13/3

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Agagpriya
Ranabir Das

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22/3/
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24/3/2K.
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Chirag Kalelkar 27/3/2K.
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Chirag. Kalelkar 29/3/2K.
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31st March
Arpan Chakraborty
Ranabir Das.
Chirag Kalelkar
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Agagpriya
Udit Raha.

Chirag Kalelkar 3rd Apr
Arpan Chakraborty
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7/4/2K

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Chirag Kalelkar 10.04.2000
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Ranakir Das

$$\vec{j}(\vec{r}) = \vec{j}_T(\vec{r}) + j_L(\vec{r})$$

$$= \vec{\nabla} \times \vec{A} + \left(-\nabla\phi\right)$$

$$\vec{\nabla} \cdot \vec{j}(\vec{r}) = \vec{\nabla} \cdot \left(\nabla \times \vec{A}\right) + \nabla \cdot (-\nabla\phi)$$

$$= \vec{\nabla} \cdot j_L(\vec{r})$$

$$= -\nabla^2 \phi$$

$$\phi = \frac{1}{4\pi} \int \frac{d\vec{r}' \; \nabla' \cdot \vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$\vec{j}_L(\vec{r}) = -\frac{1}{4\pi} \nabla \int \frac{d\vec{r}' \; \vec{\nabla} \cdot \vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$\nabla \times \vec{j} = \nabla \times (\nabla \times \vec{A}) + \nabla \times \cancel{(-\nabla\phi)}^{\,0}$$

$$= \nabla \cancel{(\nabla \cdot \vec{A})}^{\,0} - \nabla^2 \vec{A}$$

$$= -\nabla^2 \vec{A}$$

$$\vec{A} = \frac{1}{4\pi} \int \frac{\nabla' \times \vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\vec{r}'$$

$$\vec{j}_T = \nabla \times \vec{A} = \frac{1}{4\pi} \vec{\nabla} \times \int \frac{\vec{\nabla} \times \vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\vec{r}'$$

Since $\vec{\nabla} \cdot \vec{j}_T = 0$, only dipole term. $\vec{A} \sim \frac{1}{r^2}$

$$\int_{\text{all space}} \vec{J}_T(\vec{x}')\, d\tau' = \int \vec{\nabla} \times \vec{A}(\vec{x}')\, d\tau'$$

$$= \oint_{\text{sat } \infty} d\vec{S} \times \vec{A}$$

$$\neq 0$$

$e_i' = a_{ij} e_j$

$a a^T = a^T a = 1$

$\det a = \pm 1$

$V_i' = a_{ij} V_j$

$T_{ij}' = a_{ik} a_{jl} V_{kl}$

Symmetry, anti-symmetry

Form invariant by contraction T_{ii}

Lower by contraction

Raise by differentiation

$x_k = (a^T)_{ki} x_i' = a_{ik} x_i'$

$\frac{\partial}{\partial x_i} x_i' = a_{ij} \frac{\partial x_i}{\partial x_k}$ $\quad V_j' = a_{jl} V_l$

$T_{ij} = \frac{\partial}{\partial x_i} V_j$

$T_{ij}' = \frac{\partial}{\partial x_i'} V_j' = \frac{\partial}{\partial x_k} V_l' \frac{\partial x_k}{\partial x_i'}$

$= \frac{\partial}{\partial x_k} a_{jl} V_l a_{ik}$

$= a_{ik} a_{jl} \frac{\partial}{\partial x_k} V_l$

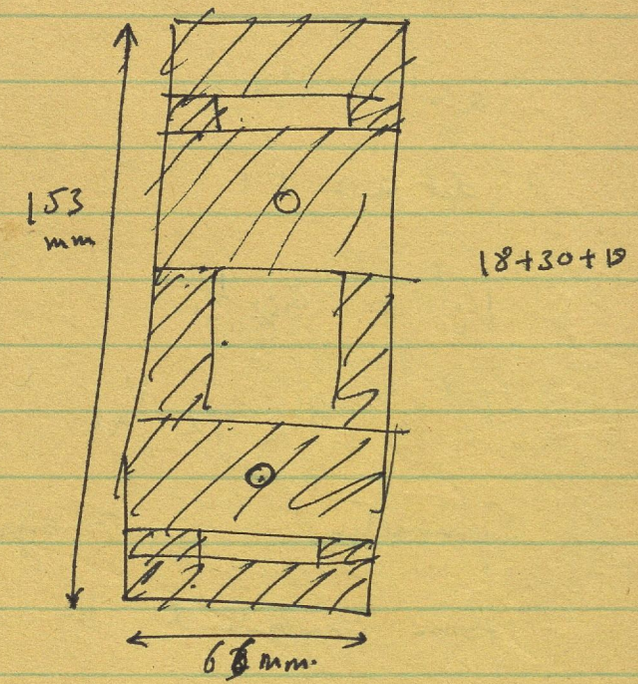
δ_{ij} is a tensor (symmetric)

ϵ_{ijk} is a pseudotensor

eg. scalar temp.
vector coordinate
tensor $x_i x_j$
pseudoscalar $\vec{A} \cdot \vec{B} \times \vec{C}$

$$C_i = \epsilon_{ijk} A_j B_k$$

pseudovector



total area.
$(153 \times 66)\,mm^2 = 10098\,mm^2$

58 mm. minor. Hexl.
42 mm min.

thickness. (0.7) mm.

$(42 \times 58)\,mm^2$
$+2(10 \times 8)\,mm^2$ } $= 2621$
$+2(\pi \times 4)\,mm^2$

total area = 7500

$7500 \times 0.7 = 5233\,mm^3$

$\frac{40.9\ gm.}{5.233\ cc^*} = 7.815$

Multipole expansion

$$V(\vec{r}) = \sum \frac{(-1)^\ell}{\ell!} \int d\vec{r}' \rho(\vec{r}') x_{i_1}' x_{i_2}' \cdots x_{i_\ell}' \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_\ell}} \frac{1}{r}$$

$$\phi_{i_1 \cdots i_\ell} = \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_\ell}} \frac{1}{r}$$

= symmetric tensor of rank ℓ

$$\nabla^2 \left(\frac{1}{r}\right) = 0 \qquad r \neq 0$$

$$\overline{Q}_{i_1 \cdots i_\ell} = \int \rho(\vec{r}) \, d\vec{r}' \, x_{i_1}' x_{i_2}' \cdots x_{i_\ell}'$$

symmetric ℓ^{th} rank tensor

$$\overline{Q}^{y}_{i_1 \cdots i_\ell} = \int \rho \, d\vec{y} \, y_{i_1} y_{i_2} \cdots y_{i_\ell}$$

$$= \int \rho \, d\vec{y}' \, a_{i_1 j_1} x_{j_1}' \cdots a_{i_\ell j_\ell} x_{j_\ell}' = a_{i_1 j_1} \cdots a_{i_\ell j_\ell} Q_{j_1 \cdots j_\ell}$$

$$V = \sum_{\ell=0}^{\infty} \frac{Q^{(\ell)}}{\ell!}$$

Make trace 0 by subtracting $\delta_{i_n i_m} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_n}} \cdots \frac{\partial}{\partial x_{i_m}} \cdots \frac{\partial}{\partial x_{i_\ell}} \frac{1}{r}$

$$Q_{i_1 i_2} = \overline{Q}_{i_1 i_2} - \frac{\delta_{i_1 i_2}}{3} Q_{kk}$$

n_1 1s n_2 2s n_3 3s $= \ell - n_1 - n_2$

$$\sum_{n_1=0}^{\ell} \sum_{n_2=0}^{\ell-n_1} (n_3 = \ell - n_1 - n_2)$$

$$\sum \ell - n_1 + 1 = \left(\ell+1\right)\left(\frac{\ell}{2}+1\right)$$

$\underline{(2\ell+1)}$

$$V = \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}}{\ell!} \rho^{\ell}_{i_1 \ldots i_\ell} \phi^{\ell}_{i_1 \ldots i_\ell} \left(\frac{\vec{r}}{R}\right)$$

$\underline{l=5}$ $\quad n_1=0$

$n_1=0$ $\quad 5+4+3+2+1+0$

$n_1=1$ $\quad 4+3+\cdots 0$

$n_1=2$ $\quad 3+2\ldots$

$n_1=4$ $\quad 2+1\ldots$

$n_1=5$

$$\frac{(l+1)(l+1) - (l+1)(l)}{2}$$

$$= (l+1)\left[l+1-\frac{l}{2}\right] = (l+1)\left[\frac{l}{2}+1\right)$$

$$f_{1on2} = \frac{q_1 q_2 (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} \qquad E = -\nabla V$$

$$V = \int \frac{d\vec{r}' \rho(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$\nabla \times E = 0 \qquad \nabla \cdot E = 4\pi\rho$$

$$V(\vec{r}) = \sum_{l=0}^{\infty} \int d\vec{r}' \rho(\vec{r}') x'_{i_1} x'_{i_2} \cdots x'_{i_l} \frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_l}} \frac{1}{r} \frac{(-1)^l}{l!}$$

$$\frac{(l-n_1)(l-n_1+1)}{2}$$

$$(l-n_1+1)\,l - (l-n_1+1)\,n_1 - (l-n_1+1)\left(\frac{l-n_1}{2}\right)$$

$$(l-n_1+1)\left[l-n_1-\frac{l-n_1}{2}\right]$$

$$= (l-n_1+1)\left(\frac{l-n_1}{2}\right)$$

$$\sum_{n_1=0}^{l} \frac{l^2-n_1 l - n_1 l + n_1^2 + l - n_1}{2}$$

$$= \cancel{n_1 l^2}$$

$$\cancel{l^3 - 2l\frac{l(l+1)}{2} + l^2}$$

$$l^2 + l - n_1(2l+1) + n_1^2$$

$$l(l+1)(l+1) - \frac{l(l+1)}{2}(2l+1) + \frac{l(l+1)(2l+1)}{6}$$

$$l(l+1)\left[l+1-\frac{2l+1}{2}+\frac{2l+1}{6}\right]$$

$$\cancel{\frac{6l+6-6l-3}{6}} + 2l+1$$

$$2l+4$$

$0+1+2\ldots l-n_1$

$$\frac{(l-n_1)(l-n_1+1)}{2}$$

$1+4+1$

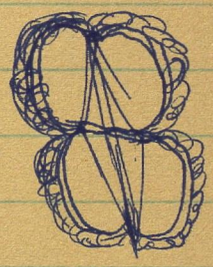
$$\sum_{n=0}^{l} n^2$$

$$\frac{d}{dn}\sum_{n=0}^{l} n^2 = \sum_{n=0}^{l} 2n$$

$$\frac{14}{2}$$

$$\frac{3\times 4\times 7}{6}$$



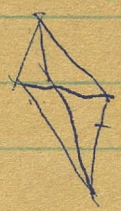


$$\nabla^2 \frac{F}{r^{l+1}} = 0$$

$$\frac{1}{r^2}\left[\frac{l(l+1)F}{r^{l+1}} + \nabla^2(\theta,\phi)\frac{F}{r^{l+1}}\right] = 0$$

$$\Rightarrow\ l(l+1)F + \nabla^2(\theta,\phi)F = 0$$

$$\nabla^2(r^l F)$$



$$\frac{1}{r^2}\left[l(l+1)r^l F + \nabla^2(\theta,\phi)\, r^l F\right]$$

$$\frac{r^l}{r^2}\left[\underbrace{l(l+1)F + \nabla^2(\theta,\phi)F}_{=0}\right] = ?$$

$$\phi_{i_1 \ldots i_l} = \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_l}} \frac{1}{r} = \frac{F_{i_1 \ldots i_l}(\theta,\phi)}{r^{l+1}}$$

$$r^l F(\theta,\phi) = \frac{F(\theta,\phi)}{r^{l+1}}$$

$(2l+1)$ polynomials $\quad \psi^l_{i_1 i_2 \ldots i_l} = r^l F^l_{i_1 i_2 \ldots i_l}(\theta,\phi)$

$$\int d\Omega\, F^l(\theta,\phi)\, F^{l'}(\theta,\phi) = 0 \quad \forall\ l \neq l'$$

Order l $\quad \sum_{m_1+m_2+m_3=l} (x+iy)^{m_1} (x-iy)^{m_2} z^{m_3} a(m_1, m_2, m_3)$

$m_1 - m_2 = m$ $\quad \sum (x+iy)^m (x^2+y^2)^{m_2} z^{m_3} a$

$$f_i = \int \rho(\vec{r})\, d\vec{r}\; E_i(\vec{r})$$

$$= \frac{1}{4\pi} \int \nabla \cdot E\; E_i(\vec{r})\; d\vec{r}$$

$$= \frac{1}{4\pi} \int \left(\frac{\partial}{\partial x_j} E_j \right) E_i\, d\vec{r}$$

$$\frac{\partial}{\partial x_j}(E_j E_i) = \frac{\partial E_j}{\partial x_j} E_i + E_j \frac{\partial E_i}{\partial x_j}$$

$$(\nabla \times \vec{E})_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} E_k$$

$$= \frac{1}{2} \epsilon_{ijk} \left(\frac{\partial}{\partial x_j} E_k - \frac{\partial E_j}{\partial x_k} \right)$$

Since $\nabla \times E = 0$

$$\frac{\partial E_k}{\partial x_j} = \frac{\partial E_j}{\partial x_k}$$

$$F_i = \frac{1}{4\pi} \int \frac{\partial}{\partial x_j}(E_j E_i) - \frac{1}{2} \frac{\partial E_j^2}{\partial x_i}$$

$$= \frac{1}{4\pi} \int d\vec{r}\, \frac{\partial}{\partial x_j} \left\{ E_j E_i - \delta_{ji} \frac{E_k^2}{2} \right\}$$

F_i

$$\delta W_i = -\int_{\infty}^{\vec{r}_i} d\vec{l} \cdot \vec{E}\, \delta q_i = \left[V(\vec{r}_i) - V(\infty)\right]^{0} \delta q_i$$

$$\delta W = \sum_i \delta W_i = \sum V(\vec{r}_i)\, \delta q_i$$

$$= \int V(\vec{r})\, \delta\rho(\vec{r})\, d\vec{r}$$

to first order in changing from $\rho(\vec{r})$ to $\rho + \delta\rho$

and $\vec{E}$ to $\vec{E} + \delta\vec{E}$ where $\nabla \cdot \delta E = 4\pi \delta\rho$

$$\delta W = \frac{1}{4\pi} \int \nabla \cdot \delta E\, \phi\, d\vec{r}$$

$$\int \nabla \cdot (\phi \delta \vec{E})\, dr - \int \nabla\phi \cdot \delta E$$

$$= \frac{1}{4\pi} \int E \cdot \delta E$$

$$\int \nabla \cdot \delta E\, d\vec{r}$$

$$=$$

$$m = m_1 - m_2$$

$$m_1 = m + m_2$$

$$m_1 + m_2 + m_3 = l \Rightarrow m + m_2 + m_2 + m_3 = l$$

$$m + 2m_2 + m_3 = l$$

$$m_2 = \frac{l - m_3 - m}{2}$$

$$\left(r^2 - z^2\right)^{\frac{l - m_3 - m}{2}} z^{m_3}$$

$$r^2 \sim m_2$$

$$\left(r^2 - z^2\right)^{m_2} z^{m_3}$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum \frac{r'^{\ell}}{r^{\ell+1}} P_\ell(w) \qquad w = \hat{r}\cdot\hat{r}'$$

$$= \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

$$\int_{-1}^{1} P_\ell^2(w)\, dw = \frac{2}{2\ell+1}$$

$$P_\ell(\hat{r}\cdot\hat{r}') = \frac{4\pi}{2\ell+1} \sum_m Y_{\ell m}(\Omega)\, Y_{\ell m}^*(\Omega')$$

$$V(\vec{r}) = \int \frac{\rho(\vec{r}')\, d\vec{r}'}{|\vec{r}-\vec{r}'|} = \sum_{\ell m} \frac{4\pi}{2\ell+1} \frac{Y_{\ell m}(\Omega)}{r^{\ell+1}} \int Y_{\ell m}^*(\Omega')\, \rho(\vec{r}')\, dr'\, r'^{\ell}$$

$$W = \int \rho(\vec{x})\, \phi(x)$$

$$\phi = \phi(0) + \vec{r}\cdot\nabla\phi\big|_0 + \frac{1}{2} x_i x_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\phi\big|_0 + \ldots$$

$$= \not{\phi}$$

15/1/97

1. $$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} + x_i' \frac{\partial}{\partial x_i'} \frac{1}{|\vec{r}-\vec{r}'|}\bigg|_{r'=0} + \frac{x_i' x_j'}{2!} \frac{\partial}{\partial x_i'} \frac{\partial}{\partial x_j'} \frac{1}{|\vec{r}-\vec{r}'|}\bigg|_{r'=0} + \ldots$$

$$= \frac{1}{r} + x_i' \frac{\partial}{\partial x_i} \frac{1}{r} + \frac{x_i' x_j'}{2!} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \frac{1}{r} + \ldots \frac{(-1)}{\ell!} x_{i_1}' \ldots x_{i_\ell}' \frac{\partial}{\partial x_{i_1}} \ldots \frac{\partial}{\partial x_{i_\ell}} \frac{1}{r}$$

2. $$\Phi(\vec{a}) = \Phi(0) + \vec{a} \cdot \nabla \Phi\big|_0 + \frac{x_i x_j}{2} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \Phi\big|_0 + \ldots$$

$$W = \int \rho(\vec{x}) \Phi(\vec{x}) \, d\vec{x}$$

$$= \int \rho(\vec{x}) \Phi(0) \, d\vec{x} + \int \vec{x} \rho(\vec{x}) \, d\vec{x} \cdot \nabla \Phi\big|_0$$

$$+ \int \frac{x_i x_j}{2} \rho(\vec{x}) \, d\vec{x} \, \frac{\partial E_j}{\partial x_i}\bigg|_0$$

$$= Q\Phi(0) - \vec{p} \cdot \vec{E}(0) -$$

$\nabla \cdot \vec{E} = 0 \quad \therefore -\frac{1}{2} x_i x_j \frac{\partial}{\partial x_i} E_j + \frac{1}{6} r^2 \delta_{ij} \frac{\partial E_j}{\partial x_i}$

$$- \frac{1}{6} \left[3 x_i x_i - r^2 \delta_{ij} \right] \frac{\partial E_j}{\partial x_i}$$

$$W = Q\Phi(0) - \vec{p} \cdot \vec{E}\big|_0 - \frac{1}{6} Q_{ij} \frac{\partial E_j}{\partial x_i}(0) + \ldots$$

3. $Y_{\ell m}$ orthogonal to each other.

$Y_{20} = az^2 + br^2$ orthogonal to Y_{00}

$Y_{30} = z(\delta z^2 + \epsilon r^2)$ orth to Y_{10}

$Y_{40} = \alpha z^4 + \beta r^2 z^2 + \gamma r^4$ orth to Y_{00} and Y_{20}

4. $$\delta W = \int \delta\rho(\vec{r})\,\phi(\vec{r})\,d\vec{r}$$

$$= \int \delta\rho(\vec{r})\,\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}\,d\vec{r}'$$

$$= \frac{1}{2}\int \frac{\delta\rho(\vec{r})\,\rho(\vec{r}') + \delta\rho(\vec{r}')\,\rho(\vec{r})}{|\vec{r}-\vec{r}'|}\,d\vec{r}\,d\vec{r}'$$

$$= \frac{1}{2}\,\delta\int \frac{\rho(\vec{r})\,\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}\,d\vec{r}'$$

$$W = \frac{1}{2}\int \rho(\vec{r})\,\phi(\vec{r})\,d\vec{r}$$

5. $$\frac{1}{\vec{r}+\vec{a}} = \frac{1}{\sqrt{\vec{r}^2+\vec{a}^2+2\vec{a}\cdot\vec{r}}}$$

$$= \frac{1}{r}\left(1 - \frac{\vec{a}\cdot\vec{r}}{r^2} + \cdots\right)$$

$$= \frac{1}{r} + \vec{a}\cdot\nabla\left(\frac{1}{r}\right) + \cdots$$

15/1/97

Electrostatics in the presence of conductors

Boundary conditions:

E field must be 0. Otherwise current would flow.

Pot. difference is 0 since

$$V_{12} = -\int_1^2 \vec{E} \cdot d\vec{l}$$

Surface is an equipotential and field normal to it.

$$E_n = 4\pi\sigma \quad \text{where } \sigma \text{ is surface charge density}$$

Can specify only total charge or potential. Surface charge will adjust themselves.

$$V_i = \phi(\vec{r}_i)$$

$$Q_i = \int \frac{dS\, E_n}{4\pi} = \int \frac{d\vec{S}\cdot\vec{E}}{4\pi} = -\int \frac{d\vec{S}\cdot\nabla\phi}{4\pi}$$

$\nabla\cdot\vec{E} = 0$ $\therefore \rho = 0$ inside conductor

Generally specify ρ $\quad \nabla^2\phi = -4\pi\rho$

Uniqueness

Given a density ρ, set of conducting surfaces S_i, with either Q_i or V_i, $\vec{E}$ is uniquely determined.

Let ϕ_1 & ϕ_2 be different solns.

~~$I = \int \nabla(\psi_1 - \psi_2)$~~ $\quad \psi = \phi_1 - \phi_2$

$$\nabla\cdot(\phi\nabla\psi) = \phi\,\nabla^2\psi + \nabla\phi\cdot\nabla\psi$$

$$\psi = \phi_1 - \phi_2$$

$$\int(\nabla\psi)^2\,d\vec{\tau} = \int \nabla\psi\cdot\nabla\psi\;d\vec{\tau}$$

$$= \int\left[\nabla\cdot(\psi\nabla\psi) - \psi\nabla^2\psi\right]d\vec{\tau}$$

$$= \sum_i \int_i d\vec{S}_i(\psi)\cdot\nabla\psi - \int\psi\nabla^2\psi\;d\vec{\tau}$$

$$\nabla^2\phi_1 = \nabla^2\phi_2 = -4\pi\rho \quad \therefore\ \nabla^2\psi = 0$$

~~Also $\psi = 0$ on surfaces of conductors~~ $\left(V_1^a - V_2^a\right)\int d\vec{S}_a\cdot\left(\vec{E}_1^a - \vec{E}_2^a\right)$

$$= \sum_i \Delta V_i\,\Delta Q_i$$

$$\therefore \int(\nabla\psi)^2\,d\vec{\tau} = 0$$

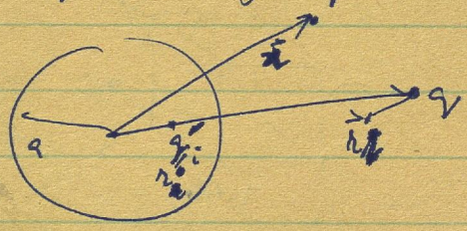
$$\therefore \phi_1 - \phi_2 = \text{const.}$$

15/1/97

Methods

1. Images
2. Factorizing $\nabla^2\phi = 0$
3. Complex variables, conformal mapping

1. Images good if you have one or two point charges in the presence of boundary surfaces.



$$\phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{r}'|} + \frac{q_i'}{|\vec{r}-\vec{r}_i'|}$$

Potential vanishes at $|r| = a$

$$\phi(\vec{r})\Big|_{r=a} = \frac{q}{r\left|\hat{r} - \frac{r'}{r}\hat{r}'\right|} + \frac{q_i}{r_i'\left|\hat{r}_i' - \frac{r}{r_i'}\hat{r}\right|} = \frac{q}{r\left(1+\frac{r'^2}{r^2} - 2\frac{r'}{r}\hat{r}\cdot\hat{r}'\right)^{1/2}} + \frac{q_i}{r_i'\left(1+\frac{r^2}{r_i'^2} - 2\frac{r}{r_i'}\hat{r}\cdot\hat{r}_i'\right)^{1/2}}$$

$$\Big|_{r=a} = \frac{q}{a\left|\hat{r} - \frac{r'}{a}\hat{r}'\right|} + \frac{q_i}{r_i'\left|\hat{r}_i' - \frac{a}{r_i'}\hat{r}\right|}$$

$$\frac{q}{a} = -\frac{q_i}{r_i'} \quad \text{and} \quad \frac{r'}{a} = \frac{a}{r_i'}$$

$$\therefore\ q_i = -q\frac{r_i'}{a} \quad \text{and} \quad r_i' = \frac{a^2}{r'}$$

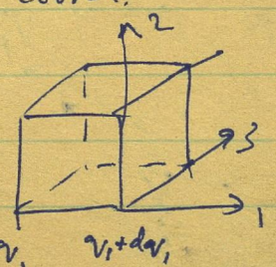
Factorizing ∇^2

$dl_i = h_i \, dq_i$ $\qquad dl_\phi = r \sin\theta \, d\phi$

$dl_\theta = r \, d\theta$

$dl_r = dr$

Only ~~orthogonal~~ dir components in orthogonal coord.



$\therefore \int \nabla^2\phi \, d\tau = \int_S \nabla\phi \cdot d\vec{s}$

$\nabla^2\phi \; h_1 h_2 h_3 \, dq_1 \, dq_2 \, dq_3 =$

$$\frac{dq_2\, dq_3}{} \; \frac{h_2 h_3}{h_1} \; \frac{\partial\phi}{\partial q_1}\Big|_{q_1+dq_1}$$

$$- \frac{dq_2\, dq_3}{} \; \frac{h_2 h_3}{h_1} \; \frac{\partial\phi}{\partial q_1}\Big|_{q_1} + \cdots$$

$$= dq_1\, dq_2\, dq_3 \left\{ \frac{\partial}{\partial q_1} \; \frac{h_2 h_3}{h_1} \; \frac{\partial}{\partial q_1} \right\} \phi + \cdots$$

$$\therefore \nabla^2 = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \; \frac{h_2 h_3}{h_1} \; \frac{\partial}{\partial q_1} + \cdots \right\}$$

Eq in sph pola

$$\nabla^2 = \frac{1}{r^2 \sin\theta} \left\{ \frac{\partial}{\partial r} \; \frac{r^2 \sin\theta}{1} \; \frac{\partial}{\partial r} + \frac{\partial}{\partial\theta} \; \frac{1 \cdot r \sin\theta}{r} \; \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi} \; \frac{1 \cdot r}{r \sin\theta} \; \frac{\partial}{\partial\phi} \right\}$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial\omega} (1-\omega^2) \frac{\partial}{\partial\omega} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\phi^2} \qquad \omega = \cos\theta$$

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Complex variables – Cauchy Riemann equations.

$w = f(z)$ is analytic $\quad w = u + iv \quad z = x + iy$

Then derivative exists indept of direction of z.

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{w(z+\Delta z) - w(z)}{\Delta z}$$

$$\frac{\Delta w}{\Delta z} = \frac{\left(\frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y\right) + i\left(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y\right)}{\Delta x + i\Delta y}$$

$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}}$ indept of $\Delta y/\Delta x$

$$\frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \frac{\Delta y}{\Delta x}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)}{1 + i\frac{\Delta y}{\Delta x}} = \frac{dw}{dz}$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{dw}{dz} \qquad \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i\frac{dw}{dz} = i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

u and v are harmonic i.e. $\nabla^2(u,v) = 0$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \quad \therefore \quad \nabla^2 u = 0$$

Charge density on $v=0$ plate, $\left.\frac{\partial v}{\partial y}\right|_x$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv = 0 \Rightarrow du = -\frac{\partial x/\partial v}{\partial x/\partial u}dv$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv = \left(\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x/\partial v}{\partial x/\partial u}\right)dv$$

$$\left.\frac{\partial v}{\partial y}\right|_x = \frac{\partial x/\partial u}{\frac{\partial y}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}}$$

$$\frac{\partial y}{\partial v} = \pi - \pi e^{2\pi u}\cos 2\pi v \qquad \frac{\partial x}{\partial v} = \pi e^{2\pi u}\sin 2\pi v$$

$$\frac{\partial y}{\partial u} = -\pi e^{2\pi u}\sin 2\pi v \qquad \frac{\partial x}{\partial u} = \pi - \pi e^{2\pi u}\cos 2\pi v$$

$$\left.\frac{\partial v}{\partial y}\right|_x = \frac{1 - \cos 2\pi v\, e^{2\pi u}}{\pi\left(1 + e^{4\pi u} - 2e^{2\pi u}\cos 2\pi v\right)}$$

At $v=0$ $= \frac{1}{\pi}\frac{1}{1-e^{2\pi u}} = -E_y = -4\pi\sigma$

At $u \to -\infty$ $E_y = -\frac{1}{\pi}$

$u \to +\infty$ $E_y \to 0$ as $e^{-2\pi u}$.

Near $u=0$ $E_y \sim \frac{1}{2\pi^2 u}$ $(E_y > 0)$ for $u > 0$ (below)

$E_y < 0$ for $u < 0$ (above)

$$x = u\pi + \frac{1}{2}\left(1 - \left[1 + 2\pi u + \frac{(2\pi u)^2}{2}\right]\right) = -\pi^2 u^2$$

$u = \sqrt{\frac{x}{\pi}}$ $\sigma \sim \frac{1}{\sqrt{x}}$ near the edge $\frac{\partial u}{\partial x}$

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$$z = w\pi + \frac{1}{2}\left(1 - e^{2\pi w}\right)$$

$$y = v\pi - \frac{1}{2} e^{2\pi u} \sin 2\pi v$$

$$x = u\pi + \frac{1}{2}\left(1 - e^{2\pi u} \cos 2\pi v\right)$$

v is potential

equipotential $v = 0$ is $y = 0$

$$x = u\pi + \frac{1}{2}\left(1 - e^{2\pi u}\right)$$

$u \to -\infty \quad x \to -\infty$

$u \to +\infty \quad x \to -\infty$

Max at $u = 0 \quad x = 0$

$$\frac{dx}{du} = \pi - \pi e^{2\pi u_m} = 0$$

$$u_m = 0$$

$v = 1 \quad y = \pi \quad , \quad x \le 0$

∴ potential of a condenser from $x = 0$ to $x = -\infty$, infinite in 3rd dimension, $\Delta v = 1$, plate $xy = \pi$.

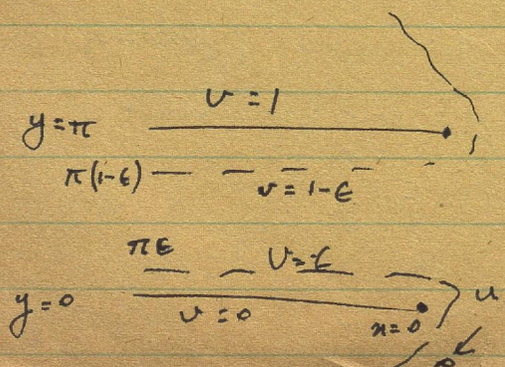
$v = 1 - \epsilon \qquad y = (1-\epsilon)\pi + e^{2\pi u}\,\pi\epsilon \qquad u \to \infty \; y \to \infty$

$u \to -\infty \; y \to (1-\epsilon)\pi$

$v = \epsilon$

$y = \epsilon\pi - e^{2\pi u}\,\pi\epsilon \qquad u \to \infty \quad y \to -\infty$

$u \to -\infty \quad y \to \epsilon\pi$



Green's function method

Finding the field for a given charge distribution in the presence of conductors with given potential V_j or charge Q_i.

Given potential V_a on surfaces S_a and/or charge Q_b on surface S_b.

Green's function is the potential produced at $\vec{r}$ by a unit charge at $\vec{r}_1$, with 0 potential on S_a and 0 charge on S_b.

$$\nabla^2 G_1(\vec{r},\vec{r}_1) = -4\pi\,\delta(\vec{r}-\vec{r}_1)$$

$$G_1(\vec{r},\vec{r}_1) \text{ on } S_a = 0 \qquad \text{and} \qquad \int d\vec{S}_b \cdot \nabla G_1(\vec{r},\vec{r}_1) = 0$$

Let ψ be the actual potential given ρ, V_a and Q_b.

$$I(\vec{r}_1') = \sum_{a\&b} \int \left\{ G_1(\vec{r},\vec{r}_1)\,\nabla\psi(\vec{r}) - \nabla G_1(\vec{r},\vec{r}_1)\,\psi(\vec{r}) \right\} \cdot d\vec{S}_i$$

On S_a, first integral is 0 and second is $+4\pi V_a Q_a(\vec{r}_1)$

On S_b, second integral is 0 and first is $+4\pi Q_b V_b(\vec{r}_1)$

$$I(\vec{r}_1) = \int \left(\nabla G_1 \cdot \nabla\psi + G_1 \nabla^2\psi - \nabla G_1 \cdot \nabla\psi - \psi \nabla^2 G_1 \right) d\vec{r}$$

$$= -4\pi \int \rho G_1(\vec{r},\vec{r}_1)\, d\vec{r} + 4\pi \int \delta(\vec{r}-\vec{r}_1)\,\psi(\vec{r})\, d\vec{r}$$

$$\psi(\vec{r}_1) = \int \rho(\vec{r}) G_1(\vec{r},\vec{r}_1)\, d\vec{r} + \sum_a V_a Q_a(\vec{r}_1) + \sum_b Q_b V_b(\vec{r}_1)$$

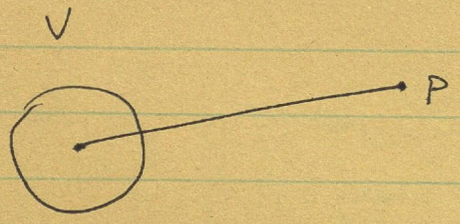
(outward normal)

$$G_1(\vec{r}_1',\vec{r}_2') = G_1(\vec{r}_2',\vec{r}_1') \qquad I(\vec{r}_1',\vec{r}_2') = \sum_i \int d\vec{S}_i \cdot \left[G_1(\vec{r},\vec{r}_1') \nabla G(\vec{r}',\vec{r}_2') - G_1(\vec{r}',\vec{r}_2') \nabla G_1(\vec{r},\vec{r}_1') \right]$$

On each surface, $G_1 = 0$ or $\int \nabla G_1 \cdot d\vec{S} = 0$

$$\therefore \quad 0 = \int \left[G_1(\vec{r},\vec{r}_1') \nabla^2 G_1(\vec{r}',\vec{r}_2') - G_1(\vec{r}',\vec{r}_2') \nabla^2 G_1(\vec{r},\vec{r}_1') \right] d\vec{r}$$

$$= -4\pi \left[G_1(\vec{r}_2,\vec{r}_1) - G_1(\vec{r}_1,\vec{r}_2) \right]$$



Use method of images to find Green's function

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Cauchy - Riemann eqns

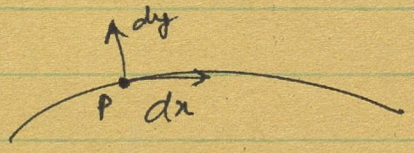
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\nabla^2 u = \nabla^2 v = 0$$

$$|\nabla u| = |\nabla v| \quad \text{and} \quad \nabla u \cdot \nabla v = 0$$

If constant v are equipotentials, constant u are lines of force.

Let v = constant be a conducting surface



E field is $-\frac{\partial v}{\partial y}$ at P, $\sigma = -\frac{1}{4\pi}\frac{\partial v}{\partial y}$

$$\sigma = -\frac{1}{4\pi}\frac{\partial u}{\partial x}$$

$\therefore$ Total charge

$$\int_{x_1}^{x_2} \sigma \, dx = \frac{1}{4\pi}\left[u(x_1) - u(x_2)\right]$$

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Systems of conductors

Linear functional dependence of V or Q.

Q_1 on conductor 1, all others 0.

$V_i^{(1)} = p_{i1} Q_1$

Q_2 on conductor 2, all others 0

$V_i^{(2)} = p_{i2} Q_2$

$\therefore V_i = \sum p_{ij} Q_j$ $\quad p_{ij}$ = coefficients of potential

Energy of configuration

1. Bring δQ_i up to i^{th} conductor

$$\delta W = \sum_i V_i \delta Q_i = \sum_{ij} p_{ij} Q_j \delta Q_i$$

2. In general

$$W = \frac{1}{2} \sum V_i Q_i = \frac{1}{2} \sum p_{ij} Q_j Q_i$$

$$\delta W = \sum_{ij} \frac{1}{2} \left(Q_i \delta Q_j + Q_j \delta Q_i \right) p_{ij}$$

$$= \frac{1}{2} \sum Q_j \delta Q_i \left(p_{ij} + p_{ji} \right)$$

$$\therefore \frac{p_{ij} + p_{ji}}{2} = p_{ij} \quad \text{or} \quad p_{ij} = p_{ji}$$

If there is a ξ,

$$V_i - V_i^0 = \sum_j p_{ij} Q_j$$

$$W = \frac{1}{2} \sum_{ij} Q_i p_{ij} Q_j > 0$$

$\therefore$ eigenvalues of the matrix p_{ij} are +ve

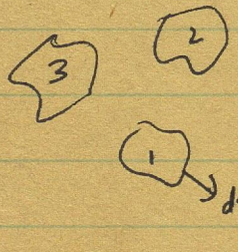
$\therefore |p_{ij}| \neq 0$ $\{p_{ij}\}^{-1}$ exists

$$\sum_i p_{ij} c_{jk} = \delta_{ik}$$

$$Q_i = \sum_j c_{ij} V_j$$

$c_{ij} = c_{ji}$ and called capacities.

Can also calculate energy by charging the conductors and bringing them together. Q's are fixed. p's change

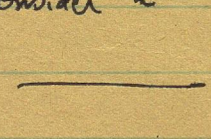


$$\vec{F}_1 \cdot d\vec{\xi}_1 = -\delta W = -\frac{1}{2} \sum_{ij} Q_i Q_j \delta p_{ij}$$

$$\delta p_{ij} = \nabla p_{ij} \cdot d\vec{\xi}$$

$$\therefore \vec{F}_1 = -\frac{1}{2} \sum_{ij} Q_i Q_j \nabla p_{ij}$$

Consider a capacitor – system of 2 conductors with equal & opp charge



$$V = \frac{Q}{C} \qquad W = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C V^2$$

$$\vec{F}\} = -\nabla W\big|_Q = +\frac{Q^2}{2}\frac{\nabla C}{C^2}$$

$$\vec{F} = -\nabla W\big|_V = -\frac{V^2}{2}\nabla C = -\frac{Q^2}{2}\frac{\nabla C}{C^2}$$

Which sign is correct? One with constant Q since the one with constant V is not an isolated system. In the latter, battery supplies twice the energy in order to change the sign

$$V_i = \sum_j p_{ij} Q_j$$

$$Q_i = \sum_j \left(\frac{1}{p}\right)_{ij} V_j = \sum_j C_{ij} V_j$$

$$W = \frac{1}{2}\sum Q_i V_i = \frac{1}{2}\sum C_{ij} V_i V_j$$

$$\vec{F}\big|_Q = -\frac{1}{2}\sum Q_i \nabla p_{ij} Q_j$$

$$\text{“}\vec{F}\text{”}\big|_V = -\frac{1}{2}\sum V_i \nabla C_{ij} V_j$$

Matrix $C = \frac{1}{p}$

$$\nabla C = ? = -\frac{1}{p}\nabla p \frac{1}{p}$$

$$\Delta C = \frac{1}{p+\Delta p} - \frac{1}{p} = \frac{1}{p+\Delta p}\left(p - p - \Delta p\right)\frac{1}{p} = \frac{1}{p+\Delta p}\left(-\Delta p\right)\frac{1}{p}$$

$$\text{“}F\text{”} = \frac{1}{2} V \frac{1}{p} \nabla p \frac{1}{p} V$$

$$= \frac{1}{2}\sum V_i \left(\frac{1}{p}\right)_{ij} (\nabla p)_{jk} \left(\frac{1}{p}\right)_{kl} V_l$$

$$V_i \left(\frac{1}{P}\right)_{ij} = \left(\frac{1}{P}\right)_{ji} V_i = Q_j$$

$$\left(\frac{1}{P}\right)_{kl} V_l = Q_k$$

$$\text{"F"} = \frac{1}{2} \sum Q_j (\nabla P)_{jk} Q_k = -\vec{F}$$

Charge transported to i^{th} conductor is δQ_0

$$\delta Q_i = \sum_j \delta C_{ij} V_j$$

Battery work $\delta W_B = \sum_i \delta Q_i V_i = \sum V_j \delta C_{ij} V_i$

$$\delta W_{Total} = \frac{1}{2} \sum V_i \delta C_{ij} V_j = \delta W_B + \delta W_{mech}$$

$$\therefore \delta W_{mech} = \delta W_{Total} - \delta W_B$$

$$= -\frac{1}{2} \sum V_i \delta C_{ij} V_j = \frac{1}{2} \sum Q_i \delta P_{ij} Q_j$$

Capacitance can be calculated for parallel plates

$$V_1 = P_{11} Q_1 + P_{12} Q_2 = (P_{11} - P_{12}) Q$$

$$V_2 = P_{21} Q_1 + P_{22} Q_2 = (P_{12} - P_{22}) Q$$

$$\Rightarrow V_1 - V_2 = (P_{11} - 2P_{12} + P_{22}) Q$$

$$C = \frac{Q}{\Delta V} = \frac{1}{P_{11} - 2P_{12} + P_{22}}$$

$C > 0$ since $P_{11} + P_{22} > |2P_{12}|$ or $W = \frac{1}{2}(Q_1^2 P_{11} + Q_2^2 P_{22} + 2Q_1 Q_2 P_{12})$ will be negative

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$$\nabla \cdot \vec{e} = 4\pi\rho$$

$$\nabla \times \vec{e} = 0$$

Average over many atoms to eliminate fluctuations.
Region contains many atoms but is small compared to scale of spatial variation of fields. Use $f(\vec{x})$

$$\int f(\vec{x})\, d\vec{x} = 1$$

Characteristic size Δ of f $\quad \Delta \ll \lambda$ (scale of variations)

$n\Delta^3 \gg 1$ $\quad n$ = no of atoms/vol

e.g. $\Delta^2 = \int f(\vec{x})\, \vec{x}^2\, d\vec{x}$ $\qquad f = \frac{1}{\frac{4\pi}{3} R^3}$ $\qquad \Delta^2 = \frac{3}{5} R^2$

$$E_i(\vec{x}) = \int f(\vec{x} - \vec{x}')\, e_i(\vec{x}')\, d\vec{x}' = \langle e_i \rangle$$

Commutes with differentiation

$$\frac{\partial}{\partial x_j} E_i = \int \frac{\partial f(\vec{x} - \vec{x}')}{\partial x_j}\, e_i(\vec{x}')\, d\vec{x}'$$

$$= -\int \frac{\partial f(\vec{x} - \vec{x}')}{\partial x_j'}\, e_i(\vec{x}')\, d\vec{x}'$$

$$= \int f(\vec{x} - \vec{x}')\, \frac{\partial e_i(\vec{x}')}{\partial x_j'}\, d\vec{x}'$$

$$= \left\langle \frac{\partial e_i}{\partial x_j} \right\rangle$$

$$\nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{E} = 4\pi \langle \rho \rangle \qquad \langle \rho \rangle \text{ average charge density}$$

ρ_{free} and ρ_{bound}

Normally microscopic average of ρ_{bound} is 0.
First non-vanishing moment is dipole moment

$$V = -\int \vec{P}(\vec{r}') \cdot \nabla \frac{1}{|\vec{r}-\vec{r}'|}\, d\vec{r}'$$

$$V_p(\vec{R}) = \int d\vec{r}\, f(\vec{R}-\vec{r}) \sum_i \vec{p}_i \cdot \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_0|^3}$$

$\vec{r}-\vec{r}_i = \vec{R}-\vec{R}'$, $\vec{R}'$ is integration variable

$$V_p(\vec{R}) = \int d\vec{R}' \left(\sum_i f(\vec{R}'-\vec{r}_i)\, \vec{p}_i \right) \cdot \frac{\vec{R}-\vec{R}'}{|\vec{R}-\vec{R}'|^3}$$

$\sum f(\vec{R}'-\vec{r}_i)\, \vec{p}_i$ counts all dipoles within the average distance of f

$\therefore = \vec{P}$, dipole moment per unit volume

$$\nabla' \cdot \left(\frac{\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) = \frac{\nabla' \cdot \vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|} + \vec{P}(\vec{r}') \cdot \nabla' \left(\frac{1}{|\vec{r}-\vec{r}'|} \right)$$

$$= \frac{\nabla' \cdot \vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|} - \vec{P}(\vec{r}') \cdot \nabla \left(\frac{1}{|\vec{r}-\vec{r}'|} \right)$$

$$V = \int \left(\nabla' \cdot \left(\frac{\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) - \frac{\vec{\nabla}' \cdot \vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|} \right) d\vec{r}'$$

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$$\vec{p} = \alpha \vec{E}$$

$\dim |E| = \frac{Q}{\text{area}}$ $\quad \dim |p| = Q \cdot \text{distance}$

$\therefore \alpha =$ dimensions of volume

Conducting sphere $R^3 f(\pi)$

Atom $a_B^3 \simeq 10^{-24}$ cm

Nucleus $\alpha \sim \frac{e^2}{g^2} R^3$ $\quad R \simeq 10^{-14}$ cm

$$\vec{Q} \simeq \alpha_Q \frac{\partial E}{\partial z} \qquad \alpha_Q \simeq a_B^5$$

$$V = \int \frac{\vec{P}(\vec{x}') \cdot d\vec{S}'}{|\vec{x} - \vec{x}'|} + \int \frac{d\vec{x}' \left(-\nabla' \cdot P(\vec{x}') \right)}{|\vec{x} - \vec{x}'|}$$

$\rho_b = -\nabla \cdot \vec{P}$ and $\sigma_b = P_{normal}$

$$\nabla \times \vec{E} = 0$$

$$\nabla \cdot \vec{E} = 4\pi\rho - 4\pi \nabla \cdot \vec{P}$$

$$\vec{D} = \vec{E} + 4\pi \vec{P}$$

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho$$

$E_{tangential}$ is cont, ΔD_n is $4\pi\sigma_{free}$

$P_i = \chi_{ij} E_j$ χ : dimensionless susceptibility

$D_i = \left(\delta_{ij} + 4\pi \chi_{ij} \right) E_j = \epsilon_{ij} E_j$

ϵ_{ij} is dielectric tensor. Isotropic systems $\epsilon_{ij} = \epsilon \delta_{ij}$

$\vec{D} = \epsilon \vec{E}$ ϵ = dielectric constant

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Uniqueness

Two solutions $\vec{E}_1$ & $\vec{E}_2$, V_1 & V_2 etc

$\nabla \times \vec{E} = 0$, $\nabla . \vec{D} = 4\pi\rho$ (ρ given), $D_i = \epsilon_{ij} E_j$ $\epsilon_{ij} = \epsilon_{ji}$ with ϵ real positive matrix

energy $\frac{1}{8\pi}\int E_i E_j \epsilon_{ij}\, d\vec{r}$ is positive

cond surfaces a V_a given, or given charge Q_b on surfaces b.

$$I = \int (\vec{E}_1 - \vec{E}_2) . (\vec{D}_1 - \vec{D}_2)\, d\vec{r}$$

$$= \int \epsilon_{ij} (E_{1i} - E_{2i}) (E_{1j} - E_{2j})\, d\vec{r}$$

> 0 unless solutions are identical.

$$I = -\int \nabla (V_1 - V_2) . (\vec{D}_1 - \vec{D}_2)\, d\vec{r}$$

$$= -\int \nabla . \left[(V_1 - V_2)(\vec{D}_1 - \vec{D}_2) \right] d\vec{r} + \int \left[(V_1 - V_2)\, \nabla . (\vec{D}_1 - \vec{D}_2) \right] d\vec{r}$$

$\nabla . \vec{D}_1 = \nabla . \vec{D}_2 = 4\pi\rho$ $\therefore \nabla . (\vec{D}_1 - \vec{D}_2) = 0$

$$I = \sum_i \int d\vec{S}_i . (V_1 - V_2)_i (\vec{D}_1 - \vec{D}_2)_i$$

$$= \sum_i (V_1 - V_2)_i \int dS_i . (\vec{D}_1 - \vec{D}_2)_i$$

$$= 4\pi \sum_i (V_1 - V_2)_i (Q_1 - Q_2)_i = 0$$

$\therefore \vec{E}_1 = \vec{E}_2$

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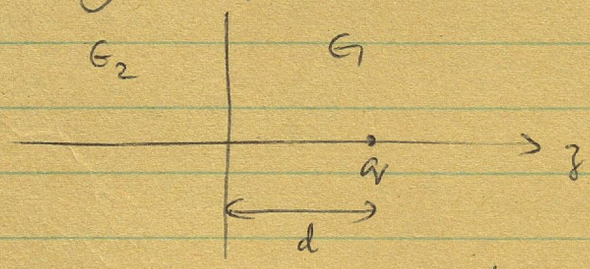
Dielectric is isotropic & uniform

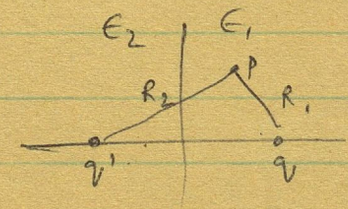
$$\nabla \cdot E = 4\pi\left(\frac{\rho}{\epsilon}\right)$$

Same as before except $\rho \to \rho/\epsilon$

Shielding due to dipoles of system

e.g. method of images





$z > 0 \qquad \phi = \frac{1}{\epsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right)$

$z < 0 \qquad \phi = \frac{1}{\epsilon_2}\frac{q''}{R_2}$

$R_1 = \sqrt{\rho^2 + (d-z)^2}$

$R_2 = \sqrt{\rho^2 + (d+z)^2}$

$$q - q' = q''$$

$$\frac{1}{\epsilon_1}(q + q') = \frac{1}{\epsilon_2} q''$$

$$\therefore \quad q' = -\left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}\right) q$$

$$q'' = \left(\frac{2\epsilon_2}{\epsilon_1 + \epsilon_2}\right) q$$

There is a surface charge density

$$\sigma = -\frac{q}{2\pi}\left(\frac{\epsilon_2 - \epsilon_1}{\epsilon_1(\epsilon_1 + \epsilon_2)}\right)\frac{d}{(\rho^2 + d^2)^{3/2}}$$

$\epsilon_2 >> \epsilon_1$ dielectric ϵ_2 behaves like a conductor

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Green's functions

$$I = \sum_i \int \left[V(\vec{r}_i) \epsilon \nabla G(\vec{r}_i, \vec{r}_i') - G(\vec{r}_i, \vec{r}') \epsilon \nabla V(\vec{r}_i) \right] \cdot d\vec{S}_i$$

where V is the assumed correct potential.

$V = V_a$ on a surfaces $\quad Q = Q_b$ on b surfaces

$$I = -4\pi \sum_a V_a Q_a(\vec{r}') + 4\pi \sum_b Q_b V_b(\vec{r}')$$

G is zero on a and charge is zero on b

$$I = \int d\vec{r} \left[\left(\nabla V \cdot \epsilon \nabla G + V \nabla \cdot \epsilon \nabla G \right) - \left(\nabla G \cdot \epsilon \nabla V + G \nabla \cdot \epsilon \nabla V \right) \right]$$

$$= \int d\vec{r} \left[V(\vec{r}) \nabla \cdot (\epsilon \nabla G) - G(\vec{r}, \vec{r}') \nabla \cdot (\epsilon \nabla V) \right]$$

$$= -4\pi V(\vec{r}') + 4\pi \int G(\vec{r}, \vec{r}') \rho(\vec{r}) d\vec{r}$$

where we have chosen

$$\nabla \cdot (\epsilon \nabla G) = -4\pi \delta(\vec{r} - \vec{r}') \quad + \text{bound. cond.}$$

$$\nabla \cdot \vec{D} = -\nabla \cdot \epsilon \nabla V = 4\pi\rho$$

The derivation goes through for a symmetric tensorial $\epsilon_{ij} = \epsilon_{ji}$

$$\frac{\partial}{\partial x_i} \epsilon_{ij} \frac{\partial V}{\partial x_j} = -4\pi\rho$$

$$\frac{\partial}{\partial x_j} \epsilon_{ij} \frac{\partial G}{\partial x_j} = -4\pi \delta(\vec{r} - \vec{r}')$$

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1. Steady Currents Charge conservation (empirical fact)

$$-\frac{d}{dt}\int_V \rho \, d\vec{\tau} = \int_S d\vec{S}\cdot\vec{j}$$

$$\int_V \frac{\partial \rho}{\partial t} d\vec{\tau} = \int_V \nabla\cdot\vec{j}\, d\vec{\tau}$$

$$\therefore \frac{\partial \rho}{\partial t} + \nabla\cdot\vec{j} = 0$$

Steady state (static) $\frac{\partial \rho}{\partial t} = 0$

$$\therefore \nabla\cdot\vec{j} = 0$$

Ohm's Law $\vec{j} = \sigma\vec{E}$

$$j_i = \sigma_{ij} E_j \quad \text{(linear media)}$$

$$\frac{dW}{dt} = \int E_i j_i \, d\vec{\tau} = \int d\vec{\tau}\, E_i \sigma_{ij} E_j$$

$$\nabla\times\vec{E} = \nabla\times\frac{\vec{j}}{\sigma} = 0 \qquad \text{for conducting}$$

$$\frac{\vec{j}}{\sigma} = -\nabla\psi$$

& $\nabla\cdot j = 0$ i.e. $\nabla\cdot(\sigma\nabla\psi) = 0$

Identical to $\nabla\cdot(\epsilon\nabla V) = 0$ or $\nabla\cdot\mathbf{D} = 0$ or $\nabla\cdot(\epsilon\vec{E}) = 0$

$D \leftrightarrow j$, $\sigma \leftrightarrow \epsilon$, $-\nabla\psi \leftrightarrow \vec{E}$

Unique if specify ψ on boundary or current flow $\int \vec{j}\cdot d\vec{S}$ through surface (should be normal so that surface is equipotential)

Magnetic fields

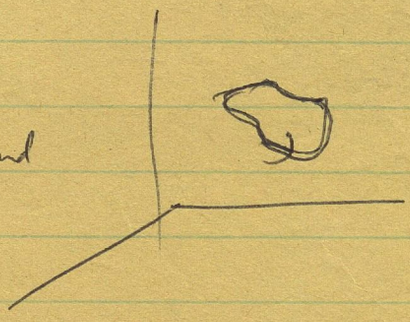
Mixed Gaussian units

esu for charge and emu (gauss) for $\vec{B}$ field.

Biot-Savart law

Magnetic field at $\vec{r}$ due to a circuit containing a current I can be obtained from

$$\vec{B}(\vec{r}) = \frac{I}{c}\oint_C d\vec{l} \times \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$



$\oint d\vec{l}$ is line integral around the circuit and
$\vec{r}'$ is the position vector of $d\vec{l}$
c is speed of light

$$I\,d\vec{l} = \vec{j}A\,dl = \vec{j}\,d\vec{r}$$

Continuous distribution of current

$$\vec{B}(\vec{r}) = \int \frac{\vec{j}(\vec{r}')}{c} \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\,d\vec{r}'$$

Steady currents, no change in charge density $\therefore \nabla\cdot\vec{j} = 0$

$$\vec{B}(\vec{r}) = -\int d\vec{r}'\,\frac{\vec{j}(\vec{r}')}{c} \times \nabla\frac{1}{|\vec{r}-\vec{r}'|}$$

$$\nabla\times(\psi\vec{V}) = \psi\nabla\times\vec{V} + \nabla\psi\times V = \psi\nabla\times\vec{V} - \vec{V}\times\nabla\psi$$

$$\vec{B}(\vec{r}) = \nabla\times\frac{1}{c}\oint\frac{\vec{j}(\vec{r}')d\vec{r}'}{|\vec{r}-\vec{r}'|} = \nabla\times\vec{A}$$

$$\vec{A} = \frac{1}{c}\int\frac{\vec{j}(\vec{r}')d\vec{r}'}{|\vec{r}-\vec{r}'|}$$

$\therefore \quad \nabla \cdot \vec{B} = 0 \qquad$ since $\vec{B} = \nabla \times \vec{A}$

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A})$$

$$= \nabla (\nabla \cdot \vec{A}) - \nabla^2 A$$

$$\nabla \cdot \vec{A} = \frac{1}{c} \int \nabla \frac{1}{|\vec{r} - \vec{r}'|} \cdot j(\vec{r}')\, d\vec{r}'$$

$$= -\frac{1}{c} \int \nabla' \frac{1}{|\vec{r} - \vec{r}'|} \cdot j(\vec{r}')\, d\vec{r}'$$

$$= \frac{1}{c} \int \frac{1}{|\vec{r} - \vec{r}'|} \nabla' \cdot j(\vec{r}')\, d\vec{r}' - \frac{1}{c} \int \nabla' \cdot \left(\frac{j(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) d\vec{r}'$$

$$-\frac{1}{c} \int \frac{j(\vec{r}')}{|\vec{r} - \vec{r}'|} \cdot d\vec{s} \to 0 \quad \text{since } j(\vec{r}') \to 0 \text{ as } \vec{r}' \to \infty$$

$$= 0$$

$$-\nabla^2 A = -\nabla^2 \frac{1}{c} \int \frac{j(\vec{r}')\, d\vec{r}'}{|\vec{r} - \vec{r}'|}$$

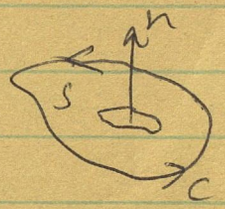
$$\nabla^2 \int \frac{\rho(\vec{r}')\, d\vec{r}'}{|\vec{r} - \vec{r}'|} = -4\pi \rho(\vec{r}) \qquad \nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = -4\pi \delta(\vec{r} - \vec{r}')$$

$$\therefore \nabla^2 \int \frac{j(\vec{r}')\, d\vec{r}'}{|\vec{r} - \vec{r}'|} = -4\pi j(\vec{r})$$

$$\therefore \boxed{\begin{aligned} \nabla \times \vec{B} &= \frac{4\pi}{c} \vec{j} \\ \nabla \cdot \vec{B} &= 0 \end{aligned}}$$

Ampere's circuital law (analogue of Gauss's law)

$$\int \nabla \times \vec{B} \cdot d\vec{S} = \frac{4\pi}{c} \int \vec{j} \cdot d\vec{S}$$



Stoke's theorem

$$\oint_C \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} I_{enclosed}$$

Gauge invariance

$\vec{A} \to \vec{A} + \nabla\psi$

$\nabla \times \vec{A} \to \nabla \times \vec{A} + \nabla \times (\nabla\psi)$ → 0

We have chosen $\nabla \cdot \vec{A} = 0$

Suppose $\nabla \cdot \vec{A}_0 \neq 0$

$$\nabla \cdot \vec{A}_1 = \nabla \cdot \vec{A}_0 + \nabla^2 \psi$$

$\therefore$ $\nabla^2 \psi = -\nabla \cdot \vec{A}_0$ makes $\nabla \cdot \vec{A}_1 = 0$

Explanation of gauge field theories

Force hypothetical pole $\vec{p}$

$$\vec{F}_{comp} = p\vec{B} = -\frac{I}{c}\oint d\vec{l} \times \frac{p(\vec{r}' - \vec{r}'')}{|\vec{r} - \vec{r}'|^3}$$

$$\frac{p(\vec{r}' - \vec{r})}{|\vec{r}' - \vec{r}|^3} = B_p(\vec{r}')$$

$$\therefore \vec{F}_{comp} = -\frac{I}{c}\oint d\vec{l} \times \vec{B}_p(\vec{r}')$$

Assuming Newton's third law

$$\vec{F}_{on\,C} = \frac{I}{c}\oint d\vec{l} \times \vec{B}(\vec{r}'')$$

$$\vec{F}_{2\,on\,1} = \frac{I_1}{c}\oint d\vec{l}_1 \times \vec{B}_2(\vec{r}_1)$$

$$= \frac{I_1 I_2}{c^2}\oint_{C_1} d\vec{l}_1 \times \oint_{C_2} \frac{d\vec{l}_2 \times (\vec{r}_1 - \vec{r}_2)}{r_{1,2}^3}$$

$$= \frac{I_1 I_2}{c^2}$$

Review

1. Continuity $\nabla \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$

2. Steady current (or static case) $\frac{\partial \rho}{\partial t} = 0$

$\therefore \nabla \cdot \vec{j} = 0$

3. $\vec{B}(\vec{r}) = \frac{I}{c} \oint_C \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|r - \vec{r}'|^3}$

$d\vec{F} = \frac{I}{c} d\vec{l} \times \vec{B}$ or $\vec{F} = \frac{I}{c} \oint d\vec{l} \times \vec{B}$

Can we get magnetic force on a single moving charge

$I = nqvA$ n = no of charges/vol
q = charge (unit)
$\vec{v}$ = velocity
A = cross-sectional area of wire

$I d\vec{l} = Nq\vec{v}$ N = no. of charges in $d\vec{l}$

$\therefore \vec{F}_q = q \frac{\vec{v}}{c} \times \vec{B}$ if only one charge

$\vec{F}_{2 \text{ on } 1} = q_1 \frac{\vec{v}_1}{c} \times \vec{B}_2(\vec{r}_1)$

$= q_1 \frac{\vec{v}_1}{c} \times \left[q_2 \frac{\vec{v}_2}{c} \times \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \right]$

$\vec{F}_{2 \text{ on } 1} = \frac{q_1 q_2}{c^2 r_{12}^3} v_1 \times \left[v_2 \times (\vec{r}_1 - \vec{r}_2) \right]$

$\vec{F}_{1 \text{ on } 2} = \frac{q_2 q_1}{c^2 r_{12}^3} v_2 \times \left[v_1 \times (\vec{r}_2 - \vec{r}_1) \right]$

$$\vec{v}_1 \times \left(\vec{v}_2 \times \vec{r}\right) = \vec{v}_2\left(\vec{v}_1 \cdot \vec{r}\right) - \vec{r}\left(\vec{v}_1 \cdot \vec{v}_2\right)$$

$$\vec{v}_2 \times \left(\vec{v}_1 \times (-\vec{r})\right) = -\vec{v}_1\left(\vec{v}_2 \cdot \vec{r}\right) + \vec{r}\left(\vec{v}_2 \cdot \vec{v}_1\right)$$

$\therefore$ They are equal and opposite only when

$$\vec{v}_2\left(\vec{v}_1 \cdot \vec{r}\right) - \vec{v}_1\left(\vec{v}_2 \cdot \vec{r}\right) = 0$$

or $\vec{r} \times \left(\vec{v}_2 \times \vec{v}_1\right) = 0$

$\therefore$ either $\vec{v}_1 \parallel \vec{v}_2$ or $\vec{r} \perp$ (plane of $\vec{v}_1$ & $\vec{v}_2$)

One must add ~~particle~~ field momentum to the particle momentum to get a final balance.

~~current carrying~~

For force between circuits carrying steady currents

$$\vec{F}_{2\text{on}1} = \frac{I_1}{c}\int d\vec{l}_1 \times \vec{B}_2(\vec{r}_1)$$

$$= \frac{I_1 I_2}{c^2}\oint_{C_1} d\vec{l}_1 \times \oint_{C_2}\left(\frac{d\vec{l}_2 \times (\vec{r}_1 - \vec{r}_2)}{r_{12}^3}\right)$$

$$\vec{F}_{2\text{on}1} = \frac{I_1 I_2}{c^2}\left[\oint_{C_1} d\vec{l}_2 \oint_{C_1} \frac{d\vec{l}_1 \cdot (\vec{r}_1 - \vec{r}_2)}{|r_1 - r_2|^3} - \int_{1,2}\frac{d\vec{l}_1 \cdot d\vec{l}_2}{r_{12}^3}(\vec{r}_1 - \vec{r}_2)\right]$$

$$\oint d\vec{l}_2 \oint d\vec{l}_1 \cdot \frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3} = -\oint d\vec{l}_2 \int d\vec{l}_1 \cdot \nabla_1 \frac{1}{|r_1 - r_2|} = 0$$

$$\int d\vec{S} \cdot \nabla \times \vec{A} = \int d\vec{l} \cdot \vec{A}$$

Magnetic Multipoles

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{\vec{j}(\vec{r}')}{|\vec{r}-\vec{r}'|}\, d\vec{r}'$$

$\nabla\cdot\vec{j}=0$ Expand $\vec{A}(\vec{r})$ in a power series in r'/r

$$\vec{A}(\vec{r}) = \frac{1}{c}\sum_l \frac{(-1)^l}{l!}\int \vec{j}(\vec{r}')(\vec{r}'\cdot\nabla)^l \frac{1}{r}\, d\vec{r}'$$

$l=0$

$$\vec{A}^{(0)}(\vec{r}) = \frac{1}{c}\frac{1}{r}\int \vec{j}(\vec{r}')\, d\vec{r}'$$

This is zero because

$$\int x_i' \frac{\partial j_k}{\partial x_k'}\, d\vec{r}' = 0 \quad *$$

Integrating by parts $x_i' j_k - \int \frac{\partial x_i'}{\partial x_k'} j_k(\vec{r}')\, d\vec{r}' = -\int j_i(\vec{r}')\, d\vec{r}'$

∴ no vector potential of order $\frac{1}{r}$ and no magnetic field of order $\frac{1}{r^2}$ for large r. Not related to absence of magnetic monopole

$l=1$

$$A^{(1)}(\vec{r}) = -\frac{1}{c}\int d\vec{r}'\, \vec{j}(\vec{r}')\,\vec{r}'\cdot\nabla\frac{1}{r}$$

$$A_i^{(1)}(\vec{r}) = -\frac{1}{c}\int d\vec{r}'\, j_i(\vec{r}')\, x_k' \frac{\partial}{\partial x_k}\frac{1}{r}$$

Decompose $j_i x_k'$ into symmetric & antisymmetric parts

$$j_i x_k' = \frac{1}{2}\left(j_i x_k' + j_k x_i'\right) + \frac{1}{2}\left(j_i x_k' - j_k x_i'\right)$$

First term integrates to 0.

$$0 = \int x_\ell' x_k' \frac{\partial}{\partial x_i} j_i\, d\vec{r}' = -\int \left(j_\ell x_k' + j_k x_\ell'\right) d\vec{r}'$$

Second term gives

$$A_i^{(1)}(\vec{r}) = -\frac{1}{c}\int d\vec{r}' \left(\frac{j_i(\vec{r}')\, x_k - j_k(\vec{r}')\, x_i}{2} \right) \frac{\partial}{\partial x_k} \frac{1}{r}$$

Define magnetic moment density

$$\vec{M}(\vec{r}') = \frac{\vec{r}' \times \vec{j}(\vec{r}')}{2c}$$

$$\frac{1}{2}(\vec{r}' \times \vec{j}) \times \nabla = \frac{1}{2}\left[-\vec{r}'\, \vec{j} \cdot \nabla + \vec{j}\, \vec{r}' \cdot \nabla \right]$$

$$A_i^{(1)}(\vec{r}) = -\int d\vec{r}'\, M(\vec{r}') \times \nabla\left(\frac{1}{r}\right)$$

$$= -\vec{m} \times \nabla\left(\frac{1}{r}\right)$$

$\vec{m}$ is the magnetic moment

$$m = \int \vec{M}(\vec{r}')\, d\vec{r}'$$

Magnetic field of the moment is

$$\vec{B}^{(1)} = \nabla \times \vec{A}^{(1)} = -\nabla \times \left(\vec{m} \times \nabla \frac{1}{r} \right)$$

$$= -\vec{m}\,\overrightarrow{\left(\nabla^2 \frac{1}{r}\right)}^{0} + (\vec{m} \cdot \nabla)\nabla \frac{1}{r}$$

$$= \nabla\left(\vec{m} \cdot \nabla \frac{1}{r} \right)$$

$$= -\nabla \frac{\vec{m} \cdot \vec{r}}{r^3}$$

We get a magnetic potential

$$\phi^*(\vec{r}) = \frac{\vec{m} \cdot \vec{r}}{r^3}$$

In general $\nabla \times \vec{B} \neq 0$, so no potential. Outside current carrying region, define a potential (magnetic pseudo potential) ϕ^* such that

$$\vec{B} = -\nabla \phi^*$$

e.g. of magnetic dipoles

1. Circuit carrying current I

$$\vec{M} = \frac{1}{2c} \int \vec{r} \times \vec{j} \, d\vec{r} = \frac{I}{2c} \oint_C \vec{r} \times d\vec{l}$$

Stokes theorem

$$\vec{M} = \frac{I}{2c} \vec{A}$$

where $\vec{A} = \int d\vec{S}$

2. Point particle of charge q, velocity v in orbit, radius $\vec{r}_p$

$$\vec{M} = \frac{1}{2c} \int \vec{r} \times \vec{j} \, d\vec{r} = \frac{1}{2c} \int q \, \delta(\vec{r} - \vec{r}_p) \, \vec{r} \times \vec{v} \, d\vec{r}$$

$$\vec{M} = \frac{q}{2c} \vec{r}_p \times \vec{v} = \frac{q}{2mc} \vec{L}$$

where $\vec{L}$ is the angular momentum of particle.

General term

$$A_i^{\ell}(\vec{x}) = \frac{(-1)^{\ell}}{c}\,\frac{1}{\ell!}\int d\vec{x}'\, j_i(\vec{x}')\, x'_{i_1}\cdots x'_{i_\ell}\,\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_\ell}}\,\frac{1}{r}$$

$$\phi^{*(\ell)}(\vec{x}) = \frac{(-1)^{\ell}}{c}\,\frac{1}{\ell!}\int d\vec{x}'$$

$$= -\frac{2\ell}{(\ell+1)!}\,(-1)^{\ell}\int d\vec{x}'\,(\vec{x}'\cdot\nabla)^{\ell-1}\,\frac{\vec{M}(\vec{x})\cdot\vec{x}}{r^3}$$

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General multipole term

$$0 = -\int x_k' \frac{\partial}{\partial x_{i_1}'} j_i(\vec{x}') x_{i_1}' x_{i_2}' \cdots x_{i_\ell}' \, d\vec{x}'$$

$$= \int \left[j_k x_{i_1}' \cdots x_{i_\ell}' + x_k' j_{i_1} x_{i_2}' \cdots x_{i_\ell}' + \cdots \right] d\vec{x}'$$

~~$A_k^\ell(\vec{x}) \frac{(-1)^\ell}{c\,\ell!} \int d\vec{x}' \left[j_k x_{i_1}' \cdots x_{i_\ell}' + x_k' j_{i_1} x_{i_2}' \cdots x_{i_\ell}' + \cdots \right] d\vec{x}'$~~

$$A_k^\ell(\vec{x}) = \frac{(-1)^\ell}{c\ell!} \int d\vec{x}' \left[j_k x_{i_1}' \cdots x_{i_\ell}' - \frac{1}{\ell+1} \left\{ j_k x_{i_1}' \cdots x_{i_\ell}' + x_k' j_{i_1} x_{i_2}' \cdots x_{i_\ell}' + \cdots \right\} \right]$$

$$\times \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_\ell}} \frac{1}{x}$$

The last ℓ terms are all equal because of the symmetry of the tensor $\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_\ell}} \frac{1}{x}$.

Therefore

$$A_k^\ell(\vec{x}) = \frac{\ell(-1)^\ell}{c(\ell+1)!} \int d\vec{x}' \left(j_k x_{i_1}' - x_k' j_{i_1} \right) x_{i_2}' \cdots x_{i_\ell}' \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_\ell}} \frac{1}{x}$$

$$x_k' j_{i_1} - x_{i_1}' j_k = 2c \epsilon_{k i_1 p} M_p \qquad \left(\vec{M} = \frac{\vec{x} \times \vec{j}}{2c} \right)$$

$$A_k^\ell(\vec{x}) = -\frac{2\ell(-1)^\ell}{(\ell+1)!} \int d\vec{x}' \epsilon_{k i_1 p} M_p x_{i_2}' \cdots x_{i_\ell}' \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_\ell}} \frac{1}{x}$$

$$B_i^\ell = (\nabla \times \vec{A}^\ell)_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k^\ell$$

$$= -\frac{2\ell(-1)^\ell}{(\ell+1)!} \int d\vec{x}' \epsilon_{ijk} \epsilon_{k i_1 p} M_p x_{i_2}' \cdots x_{i_\ell}' \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_\ell}} \frac{1}{x}$$

$$\epsilon_{ijk}\epsilon_{ki_1p} = \delta_{ii_1}\delta_{jp} - \delta_{ip}\delta_{ji_1}$$

So

$$B_i^{\ell} = -\frac{2\ell(-1)^{\ell}}{(\ell+1)!}\int d\vec{x}'\left\{\frac{\partial}{\partial x_i} M_j x'_{i_2}\cdots x'_{i_\ell}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_{i_2}}\cdots\frac{\partial}{\partial x_{i_\ell}}\frac{1}{r}\right.$$

$$\left. - M_i x'_{i_2}\cdots x'_{i_\ell}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_{i_2}}\cdots\frac{\partial}{\partial x_{i_\ell}}\frac{1}{r}\right\}$$

Second term is 0 because $\nabla^2\frac{1}{r} = 0$

$$\phi^{*\ell}(\vec{x}) = \frac{2\ell(-1)^{\ell}}{(\ell+1)!}\int d\vec{x}'\, M_j(\vec{x}')\cdot x'_{i_2}\cdots x'_{i_\ell}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_{i_2}}\cdots\frac{\partial}{\partial x_{i_\ell}}\frac{1}{r}$$

Current loop

$$\vec{M}_{loop} = \frac{I}{c}\oint\frac{\vec{x}'\times d\vec{l}'}{2}$$

$$\oint_C \vec{V}\cdot d\vec{l} = \int_S \nabla\times\vec{V}\cdot d\vec{S}$$

$$\frac{1}{2}\oint\vec{a}\cdot\vec{x}\times d\vec{l} = \frac{1}{2}\oint\vec{a}\times\vec{x}\cdot d\vec{l}$$

$$= \frac{1}{2}\int_S d\vec{S}\cdot\nabla\times(\vec{a}\times\vec{x})$$

$$= \frac{1}{2}\int_S d\vec{S}\cdot\left[(\nabla\cdot\vec{x})\vec{a} - (\vec{a}\cdot\nabla)\vec{x}\right]$$

$$= \frac{1}{2}\int_S d\vec{S}\cdot\left[3\vec{a} - \vec{a}\right]$$

$$= \vec{a}\cdot\int_S d\vec{S}$$

$$\therefore \vec{M}_{loop} = \frac{I}{c}\int d\vec{S}'$$

Magnetic fields & matter

$$\vec{B} = \int f(\vec{x}-\vec{x}')\, b(\vec{x}')\, d\vec{x}'$$

$$\nabla \times \vec{b} = \frac{4\pi}{c} j$$

$$\nabla \cdot \vec{b} = 0$$

$$\Rightarrow \quad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \langle \vec{j} \rangle$$

Applied magnetic field will induce a magnetic dipole density, the dipole moment per unit volume $\vec{M}$. Vector potential due to $\vec{M}$ will be

$$\vec{A}_M = \int d\vec{x}' \, \nabla \frac{1}{|\vec{x}-\vec{x}'|} \times M(\vec{x}')$$

$$= \int \frac{d\vec{x}'}{|\vec{x}-\vec{x}'|} \nabla' \times M(\vec{x}')$$

$$-\nabla' \times \left(\frac{M(\vec{x}')}{|\vec{x}-\vec{x}'|} \right) + \frac{\nabla' \times M(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$-\nabla' \cdot \left(\frac{M(\vec{x}') \times \vec{a}}{|\vec{x}-\vec{x}'|} \right)$$

$$-\int d\vec{S} \cdot \frac{M(\vec{x}') \times \vec{a}}{|\vec{x}-\vec{x}'|}$$

$$-\int \frac{d\vec{S} \times \vec{M}}{|\vec{x}-\vec{x}'|} \cdot \vec{a}$$

$\nabla' \times \vec{M}(\vec{x}') \leftrightarrow \frac{\langle \vec{j} \rangle}{c}$ as $-\vec{\nabla} \cdot \vec{P} \leftrightarrow \rho$

Surface terms neglected.

$$\langle \vec{j} \rangle = c \nabla \times \vec{M} + \vec{j}_f$$

where $\vec{j}_f$ is the averaged conduction current of moving charges. Calling $\vec{j}_f = \vec{j}$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} + 4\pi \nabla \times \vec{M}$$

Analogy $B \leftrightarrow E$, $D \leftrightarrow H$, $\nabla \cdot \leftrightarrow \nabla \times$, $\rho \leftrightarrow j/c$

$$\vec{D} = \vec{E} + 4\pi \vec{P}$$

Analogous to $\vec{D}$, define $\vec{H} = \vec{B} - 4\pi \vec{M}$

$$\nabla \times \vec{H} = \frac{4\pi}{c} \vec{j}$$

$\nabla \cdot \vec{B} = 0$ $\xrightarrow{\text{Gauss theorem}}$ B_{normal} is continuous

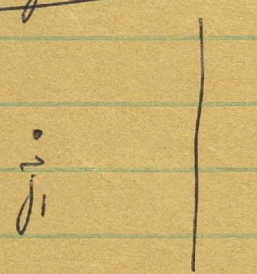
$\nabla \times H = \frac{4\pi}{c} \vec{j}$ $\xrightarrow{\text{Stokes theorem}}$ $\Delta H_{\text{tangential}} = \frac{4\pi}{c} K$ (surface current)

If $\vec{B}$ is a linear function of $\vec{H}$

$B_i = \mu_{ij} H_j$, μ_{ij} is permeability tensor

Isotropic case $\mu_{ij} = \mu \delta_{ij}$

Image currents



$\vec{j}$ and $\vec{j}_2$ in same place

$\vec{j} + \vec{j}_1$ for fields on right

$\vec{j}_2$ for fields on left

Diamagnetic susceptibility

Arises from Lenz's law. Order of magnitude estimate

$$\vec{\mu} = \frac{q\vec{L}}{2mc}$$

$\vec{L}$ is angular momentum which will have a component proportional to B.

$$L \propto m R^2 \omega \qquad \omega = \omega_c \sim \frac{qB}{mc}$$

$$\mu \sim \frac{q}{2mc}\, m R^2 \frac{q}{mc} B$$

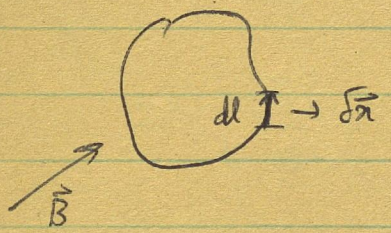
$$= \frac{q^2}{2mc^2} R^2 B$$

$$\therefore \chi_D \sim \frac{e^2}{mc^2} a_B^2 \sim \alpha^2 a_B^3$$

$$\frac{e^2}{mc^2} \Big/ a_B = \left(\frac{1}{137}\right)^2 \quad \therefore \chi_D \text{ very small}$$

$\chi_{paramagnetic} \sim \frac{1}{T} a_B^3$ Small compared to electric values except at low T.

Motional emf and electromagnetic induction



Element $d\vec{l}$ is displaced $\delta\vec{x}$ in time δt

$$\vec{v} = \frac{\delta\vec{x}}{\delta t}$$

If charge in element is q, $F_q = \frac{q}{c}\frac{\delta\vec{x}}{\delta t} \times \vec{B}$

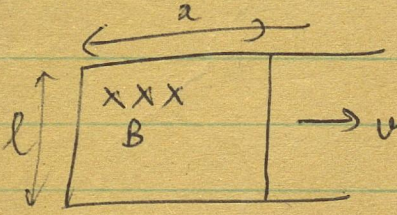
$$\vec{E}_{ef} = \frac{1}{c}\frac{\delta\vec{x}}{\delta t} \times \vec{B}$$

This equivalent electric field will drive more current through the circuit.

$$\mathcal{E}_{ef} = \frac{1}{c}\int \frac{\delta\vec{x}}{\delta t} \times \vec{B} \cdot d\vec{l}$$

Define flux $\Phi = \int_S \vec{B} \cdot d\vec{s}$ independent of surface used to calculate as long as S is bounded by C.

$\nabla \cdot \vec{B} = 0$



$$\mathcal{E} = -\frac{Blv}{c} = -\frac{1}{c}\frac{d\Phi}{dt} \quad \text{since } \Phi = Bxl$$

$$\frac{1}{c}\frac{d\Phi}{dt} = Bl\frac{dx}{dt} = Blv$$

In general,

$$\Phi = \int \vec{B} \cdot d\vec{s} = \int \nabla \times \vec{A} \cdot d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

Change from contour C_1 $(x_1(\tau))$ to contour C_2 $(x_2(\tau))$

Parametrize integration path with τ

$x_{1i} = x_i(\tau)$ $\quad 0 \le \tau \le 1 \quad x_i(1) = x_i(0)$

$x_{2i} = x_{0i}(\tau) + \delta x_i(\tau)$

$$dl_i = \frac{dx_i}{d\tau}\, d\tau$$

$$\vec{r} = \vec{r}(\tau)$$

$$\delta\Phi = \Phi_2 - \Phi_1$$

$$= \int_0^1 \left[\vec{A}_0(\vec{r}+\delta\vec{r}) \cdot \left(\frac{d\vec{r}}{d\tau} + d\,\frac{\delta\vec{r}}{d\tau} \right) - \vec{A}(\vec{r}) \cdot \frac{d\vec{r}}{d\tau} \right] d\tau$$

Tensor notation

$$= \int_0^1 \left\{ \left[A_i(\vec{r}) + \frac{\partial A_i}{\partial x_j}\,\delta x_j \right] \left(\frac{dx_i}{d\tau} + \frac{d}{d\tau}\,\delta x_i \right) d\tau - A_i(\vec{r})\,\frac{dx_i}{d\tau}\,d\tau \right\}$$

$$= \int_0^1 d\tau \left[A_i \frac{d}{d\tau}\,\delta x_i + \frac{\partial A_i}{\partial x_j}\,\delta x_j \frac{dx_i}{d\tau} \right]$$

Integrate by parts

$$= \int d\tau \left\{ \frac{d}{d\tau}\left[A_i\,\delta x_i \right] - \delta x_i \frac{dA_i}{d\tau} + \frac{\partial A_i}{\partial x_j}\,\delta x_j \frac{dx_i}{d\tau} \right\}$$

$A_i\,\delta x_i \to 0$ on loop

change $i \to j$ $\quad \delta x_j \frac{dA_j}{d\tau} = \delta x_j \frac{\partial A_j}{\partial x_i} \frac{dx_i}{d\tau}$

$$= \int d\tau\, \frac{dx_i}{d\tau}\,\delta x_j \left[\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right]$$

$$= \int \delta x_j\, \epsilon_{mji}\, B_m\, dx_i \qquad \epsilon_{mji}\,\delta x_j\, dx_i = (\delta\vec{x} \times d\vec{x})_m$$

$$\therefore \quad \delta\Phi = \int (\delta\vec{x} \times d\vec{x}) \cdot \vec{B} = \int \vec{B} \cdot \delta\vec{x} \times d\vec{l}$$

$$\delta\Phi = -\int \delta\vec{x} \times \vec{B} \cdot d\vec{l}$$

$$\therefore \quad \frac{1}{C}\frac{\delta\Phi}{\delta t} = -\frac{1}{C}\oint \frac{\delta\vec{x}}{\delta t} \times \vec{B} \cdot d\vec{l} = -\mathcal{E}$$

$$\therefore \quad \boxed{\mathcal{E} = -\frac{1}{C}\frac{d\Phi}{dt}}$$

$$-\frac{1}{C}\int_S d\vec{S} \cdot \frac{\partial \vec{B}}{\partial t} = \oint_C \vec{E} \cdot d\vec{l} = \int_S d\vec{S} \cdot \nabla \times \vec{E}$$

$$\therefore \quad \nabla \times E = -\frac{1}{C}\frac{\partial \vec{B}}{\partial t}$$

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Magnetic energy & force

Rate of energy loss by magnetic field acting on current

$$-\frac{dW}{dt} = \int \vec{E}\cdot\vec{j}\, d\vec{r}$$

$$= \frac{c}{4\pi}\int \vec{E}\cdot\nabla\times\vec{H}\, d\vec{r}$$

$$= \frac{c}{4\pi}\int \vec{E}\times\nabla\cdot\vec{H}\, d\vec{r}$$

$$= -\frac{c}{4\pi}\int \vec{E}\times\overleftarrow{\nabla}\cdot\vec{H}\, d\vec{r}$$

$$= \frac{c}{4\pi}\int (\nabla\times\vec{E})\cdot\vec{H}\, d\vec{r}$$

$$= -\frac{1}{4\pi}\int \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\, d\vec{r}$$

If system is linear & permeability tensor symmetric

$$W = \frac{1}{8\pi}\int \vec{H}\cdot\vec{B}\, d\vec{r}$$

System of conducting circuits, non-magnetic media

$$W = \frac{1}{8\pi}\int \vec{B}^2 d\vec{r}$$

$$= \frac{1}{8\pi}\int \nabla\times\vec{A}\cdot\vec{B}\, d\vec{r}$$

$$= \frac{1}{8\pi}\int \vec{A}\cdot\nabla\times\vec{B}\, d\vec{r}$$

$$= \frac{1}{2c}\int \vec{A}\cdot\vec{j}\, d\vec{r}$$

Analogues

charge → current
potential → flux

For circuit

$$W = \frac{1}{2c} \sum_i I_i \oint_{C_i} \vec{A} \cdot d\vec{l} \,.$$

$$= \frac{1}{2c} \sum_i I_i \Phi_i \qquad \text{using Stokes' theorem}$$

I_i is current & Φ_i is flux

analogue to charge | analogue to potential

Fluxes and currents are linear functions

$$\Phi_i = c \sum_j L_{ij} I_j = \oint d\vec{l}_i \cdot A(\vec{r}_i)$$

$$\vec{A}(\vec{r}_i) = \frac{1}{c} \sum_j I_j \oint \frac{d\vec{l}_j}{|\vec{r}_i - \vec{r}_j|}$$

$$= c \sum_j \frac{1}{c^2} \oint_i \oint_j \frac{d\vec{l}_i \cdot d\vec{l}_j}{|\vec{r}_i - \vec{r}_j|} I_j$$

L_{ij} are coefficients of induction. L_{ij} is defined such that the emf in C_i induced by a changing current in C_j

$$\mathcal{E}_i = -L_{ij} \frac{dI_j}{dt} \qquad \text{(no sum)}$$

$$W = \frac{1}{8\pi} \int \vec{B}^2 \, d\vec{x} = \frac{1}{2c} \sum_i I_i \Phi_i$$

$$= \frac{1}{2} \sum_{ij} I_i L_{ij} I_j$$

and with $GL = 1$

$$W = \frac{1}{c^2} \sum_{ij} \Phi_i G_{ij} \Phi_j$$

Diamagnetism

$$m_e \frac{d^2\vec{r}}{dt^2} = \vec{F}_e + \frac{e}{c}\vec{v}_e \times \vec{B}_0$$

Equation of motion for an electron in an atom placed in a constant magnetic field $\vec{B}_0$. $\vec{F}_e$ is the total remaining force on the eth electron, assumed to be velocity independent and angular momentum conserving. e.g. electrostatic.

What happens when we rotate the system about a point with angular velocity $\vec{\omega}$.

$$\vec{A} = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z \quad \text{components along rotating axes.}$$

Rotation of coordinate axes is defined as

$$\frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i} \qquad \frac{d\hat{j}}{dt} = \vec{\omega} \times \hat{j} \qquad \frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}$$

$$\boxed{\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A} + \frac{d\vec{A}}{d[t]}}$$

$$\frac{d\vec{A}}{d[t]} = \hat{i}\frac{dA_x}{dt} + \hat{j}\frac{dA_y}{dt} + \hat{k}\frac{dA_z}{dt}$$

$\frac{d\vec{A}}{d[t]}$ is the rate of change of $\vec{A}$ as seen by an observer in the rotating coordinate system

$$\therefore m_e\left(\frac{d^2\vec{r}}{d[t]^2} + 2\vec{\omega} \times \frac{d\vec{r}}{d[t]} + \vec{\omega} \times (\vec{\omega} \times \vec{r})\right) = \vec{F}_e + \frac{e}{c}\left[\frac{d\vec{r}}{d[t]} + \vec{\omega} \times \vec{r}\right] \times \vec{B}_0$$

Set $\vec{\omega} = \vec{\omega}_L = -\frac{e\vec{B}_0}{2m_e c}$ and neglect quadratic terms in $\vec{\omega}$ or $\vec{B}$.

$$m_e \frac{d^2 \vec{r}_e}{d[t]^2} = \vec{F}_e$$

Solution as if there were no field and no rotation.
Neglecting quadratic terms okay if $\omega_L \vec{r} \ll \frac{d\vec{r}}{d[t]}$ & $\omega_L \frac{d\vec{r}}{d[t]} \ll \frac{d^2\vec{r}}{d[t]^2}$
which is true for ordinary atoms.

Velocity of atoms is

$$\frac{d\vec{r}_e}{dt} = \frac{d\vec{r}_e}{d[t]} + \vec{\omega}_L \times \vec{r}_e$$

Angular momentum is

$$\vec{l}_e = m_e \vec{r}_e \times \frac{d\vec{r}_e}{dt} = m_e \vec{r}_e \times \frac{d\vec{r}_e}{d[t]} + m_e \vec{r}_e \times (\vec{\omega}_L \times \vec{r}_e)$$

Magnetic moment is

$$\vec{m} = \frac{e}{2m_e c} \vec{l}_e = \frac{e}{2c} \vec{r}_e \times \frac{d\vec{r}_e}{d[t]} + \frac{e}{2c} \vec{r}_e \times (\vec{\omega}_L \times \vec{r}_e)$$

We have taken the origin of $\vec{r}$ (i.e. center of rotation) at the center of the atom. Adding up all the electrons in the atom and averaging (assume first term is 0)

$$\langle \vec{m} \rangle = -\frac{e^2}{4m_e c^2} \sum_e \langle \vec{r}_e \times (\vec{B}_0 \times \vec{r}_e) \rangle$$

$$= -\frac{e^2}{4m_e c^2} \sum_e \langle r_e^2 \vec{B}_0 - (\vec{r}_e \cdot \vec{B}_0) \vec{r}_e \rangle$$

$$= -\frac{e^2}{6m_e c^2} \left(\sum_e \langle r_e^2 \rangle \right) \vec{B}_0$$

$$\therefore \alpha_D = -\frac{e^2}{6m_e c^2} \sum_e \langle r_e^2 \rangle$$

For an extended current $\vec{j}(\vec{r})$

$$W = \frac{1}{2c}\int \vec{A}\cdot\vec{j}\, d\vec{r}$$

$$= \frac{1}{2c^2}\int \vec{j}(\vec{r}\,') \frac{1}{|\vec{r}-\vec{r}\,'|}\cdot\vec{j}(\vec{r})\, d\vec{r}\, d\vec{r}\,'$$

Contribution to W from two separated circuits 1 & 2 is

$$W_{12} = \oint\oint \frac{\vec{dl}_1\cdot\vec{dl}_2}{|\vec{r}_1-\vec{r}_2|}\,\frac{I_1 I_2}{2c^2}$$

so that

$$L_{12} = \frac{1}{c^2}\oint\oint \frac{\vec{dl}_1\cdot\vec{dl}_2}{|\vec{r}_1-\vec{r}_2|}$$

<u>Generalized force</u> for a displacement $\delta\xi_k$

$$F_k = -\left.\frac{\delta W}{\delta\xi_k}\right|_{\text{constant }\Phi} = \left.\frac{\delta W}{\delta\xi_k}\right|_{\text{constant } I}$$

as in electrostatic case.

$$F_k = \frac{1}{2c^2}\sum_{ij} I_i I_j \frac{\delta L_{ij}}{\delta\xi_k}$$

same as

$$F_{2\text{on}1} = -\frac{I_1 I_2}{c^2}\oint_{1,2} \frac{\vec{dl}_1\cdot\vec{dl}_2}{r_{12}^3}(\vec{r}_1-\vec{r}_2)$$

Can find the total force on all currents in a region via the magnetic stress tensor. Total force in a volume V is

$$\vec{F} = \frac{1}{c}\int_V d\vec{r}\, \vec{j} \times \vec{B}$$

$$= \frac{1}{4\pi}\int (\nabla \times \vec{B}) \times \vec{B}\, d\vec{r}$$

Tensor notation

$$F_i = \frac{1}{4\pi}\int d\vec{r}\left(B_k \partial_k B_i - B_k \partial_i B_k\right)$$

Using $\partial_k B_k = \nabla \cdot B = 0$

$$F_i = \frac{1}{4\pi}\int d\vec{r}\, \partial_k \left(B_k B_i - \frac{\delta_{ik}}{2}\vec{B}^2\right)$$

Magnetic stress tensor

$$T_{ik}^M = \frac{1}{4\pi}\left(B_k B_i - \delta_{ik}\frac{\vec{B}^2}{2}\right)$$

and force on the currents in V

$$F_i = \int_S dS_k\, T_{ik}$$

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Maxwell's equations

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad \text{——— (1)}$$

$$\nabla . E = 4\pi\rho \quad \text{——— (2)}$$

$$\nabla . \vec{B} = 0 \quad \text{——— (3)}$$

$$\nabla \times \vec{B} = \frac{4\pi \vec{j}}{c} \quad \text{——— (4)}$$

(1) & (3) consistent

But (4) is inconsistent, if ρ is time varying.

$\nabla . \vec{j} \neq 0$ and (4) is wrong

Simplest correction, add a term $\frac{4\pi \vec{j}'}{c}$ to RHS of (4)

$$\nabla \times \vec{B} = \frac{4\pi}{c}(\vec{j} + \vec{j}')$$

such that $\nabla . (\vec{j} + \vec{j}') = 0$

$$\nabla . \vec{j} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla . \vec{j} = -\frac{\partial \rho}{\partial t}$$

$$\therefore \nabla . \vec{j}' - \frac{\partial \rho}{\partial t} = 0 \Rightarrow \nabla . \vec{j}' = \frac{\partial \rho}{\partial t}$$

from (2) $\frac{\partial \rho}{\partial t} = \frac{1}{4\pi}\frac{\partial}{\partial t}(\nabla . E) = \nabla . \vec{j}'$ (simplest assumption)

$$\therefore \frac{1}{4\pi}\nabla . \left(\frac{\partial E}{\partial t}\right) = \nabla . \vec{j}'$$

$$\Rightarrow \vec{j}' = \frac{1}{4\pi}\frac{\partial E}{\partial t} + \nabla \times \vec{Q}$$

Simplest assumption $\vec{Q} = 0$

$$\therefore \nabla \times \vec{B} = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi \vec{j}}{c} \quad \text{——— (4')}$$

Maxwell called $\vec{j}'$ the displacement current. Together with $\vec{j}$ the total current is "conserved" in the sense $\nabla\cdot(\vec{j}+\vec{j}') = 0$

$$\left.\begin{aligned} \frac{\partial \vec{E}}{\partial t} &= c\,\nabla\times\vec{B} - 4\pi\vec{j} \\ \frac{\partial \vec{B}}{\partial t} &= -c\,\nabla\times\vec{E} \end{aligned}\right\} \Rightarrow$$

specifying $\vec{B}, \vec{E}, \vec{j}$ as functions of position tells us how $\vec{B}$ and $\vec{E}$ evolve with time.

Also taking divergence of these equations

$$\nabla\cdot\frac{\partial \vec{E}}{\partial t} = -4\pi\,\nabla\cdot\vec{j} = 4\pi\frac{\partial \rho}{\partial t} \Rightarrow \frac{\partial}{\partial t}\left(\nabla\cdot\vec{E} - 4\pi\rho\right) = 0$$

$$\nabla\cdot\frac{\partial \vec{B}}{\partial t} = 0 \Rightarrow \frac{\partial}{\partial t}\left(\nabla\cdot B\right) = 0$$

so that if div equation hold at one time they hold at all times. These equations are the more fundamental dynamical eqns and the div equations enforce the boundary condition

Give E & B subject to $\nabla\cdot E = 4\pi\rho$ & $\nabla\cdot B = 0$, the other eqns predict past & future.

Momentum & Energy

Consider the force on charges and currents in the absence of matter.

$$F_i = \int_V d\vec{r}\left[\rho E_i + \frac{1}{c}(\vec{j}\times\vec{B})_i\right]$$

Substitute $\nabla\cdot\frac{\vec{E}}{4\pi}$ for ρ and $c\frac{\nabla\times\vec{B}}{4\pi} - \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}$ for $\vec{j}$

$$F_i = \int_V \frac{d\vec{r}}{4\pi}\left[E_i\partial_k E_k - \frac{1}{c}\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right)_i - \left[\vec{B}\times(\nabla\times\vec{B})\right]_i\right]$$

$$E_i\partial_k E_k = \partial_k(E_iE_k) - E_k(\partial_k E_i - \partial_i E_k) - E_k\partial_i E_k$$

$$\left[\vec{B}\times(\nabla\times\vec{B})\right]_i = B_k\partial_i B_k - \vec{B}\cdot\nabla B_i = \partial_i\frac{B_k^2}{2} - \partial_k(B_kB_i)$$

$$E_k(\partial_k E_i - \partial_i E_k) = -\left[\vec{E}\times(\nabla\times\vec{E})\right]_i = \left[\frac{1}{c}\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right]_i$$

$$F_i = \int_V \frac{d\vec{r}}{4\pi}\left[\partial_k(E_iE_k) - E_k\partial_i E_k - \frac{1}{c}\left[\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right]_i - \frac{1}{c}\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right)_i - \partial_i\frac{B_k^2}{2} + \partial_k(B_kB_i)\right]$$

$$= -\int_V \frac{d\vec{r}}{4\pi c}\left[\left(\frac{\partial\vec{E}}{\partial t}\times\vec{B}\right)_i + \left(\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right)_i\right] + \int d\vec{r}\,\partial_k\left[T^E_{ik} + T^M_{ik}\right]$$

$$\vec{p}_{EM} = \int_V d\vec{r}\,\frac{\vec{E}\times\vec{B}}{4\pi c}$$ time derivative of this

Transpose $-\frac{d\vec{p}_{EM}}{dt}$ to left and notice $\vec{F} = \frac{d\vec{p}_M}{dt}$, material momentum of charges & currents acted on by fields

$$\therefore \frac{d}{dt}\left[\vec{P}_{EM} + \vec{P}_{m}\right]_i = \int_S dS_k\, T_{ik}$$

Energy $\quad \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \nabla\times\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi\vec{j}}{c}$

$$\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t} + \vec{B}\cdot\frac{\partial\vec{B}}{\partial t}\right) = \vec{E}\cdot\left(c\nabla\times\vec{B} - 4\pi\vec{j}\right) + \vec{B}\cdot\left(-c\nabla\times\vec{E}\right)$$

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\vec{E}^2+\vec{B}^2\right) = -\vec{j}\cdot\vec{E} + \frac{c}{4\pi}\left(\vec{E}\cdot\nabla\times\vec{B} - \vec{B}\cdot\nabla\times\vec{E}\right)$$

$$\nabla\cdot(\vec{E}\times\vec{B}) = \nabla\times(\vec{E}\cdot\vec{B}) = (\nabla\times\vec{E})\cdot\vec{B} - (\vec{E}\times\vec{\nabla})\cdot\vec{B}$$

$$= (\nabla\times\vec{E})\cdot\vec{B} - \vec{E}\cdot(\nabla\times\vec{B})$$

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\vec{E}^2+\vec{B}^2\right) = -\vec{j}\cdot\vec{E} - \nabla\cdot\frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right)$$

$$W_F = \frac{1}{8\pi}\int d\vec{\tau}\left(\vec{E}^2+\vec{B}^2\right)$$

$$\frac{\partial W_F}{\partial t} = -\int_V \vec{j}\cdot\vec{E}\, d\vec{\tau} - \int_S d\vec{S}\cdot\vec{P}$$

$\vec{P}$ = Poynting vector $= \frac{c}{4\pi}\vec{E}\times\vec{B}$

Rate of work done on charges $(\vec{j}\cdot\vec{E})$ + rate of energy loss through surface surrounding V is rate of energy lost by field.

Electromagnetic fields in matter

$$\nabla \cdot \vec{E} = 4\pi \left(\langle \rho_f \rangle + \langle \rho_b \rangle \right) \qquad \langle \rho_f \rangle = \rho$$

$$\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \left(\langle \vec{j}_f \rangle + \langle \vec{j}_b \rangle \right) \qquad \langle \vec{j}_f \rangle = \vec{j}$$

Simplest consistent extension of earlier results

$$\langle \rho_b \rangle = - \nabla \cdot \vec{P}$$

$\vec{P}$ is electric dipole moment / unit vol.

If $\vec{P}$ is time-dependent, there must be a current j_b to balance the change of charge density.

$$\frac{\partial}{\partial t} \langle \rho_b \rangle = - \nabla \cdot \langle \vec{j}_b \rangle = - \nabla \cdot \frac{\partial \vec{P}}{\partial t}$$

$$\therefore \langle \vec{j}_b \rangle = \frac{\partial \vec{P}}{\partial t} + c \nabla \times \vec{M}$$

$\vec{M}$ is most simply magnetic dipole density / vol.

$$\therefore \nabla \cdot \vec{E} = 4\pi \rho - 4\pi \nabla \cdot \vec{P}$$

$$\boxed{\nabla \cdot \vec{D} = 4\pi \rho} \qquad \vec{D} = \vec{E} + 4\pi \vec{P}$$

$$\nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} \vec{j} + \frac{4\pi}{c} \frac{\partial \vec{P}}{\partial t} + \frac{4\pi}{c} \nabla \times \vec{M}$$

$$= \frac{1}{c} \frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c} \vec{j} + \frac{4\pi}{c} \nabla \times \vec{M}$$

$$\therefore \boxed{\nabla \times \vec{H} = \frac{1}{c} \frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c} \vec{j}} \qquad \vec{H} = \vec{B} - 4\pi \vec{M}$$

Boundary conditions

$\Delta D_{\text{normal}} = 4\pi\sigma$ (surface charge)

$\Delta B_{\text{normal}} = 0$

$\Delta E_{\text{tangential}} = 0$

$\Delta H_{\text{tangential}} = \frac{4\pi}{c} K$ (surface current)

12/2/97

EM fields in matter

Model of non-interacting atoms

$$\rho_b = \sum_n \rho_n (\vec{r} - \vec{r}_n, t)$$

ρ_n falls to 0 for $r >$ atomic radii

$$\int \rho_n(\vec{r})\, d\vec{r} = 0 \qquad \text{(neutral atoms)}$$

$$\vec{J}_b = \sum_n \vec{J}_n (\vec{r} - \vec{r}_n, t)$$

Average $\langle \rho_b \rangle = \sum_n \int f(\vec{r} - \vec{r}\,') \rho_n (\vec{r}\,' - \vec{r}_n)$

$\vec{r}\,' - \vec{r}_n = \vec{x} \qquad \vec{r}\,' = \vec{r}_n + \vec{x}$

$$\langle \rho_b \rangle = \sum_n \int f(\vec{r} - \vec{r}_n - \vec{x})\, \rho_n(\vec{x})\, d\vec{x}$$

Since x is restricted by ρ_n to be $\lesssim a_B$ but f varies on a scale that includes many atoms, we expand f in powers of $\vec{x}$

$$\langle \rho_b \rangle = \sum_n \int d\vec{x}\, \rho_n(\vec{x}) \left\{ f(\vec{r} - \vec{r}_n) - \vec{x} \cdot \vec{\nabla}_r f(\vec{r} - \vec{r}_n) + \dots \right\}$$

First term is 0 because of neutrality. Second term

$$= -\nabla \cdot \vec{P}(\vec{r}) \qquad \vec{P}(\vec{r}) = \sum f(\vec{r} - \vec{r}_n) \int d\vec{x}\, \vec{x}\, \rho_n(\vec{x})$$

$\vec{P}$: Average dipole moment per unit volume

$\vec{p}_n$ is the dipole moment of nth atom

For $\langle \vec{j}_b \rangle$, we carry out a number expansion

$$\langle \vec{j}_b \rangle = \sum_n \int f(\vec{r} - \vec{r}_n - \vec{x})\, \vec{j}_n(\vec{x})\, d\vec{x}$$

$$= \sum_n f(\vec{r} - \vec{r}_n) \int \vec{j}_n(\vec{x})\, d\vec{x} - \int \vec{x} \cdot \nabla_r \sum_n f(\vec{r} - \vec{r}_n)\, \vec{j}_n(\vec{x})\, d\vec{x}$$

Neglecting surface integrals

$$\int \nabla \cdot \vec{j}_n(\vec{x})\, \vec{x}\, d\vec{x} = \cancel{\vec{j}_n(\vec{x})\, \vec{x}}^{\,0} - \int \vec{j}_n(\vec{x})\, d\vec{x}$$

$$= + \int \frac{\partial \rho_n}{\partial t}\, \vec{x}\, d\vec{x}$$

$$= \frac{d\vec{p}_n}{dt}$$

$$\therefore \langle \vec{j}_{b_1} \rangle = \sum_n f(\vec{r} - \vec{r}_n)\, \frac{d\vec{p}_n}{dt} = \frac{\partial \vec{P}}{\partial t}$$

$$\langle \vec{j}_{b_2} \rangle = - \int \vec{x} \cdot \vec{\nabla}_r \sum_n f(\vec{r} - \vec{r}_n)\, j_n(\vec{x})\, d\vec{x}$$

$$\langle j_{b_2 i} \rangle = - \int x_k \frac{\partial}{\partial r_k} \sum_n f(\vec{r} - \vec{r}_n)\, j_{n_i}(\vec{x})\, d\vec{x}$$

$x_k j_i$ into symmetric & antisymmetric parts. Symmetric part will give rise to a quadrupole-like contribution, hence neglected

Antisymmetric part will give magnetic dipole density

$$\langle j_{b_2 i} \rangle = - \frac{1}{2} \sum_n \frac{\partial}{\partial r_k} f(\vec{r} - \vec{r}_n) \int (x_k j_i - x_i j_k)\, d\vec{x}$$

$$\langle \vec{j}_{b_2} \rangle = -\frac{1}{2} \sum_n \int \left(\vec{x} \cdot \nabla_x f \vec{j} - \vec{j} \cdot \nabla_x f \, \vec{x} \right) d\vec{x}$$

$$= \frac{1}{2} \sum_n \nabla_x f \times \int \vec{x} \times \vec{j} \, d\vec{x}$$

$$= \sum_n \nabla_x f (\vec{x} - \vec{x}_n) \times c \vec{m}_n$$

mag ~~dipole~~ moment of nth atom

$$\therefore \langle \vec{j}_{b_2} \rangle = c \nabla \times \sum_n f(\vec{x} - \vec{x}_n) \vec{m}_n = c \nabla \times \vec{M}$$

$\vec{M}$ = magnetic dipole moment/vol

$$\langle \rho_b \rangle = -\nabla \cdot \vec{P} \qquad \langle \vec{j}_b \rangle = \frac{\partial \vec{P}}{\partial t} + c \nabla \times \vec{M}$$

Energy

$$\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} = c \vec{E} \cdot (\nabla \times \vec{H}) - \vec{E} \cdot 4\pi \vec{j}$$

$$\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = -c \vec{H} \cdot (\nabla \times \vec{E})$$

$$\frac{1}{4\pi} \left[\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right] = -\vec{j} \cdot \vec{E} + \frac{c}{4\pi} \left(\vec{E} \cdot \nabla \times \vec{H} - \vec{H} \cdot \nabla \times \vec{E} \right)$$

$$= -\vec{j} \cdot \vec{E} - \frac{c}{4\pi} \vec{\nabla} \cdot (\vec{E} \times \vec{H})$$

$\vec{P}$ = Poynting vector = $\frac{c}{4\pi} \vec{E} \times \vec{H}$

- Frequency dependent dielectric constant

1. Fourier expansion

a. Reality

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dw\, e^{-iwt} g(w)$$

$$g(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, e^{iwt} f(t)$$

If f is real

$$f(t) = f^*(t) = \frac{1}{\sqrt{2\pi}} \int e^{iwt}\, dw\, g^*(w)$$

$$= \frac{1}{\sqrt{2\pi}} \int dw\, e^{-iwt} g^*(-w)$$

$$\therefore g(w) = g^*(-w)$$

b. Completeness Find an expression for singular function $\delta(t-t')$ by substituting.

$$f(t) = \frac{1}{2\pi} \int dw\, e^{-iwt} \int dt' e^{iwt'} f(t')$$

$$= \int dt' f(t') \underbrace{\left\{ \frac{1}{2\pi} \int dw\, e^{-iw(t-t')} \right\}}$$

$$= \delta(t-t')$$

$$\int f_1^*(t) f_2(t)\, dt = \int \frac{dt}{2\pi} \int dw\, e^{-iwt} g_1^*(w) \int dw'\, e^{-iw't} g_2(w')$$

$$= \int g_1^*(w) g_2(w)\, dw = \int g_1^*(w) g_2(w)\, dw$$

c. Causality

Suppose $f(t) = 0$ for $t < 0$. Then

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{i\omega t} f(t)\, dt$$

is an analytic function of ω in the upper half complex ω plane, the limit as $\mathrm{Im}\,\omega \to 0$ from above is $g(\omega)$ for real ω

Consider an electric field $E_i(t)$ with Fourier transform $e_i(\omega)$. If $E_i(t) = 0$ for $t < 0$, $e_i(\omega)$ is analytic in upper half ω plane. If polarization linearly related to $E_i(t)$, then in a time independent (passive) situation

$$P_i(t) = \int dt'\, \chi_{ij}(t-t')\, E_j(t')\, dt'$$

Causality put in by requiring $\chi_{ij}(t-t') = 0$ for $t < t'$

~~$P_i(t) = \frac{1}{\sqrt{2\pi}} \int$~~ $E_j(t') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t'} e_j(\omega)$

$$P_i(t) = \int dt' \frac{1}{\sqrt{2\pi}} \int \chi_{ij}(t-t')\, d\omega\, e^{-i\omega t'} e_j(\omega)$$

$$= \int d\omega\, e^{+i\omega t} e_j(\omega) \underbrace{\int \frac{dt'}{\sqrt{2\pi}}\, e^{-i\omega(t-t')}\, \chi_{ij}(t-t')}_{\Psi_{ij}(\omega)}$$

$$= \int d\omega\, e^{-i\omega t} P_i(\omega)$$

$\therefore P_i(\omega) = e_j(\omega)\, \Psi_{ij}(\omega)$

Causality requires $\chi_{ij}(t-t') = 0$ $t < t'$ $\therefore \Psi_{ij}(\omega)$ must be analytic for $\mathrm{Im}\,\omega > 0$.

$$\frac{\delta W}{\delta t} = \frac{1}{4\pi} \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\int_{-\infty}^{\infty} dt \, \frac{\delta W}{\delta t} = \frac{1}{4\pi} \int_{-\infty}^{\infty} \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \, dt$$

Parseval's theorem $\int dt \, H(t) \, K(t) = \int d\omega \, h^*(\omega) \, k(\omega)$

$$\int_{-\infty}^{\infty} dt \, \frac{\delta W}{\delta t} = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \, e_i^*(\omega) \, (-i\omega) \, \epsilon_{ij}(\omega) \, e_j(\omega)$$

$e_i = R_i + i I_i$

$$= \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \, (-i\omega) \, (R_i + i I_i) \, \epsilon_{ij} \, (R_j + i I_j)$$

$$\Delta W = \frac{1}{4\pi} \int_{-\infty}^{\infty} d\omega \, (-i\omega) \left[(R_i R_j + I_i I_j) \epsilon_{ij} + i (R_i I_j - R_j I_i) \epsilon_{ij} \right]$$

Let $\epsilon_{ij} = \epsilon_{ij}^S + \epsilon_{ij}^A$

$$\Delta W = \frac{2}{4\pi} \int_0^{\infty} d\omega \, \omega \left[(R_i R_j + I_i I_j) \, \mathrm{Im} \, \epsilon_{ij}^S + (R_i I_j - R_j I_i) \, \mathrm{Re} \, \epsilon_{ij}^A \right]$$

only +ve ω

$\therefore$ absorption due to $\mathrm{Im} \, \epsilon_{ij}^S$ (standard) but also $\mathrm{Re} \, \epsilon_{ij}^A$.

Feb 14, 1997

Spatially displaced response

For ordinary systems the characteristic length is a_B

$$P_i(\vec{x}) = \int \chi_{ij}(\vec{x}-\vec{x}')\, E_j(\vec{x}')\, d\vec{x}'$$

This goes to zero rapidly for $|\vec{x}-\vec{x}'| > a_B$. Since spatial averaging is over many atoms, response looks local

$$P_i(\vec{x}) = \chi_{ij}(\vec{x})\, E_j(\vec{x})$$

Can safely ignore spatially displaced response except for anomalously long correlation lengths. These occur in superfluids and near critical points. We can easily deal with visible light whose wavelengths are many atomic diameters. (page 308 Jackson). In conductors at low temperatures, mean free path is much longer than skin depth. Ohms law is a function of $\vec{k}$ & ω, nonlocal. Called Anomalous skin ~~depth~~ effect.

Free fields in isotropic materials

Isotropic materials with scalar dielectric constant $\epsilon(\omega)$, permeability $\mu(\omega)$ and conductivity $\sigma(\omega)$

All quantities depend on time like $e^{-i\omega t}$

$$\nabla \times \vec{H} = \frac{1}{c}\left(-i\omega \vec{D} + 4\pi \vec{j}\right)$$

$$\nabla \times \vec{E} = i\frac{\omega}{c}\vec{B}$$

for $\omega \neq 0$, constraint equations are automatically satisfied.

$$\vec{j} = \sigma \vec{E}, \quad \vec{D} = \epsilon \vec{E} \quad \text{and} \quad \vec{H} = \frac{\vec{B}}{\mu}$$

$$\nabla \times \frac{\vec{B}}{\mu} = \frac{1}{c}\left(-i\omega\epsilon + 4\pi\sigma\right)\vec{E}$$

$$= -\frac{i\omega}{c}\left(\epsilon + \frac{4\pi i\sigma}{\omega}\right)\vec{E}$$

We see that conductivity is equivalent to an imaginary part of ϵ with a pole at $\omega = 0$. Assuming $\frac{4\pi\sigma}{\omega}$ to be included in ϵ

$$\nabla \times \frac{\vec{B}}{\mu} = -i\frac{\omega\epsilon}{c}\vec{E}$$

We can either eliminate $\vec{E}$ or $\vec{B}$ and find propagation equations for $\vec{E}$ & $\vec{B}$ alone. Using $\vec{B} = \frac{c}{i\omega}(\nabla \times \vec{E})$

$$\nabla \times \left(\frac{\nabla \times \vec{E}}{\mu}\right) = \frac{\omega^2}{c^2}\epsilon\vec{E}$$

or using $\nabla \times E$ expression in $\nabla \times \frac{\vec{B}}{\mu}$

$$\nabla \times \frac{1}{\epsilon}\left(\frac{\nabla \times \vec{B}}{\mu}\right) = \frac{\omega^2}{c^2}\vec{B}$$

Monochromatic em signal in a linear, isotropic, material medium

The equations are equivalent. If $\vec{E}$ is a solution of (1), then $\vec{B} = \text{const.}\ \nabla \times \vec{E}$ satisfies (2) and if $\vec{B}$ satisfies (2) then $\vec{E} = \text{const.}\ \frac{1}{\epsilon}\left(\nabla \times \frac{\vec{B}}{\mu}\right)$ satisfies (1).

If we consider homogeneous media, so that ϵ & μ are spatial constants

$$\nabla^2 \vec{E} = -\frac{\omega^2}{c^2}\epsilon\mu\,\vec{E} \qquad \text{with } \nabla\cdot\vec{E} = 0$$

and $$\nabla^2 \vec{B} = -\frac{\omega^2}{c^2}\epsilon\mu\,\vec{B} \qquad \text{with } \nabla\cdot\vec{B} = 0$$

and $$\vec{B} = \frac{c}{i\omega}\nabla\times\vec{E} \quad \& \quad E = -\frac{c}{i\omega}\frac{\nabla\times\vec{B}}{\epsilon\mu}$$

For definite wave number $\vec{k}$ (plane wave)

$$\vec{E} = \vec{c}\, e^{i\vec{k}\cdot\vec{x}}$$

$$\vec{B} = \vec{b}\, e^{i\vec{k}\cdot\vec{x}}$$

$$\omega^2 = \frac{c^2k^2}{\epsilon\mu}\,, \qquad \vec{b} = \frac{c\vec{k}\times\vec{c}}{\omega}\,, \qquad \vec{c} = -\frac{c}{\epsilon\mu\omega}\,\vec{k}\times\vec{b}$$

Thus the solution, for given $\vec{k}$ & ω, with $\frac{c^2k^2}{\epsilon\mu} = \omega^2$, is determined by the vector $\vec{c}$ which must be TRANSVERSE ie. $\vec{c}\cdot\vec{k} = 0$ Two direction or modes

Most general $\vec{e}_\lambda = \vec{e}_1 + i\vec{e}_2$

$$\vec{E} = \mathrm{Re}\left[(\vec{e}_1 + i\vec{e}_2)\, e^{i\omega t}\right] = \vec{e}_1 \cos\omega t + \vec{e}_2 \sin\omega t$$

If $\vec{e}_1 \parallel \vec{e}_2$, plane polarized

$\vec{e}_1 \perp \vec{e}_2$ and mag. equal circularly polarized

Otherwise elliptically polarized

General soln of propagation equation

$$\vec{E}(\vec{x},t) = \int d\omega\, e^{-i\omega t} \int_{k^2 = \frac{\epsilon\mu\omega^2}{c^2}} d\vec{k} \sum_{\lambda_i} \vec{e}_{\lambda_i} e^{i(\vec{k}\cdot\vec{x} - \omega t)} a(\vec{k}, \omega, \lambda_i)$$

$$\vec{B}(\vec{x},t) = \int d\omega\, e^{-i\omega t} \int_{k^2 = \frac{\epsilon\mu\omega^2}{c^2}} d\vec{k}\, e^{i(k\cdot x - \omega t)} \frac{c\vec{k}}{\omega} \times \sum_{\lambda_i} \vec{e}_{\lambda_i} a(\vec{k}, \omega, \lambda_i)$$

Consider single frequency signal

$$f(t) = \mathrm{Re}\left(f_0 e^{-i\omega t}\right)$$

$$g(t) = \mathrm{Re}\left(g_0 e^{-i\omega t}\right)$$

Time integrated flux like $\int f(t) g(t)\, dt$ will be ∞ but averages okay

$$\mathrm{Re}\, f = (\mathrm{Re}\, f_0)\cos\omega t + (\mathrm{Im}\, f_0)\sin\omega t$$

$$\mathrm{Re}\, g = (\mathrm{Re}\, g_0)\cos\omega t + (\mathrm{Im}\, g_0)\sin\omega t$$

$$\overline{(\mathrm{Re}\, f \cdot \mathrm{Re}\, g)} = \frac{1}{2}\left[\mathrm{Re}\, f_0\, \mathrm{Re}\, g_0 + \mathrm{Im}\, f_0\, \mathrm{Im}\, g_0\right]$$

$$= \frac{1}{2}\mathrm{Re}\left(f_0^* g_0\right)$$

$$\vec{E} = \mathrm{Re}\, e_\lambda e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

$$\vec{B} = \mathrm{Re}\, \frac{c\vec{k}}{\omega} \times e_\lambda e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

Energy density

$$u = \epsilon \frac{\vec{E}^2}{8\pi} + \frac{\vec{B}^2}{8\pi\mu}$$

$$\langle u \rangle = \frac{1}{2} \cdot \frac{1}{8\pi} \left\{ \epsilon \vec{e}_\lambda^{\,*} \cdot \vec{e}_\lambda + \frac{c^2 k^2}{\omega^2 \mu} \vec{e}_\lambda^{\,*} \cdot \vec{e}_\lambda \right\}$$

$$= \frac{1}{8\pi} \epsilon \, \vec{e}_\lambda^{\,*} \cdot \vec{e}_\lambda \qquad \text{(equal contributions)}$$

Energy flux

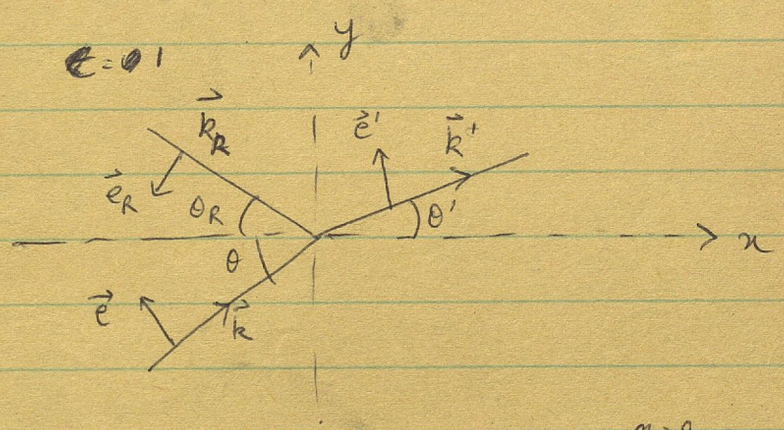
$$\langle \vec{P} \rangle = \frac{c}{4\pi} \langle \vec{E} \times \vec{H} \rangle$$

$$= \frac{1}{2} \frac{c}{4\pi} \vec{e}_\lambda^{\,*} \cdot \frac{\vec{e}_\lambda \vec{k}}{\omega\mu}$$

$$= \frac{\epsilon}{8\pi} \vec{e}_\lambda^{\,*} \cdot \vec{e}_\lambda \frac{c\vec{k}}{\epsilon\mu\omega} = \langle u \rangle \cdot \frac{c}{\sqrt{\epsilon\mu}} \vec{k}$$

Reflection & Reflaction

$n = \sqrt{\epsilon\mu} = \pm(n_1 + i n_2)$ $\qquad \frac{n_2}{n_1} > 0$



Boundary conditions at interface, $n=0$. Therefore phase is continuous across boundary. k_y & k_z are continuous

$$\omega^2 = c^2k^2 = \frac{c^2 k'^2}{\epsilon\mu} \qquad \therefore \quad k_x'^2 + k_y^2 = (k_x^2 + k_y^2)\epsilon\mu$$

$$\therefore \quad k_y = k\sin\theta \qquad k_y' = k'\sin\theta'$$

$$\frac{\sin\theta'}{\sin\theta} = \frac{k}{k'} = \frac{1}{\sqrt{\epsilon\mu}} \qquad \text{if } \epsilon\mu \text{ is real}$$

$$k_y = k\sin\theta \qquad k_{Ry} = k_R \sin\theta_R = k\sin\theta_R$$

$$\therefore \quad \theta = \theta_R$$

For complex $\epsilon\mu$, we just take $\frac{4\pi\sigma}{\omega} \gg 1$

$$\epsilon \sim \frac{4\pi i\sigma}{\omega}$$

$$k_x'^2 \sim k^2 \frac{4\pi i\sigma}{\omega}\mu$$

$$k_x' \sim \frac{1+i}{\sqrt{2}} k \sqrt{\frac{4\pi\sigma}{\omega}}\,\mu$$

All fields damp out like $e^{-ik\mu x\sqrt{\frac{2\pi\sigma}{\omega}}}$

Beyond Skin depth

$$\delta = \frac{1}{\mu k}\sqrt{\frac{\omega}{2\pi\sigma}}$$

fields go to 0.

If $\mathrm{Im}\,\epsilon\mu$ is small

$$\left(\mathrm{Re}\,k_x' + \mathrm{Im}\,k_x'\right)^2 + k_y^2 = k^2\epsilon\mu$$

$$\mathrm{Im}\,k_x' \simeq \frac{k^2}{2\,\mathrm{Re}\,k_x'}\,\mathrm{Im}\,\epsilon\mu$$

$$= \frac{k}{2}\,\frac{\mathrm{Im}\,\theta'}{\cos\theta'\sin\theta}\,\mathrm{Im}\,\epsilon\mu$$

$\mu = 1$

$$\vec{E} = e^{i(k_x x + k_y y - \omega t)}\,\vec{e} + c\,e^{i(-k_x x + k_y y - \omega t)}\,\vec{e}_R \qquad x < 0$$

$$\vec{E} = e^{i(k_x' x + k_y y - \omega t)}\,\vec{e}'$$

$\vec{B}$

$$i\frac{\omega}{c}\vec{B} = \nabla \times \vec{E}$$

$$\vec{B} = \frac{c\vec{k}}{\omega} \times \vec{E}$$

$$b_z = \frac{c}{\omega}\left(k_x e_y - k_y e_x\right)$$

$k_x = k\cos\theta \qquad k_y = k\sin\theta \qquad e_x = -e\sin\theta \qquad e_y = e\cos\theta$

$$b_z = \frac{ck}{\omega}e$$

$$b_{R_z} = \frac{ck}{\omega} e_R$$

$$b'_z = \frac{c}{\omega}\left(k'_x e'_y - k'_y e'_x\right)$$

E_y cont $\qquad e\cos\theta - e_R\cos\theta = e'_y$

$$k'_x e'_x + k'_y e'_y = 0$$

B_z cont $\qquad k(1+e_R) = k'_x e'_y - k'_y e'_x$

$$e_R = \frac{k'^2\cos\theta - kk'_x}{k'^2\cos\theta + kk'_x}$$

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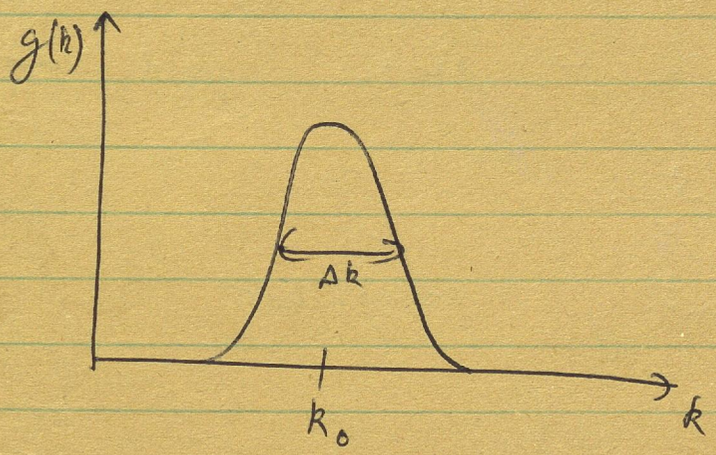
$$E(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)\, e^{i(kx - \omega(k)t)}\, dk$$

$$E(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)\, e^{ikx}\, dk$$

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(x,0)\, e^{-ikx}\, dx$$



$k_0,\ k_0 \pm \frac{\Delta k}{2}$

1, 1/2

$$E(x) = \frac{g(k_0)}{\sqrt{2\pi}} \left[e^{ik_0 x} + \frac{1}{2} e^{i\left(k_0 - \frac{\Delta k}{2}\right)x} + \frac{1}{2} e^{+i\left(k_0 + \frac{\Delta k}{2}\right)x} \right]$$

$$= \frac{g(k_0)}{\sqrt{2\pi}}\, e^{ik_0 x} \left\{ 1 + \cos \frac{\Delta k}{2} x \right\}$$

Max when $x = 0$. Waves in phase interfere constructively

Stationary phase for general case

$$g(k) = |g(k)|\, e^{i\alpha(k)}$$

$$\alpha(k) \approx \alpha(k_0) + (k-k_0)\left.\frac{d\alpha}{dk}\right|_{k=k_0}$$

$$E(x,0) = \frac{e^{i(k_0 x + \alpha(k_0))}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |g(k)|\, e^{i(k-k_0)(x-x_0)}\, dk$$

$$x_0 = -\left[\frac{d\alpha}{dk}\right]_{k=k_0}$$

x_0 ~~[illegible]~~ $x \gg x_0$ or $x \ll x_0$, rapid variation in phase

if $x \simeq x_0$, slow variation

$x_0 = -\left.\frac{d\alpha}{dk}\right|_{k=k_0}$ is center of wave packet

Phase velocity $v_\varphi = \frac{\omega}{k} = c$

Group velocity

$$v_\varphi = \frac{c}{n(k)}$$

$$E(x,t) = \frac{g(k_0)}{\sqrt{2\pi}} \left\{ e^{i(k_0 x - \omega_0 t)} + \frac{1}{2} e^{i\left[\left(k_0 - \frac{\Delta k}{2}\right)x - \left(\omega_0 - \frac{\Delta\omega}{2}\right)t\right]} + \frac{1}{2} e^{i\left[\left(k_0 + \frac{\Delta k}{2}\right)x - \left(\omega_0 + \frac{\Delta\omega}{2}\right)t\right]} \right\}$$

$$= \frac{g(k_0)}{\sqrt{2\pi}}\, e^{i(k_0 x - \omega_0 t)} \left[1 + \cos\left(\frac{\Delta k}{2} x - \frac{\Delta\omega}{2} t \right) \right]$$

$x_M(t) = \frac{\Delta\omega}{\Delta k} t$ not at $\frac{\omega_0}{k_0} t$

$\alpha(k) \rightarrow \alpha(k) - \omega(k) t$ $\qquad g(k) \rightarrow g(k) e^{-i(\omega/k)t}$

$x_M(t) = \left.\frac{d\omega}{dk}\right|_{k=k_0} t - \left.\frac{d\alpha}{dk}\right|_{k=k_0}$

$\therefore v_g|_{k_0} = \left.\frac{d\omega}{dk}\right|_{k_0}$

At any point phase difference

$\phi_{rel} = \phi_{fast} - \phi_{slow}$

$= \frac{\omega L}{c} (n_{fast} - n_{slow})$

Propagation in anisotropic media

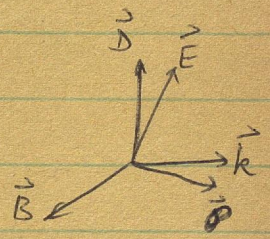
Assume $\epsilon_{ij} = \epsilon_{ji}^*$ i.e. real dielectric tensor & symmetric

$\mu = 1$

$$\frac{\omega}{c}\vec{b} = \vec{k}\times\vec{e} \qquad \text{——— ①}$$

$$\vec{k}\times\vec{b} = -\frac{\omega}{c}\vec{d} \qquad \text{——— ②}$$

$$d_i = \epsilon_{ij} e_j$$



$\vec{k}, \vec{d}$ & $\vec{b}$ form a right handed coordinate system

$\vec{e}, \vec{b}$ & $\vec{S} = \frac{c}{4\pi}\vec{e}\times\vec{b}$ form another R.H. coordinate system

$\vec{k}, \vec{E}$ & $\vec{d}$ are all orthogonal to $\vec{b}$ and are coplanar.

∴ direction of energy flow & that of $\vec{k}$ are not necessarily the same.
Exception is ordinary ray of monoaxial crystal. Substitute ① in ②.

$$\left[\vec{k}\times(\vec{k}\times\vec{e})\right]_i = -\frac{\omega^2}{c^2}\epsilon_{ij} e_j$$

$$k^2 e_i - \vec{k}\cdot\vec{e}\, k_i = \frac{\omega^2}{c^2}\epsilon_{ij} e_j$$

Diagonalize ϵ_{ij} in some coordinate axes x_α.

$$\sum_\beta \epsilon_{\alpha\beta} e_\beta = \epsilon_\alpha e_\alpha \quad \text{(no sum)}$$

where ϵ_α are the ~~eigenfunctions of~~ eigenvalues of ϵ_{ij}, considered to be known functions of ω. In the x_α coordinates

$$\vec{k}\cdot\vec{e}\, k_\alpha - \vec{k}^2 e_\alpha = -\frac{\omega^2}{c^2}\epsilon_\alpha e_\alpha \quad \text{(no sum)}$$

If $k_\alpha = e_\alpha$, $\omega = 0$ — trivial.
other 2 solutions?

$$e_\alpha = \frac{\vec{k}\cdot\vec{e}\; k_\alpha}{k^2 - \frac{\omega^2}{c^2}\epsilon_\alpha}$$

Multiply by k_α and sum over α

$$\vec{k}\cdot\vec{e} = \sum_\alpha k_\alpha e_\alpha = \vec{k}\cdot\vec{e}\sum_\alpha \frac{k_\alpha^2}{k^2 - \frac{\omega^2}{c^2}\epsilon_\alpha}$$

For $\vec{k}\cdot\vec{e} \neq 0$, $\sum_\alpha \frac{k_\alpha^2}{k^2 - \frac{\omega^2}{c^2}\epsilon_\alpha} = 1$

$$\sum_\alpha \frac{k_\alpha^2}{k^2} = 1$$

$$\therefore \sum_\alpha \left(\frac{k_\alpha^2}{k^2 - \frac{\omega^2}{c^2}\epsilon_\alpha} - \frac{k_\alpha^2}{k^2} \right) = 0$$

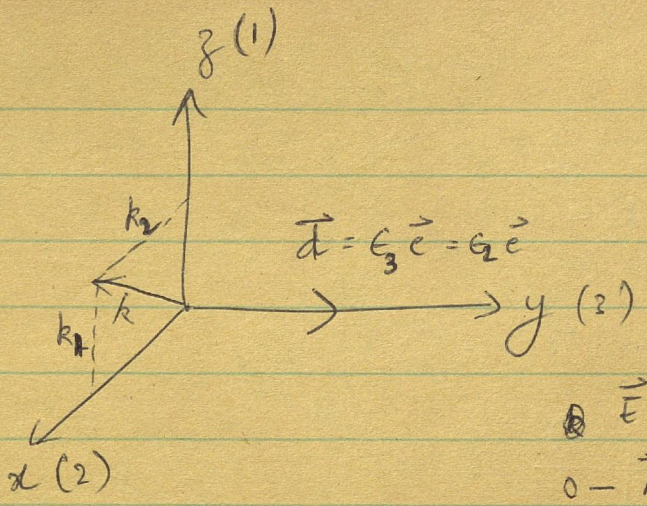
$$\sum_\alpha \frac{k_\alpha^2 \epsilon_\alpha}{k^2 - \frac{\omega^2}{c^2}\epsilon_\alpha} = 0$$

Multiplying by $\left(k^2 - \frac{\omega^2}{c^2}\epsilon_1\right)\left(k^2 - \frac{\omega^2}{c^2}\epsilon_2\right)\left(k^2 - \frac{\omega^2}{c^2}\epsilon_3\right)$ gives a quadratic equation for ω with 2 roots. Ignoring dispersion over the width of the wavepacket.

Monoaxial crystal — $\epsilon_x = \epsilon_y = \epsilon_2$; $\epsilon_z = \epsilon_1 \neq 0$

Triaxial crystal — $\epsilon_x \neq \epsilon_y \neq \epsilon_z \neq 0$

→ Choose optic axis to be in z direction. Any pair of orthogonal axes in the x-y plane are eigenvectors of ϵ matrix and we can construct a solution of ① & ② by choosing $\vec{k}$ in x-z plane, $\vec{e}$ along $\hat{y}$ axis, $\vec{b}$ perpendicular to $\vec{k}$ in x-z plane. Since $\hat{e}_y$ is an eigenvector of ϵ_{ij} with eigenvalue ϵ_1, $\vec{d}$ is also along y axis with value $\vec{d} = \epsilon_2 \vec{e}$



$\vec{E} \times \vec{H} \propto \vec{k}$

$\vec{k} \cdot \vec{e} = 0$ for this mode

$$\therefore \omega_0^2 = \frac{c^2 k^2}{\epsilon_1}$$

0 for ordinary. The direction of propagation energy flow is in the direction of $\vec{k}$, so the ordinary ray behaves as if it were propagating in an isotropic medium.

$\vec{E} = \vec{e}_y$

$$0 - \vec{k}^2 \vec{e}_y = -\frac{\omega^2}{c^2} \epsilon_2 \vec{e}_y$$

$$\omega^2 = \frac{c^2}{\epsilon_2} k^2 \qquad \omega = \frac{ck}{\sqrt{\epsilon_2}}$$

$$\omega \nabla_k \omega = \frac{c^2}{\epsilon_2} \vec{k}$$

$$v_g = \nabla_k \omega = \frac{c^2}{\omega \epsilon_2} \vec{k} = \frac{c^2}{\epsilon_2} \frac{\sqrt{\epsilon_2}}{ck} \vec{k} = \frac{c}{\sqrt{\epsilon_2}} \hat{k}$$

The extraordinary ray $\vec{d}_e$ will have $\vec{d}_e \perp$ to $\vec{k}$ and $\perp$ to $\vec{e}_o$

$$\frac{k_1^2 \epsilon_1}{k^2 - \frac{\omega^2}{c^2}\epsilon_1} + \frac{k_2^2 \epsilon_2}{k^2 - \frac{\omega^2}{c^2}\epsilon_2} = 0$$

$$k_1^2 \epsilon_1 \left(k^2 - \frac{\omega^2}{c^2}\epsilon_2\right) + k_2^2 \epsilon_2 \left(k^2 - \frac{\omega^2}{c^2}\epsilon_1\right) = 0$$

$$k^2 \left(k_1^2 \epsilon_1 + k_2^2 \epsilon_2\right) = \frac{\omega^2}{c^2} \epsilon_1 \epsilon_2 \left(k_1^2 + k_2^2\right) = \frac{\omega^2}{c^2} \epsilon_1 \epsilon_2 k^2$$

$$\frac{\omega^2}{c^2} = \frac{k_1^2 \epsilon_1 + k_2^2 \epsilon_2}{\epsilon_1 \epsilon_2}$$

$$\nabla_k \omega = \frac{c^2}{\omega \epsilon_1 \epsilon_2} \left(k_1 \epsilon_1 \hat{i}_1 + k_2 \epsilon_2 \hat{i}_2\right)$$

Fields

$$\vec{k} = k_1 \hat{i}_1 + k_2 \hat{i}_2$$

$$\vec{k} \cdot \vec{D} = 0 \qquad \vec{D} = \left(-\hat{i}_1 k_2 + \hat{i}_2 k_1\right) \epsilon_1 \epsilon_2$$

$$\vec{E} = \left(-\hat{i}_1 k_2 \epsilon_2 + \hat{i}_2 k_1 \epsilon_1\right)$$

$$\cos\left(\vec{k}, \vec{E}\right) = \frac{k_1 k_2 \left(\epsilon_1 - \epsilon_2\right)}{|k|\,|E|}$$

$$W_E = \frac{\vec{E} \cdot \vec{D}}{8\pi} = \frac{\epsilon_1 \epsilon_2}{8\pi}\left(k_2^2 \epsilon_2 + k_1^2 \epsilon_1\right)$$

$$W_H = \frac{H^2}{8\pi}$$

$$\vec{k} \times \vec{E} = \frac{\omega}{c} \vec{H} \qquad \vec{H} = \frac{c}{\omega} \vec{k} \times \vec{E}$$

$$\vec{H} = \frac{c}{\omega}\left(\hat{i}_1 k_1 + \hat{i}_2 k_2\right) \times \left(-\hat{i}_1 k_2 \epsilon_2 + \hat{i}_2 k_1 \epsilon_1\right)$$

$$= \frac{c}{\omega} \hat{i}_3 \left(k_1^2 \epsilon_1 + k_2^2 \epsilon_2\right)$$

$$\frac{\vec{H}^2}{8\pi} = \frac{c^2}{\omega^2}\left(k_1^2 \epsilon_1 + k_2^2 \epsilon_2\right)^2 = \frac{\epsilon_1 \epsilon_2}{8\pi}\left(k_1^2 \epsilon_1 + k_2^2 \epsilon_2\right)$$

$$\therefore W_E = W_B = \frac{\epsilon_1 \epsilon_2}{8\pi}\left(k_1^2 \epsilon_1 + k_2^2 \epsilon_2\right)$$

$$W = W_E + W_B = \frac{\epsilon_1 \epsilon_2 \left(k_1^2 \epsilon_1 + k_2^2 \epsilon_2\right)}{4\pi}$$

Energy flow $\vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{H}$

$$\vec{S} = \frac{c^2}{4\pi\omega}\left[\hat{i}_2 k_2 \epsilon_2 + \hat{i}_1 k_1 \epsilon_1\right]\left(k_1^2 \epsilon_1 + k_2^2 \epsilon_2\right)$$

$$\vec{v}_g = \nabla_k \omega = c^2 \frac{\hat{i}_1 k_1 \epsilon_1 + \hat{i}_2 k_2 \epsilon_2}{\epsilon_1 \epsilon_2 \omega} \qquad \therefore \quad \vec{S} = \vec{v}_g W$$

Mar 5, 1997

RADIATION

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \qquad \nabla \cdot \vec{E} = 4\pi\rho$$

Using $\nabla \cdot \vec{B} = 0$, there exists a vector potential such that $\vec{B} = \nabla \times \vec{A}$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}(\nabla \times \vec{A}) \Rightarrow \nabla \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0$$

$$\therefore \vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\nabla\phi$$ i.e. there exists a scalar potential

$\vec{E}$ & $\vec{B}$ from

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi$$

$\vec{A}$ & ϕ have usual gauge invariance ambiguity

$$\vec{A}' = \vec{A} + \nabla\chi$$

$$\phi' = \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}$$

leave $\vec{E}$ & $\vec{B}$ unchanged. Gauge chosen for calculational convenience.

Inhomogeneous eqns are

$$\nabla \cdot \vec{E} = -\frac{1}{c}\nabla \cdot \frac{\partial \vec{A}}{\partial t} - \nabla^2\phi = 4\pi\rho$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \nabla \times (\nabla \times \vec{A}) + \frac{1}{c}\left(\frac{1}{c}\frac{\partial^2 \vec{A}}{\partial t^2} + \nabla\frac{\partial \phi}{\partial t}\right) = \frac{4\pi}{c}\vec{j}$$

Radiation, Coulomb or transverse gauge

$$\nabla \cdot \vec{A} = 0$$

Start from any $\vec{A}_0$ and solve $\nabla \cdot \vec{A}_0 + \nabla^2 \chi = 0$

$$\nabla^2 \phi = -4\pi\rho$$

$$\therefore \quad \phi(\vec{r}, t) = \int d\vec{r}\,' \frac{\rho(\vec{r}\,', t)}{|\vec{r} - \vec{r}\,'|}$$

instantaneous connection. Violates causality. But no - when calculating field strength, $\vec{A}$ will make a contribution that cancels $\nabla\phi$.

$$\nabla \times (\nabla \times \vec{A}) + \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = \frac{4\pi \vec{j}}{c} - \frac{1}{c} \nabla \frac{\partial \phi}{\partial t}$$

Purely transverse

$$\nabla \cdot \left(\vec{j} - \frac{\nabla}{4\pi} \frac{\partial \phi}{\partial t} \right) = \nabla \cdot \vec{j} - \frac{\partial}{\partial t} \frac{\nabla^2 \phi}{4\pi}$$

$$= -\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial t}(-\rho) = 0$$

We see that $\frac{1}{4\pi} \nabla \frac{\partial \phi}{\partial t}$ is the long part of $\vec{j}$, so even with $\vec{j}$ & $\vec{\rho}$ spatially confined, long part falls off as $\frac{1}{r^2}$ as $\vec{r} \to \infty$. The contribution of this term to $\vec{A}$ and $\frac{\partial \vec{A}}{\partial t}$ at large $\vec{r}$ will have to cancel the unwanted contribution to $\vec{E}$ from $-\nabla\phi$.

Lorentz gauge

$$-\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} + \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial\phi}{\partial t} - \nabla\cdot\vec{A}\right) = 4\pi\rho$$

$$-\nabla^2\vec{A} + \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} + \nabla\left(\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t}\right) = \frac{4\pi\vec{j}}{c}$$

$$\therefore \quad \nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$$

Mar 10, 1997

Vector & Scalar potentials for radiation.

Gauge invariance

$$\left.\begin{aligned} \vec{A}' &= \vec{A} + \nabla\chi \\ \phi' &= \phi - \frac{1}{c}\frac{\partial\chi}{\partial t} \end{aligned}\right\} \text{ leave } \vec{B} \text{ \& } \vec{E} \text{ unchanged}$$

Radiation gauge $\quad \nabla\cdot\vec{A} = 0$

If $\nabla\cdot\vec{A}_0 \neq 0$, $\quad \vec{A} = \vec{A}_0 + \nabla\chi$

$$\nabla\cdot\vec{A} = \nabla\cdot\vec{A}_0 + \nabla^2\chi = 0 \quad \rightarrow \text{ Solve for } \chi$$

Lorentz gauge

$$\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$$

If $(\vec{A}_0, \phi_0)$ such that $\nabla\cdot\vec{A}_0 + \frac{1}{c}\frac{\partial\phi_0}{\partial t} \neq 0$, then $(\vec{A}, \phi)$ $\vec{A}_0 + \nabla\chi$, $\phi_0 - \frac{1}{c}\frac{\partial\chi}{\partial t}$

$$\nabla\cdot\left(\vec{A}_0 + \nabla\chi\right) + \frac{1}{c}\frac{\partial}{\partial t}\left(\phi_0 - \frac{\partial\chi}{\partial t}\right) = 0$$

$$\nabla^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = -\left(\nabla\cdot\vec{A}_0 + \frac{1}{c}\frac{\partial\phi_0}{\partial t}\right)$$

Solve for χ. Not unique because can always add any solution of

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\chi = 0$$

In Lorentz gauge need to solve

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \Box\vec{A} = -\frac{4\pi}{c}\vec{j}$$

$$\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \Box\phi = -4\pi\rho$$

Green's functions

General equation $\Box \psi = -4\pi f(\vec{r}, t)$

f is a source distribution given as a function of space & time.

Green's function $G(\vec{r}, t; \vec{r}', t')$ which gives

$$\psi = \int G(\vec{r}, t; \vec{r}', t') f(\vec{r}', t') \, d\vec{r}' \, dt'$$

satisfies the equation

$$\Box G = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r}, t; \vec{r}', t') = -4\pi \delta^3(\vec{r} - \vec{r}') \, \delta(t - t')$$

The solution is only a function of $\vec{r} - \vec{r}'$ and $t - t'$

$$\Box G(\vec{r}, t) = -4\pi \delta^3(\vec{r}) \, \delta(t)$$

Taking Fourier transforms

$$G(\vec{r}, t) = \int d\vec{k} \, d\omega \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \, g(\vec{k}, \omega)$$

$$\delta^3(\vec{r}) \, \delta(t) = \int \frac{d\vec{k} \, d\omega}{(2\pi)^4} \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G = \int \frac{d\vec{k} \, d\omega}{(2\pi)^4} \left(-\vec{k}^2 + \frac{1}{c^2}\omega^2\right) g \; e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\left(\frac{\omega^2}{c^2} - \vec{k}^2\right) g = -\frac{4\pi}{(2\pi)^4}$$

$$= -\frac{4\pi}{(2\pi)^4} \int d\vec{k} \, d\omega \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$g = -\frac{1}{4\pi^3} \, \frac{1}{\frac{\omega^2}{c^2} - \vec{k}^2}$$

$$G(\vec{r}, t) = -\frac{c^2}{4\pi^3} \int d\vec{k} \, e^{i\vec{k}\cdot\vec{r}} \int d\omega \, \frac{e^{-i\omega t}}{\omega^2 - c^2\vec{k}^2}$$

Causality is maintained if G vanishes for negative t, i.e. if we take ω to approach the real axis from above – RETARDED POTENTIAL.

Requiring G to vanish for $t<0$ is equivalent to saying that the radiation from a source delta function at $t=0$, $\vec{r}=0$ comes after the pulse.

Retarded

$$G_R(\vec{r},t) = -\lim_{\epsilon\to 0} \frac{c^2}{4\pi^3}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{r}} \int_{-\infty}^{\infty} \frac{d\omega\, e^{-i\omega t}}{(\omega+i\epsilon)^2 - c^2\vec{k}^2}$$

Advanced

$$G_A(\vec{r},t) = -\lim_{\epsilon\to 0} \frac{c^2}{4\pi^3}\int d\vec{k}\, e^{i\vec{k}\cdot\vec{r}} \int_{-\infty}^{\infty} \frac{d\omega\, e^{-i\omega t}}{(\omega-i\epsilon)^2 - c^2\vec{k}^2}$$

For G_R, $t>0$ close contour in lower half plane

$t<0$ close contour in upper half plane

$$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{d\omega\, e^{-i\omega t}}{(\omega+i\epsilon)^2 - c^2\vec{k}^2} = \begin{cases} -\frac{2\pi}{ck}\sin ckt & t>0 \\ 0 & t<0 \end{cases}$$

Integrate over angles of $\vec{k}$

$\vec{k}$, z axis along $\vec{r}$

$\vec{k}\cdot\vec{r} = kr\cos\theta$

$$G_R = \frac{c}{2\pi^2}\int \frac{d\vec{k}\, e^{i\vec{k}\cdot\vec{r}}}{k}\sin ckt$$

$$= \frac{c}{2\pi^2}\iint \frac{dk\, k\sin\theta\, d\varphi\, k\, d\theta\, e^{ikr\cos\theta}}{k}$$

$$= \frac{c}{2\pi^2}\int dk\, k \int_0^{\pi}\sin\theta\, d\theta\, e^{ikr\cos\theta}\int_0^{2\pi} d\varphi$$

$$= \frac{2\pi c}{2\pi^2}\int dk\, k\int_0^{\pi} e^{ikr\cos\theta}\, d(-\cos\theta)$$

$$= \frac{c}{\pi}\int dk\, k\, \frac{e^{ikr\cos\theta}}{ikr}\Big|_0^{\pi}$$

$$= \frac{c}{\pi r}\int dk\, \frac{e^{ikr} - e^{-ikr}}{i}$$

$$= \frac{2c}{\pi r}\int_0^{\infty} dk \sin kr \sin ckt$$

$$= \frac{c}{r}\left[\delta(r-ct) - \delta(r+ct)\right]$$

($\delta(r+ct) = 0$ for $r, t > 0$)

$$= \frac{c}{\pi r}\int_0^{\infty} dk\left[\cos k(r-ct) - \cos k(r+ct)\right]$$

$$= \frac{c}{2\pi r}\int_{-\infty}^{\infty} dk\left(e^{ik(r-ct)} - e^{ik(r+ct)}\right)$$

Green's function

$$G_R(\vec{r},t) = \frac{c}{r}\delta(r-ct)$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{c}{r}\delta(r-ct) = -4\pi\delta^3(\vec{r})\delta(t) \quad ?$$

$$\nabla^2\left(\frac{\delta(r-ct)}{r}\right) = f\nabla^2\frac{1}{r} + 2\nabla f.\nabla\frac{1}{r} + \frac{1}{r}\nabla^2 f \qquad f = \delta(r-ct)$$

$$= -4\pi\delta^3(\vec{r})f(0) + 2\nabla f.\left(-\frac{\vec{r}}{r^3}\right) + \frac{1}{r}\left(\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}\right)$$

$$= -4\pi\delta^3(\vec{r})f(0) + \frac{1}{c^2 r}\frac{\partial^2 f}{\partial t^2}$$

$$f(0) = \delta(-ct) = \frac{1}{c}\delta(t)$$

$$\therefore \nabla^2\left(\frac{\delta(r-ct)}{r}\right) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\delta(r-ct)}{r}\right) = -\frac{4\pi}{c}\delta^3(\vec{r})\delta(t)$$

$$\therefore \psi(\vec{r},t) = \int \frac{d\vec{r}'\,dt'}{|\vec{r}-\vec{r}'|}\,\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)f(\vec{r}',t')$$

$$\psi(\vec{r},t) = \int \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|}\,f(\vec{r}',t_R)$$

$t_R = t - \dfrac{|\vec{r}-\vec{r}'|}{c}$ is the time at which a signal must be emitted at $\vec{r}'$ to reach $\vec{r}$ at t

Imagine a spherical wave converging at the ~~point~~ time t on the point $\vec{r}$. The retarded time t_R is the time at which the wave crosses the point $\vec{r}'$, picking up its contribution from the source $f(\vec{r}',t_R)$ as it does so.

$$\vec{A} = \frac{1}{c}\int \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|}\,\vec{j}(\vec{r}',t_R) \quad \& \quad \phi = \int \frac{d\vec{r}'}{|\vec{r}-\vec{r}'|}\,\rho(\vec{r}',t_R)$$

Mar 12 1997

Radiation from a fixed frequency source:

$$\vec{j}(\vec{x},t) = \vec{j}(\vec{x})\, e^{-i\omega t}$$

$$\rho(\vec{x},t) = \rho(\vec{x})\, e^{-i\omega t}$$

and $\nabla\cdot\vec{j} - i\omega\rho = 0$

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int \frac{d\vec{x}'}{|\vec{x}-\vec{x}'|}\,\vec{j}(\vec{x}')\,e^{-i\omega t_R} = e^{-i\omega t}\int \frac{d\vec{x}'}{|\vec{x}-\vec{x}'|}\,\frac{\vec{j}(\vec{x}')}{c}\,e^{i\frac{\omega}{c}|\vec{x}-\vec{x}'|}$$

$$\phi(\vec{x},t) = \int \frac{d\vec{x}'}{|\vec{x}-\vec{x}'|}\,\rho(\vec{x}')\,e^{-i\omega t_R} = e^{-i\omega t}\int \frac{d\vec{x}'}{|\vec{x}-\vec{x}'|}\,\rho(\vec{x}')\,e^{i\frac{\omega}{c}|\vec{x}-\vec{x}'|}$$

In the wave zone, fields $\vec{E}$ & $\vec{B}$ go as $\frac{1}{r}$ for large r. This is the signature of radiation, so that

$$\vec{P}\cdot d\vec{S} \to \text{constant as } r\to\infty$$

$$= \frac{c}{4\pi}\,\vec{E}\times\vec{B}\cdot\hat{r}\, r^2 d\Omega$$

$$= \text{const indep}^{t} \text{ of } r.$$

Only keep leading term in $1/r$.

$$\frac{1}{|\vec{x}-\vec{x}'|} \simeq \frac{1}{r} \qquad \text{(wave zone)}$$

$$|\vec{x}-\vec{x}'| = \sqrt{r^2 - 2\vec{x}\cdot\vec{x}' + x'^2} = r\left(1 - \frac{2\vec{x}\cdot\vec{x}'}{r^2} + \frac{x'^2}{r^2}\right)^{1/2}$$

$$= r\left(1 - \frac{\vec{x}\cdot\vec{x}'}{r^2}\right) + O\left(\frac{x'^2}{r}\right)$$

$$e^{ik|\vec{r}-\vec{r}'|} = e^{ikr - ik\frac{\vec{r}\cdot\vec{r}'}{r} + i\,O\left(\frac{kr'^2}{r}\right)}$$

Last term can be neglected if $\frac{kr'^2}{r} \ll 1 \Rightarrow \lambda \gg \frac{r'^2}{r}$

$$\psi \simeq \frac{e^{i(kr-\omega t)}}{r}\int d\vec{r}'\, e^{-ik\hat{r}\cdot\vec{r}'} S_\omega(\vec{r}')$$

$$= \frac{e^{i(kr-\omega t)}}{r} S_{\omega,\vec{k}}$$

$$S_{\omega,\vec{k}} = \int d\vec{r}'\, e^{-i\vec{k}\cdot\vec{r}'} S_\omega(\vec{r}') \qquad \vec{k} = k\hat{r}$$

$\psi = \vec{A} \qquad S = \vec{j}/c$

$\psi = \phi \qquad S = \rho$

$$\vec{A} = \frac{e^{i(kr-\omega t)}}{r}\,\frac{\vec{j}_{\omega,\vec{k}}}{c}$$

$$\phi = \frac{e^{i(kr-\omega t)}}{r}\,\rho_{\omega,\vec{k}}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \nabla\phi = \frac{i\omega}{c^2 r}\vec{j}_{\omega,\vec{k}}\, e^{i(kr-\omega t)} - \frac{i\vec{k}}{r}\rho_{\omega,\vec{k}}\, e^{i(kr-\omega t)}$$

$$\vec{B} = \nabla\times\vec{A} = i\vec{k}\times\frac{\vec{j}_{\omega,\vec{k}}}{cr}\, e^{i(kr-\omega t)}$$

$$\frac{\partial \vec{A}}{\partial x} = \frac{e^{i(kr-\omega t)}}{cr} \vec{j}_{\omega,\vec{k}} \frac{\partial}{\partial x} ik\sqrt{x^2+y^2+z^2} \qquad \frac{2x}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{e^{i(kr-\omega t)}}{cr} i\vec{j}_{\omega,\vec{k}} \frac{kx}{r}$$

$$\frac{\partial \vec{A}}{\partial x_i} = \frac{e^{i(kr-\omega t)}}{cr} \vec{j}_{\omega,\vec{k}} \frac{ikx_i}{r}$$

$$\frac{\partial \phi}{\partial x_i} = \frac{e^{i(kr-\omega t)}}{r} \rho_{\omega,\vec{k}} \frac{ikx_i}{r}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi$$

$$= -\frac{1}{c}(-i\omega)\frac{e^{i(kr-\omega t)}}{cr}\vec{j}_{\omega,\vec{k}} - \frac{e^{i(kr-\omega t)}}{r}\rho_{\omega,\vec{k}}\frac{ik\vec{r}_i}{r}$$

$$= \left[\frac{i\omega}{c^2}\vec{j}_{\omega,\vec{k}} - i\vec{k}\rho_{\omega,\vec{k}}\right]\frac{e^{i(kr-\omega t)}}{r}$$

$$\vec{B} = \nabla \times \vec{A} = i\vec{k} \times \vec{j}_{\omega,\vec{k}} \frac{e^{i(kr-\omega t)}}{cr}$$

$\therefore$ $\vec{B}$ is transverse obviously.

$\nabla' \cdot \vec{j}_{\omega,\vec{k}} = 0$

$$= -i\vec{k} \cdot \int d\vec{r}\,' e^{-i\vec{k}\cdot\vec{r}\,'} \vec{j}(\vec{r}\,')$$

$$+ \int d\vec{r}\,' e^{-i\vec{k}\cdot\vec{r}\,'} \nabla' \cdot \vec{j}(\vec{r}\,')$$

$$= -i\vec{k} \cdot \vec{j}_{\omega,\vec{k}} + \int d\vec{r}\,' e^{-i\vec{k}\cdot\vec{r}\,'} i\omega\rho(\vec{r}\,')$$

$$= -i\vec{k} \cdot \vec{j}_{\omega,\vec{k}} + i\omega\rho_{\omega,\vec{k}}$$

$$\therefore \vec{k} \cdot \vec{j}_{\omega,\vec{k}} = \omega\rho_{\omega,\vec{k}}$$

$$\vec{E} = i\frac{e^{i(kr-\omega t)}}{r}\left[\frac{k}{c}\vec{j}_{\omega,\vec{k}} - \vec{k}\rho_{\omega,\vec{k}}\right]$$

$$\left[\frac{k}{c}\vec{j}_{\omega,\vec{k}} - \frac{k\hat{k}}{\omega}\vec{j}_{\omega,\vec{k}} \cdot \vec{k}\right]$$

$$\left(\vec{j}_{\omega,\vec{k}}\right)_{\perp} = \vec{j}_{\omega,\vec{k}} - \hat{k} \cdot \vec{j}_{\omega,\vec{k}}\,\hat{k}$$

$\therefore \vec{E}$ is also transverse.

Another way

$$\vec{k} \cdot \vec{E} = \vec{k} \cdot \left[\frac{i\omega}{c}\vec{A} - i\vec{k}\phi\right]$$

$$= \frac{i\omega}{c}\vec{k} \cdot \vec{A} - i\frac{\omega^2}{c^2}\phi$$

$$= i\frac{\omega}{c}\left[i\vec{k} \cdot \vec{A} - \frac{i\omega}{c}\phi\right] = 0 \text{ from Lorentz condition.}$$

$\vec{\nabla} \to i\vec{k}$

$\frac{\partial}{\partial t} \to -i\omega$

$\vec{k} = \frac{\omega}{c}\hat{r}$

$\vec{E}$ is transverse and second term (2nd term) is only cancelling longitudinal part of $\vec{j}_{\omega,\vec{k}}$ leaving only transverse part.

Average power radiated (over a cycle)

$$\frac{dW}{dt\,d\Omega} = \langle \vec{P} \cdot \hat{r}\, r^2 \rangle$$

$$= \frac{c}{4\pi} \langle \vec{E} \times \vec{B} \cdot \hat{r}\, r^2 \rangle$$

$$= \frac{c}{8\pi} \frac{\omega}{c^3} \vec{j}_T^{\,*} \times \left(\vec{k} \times \vec{j}_T \right) \cdot \hat{r}$$

$$= \frac{\omega}{8\pi c^2} \left[\vec{k} \left(\vec{j}_T^{\,*} \cdot \vec{j}_T \right) - \vec{j}_T \left(\vec{k} \cdot \vec{j}_T^{\,*} \right)^{0} \right] \cdot \hat{r}$$

$$= \frac{\omega^2}{8\pi c^3} \vec{j}_T^{\,*} \cdot \vec{j}_T \; \hat{k} \cdot \hat{r}$$

$$\vec{j}_T^{\,*} \cdot \vec{j}_T = \left(\vec{j}^{\,*} - \vec{j}^{\,*} \cdot \hat{r}\, \hat{r} \right) \cdot \left(\vec{j} - \vec{j} \cdot \hat{r}\, \hat{r} \right)$$

$$= \vec{j}^{\,*} \cdot \vec{j} - \left(\vec{j}^{\,*} \cdot \hat{r} \right) \left(\vec{j} \cdot \hat{r} \right)$$

Mar 14, 1997

Radiation by a slowly moving point particle

Slow $\Rightarrow \frac{v}{c} \ll 1$

$$\vec{j}(\vec{r},t) = q\,v(t)\,\delta(\vec{r}-\vec{r}(t))$$

$\vec{r}(t)$ is position and $v(t)$ is velocity at time t, q is charge

Fourier transform

$$\vec{j}(\vec{r},t) = \int d\omega\, e^{-i\omega t}\, \vec{j}_\omega(\vec{r})$$

From before

$$\vec{A}(\vec{r},t) = \int \frac{e^{-i\omega(t-r/c)}}{rc}\, d\omega \int \vec{j}_\omega(\vec{r}')\, e^{-i\vec{k}\cdot\vec{r}'}\, d\vec{r}'$$

$\vec{k} = \frac{\omega}{c}\hat{r}$

$e^{-i\vec{k}\cdot\vec{r}'} \simeq 1$ for slowly moving particle since $kr' \sim \frac{\omega}{c} r' \sim \frac{v}{c}$

$$\therefore \vec{A}(\vec{r},t) = \frac{1}{rc}\int d\omega\, e^{-i\omega(t-r/c)} \int \vec{j}_\omega(\vec{r}')\, d\vec{r}'$$

$$= \frac{1}{rc}\int \vec{j}(\vec{r}', t-\tfrac{r}{c})\, d\vec{r}'$$

$$\vec{A}(\vec{r},t) = \frac{1}{rc}\, q\,\vec{v}(t-r/c) + O\left(\tfrac{1}{r^2}\right)$$

$\therefore \vec{A}$ is given by radiating particle's velocity at its retarded time. Since $\vec{r}'$ is small, it does not matter which of the particles positions is associated with retarded time.

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}\bigg|_T = -\frac{q}{Rc^2}\,\vec{a}_T(t - R/c) + O(1/R^2)$$

where $\vec{a}_T = \vec{a} - \vec{a}\cdot\hat{R}\,\hat{R}$ is transverse component of $\vec{a}$.
$\hat{R}$ is direction of observation & $\vec{k} = \frac{\omega}{c}\hat{R}$.

$$\vec{B} = \nabla \times \vec{A} = \frac{q}{Rc^2}(-\hat{R}) \times \frac{\partial}{\partial R}\vec{v}(t - R/c)$$

$$= \hat{R} \times \vec{E}$$

$$\vec{P} = \frac{c}{4\pi}\,\vec{E} \times \vec{B}$$

$$= \frac{q^2}{4\pi R^2 c^3}\left\{\vec{a}_T(t - R/c)\right\}^2 \hat{R}$$

$$= \frac{q^2}{4\pi R^2 c^3}\left[\vec{a}^2 - (\vec{a}\cdot\hat{R})^2\right]\hat{R} \qquad [\ ] \text{ retarded time}$$

Average over cycle

$$\left\langle \frac{dW_{ED}}{dt\,d\Omega} \right\rangle = \frac{q^2}{4\pi c^3}\left[\langle \vec{a}^2 \rangle - \langle (\vec{a}\cdot\hat{R})^2 \rangle\right] \Rightarrow \text{indep of } R.$$

Total power radiated

$$\frac{dW_{ED}}{dt} = \int_S d\vec{S}\cdot\vec{P} = \frac{q^2}{4\pi c^3}\int d\Omega \langle \vec{a}^2 \rangle (1 - \cos^2\theta)$$

(θ: angle between trajectory dirn and observation dirn)

$$\langle \cos^2\theta \rangle = \frac{1}{3}$$

$$\therefore \frac{dW_{ED}}{dt} = \frac{2}{3}\frac{q^2}{c^3}\langle \vec{a}^2 \rangle$$

Larmor formula for dipole radiation (non-relativistic)

March 14, 1997

ED, MD, & EQ

$$\vec{J}_{\vec{k}} = \int d\vec{r}\,' e^{-i\vec{k}\cdot\vec{r}\,'} \vec{j}(\vec{r}\,')$$

$$\nabla\cdot\vec{j}(\vec{r}\,') = i\omega\rho(\vec{r}\,')$$

$$e^{-i\vec{k}\cdot\vec{r}\,'} = 1 - i\vec{k}\cdot\vec{r}\,' + \ldots$$

$$\vec{J}^{(1)}_{\vec{k}} = \int d\vec{r}\,' \vec{j}(\vec{r}\,')$$

Integrating by parts, for confined current

$$\vec{J}_{\vec{k}} = -\int d\vec{r}\,' \vec{r}\,' \nabla'\cdot\vec{j}(\vec{r}\,')$$

$$\vec{J}^{(1)}_{\vec{k}} = -i\omega \int \rho(\vec{r}\,')\,\vec{r}\,'\,d\vec{r}\,' = -i\omega\vec{P}_E$$

$$\vec{E}^{(1)} = \frac{i\omega}{c^2 r} e^{i(kr-\omega t)} (-i\omega)\left(\vec{P}_E - \vec{P}_E\cdot\hat{r}\,\hat{r}\right)$$

$$\vec{B}^{(1)} = \hat{r}\times\vec{E}^{(1)}$$

$$\frac{dW}{dt\,d\Omega} = \frac{\omega^4}{8\pi c^3}\left(|\vec{P}_E|^2 - \left(\vec{P}_E^{\,*}\cdot\hat{r}\right)\left(\vec{P}_E\cdot\hat{r}\right)\right)$$

Point particle $\vec{r}(t) = \mathrm{Re}\left(\vec{r}_0 e^{-i\omega t}\right)$

$$\vec{a}(t) = -\omega^2\,\mathrm{Re}\left(\vec{r}_0 e^{-i\omega t}\right)$$

$$\vec{P}_E = q\vec{r}_0$$

$$\vec{j}_{\vec{k}}^{(2)} = -i \int \vec{k} \cdot \vec{r}' \, \vec{j}(\vec{r}') \, d\vec{r}'$$

generally smaller by $kr' \sim \frac{v}{c}$

$$-i\vec{k} \cdot \vec{r}' \, j_\ell(\vec{r}') = -i k_i x_i' j_\ell(\vec{r}')$$

$$= -i k_i \left[\frac{x_i' j_\ell - x_\ell' j_i}{2} + \frac{x_i' j_\ell + x_\ell' j_i}{2} \right]$$

$$= \cancel{\text{[illegible]}}$$

$$\vec{j}_{\vec{k}\,\ell}^{(2)} = \cancel{\text{[illegible]}}$$

$$= \int -i\vec{k} \cdot \vec{r}' \, j_\ell(\vec{r}') = i \left[\vec{k} \times \int \frac{\vec{r} \times \vec{j}(\vec{r}')}{2} d\vec{r}' \right]_\ell + i \frac{k_i}{2} \int x_i' x_\ell' \nabla \cdot \vec{j} \, d\vec{r}'$$

$$= i \left(c\vec{k} \times \vec{p}_M \right)_\ell - \omega k_i \int x_i' x_\ell' \rho(\vec{r}') \, d\vec{r}'$$

$\vec{p}_M$ is magnetic dipole moment $\int \frac{\vec{r}' \times \vec{j}(\vec{r}')}{2c} d\vec{r}'$

$$Q_{E_{ij}} = \int \rho \left(x_i' x_j' - \frac{1}{3} \delta_{ij} \vec{r}'^2 \right) d\vec{r}'$$

The δ_{ij} term only produces a longitudinal contribution to $\vec{j}^{(2)}$ and does not contribute to radiation.

$$\vec{B} = i\vec{k} \times \vec{A} = \cancel{\text{[illegible]}} \frac{e^{i(kr - \omega t)}}{r} \frac{\omega^2}{c^2} \cancel{\text{[illegible]}} \vec{p}_M$$

ED → MD
$\vec{E} \to \vec{B}$
$\vec{p}_E \to \vec{p}_M$
$\vec{B} \to -\vec{E}$ ← − sign important if energy is to flow in correct direction

March 17, 1997

RELATIVITY

Invariance under

Translation in space	3	} 10 parameter group
Translation in time	1	
Rotation in space	3	
Uniform motion	3	

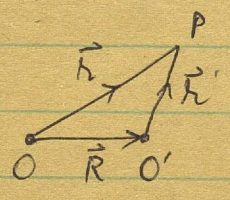
Newton's laws invariant under

$$m\frac{d^2x_i}{dt^2} = F_i$$

$$F_i = GMm\frac{x_i - x_m}{|\vec{r}-\vec{r}_m|^3}$$

$x_i \to x_i + u_i t$
$x_m \to x_m + u_i t$ } Galilean transformations

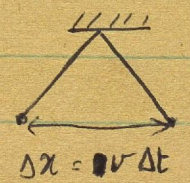
Maxwell's equations : EM waves propagate with velocity c



$$\vec{r} = \vec{r}' + \vec{R}$$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}'}{dt} + \frac{d\vec{R}}{dt}$$

∴ If this is to be consistent with same c in all frames, $\frac{d}{dt}$ is not absolute but according to observer.



$$c = \frac{2h}{\Delta t'} = \frac{2\sqrt{\left(\frac{v\Delta t}{2}\right)^2 + h^2}}{\Delta t}$$

$$1 = \frac{4}{c^2\Delta t^2}\left(\frac{v^2\Delta t^2}{4} + h^2\right) = \frac{v^2}{c^2} + \left(\frac{2h}{c\Delta t}\right)^2$$

$$\therefore \frac{2h}{c\Delta t} = \sqrt{1-v^2/c^2} \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\therefore \Delta t = \frac{2h}{c}\gamma = \gamma \Delta t'$$

$$\Delta x' = \frac{\Delta x - v\Delta t}{\sqrt{1-v^2/c^2}} \qquad \Delta t' = \frac{\Delta t - v\Delta x/c^2}{\sqrt{1-v^2/c^2}}$$

Can be derived by just assuming Lorentz transformations form a group.

$$x' = \gamma(x - vt) \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

Addition of velocities $\dfrac{u+v}{1+\frac{uv}{c^2}}$

Invariant $\qquad ds^2 = \Delta x^2 - c^2(\Delta t)^2 = -dc^2$

$$= (\Delta x')^2 - c^2(\Delta t')^2$$

$$\frac{\Delta x^2 - 2\Delta x \Delta t\, v + v^2\Delta t^2}{1-\frac{v^2}{c^2}} - \frac{c^2}{1-\frac{v^2}{c^2}} \frac{\Delta t^2 - 2\frac{v}{c^2}\Delta x\Delta t + \frac{v^2}{c^4}\Delta x^2}{1}$$

$$= \frac{c^2}{c^2-v^2}\left[\Delta x^2 - 2v\Delta x\Delta t + v^2\Delta t^2 - c^2\Delta t^2 + 2v\Delta x\Delta t - \frac{v^2}{c^2}\Delta x^2\right]$$

$$= \frac{c^2}{c^2-v^2}\left(\frac{c^2-v^2}{c^2}\right)\Delta x^2 - \frac{c^2}{c^2-v^2}\left(c^2-v^2\right)\Delta t^2 = \Delta x^2 - c^2\Delta t^2$$

$d\tau^2 > 0$: time like

$ds^2 > 0$: space like

$d\tau = ds = 0$: light like

} Intervals

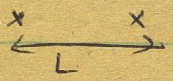
Point clock :

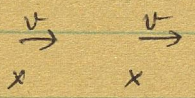
$$(d\tau)^2 = c^2(\Delta t)^2 \left(1 - \frac{(\Delta x)^2}{c^2(\Delta t)^2}\right)$$

$\Delta t_{Rest} = \Delta t \sqrt{1 - \frac{v^2}{c^2}}$ Clock is slowest in rest frame

$$\Delta t = \frac{\Delta t_R}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Lorentz – Fitzgerald contraction





O $\Delta x = L$ O' $\Delta x = \frac{\Delta x' - v\Delta t'}{\sqrt{1 - v^2/c^2}}$

For measurement by O', $\Delta t' = 0$

$\therefore \Delta x' = \sqrt{1 - \frac{v^2}{c^2}}\, \Delta x = \gamma^{-1} L$

Let $c^2 = 1$

In 3-d $\quad d\vec{x}^2 = dx^2 + dy^2 + dz^2 \qquad$ invariant

$$dx_j' = a_{jk}\, dx_k$$

$$ds^2 = (dx_i)^2 - (\Delta t)^2 \qquad i = 1,2,3$$

$$= (dx^i)^2 - (dx^0)^2$$

$$\neq \sum_\alpha (dx^\alpha)^2 \qquad (\alpha = 0,1,2,3)$$

Introduce covariant & contravariant

$dx^\alpha = (dx^i, dx^0) \qquad$ contravariant

$dx_\alpha = (dx^i, -dx^0) \qquad$ covariant

$$ds^2 = dx_\alpha\, dx^\alpha = (dx^i)^2 - (dx^0)^2$$

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\, dx^\nu \qquad \text{(contra)}$$

$$\frac{\partial \phi}{\partial x'^\mu} = \frac{\partial \phi}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\mu} \qquad (\phi = \text{scalar})$$

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu \qquad \text{(covariant)}$$

$$A'_\mu\, dx'^\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu \frac{\partial x'^\mu}{\partial x^\lambda} dx^\lambda$$

$$\sum_\mu \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x^\nu}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x^\lambda} = \delta_\lambda{}^\nu$$

$$\therefore\ A'_\mu\, dx'^\mu = A_\nu\, dx^\nu$$

March 19, 1997

Contravariant tensor of rank 2

$$T'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} T^{\lambda\sigma}$$

Covariant tensor of rank 2

$$T'_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} T_{\lambda\sigma}$$

Mixed tensor

$$T'^{\mu}_{\ \nu} = \frac{\partial x'^{\mu}}{\partial x^{\lambda}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} T^{\lambda}_{\ \sigma}$$

$dx^{\mu} - dx^{i}, dx^{0}$

$dx_{\mu} - dx^{i}, -dx^{0}$

$$dx^{i\,2} - dx^{0\,2} = g_{\mu\nu}\, dx^{\mu} dx^{\nu}$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu} = \Lambda^{\mu}_{\ \nu}\, dx^{\nu}$$

$$ds^2 = g_{\mu\nu}\, dx^{\mu} dx^{\nu} = g'_{\sigma\lambda}\, dx'^{\sigma} dx'^{\lambda}$$

$$\therefore g_{\mu\nu} = \frac{\partial x'^{\sigma}}{\partial x^{\mu}} \frac{\partial x'^{\lambda}}{\partial x^{\nu}} g'_{\sigma\lambda} \Rightarrow g_{\mu\nu} \text{ is a tensor}$$

For Lorentz transformation : $g'_{\sigma\lambda} = g_{\sigma\lambda} = g_{\lambda\sigma}$

$$g_{\mu\nu} = \frac{\partial x'^{\sigma}}{\partial x^{\mu}} \frac{\partial x'^{\lambda}}{\partial x^{\nu}} g_{\sigma\lambda}$$

$$= \Lambda^{\sigma}_{\mu} \Lambda^{\lambda}_{\nu} g_{\sigma\lambda}$$

$$= \Lambda_{\mu\sigma} \Lambda_{\nu\lambda} g_{\sigma\lambda}$$

$$= \Lambda_{\mu\sigma} g_{\sigma\lambda} \Lambda^{T}_{\lambda\nu}$$

$$\| \; g = \Lambda^{T} g \Lambda \; \|$$

$$\det g = \det \Lambda^{T} \det g \det \Lambda$$

$$= (\det \Lambda)^2 \det g$$

$$(\det \Lambda)^2 = 1 \Rightarrow \det \Lambda = \pm 1$$

$+1$ — proper

-1 — improper ($t' \to -t$ and mirror reflections)

If for any A^{μ}, $V_{\mu} A^{\mu}$ = invariant for all vectors V_{μ}, then A^{μ} is a vector

Generalizing $V_{\mu\nu\lambda} A^{\mu} = T_{\nu\lambda}$ for all tensors $V_{\mu\nu\lambda}$, A^{μ} is a vector

From any contravariant vector, we can form a covariant vector

$$g_{\mu\nu} V^{\mu} = V_{\nu}$$

$g^{\lambda\sigma} g_{\sigma\mu} = \delta^{\lambda}_{\mu}$ (Kronecker delta)

$\delta'^{\lambda}_{\mu} = ?$

$$\frac{\partial x'^{\lambda}}{\partial x^{\sigma}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \delta^{\sigma}_{\tau} = \frac{\partial x'^{\lambda}}{\partial x^{\sigma}} \frac{\partial x^{\sigma}}{\partial x'^{\mu}} = \frac{\partial x'^{\lambda}}{\partial x'^{\mu}} = \delta'^{\lambda}_{\mu} \Rightarrow \text{tensor}$$

Pseudo-tensor generated by $\epsilon_{\mu\nu\lambda\eta}$

Antisymmetric - no two indices same

$\epsilon_{1230} = 1$

$\epsilon'_{\mu\nu\lambda\eta} = \flat$

In 3d, cyclic permutation no change in ϵ

In 4d, $\epsilon_{\mu\nu\sigma\lambda} = -\epsilon_{\lambda\mu\nu\sigma}$

$$\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha} = \left(\frac{-\partial}{\partial x^0}, +\vec{\nabla}\right)$$

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla}\right)$$

$\therefore$

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \nabla \cdot \vec{A}$$

(continuity)

similar to conservation of charge or Lorentz gauge condition

Conservation of charge

$$\rho' d\vec{x}' = \rho d\vec{x}$$

$d^4x = dx^0 d\vec{x}$ is a Lorentz inv.

$d^4x' = \text{Jacobian}\ d^4x = \det A\, d^4x = d^4x$

$$J\left(\frac{\partial x'}{\partial x}\right) = \det\left(\frac{\partial x'^\nu}{\partial x^\mu}\right)$$

$\therefore$ $\rho' d\vec{x}' = \rho^0 d\vec{x}$

$\rho' d\vec{x}' dx'^0 = \rho d\vec{x}\, dx'^0$

$= \rho\, d\vec{x} \frac{\partial x'^0}{\partial x^\nu} dx^\nu = \rho \frac{\partial x'^0}{\partial x^0} d\vec{x}' dx^1$

$\therefore \rho' = \frac{\partial x'^0}{\partial x^0} \rho \Rightarrow$ transforms like the time component of 4-vector

$$d\vec{x}'\,dt' = d\vec{x}\,dt$$

$$d\vec{x}'\,\rho' = d\vec{x}\,\rho$$

$$\therefore \quad \int dt' = \frac{\partial t'}{\partial x^\nu}\,dx^\nu$$

$$\rho' = \frac{\partial \rho'}{\partial x^\nu}\,dx^\nu$$

$$(a_{11}x + a_{12}y + a_{13}z).(a_{21}x + a_{22}y + a_{23}z)(a_{31}x + a_{32}y + a_{33}z)\,dt' = x.y.z\,dt$$

$$\rho' = x.y.z\,\rho$$

ρ

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\mu A^\nu$$

$$F^{0i} = -F^{i0} = \partial^0 A^i - \partial^i A^0$$

$$= -\frac{\partial}{\partial t} A^i - \partial_i \phi$$

$$= E_i$$

$$F^{ij} = \partial^i A^j - \partial^j A^i = B^{ij} = B_{ij} \qquad B_{12} = B_3 \ldots$$

$$F^{0i} = E_i \qquad F^{ij} = B_{ij}$$

$$F^{\mu\nu} = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \end{array}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} = g_{\mu\sigma} F^{\sigma\tau} g_{\tau\nu}$$

$$\partial_\mu F^{\mu\nu} = -4\pi j^\nu$$

$\nu = 0$ $\quad \partial_\mu F^{\mu 0} = \partial_i F^{i0} = -\nabla \cdot \vec{E} = -4\pi\rho$

$\nu = i$ $\quad \partial_\mu F^{\mu i} = \partial_0 F^{0i} + \partial_j F^{ji} = \frac{\partial E_i}{\partial t} + \partial_j B_{ji} \stackrel{?}{=} -4\pi j^i$

$\nu = 1 \quad \frac{\partial E_1}{\partial t} + \partial_2 B_{21} + \partial_3 B_{31} \stackrel{?}{=} -4\pi j^1$

$\frac{\partial E_1}{\partial t} - \partial_2 B_3 + \partial_3 B_2 \stackrel{?}{=} -4\pi j^1$

$\frac{\partial E_1}{\partial t} - (\nabla \times \vec{B})_1 = -4\pi j^1$

$\nabla \times \vec{B} = 4\pi \vec{j} + \frac{\partial \vec{E}}{\partial t}$

What about homogeneous equations.

$\partial^\mu F^{\nu\sigma} + \partial^\nu F^{\sigma\mu} + \partial^\sigma F^{\mu\nu} = 0$

Inhomogeneous eqn gave $-\nabla \cdot E = -4\pi\rho$

$-\nabla \times \vec{B} + \frac{\partial \vec{E}}{\partial t} = -4\pi \vec{j}$

Homogeneous, we want $\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

$\therefore \vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}$$

$\partial_\mu \tilde{F}^{\mu\nu} = 0$

$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma}$

$$\tilde{F}^{00} = \frac{1}{2}\epsilon^{0ijk} F_{jk}$$

$$= \frac{1}{2}\epsilon_{ijk} F_{jk}$$

$$\tilde{F}^{01} = \frac{1}{2}\left[F_{23} - (-F_{32}) \right] = \frac{1}{2}\left[2F_{23} \right] = B_{23} = B_1$$

$$\tilde{F}^{ij} = \frac{1}{2}\epsilon^{ij\sigma\lambda} F_{\sigma\lambda}$$

$$= -\frac{1}{2}\left[\epsilon^{ijk0} F_{k0} + \epsilon^{ij0k} F_{0k} \right] \qquad \epsilon^{ijk0} = -\epsilon^{0ijk}$$

$$= -0 \quad \epsilon_{ijk} F_{k0}$$

$$\tilde{F}^{12} = F_{30} = -E_3 \, , \ldots$$

$$\partial_\alpha \tilde{F}^{\alpha\beta} = \frac{1}{2}\partial_\alpha \epsilon^{\alpha\beta\lambda\mu} F_{\lambda\mu}$$

$$= \frac{1}{2}\partial_\alpha \epsilon^{\alpha\beta\lambda\mu}\left[\partial_\lambda A_\mu - \partial_\mu A_\lambda \right]$$

$$= \partial_\alpha \epsilon^{\alpha\beta\lambda\mu} \partial_\lambda A_\mu$$

$$= \epsilon^{\alpha\beta\lambda\mu} \partial_\alpha \partial_\lambda A_\mu$$

$$\partial_\mu F^{\mu\nu} = -4\pi j^\nu$$

$$\partial_\nu \partial_\mu F^{\mu\nu} = -4\pi \partial_\nu j^\nu$$

Classical Fields

Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

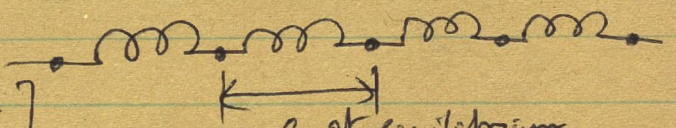
Derived from a variational principle

$$\delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt = 0$$

Lagrangian L (assumed not to depend on time explicitly) is given by difference between KE & PE

$$L = T - V.$$

Consider a collection of N particles connected with identical springs of spring constant k in one dimension



$$L = \frac{1}{2}\sum_{i=1}^{N}\left[m\dot{\eta}_i^2 - k(\eta_{i+1} - \eta_i)^2\right]$$

$$= \sum^{N} a\,\frac{1}{2}\left[\frac{m}{a}\dot{\eta}_i^2 - ka\left(\frac{\eta_{i+1} - \eta_i}{a}\right)^2\right]$$

$$= \sum^{N} a\,\mathcal{L}_i$$

$\mathcal{L}_i$ = Linear Lagrangian density

Discrete to continuous as $N \to \infty$ while a becomes infinitesimal

$a \to dx$, $\frac{m}{a} \to \mu$ = linear mass density

$\frac{\eta_{i+1} - \eta_i}{a} \to \frac{d\eta}{dx}$, $ka \to Y$ = Young's modulus

$$L = \int \mathcal{L}\, dx$$

$$\mathcal{L} = \frac{1}{2}\left[\mu \dot{\eta}^2 - Y\left(\frac{d\eta}{dx}\right)^2\right]$$

η is a function of cont. parameters x & t.

Yet η is like a generalized coordinate analogous to q.

$$\delta \int_{t_1}^{t_2} L\, dt = \delta \int_{t_1}^{t_2} dt \int dx\, \mathcal{L}\left(\eta, \dot{\eta}, \frac{\partial \eta}{\partial x}\right)$$

Variation of η is assumed to vanish at t_1 & t_2 and also at extremities of space integration.

$$\delta \int L\, dt = \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial \eta}\delta\eta + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial x}\right)} \delta\left(\frac{\partial \eta}{\partial x}\right) + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)} \delta\left(\frac{\partial \eta}{\partial t}\right) \right\}$$

$$= \int dt \int dx \left\{ \frac{\partial \mathcal{L}}{\partial \eta}\delta\eta + \frac{\partial}{\partial x}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial x}\right)} \delta\eta \right] - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial x}\right)}\right)\delta\eta \right.$$

$$\left. + \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)} \delta\eta \right] - \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)} \right]\delta\eta \right\}$$

($\delta\eta \to 0$ B.C)

$$= \int dt \int dx\, \delta\eta \left[\frac{\partial \mathcal{L}}{\partial \eta} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial x}\right)} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \eta}{\partial t}\right)} \right].$$

If $\delta \int L\, dt = 0$

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial x)} + \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial\eta/\partial t)} - \frac{\partial \mathcal{L}}{\partial \eta} = 0$$

For our case $\quad Y\frac{\partial^2 \eta}{\partial x^2} - \mu \frac{\partial^2 \eta}{\partial t^2} = 0$

Wave equation for one-dimensional propagation of disturbance with velocity $\sqrt{\frac{Y}{\mu}}$.

In 3-d,

$$\mathcal{L} = \mathcal{L}\left(\phi, \frac{\partial\phi}{\partial x_k}, \frac{\partial\phi}{\partial t}\right)$$

$$\frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x_k)} + \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial t)} - \frac{\partial \mathcal{L}}{\partial\phi} = 0$$

$$\frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial(\partial\phi/\partial x^\mu)}\right] - \frac{\partial \mathcal{L}}{\partial\phi} = 0$$

Covariant if $\mathcal{L}$ is relativistic scalar density

Let $\phi(x)$ be a scalar field

$$\phi'(x') = \phi(x)$$

Possible candidate for $\mathcal{L}$

$$\mathcal{L} = -\frac{1}{2}\left(\frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x_\mu} + \mu^2\phi^2\right)$$

E-L eqns give

$$-\frac{1}{2}\frac{\partial}{\partial x^\mu}\left(2\frac{\partial\phi}{\partial x_\mu}\right) + \mu^2\phi = 0$$

$$\frac{\partial}{\partial x_\mu} = g^{\mu\nu}\frac{\partial}{\partial x^\nu}$$

$$-\frac{1}{2}\frac{\partial}{\partial x^\mu}\left[\frac{\partial\phi}{\partial x_\mu} + g^{\mu\nu}\frac{\partial\phi}{\partial x^\mu}\right] + \frac{1}{2}\mu^2 2\phi = 0$$

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\phi - \mu^2\phi = 0$$

$$\Box\phi - \mu^2\phi = 0 \qquad \Box = \nabla^2 - \frac{\partial^2}{\partial t^2}$$

Klein-Gordon

Similar to $E^2 - p^2c^2 = m^2c^4$

In the presence of interactions

$$\mathcal{L}_{int} = -\phi\rho \qquad \rho = \text{source density}$$

Field eq. $\Box\phi - \mu^2\phi = \rho$

Static (time-indep^t) solution

$$\nabla^2\phi - \mu^2\phi = G\,\delta^3(\vec{x})$$

(G: strength of coupling)

$$\phi(\vec{x}) = -\frac{G}{4\pi}\,\frac{e^{-\mu r}}{r}$$

$$\phi(\vec{x}) = -\frac{G}{4\pi}\,\frac{e^{-\mu|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$$

Short-ranged as opposed to Coulomb case

$$\phi \to 0 \quad \text{as} \quad |\vec{x}-\vec{x}'| \gg \frac{1}{\mu}$$

EM field $\quad \mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}j_\alpha A^\alpha$

1. $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2$

$\Box\phi - \mu^2\phi^2 = 0$

$p^2 - \frac{\partial^2}{\partial t^2} - \mu^2\phi^2 = 0$ $\qquad E \to i\hbar\frac{\partial}{\partial t}$ $\qquad \vec{p} \to -i\hbar\nabla$

Similar to $E^2 - p^2 = m^2 \Rightarrow \mu = \frac{mc}{\hbar}$

Also solution

$$\phi(\vec{r}) = -\frac{G}{4\pi}\frac{e^{-\mu r}}{r} = -\frac{G}{4\pi}\frac{e^{-\mu|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$$

Vanishes for $|\vec{r}-\vec{r}'| \gg \frac{1}{\mu}$.

2. Complex scalar field

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu\phi_1\,\partial^\mu\phi_1 + \mu^2\phi_1\right) - \frac{1}{2}\left(\partial_\mu\phi_2\,\partial^\mu\phi_2 + \mu^2\phi_2\right)$$

$$= -\left(\partial_\mu\phi^*\,\partial^\mu\phi + \mu^2\phi^*\phi\right)$$

Treat ϕ & ϕ^* as indept fields to get from E-L equations

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0 \Rightarrow \Box\phi^* - \mu^2\phi^* = 0$$

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\right] - \frac{\partial\mathcal{L}}{\partial\phi^*} = 0 \Rightarrow \Box\phi - \mu^2\phi = 0$$

Gauge invariance. $\mathcal{L}$ remains unchanged

$$\phi' = e^{i\lambda}\phi = \phi_1 \cos\lambda - \phi_2 \sin\lambda$$

$$\phi^{*\prime} = e^{-i\lambda}\phi^* = \phi_1 \sin\lambda + \phi_2 \cos\lambda$$

If λ is infinitesimally small

$$\delta\phi = i\lambda\phi \quad , \quad \delta\phi^* = -i\lambda\phi^*$$

$$\delta\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \right] + \text{c.c.}$$

$$= \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) \right]\delta\phi + \text{c.c.}$$

$$+ \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* \right]$$

$$= -i\lambda\,\partial_\mu\left(\phi\,\partial^\mu\phi^* - \phi^*\partial^\mu\phi \right)$$

Since $\delta\mathcal{L}$ is zero,

$$\partial_\mu s^\mu = 0$$

where $s^\mu = i\left(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right)$
is a conserved four-current which changes sign $\phi \leftrightarrow \phi^*$.
It is a charge current density.

Maxwell field

$$\partial_\mu F^{\mu\nu} = -4\pi j^\nu \quad , \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$\partial_\nu j^\nu = 0$ and $\partial_\mu \tilde{F}^{\mu\nu} = 0$ using antisymmetry.

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + j_\mu A^\mu$$

Regard each component of A_μ as a separate field.

$$\mathcal{L} \sim \vec{B}^2 - \vec{E}^2 \qquad \mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_4 A_\mu)} \partial^4 A_\mu - \mathcal{L}$$

$$\sim \vec{B}^2 + \vec{E}^2$$

$$\mathcal{L} = -\frac{1}{16\pi} g_{\lambda\alpha} g_{\beta\sigma} \left(\partial^\alpha A^\sigma - \partial^\sigma A^\alpha\right)\left(\partial^\lambda A^\beta \partial^\beta A^\lambda\right) - j_\mu A^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} = \mp \frac{1}{4\pi} F_{\nu\mu} = +\frac{1}{4\pi} F_{\mu\nu}$$

$$\frac{\partial \mathcal{L}}{\partial A^\mu} = \mp j_\mu$$

$$\partial^\nu F_{\nu\mu} = \mp 4\pi j_\mu$$

Generalized force $Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$

U = generalized or velocity dependent potential

$$\vec{F} = q\left\{\vec{E} + \frac{1}{c}(\vec{v} \times \vec{B})\right\}$$

$$\vec{B} = \nabla \times \vec{A} \qquad \vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\vec{F} = q\left\{-\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \frac{1}{c}(\vec{v} \times (\vec{\nabla} \times \vec{A}))\right\}$$

$$F_x = q\left\{-\frac{\partial}{\partial x}\left(\phi - \frac{1}{c}\vec{v}\cdot\vec{A}\right) - \frac{1}{c}\frac{d}{dt}\left(\frac{\partial}{\partial v_x}(\vec{A}\cdot\vec{v})\right)\right\}$$

$$= -\frac{\partial}{\partial x}\left[q\phi - \frac{q}{c}\vec{A}\cdot\vec{v}\right] + \frac{d}{dt}\left\{\frac{\partial}{\partial v_x}\left[q\phi - \frac{q}{c}\vec{A}\cdot\vec{v}\right]\right\}$$

$$U = q\phi - \frac{q}{c}\vec{A}\cdot\vec{v}$$

$$L = T - q\phi + \frac{q}{c}\vec{A}\cdot\vec{v} = \frac{1}{2}mv^2 - q\phi + \frac{q}{c}\vec{A}\cdot\vec{v}$$

$$L = \frac{1}{2}m\dot{x}_i\dot{x}_i + \frac{q}{c}A_i\dot{x}_i - q\phi$$

$$p_i = m\dot{x}_i + \frac{q}{c}A_i = \frac{\partial L}{\partial \dot{q}_i}$$

$$H = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\phi$$

$\vec{p}$ is momentum conjugate to $\vec{x}$ only ~~canonical momentum~~ not $m\dot{\vec{x}}$

Liénard Wiechart Potentials

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int \frac{d\vec{r}'\,dt'}{|\vec{r}-\vec{r}'|}\,\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)\vec{j}(\vec{r}',t')$$

$$\phi(\vec{r},t) = \int \frac{d\vec{r}'\,dt'}{|\vec{r}-\vec{r}'|}\,\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)\rho(\vec{r}',t')$$

$$\vec{j}(\vec{r}',t') = q\,\vec{v}(t')\,\delta^3(\vec{r}'-\vec{r}(t'))$$

$$\rho(\vec{r}',t') = q\,\delta^3(\vec{r}'-\vec{r}(t'))$$

where $\vec{r}(t')$ and $\vec{v}(t')$ are coordinate and vel at time t'.
Carry out $d\vec{r}'$ integral

$$\vec{A}(\vec{r},t) = \frac{v}{c}\int \frac{dt'}{|\vec{r}'-\vec{r}(t')|}\,\delta\left(t-t'-\frac{|\vec{r}-\vec{r}(t')|}{c}\right)\vec{v}(t')$$

$$\phi(\vec{r},t) = q\int \frac{dt'}{|\vec{r}-\vec{r}(t')|}\,\delta\left(t-t'-\frac{|\vec{r}-\vec{r}(t')|}{c}\right)$$

Calculate retarded time

$$t_R = t - \frac{|\vec{r}-\vec{r}(t_R)|}{c}$$

$$\vec{A}(\vec{r},t) = \frac{q}{c}\int \frac{dt'}{|\vec{r}-\vec{r}(t')|}\,\delta\left(t-t'-\frac{|\vec{r}-\vec{r}(t')|}{c}\right) v(t')$$

$$\phi(\vec{r},t) = q\int \frac{dt'}{|\vec{r}-\vec{r}(t')|}\,\delta\left(t-t'-\frac{|\vec{r}-\vec{r}(t')|}{c}\right)$$

$t_R = t - \dfrac{|\vec{r}-\vec{r}(t_R)|}{c}$ has solution

$$\tau = t' + \frac{|\vec{r}^{\,*}-\vec{r}(t')|}{c}$$

$$d\tau = dt'\left(1 - \frac{\vec{r}-\vec{r}(t')}{c|\vec{r}-\vec{r}(t')|}\cdot\frac{d\vec{r}(t')}{dt'}\right)$$

$\dfrac{d\vec{r}(t')}{dt'} = \vec{v}$ = velocity at t'

$\hat{r}$ is unit vector pointing from $\vec{r}(t')$ (retarded position) to $\vec{r}$ (field point)

$$\frac{dt'}{d\tau} = \frac{1}{1-\dfrac{\hat{r}\cdot\vec{v}}{c}}$$

$$\vec{A}(\vec{r},t) = \frac{q}{c}\int \frac{d\tau}{|\vec{r}-\vec{r}(t_R)|}\,\delta(\tau)\,\frac{\vec{v}(t_R)}{1-\dfrac{v(t_R)\cdot\hat{r}}{c}}$$

$$\vec{A} = \frac{q\vec{v}}{cs}\quad,\quad \phi = \frac{q}{s}$$

$$s = |\vec{r}-\vec{r}(t_R)| - \frac{(\vec{r}-\vec{r}(t_R))\cdot\vec{v}(t_R)}{c}$$

$\vec{E}$ & $\vec{B}$ fields. Need derivatives of t_R. $\vec{x}$ is field point

$t_R = t - |\vec{x} - \vec{r}(t_R)|$

$\vec{r}$ is $\vec{x} - \vec{r}(t_R)$ $\quad |\vec{r}| = |\vec{x} - \vec{r}(t_R)|$

$\hat{r}$ is $\dfrac{(\vec{x} - \vec{r}(t_R))}{r}$

$$\frac{\partial t_R}{\partial t} = 1 + \hat{r} \cdot \vec{v} \frac{\partial t_R}{\partial t}$$

$$\therefore \frac{\partial t_R}{\partial t} = \frac{1}{1 - \hat{r} \cdot \vec{v}} = \frac{r}{s}$$

$$\nabla t_R = -\hat{r} + \hat{r} \cdot \vec{v} \nabla t_R$$

$$\therefore \nabla t_R = -\frac{\hat{r}}{1 - \hat{r} \cdot \vec{v}} = -\frac{\vec{r}}{s}$$

For $\vec{E}$ & $\vec{B}$ we now need ϕ & A derivatives

$$-\nabla\phi = \frac{q}{s^2} \nabla s = \frac{q}{s^2} \left[\hat{r} - \vec{v} + \frac{\partial s}{\partial t_R} \nabla t_R \right]$$

$$-\frac{\partial \vec{A}}{\partial t} = -q \left(\frac{\vec{a}}{s} - \frac{\vec{v}}{s^2} \frac{\partial s}{\partial t_R} \right) \frac{\partial t_R}{\partial t} \quad , \quad \vec{a} = \frac{d\vec{v}(t_R)}{dt_R}$$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

$$= \frac{q}{s^2} \left(\hat{r} - \vec{v} + \frac{\partial s}{\partial t_R} \left(-\frac{\vec{r}}{s} \right) \right) - q \left(\frac{\vec{a}}{s} - \frac{\vec{v}}{s^2} \frac{\partial s}{\partial t_R} \right) \frac{r}{s}$$

$$\frac{\partial s}{\partial t_R} = -\hat{r} \cdot \vec{v} + \vec{v}^2 - \vec{r} \cdot \vec{a}$$

$$\frac{S^3 \vec{E}}{q} = (\vec{r} - r\vec{v})(1-v^2) + \vec{r} \times \left[(\vec{r} - r\vec{v}) \times \vec{a} \right]$$

First contribution goes as $\frac{1}{r^2}$, second as $\frac{1}{r}$ & transverse

$$\frac{\vec{B}}{q} = \frac{\nabla \times \vec{A}}{q} = \nabla \times \left(\frac{\vec{v}(t_R)}{s} \right)$$

$$= \frac{\nabla t_R \times \vec{a}}{s} - \frac{\nabla s}{s^2} \times \vec{v}$$

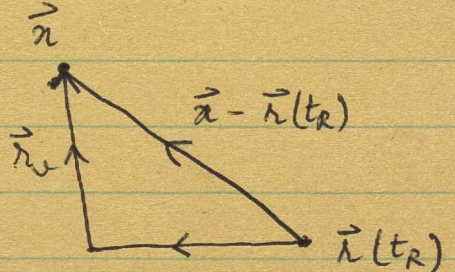
$$\nabla t_R = -\frac{\vec{r}}{s} \quad , \quad \frac{\partial t_R}{\partial t} = \frac{r}{s} \quad , \quad \nabla s = \hat{r} - \vec{v} + \frac{\partial s}{\partial t_R} \nabla t_R$$

we find $\vec{B} = \hat{r} \times \vec{E}$

Define "virtual present radius" $\vec{r}_v = \vec{r} - r\vec{v}$

$$\frac{S^3 \vec{E}}{q} = \vec{r}_v (1-v^2) - \vec{r} \times (\vec{r}_v \times \vec{a})$$

$$\vec{B} = \hat{r} \times \vec{E}$$



$$\vec{v} \cdot (t - t_R)$$
$$= \vec{v} \cdot \frac{|\vec{x} - \vec{r}(t_R)|}{c}$$
$$= \vec{v}\, r$$

we call $\vec{r}_v$ as virtual present radius because it is the location of the radiating particle at time t if it had kept on the course it was following at time t_R for the time $t - t_R = \frac{r}{c}$.

Radiation fields are given by

$$\vec{E}_R = \frac{q}{S^3}\,\vec{R}\times\left[\vec{R}_v\times\vec{a}\right]$$

$$\vec{B}_R = \hat{R}\times\vec{E}_R$$

$\vec{E}_R$ and $\vec{B}_R$ are transverse to $\hat{R}$ and orthogonal to each other.

For small velocities, these equations reduce to our previous relations for electric dipole radiation

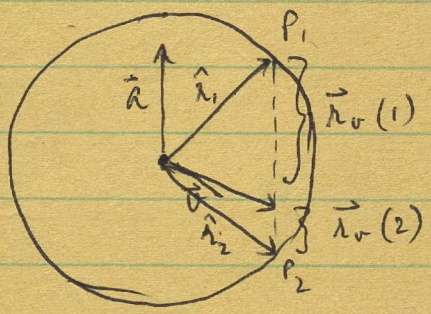
$$\vec{E} = -\frac{q}{Rc^2}\left[\vec{a} - \hat{R}\,\hat{R}\cdot\vec{a}\right]_{t-R/c}$$

$$\vec{B} = \hat{R}\times\vec{E}$$

$$\vec{E}_R = \frac{q}{s^3}\,\vec{n}\times\left[\vec{n}_v \times \vec{a}\right]$$

$$\vec{B}_R = \hat{n}\times\vec{E}_R$$

For $v\sim 1$, the factor $s^3 = (1-\vec{v}\cdot\hat{n})^3 R^3$ peaks the radiation sharply in the direction of $\vec{v}$, even though the amplitude vanishes quite close to $\theta = 0$.
$\vec{E}$ & $\vec{B}$ vanish whenever $\vec{a}$ & $\vec{n}_v$ are parallel.
This always happens for two values of $\hat{n}$.



As $v\to 1$, point P_2 and $\hat{n}_2$ move to the direction of $\vec{v}$
$\vec{E}$ vanishes near $\theta = 0$ does not prevent radiation from peaking forward. Simple case of $\vec{a}$ parallel to $\vec{v}$.

$$\vec{E}_R = \frac{q}{s^3}\,\vec{n}\left((\vec{n}-\vec{v}_R)\times\vec{a}\right) = \frac{q}{s^3}\,\vec{n}\times(\vec{n}\times\vec{a})$$

Very much like low v electric dipole radiation but with the factor $(1-v\cos\theta)^3$ in the denominator.
Intensity of radiation crossing distant sphere is

$$\frac{dW}{dt\,d\Omega} = R^2\,\vec{P}\cdot\hat{n} = \frac{1}{4\pi}\,\vec{a}^2\,\frac{q^2}{s^6}\,R^6\sin^2\theta$$

$$P = \frac{\vec{E}\times\vec{B}}{4\pi} = \frac{\vec{E}\times\hat{n}\times\vec{E}}{4\pi} = \frac{\hat{n}\,\vec{E}^2}{4\pi} \quad \text{since } \vec{E}\cdot\hat{n} = 0$$

(c=1)

Vanishes for $\theta = 0, \pi$, but angular dependence for small θ and $v \sim 1$ is

$$\frac{dW}{dt\, d\Omega} \propto \frac{\theta^2}{\left(1 - v + \frac{\theta^2}{2}\right)^6}$$

Sharp maximum at $\theta^2 = \frac{2(1-v)}{5}$

$$y = \frac{x}{(a+x)^6} \qquad \frac{dy}{dx} = \frac{1}{(a+x)^6} - \frac{6x}{(a+x)^7} = 0$$

$$\therefore \quad a + x = 6x \quad \Rightarrow \quad x = \frac{a}{5}$$

$$\frac{\theta^2}{2} = \frac{1-v}{5} \Rightarrow \quad \theta^2 = \frac{2(1-v)}{5}.$$

Total energy radiated ?

Choose to calculate $\frac{dW}{dt_R\, d\Omega}$ — rate of radiation by particle rather than $\frac{dW}{dt\, d\Omega}$ which is rate of radiation thro' distant sphere.

Rates different because $\frac{\partial t_R}{\partial t} = \frac{1}{5}$. Integrated over time both are same

$$\int \frac{dW}{dt}\, dt = \int \frac{dW}{dt_R}\, dt_R$$

Rate of energy loss by particle is more important in accelerator design.

$$\int d\Omega\, \frac{dW_{\parallel}}{dt_R\, d\Omega} = \int d\Omega\, \frac{dW}{dt\, d\Omega}\, \frac{\partial t}{\partial t_R}$$

$$= \frac{q^2 \vec{a}^2}{4\pi} \int \frac{d\Omega \sin^2\theta}{(1 - v\cos\theta)^5} = \frac{2}{3}\, \frac{\vec{a}^2 q^2}{(1-v^2)^3}$$

Perpendicular component not so simple

$$\int d\Omega \frac{dW_\perp}{dt_R d\Omega} = \frac{2}{3} \frac{\vec{a}^2 q^2}{(1-v^2)^2}$$

CERENKOV RADIATION

Radiation from a charged particle moving at CONSTANT velocity v in a medium in which the phase velocity of light c_M is smaller than v.

Lienard-Wiechart potentials denominator

$$1 - \frac{\hat{n}\cdot\vec{v}}{c_M} = 0 \quad \text{when} \quad \cos\theta_C = \frac{c_M}{v} < 1$$

Potentials become infinite. Singularity not really there. Consequence of assuming no dispersion; so that all frequencies can be radiated. In practice as $\omega \to \infty$, dielectric constant $\epsilon \to 1$, and $c_M \to c > v$, radiation finite. Must consider freq dependent ϵ.

$\hat{R} \cdot \vec{A} = A\cos\theta$ $\hat{R}$

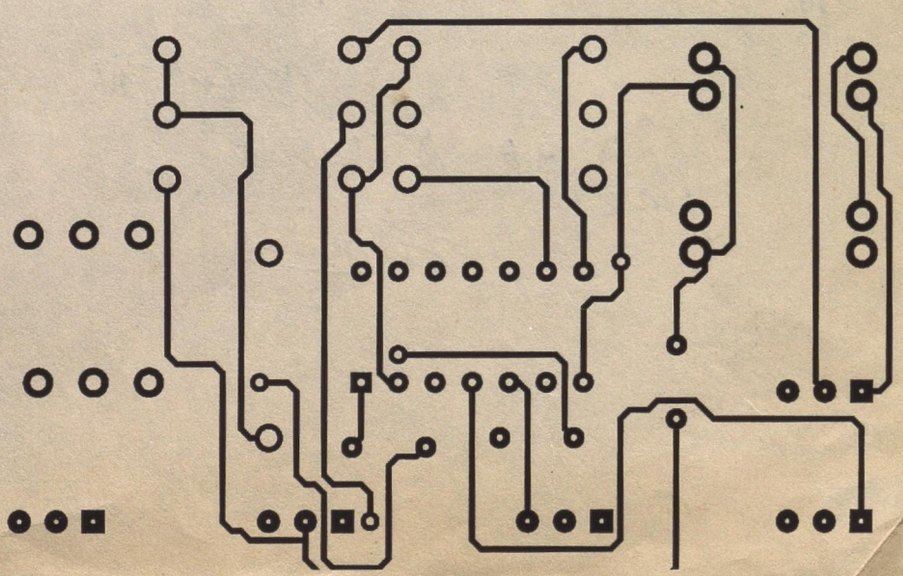
$$\vec{A} \cdot \int \hat{R} \cdot \vec{A}\, \hat{n}\, d\Omega = \int (\hat{R} \cdot \vec{A})^2 d\Omega$$

$$= \int A^2 \cos^2\theta \sin\theta \, d\theta \, d\phi$$

$$= A^2 2\pi \left(-\frac{\cos^3\theta}{3} \right)_0^{\pi}$$

$$= 2\pi A^2 \frac{2}{3}$$

$$= \frac{4\pi}{3} \vec{A} \cdot \vec{A}$$



$$\int \frac{\nabla' \cdot \vec{j}\, d\vec{\tau}'}{|\vec{\tau} - \vec{\tau}'|} = -\int \vec{j}(\vec{\tau}') \cdot \nabla' \frac{1}{|\vec{\tau} - \vec{\tau}'|}\, d\vec{\tau}'$$

$$= -\int \vec{j}(\vec{\tau}') \cdot \frac{(\vec{\tau} - \vec{\tau}')}{|\vec{\tau} - \vec{\tau}'|^3}\, d\vec{\tau}'$$

$$\vec{j}_L = \frac{1}{4\pi} \nabla \int \frac{\vec{\tau} - \vec{\tau}'}{|\vec{\tau} - \vec{\tau}'|^3} \cdot \vec{j}(\vec{\tau}')\, d\vec{\tau}'$$

$$\int_V \vec{j}_L\, d\vec{\tau}' = \frac{1}{4\pi} \int_S d\vec{S} \int \frac{\vec{\tau} - \vec{\tau}'}{|\vec{\tau} - \vec{\tau}'|^3} \cdot \vec{j}(\vec{\tau}')\, d\vec{\tau}'$$

$$\int j_L\, d\vec{\tau}' \Longrightarrow \frac{1}{4\pi} \int \tau^2\, \hat{\tau}\, d\Omega\, \frac{\vec{\tau}}{\tau^3} \cdot \int \vec{j}(\vec{\tau}')\, d\vec{\tau}'$$

$$\int \hat{\tau}\, \hat{\tau} \cdot \vec{A}\, d\Omega = \frac{4\pi}{3} \vec{A}$$

$$\therefore \int \vec{j}_L\, d\vec{\tau}' = \frac{1}{3} \int \vec{j}(\vec{\tau}')\, d\vec{\tau}'$$

6. $\int B\hat{\tau}\, \hat{\tau} \cdot \vec{A}\, d\Omega = \int \vec{B} \cdot \hat{\tau}\, \vec{A} \cdot \hat{\tau}\, d\Omega$

$$= \int \hat{A} \cos\theta\, A \cos\theta\, \sin\theta\, d\theta\, d\phi$$

$$2\pi \frac{\sin^2\theta}{2} \Big|_0^{\pi}$$