

Spin and the Statistics of the Nuclei

Spin of the nucleus may be deduced from

1) the no. of components in the hyperfine structure.

Analysis
difficult

= $2I + 1$ or $2J + 1$, whichever is smaller
 J : angular momentum of the electron I : of nucleus.

2) from Paschen Back effect - no. = $2I + 1$

Jackson's expts on the components in absorption of the resonance lines of K and Na.
 How Doppler broadening is avoided by limiting vapour absorbing the light to a narrow beam limited by 2 slits and observing transversely.

is hyperhyperfine structure = broad even when $I=0$ for all the isotopes. not due to mass. Probably h volumes + separation different. from Coulomb last suff. to explain spread.

But the most elegant method of determining the spin moment (and also the statistics of the nucleus) is from study of the alternating intensities of the rotational components in ^{electronic} bands or Raman spectra.

Statistics

Consider any given system, and we form the Schrödinger equation $H(\psi, \psi) = E$ ^{classical Hamiltonian}

$$[H - \frac{\hbar^2 \nabla^2}{2m}] \psi = 0$$

where H is the well known differential operator obtained from the classical Hamiltonian of the system.

$E_0 \dots$ Eigen energies

$\psi_0 \dots$ correspond Eigen fns or wave fns.

If we have two such similar systems with feeble interactions then the Hamiltonian of the pair, to a 1st approximation will be the sum of the 2 Hamiltonians:

$$\Psi = \psi_0(q_1) * \psi_1(q_2) \quad \alpha_1, \alpha_2, \dots, \alpha_n$$

similar for M similar systems; we shall denote the wavefn by $\Psi(1, 2, 3 \dots M)$. The order of the no. being in general significant.

Then Ψ is said to be symmetrical about

two or more systems α_1, α_2 if

$$\Psi(1, 2, 3, 4 \dots M) = \Psi(2, 1, 3, 4 \dots M)$$

and symmetrical in three systems $\alpha_1, \alpha_2, \alpha_3$ if

$$\Psi(1, 2, 3, 4, 5 \dots M) = \Psi(1, 3, 2, 4 \dots) = \Psi(3, 1, 2, 4 \dots M)$$

and symmetrical in $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n$ if

~~the~~ numbers are interchanged.

Similarly if we were for Ψ said to be asymmetric

in $\alpha_1, \alpha_2, \alpha_n$ if on interchange of any two of the numbers Ψ becomes $-\Psi$.

Two special cases. when the wave fn is symmetric in all the systems, or asymmetric in all the systems.

If an assembly is originally represented by a wave fn symmetrical in the systems $\alpha_1, \alpha_2, \dots, \alpha_n$ will for this confined to this group. Similarly for antisymmetry. The two groups do not combine.

are of great interest - 3

∴ Actual assemblies of similar systems in Nature are indeed of this type.

Take an ~~assembly~~ ^{representing} assembly of electrons. We can show that the wave function of the assembly should be anti-symmetric in all the electrons. ∴ according to Pauli's Principle, no 2 electrons can have all the quantum numbers identical, i.e. can have the same wave function. i.e. if any two electrons are interchanged ψ can not be the same.

Similarly, an assembly of protons, neutrons, pi-mesons, electrons; in fact with all two particles which obey Fermi-Statistics.

A ~~system~~ ^{an assembly} Consider 2 similar systems - each complex consisting of a neutrons, e electrons, p protons, n mesons. The wave functions representing the assembly will of course be anti-symmetrical in all the electrons; but will be symmetrical in the systems 1 + 2, or anti-symmetrical, if according to the no. of $(e + p + a)$ is odd or even.

∴ if ~~odd~~ we have M similar systems each having the same ψ and even no. of $(e + p + a)$, the ^{wave function of} assembly will be symmetrical in all the systems: ~~and~~ Such an assembly, ~~which~~ will obey Bose statistics though the assembly will be ^{anti} symmetrical in all the ~~protons, electrons & neutrons~~.

Interchange between the 2 systems = interchange of $(m+n+a)$ pairs; ~~the~~ sign changes $m+n+p+a$ times Hence the result-

We can \therefore talk about the spin of the nucleus: if $(Z+N+0)$ odd Fermi
 even Bose

The influence of the spin of the nucleus in the alternation of intensities of the rotational components in a homonuclear diatomic molec. comes in this way.

The wave-fn for a diatomic molecule can be split up into

$\Psi = \underbrace{\Phi(x, y, z, S_e, \dots)}_{\substack{\text{Electronic} \\ \text{probab. condns. of electrons} \\ \text{spin elect.}}} \underbrace{P(\rho)}_{\text{vib.}} \underbrace{\Theta(\theta, \chi)}_{\text{rotat.}} \times \underbrace{F(I, J)}_{\substack{\text{spin} \\ \text{nuclei}}}$

Labels: Φ (Electronic), $P(\rho)$ (vib.), $\Theta(\theta, \chi)$ (rotat.), $F(I, J)$ (spin nuclei). Additional notes: "dist. betw nuclei", "orientation", "complete along", "spin nuclei".

~~Exp.~~ F represents 2 spins altogether $(2I+1)$ or some of which are not eq. in num. They are not eq. in num. Let their num. be $N_1 + N_2$ or $N_1 + N_2$ respectively. As we shall see $N_1 > N_2$.

N_3 in consist of F_{11}, F_{12} from Ψ in Ψ_{12} & F_{21} & F_{22} altogether $(2I+1) + 2I(2I+1)^{1/2}$.

$N_1 = (2I+1)(I+1)$
 $N_2 = (2I+1)I$
 $N_3/N_2 = \frac{I+1}{I}$ (2)

For any given electronic transition, let us consider only the effects of the nuclear spins & of rotations; others remaining the same.

$\Psi = \Phi(\rho) \Theta(\theta, \chi) \times F(I, J)$

Now we know 2 results - definite

The fn Θ is symmetrical with respect to the two nuclei when the rotational quantum no. J is even and, antisymmetrical when it is odd.

If the other wave fns do not change, the resulting wave fn will be alternately Sym(S) or anti-sym(A) with respect to the 2 nuclei depending on the value of J .

That the Ψ should be sym or A depending on the statistics applicable to the nucleus.

Now for a given diatomic homonuclear molec. X_2 ,
~~Now~~ the symmetry of Ψ is uniquely
 fixed by the statistics applicable to X ; that of
 Φ' also remains fixed when we are considering
~~only~~ only the effects of the rotation of the nuclei
 and of the spins of the nuclei.

Now the product of any function f with a symmetric
 f will not change the sym. of the f ; when multiplied
 by an A.S. f the symmetry will change from
 S to A or A to S.

Since the symmetry of Ψ and Φ' are fixed
 and $\Theta(n)$ alternates from S to A, we have
 to choose alternately the appropriate $F(\Omega_1, \Omega_2)$
 of the appropriate symmetry which will give conform
 to the known symmetry of this.

Since the symmetry of Ψ is fixed by the nature
 of the diatomic molecule under consid, and that
 of Φ' also is fixed for a given set of rotation
 bands, the product of $\Theta(n)$ and of $F(\Omega_1, \Omega_2)$
 should have a specific symmetry determined by these
 above conditions.

Now the sym. of $\Theta(n)$ alternates between S and A.
 for even + odd values of n . The ~~odd~~ or sym. ~~is~~
 odd values of F , or the ~~sym.~~ ^{sym.} which is
 appropriate for the particular n for giving the reqd
 sym. product, will be involved. ~~is i.e.~~ for odd

Further found odd ⁷ ~~n's~~ strongly intensify
 $\therefore \psi$ anti-sym. \therefore ~~max~~ statistics, ~~of~~ F.

N₂ Alternate ones missing $I = 0$
 may odd + statistics Bose Ein.

with Proton + Elec. $N = 14$
 $I = 14p + 7e$
 21 particles odd
 statistics ~~odd~~ F.
 whereas actually Bose

\therefore the total no. of particles in ψ state take F. stats
 applies should be even not odd
 \therefore no electrons 7 protons + 7 neutrons.

ortho and para hydrogen; from considerations
 of Sp. heat - : non combination!
 feeble interchange betw ψ_1 & ψ_2 .

~~Had there~~ Alternating intensities odd absent $I = 0$
He₂ Bose statistics:

When with $4p + 2e$ } fits.
 $\alpha 2p + 2n$

$d^{35} d^{35}$ $S = 7$ $I = 7/2$ Fer.

$\left| \begin{array}{cc} O^{16} & O^{17} \\ C^{12} & C^{13} \\ \cdot & \cdot \end{array} \right|$

no missing lines

$\left| \begin{array}{cc} O^{16} & O^{16} \\ \cdot & \cdot \\ C^{12} & C^{12} \end{array} \right|$

alternate ones missing
 may odd

§ D has spin $I = 1$
Alternation $2:1$; Even predominately.

Magnetic moment of nucleus

Nuclear magneton $\beta_n = \beta_e / 1840$.

Stern + Rabi proton $2.85 \beta_n$.

Zn Hg Sn etc β_n i.e. $g \approx 1$.

Homonuclear diatomic molecules

If n (no. of neutrons) + p (no. of protons) is odd

Ψ antisymmetric in the 2 two nuclei.

If in ground state, $\Sigma \Lambda = 0 \quad S = 0$

Φ is symmetrical. Then

Rotational $(-1)^n \times F$ should be ~~an~~ antisym.

of type $\psi_n^{(a)} \psi_n^{(b)} - \psi_n^{(a)} \psi_n^{(b)}$ and $(2I+1)I$ antisymmetric values } of F

and also $\psi_n^{(a)} \psi_n^{(b)} + \psi_n^{(a)} \psi_n^{(b)}$ and $(2I+1)(I+1)$ symmetric values }

Total $(2I+1)(2I+1)$

Rotational quantum nos r, s .

Here Homonuclear odd mass nos.

$$E_r = r(r+1) \frac{h^2}{8\pi^2 I}$$

$$-E_r/RT$$

Partition fn $Z_r = (2I+1)I \sum_{r=0,2,4}^{\infty} (2r+1) e^{-E_r/RT}$

+ $(2I+1)(I+1) \sum_{r=1,3}^{\infty} (2r+1) e^{-E_r/RT}$

In absorption in ground state odd r 's correspond to larger intensities

Homonuclear even mass nos.

even r 's larger intensities.

Partn. fn = $(2I+1)(I+1) \sum_{r=0,2,4}^{\infty} (2r+1) e^{-E_r/RT}$

+ $(2I+1)I \sum_{r=1,3}^{\infty} (2r+1) e^{-E_r/RT}$

Probability of transition depends on both upper & lower r 's. Effect is to replace $2r+1$.

Heteronuclear The two atoms a, b.

$$\text{Partition } p_r = (2I_1 + 1)(2I_2 + 1) \sum_{r=0,1,2,\dots}^{\infty} (2r+1) e^{-E_r/RT}$$

For H_2 $I = 1/2$.

Coming back to the partition function for H_2 , if exchange between ortho and para were free, the

$$E = \frac{\partial(F_a/T)}{\partial(1/T)} \quad E_{\text{rot}} = N \times RT^2 \frac{\partial}{\partial T} \log P(T)$$

spherical
const. vol.
Rotational

where $P(T)$ is given by (1) on the previous p.;

with $I = 1/2$.

$$\rightarrow c = \text{pr particle} = R \frac{\partial}{\partial T} \left[T^2 \frac{\partial \log P(T)}{\partial T} \right]$$