ON A CONJECTURE OF RAMANUJAN AND SOME SIMPLE DEDUCTIONS

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1. *Introduction*

The conjecture referred to is that the equation

$$2^{n+2}-7=x^2 \tag{1}$$

has just five solutions with $n=1, 2, 3, 5, 12$, and the corresponding values of $x=1, 3, 5, 11, 181$. Proofs of this conjecture have been given by several authors[1], but all these proofs are based on the properties of the field $F(\sqrt{-7})$, and are quite complicated. I had an elementary proof of (1) many years back, based only on the properties of the convergents of the continued fraction for $\sqrt{2}$, and have given it here for the first time. The proof is contained in §3, and before indicating this, I have pointed out in §2 three simple deductions from (1) which include the results of M. Satyanarayana[2], and U. V. Satyanarayana[3] about triangular numbers 13/

2. *Three simple deductions*

M. Satyanarayana has proved that

(2) (i) no Fermat number, i.e. 2^{2^n} $(n\geq 1)$, is triangular; +1/

(3) (ii) no Mersenne number $M_n=2^n-1$ can be triangular for n odd and >1.

U. V. Satyanarayana has proved that

(4) (iii) there are infinitely many even n for which M_n is not triangular.

These results can be deduced by using (1). Since x is obviously odd, we put $x=2y+1$, and easily bring (1) to the form

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[1] See Math. Rev. July 1963, p. 15, No. 74.

[2] M. Satyanarayana (3).

[3] U. V. Satyanarayana (4).

$$(5) \qquad 1+\Delta_y=2^{n-1},$$

where Δ_y is the triangular number $\frac{1}{2}y(y+1)$. Putting $m=n-1$ in (5), we have from Ramanujan's conjecture that $M_m=\Delta_y$ is true only for $m=0$, makes/ 1, 2, 4, 12 which readily proves (3), and ~~also~~ (4), ~~further making it more exact~~ plausible/ by showing that there are *only four* even numbers $m=0, 2, 4, 12$ for which M_m is triangular.

As regards (2), we have from (5), $2^{m+1}=\Delta_y+2=\Delta_z$, say, and $\Delta_z-\Delta_y=2$ is equivalent to $(z-y)(z+y+1)=4$, which obviously holds only for $y=1$, $z=2$. Hence 2^m+1 can only be equal to $\Delta_2=3$, and if $m=2^l$, this means that $l=0$, so that the Fermat number 2^{2^n} cannot be triangular +1/ for $n\geqslant 1$, which is (2).

3. *Proof of Ramanujan's conjecture*

The following elementary proof is based on a remark of A. Boutin quoted by Dickson[1] that the relation

$$(6) \qquad 1+\Delta_y=x^2$$

can be reduced to

$$(7) \qquad p^2-2q^2=-1$$

by setting

$$(8) \qquad 2x=3q\mp p;\ y=\tfrac{1}{2}(k-1): k=3p\mp 2q,$$

a remark which can be easily verified.

We notice, first of all, that (6) comes under (5) not for all the five values of the conjecture, but only for four of them, viz. $n=1, 3, 5, 13$, with the corresponding $x=1, 2, 2^2, 2^6$. Hence, if we prove that (6) is true *only* for these four values of x, with the corresponding n in (5) being odd, and that the *only* even value of n for which (5) is true, is given by $n=2$, then the proof of the conjecture is complete.

Solutions of (7) are given by expressing $\sqrt{2}$ as a continued fraction, viz :

$$(9) \qquad \sqrt{(2)}=1+\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\dots$$

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[1] Dickson (2).

and since the cycle consists of a single element, the solutions of (7) are the odd convergents (p_{2n+1}, q_{2n+1}), $(n=0, 1, 2, \dots)$ of the continued fraction.[1] Also the following recurrence relations hold between the convergents :

(10) $$p_n = q_n + q_{n-1}; \quad 2q_n = p_n + p_{n-1},$$

from which follow

(11) $$p_n - 2p_{n-1} - p_{n-2} = 0; \quad q_n - 2q_{n-1} - 2q_{n-2} = 0$$

so that if α and β are the roots of the quadratic equation $x^2 - 2x - 1 = 0$, i.e.

(12) $$\alpha = 1 + \sqrt{2}, \ \beta = 1 - \sqrt{2},$$

p_n and q_n are given by

$$p_n = A\alpha^n + B\beta^n; \quad q_n = C\alpha^n + D\beta^n,$$

with the constants A, B, C, D being determined from $p_1 = 1 = A\alpha + B\beta$, $p_2 = 3 = A\alpha^2 + B\beta^2$, and $q_1 = 1 = C\alpha + D\beta$, $q_2 = 2 = C\alpha^2 + D\beta^2$, leading by use of (12) to

(13) $$\begin{cases} q_n = (\alpha^n - \beta^n)/(\alpha - \beta), \\ p_n = \frac{1}{2}(\alpha^n + \beta^n). \end{cases}$$

Since the solutions of (7) are the odd convergents, the values of x in (8) can be written as

(14) $$x = \tfrac{1}{2}(3q_{2n+1} - p_{2n+1}) \text{ or } x = \tfrac{1}{2}(3q_{2n+1} + p_{2n+1}).$$

From (13) we have

(15) $$q_{2n+1} = \frac{\alpha^{2n+1} - \beta^{2n+1}}{\alpha - \beta}; \quad p_{2n+1} = \tfrac{1}{2}(\alpha^{2n+1} + \beta^{2n+1}).$$

Putting in the values of α and β from (12) in (15), and simplifying, we can find expressions for q_{2n+1} and p_{2n+1}, and using these, we can write the two cases in (14) as follows :—

Case A.

(16) $$\begin{aligned} x = {} & (3n+1) + 2n(n-1)(2n+1) + \{3 \,.\, (2n+1)C_5 - (2n+1)C_4\}2 \\ & \{3 \,.\, (2n+1)C_7 - (2n+1)C_6\}2^2 + \dots \\ & + \{3 \,.\, (2n+1)C_{2n+1} - (2n+1)C_{2n}\}2^{n-1}. \end{aligned}$$

(i) If n be *even*, then x=odd, and hence $x \neq 2^m$ except in the trivial case of $n=0$, when $x = 2^0 = 1$.

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[1] Barnard and Child (1), 355.

(ii) If n be *odd*, then x=even, and when $n=1$, $x=2^2$, since the second term in (16) is equal to 0 when $n=1$.

For odd $n>1$, each term in (16) is of the form 2^p (p odd), and hence $x=2^l$ (l odd), where l is the least of the p's, and hence $x \neq 2^m$.

Case B.

$$x=(3n+2)+2n^2(2n+1)+\{3\,.\,(2n+1)C_5+(2n+1)C_4\}\,.\,2+\ldots \\ +\{3\,.\,(2n+1)C_{2n+1}+(2n+1)C_{2n}\}2^{n-1}. \tag{17}$$

(i) If n be odd, x=odd, and hence $x \neq 2^m$.

(ii) If n be even, we have

$$x=2, \text{ for } n=0, \text{ and}$$

$$x=8+40+16=64=2^6, \text{ for } n=2.$$

For even $n>2$, it can be shown by an argument similar to that used in Case A for odd $n>1$ that $x \neq 2^m$.

We have thus the *only* four values $x=1, 2, 2^2, 2^6$ of (8), giving the four odd values of n for which (5) holds. To complete the proof of Ramanujan's conjecture, we have to show that the only even value of n for which (5) holds is given by $n=2$. Hence writing (5) in the form $1+\triangle_y=2^{2m-1}$, and solving the quadratic equation in y arising therefrom, we have

$$y=\tfrac{1}{2}\{-1\pm\sqrt{(2^{2m+2}-7)}\}. \tag{18}$$

This requires that $2^{2m+2}-7=\square=v^2$, say, i.e.

$$2^{2m+2}-v^2=7, \text{ or } (2^{m+1}+v)(2^{m+1}-v)=7,$$

leading to the only possibility of $2^{m+1}+v=7$, and $2^{m+1}-v=1$. Adding we get, $2^{m+2}=8$ i.e. $m=1$ or $n=2$, which conpletes the proof of the conjecture.

I deem it a privilege to contribute this small piece of work to the Memorial Volume being published in honour of the late Prof. B. N. Prasad.

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4. U. V. Satyanarayana, "A note on Mersenne's numbers", *Math. Student*, **28** (1960), 7.

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D. p.381 – Chap. 16

(1) $(2^p+1)/3$ has no factors other than $2kp+1$ if p is an odd prime

(2) a^4+4b^4 has the factors $a^2 \pm 2ab + 2b^2$

(3) If m is prime & a, b rel. prime, a factor of $a^m - b^n$ not a divisor of $a-b$ is of the form $kn+1$. If $p = kn+1$ is a prime & ~~k=mn~~ and $a = f^n \pm p\alpha$, then $a^k - 1$ is divisible by p. If $af^n - bg^n$ is divisible by a prime $p = mn+1$, while f & g are ~~both~~ not both divisible by p, then $a^m - b^m$ is divisible by p; the converse is true if m & n are rel. prime

(4) For q an odd prime, any prime divisor of $a^q - 1$, not a divisor of $a-1$, is of the form $2nq+1$. If $a^m - 1$ is divisible by the prime $p = mn+1$, we can find integers x, y not divisible by p such that $A = ax^n - y^n$ is divisible by p

(5) If a^2+b^2 is divisible by p^2+q^2, (p^2+q^2 = prime of form $4n+1$) there exist integers r and s such that $a = pr+qs$, $b = ps-qr$. With $b = \pm 1$, we take the convergent r/s preceding p/q in the continued fraction for p/q; thus $ps - qr = \pm 1$ and hence all integers a for which a^2+1 is divisible by a given prime $4n+1$ are of the form $a = \pm(pr+qs)$

(6) Every prime divisor of a^n+1 is either of the form $2nx+1$ or divides $a^\omega + 1$, where ω is the quotient of n by an odd factor. Every prime divisor of $a^n - 1$ is either of the form $nx+1$ or divides $a^\omega - 1$, where ω is a factor n

(7) $2^{4i+2}+1$ has the two factors $2^{2i+1} \pm 2^{i+1} + 1$

(8) If p is an odd prime every odd prime factor of $a^p - 1$ is either of the form $2pz+1$ or is a factor of $a-1$, and moreover is a divisor of $x^2 - ay^2$. Hence for $a = 2$, it is of the form $2pz+1$ and also of one of the forms $8m \pm 1$. Every odd prime factor of $a^{2n+1}+1$ is either of the forms $2(2n+1)z+1$ or a divisor of ~~the form~~ $a-1$.

(9) If $F_a(b) = (x^{ab}-1)/(x^a-1)$ & putting $a = \alpha b + b_1$, $b = \alpha_1 b_1 + b_2$, $b_1 = \alpha_2 b_2 + b_3$, … & if a and b are rel. prime, a formula for $\frac{1-x^{ab}}{(x^a-1)(x^b-1)}$ … (too huge)

(10) $x^{p-1} + \dots + x + 1$ has no prime divisor other than the prime p and numbers of the form $kp+1$.

(11) If $n = a \mp 1$ is odd, $a^n \mp 1$ is divisible by n^2, but not by n^3

(12) $6^{40}+1)$ / [illegible]

(13) Proper divisors of $2^{4n}+1$ are of the form $16nq+1$, those of $a^{2abn}+b^{2abn}$ are of the form $8abnq+1$; for n odd those of $a^{abn}+b^{abn}$ are of the form $4abnq+1$ if $ab=4h+1$; those of $a^{abn}-b^{abn}$ are of the form $4abnq+1$ if $ab=4h+3$.

(14) $x^4+2(q-r)x^2+q^2$ for $x^2=(2r)^{2K+1}$ has the rational factors $(2r)^{2K+1}\pm(2r)^{K+1}+q$. The case of $r=q=1$ gives (7). $3^{6K+3}+1$ has the factors $3^{2K+1}+1$, $3^{2K+1}\pm 3^{K+1}+1$

(15) $2^{4n}+2^{2n}+1=\Pi\left(2^{2n}\pm 2^{n}+1\right)$

(16) If $F_n(A,B)=\dfrac{A^n-B^n}{A-B}$ (A, B, rel. prime), F_n has besides n, no prime factors except those of the form Hn^t+1 when A and B are exact n^{t-1}th powers of integers. Second for $T=n^i m^h$

($T=n^t$) n a prime

where n and m are distinct primes, the integral quotient of $F_n(u^m,v^m)$ by $F_n(u,v)$ has only prime factors of the form $Hn^i m^h+1$ if u & v are powers of rel. prime integers with the exponent $m^{h-1}n^{t-1}$

(17) If n and $(2n+1)$ are primes, $2n+1$ is a factor of 2^n-1 or 2^n+1 according as $n\equiv 3$ or $n\equiv 1$ (mod 4). If n and $4n+1$ are primes, then $4n+1$ is a factor of $2^{2n}+1$. If n and $8n+1$ ~~and primes~~ $=A^2+16B^2$ are primes, then $8n+1$ is a factor of $2^{2n}+1$ if B is odd, of $2^{2n}\pm1$ if B is even. ~~when~~ $6n+1=4L^2+3M^2$, $12n+1=L^2+12M^2$, $24n+1=L^2+48M^2$ ~~[illegible]~~ are prime factors of $2^{Kn}\pm1$ for certain K's.

(18) If p is an odd prime dividing $a-1$ & p^r divides a^p-1, then p^{r-1} divides $a-1$

(19) If $P=1+p+\cdots+p^{r-1}$ is divisible by q, and p, r primes, either r divides $q-1$ or $r=q$ divides $p-1$. If $P=q^j$ and p, r, j primes, j is a divisor of $q-r$.

(20) If $m=2^{n-1}$, $1+2^{(2m+1)n}=(1+2^n)^2\left\{1-2m+(2m-1)2^n-(2m-3)2^{2n}+\cdots-2\cdot 2^{(2m-2)n}+2^{(2m-1)n}\right\}$

(21) n^n-1 is divisible by $4n+1$ if $4n+1$ is prime (p. 387).