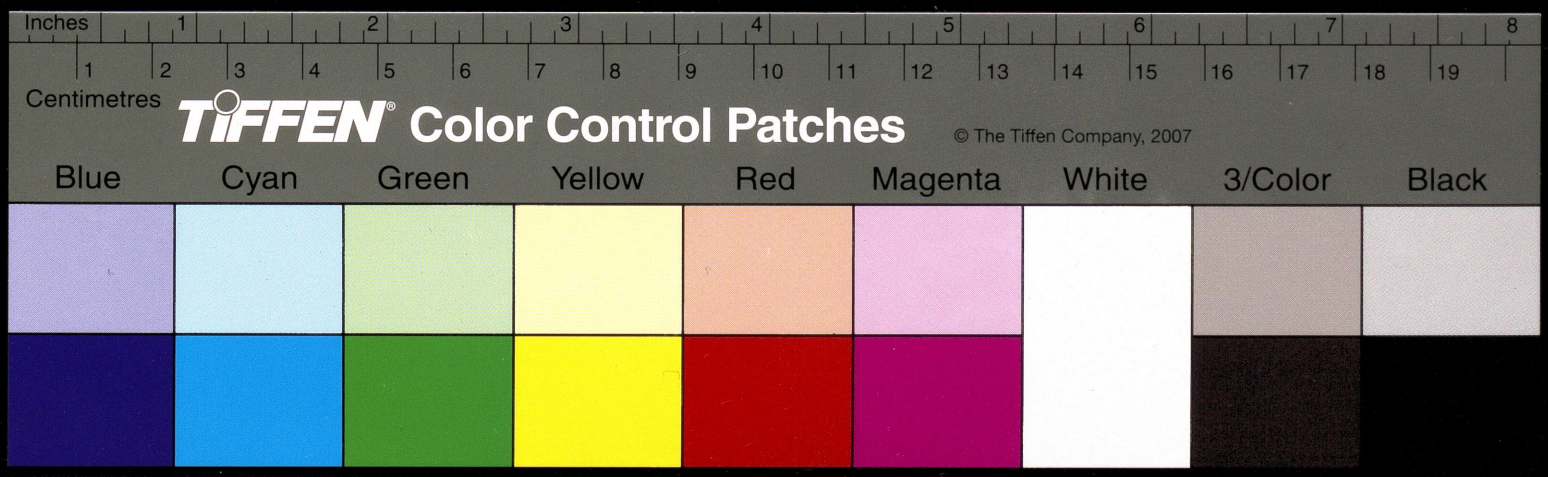
James D. Louck first pointed out to us the usefulness of the symmetric functions for doing calculations of the type contained in Sec. V, and he also made a number of helpful suggestions for the calculations.

David C. Peaslee commented on the implications of his DKP second-quantized renormalization calculations.[75]

A. S. Wightman kindly took the time, both by mail and in person, to ciscuss in detail interacting field theory problems *vis a vis* the Bhabha fields.

D. Zwanziger made many useful comments on the c-number causality theory for first-order wave equations.

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## APPENDIX: SYMMETRIC FUNCTIONS

The elementary symmetric functions $\phi_r(x_{\{n\}})$ of the n quantities $x_j$ are well know.[52-54] They can be obtained from the product formula

$$0 = \prod_{j=1}^{n} (A - x_j) \equiv \sum_{k=0}^{n} A^k (-1)^{n-k} \phi_{n-k}(x_{\{n\}}) , \qquad (A\text{-}1)$$

and thus are given by

$$\phi_0 = 1 , \qquad (A\text{-}2a)$$

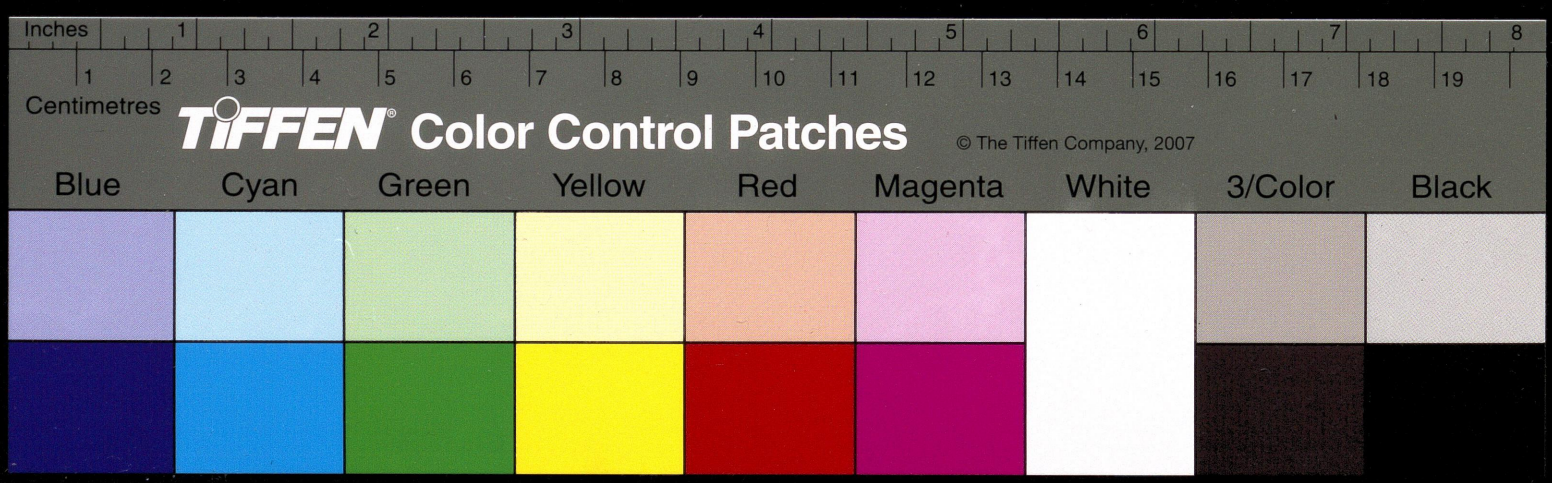
$$\phi_{r\neq 0}(x_{\{n\}}) = \left\{ \sum_{i(1) > i(2) > \ldots > i(r) = 1}^{n} \left[ x_{i(1)} x_{i(2)} \cdots x_{i(r)} \right] \right\} . \qquad (A\text{-}2b)$$

That is, the function $\phi_r$ "is the sum of the $\binom{n}{r}$ products of r different quantities" $x_i$.[53]

We also deal with the functions $h_a$, where $\underline{a}$ can be a positive or negative integer, or zero. These functions are defined by

$$h_{q-n+1}(y_{\{n\}}) = \sum_{k=1}^{n} \frac{y_k^q}{\prod_{\substack{j=1 \\ j\neq k}}^{n} (y_k - y_j)} , \qquad (A\text{-}3)$$

and have the properties[54]



$$
\begin{aligned}
h_{-|a|} &= 0 \ , \\
h_0 &= 1 \ , \\
h_1 &= \phi_1 \ , \\
h_2 &= h_1\phi_1 - h_0\phi_2 \ , \qquad \text{(A-4)} \\
&\ \ \vdots \\
h_n &= -\sum_{b=0}^{n-1} (-1)^{n-b} \phi_{n-b}\, h_b \ .
\end{aligned}
$$

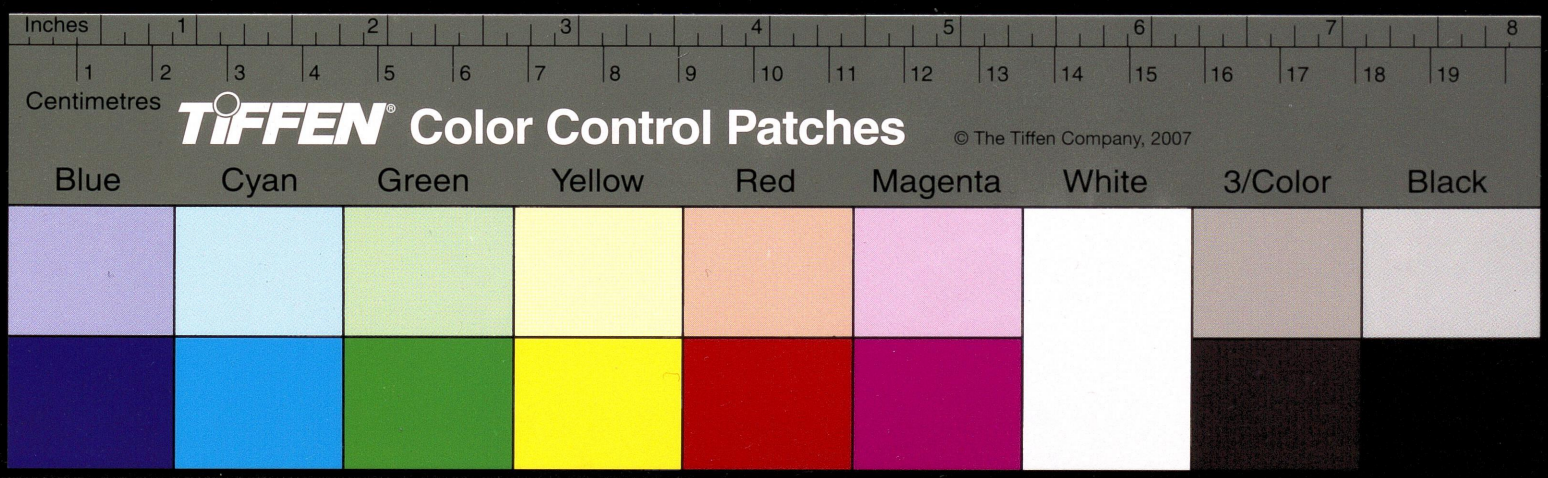
For $a \geq 0$, these functions equal the "homogeneous symmetric functions" $h_{|a|}$ of Littlewood,[53] so we call the entire set (A3-A4) this name. (Louck and Biedenharn[54] denote these $h_a$ by $\beta_a$, a notation we avoid, so as not to confuse them with the DKP matrices $\beta_\lambda$.)

The result quoted in Eq. (4.31) for the form of the multimass invariant $\Delta$ function comes from considering a special case of Eqs. (A3-A4),

$$
h_{2-n}\,(y_{\{n+1\}}) = 0 = \sum_{k=1}^{n+1} \frac{1}{\prod_{\substack{j=1 \\ j\neq k}}^{n+1} (y_k - y_j)} \ . \qquad \text{(A-5)}
$$

Separate the (n+1)st term in the sum, to give

$$
\frac{-1}{\prod_{j=1}^{n} (y_{n+1} - y_j)} = \sum_{k=1}^{n} \frac{1}{(y_k - y_{n+1}) \prod_{\substack{j=1 \\ j\neq k}}^{n} (y_k - y_j)} \ . \qquad \text{(A-6)}
$$



Now let $y_{n+1} = -p^2$ and $y_{j\neq n+1} = \chi^2/j^2$, and you have

$$\frac{1}{\prod_{j=1}^{n}(p^2 + \chi^2/j^2)} = \sum_{k=1}^{n} \frac{1}{(p^2 + \chi^2/k^2)\prod_{\substack{j=1\\ j\neq k}}^{n}\left(\frac{\chi^2}{k^2} - \frac{\chi^2}{j^2}\right)} \quad . \qquad \text{(A-7)}$$

But $(p^2 + \chi^2/k^2)^{-1}$ is just the momentum space representation of the standard invariant $\Delta(x - x', \chi/k)$ function for a single mass $\chi/k$, and the left hand side of Eq. (A-7) is the momentum space representation of the multimass invariant $\Delta$ function. Thus, integrating around the appropriate contour[81] yields the desired expression for $\Delta_S(x - x')$ in Eq. (4.31),

$$\Delta_S(x - x') = \sum_{k=(\frac{1}{2},1)}^{S} \left[ \frac{\Delta(x - x', \chi/k)}{\prod_{\substack{j=(\frac{1}{2},1)\\ j\neq k}}^{S}\left(\frac{\chi^2}{k^2} - \frac{\chi^2}{j^2}\right)} \right] \quad . \qquad \text{(A-8)}$$

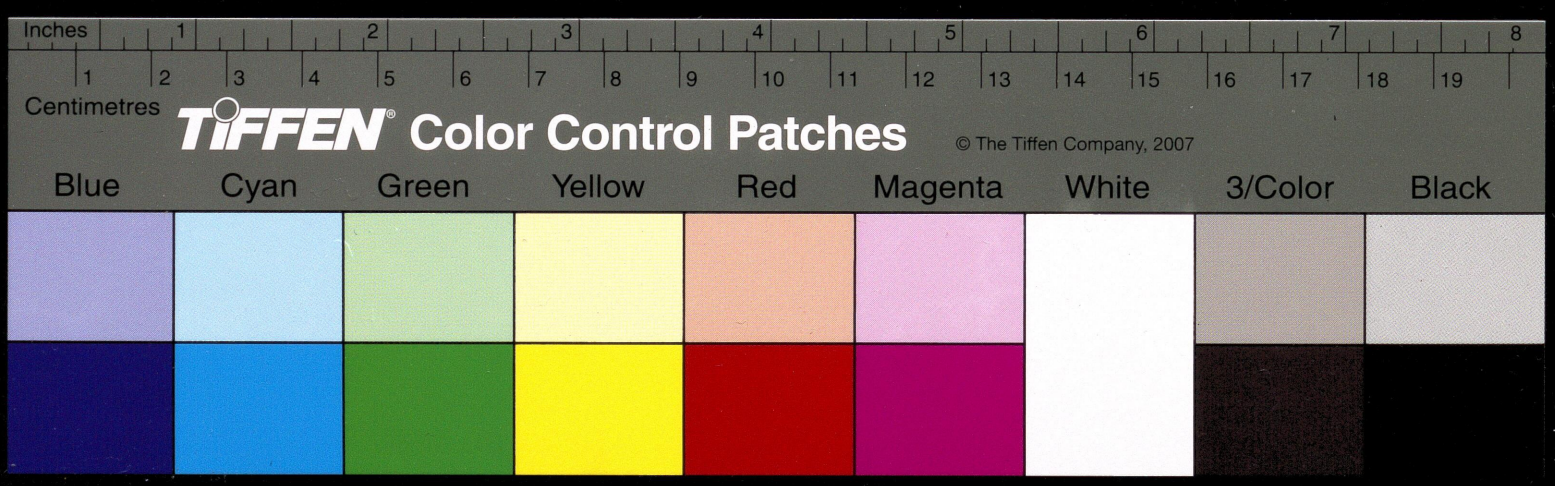
## REFERENCES


1. R. A. Krajcik and M. M. Nieto, Phys. Rev. D 10, 4049 (1974), paper I of this series, on C, P, and T. (References to equations in this paper will be prefixed with an I; for example, (I3.47).)

2. R. A. Krajcik and M. M. Nieto, Phys. Rev. D 11, 1442 (1975), paper II of this series, on the mass and spin content, the Hamiltonians, and the general Sakata-Taketani reductions of Bhabha fields. (References to equations in this paper will be prefixed with a II; for example, (II3.8).)

3. R. A. Krajcik and M. M. Nieto, Phys. Rev. D 11, 1459 (1975), paper III of this series, on the Poincaré generators. (References to equations in this paper will be prefixed with a III; for example (III1.5).)

4. H. J. Bhabha, Rev. Mod. Phys. 17, 200 (1945).

5. H. J. Bhabha, Proc. Indian Acad. Sci. A21, 241 (1945).

6. H. J. Bhabha, Rev. Mod. Phys. 21, 451 (1949).

7. E. Fischbach, J. Louck, M. M. Nieto and C. K. Scott, J. Math. Phys. 15, 60 (1974).

8. B. S. Madhava Rao (Madhavarao), Proc. Indian Acad. Sci. A26, 221 (1947).

9. J. K. Lubański, Physica 9, 310, 325 (1942) first proposed the so(5) algebra and recognized the type of spin and mass spectrum which would be obtained. However, his papers were published in occupied Holland and so did not receive the recognition they deserved. B. S. Madhavarao (Madhava Rao) inspired by the DKP algebra, began looking at first order equations in Proc. Indian Acad. Sci. A15, 139 (1942), concentrating on obtaining what turned out to be the $S = 3/2$ and $S = 2$ commutation relations of the general so(5) algebra. In J. Mysore Univ. 3, 59 (1942) he tried to obtain restrictions on the general algebra which would yield a single mass KG equation. However, as we now know, that only is possible for the Dirac and DKP cases. In J. Mysore Univ. 6, 57 (1945) Madhavarao acknowledged that H. J. Bhabha, in Current Science 14, 89 (1945), had observed that for higher spin what we call the so(5) equations will satisfy a multimass KG equation. Further work appeared in our Ref. 8, and Ref. 96 below.

10. J. M. Jauch and F. Rohrlich, The Theory of Photons and Electrons (Addison-Wesley, Reading, Mass., 1955). See Sec. 4-2, p. 73, Sec. 5-1, p. 82, and places referred to there.

11. K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N.Y.) 13, 126 (1961).

12. W. Rarita and J. Schwinger, Phys. Rev. 60, 61 (1941).

13. H. J. Bhabha, Philos. Mag. 43, 33 (1952).

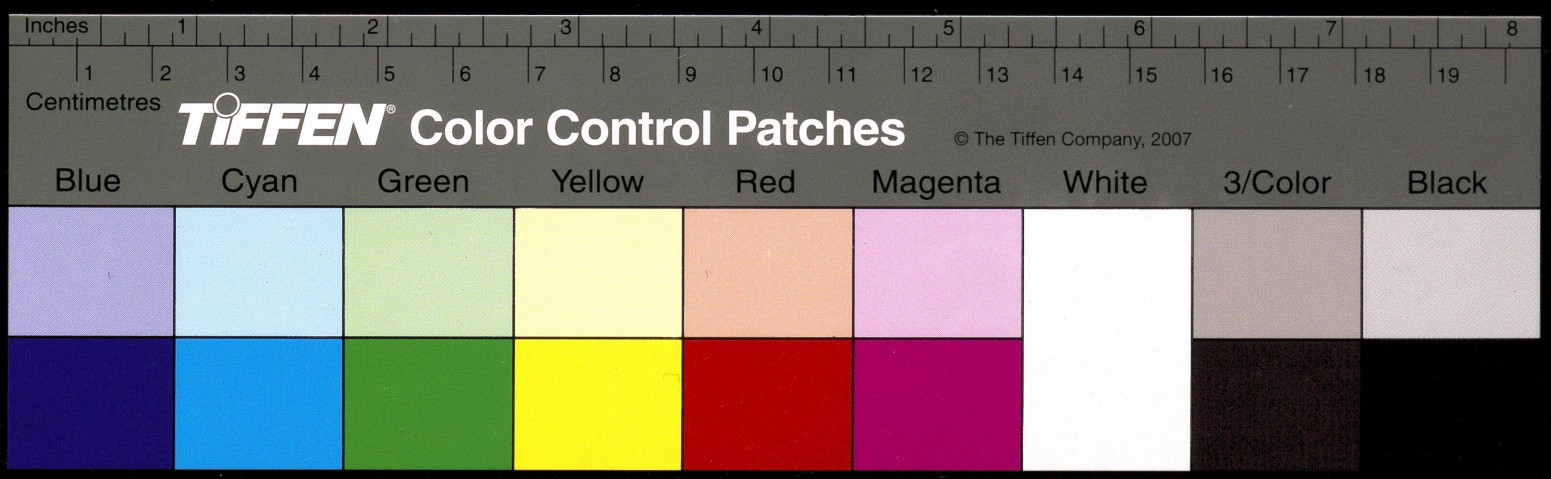
14. K. K. Gupta, Proc. R. Soc. A222, 118 (1954).

15. H. L. Baisya, Nucl. Phys. B23, 633 (1970).

16. A. K. Nagpal, Nucl. Phys. B53, 634 (1971); B64, 544(E) (1973). See the errata for the author's final conclusions that the q-number Bhabha spin-2 field is noncausal.

17. G. Velo and D. Zwanziger, Phys. Rev. 186, 1337 (1969).

18. G. Velo and D. Zwanziger, Phys. Rev. 188, 2218 (1969).



19. A. S. Wightman, in Proceedings of the Fifth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1968, edited by A. Perlmutter, C. Angas Hurst, and B. Kurşunoğlu, (Benjamin, New York, 1968), p. 291. With reference to the discussion in this paper of the theory of S. N. Gupta, Phys. Rev. 95, 1334 (1954), see footnote 36 in our paper III (Ref. 3 above).

20. Troubles in the External Field Problem for Invariant Wave Equations, Vol. 4 of the Lectures from the Coral Gables Conference on Fundamental Interactions at High Energy, 1971, reviewer A. S. Wightman, (Gordon and Breach, New York, 1971). See the articles by A. S. Wightman, p. 1, G. Velo and D. Zwanziger, p. 8, and R. Seiler, p. 23.

21. A. S. Wightman, in Lectures in Differential Equations, Vol. II, Ed. by A. K. Aziz (Van Nostrand Reinhold, New York, 1969), p. 1.

22. H. L. Baisya, Trinity College, University of Dublin, thesis (1972).

23. A. S. Wightman, in Partial Differential Equations, proceedings of the Symposium in Pure Mathematics, Berkeley, Vol. 23, 1971 (American Mathematical Society, Providence, 1973), p. 441.

24. A. S. Wightman, in Fundamental Interactions in Physics and Astrophysics, Lectures from the 1972 Coral Gables Conference on Fundamental Interactions at High Energy, ed. by G. Iverson, A. Perlmutter, and S. Mintz (Plenum Press, New York, 1973), p. 1.

25. L. P. S. Singh and C. R. Hagen, Phys. Rev. D 9, 898 (1974).

26. L. P. S. Singh and C. R. Hagen, Phys. Rev. D 9, 910 (1974).

27. W. J. Hurley, Phys. Rev. D 4, 3605 (1971); Phys. Rev. Lett. 29, 1475 (1972); Phys. Rev. D 10, 1185 (1974). With respect to the above, also see earlier discussions of this type of wave equation given by J. D. Harris, Ph.D. thesis, Purdue University, 1955 (unpublished); S.-J. Chang, Phys. Rev. Lett. 17, 1024 (1966); and W.-K. Tung, Phys. Rev. 156, 1385 (1967).

28. A. K. Nagpal, Nucl. Phys. B72, 359 (1974).

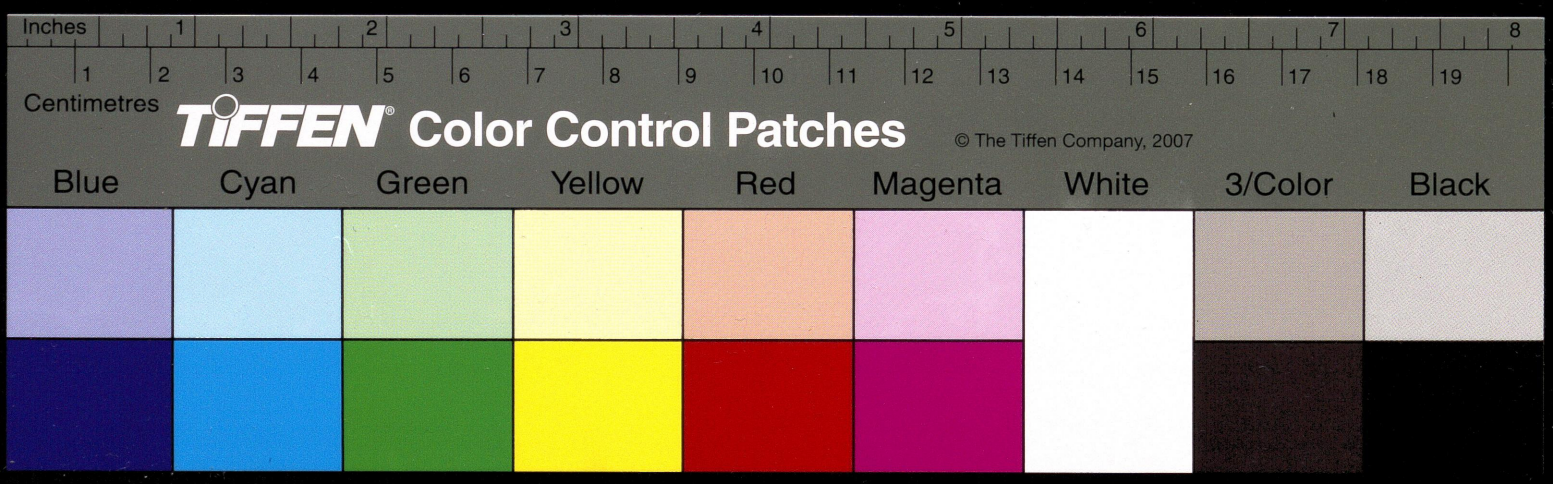
29. A. Shamaly and A. Z. Capri, Ann. Phys. 74, 503 (1972).

30. A. K. Nagpal, Nucl. Phys. B80, 206 (1974). Nagpal's paper appeared while this work was in progress. The author himself pointed out to us its existence while the present manuscript was being written, for which we express our gratitude.

31. V. Amar and U. Dozzio, Lett. Nuovo Cimento 5, 355 (1972).

32. V. Amar and U. Dozzio, Lett. Nuovo Cimento 12, 659 (1975).

33. H. Biritz, Nuovo Cimento 25B, 449 (1975); Phys. Rev. 11, 2862 (1975).



34. R. W. Hayward, Ann. Phys. (N. Y.), submitted.

35. Y. Takahashi and R. Palmer, Phys. Rev. D 1, 2974 (1970).

36. A. Shamaly and A. Z. Capri, Can. J. Phys. 52, 917 (1974).

37. G. Velo, Nucl. Phys. B65, 427 (1973).

38. P. M. Mathews and M. Seetharaman, Phys. Rev. D 8, 1815 (1973); C. R. Hagen, Phys. Rev. D 9, 498 (1974).

39. J. D. Jenkins, J. Phys. A 6, 1935 (1973).

40. S. N. Gupta and W. W. Repko, Phys. Rev. 177, 1921 (1969); J. D. Kimel and L. M. Nath, Phys. Rev. D 6, 2132 (1972); A. K. Nagpal, Lett. Nuovo Cimento 9, 137 (1974); G. B. Mainland and E. C. G. Sudarshan, Phys. Rev. D 10, 3343 (1974).

41. L. M. Nath, B. Etemadi, and J. D. Kimel, Phys. Rev. D 3, 2153 (1971); C. R. Hagen, Phys. Rev. D 4, 2204 (1971); A. K. Nagpal, Lett. Nuovo Cimento 3, 698 (1972); C. R. Hagen, Lett. Nuovo Cimento 4, 816 (1972).

42. H. L. Baisya, Nucl. Phys. B29, 104 (1971).

43. J. Bellissard and R. Seiler, Lett. Nuovo Cimento 5, 221 (1972).

44. L. P. S. Singh, Phys. Rev. D 7, 1256 (1973).

45. A. Z. Capri and A. Shamaly, Can. J. Phys. 52, 919 (1974).

46. A. Z. Capri, J. Math. Phys. 15, 2093 (1974); 16, 1201 (E) (1975).

47. C. R. Hagen, Phys. Rev. D 10, 3972 (1974).

48. C. R. Hagen, Phys. Rev. D 6, 984 (1972).

49. J. F. Reilly, Nucl. Phys. B70, 356 (1974).

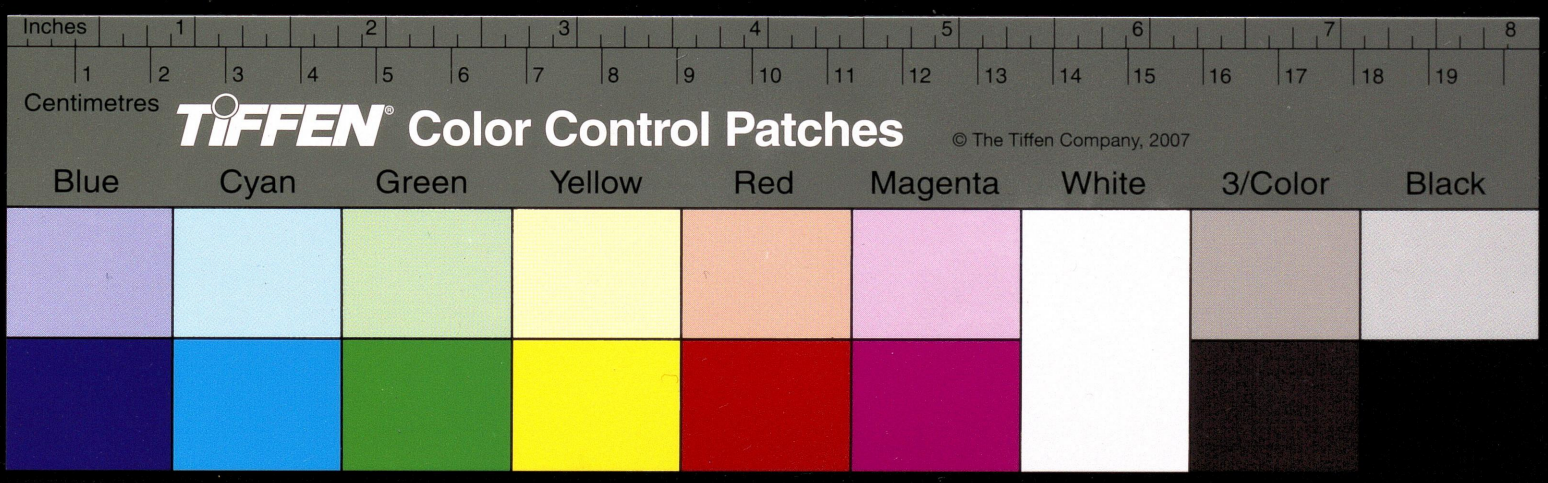
50. A. S. Wightman in Aspects of Quantum Theory, ed. by A. Salam and E. P. Wigner, (Cambridge University Press, Cambridge, 1972), p. 95.

51. A. S. Wightman in Bargmann Festschrift (to be published).

52. O. Perron, Algebra I, (Walter de Gruyter, Berlin, 1951), p. 153.

53. D. E. Littlewood, The Theory of Group Characters and Matrix Representations of Groups, 2nd ed. (Oxford University Press, London, 1950), p. 82.

54. J. D. Louck and L. C. Biedenharn, J. Math. Phys. 11, 2368 (1970). See Appendix A.



55. L. Hörmander, Linear Partial Differential Operators, (Springer-Verlag, Berlin, 1963), Chaps. 7 and 8.

56. A. S. Wightman (private communication) has expressed the opinion that as of yet no widely read and understood thorough description of the singular matrix case exists in the physics literature.

57. P. A. M. Dirac, Sgríbhinní Institiúid Árd-Léighinn Bhaile Átha Cliath (Comm. Dublin Inst. Adv. Studies), Series A, No. 1 (1943); Series A, No. 3 (1946); Proc. R. Soc. A180, 1 (1942).

58. W. Pauli, Rev. Mod. Phys. 15, 175 (1943).

59. E. Wild, Proc. R. Soc. A191, 253 (1947).

60. K. L. Nagy, State Vector Spaces with Indefinite Metric in Quantum Field Theory (Akadémeai Kiadó, Budapest, 1966), and references therein.

61. J. G. Taylor, in Lectures on Particles and Fields, ed. by H. H. Aly, (Gordon and Breach, New York, 1970), p. 213.

62. K. J. Le Couteur, Proc. Camb. Phil. Soc. 44, 63 (1948).

63. K. J. Le Couteur, Proc. R. Soc. A196, 251 (1949).

64. H. J. Bhabha, in Report of an International Conference on Fundamental Particles and Low Temperatures held at the Cavendish Laboratory, Cambridge on 22-27 July 1946, Vol. I (Physical Society, London, 1947), p.22.

65. R. A. Krajcik and M. M. Nieto, Phys. Rev. (to be published). V of this series, on the indefinite metric and Foldy-Wouthuysen transformations.

66. H. Umezawa and A. Visconti, Nucl. Phys. 1, 348 (1956).

67. A. S. Glass, Commun. Math. Phys. 23, 176 (1971).

68. Harish-Chandra, Proc. Roy. Soc. A186, 502 (1946).

69. Harish-Chandra, Proc. Camb. Phil. Soc. 43, 414 (1947).

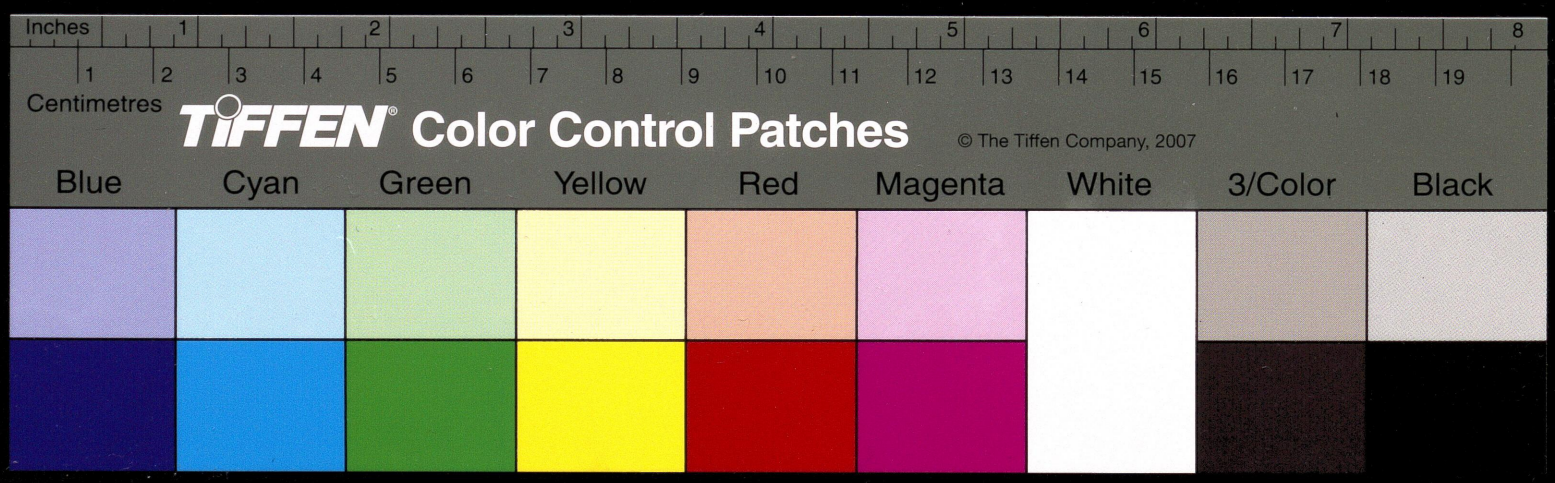
70. Harish-Chandra, Phys. Rev. 71, 793 (1947).

71. Harish-Chandra, p. 185 in Ref. 64.

72. Harish-Chandra, Proc. Roy. Soc. A192, 195 (1948).

73. As first pointed out by Nagpal (Ref. 16) concerning Baisya's calculations of Ref. 15, the formulas proposed by Baisya in Refs. 15 and 22 for the KG divisors do not have the proper number of terms for $S > 3/2$.

74. In footnote 38 of our Ref. (50), Wightman attributes the method to L. Svensson, communicated privately by L. Gårding.



75. D. C. Peaslee, Phys. Rev. 81, 94, 107 (1951).

76. M. Neuman and W. H. Furry, Phys. Rev. 76, 1677 (1949).

77. R. G. Moorhouse, Phys. Rev. 76, 1691 (1949).

78. T. Kinoshita, Prog. Theor. Phys. (Kyoto) 5, 473 (1950).

79. T. Kinoshita and Y. Nambu, Prog. Theor. Phys. (Kyoto) 5, 749 (1959).

80. In Sec. 5 of Ref. 23, Wightman describes the current limited theorem dealing with constant coefficients as being due to a "folk lemma" he learned from L. Gårding, the work of Hörmander in the book listed in Ref. 55, and a recent contribution from L. Svensson.

81. F. Mandl, Introduction to Quantum Field Theory (Interscience, New York, 1959). Chap. 6 contains a nice discussion of the invariant Δ-functions.

82. N. N. Bogoliubov and D. V. Shirkov, Introduction to the Theory of Quantized Fields (Interscience, New York, 1959). Note Appendix I on the singular functions.

83. Y. Katayama, Prog. Theor. Phys. (Kyoto) 10, 31 (1953).

84. H. Umezawa, Quantum Field Theory (North Holland, Amsterdam, 1956).

85. Y. Takahashi and H. Umezawa, Prog. Theor. Phys. (Kyoto) 9, 14 (1953).

86. Y. Takahashi and H. Umezawa, Nucl. Phys. 51, 193 (1964).

87. Y. Takahashi, in Lectures in Theoretical Physics, Vol. 10A, from the 1967 Boulder Summer School, ed. by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), p. 351.

88. Y. Takahashi, An Introduction to Field Quantization (Pergamon Press, Oxford, 1969).

89. S. S. Schweber, An Introduction to Relativistic Quantum Field Theory (Fox, Peterson, and Co., Evanston, Ill., 1961).

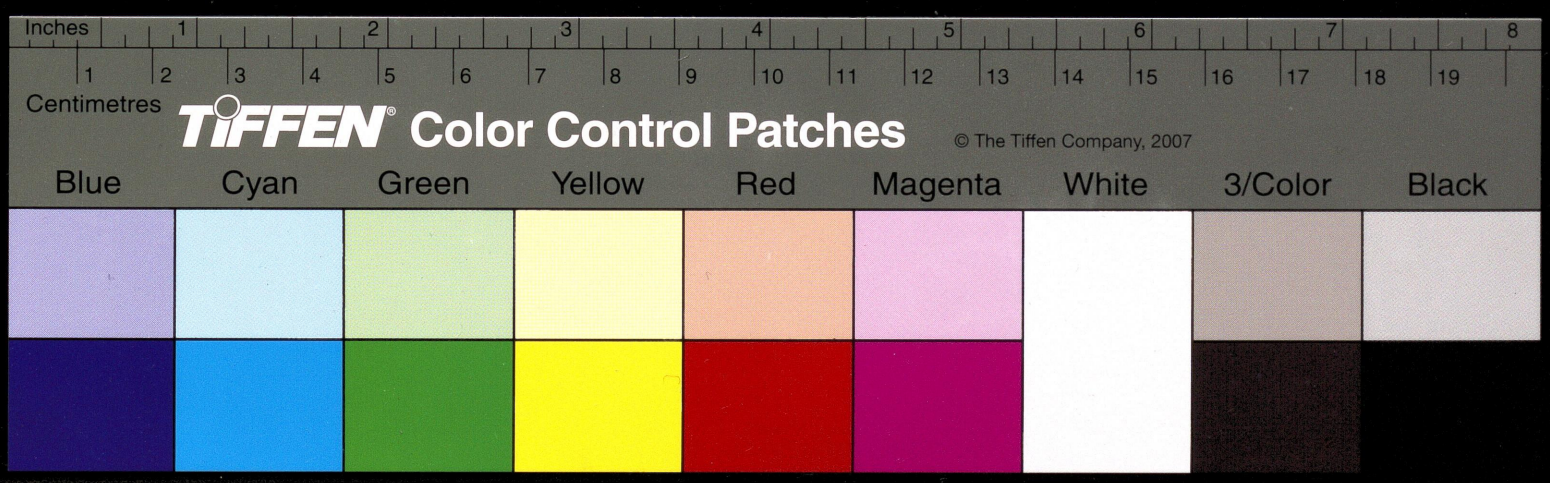
90. A. I. Akhiezer and V. B. Berestetskii, Quantum Electrodynamics, trans. from 2nd Russian edition by G. M. Volkoff (Interscience, New York, 1965). Articles 17, 18, and 56-61 describe the quantization of the electromagnetic, Dirac, and DKP interacting fields, respectively.

91. F. J. Belinfante, Physica 6, 870 (1939).

92. W. Pauli and F. J. Belinfante, Physica 7, 177 (1940).

93. W. Pauli, Phys. Rev. 58, 716 (1940).

94. M. Fiertz, Helv. Phys. Acta 12, 3 (1939).



95. R. F. Streater and A. S. Wightman, PCT, Spin and Statistics, and All That (Benjamin, New York, 1964). See p. 146 and the original papers given in the bibliography on pp. 175-177.

96. B. S. Madhavarao (Madhava Rao), V. R. Thiruvenkatachar, and K. Venkatachaliengar, Proc. R. Soc. A187, 385 (1946).

97. W. Pauli and V. Weisskopf, Helv. Phys. Acta 7, 709 (1934).

98. S. N. Gupta, Proc. Phys. Soc. 63, 681 (1950).

99. S. N. Gupta, Proc. Phys. Soc. 64, 850 (1951).

100. K. Bleuler, Helv. Phys. Acta 23, 567 (1950).

101. K. Bleuler and W. Heitler, Prog. Theor. Phys. (Kyoto) 5, 600 (1950).

102. G. Källen, Arkiv för Fysik 2, 371 (1950).

103. C. N. Yang and D. Feldman, Phys. Rev. 79, 972 (1950).

104. In the Appendix to Katayama's Ref. 83, the factor $(\Delta\text{-}m^2)^{q/2}$ of our Eq. (5.3) was inadvertently left out of his analogous equation.

105. S. Sakata and M. Taketani, Sci. Pap. Inst. Phys. Chem. Res. Tokyo 38, 1 (1940).

106. M. Taketani and S. Sakata, Proc. Phys. Math. Soc. Jap. 22, 757 (1940).

107. Refs. 90 and 91 have been reprinted in Prog. Theor. Phys. Suppl. No. 1, 84, 98 (1955).

108. W. Heitler, Proc. R. Irish Acad. 49, 1 (1943).

109. E. Fischbach, M. M. Nieto, and C. K. Scott, Prog. Theor. Phys. (Kyoto) 48, 574 (1972).

110. In Appendix D of Ref. 88, the special case formula (D.23) for

$\frac{1}{2}\ \varepsilon(x_0 - x'_0),_{\mu}\partial_{\nu}\partial_{\lambda}\partial_{\rho}$ should have a $+ n_{\rho}n_{\mu}n_{\nu}n_{\lambda}(\Box - m^2 + 4(\partial\cdot n)^2)$ instead of $- n_{\rho}n_{\mu}n_{\nu}n_{\lambda}(\Box - m^2)$ for its last term.

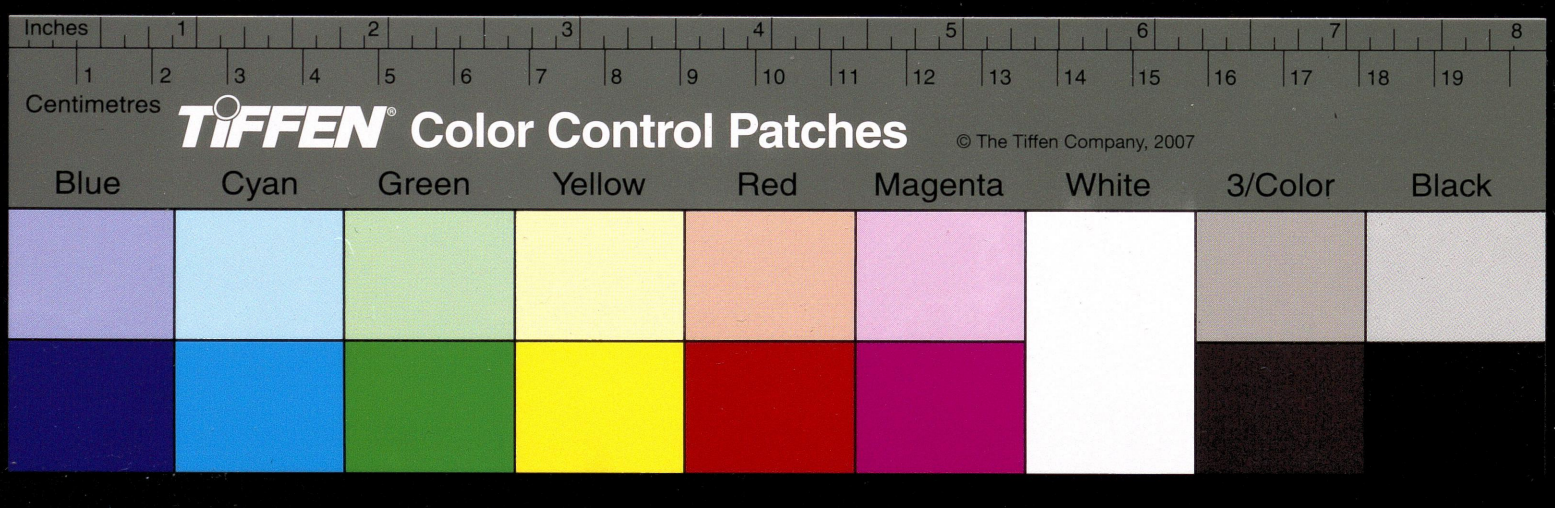
111. M. H. L. Pryce, Proc. R. Soc. A150, 166 (1935) and R. Becker, Nachr. Akad. Wiss. Göttingen, Math. Physik Kl. II, No. 1, 39 (1945), in little known papers, preceded some of the results of Foldy and Wouthuysen in Ref. 112.

112. L. L. Foldy and S. A. Woulhuysen, Phys. Rev. 78, 29 (1950).

113. S. Tani, Prog. Theor. Phys. (Kyoto) 6, 267 (1951).

114. K. M. Case, Phys. Rev. 95, 1323 (1954).

115. C. G. Bollini and J. J. Gianbiagi, Nuovo Cimento 21, 107 (1961).



116. F. Strocchi, Nuovo Cimento 26, 955 (1962).

117. D. L. Weaver, C. L. Hammer and R. H. Good, Jr., Phys. Rev. B241 (1964).

118. S. A. Williams, J. P. Draayer, and T. A. Weber, Phys. Rev. 152, 1207 (1966).

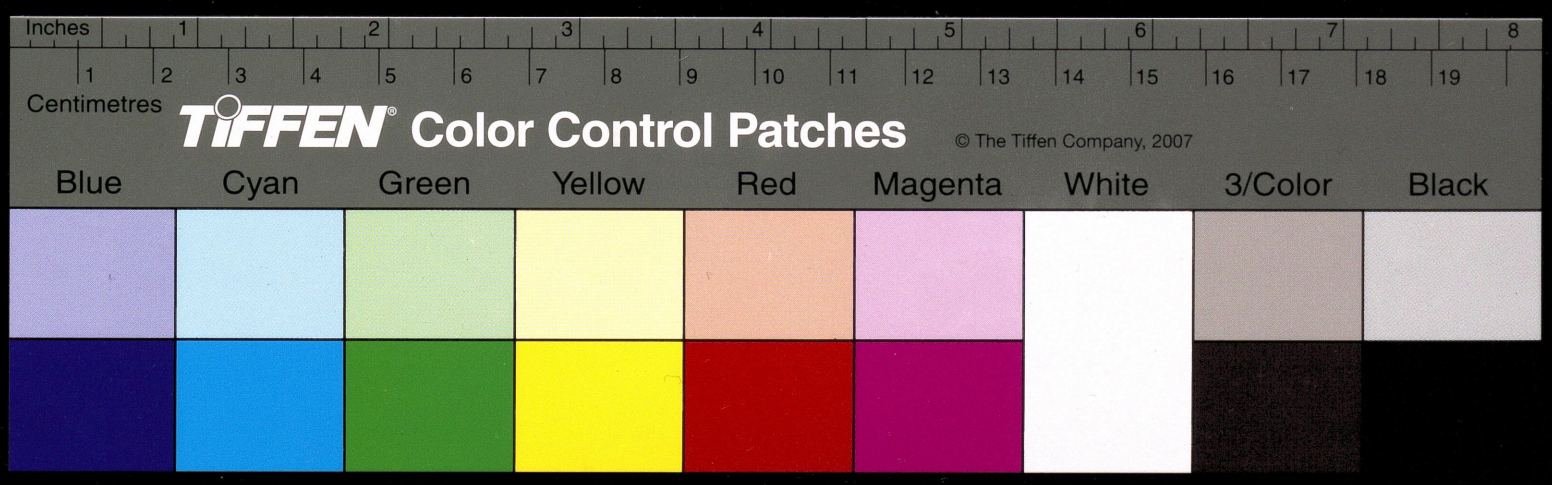
119. P. M. Matthews, Phys. Rev. 143, 985 (1966).

120. D. L. Weaver, Nuovo Cimento 53A, 665 (1968).

121. G. Alagar Ramanujam, Nuovo Cimento 20A, 27 (1974).

122. A. K. Nagpal and R. S. Kaushal, Lett. Nuovo Cimento 9, 391 (1974).

123. R. F. Guertin, Ann. Phys. (NY), (to be published).



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TITLE: Bhabha first-order wave equations. IV. Causality with minimal electromagnetic coupling.

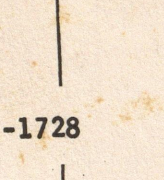
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