Mis-cellaneous notes on

Quantum mechanics.

C.G. Darwin's New Conceptions of Matter:

Chapter I.

(A) Building materials

(B) " site

(C) Principles of Architecture.

(A) Electrons, Protons, Photons.

Photon is a light-corpuscle differing from electron & protons in (i) Photon moves always with vel c

(ii) ~~Two~~ Photons do not affect one another in any wa, but photons affect electrons & protons.

Modern views bring about similarity between the three units by denying individuality to electrons & protons rather than by endowing individuality to photons.

(B) The aether is taken as the subject of the verb "to undulate".

~~This is all the use of relativ~~'

(C) Conservation of linear and angular momentum

Conservation of energy – For matter in bulk heat energy in addition to kinetic & potential is to be consider

2) but for individual atoms there is no room for heat energy.

Chapter II.

Article by V. Karapetoff in Mechanical Engineering April, May & June 1933

On The Fermi-Dirac statistical theory of gas degeneration:

I Introduction A quantity of gas ~~of vo~~ occupying vol V and consisting of n identical particles. At a given instant vector momentum of each molecule is $p = mv$. Space representing p-vectors is called momentum space. Assumptions introduced into this momentum space and not in the real space occupied by the gas are as follows:—

(A) Cells in representative space. Divide the momentum space by shells of radii p & $p+dp$ and subdivide further into cells of equal volume w each. No distinction is made between vectors whose ends lie in one and the same cell

(B) Size of unit cells. The assumption of size is given by

$$w = h^3/V \qquad (h = \text{Planck's const}) \qquad \text{———} (1)$$

Eqn (1) can be deduced on the basis of de Broglie's Waves or can be taken to be by itself a postulate.

(C) Number of vectors per cell. Last statement in (A) might suggest possibility of two vectors in one cell; but the Pauli's exclusion principle (originally stated for electrons in an atom) extended to molecules of gas however states that not more than one vector can terminate at a given instant in a given cell and many cells must remain empty for some of the time.

This third assumption gives an immediate explanation of the "zero-point energy" associated with Nernst's heat theorem. At extremely low temperatures, as the magnitudes of the individual momenta decrease, the cells near the origin become more and more occupied. At the absolute zero all the cells nearest the origin are occupied by vectors of momenta and the gas is said to be completely degenerate. To account for all the particles of gas, (since there are only a few cells right at the origin) certain vectors of momentum must be of considerable magnitude to extend to empty cells; so that the gas itself is endowed with a certain amount of energy called the zero point energy.

Another important consequence of the cell theory is that an electron gas obeying the Fermi-Dirac Statistics is completely degenerate even at very high temperatures. Consider a mixture of H_2 mols & electrons in thermodynamic equilibrium at about 50000°C (at which temp. electron gas might be considered an ideal gas. Note that <u>ideal</u> and <u>degenerate</u> are quite different attributes) Assuming law of equipartition of energy to hold good $m\overline{v^2}$ is the same for H_2 mols & electrons; if $\sqrt{\overline{v^2}} = 1$ for H_2 mols; we have

$$m_{H_2} = m_e \times \overline{v^2} \qquad \& \; m_{H_2} = 3600\, m_e$$

$$\therefore \; v_e = \sqrt{\overline{v^2}} = 60\, v_H \qquad (v_H = 1)$$

but $p_{H_2} = 3600 \times v_H = 3600$ & $p_e = 1 \times 60 = 60$

Thus momentum vectors of H_2 mols occupy mostly cells remote from the origin while those of electrons will occupy cells near the origin. The electron gas is degenerate whereas the H_2 is very far from degeneration.

II Fundamental Equations

The number of shells in momentum space between radii p & $p+dp$ is given by

$$C(p, p+dp) = 4\pi p^2\, dp\, V/h^3 \qquad \text{——— (2)}$$

The Kinetic energy $E = p^2/2m$ ——— (3)

and $dE = p\,dp/m$ ——— (4)

Eliminating p and dp from (2), (3) and (4) we have

$$C(E, E+dE) = 4\pi \cdot 2mE \cdot m\,dE\,(2mE)^{-\frac{1}{2}} \cdot V/h^3$$

$$= (V/h^3)\, 2\pi \,(2m)^{\frac{3}{2}}\, E^{1/2} \cdot dE \qquad \text{——— (5)}$$

C in eqn (5) might also be considered to give the number of "energy states" at E over the range $E + dE$ in a representative "energy space" having E for radii.

Let N be the actual number of molecules in these energy states. Then by Pauli's principle (vide (C) above) $N \ngtr C$. Hence the number of ways in which these N molecules can be distributed among the C states is

$$W = \frac{\lfloor C}{\lfloor N \, \lfloor C-N} \quad \text{———} \quad (6)$$

Let N_t = total no of molecules (taking all shells

E_t = " Kinetic energy

Then $$\Sigma N = N_t = \text{given} \quad \text{———} \quad (7)$$

$$\Sigma EN = E_t = \text{"} \quad \text{———} \quad (8)$$

Number of distributions over all the shells is next given by

$$W_t = \Pi W \quad \text{———} \quad (9)$$

W_t giving the thermodynamic probability. The problem is to find values of N for each shell so as to make (9) a maximum subject to (7) and (8). The solⁿ of this problem gives

$$N = \frac{C}{e^{\alpha + \beta E} + 1} \quad \text{———} \quad (10)$$ (α & β being undetermined constants).

III The next step is to determine α and β.

(~~a) To find β~~ Substituting for ~~(C)~~ C in (10) from (5) we have

$$N = (V/h^3)\, 2\pi (2m)^{3/2} E^{1/2} . dE \Big/ \left(e^{\alpha + \beta E} + 1\right) \quad \text{———} \quad (11)$$

6) (a) To determine β, we make use of the fact that in the limiting case of large E, the results obtained must tend towards the classical value

$$N = KE^{1/2} dE \Big/ e^{E/kT} \quad (12) \quad (k = \text{Boltzmann's constant})$$

[This is seen immediately seen from the form of the Zustands-Integral

$$\bar{\epsilon} = \frac{\sum \epsilon_i \, e^{-\epsilon_i/kT}}{\sum e^{-\epsilon_i/kT}} \Big]$$

Hence writing (11) for large values of E in the form

$$N = (V/h^3) \, 2\pi (2m)^{3/2} E^{\frac{1}{2}} dE \Big/ e^{\alpha + \beta E}$$

Comparing this with (12) we must have

$$\beta = 1/(kT) \text{ ——————— } (13)$$

(b) To find α. Substitute for N from (11) in

$$\sum N = N_t \;\; \& \;\; \sum NE = E_t$$

and then replace summations by integrations (this is permissible since a transition from shell to shell is a macroscopic transition unlike transitions from cell to cell in the same shell) with limits of E from 0 to ∞. The final results are given in two cases

Case (i) – Large positive values of α (corresponding to temperatures & densities at which gas is non-degenerate), then

$$e^{\alpha} = V(2\pi mkT)^{3/2} \Big/ (N_t h^3) \text{ ——— } (14)$$

and $E_t = \frac{3}{2} N_t k T$ ———— (15)

(15) is the usual classical formula deduced by the aid of equipartition of energy

$$\left[\ \overline{\epsilon} = \frac{s}{2}\,\Theta \quad \text{ie} \quad \frac{E_t}{N_t} = \frac{s}{2}\cdot kT \quad \& \ s = \text{no of degrees of freedom} = 3\ \right]$$

Case (ii) Large negative values of α (corresponding to strong degeneration). In this case

$$\alpha = -\left(1/\theta T\right)\left\{1 - (\pi\theta T)^2/12 + \cdots\right\} \qquad (16)$$

where $\theta = 8\left(\pi V/6N_t\right)^{2/3}\left(mk/h^2\right)$ . . . . (17)

and as an approximate result for E_t

$$E_t = 0.6\left(N_t k/\theta\right)\left\{1 - (\pi\theta T)^2/12 + \cdots\right\}\left\{1 + (\pi\theta T)^2/2 + \cdots\right\} \qquad (18)$$

Putting $T=0$ in (18) we get the zero point energy

$$E_{T=0} = 0.6\, N_t k/\theta \qquad (19)$$

For sufficiently low temperatures we might retain in (18) terms only up to T^2. θ is referred to any volume V. For a unit volume we put $V = 1$ & $N_t = n$ = no of mols per unit volume. For a gm-mol, we multiply the result by L/n (L = Loschmidt's no). Thus energy per gm-mol is given from (11)

by $E_t = \frac{L}{n}\cdot(0.6)\left(\cancel{N_t} n k/\theta\right)\left\{\cdots\right\}\left\{\cdots\right\}$

where $\theta = 8\left(\pi/6n\right)^{2/3}\left(mk/h^2\right)$

8)

$$\text{ie}\quad E_t = 0.6L\cdot\frac{K}{8}\left(\pi/6n\right)^{-2/3}\left(mk/h^2\right)^{-1}\left\{1-(\pi\theta T)^2/12+\cdots\right\}\left\{\cdots\right\}$$

$$= \qquad \text{"} \qquad \left\{1+\frac{5}{12}\pi^2\theta^2T^2+\cdots\right\}$$

$\therefore\ \dfrac{dE_t}{dT}$ = Specific heat at constant volume per gm mol of degenerate gas = molar heat at constant vol of degenerate gas

$$= C_{vd} = \frac{0.6L}{8}\left(\pi/6n\right)^{-2/3}\left(m/h^2\right)^{-1}\cdot\frac{5}{6}\pi^2T\cdot\left(\frac{\pi}{6n}\right)^{4/3}\left(mk/h^2\right)^2\cdot 8^2$$

$$= 4L\left(\pi/6n\right)^{2/3}\left(m/h^2\right)\pi^2K^2T$$

$$= \left(\frac{4\pi}{3}\right)^{2/3}\left[\pi^2Lmk^2T/n^{2/3}h^2\right] \qquad \text{———} (20)$$

(19) and (20) giving the behaviour of gases at zero temperature (viz a horizontal tangent and a finite ordinate of the E-T curve) is called Nernst's heat theorem which states that "it is impossible to devise an engine which will completely deprive a body of its heat content".

(20) is in direct contradiction with the result for a perfect gas which requires C_v constant proportional to T and become equal to zero for T = 0.

Eqn (19) for zero point energy could also be deduced directly from assumptions (A), (B) & (C) in $\underline{I}$. Let r be the radius of the sphere within which all the occupied cells at T = 0 are located.

$$\therefore \left(\tfrac{4}{3}\pi r^3\right)\left(V/h^3\right) = N_t \qquad \text{——— (21)}$$

The number of cells in the shell $(p, p+dp) = (4\pi p^2 dp)(V/h^3)$ and energy per mol $= p^2/2m$

$$\therefore E_t = \int_0^r (4\pi p^2 dp)(V/h^3)(p^2/2m) = 0.4\, V r^5/(mh^3) \qquad \text{——— (22)}$$

Eliminating r between (21) and (22) we have

$$\left(\frac{3N_t}{4\pi}\right)^5 \left(h^3/V\right)^5 = \left(\frac{5mh^3E_t}{2V}\right)^3$$

ie $$\frac{5}{2}\frac{mh^3E_t}{2V} = \left(\frac{3N_t}{4\pi}\right)^{5/3}\left(h^3/V\right)^{5/3}$$

ie $$E_{T=0} = \frac{2V}{5mh^3}\cdot\left(\frac{3}{4}\right)^{5/3}\cdot\frac{N_t^{5/3}}{\pi^{5/3}}\cdot\frac{h^5}{V^{5/3}} = \frac{3}{5}N_t k/\theta$$

which is eqn (19).

III Degenerate Electron gas in a metal

Phenomena in metals may be explained by assuming a piece of metal to be "permeated" by electrons forming a so called electron-gas. It is further assumed that the electrongas is kept within the piece of metal by the retarding effect of some sort of wall or barrier, such as an intrinsic ~~poten~~ electric potential. The phenomena that can be qualitatively explained by these assumptions are

(1) electrical conductivity (2) thermal conductivity (3) contact potential (4) The Thomson effect (5) Thermoelectric currents

10)

(6) photo-electric effect (7) glow discharge (8) thermionic emission (9) magnetic properties of alkali metals. Thus

(1) is an actual drift of the electrons under the influence of an applied voltage.

(2) may be imagined to be due to a kind of "electronic convection" or a drift of the kinetic energy associated.

(3) ~~wht~~ is due to an interchange of slower for faster electrons (& vice versa) between the two metals if we postulate that the "density" & "pressure" of the electron gas depends upon the number of atoms per unit volume.

(4) is due to the fact that if two adjacent portions of a homogeneous metal bar are kept at different temperature there is a resultant electronic pressure (difference of electric potential) so that if an electric current is sent along such a bar, the heat liberated in its individual portions do not obey Joule's law

(5) is explained by difference in the kinetic energy of the electron gas.

(6) is due to the energy communicated by the light to some electrons just behind the surface accelerating them past the barrier.

(7) is the pulling out of electrons by an applied voltage which modifies the barrier voltage

(8) is the result of bringing the metal to a high temperature, the kinetic energy of the fastest electrons being increased sufficiently to overcome the restraining potential

(9) is explained by assuming that each electron is spinning about its axis & thus forms an elementary magnet

In addition to the two assumptions above made, a third assumption as regards the property of the electron gas is made viz that it behaves (even at the highest temperatures) like a completely degenerate gas.

In addition to these three specific assumptions relative to the electron theory of metals, it is necessary, to fully grasp this theory, to be able to appreciate other concepts of atomic theory. The following are some ideas or "pictures" not to be taken too literally however.

(a) Atoms: Considering for simplicity an alkali metal with a single valence electron. This electron is easily detached and then converts the atom into a positive ion. The atoms of such a metal are assumed to form a ~~space lattice~~ a regular space lattice, there being a cohesive force keeping these atoms at fixed distances in the solid state. Each atom is assumed to oscillate irregularly about its mean position which accounts for the temperature + heat energy stored by virtue of its specific heat. Higher temperature increases intensity of oscillations and the amplitude, frequency and length of path may be thought of as being functions of temperature. The valence electrons are assumed to possess much more freedom of motion than the rest of the atom & are called free electrons; these free electrons are considerably affected when approaching a positive ion.

(b) Electrons: The assumption is made that there are as many ~~elec~~ free electrons as atoms which is true for the alkali metals. The number of atoms in a piece of metal is vastly greater than in ~~a~~ an equal volume of gas. Therefore the electron gas in a metal possesses vastly greater number of discrete particles per unit volume than usual gases.

(c) Degeneracy of electron gas. Considering eqn (14) let us suppose that α is a critical value at which degeneracy just sets in. We might write

$$T \propto e^{2/3\alpha} \cdot N_t^{2/3} / m$$

The small value for m for an electron and the large value for N_t for electron gas in a metal show that even at very high temperatures the electron-gas is degenerate. Hence in the electron theory of metals eqns (19) & (20) could be used with justification

(d) Specific heat of electrons. This is made up of the Sp. heat for the +ve ions plus that for electrons. From (20) the latter portion at ordinary temperatures is very small and may be disregarded. Thus we have a logical explanation of Dulong & Petit's Law for metals & that electrons contribute little to the specific heat

(e) Electron Spin. In addition to the vectorial velocity of translation an electron is also endowed with a spin, this assumption being equivalent to assuming that each electron is an elementary magnet of definite magnetic moment. For two electrons with the same vector of velocity of translation it is further assumed that they must have their axes of spin anti-parallel to each other. This is also in accordance with the Pauli Exclusion Principle if we associate three quantum numbers with translation and one quantum number with spin.

In the previous investigations with gas molecules, one point alone is associated with a cell in phase space. If the same investigation is to hold good here, we must allow two points in one cell (corresponding ~~two~~ to ~~parallel & anti-parallel directions of spin~~) in the phase space corresponding to the electron gas. Hence in applying the Fermi Dirac Statistics to an electron gas we replace N_t and n in formulae (17), (18), (19) & (20) by $\frac{1}{2}N_t$ and $\frac{1}{2}n$.

(f) Pressure of electron gas. The relation pressure = $\frac{2}{3}$ energy density is also assumed valid for an electron gas just as for a classical gas.

(g) Electrons as waves. This conception is also useful in the theory of the electron gas. In view of the low speeds of electrons within metals, the associated de ~~Brog~~ Broglie wavelengths are several times the lattice spacing of a metal crystal. The question of collision of ~~electr~~ free electrons and the positive ions is different when viewed from the particle & wave aspects. According to the former, there may be no collision in view of the relative sizes of electrons & atoms; according to the latter it is a question of the scattering of a wave by a large particle.

Since the de Broglie wave length of the electron is several times greater than the lattice spacing, it follows from the electro-magnetic theory of waves (~~Kinematic~~ that there will be no scattering. If the particles forming the lattice are slightly displaced, dispersion ensues. This explains the fact of lowering of thermal conductivity by presence of even minute amounts of impurities in a metal.

(h) mean free path – on the particle aspect this is defined as the ratio of the total path over a long interval of time to the number of collisions in the interval with the ions. On the wave aspect it is measured by the rate at which an electron beam is dispersed by the ions or as put by Fowler "The mean free path measures the space rate of loss of directed momentum in a given electron beam". Thus dispersion causes a decrease in the mean free path. ~~This~~

Consider the fact that conductivity of pure metals decreases with increasing temperature. On the wave aspect this is explained by the departure from geometrical irregularity of the ionic lattice and the consequent increase of the scattering of electron waves. This also means a decrease in the mean free path. On the particle aspect we might say that in view of degeneration the total energy of the electrons is independent of the temperature being the zero point energy given by (19) while the ions vibrate more violently with higher temprature. Thus under these conditions chances of collision are increased & hence no of collisions per unit time increases decreasing the mean free path. This means decrease of ~~ele~~ conductivity.

(i) <u>Maximum energy of electrons _ Photoelectric threshhold.</u>

In Eqn (19) $$E_{T=0} = 0.6 \, N_t \, K/\theta$$

the factor 0.6 can be shown to be the ratio between mean & max. Kinetic energies of the electron. Omitting this factor the energy of the fastest electrons is given on the average (by dividing by N_t) by K/θ, where θ is found from (18) taking account of electron spin. From (18) it is found that $1/\theta \propto N_t$ and hence $\propto$ compactness of the ionic lattice. If the average max. K.E is measured in volts it is found that this equivalent voltage of fastest electrons varies from 2.1 to 6 volts (from K to Pt) in contradiction to the classical theory where these were computed to within fractions of a volt. Further the value of the equivalent voltage does not depend on the temperature as long as the electron gas is degenerate. This fact can be made to account for the photoelectric threshhold thus:—

Let the frequency of the incident light be gradually increased until the quanta of light impinge upon the fastest electrons just below the surface, communicate to them their full energy such that the fastest electrons just begin to be emitted. This is known as the photoelectric threshhold. Since the speed of the fastest electrons in a degenerate gas is independent of the temperature, the threshhold frequency should also be the same. According to the classical theory if electrons behaved like a perfect gas it would require less energy to eject them at a higher temperature & thus the photoelectric thresh hold frequency would be lower at higher temperatures which is not true.

(f) Energy distribution in degenerate Electron gas. The state of an aggregation of particles at a given temperature may be characterised by the fraction of total number of particles which have energies between E and $E + dE$. Let this "distribution - in - energy" function be $F(E)$ such that

$$n = \int_0^\infty F(E)\, dE \qquad (23)$$

n being the total number of particles per unit volume. F is also defined by

$$dn = F(E)\, dE \qquad (24)$$

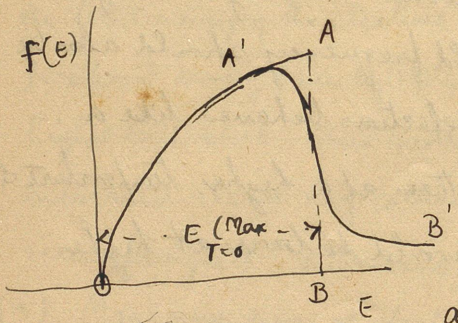
or again by $F = dn/dE$ ——————— (25)

For a completely degenerate electron gas obeying the Fermi - Dirac statics we have from (11)

$$dn = N = (V/h^3)\, 2\pi\, (2m)^{3/2} E^{1/2}\, dE \Big/ e^{\alpha+\beta E} + 1$$

where $e^{\alpha+\beta E}$ can be neglected in comparison with unity since α has large ~~positive~~ negative values and E has small values

Hence $F(E) \propto E^{1/2}$ which gives a parabola



For values of E greater than the maximum $p'e$ (section i), $F = 0$. Thus the form of the curve is OAB. At higher temp the factor $\beta = 1/KT$ has an influence and the curve has really the form $OA'B'$.

The curve shows that there some electrons which acquire high velocities & this fact accounts for thermionic emission.

(K) <u>Momentum - distribution</u> :— $f(p)$ is defined as the number of gas particles per unit volume in the momentum space so chosen as to refer to the total number of particles in unit volume ~~for~~ in the actual space. ie it is also the number of particles per unit vol in real space possessing momenta between p and $p + dp$.

volume of shell of thickness dp at p in momentum space is $4\pi p^2 dp$. Hence the total number of particles in real space is

$$n = \int_0^\infty 4\pi f(p) p^2 dp \quad \text{———} \quad (26).$$

$$\text{or} \quad dn = 4\pi f(p) p^2 dp \quad \text{———} \quad (27)$$

Comparing (24) and (27)

$$F(E)\, dE = 4\pi f(p) p^2 dp \quad \text{———} \quad (28)$$

$$\text{but } p^2 = 2mE \quad \text{———} \quad (29)$$

$$\therefore 2p\,dp = 2m\,dE \text{ i.e. } p\,dp = m\,dE \quad \text{———} \quad (30)$$

$$\therefore F(E) = 4\pi m p f(p) = 2\pi (2m)^{3/2} E^{1/2} f(p) \quad \text{———} \quad (31)$$

but from (11) $F(E)\, dE = N = \left(\frac{1}{h^3}\right) 2\pi (2m)^{3/2} E^{1/2} dE \Big/ e^{\alpha+\beta E}+1$

[putting $V = 1$

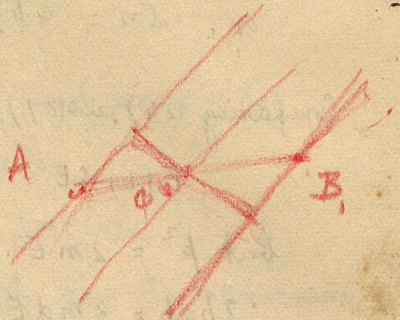
$$\text{Hence} \quad f(p) = (1/h^3) \Big/ [\exp(\alpha+\beta E)+1] \quad \text{———} \quad (32)$$

If applied to electrons, taking electron spin into consideration we replace $(1/h^3)$ by $(2/h^3)$ in (32).

For a completely degenerate electron gas we have $f(p) = 2/h^3$ and the function is represented by a horizontal st: line. At very high

Ramsey Statics p-292 Ex 9

$$\cancel{l}\sin\theta < \frac{\cancel{l}}{\sqrt{2}}\sec\tfrac{1}{2}\phi$$



$$\sqrt{2}\sin\theta < \sec^2\tfrac{1}{2}\phi$$

$$2\sin^2\theta\cos^2\tfrac{1}{2}\phi < 1$$

$$\sin^2\theta\,(1+\cos\phi) < 1$$

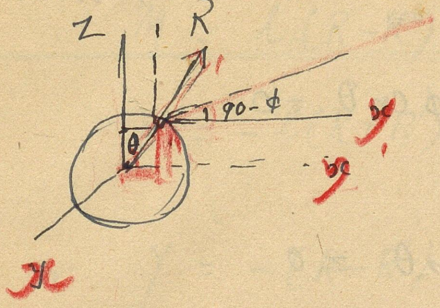
$$\sin^2\theta\cos\phi < \cos^2\theta.$$

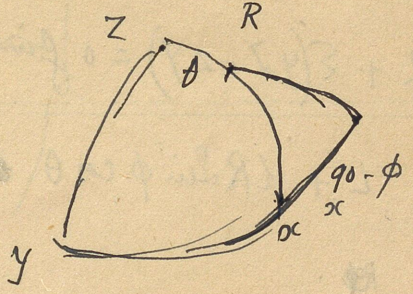
$$\cos\phi. < \cot^2\theta.$$

$$\tan^2\theta < \sec\phi$$

$$1+\sec^2\theta < \sec\phi$$

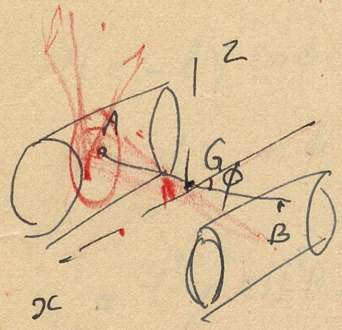
Ramsey Statics p. 292. Ex. 9.





$$2l\sin\phi = 2c - 2a\sin\theta$$

$$l\sin\phi = c - a\sin\theta$$



R at A $(0, R\sin\theta, R\cos\theta)$

R' at B $(0, -R'\sin\theta, R'\cos\theta)$

P at G (X, Y, Z)

W at G $(0, 0, -Mg)$

Let couple be L, M, N.

$X = 0, \quad Y + (R - R')\sin\theta = 0$

$-\cancel{Mg} + Z + (R + R')\cos\theta = 0.$

$L + \Sigma yZ - zY$

Coords of A are $(l\cos\phi, l\sin\phi, 0)$, $B(-l\cos\phi, -l\sin\phi, 0)$

$$L + \Sigma(yZ - zY) = 0 \text{ gives, } M + \Sigma(zX - xZ),\ N + \Sigma(xY - yX) = 0$$

$$L + lR\sin\phi\cos\theta - lR'\sin\phi\cos\theta = 0.$$

$$M - lR\cos\phi\cos\theta + lR'\cos\phi\cos\theta = 0$$

$$N + lR\sin\theta\cos\phi + lR'\sin\theta\cos\phi = 0.$$

~~When P acts vertically~~ $Y = 0$, ie $R = R'$

~~then $L = M = 0$~~

$$\cancel{N = -2lR\sin\theta\cos\phi}$$

$$\cancel{P = -2R\cos\theta}.$$

$$\cancel{N = -2l\sin\theta\cos\phi \cdot \frac{P}{2\cos\theta}}$$

$$\cancel{= Pl\tan\theta\cos\phi}.$$

$$\left.\begin{aligned} L &= -l(R-R')\sin\phi\cos\theta \\ M &= l(R-R')\cos\phi\cos\theta. \\ N &= -l(R+R')\sin\theta\cos\phi \end{aligned}\right\}$$

or $Y = -(R-R')\sin\theta$

$Z = -(R+R')\cos\theta$

$$\left.\begin{aligned} L &= lY\sin\phi\cot\theta \\ M &= -lY\cos\phi\cot\theta \\ N &= lZ\cos\phi\tan\theta \end{aligned}\right\}$$

$P = \sqrt{Y^2+Z^2}$

$\frac{\cos\theta}{\sin\theta} = 0$

$\theta = \pi/2$;

$G^2 = l^2Y^2\cot^2\theta + l^2Z^2\cos^2\phi\tan^2\theta.$

~~$l^2Y^2\cot\theta\operatorname{cosec}^2\theta = l^2Z^2\cos^2\phi\tan\theta\sec^2\theta.$~~

$l^2Y^2\frac{\cos\theta}{\sin^3\theta} = l^2Z^2\cos^2\phi\frac{\sin\theta}{\cos^3\theta}$

when $Y=0$, $G^2 = l^2Z^2\cos^2\phi\tan^2\theta$

$Z=0$, $G^2 = l^2Z^2\cot^2\theta.$

$Y^2\cos^4\theta = Z^2\cos^2\phi\sin^4\theta.$

P^2

i.e. $Y^2\cot^2\theta = Z^2\cos^2\phi\tan^2\theta.$

Sin $p = \sqrt{y^2+z^2}$

When $y = 0$ ie $z = p$,

$$G^2 = l^2p^2\cos^2\phi\tan^2\theta$$

When $z = 0$ ie $y = p$

$$G^2 = l^2p^2\cot^2\theta.$$

the former is less than latter

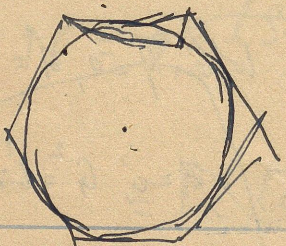
if $\cos^2\phi\tan^2\theta < \cot^2\theta$

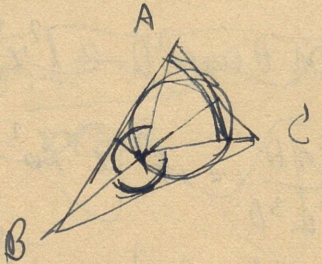
$$\cos\phi < \cot^2\theta$$

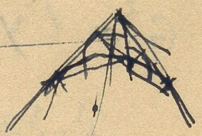
this reduces to

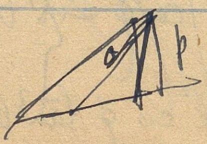
$$c < l\sin\phi + \frac{a}{\sqrt{2}}\sec\tfrac{1}{2}\phi.$$

(6)

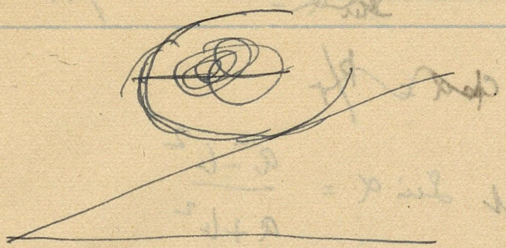


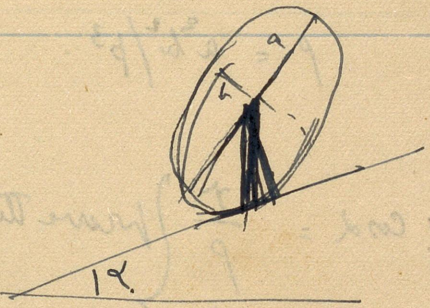






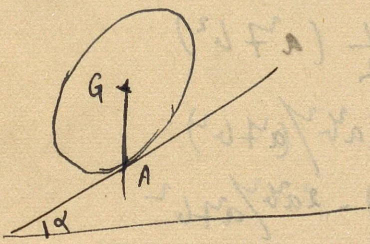
$\frac{2m}{p}\sin 2\theta$



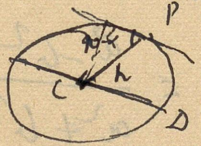


Elliptic cylinder eqm:

If $\frac{\cos\alpha}{h} = \frac{1}{\rho}$.
eqm is neutral.



$$\rho = \frac{CD^3}{ab} = \frac{a^2b^2}{p^3}.$$



$$p = h \sin\alpha$$

$$\cos\alpha = \frac{h}{\rho} = \frac{hp^3}{a^2b^2}.$$

$$2\sin\alpha\cos\alpha = \frac{2p^4}{a^2b^2}$$

$$\sin 2\alpha = \frac{2p^4}{a^2b^2}.$$

$$\frac{\cos\alpha}{h} = \frac{ab}{CD^3} \qquad CD^3\cos\alpha = abh.$$

$$\frac{a^2b^2}{p^2} = \frac{1}{2}(a^2+b^2)$$

$$a^2+b^2-r^2 = a^2b^2/p^2$$

$$\cancel{\sin\alpha} = \qquad 2a^2b^2/a^2+b^2$$

$$\rho = a^2b^2/p^3 .$$

$$\cancel{\cos\alpha = p/r}$$

If $\cos\alpha = \dfrac{r}{\rho} = \dfrac{p}{r}$ prove that $\sin\alpha = \dfrac{a^2-b^2}{a^2+b^2}$

$$\frac{r}{\rho} = \frac{p}{r} \qquad r^2 = p\rho$$

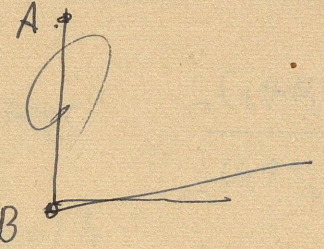
$$r^2 = p \cdot a^2b^2/p^3 \quad \text{ie} \quad r = \frac{ab}{p} .$$

$$a^2+b^2-r^2 = r^2 \qquad r^2 = \frac{1}{2}(a^2+b^2)$$

$$p^2 = 2a^2b^2/(a^2+b^2)$$

$$\sin^2\alpha = \frac{r^2-p^2}{r^2} = \frac{\frac{1}{2}(a^2+b^2) - 2a^2b^2/a^2+b^2}{\frac{1}{2}(a^2+b^2)}$$

$$= \frac{(a^2-b^2)^2}{(a^2+b^2)^2} \qquad \sin\alpha = \frac{a^2-b^2}{a^2+b^2}$$

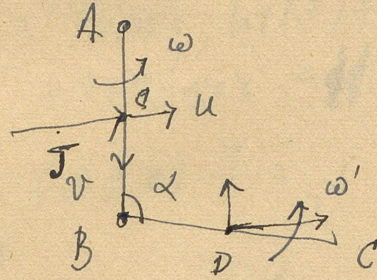


$C(u, v)$.

$B(u + a\omega, v)$

$D(u + a\omega - b\omega' \sin\alpha,$

$v + b\omega' \cos\alpha)$.

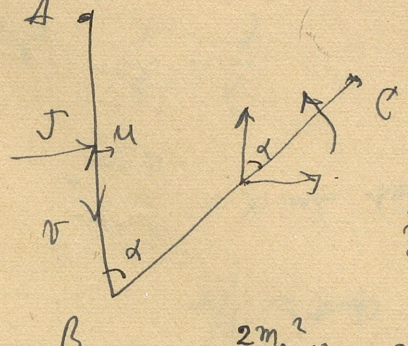


$$2T = mu^2 + mv^2 + \frac{m}{3}a^2\omega^2$$

$$+ m'(u + a\omega - b\omega'\sin\alpha)^2$$

$$+ m'(v + b\omega'\cos\alpha)^2$$

$$+ \frac{1}{3}m'b^2\omega'^2$$



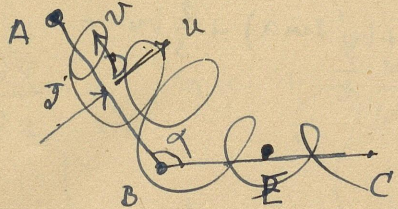
$$\frac{\partial T}{\partial \omega} = 0, \quad \frac{\partial T}{\partial \omega'} = 0.$$

$$\frac{2m}{3}a^2\omega + 2m'a(u + a\omega - b\omega'\sin\alpha) = 0.$$

$$2b\cos\alpha\, m'(v + b\omega'\cos\alpha)$$

$$- 2b\sin\alpha\, m'(u + a\omega - b\omega'\sin\alpha) + \frac{2}{3}m'b^2\omega' = 0.$$

Ramsey p. 246. Ex 12:

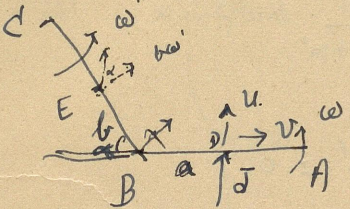


Vel B = vel of D + vel of B rel to D

$= (u - a\omega,\ v)$.

vel of E = vel of B + vel of E rel to B

$= (u - a\omega \pm b\omega'\cos\alpha,$
$\quad v + b\omega'\sin\alpha$.



~~$u + u - a\omega \pm b\omega'\cos\alpha =$~~

$$\left.\begin{aligned} mu + m'(u - a\omega \pm b\omega'\cos\alpha) &= J \\ mv + m'(v + b\omega'\sin\alpha) &= 0 \end{aligned}\right\}\ \begin{matrix}(1)\\(2)\end{matrix}$$

$$T = \tfrac{1}{2}mu^2 + \tfrac{1}{2}mv^2 + \tfrac{1}{3}ma^2\omega^2 + \tfrac{1}{2}m'(u - a\omega \pm b\omega'\cos\alpha)^2$$
$$+ \tfrac{1}{2}m'(v + b\omega'\sin\alpha)^2 + \tfrac{1}{3}m'b^2\omega'^2$$

$$\frac{\partial T}{\partial\omega} = 0, \quad \tfrac{2}{3}ma^2\omega + m'(u - a\omega \pm b\omega'\cos\alpha)(-a) = 0 \quad \text{--- (3)}$$

~~[illegible]~~

~~[illegible]~~

$$\frac{\partial T}{\partial\omega'} = 0, \quad m'(u - a\omega \pm b\omega'\cos\alpha)(\pm b\cos\alpha)$$
$$+ m'(v + b\omega'\sin\alpha)(b\sin\alpha) + \tfrac{2}{3}m'b^2\omega' = 0.$$

--- (4).

$m u + m'($

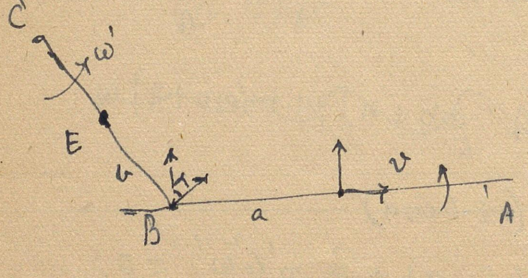
$$\begin{cases} 2ma\omega - 3m'(u - a\omega \pm b\omega'\cos\alpha) = 0 \\ \pm \cos\alpha\,(u - a\omega \pm b\omega'\cos\alpha) + \sin\alpha\,(v + b\omega'\sin\alpha) + \frac{2}{3}b\omega' = 0. \\ mu + m'(u - a\omega \pm b\omega'\cos\alpha) = J \\ mv + m'(v + b\omega'\sin\alpha) = 0. \end{cases}$$

$$v(m+m') = -m'b\omega'\sin\alpha \qquad v = -\frac{m'}{m+m'}b\omega'\sin\alpha.$$

$$\pm\cos\alpha\cdot\frac{2ma\omega}{3m'} + \sin\alpha\left(-\frac{mv}{m'}\right) + \frac{2}{3}b\omega' = 0.$$

$$\pm\cos\alpha\cdot\frac{2ma\omega}{3} + \frac{m\sin\alpha\cdot m'}{m+m'}b\omega'\sin\alpha + \frac{2}{3}m'b\omega' = 0,$$

$$\pm 2ma\omega\cos\alpha\,(m+m') + 3mm'\sin^2\alpha\,b\omega' + 2m'b\omega'(m+m') = 0.$$



$$T = \frac{1}{2}mu^2 + \frac{1}{2}mv^2 + \frac{1}{3}ma^2\omega^2 + \frac{1}{3}m'b^2\omega'^2$$

$u - a\omega,\ v$

$$\left.\begin{aligned} u - a\omega &= b\omega'\cos\alpha \\ v &= b\omega'\sin\alpha \end{aligned}\right\}$$

$$T = \frac{1}{2} m u^2 + \frac{1}{2} m v^2 + \frac{1}{3} m a^2 \omega^2$$

$$+ \frac{1}{3} m' b' \frac{(u - a\omega)^2}{b^2 \cos^2\alpha}.$$

$$\frac{\partial T}{\partial u} = 0, \quad m u + \frac{2}{3} \frac{m'}{b' \cos^2\alpha} \cdot (u - a\omega) = 0. \qquad , v = 0$$

$$\frac{\partial T}{\partial \omega} = 0, \quad \cancel{\tfrac{2}{3}} m a^2 \omega - \frac{a \cancel{2} m'}{\cancel{3} b' \cos^2\alpha} (u - a\omega) = 0.$$

$$m a \omega = \frac{m'}{b' \cos^2\alpha} (u - a\omega) = 0.$$

$$a\omega \left(m + \frac{m'}{b' \cos^2\alpha} \right) = \frac{m' u}{b' \cos^2\alpha}.$$

$$v = \frac{\cancel{b} \sin\alpha (u - a\omega)}{\cancel{b} \cos\alpha} = (u - a\omega) \tan\alpha$$

$$T = \frac{1}{2} m u^2 + \frac{1}{2} m \tan^2\alpha (u - a\omega)^2 + \frac{1}{3} m a^2 \omega^2$$

$$+ \frac{1}{3} m' b^2 \frac{(u - a\omega)^2}{b^2 \cos^2\alpha}.$$

$m u$

$2T =$

$$m(b\omega'\cos\alpha - a\omega)^2 + m'(b\omega'\sin\alpha)^2$$

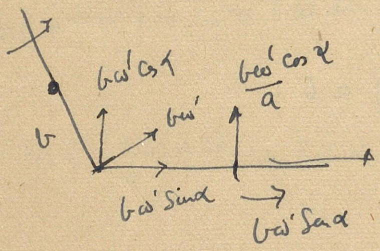
$$+\frac{1}{3}ma^2(\omega-\omega_0)^2 + \frac{1}{3}m'b^2(\omega'-\omega_0')^2$$

Stet

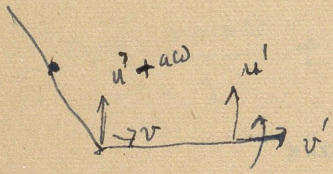
$$\frac{\partial T}{\partial \omega} = 0, \quad -a(b\omega'\cos\alpha - a\omega) + \omega'.\, b^2\sin^2\alpha$$

$$+\frac{1}{3}ma^2(\omega-\omega_0) + \cancel{\frac{1}{3}m'b^2(\omega'-\omega_0')}$$

$$= 0.$$



$$u - a\omega - b\omega'\cos\alpha = 0.$$
$$v + b\omega'\sin\alpha = 0.$$



$$2T = m(u-u_0)^2 + m(v-v_0)^2 + \frac{1}{3}ma^2(\omega-\omega_0)^2$$

$$+\frac{1}{3}m'b^2(\omega'-\omega_0')^2 + \cancel{f}(u, v, \ldots)$$

$$\frac{\partial T}{\partial \omega} = 0, \quad \frac{\partial T}{\partial \omega} = 0, \quad \frac{\partial T}{}$$

$$= (Mgh + mgl) \sin\theta$$

Putting $\theta = \frac{1}{2}\pi + x$

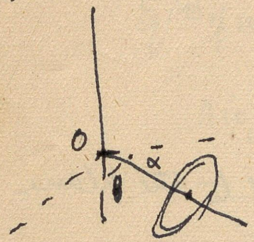
Let α be the ~~[illegible]~~ dip of the axis after the blow

$$R = Mg \Big/ \left\{ 1 + \frac{(\nu y - \mu x)^2}{a^2} + \cdots \right\}$$

In the problem the coordinates of the normal are ~~a~~ meant to be what we have here called $(\nu y - \mu x)$ etc

(22) See Loney's Treatise

(23) After the disturbance



$$A\ddot{\theta} + Cn\dot{\psi}\sin\theta - A\dot{\psi}^2 \sin\theta\cos\theta$$

$$= (Mgh + mgl)\sin\theta$$

$$A\dot{\psi}\sin^2\theta + Cn\cos\theta = A\Omega$$

~~If the axis descends through a small angle α, then expressing the change of A.M. about Oz, we have~~

$$\cancel{mvl = Cn}$$

~~Also since initial motion was steady with $\theta = \pi/2$~~

$$\cancel{Cn\Omega = Mgh.}$$

Eliminating $\dot{\psi}$ from $\dot{\psi} = (\Omega - \frac{C}{A} n\cos\theta)/\sin^2\theta$

$$A\ddot{\theta} + Cn\sin\theta\left(\Omega - \frac{Cn}{A}\cos\theta\right) - A\left(\Omega - \frac{Cn}{A}\cos\theta\right)^2 \frac{\sin\theta\cos\theta}{\sin^4\theta}$$

Resolving vertically

$$M\ddot{p} = R - Mg \qquad - \quad - \quad (2)$$

Also since λ, μ, ν are direction cosines of a fixed direction viz

the vertical

$$\left.\begin{aligned} \dot{\lambda} - \mu\omega_3 + \nu\omega_2 &= 0 \\ \dot{\mu} - \nu\omega_1 + \lambda\omega_3 &= 0 \\ \dot{\nu} - \lambda\omega_2 + \mu\omega_1 &= 0 \end{aligned}\right\} \qquad (3)$$

from eq^ns of moving axes.

Velocity of point of contact resolved normal to the plane

should be zero i.e.

$$\dot{p} + (y\omega_3 - z\omega_2)\lambda + (z\omega_1 - x\omega_3)\mu + (x\omega_2 - y\omega_1)\nu = 0.$$

$$- \, - \, (4)$$

$$\ddot{p} + (y\dot{\omega}_3 - z\dot{\omega}_2)\lambda + (y\omega_3 - z\omega_2)\dot{\lambda} + \cdots = 0.$$

$$+ (z\dot{\omega}_1 - x\dot{\omega}_3)\mu + (z\omega_1 - x\omega_3)\dot{\mu}$$

$$+ (x\dot{\omega}_2 - y\dot{\omega}_1)\nu + (x\omega_2 - y\omega_1)\dot{\nu} = 0$$

i.e. $\ddot{p} + \dot{\omega}_1(\mu z - \nu y) + \dot{\omega}_2(\nu x - \lambda z) + \dot{\omega}_3(\lambda y - \mu x) = 0$, putting ω's $= 0$

i.e. $\dfrac{R}{M} - g + \dfrac{R(\nu y - \mu z)^2}{A} + \cdots = 0$

Proceeding exactly as in 12·23 we get the result required.

(20) obvious appln of bookwork.

(21) Let $\omega_1, \omega_2, \omega_3$ denote angular velocities about the principal axes at the centre of inertia, x, y, z coords of the pt of contact, p the $\perp^r$ from the C.G on the given plane

Eqns of moments about the axes are

$$\left. \begin{array}{l} A\dot{\omega}_1 - (B-C)\,\omega_2\omega_3 = p(yZ - zY) \\ \text{— — —} \\ \text{— — —} \end{array} \right\}$$

X, Y, Z being the components of the reaction at (x, y, z)

and hence writing $X = -\lambda R,\ Y = -\mu R,\ Z = -\nu R$

the eqns become

$$\left. \begin{array}{l} A\dot{\omega}_1 - (B-C)\,\omega_2\omega_3 = -R(\nu y - \mu z) \\ \text{— — —} \\ \text{— — —} \end{array} \right\}$$

Since the initial velocities are zero

$$A\dot{\omega}_1 = -R(\nu y - \mu z), \text{ etc} - . \qquad (1)$$

(15) The first part is bookwork. For second part See Loney § 262-63.

(16) Let a small deviation θ of the diameter OC from the direction of the earth's axis. The second eq^n of (3) of 12.2 gives

$$A\ddot{\theta} + (Cn - A\Omega\cos\theta)\Omega\sin\theta = 0$$

In order that OC be always in the meridian $\Omega = \omega$. Next

Also $C = A$ and for small θ, therefore

$$\ddot{\theta} + (n-\omega)\omega\theta = 0$$

i.e. period is $\dfrac{2\pi}{\sqrt{n\omega - \omega^2}}$.

(17) This is identically the same as Ex 5, p. 311

(18) This is § 12.6 only in different notation

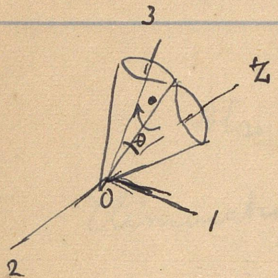
(19) Same ~~$\cos\beta = \lambda - \sqrt{1-2\lambda\cos\alpha+\lambda^2}$~~

. Initial conditions $\dot{\theta} = 0$, $\theta = \alpha$, $\dot{\psi} = \Omega$. Eq^ns (4), (5) of 12.2 become

$$\left.\begin{aligned} A\dot{\psi}\sin^2\theta + Cn\cos\theta &= A\Omega\sin^2\alpha + Cn\cos\alpha \\ A(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2) + 2Mgh\cos\theta &= A(\Omega^2\sin^2\alpha) + 2Mgh\cos\alpha \end{aligned}\right\}$$

The angular velocity Ω about the axis of the fixed cone is given by

$$\mu \sin 2\alpha \text{ (about } 01) = \omega \sin \alpha$$



$$\text{i.e. } \mu = \omega \sin\alpha / \sin 2\alpha$$

Rate of change of A. M about 0 Z

$[\theta = 2\alpha]$
$$= (Cn - A\Omega\cos\theta)\Omega\sin\theta \qquad (\S 12.13 \text{ or } 12.12)$$

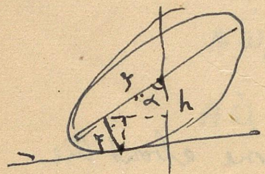
$$= \left(C\omega\cos\alpha - A\omega\frac{\sin\alpha}{\sin}\cos 2\alpha\right)\omega\frac{\sin\alpha}{\sin 2\alpha}\cdot\sin 2\alpha$$

$$= \frac{1}{2}\omega^2\tan\alpha\left\{C + (C-A)\cos 2\alpha\right\}.$$

(14) From § 12.41, cond^n for steady motion is

$$Cn\Omega\sin\alpha - A\Omega^2\sin\alpha\cos\alpha = mg(\bar{y}\sin\alpha - \bar{x}\cos\alpha)$$

$$h = \bar{x}\cos\alpha + \bar{y}\sin\alpha.$$



Substituting the right values for C & A

the cond^n for real roots of Ω gives

limiting value for n^2 etc

Multiply from (11)

$$A(\dot{\omega}_1 + i\dot{\omega}_2) = (C-A)(\omega_1 + i\omega_2)\, i\,\frac{Nt}{C}$$

$$\frac{d}{dt}\left\{\log(\omega_1 + i\omega_2)\right\} = \frac{C-A}{A}\cdot\frac{iNt}{C}$$

giving $\quad \omega_1 + i\omega_2 = e^{\frac{C-A}{A}\cdot\frac{iNt^2}{2C}}$

$$\tan^{-1}(\omega_2/\omega_1) = \frac{C-A}{A}\cdot\frac{Nt^2}{2C}$$

For the instantaneous axis

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3} = \frac{(x^2+y^2)^{1/2}}{\Omega}$$

$$\therefore \frac{Nt^2}{2C} = \frac{C\,\Omega^2 z^2}{2N(x^2+y^2)}$$

$$\therefore \tan^{-1}\left(\frac{y}{x}\right) = \frac{C-A}{A}\cdot\frac{\Omega^2 C}{2N}\cdot\frac{z^2}{x^2+y^2}$$

(12)

(13) The resultant angular velocity is that of the rolling cone about the generator in contact with the fixed cone. Hence angular velocity about the axis of revolution $= \omega\cos\alpha$

$$\therefore Cn = C\omega\cos\alpha.$$

From the Condn of rolling (2) in § 12.6

$$v - a\Omega \sin\alpha\cos\alpha + an\sin\alpha = 0 \quad (2)$$

Also $v = -\Omega r$ (3)

Eliminating n and Ω from (1) – (3)

$$n = -\frac{v(a\sin\alpha\cos\alpha + r)}{ar\sin\alpha}, \quad \Omega = -\frac{v}{r}$$

$$K^2\omega - \frac{a^2 v(a\sin\alpha\cos\alpha + r)}{ar\sin\alpha} = -(k^2+a^2)\frac{v\cos\alpha}{r}.$$

$$aK^2\omega r\sin\alpha - \cancel{a^3 v\sin\alpha\cos\alpha} - a^2vr = \cancel{a^3 v\sin\alpha\cos\alpha} - ak^2 v\sin\alpha\cos\alpha$$

$$k^2\sin\alpha(v\cos\alpha - r\omega) = var.$$

(11) The eqns of motion are

$$\left.\begin{aligned} A\dot\omega_1 - (A-C)\omega_2\omega_3 &= 0 \\ A\dot\omega_2 - (C-A)\omega_3\omega_1 &= 0. \\ C\dot\omega_3 &= N \end{aligned}\right\} \quad (1)$$

The third gives $C\omega_3 = Nt$ (2)

& the first two combine $\omega_1^2 + \omega_2^2 = \Omega^2$ (3)

(10) This is a special case of § 12·6 where the inertia of the outer shell is neglected, the inner gyrostat is a sphere. From eqn (8) of ~~the~~ § 12·6, putting $C_\phi = A_\phi = 0$ we get, since further

$C_1 = A_1$,

$$n \cos\theta + \dot\psi \sin^2\theta = \text{const}$$

expressing constancy of A·M of inner sphere about vertical.

Again eqn (9) of same article reduces in case of steady motion with $m_\phi = 0$, etc, $\dot\theta' = 0$, $\theta = \alpha$, $\dot\psi = \Omega$,

$$ma^2(-\Omega^2 \sin\alpha\cos\alpha + \Omega\omega\sin\alpha)$$
$$+ A(-\Omega^2 \sin\alpha\cos\alpha) + C\omega\,\Omega\sin\alpha = 0 \qquad A = \cancel{C = mk^2}$$
$$- (k^2+a^2)\Omega^2\cos\alpha$$

$$m_1 a^2(-\Omega^2 \sin\alpha\cos\alpha + \Omega n \sin\alpha)$$
$$+ A_1(-\Omega^2 \sin\alpha\cos\alpha) + C_1\omega\Omega\sin\alpha = 0$$

(n = spin of the shell, ω = that of the sphere)

ie $\quad k^2\omega + a^2 n = (k^2+a^2)\Omega\cos\alpha \qquad$ ——— (1)

~~For motion round a vertical axis through G~~

For rotational motion taking G3, as axis of gyrostat, G1 in the vertical plane, the eqⁿ of

$$\omega_1 = \theta_1 = -n \sin\phi,\ \theta_2 = 0 = \omega_2,\ \theta_3 = n\cos\phi$$

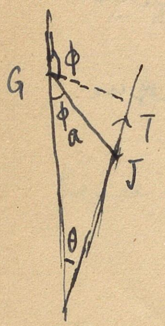
Since motion is steady and

$$A\dot{\omega}_2 - C\omega_3\theta_1 + A\omega_1\theta_3 = \text{moment of forces about } G2$$

gives since $\dot{\omega}_2 = 0$, $\omega_3 = \omega = \text{const}$

$$C\omega n \sin\phi - An^2 \sin\phi\cos\phi$$

$$= Ta\sin(\phi - \theta) \quad \text{——} (3)$$



If ϕ be angle made by JG with the upward vertical

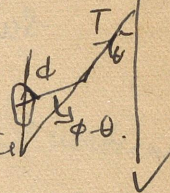
Using (2), eqⁿ (3) becomes

$$n\sin\phi\,(C\omega - An\cos\phi)\cos\theta = Mga\sin(\phi - \theta)$$

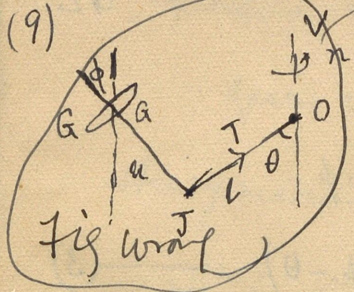
which is the other eqⁿˢ required.

and another $(A-B)n\omega_1$ in another axial plane. The components of the latter are periodic gyrostatic couples passing through all phases twice as ϕ varies from 0 to 2π ie the axial plane revolves twice as fast as the disc on its axis ($2\dot{\phi} = 2n$).

In the present example replace ω_1 by $\dot{\psi}$



(9)



Let J be the universal joint, JO the string, JG the axis of the gyrostat G being the centre of gravity. $OJ = l$, $JG = a$

G is turning about the vertical OV with uniform angular velocity n, and when the motion is steady OJ, JG remain in the vertical plane through OV.

For the equations of motion of the centroid G

$$Mn^2(l\sin\theta + a\sin\phi) = T\sin\theta \quad \text{——(1)}$$

$$Mg = T\cos\theta \quad \text{——(2)}$$

These together give $n^2(l\sin\theta + a\sin\phi) = g\tan\theta$

one of the eq^{ns} required.

Eulerian eqns w.r.t O (A, B, C) give

$$A\dot{\omega}_A - (B-C)\,\omega_B n = L$$

$$B\dot{\omega}_B - (C-A)\,n\omega_A = M$$

$$C\dot{n} - (A-B)\,\omega_A\,\omega_B = N$$

as the couples called into play. Suppose one of the axes OD is fixed at right angles to OC so that $\dot{\phi} = n$, and also let $\omega_2 = 0$, $\dot{\omega}_2 = 0$ and $\dot{\omega}_1 = 0$

then $\omega_A = \omega_1 \cos\phi$; $\omega_B = -\omega_1 \sin\phi$; $\dot{\omega}_A = -\omega_1 \sin\phi \cdot n$
$\dot{\omega}_B = -\omega_1 \cos\phi \cdot n$

and the above eqns becomes

$$\left.\begin{aligned} &-An\omega_1 \sin\phi = L \\ &-Bn\omega_1 \cos\phi \mp (C-A)\,n\omega_1 \cos\phi = M \end{aligned}\right\}$$

Hence for couples about OD & OE we get

$$L\cos\phi - M\sin\phi \quad \text{and} \quad L\sin\phi + M\cos\phi$$

ie $-(A-B)\,\omega_1 n \sin 2\phi$ about OD

and $(A-B)\,\omega_1 n \cos 2\phi - Cn\omega_1$ about OE

These might be combined into two couples $Cn\omega_1$ in a vertical axial plane (OE horizontal) and the first equation becomes

$$\dot{\theta} \tan \chi \sin \theta \propto \lambda \quad \text{— (5)}$$

Dividing (5) by (4) we have

$$\sin \chi \sin \theta \propto \sqrt{\lambda}$$

or using (7) $\quad \sin^2 \omega \propto \lambda$

$$\propto (\cos \alpha - \cos \theta).$$

(7) Equivalent to rise & fall of a top. $\dot{\psi}$ constant but $\theta \neq \alpha$ throughout

(8) Consider the ~~disc~~ ~~as a~~ propeller as ~~a~~ disc rotating with angular speed n about an axis OC at rt L^s to its plane. Let OA, OB be two other pr. axes in plane. Let OD, OE be two other axes fixed w. r. t in the aeroplane and let ~~disc~~ the disc be turning with angular speeds ω_1, ω_2 about space axes coinciding at the instant with OD, OE. If at instant considered OA has turned through ϕ from OD angular speeds ~~[illegible]~~ about OA, OB (ω_A, ω_B) are

$$\left.\begin{aligned} \omega_A &= \omega_1 \cos\phi + \omega_2 \sin\phi \\ \omega_B &= -\omega_1 \sin\phi + \omega_2 \cos\phi \end{aligned}\right\}$$

$$U \frac{\partial \vec{q}}{dx} = -\frac{1}{\rho} \operatorname{grad} p + \nu \nabla^2 \vec{q}. \qquad (1)$$

$$\operatorname{div} \vec{q} = 0 \qquad (2)$$

Taking divergence of (1), we have $\nabla^2 p = 0$ (3)

$$p = \rho U \frac{\partial \phi}{\partial x} \qquad (4)$$

satisfies (1) & (2) if $\vec{q} = -\operatorname{grad}\phi$ (5)

and $\nabla^2 \phi = 0$ (6)

(4) gives a particular solution, and general solution is given by

$$\vec{q} = -\operatorname{grad}\phi + \vec{q}\,' \qquad (7)$$

where $-U \operatorname{grad} \frac{\partial \phi}{\partial x} + U \frac{\partial \vec{q}\,'}{\partial x} = -U \operatorname{grad} \frac{\partial \phi}{\partial x} + \nu \nabla^2(-\operatorname{grad}\phi) + \nu \nabla^2 \vec{q}\,' = 0$

ie $U \frac{\partial \vec{q}\,'}{\partial x} = \nu \nabla^2 \vec{q}\,' = 0$ since $\nabla^2(\operatorname{grad}\phi) = 0$.

and from (2) $\operatorname{div}\vec{q}\,' = 0$ ie $\vec{q}\,'$ must satisfy

$$\left(\nabla^2 - \frac{U}{\nu}\frac{\partial}{\partial x}\right)\vec{q}\,' = 0 \qquad \cancel{(8)}$$

or $\left(\nabla^2 - 2k\frac{\partial}{\partial x}\right)\vec{q}\,' = 0$ (8)

and $\operatorname{div}\vec{q}\,' = 0$ (9)

The motion considered is that of sphere along Ox, and hence is symmetrical about Ox & hence vortex lines are circles having centres on Ox. Hence we may assume

$$\xi = 0, \quad \eta = -\frac{\partial \chi}{\partial z}, \quad \zeta = \frac{\partial \chi}{\partial y} \tag{10}$$

where χ is a function of x and ϖ.

From (8), $\nabla^2 \dfrac{\partial u'}{\partial z} - 2k\dfrac{\partial^2 u'}{\partial z \partial x} = 0$ & $\nabla^2 \dfrac{\partial w'}{\partial x} - 2k \dfrac{\partial^2 w'}{\partial x^2} = 0$

Hence $\nabla^2\left(-\dfrac{\partial \chi}{\partial z}\right) - 2k\dfrac{\partial}{\partial x}\left(-\dfrac{\partial \chi}{\partial z}\right) = 0$, $\left(\eta = \dfrac{\partial u'}{\partial z} - \dfrac{\partial w'}{\partial x}\right)$

Simly $\nabla^2\left(\dfrac{\partial \chi}{\partial y}\right) - 2k\dfrac{\partial}{\partial x}\left(\dfrac{\partial \chi}{\partial y}\right) = 0$

$$\therefore \quad \left(\nabla^2 - 2k\frac{\partial}{\partial x}\right)\chi = 0 \tag{11}$$

plus a fn independent of y & z

ie a function of x which is irrelevant for determination of η & ζ.

Since $\operatorname{div} \vec{q}\,' = 0$, we have $\nabla_\wedge \nabla_\wedge \vec{q}\,' = \nabla^2 \vec{q}\,'$

$$\therefore \quad 2k\frac{\partial u'}{\partial x} = \nabla^2 u' = \frac{\partial \eta}{\partial z} - \frac{\partial \zeta}{\partial y} = -\left(\frac{\partial^2 \chi}{\partial z^2} + \frac{\partial^2 \chi}{\partial y^2}\right)$$

$$= -\left(\nabla^2 \chi - \frac{\partial^2 \chi}{\partial x^2}\right) = \frac{\partial^2 \chi}{\partial x^2} - 2k\frac{\partial \chi}{\partial x} \quad \text{from (11)}$$

$$2k\frac{\partial v'}{\partial x} = \nabla^2 v' = \frac{\partial \xi}{\partial x} - \frac{\partial \xi}{\partial z} = \frac{\partial \zeta}{\partial x} = \frac{\partial^2 \chi}{\partial x \partial y}$$

$$2k\frac{\partial w'}{\partial x} = \nabla^2 w' = \frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial x} = -\frac{\partial \eta}{\partial x} = \frac{\partial^2 \chi}{\partial x \partial z}$$

(12)

Hence from (12)

$$\left.\begin{aligned} u' &= \frac{1}{2k}\frac{\partial \chi}{\partial x} - \chi \\ v' &= \frac{1}{2k}\frac{\partial \chi}{\partial y} \\ w' &= \frac{1}{2k}\frac{\partial \chi}{\partial z} \end{aligned}\right\} \qquad (13)$$

Equation (11) may be written

$$(\nabla^2 - k^2)\, e^{-kx}\chi = 0 \qquad (14)$$

since $\nabla^2(e^{-kx}\chi) = \frac{\partial^2}{\partial x^2}(e^{-kx}\chi) + \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(e^{-kx}\chi)$

$$= k^2 e^{-kx}\chi - 2ke^{-kx}\frac{\partial \chi}{\partial x} + e^{-kx}\frac{\partial^2 \chi}{\partial x^2} + e^{-kx}\left(\frac{\partial^2 \chi}{\partial y^2} + \frac{\partial^2 \chi}{\partial z^2}\right)$$

$$= e^{-kx}\left(\nabla^2 - 2k\frac{\partial}{\partial x}\right)\chi + k^2 e^{-kx}\chi .$$

The equation $(\nabla^2 - k^2)f = 0$ where $f = f(x, \varpi)$ can be written

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial \varpi^2} + \frac{1}{\varpi}\frac{\partial f}{\partial \varpi} - k^2 f = 0$$

or transforming into polar coords r, θ with $x = r\cos\theta$, $\varpi = r\sin\theta$

$$\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial \theta}\cos\theta - k^2 f = 0$$

A simple solution independent of θ is given as soln of

$$\partial^2 f/\partial r^2 + \frac{2}{r}\,\partial f/\partial r - k^2 f = 0$$

i.e. $$\frac{\partial^2(rf)}{\partial r^2} - K^2(rf) = 0$$

The simplest type of solution of this is given by

$$rf = Ce^{-Kr} \quad \text{or} \quad f = Ce^{-Kr}/r$$

and with $f = e^{-Kx}\chi$, $\quad e^{-Kx}\chi = Ce^{-Kr}/r$.

Adopting this soln for χ, we get

$$\left.\begin{aligned} u &= -\frac{\partial\phi}{\partial x} + \frac{1}{2K}\frac{\partial\chi}{\partial x} - \chi \\ v &= -\frac{\partial\phi}{\partial y} + \frac{1}{2K}\frac{\partial\chi}{\partial y} \\ w &= -\frac{\partial\phi}{\partial z} + \frac{1}{2K}\frac{\partial\chi}{\partial z} \\ \text{where } \chi &= Ce^{-K(r-x)}/r \end{aligned}\right\} \qquad (15)$$

Since there is symmetry about Ox, and ϕ vanishes at infinity ϕ involves only zonal harmonics of negative degree & we can write $\phi = A_0 \cdot \frac{1}{r} + A_1 P_1/r^2 + A_2 P_2/r^3 + \cdots$ or alternatively

$$\phi = \frac{A_0}{r} + A_1\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + A_2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \cdots \qquad (16)$$

Writing series for $e^{-K(r-x)}$ we have for small values of Kr

$$\chi = C\left(\frac{1}{r} - K + \frac{Kx}{r} + \cdots\right) \qquad (17)$$

since $K(r-x)$ can be written $Kr(1-\cos\theta)$ and expansion is obtained in powers of Kr.

From (17) $$\frac{1}{2K}\frac{\partial \chi}{\partial x} - \chi = \frac{C}{2K}\left(-\frac{x}{r^3} + \frac{K}{r} - \frac{Kx^2}{r^3}\right) - C\left(\frac{1}{r} - K + \frac{Kx}{r}\right)$$

$$= -\frac{C}{2K}\left(\frac{K}{r} + \frac{x}{r^3} + \frac{Kx^2}{r^3} + \cdots\right) \text{ neglecting } K^2 \text{ etc}$$

$$= -\frac{C}{2K}\left\{\frac{4}{3}\frac{K}{r} + \frac{x}{r^3} + \frac{1}{3}Kr^2\frac{(3x^2-r^2)}{r^5} + \cdots\right\}$$

$$\left.\begin{aligned}
&= -\frac{C}{2K}\left\{\frac{4}{3}\frac{K}{r} - \frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \frac{1}{3}Kr^2\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \cdots\right\} \\
\text{Simly } \frac{1}{2K}\frac{\partial \chi}{\partial y} &= -\frac{C}{2K}\left\{ -\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \frac{1}{3}Kr^2\frac{\partial^2}{\partial y\partial x}\left(\frac{1}{r}\right) + \cdots\right\} \\
\frac{1}{2K}\frac{\partial \chi}{\partial z} &= -\frac{C}{2K}\left\{ -\frac{\partial}{\partial z}\left(\frac{1}{r}\right) + \frac{1}{3}Kr^2\frac{\partial^2}{\partial x\partial z}\left(\frac{1}{r}\right) + \cdots\right\}
\end{aligned}\right\} \quad (18)$$

Comparing with (15) and using (16) the boundary conditions

$u = -U,\ v = 0,\ w = 0$ for $r = a$ are satisfied

provided $$C = \frac{3}{2}Ua,\quad A_0 = \frac{3}{2}\nu a,\quad A_1 = -\frac{1}{4}Ua^3 \qquad (19)$$

If l, m, n be directions of radius vector in (x, ϖ) plane relative to the axes, the radial velocity

$$= lu + mv + nw$$

$$= -\left(l\frac{\partial \phi}{\partial x} + \cdots\right) + \frac{1}{2K}\left(l\frac{\partial \chi}{\partial x} + \cdot\right) - l\chi$$

$$= -\left(\frac{\partial \phi}{\partial x}\cdot\frac{\partial x}{\partial r} + \cdot\right) + \frac{1}{2K}\left(\frac{\partial \chi}{\partial x}\cdot\frac{\partial x}{\partial r} + \cdot\right) - l\chi$$

$$= -\frac{\partial \phi}{\partial r} + \frac{1}{2K}\frac{\partial \chi}{\partial r} - \chi\cos\theta$$

Also the radial velocity in terms of the stream f^n ψ is

$$= -\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta}$$

$$\therefore \quad \psi = r^2\int_0^\theta \left(\frac{\partial \phi}{\partial r} - \frac{1}{2K}\frac{\partial \chi}{\partial r} + \chi\cos\theta\right)\sin\theta\cdot d\theta$$

$\phi = \frac{A_0}{r} - \frac{A_1\cos\theta}{r^2}$, neglecting other coeffts. not determined by the boundary conditions

$$\frac{\partial \phi}{\partial r} = -\frac{A_0}{r^2} + \frac{2A_1\cos\theta}{r^3}$$

$$\chi = \frac{C}{r}e^{-Kr(1-\cos\theta)}$$

$$\frac{\partial \chi}{\partial r} = -\frac{C}{r^2}e^{-Kr(1-\cos\theta)}\left\{1 + Kr(1-\cos\theta)\right\}$$

$$\therefore\ \psi = r^2\int_0^\theta\Big[-\frac{A_0}{r^2}+\frac{2A_1\cos\theta}{r^3}+\frac{C}{2Kr^2}e^{-Kr(1-\cos\theta)}\left\{1+Kr(1-\cos\theta)\right\}$$

$$+\frac{C\cos\theta}{r}e^{-Kr(1-\cos\theta)}\Big]\sin\theta\cdot d\theta$$

$$=\int_0^\theta\Big[-A_0\sin\theta+\frac{A_1\sin 2\theta}{r}+\frac{C}{2K}e^{-Kr(1-\cos\theta)}(1+Kr+Kr\cos\theta)\sin\theta\Big]d\theta$$

$$=A_0(\cos\theta-1)-\frac{A_1(\cos 2\theta-1)}{2r}+\frac{C}{2K}\int_r^x e^{-K(r-x)}(1+Kr+Kx)\left(-\frac{dx}{r}\right)$$

$$=\frac{3}{2}\nu a(\cos\theta-1)-\frac{1}{4}Ua^3\cdot\frac{2\sin^2\theta}{2r}-\frac{Ce^{-Kr}}{2Kr}\int_r^x e^{Kx}(1+Kr+Kx)\,dx$$

$$=\frac{3}{2}\nu a(\cos\theta-1)-\frac{1}{4}\frac{Ua^3}{r}\sin^2\theta-\frac{Ce^{-Kr}}{2Kr}(1+Kr)\int_r^x e^{Kx}dx$$

$$-\frac{Ce^{-Kr}}{2r}\int_r^x xe^{Kx}dx$$

The integral in the third term $= \frac{1}{K}(e^{Kx}-e^{Kr})$

" " last term $= \frac{1}{K}(xe^{Kx}-re^{Kr})-\frac{1}{K}\int e^{Kx}dx$

$$=\frac{1}{K}(xe^{Kx}-re^{Kr})-\frac{1}{K^2}(e^{Kx}-e^{Kr})$$

$$\therefore\ \psi=\frac{3}{2}\nu a(\cos\theta-1)-\frac{1}{4}\frac{Ua^3}{r}\sin^2\theta+\frac{C}{K}-\frac{C}{2K}e^{-K(r-x)}(1+x/r)$$

$$=\frac{3}{2}\nu a(\cos\theta-1)-\frac{1}{4}\frac{Ua^3}{r}\sin^2\theta+3\nu a-\frac{3}{2}\nu a\,e^{-K(r-x)}(1+x/r)$$

$$= \frac{3}{2}\nu a(1+\cos\theta) - \frac{1}{4}\frac{Ua^3}{r}\sin^2\theta - \frac{3}{2}\nu a e^{-kr(1-\cos\theta)}(1+\cos\theta)$$

$$= \frac{3}{2}\nu a(1+\cos\theta)\left\{1 - e^{-kr(1-\cos\theta)}\right\} - \frac{1}{4}\frac{Ua^3}{r}\sin^2\theta.$$

For small values of kr this becomes

$$\psi = \frac{3}{2}\nu a(1+\cos\theta)\left\{1 - \left[1 - kr(1-\cos\theta)\right]\right\} - \frac{1}{4}\frac{Ua^3}{r}\sin^2\theta$$

$$= \frac{3}{2}\nu a(1+\cos\theta)\,kr(1-\cos\theta) - \frac{1}{4}\frac{Ua^3}{r}\sin^2\theta$$

$$= \frac{3}{4}Uar\sin^2\theta - \frac{1}{4}\frac{Ua^3}{r}\sin^2\theta \qquad [k = U/2\nu$$

$$= \frac{3}{4}Ua\left(r - \frac{1}{3}\frac{a^2}{r}\right)\sin^2\theta$$

which coincides with the case of Stokes. (§ 19.64).

The vorticity is given by

$$\omega^2 = \left(\frac{\partial\chi}{\partial y}\right)^2 + \left(\frac{\partial\chi}{\partial z}\right)^2$$

where χ is a fn of x and ϖ, and $\varpi\cos\phi = y$, $\varpi\sin\phi = z$

so that $\dfrac{\partial\chi}{\partial y} = \dfrac{\partial\chi}{\partial\varpi}\cos\phi$, $\dfrac{\partial\chi}{\partial z} = \dfrac{\partial\chi}{\partial\varpi}\sin\phi$

$$\therefore\quad \omega = \pm\frac{\partial\chi}{\partial\varpi} = -C\left\{-\frac{1}{r^2}\frac{\partial r}{\partial\varpi}e^{-k(r-x)} + \frac{1}{r}e^{-k(r-x)}\left[-k\frac{\partial r}{\partial\varpi}\right]\right\}$$

with the − sign

$$= \frac{C\varpi}{r^3}e^{-k(r-x)}(1+kr) \qquad \frac{\partial r}{\partial\varpi} = \frac{\varpi}{r}$$

$$= \frac{3}{2}Ua(1+kr)\frac{\varpi}{r^3}e^{-k(r-x)}.$$

and the first equation becomes

$$\dot{\theta} \tan \chi \sin \theta \propto \lambda \quad \text{— (5)}$$

Dividing (5) by (4) we have

$$\sin \chi \sin \theta \propto \sqrt{\lambda}$$

or using (7) $\sin^2 \omega \propto \lambda$

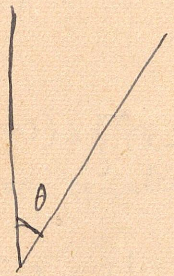
$$\propto (\cos \alpha - \cos \theta).$$

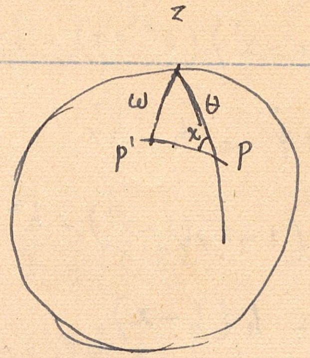
(7) Equivalent to rise & fall of a top. $\dot{\psi}$ constant but $\theta \neq \alpha$ throughout

(8) Consider the ~~disc~~ ~~as a~~ propeller as ~~a~~ disc rotating with angular speed n about an axis OC at rt L^s to its plane. Let OA, OB be two other pr. axes in plane. Let OD, OE be two other axes fixed w.r.t ~~in~~ the aeroplane and let ~~disc~~ the disc be turning with angular speeds ω_1, ω_2 about space axes coinciding at the instant with OD, OE. If at instant considered OA has turned through ϕ from OD angular speeds ~~[illegible]~~ about OA, OB (ω_A, ω_B) are

$$\left.\begin{aligned} \omega_A &= \omega_1 \cos\phi + \omega_2 \sin\phi \\ \omega_B &= -\omega_1 \sin\phi + \omega_2 \cos\phi \end{aligned}\right\}$$

(6) First part is bookwork.





From (12.23) (1) & (2) calling $\cos\alpha - \cos\theta = \lambda$, we have

$$\left.\begin{aligned} \dot\psi \sin^2\theta &\propto \lambda \\ \dot\psi^2 \sin^2\theta + \dot\theta^2 &\propto \lambda \end{aligned}\right\} \quad \text{——— (1)}$$

From (12.22) if χ be angle made by path of the axis PP' on the unit sphere with the meridian

$$\tan\chi = \sin\theta \frac{d\psi}{d\theta} = \sin\theta \frac{\dot\psi}{\dot\theta}$$

$$\text{ie} \quad \dot\psi \sin\theta = \dot\theta \tan\chi \quad \text{——— (2)}$$

Here ω is the $\perp^r$ from Z on PP' & the sph. $\triangle^{le}$ ZPP' (rt L^d) gives

$$\sin\omega = \sin\chi \sin\theta. \quad \text{——— (3)}$$

Using (2) the second eqn of (1) becomes $\dot\theta \sec\chi \propto \sqrt{\lambda}$ — (4)

and the eq^{ns} become

$$\dot{x}^2 = (1-x^2)(e-ex) - (h-hx)^2$$

$$(1-x^2)\dot{\psi} = h - hx$$

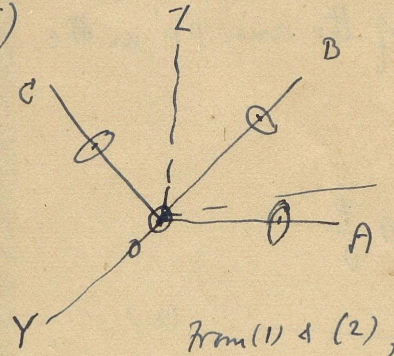
i.e. $\dot{x}^2 = e(1-x)(1-x^2) - h^2(1-x)^2 \equiv (1-x)^2\{e(1+x) - h^2\}$.

$$(1-x^2)\dot{\psi} = h(1-x).$$

i.e. $\dot{x} = (1-x)\sqrt{e-h^2+ex}$. which can be integrated without

elliptic f^{ns}. etc —

(5) As in 12·27: with $X=0, Y=0$ for each rod.



$$C\omega p - (A+Ma^2)l = -Mga \quad (1)$$

$$C\omega q - (A+Ma^2)m = -Mga \quad (2)$$

$$C\omega r - (A+Ma^2)n = -Mga \quad (3)$$

From (1) & (2); $C\omega(pm - ql) = -Mga(m-l)$

i.e. $$\frac{ql-pm}{l-m} = -\frac{Mga}{C\omega}$$

Similarly from (2), (3) & (3), (1) the other eq^{ns} are obtain[ed]

This separate consideration is possible because three axes are mutually at rt L^s

ie $M^2c^4n^2 < 4\cdot\frac{1}{2}M^2(2a^2-c^2)g\sqrt{a^2-c^2}$

$$\text{ie } n < \left\{2g\cdot\frac{2a^2-c^2}{c^4}\sqrt{a^2-c^2}\right\}^{1/2}$$

(3) These are equations (3) and (1) of §12.22

$$\text{with } \left.\begin{array}{l} x = \chi \\ \phi = \psi \end{array}\right\}; \quad \left.\begin{array}{ll} e = E/A, & q = Mgh/A \\ h = D/A, & p = Cn/A \end{array}\right\}$$

and D, E are given by

$$\left.\begin{array}{r} A\dot\psi\sin^2\theta + Cn\cos\theta = D \\ A(\dot\psi^2\sin^2\theta + \dot\theta^2) + 2Mgh\cos\theta = E \end{array}\right\} \quad \text{— (2)}$$

If $\theta = \pi/3$, $\dot\theta = 0$, $\dot\psi = \frac{2}{3}p$ give

$$A\cdot\frac{2}{3}p\cdot\frac{3}{4} + \frac{Cn}{2} = D$$

$$A\left(\frac{4}{9}p^2\cdot\frac{3}{4}\right) + 2Mgh\cdot\frac{1}{2} = E$$

or $\quad Ap + Cn = 2D. \quad$ or $Cn = D$ ie $\underline{h = p}$

$\frac{Ap^2}{3} + Mgh = E \quad$ and if $p^2 = 3q$ this gives

$Aq + Mgh = E$ ie $\underline{e = 2q}$

$C = 0$, and the rotational eqns are

$$A \frac{d}{dt}\left(-\dot\psi \sin\theta\right) - A\,\dot\theta\dot\psi\cos\theta = 0 \quad \text{—— (1)}$$

$$A\ddot\theta - A\dot\psi^2 \sin\theta\cos\theta = 0. \quad \text{—— (2)}$$

The third is identically satisfied

(1) can be written

$$\ddot\psi \sin\theta + 2\dot\psi\dot\theta\cos\theta = 0$$

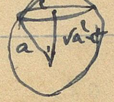
or $\quad \dfrac{d}{dt}\left(\dot\psi \sin^2\theta\right) = 0$ giving $\dot\psi = b\,\mathrm{cosec}^2\theta$

(2) gives $\ddot\theta = \dot\psi^2 \sin\theta\cos\theta = b\,\mathrm{cosec}^3\theta\cot\theta = b\,\mathrm{cosec}^2\theta\cot\theta$

integrating $\left(\dfrac{d\theta}{dt}\right)^2 = a - b\,\mathrm{cosec}^2\theta$.

(2) The ring moves on the surface of sphere and may be regarded as a top turning about the centre of the sphere. The motion of the top is unstable and if

$$C^2 n^2 < 4AMgh\cos\theta.$$



Here $\theta = 0$, $C = Mc^2$, $A = \frac{1}{2}Mc^2 + M(a^2 - c^2) = M(a^2 - \frac{1}{2}c^2) = \frac{1}{2}M(2a^2 - c^2)$

$h = \sqrt{a^2 - c^2}$

p. 316:

(1)

$$\frac{d^2x}{dt^2}=0,\quad \frac{d^2y}{dt^2}=0,\quad \frac{d^2z}{dt^2}=-g$$

$$\cancel{\frac{d\theta}{dt}=A}$$

$$\omega_1=\theta_1=-\dot\psi\sin\theta,\ \omega_2=\theta_2=\dot\theta,\ \theta_3=\dot\psi\cos\theta$$

$$C=0,\quad A\frac{d}{dt}(-\dot\psi\sin\theta)-A\dot\theta\dot\psi\cos\theta=0 \quad (1)$$

$$A\ddot\theta-A\dot\psi\sin\theta\,\dot\psi\cos\theta=0$$

$$\dot p_1-p_2\dot\psi\cos\theta+p_3\dot\theta=0$$

$$\dot p_2+p_3\dot\psi\sin\theta+p_1\dot\psi\cos\theta=0$$

$$\dot p_3-p_1\dot\theta-p_2\dot\psi\sin\theta=-Mg$$

Eqns (1) give $\frac{d}{dt}(A\dot\psi\sin\theta)=0$ (2)

$$A\ddot\theta=\dot\psi^2\sin\theta\cos\theta$$

i.e. $\dot\psi=k\operatorname{cosec}\theta$.

i.e. $\ddot\theta=k^2\operatorname{cosec}^2\theta\sin\theta\cos\theta=b\operatorname{cosec}\theta\cos\theta=b\cot\theta$.

Integrating

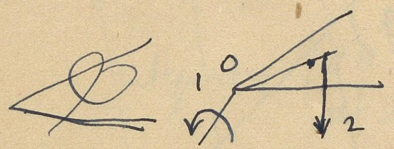
(5) The eqns in this case are

$$\left.\begin{aligned} C(\omega_3 - n) &= G\sin\alpha \\ A\omega_1 &= -G\cos\alpha \\ \omega_3\sin\alpha - \omega_1\cos\alpha &= \Omega \end{aligned}\right\}$$

n = spin of cone about its axis before impact.

In order that the cone may rise off the disc the rate of change of change of momentum about a ‖ˡ to T thro' O (§ 12.8) is less than ~~exceeds~~ the moment of the weight. The condition for this

$\cancel{A\Omega_3\omega_1} + A\omega_1$ i.e. $1 < 2$



If the cone is not to rise with the point O however fixed.

$$\cancel{= \frac{1}{2} A \cdot \frac{G^2 \cos^2\alpha \cos^2\phi}{A^2} + \frac{1}{2} I' \Omega^2}$$

$$\cancel{I' = \frac{AC}{A \sin^2\alpha + C\cos^2\alpha}} \qquad \cancel{\frac{1}{2} I \Omega^2} \qquad \cancel{\frac{\frac{1}{2} I \Omega^2}{\sin^2\alpha}}$$

(4) Let axis of disc be Oz & let G = couple of reaction between the cone & the lamina its axis being also Oz. Let Ox be the generator in contact. Taking moments for the cone about principal axes

$$C\omega_3 = G \sin\alpha \quad \text{—(1)}$$

$$A\omega_1 = -G\cos\alpha \quad \text{—(2)}$$

Since the cone rolls on the lamina

$$\omega_3 \sin\alpha - \omega_1 \cos\alpha = \Omega \quad \text{—(3)}$$

$$\frac{G\sin^2\alpha}{C} + \frac{G\cos^2\alpha}{A} = \Omega$$

$$G = \Omega \Big/ \left(\frac{\sin^2\alpha}{C} + \frac{\cos^2\alpha}{A}\right)$$

K.E. generated $= \frac{1}{2} G \cdot \Omega$ (from theorem ~~..~~ in 8.4)

[9 is a misprint for 6 in the answer]

(4.) $K.E = \frac{1}{2}(A - Mh^2)\omega_1^2 + \frac{1}{2}C\omega_3^2$ = that of rotation

$K.E$ of tr^n $= \frac{1}{2}M(h\cos\alpha.\omega)^2$

$K.E = \frac{1}{2}(A - \cancel{Mh^2})\omega^2\cos^2\alpha + \frac{1}{2}C\omega^2\sin^2\alpha + \cancel{\frac{1}{2}Mh^2\cos^2\alpha\omega^2}$

$= \frac{1}{2}\omega^2(A\cos^2\alpha + C\sin^2\alpha)$.

Since eqn $= \frac{1}{2}AC\Omega^2/I =$

If G be A.M about vertical

$G\sin\alpha$ = about axis of cone

$G\cos\alpha\cos\phi$, $G\cos\alpha\sin\phi$ about other pr. axes

ie ang. velocities about these are

$\frac{G\sin\alpha}{C}$, $\frac{G\cos\alpha\cos\phi}{A}$, $\frac{G\cos\alpha\sin\phi}{A}$

$K.E = \frac{1}{2}A\omega_1^2 + \frac{1}{2}A\omega_2^2 + \frac{1}{2}C\omega_3^2$

See if (1) is correct.

$$-C\omega_3 \cdot \omega \cos\alpha + (A\omega_1 - Mh^2\omega_1)\,\omega \sin\alpha$$

$$+ M\omega^2 h^2 \cos\alpha \sin\alpha > Mg\cos\beta\,(l - h\cos\alpha).$$

$$C\omega^2 \cos^2\alpha \cot\alpha + (A - \cancel{Mh^2})\,\omega^2 \sin\alpha\cos\alpha + \cancel{M\omega^2 h^2 \cos\alpha\sin\alpha}$$
$$> \cdots$$

$$C\omega^2 \cos^3\alpha + A\omega^2 \sin^2\alpha\cos\alpha > Mg\cos\beta\sin\alpha\,(l - h\cos\alpha)$$

$$\omega^2 (C\cos^2\alpha + A\sin^2\alpha) > Mg\cos\beta\,(l - h\cos\alpha)\tan\alpha.$$

So (3) is correct. I in (1) & (2) are the same.

(3) reduces to

$$2\cancel{M}gh\sin\beta\sin\alpha \not> \cancel{Mg}\cos\beta\,(l - h\cos\alpha)$$

$$\tan\beta < \frac{(l - h\cos\alpha)}{2h\sin\alpha}$$

$$< \frac{l - \frac{3}{4}l\cos^2\alpha}{\frac{6}{4}l\cos\alpha\sin\alpha} < \frac{4 - 3\cos^2\alpha}{6\cos\alpha\sin\alpha}.$$

~~(3) $\frac{1}{2}\dot{\theta}^2 = k\cos\theta + C$~~

~~Let $\dot{\theta} = \omega$, when $\theta = 0$~~

~~$\frac{1}{2}\omega^2 = k + C \qquad C' = \frac{1}{2}\omega^2 - k$~~

~~i.e. $\frac{1}{2}\dot{\theta}^2 = k\cos\theta + \frac{1}{2}\omega^2 - k$~~

~~$= \frac{1}{2}\omega^2 - k(1-\cos\theta)$~~

(3) condn of § 12.8 gives

$$I\omega^2 > Mg\cos\beta\,(l - h\cos\alpha)\tan\alpha \qquad \text{—— (1)}$$

taking moment about PT which is horizontal on the inclined plane

ω^2 from previous problem in the position where ~~axis is vertical~~ generator is along the line of greatest slope $= \dfrac{2Mgh\sin\beta\cos\alpha}{I\cot^2\alpha}$ —— (2)

$$\frac{2Mgh\sin\beta\cos\alpha}{\cot^2\alpha} \ngtr Mg\cos\beta\tan\alpha\,(l - h\cos\alpha) \qquad \text{—— (3)}$$

instantaneous axis of rotation of the cone. If Ω be angular velocity of cone about this generator, the K.E of the cone is

$$\frac{1}{2} I \Omega^2$$

but $\Omega = \dot{\theta} \cot\alpha$ (Since θ is fixed & the cone rolls $\dot{\theta} = n$).

$$\therefore \quad T = \frac{1}{2} I \dot{\theta}^2 \cot^2\alpha$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta} \text{ gives}$$

$$\frac{d}{dt}\left(I \dot{\theta} \cot^2\alpha\right) = -Mgh \sin\beta \cos\alpha \sin\theta .$$

$$I \cot^2\alpha \cdot \ddot{\theta} + Mgh \sin\beta \cos\alpha \sin\theta = 0.$$

$$\text{or} \quad \ddot{\theta} + K \sin\theta = 0$$

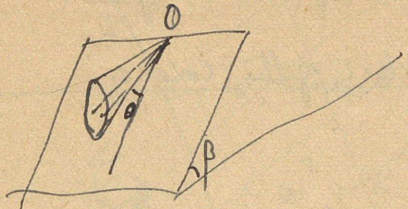
$$\frac{1}{2}\dot{\theta}^2 - K\cos\theta = C \qquad \theta = 90^\circ,\ \dot{\theta} = 0,\ C = 0$$

$$\frac{1}{2}\dot{\theta}^2 = K\cos\theta .$$

when $\theta = 0$, $\frac{1}{2}\dot{\theta}^2 = K$, $\dot{\theta} = \sqrt{2K}$. ~~as say~~

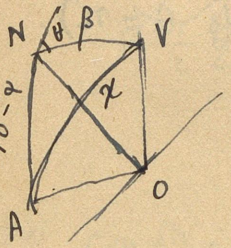
$n^2 = \dot{\theta}^2 \cos^2\alpha \cot^2\alpha$] i.e. $\left\{2 Mgh \sin\beta \cos^3\alpha / I\right\}^{1/2} = n$

(2)



Let χ = angle made by axis of the cone with upward vertical

Consider spherical Δ^{le} defined by dirns OV of vertical, ON normal to the plane, OA axis of cone. The sides are as shown & also angle θ. Hence



$\cancel{\cos\chi = \cos\beta\sin\alpha - \sin\alpha}$

$$\cos\chi = \cos\beta\sin\alpha - \cos\alpha\sin\beta\cos\theta \qquad (1)$$

The vertical height of G above O is $h\cos\chi$ and potential energy of cone is (neglecting a constant term).

$\cancel{V = -\frac{3}{4}Mgh\cos\alpha}$

$$V = -Mgh\sin\beta\cos\alpha\cos\theta \qquad (2)$$

$\cancel{V = -\frac{3}{4}\ldots}$

Next let us find the K.E. ~~Let M.I. of cone about axis and about an axis at right angles of I to axis be~~ $\cancel{I_1 \,\&\, I_2}$ The M.I. about a generator is given to be I_1 and this generator in contact is the

$$= 2\pi \cdot \frac{C\omega \sin\alpha}{Mg(\xi \sin\alpha - \zeta\cos\alpha)} = 2\pi \cdot \frac{\frac{2}{5} m a^2 \omega \sin\alpha}{g \frac{m(n+1)}{n} \cdot \frac{a \sin\alpha}{n}}$$

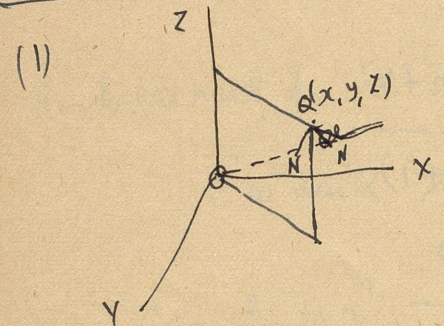
$$= \frac{4\pi n a \omega}{5 g}.$$

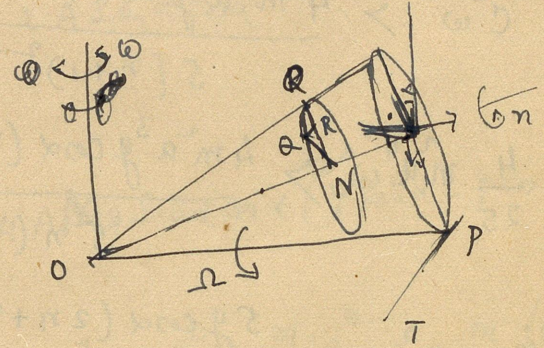
(8) Follows directly from eq^n (7) of § 12.4 by making the necessary specialisation

---

p. 315:

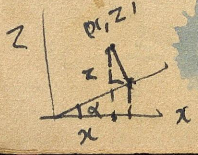
(1)





If Q be (x, y, z) let QN be $\perp^r$ on axis of the cone

N is foot of $\perp^r$ from $(x, 0, z)$ on line $z = x\tan\alpha$.



~~i.e. N is x of N be (x', z'). length of $\perp^r$ = ON tan α~~

$$A = ma^2\left\{\frac{2}{5} + \frac{1}{(n+1)^2} + \frac{1}{\cancel{n}}\cdot\frac{n^2}{(n+1)^2}\right\}$$

$$= ma^2\left\{\frac{2}{5} + \frac{1}{n+1}\right\} = \frac{2n+7}{5(n+1)}\,ma^2$$

$$\xi\sin\alpha - \zeta\cos\alpha = \left(a\cos\alpha + \frac{a}{n+1}\right)\sin\alpha - a\sin\alpha\cos\alpha$$

$$= \frac{a\sin\alpha}{n+1}.$$

ie $$C^2\omega^2\sin^2\alpha > \frac{4\cdot(2n+7)}{5(n+1)}\cdot ma^2\cdot Mg\sin\alpha\cos\alpha\cdot\frac{a\sin\alpha}{n+1}.$$

ie $$C^2\omega^2 > \frac{4m^2a^3g\cos\alpha\,(2n+7)}{5(n+1)^2}\cdot m\left(1+\frac{1}{n}\right).$$

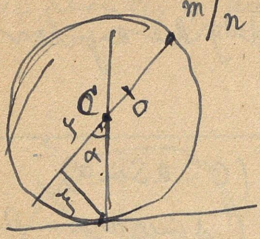
ie $$\frac{4}{25}m^2a^4\omega^2 > \frac{4m^2a^3g\cos\alpha\,(2n+7)}{5n(n+1)}$$

ie $$\omega^2 > \frac{5g\cos\alpha\,(2n+7)}{n(n+1)}$$

If this cond^n is satisfied the two periods are $\frac{2\pi}{\Omega_1}$ & $\frac{2\pi}{\Omega_2}$

Sum $= \frac{2\pi(\Omega_1+\Omega_2)}{\Omega_1\Omega_2} = 2\pi\cdot\sqrt{Mg(\xi\sin\alpha - \zeta\cos\alpha)}$

(7)



from (12·41) the condⁿ for steady motion

is $Cn\Omega\sin\alpha - A\Omega^2\sin\alpha\cos\alpha$

$$= mg\,(\eta\sin\alpha - \xi\cos\alpha)$$

or $A\Omega^2\sin\alpha\cos\alpha - C\omega\Omega\sin\alpha + mg\,(\eta\sin\alpha - \xi\cos\alpha) = 0$

$[\omega = n$

condⁿ The two periods are given by

$$\frac{C\omega\sin\alpha \pm \sqrt{C^2\omega^2\sin^2\alpha - 4A\sin\alpha\cos\alpha\cdot mg\,(\eta\sin\alpha - \xi\cos\alpha)}}{2A\sin\alpha\cos\alpha}$$

Roots are real if

$$C^2\omega^2\sin^2\alpha > 4AMg\sin\alpha\cos\alpha\,(\eta\sin\alpha - \xi\cos\alpha).$$

$$C = \frac{2}{5}ma^2,\quad A = \frac{2}{5}ma^2 + \frac{m}{n}\cdot\frac{a^2}{(n+1)^2} + \frac{m}{n}\left(a - \frac{a}{n+1}\right)^2$$

$\frac{aM/n}{M+M/n}$

$M = m\left(1+\frac{1}{n}\right)$. $\quad\xi = a\sin\alpha$

$\eta = a\cos\alpha + \frac{a}{n+1}$ $\quad\frac{aM}{nM+M}$

ie $\cancel{\frac{2}{5}ma^2\omega^2\sin^2\alpha > 4ma^2\left(\frac{2}{5}+\frac{1}{n}\right)}$

~~Since the sphere has no intrinsic spin, equating the angular velocity~~ §12.4. From (2) with $\xi = 0$.

$$\cancel{v = -a\sin\alpha} \quad v = -a\omega_1 = -\Omega(c - a\sin\alpha)$$

since $v = -\Omega r$

& $r = c - a\sin\alpha$

$$A\omega_1\theta_3 = -Fz + Ra\sin\alpha \quad , \quad m a\omega_1\Omega = F$$

$$= -m a\omega_1 z\Omega + mg\,a\sin\alpha \qquad \text{with } \xi = 0,\ \zeta = a$$

$$\frac{\Omega}{a}(c - a\sin\alpha)(A\theta_3 + ma\,z\Omega) = mga\sin\alpha$$

$$\theta_3 = \Omega\cos\alpha$$

$$z = a\cos\alpha$$

$$\therefore \frac{\Omega^2}{a}(c - a\sin\alpha)\left(\frac{2}{5}ma^2\cos\alpha + ma^2\cos\alpha\right) = mg\,a\sin\alpha$$

$$\frac{7}{5}\Omega^2\cos\alpha(c - a\sin\alpha) = g\sin\alpha$$

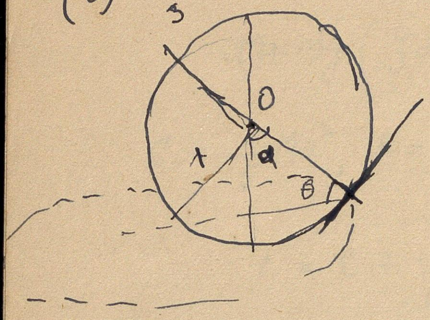
$$\Omega^2 = \frac{5}{7}g\tan\alpha \Big/ (c - a\sin\alpha).$$

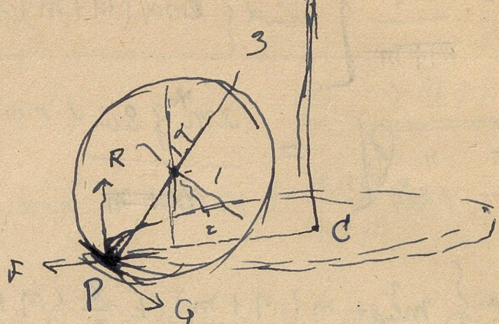
[ie the only thing we have used is $a\omega_1 = \Omega c - \Omega a\sin\alpha$].

Which is also obvious by considering motion about $O1$.

This reduces to the given expⁿ.

(b)





~~Using § R.4 with n = 0~~

$$\cancel{-A\Omega^2 \sin\alpha\cos\alpha = mv\Omega z + mg(\xi \sin\alpha - \zeta \cos\alpha)}$$

$a\sin\alpha - c = a\sin\alpha$

$$\cancel{v = \Omega\zeta \sin\alpha}$$

$-\Omega(c - a\sin\alpha)$

$= \Omega a \sin\alpha$

$$\cancel{\Omega^2(m\zeta z \sin\alpha + A\sin\alpha\cos\alpha) = mg(\xi\cos\alpha - \zeta\sin\alpha)}.$$

$$\cancel{\xi = 0,\ \zeta = a,\ z = a\cos\alpha}$$

$$\cancel{\xi = 0,\ \zeta = r,\ z = a\cos\alpha}$$

$$-\Omega r = v, \qquad \cancel{r = c - a\sin\alpha}$$

$$Cn\Omega\sin\alpha - A\Omega^2 \sin\alpha\cos\alpha = -m\Omega^2 r z + mg\, a\sin\alpha$$

$$\Omega^2\left\{ma\cos\alpha\,(c - a\sin\alpha) - A\sin\alpha\cos\alpha\right\} = mga\sin\alpha$$

$$\Omega^2\left\{mac\cos\alpha - a^2 m\sin\alpha\cos\alpha - \frac{2}{5}ma^2\sin\alpha\cos\alpha\right\}$$

$$= \frac{a m g}{\cancel{M+m}} \left[\sin\alpha \left\{ \cos\alpha (M+m) + m \right\} - \sin\alpha\cos\alpha (M+m) \right]$$

$$= \text{"} \left[= \frac{a m^2 g \sin\alpha}{\cancel{M+m}} \right.$$

$$\alpha^2 \left[c \left\{ m^2\cos\alpha + m(M+m) + \frac{2}{5}(M+m)M \right\} \right.$$

$$+ \frac{2}{5} M a \sin\alpha (M+m)\cos\alpha + \frac{2}{5} M a \sin\alpha \cdot m$$

$$+ \frac{a m \sin\alpha}{M+m} \left\{ m^2\cos\alpha + m(M+m) + \frac{2}{5}M(M+m) \right\}$$

$$- a\sin\alpha\cos\alpha \left(\frac{2M+5m}{5}\right)(M+m) = m^2 g \sin\alpha$$

~~Replacing M+m by M since m is small~~

$$\cancel{\alpha^2 \left[\left(\frac{7}{5} Mm + m^2 \right) \cos\alpha \right.}$$ Replacing $M+m$ by M

$$\alpha^2 \left[\frac{2}{5} M a^2 \sin^2\alpha \, \frac{M\cos\alpha + m}{M\sin\alpha} + \frac{cM + am\sin\alpha}{M} \left\{ (M+m)a + am\cos\alpha + \frac{2}{5} M a \right\} \right.$$

$$- a^2 \left(\frac{2}{5} M + m \right) \sin\alpha\cos\alpha$$

$$= a m^2 g \sin\alpha$$

$$\text{i.e. } \lambda^2\left[C\sin^2\alpha\cdot\frac{\zeta}{\xi}+r\left(\frac{m}{M+m}z+\frac{C\sin\alpha}{\xi}\right)-A\sin\alpha\cos\alpha\right]$$

$$=(M+m)g\left(\zeta\sin\alpha-\xi\cos\alpha\right).\qquad \underline{(m\rightarrow M+m)}$$

$$C=\frac{2}{5}Ma^2,\; A=\frac{2}{5}Ma^2+ma^2,\quad z=a+\frac{am}{M+m}\cos\alpha$$

$$r=c+\frac{am}{M+m}\sin\alpha,\quad \xi=a\sin\alpha,\; \zeta=a\cos\alpha+\frac{am}{M+m}$$

$$\therefore \lambda^2\left[\frac{2}{5}Ma^2\sin^2\alpha\cdot\frac{(M+m)a\cos\alpha+am}{(M+m)a\sin\alpha}+\frac{c(M+m)+am\sin\alpha}{M+m}\times\right.$$

$$\left\{\frac{ma(M+m)+am^2\cos\alpha}{M+m}+\frac{2}{5}\cdot\frac{Ma^2\sin\alpha}{a\sin\alpha}\right\}$$

$$\left.-\left(\frac{2}{5}Ma^2+ma^2\right)\sin\alpha\cos\alpha\right]$$

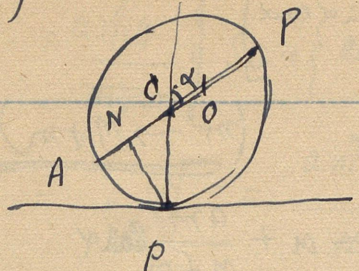
$$=mg\left\{\frac{a\cos\alpha(M+m)+am}{M+m}\sin\alpha-a\sin\alpha\cos\alpha\right\}$$

$$\lambda^2\left[\frac{2}{5}Ma^2\sin\alpha\cdot\frac{(M+m)\cos\alpha+m}{M+m}\right.$$

$$+ca\left\{\frac{m(M+m)+m^2\cos\alpha}{M+m}+\frac{2}{5}M\right\}$$

$$\left.+\frac{a^2m\sin\alpha}{M+m}\left\{\qquad ''\qquad\right\}-a^2\sin\alpha\cos\alpha\left(\frac{2}{5}M+m\right)\right]$$

(5)



If O be C.G. of composite body

$$CO \cdot M = PO \cdot m$$
$$= (a - CO)\,m$$

i.e. $$CO = \frac{am}{M+m}.$$

$$\xi = a\sin\alpha,\ \zeta = a\cos\alpha + \frac{am}{M+m}.$$

Eq^n of steady motion

$$Cn\lambda\sin\alpha - A\lambda^2\sin\alpha\cos\alpha = m v\lambda z + mg(\zeta\sin\alpha - \xi\cos\alpha)$$

~~$Cn\lambda\sin$~~ $\quad -\lambda r = v$

$$v = -n\xi + \lambda\zeta\sin\alpha$$

i.e. $$C\lambda\sin\alpha\left(\frac{\lambda\zeta\sin\alpha}{\xi} - \frac{v}{\xi}\right) - A\Omega^2\sin\alpha\cos\alpha =$$

$$\frac{C\lambda\sin\alpha}{\xi}(\lambda\zeta\sin\alpha) - A\Omega^2\sin\alpha\cos\alpha$$
$$= v\lambda\left((M+m)z + \frac{C\sin\alpha}{\xi}\right) + (M+m)g(\zeta\sin\alpha - \xi\cos\alpha)$$
$$= -\lambda^2 r(\quad '' \quad) + \quad ''$$

or $\omega^2\left[m(a\sin\alpha + l\cos\alpha)(c + l\sin\alpha - a\cos\alpha) - A\sin\alpha\cos\alpha\right] =$

$$= mg(l\sin\alpha - a\cos\alpha)$$

$$\omega^2\left[(a\sin\alpha + l\cos\alpha)c + (a\sin\alpha + l\cos\alpha)(l\sin\alpha - a\cos\alpha) - \left(\frac{a^2}{4} + \frac{l^2}{3}\right)\sin\alpha\cos\alpha\right] = mg(l\sin\alpha - a\cos\alpha)$$

or $\omega^2\left[c(a\sin\alpha + l\cos\alpha) -\right.$

$$v = -n\xi + \Omega\zeta\sin\alpha$$

$$v = -na + \omega l\sin\alpha \qquad \frac{Cv\omega}{a} = -Cn\omega + \frac{C\omega^2 l\sin\alpha}{a}$$

$$Cn\omega\sin\alpha = \frac{C\omega^2 l\sin^2\alpha}{a} - \frac{Cv\omega\sin\alpha}{a}$$

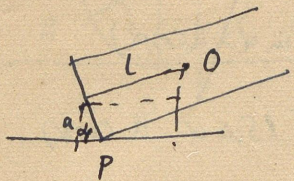
$$= \frac{C\omega^2 l\sin^2\alpha}{a} + \frac{C\sin\alpha}{a}\cdot\omega^2(c + l\sin\alpha - a\cos\alpha)$$

$$\cancel{C\omega^2\sin^2\alpha} = \frac{\omega^2\sin\alpha}{a}\left(Cl\sin\alpha + c\ldots\right)$$

Substituting these values & dividing throughout by $\cos\alpha$ result follows.

$-\Omega r = v$ Here,

$$\left.\begin{aligned} \xi &= a \\ \eta &= l \end{aligned}\right\}$$



$X = a\cos\alpha +$

$Z = a\sin\alpha + l\cos\alpha.$

$\Omega = \omega$

horl distance between P and O $= l\sin\alpha - a\cos\alpha$

$\therefore \quad r = c + l\sin\alpha - a\cos\alpha$

$\therefore \quad -\omega(c + l\sin\alpha - a\cos\alpha) = v$

& eqⁿ for steady motion becomes

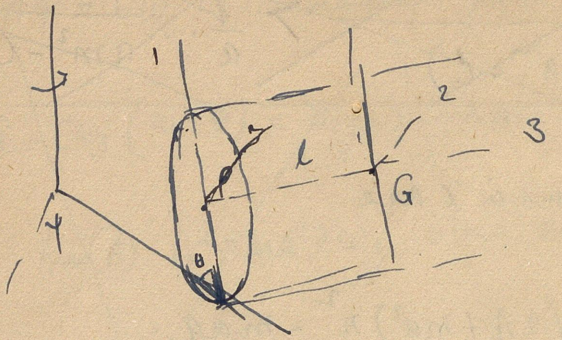
$\cancel{C\cos} \; Cn\omega\sin\alpha - A\omega^2\sin\alpha\cos\alpha = mv\omega(a\sin\alpha + l\cos\alpha)$
$\qquad + mg(l\sin\alpha - a\cos\alpha)$

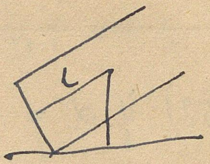
$C = \frac{1}{2}ma^2; \; A = m\left(\frac{a^2}{4} + \frac{l^2}{3}\right)$

~~Cn sin α~~ Since $n = 0$ in this case, we get

$-A\omega^2\sin\alpha\cos\alpha = m(a\sin\alpha + l\cos\alpha)\{-\omega^2(c + l\sin\alpha - a\cos\alpha)\}$
$\qquad + mg(l\sin\alpha - a\cos\alpha).$

(4)





$\omega_1 = -\omega \sin\theta$, $\omega_2 = \dot\theta$, $\theta_3 = \omega\cos\theta$. (1)

Condⁿ for rolling are

$$u - a\omega_2 \sin\theta, \quad v + a\omega_3 = 0 \qquad (2)$$

Translation eqⁿˢ are

$$m(\dot u - v\dot\psi) \quad m(\dot u - v\omega) = F$$

$$m(\dot v + u\omega) = G$$

$$m\ddot z = R - mg$$

We could directly use the result of § 12.4. Condⁿ:

$$Cn\,\Omega \sin\alpha - A\Omega^2 \sin\alpha\cos\alpha = mv\Omega Z + mg(\xi\sin\alpha - \zeta\cos\alpha)$$

and if r be radius of ○ˡᵉ described by the C G

$$\text{ie}\quad \cancel{\omega^2 > \frac{g(C+ma^2)}{a(ma^2-C)} > \frac{g}{a}\cdot\frac{am^2+C}{am^2-C}}.$$

In (12·52) if $\alpha = 90^\circ$, terms in χ are

$$(A+ma^2)\Omega^2 + 2(2A+ma^2)n^2 - mag$$

or with $n=0$, $(A+ma^2)\Omega^2 - mag$

& condⁿ for stability is $\Omega^2 > \dfrac{mag}{A+ma^2} = \dfrac{mag}{\frac{1}{4}ma^2+ma^2}$

$$\text{or } \Omega^2 > \frac{4}{5}\frac{g}{a}.$$

(3) The first eqⁿ of motion of Ex 2 is

$$(C+Ma^2)\dot\omega_3 = Ma^2\dot\theta\dot\psi\sin\theta \qquad (\omega_3 = \omega$$

$C = \frac{1}{2}Ma^2$ & hence $\frac{3}{2}Ma^2\dot\omega = Ma^2\dot\theta\dot\psi\sin\theta$

$$\text{ie}\quad 3\dot\omega = 2\sin\theta\cdot\dot\theta\dot\psi$$

Simly the other eqⁿˢ of Ex 2, give the reqᵈ eqⁿˢ.

$A\dot\psi = C\omega_3 \chi$. Eliminating F gives

$$A\ddot{\chi} - C\omega_3(\omega - \dot\psi) = Mga\chi - am(a\ddot\chi + a\omega_3\dot\psi + a\omega_3\omega)$$

from (A) $(C+ma^2)\omega_3 = ma^2\chi\omega$

$\therefore$ $(A+ma^2)\ddot\chi + \cancel{C\omega_3\omega}$

$$-\frac{C\cdot ma^2\chi\omega^2}{(C+ma^2)} + C\omega_3\cdot\frac{C\omega_3\chi}{A} = Mga\chi - a^2m\cdot\frac{C\omega_3^2\chi}{A}$$

$$-\ a^2m\cdot\frac{ma^2\chi\omega^2}{C+ma^2} = 0.$$

Terms in χ brought to L.H.S. are

$$\frac{Ca^2\omega_3^2}{A}m + \frac{m^2a^4\omega^2}{C+ma^2} + \frac{C^2\omega_3^2}{A} - mga - \frac{Ca^2m\omega^2}{C+ma^2}$$

i.e. $$\frac{mCa^2\omega_3^2(C+ma^2) + Am^2a^4\omega^2 + C^2\omega_3^2(C+ma^2) - ACa^2m\omega^2}{A(C+ma^2)} > mga.$$

with $\omega_3 = 0$, this reduces to

$$Am^2a^4\omega^2 - ACa^2m\omega^2 > Amga(C+ma^2).$$

$$\omega^2 > Amga(C+ma^2)/(Am^2a^4 - AmCa^2)$$

~~the eq^{ns} (4), (5), (6) become~~

$\cancel{A\ddot{\chi} - A\omega} \quad \cancel{A\ddot{\psi} - Cn\dot{\chi} = 0}$

$\cancel{A\ddot{\chi} - Cn(\omega - \dot{\psi})} = mga\chi - F\cdot a$

$\cancel{C\dot{\omega}_3 - A}$

$\omega_1 = \omega - \dot{\psi}, \quad \omega_2 = \dot{\theta}, \quad \theta_3 = 0.$ $\qquad [\theta = 90^\circ + \chi$

$u - a\omega_2 = 0, \quad v + a\omega_3 = 0.$

$u = a\dot{\chi}, \quad v = -a\omega_3, \quad z = a$

$$\left.\begin{aligned} m\{a\ddot{\chi} + a\omega_3(\dot{\psi} + \omega)\} &= F \\ m\{-a\dot{\omega}_3 + a\dot{\chi}(\dot{\psi} + \omega)\} &= G \\ 0 &= R - mg \end{aligned}\right\}$$

$m(a\ddot{\chi} + a\omega_3\dot{\psi} + a\omega_3\omega) = F$

$m(-a\dot{\omega}_3 + a\dot{\chi}\omega) = G$

$0 = R - mg.$

$$\left.\begin{aligned} -A\dot{\psi} + C\omega_3\dot{\theta} &= 0 \\ A\ddot{\theta} - C\omega_3(\omega - \dot{\psi}) &= Mga\chi - Fa \\ C\dot{\omega}_3 &= Ga \end{aligned}\right\}$$

Eleminating G gives $\quad C\dot{\omega}_3 = ma(-a\dot{\omega}_3 + a\dot{\chi}\omega)$

i.e. $(C + ma^2)\dot{\omega}_3 = ma^2\dot{\chi}\omega.$ ——— (A)

(2)

The rotational eq^ns are ~~writing them~~ (12.5)

$$\left.\begin{aligned} A\ddot{\psi}\sin\theta + 2A\dot{\psi}\dot{\theta}\cos\theta - C\omega_3\dot{\theta} &= 0 \\ A\ddot{\theta} + C\omega_3\dot{\psi}\sin\theta - A\dot{\psi}^2\sin\theta\cos\theta &= -Ra\cos\theta - Fa\sin\theta \\ C\dot{\omega}_3 &= Ga \end{aligned}\right\}$$

$$\left.\begin{aligned} F &= M(\dot{u} - v\dot{\psi}) \\ G &= M(\dot{v} + u\dot{\psi}) \\ R - Mg &= M\ddot{z} \end{aligned}\right\} \qquad \begin{aligned} u - a\omega_2\sin\theta &= 0 \\ v + a\omega_3 &= 0. \\ z &= a\sin\theta. \end{aligned}$$

$a\cos\theta\cdot\dot{\theta}$

$$F = M\left(a\dot{\omega}_2\sin\theta + a\omega_2\cos\theta\cdot\dot{\theta} - a\omega_3\dot{\psi}\right)$$

$$G = M\left(-a\dot{\omega}_3 + a\omega_2\sin\theta\cdot\dot{\psi}\right)$$

$$R = Mg + M\left(a\cos\theta\cdot\ddot{\theta} - a\sin\theta\cdot\dot{\theta}^2\right)$$

The third eq^n is

$$C\dot{\omega}_3 = Ma\left(-a\dot{\omega}_3 + a\omega_2\sin\theta\cdot\dot{\psi}\right)$$

i.e. $(C + Ma^2)\dot{\omega}_3 - Ma^2\dot{\theta}\dot{\psi}\sin\theta = 0 \quad (\because \omega_2 = \dot{\theta})$

Similarly the other two eq^ns can be derived.

Proceeding as in 12.51 with $\omega_1 = \omega - \dot{\psi}$

Oscillation about $\phi = 0$ ie vertical (in the plane of the meridian) with period

$$2\pi\sqrt{\left(A/Cn\omega\sin\lambda\right)}.$$

& ratio of squares of periods in the two cases.

$$\cos\lambda/\sin\lambda = \cot\lambda.$$

Page 311: (1) As given in §12.5, condn of steady motion for hoop

is $\Omega^2(4r + a\cos\alpha) = 2g\cot\alpha$, α = angle with horl plane

with $an = -r\Omega$ ie

$$-4a\Omega n + a\Omega^2\cos\alpha = 2g\cot\alpha$$

or $-4a\Omega n + a\Omega^2\sin\alpha = 2g\tan\alpha$, $\alpha = \angle^{r}$ with vertical

ie $4a\Omega n\cos\alpha - a\Omega^2\sin\alpha\cos\alpha + 2g\sin\alpha = 0$

~~$Cn\dot{\theta} - 2A\omega\cos\lambda\cos\theta\,\dot{\theta} = Cn\omega\sin\lambda$ and by integration~~

~~$Cn\theta - 2A\omega\cos\lambda\sin\theta = Cn\omega\sin\lambda\cdot t$.~~

In the case (i) ~~call the vertical~~ as Ox, direction of AB as OZ and axis of gyroscope as O3,

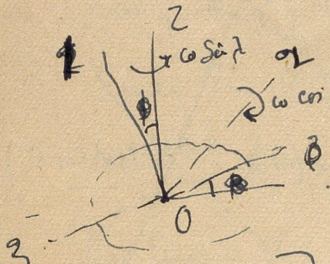
O3 has no motion in θ but angle between O3 and vertical (say ϕ) varies. In this case

(ϕ = angle of O3 with vertical)

$\omega_1 = -\omega\sin\lambda\sin\phi$

$\omega_2 = \omega\cos\lambda - \dot{\phi}$

$\theta_3 = +\omega\sin\lambda\cos\phi$



rt handed system

& the eqns of motion are

$$\left.\begin{aligned} A\dot{\omega}_1 - A\omega_2\theta_3 + C\omega_3\theta_3 &= M \\ A\dot{\omega}_2 - C\omega_3\theta_1 + A\omega_1\theta_3 &= 0 \end{aligned}\right\}\quad \omega_3 = n$$

The second eqn gives

$$A\ddot{\phi} + Cn\omega\sin\lambda\sin\phi - A\omega^2\sin^2\lambda\sin\phi\cos\phi = 0$$

neglecting ω^2, this gives

$$-A\omega\cos\lambda\cos\theta\cdot\dot\theta - Cn(\omega\sin\lambda - \dot\theta) + A(\omega\sin\lambda - \dot\theta)\,\omega\cos\lambda\cos\theta = 0.$$

$$(2A\omega\cos\lambda\cos\theta - Cn)\,\dot\theta = \omega\sin\lambda\,(A\omega\cos\lambda\cos\theta - Cn)$$

$$-2A\omega\cos\lambda\sin\theta\cdot\dot\theta^2 + (2A\omega\cos\lambda\cos\theta - Cn)\,\ddot\theta$$

$$= -\omega\sin\lambda\,(A\omega\cos\lambda\sin\theta\cdot\dot\theta)$$

$$-2A\omega\cos\lambda\sin\theta\cdot\frac{\omega^2\sin^2\lambda\,(A\omega\cos\lambda\cos\theta - Cn)^2}{2A\omega\cos\lambda\cos\theta - Cn}$$

i.e. $$\dot\theta = \frac{-Cn\,\omega\sin\lambda}{2A\omega\cos\lambda\cos\theta - Cn}$$ [neglecting ω^2.

$$= \frac{\omega\sin\lambda}{1 - \dfrac{2A\omega\cos\lambda\cos\theta}{Cn}} = \omega\sin\lambda\left(1 + \frac{2A\omega\cos\lambda\cos\theta}{Cn}\right)$$

$$\ddot\theta = -\frac{2A\omega^2\sin\lambda\cos\lambda}{Cn}\sin\theta\cdot\dot\theta$$

$$= -$$

$$\frac{A\omega\cos\lambda\cos\theta\cdot\dot\theta}{A\omega\cos\lambda\cos\theta - Cn} + \dot\theta = \omega\sin\lambda.$$

or $$\dot\theta = \frac{Cn\,\omega\sin\lambda}{Cn - 2A\omega\cos\lambda\cos\theta}$$ neglecting ω^2.

this reduces to

$$\ddot{\theta} + R\sin(\theta+\alpha).$$

ie posn of eqm is inclined at α to the meridian

$$\text{where } \tan\alpha = \frac{G/A}{Cn\omega\cos\lambda/A} = \frac{G}{Cn\omega\cos\lambda}$$

$$\text{or in the notation of the example} = \frac{G}{I\omega\Omega\cos\lambda}$$

The second part of the problem is exactly what is worked out in the book. See eqn (7) p. 304

(2) Since the axis of the gyroscope is a ~~the~~ diameter of the ring, it lies in the plane of the ring, so that in both the cases (i) & (ii) the axis is horizontal & hence the first part of the Qn follows.

~~In one case M appears in eqn (1) (p. 303) & in the other in (2). Consider the case where it appears in (1). Then taking eqn (2)~~

$$\cancel{\dot{\omega}_2 = \bar{\Omega}\,\omega\cos\lambda\cos\theta\cdot\dot{\theta}}$$

p. 364: (1) with the notation of 12.3.

$$\left.\begin{aligned} A\dot\omega_1 - A\omega_2\theta_3 + C\omega_3\theta_2 &= G \\ A\dot\omega_2 - C\omega_3\theta_1 + A\omega_1\theta_3 &= M \\ C\dot\omega_3 &= 0. \end{aligned}\right\}$$

$$\omega_1 = \omega\sin\lambda - \dot\theta = \theta_1 \;;\; \omega_2 = \theta_2 = -\omega\cos\lambda\sin\theta,\; \theta_3 = \omega\cos\lambda\cos\theta$$

$$-A\ddot\theta - A(-\omega\cos\lambda\sin\theta)(\omega\cos\lambda\cos\theta) + Cn(-\omega\cos\lambda\sin\theta) = G$$

$$A\ddot\theta - A\omega^2\cos^2\lambda\sin\theta\cos\theta + Cn\omega\cos\lambda\sin\theta + G = 0.$$

neglecting 2nd term in ... third

$$A\ddot\theta + Cn\omega\cos\lambda\sin\theta + G = 0.$$

or for small θ,

$$\ddot\theta + \frac{Cn\omega\cos\lambda\sin\theta}{A} + \frac{G\cos\theta}{A} = 0.$$

$$\ddot\theta \quad \text{if} \quad \frac{Cn\omega\cos\lambda}{A} = R\cos\alpha, \quad \frac{G}{A} = R\sin\alpha$$

Similar for gyrostat on PQ by taking moments about R

$$C\Omega n \sin\alpha - (A + Ma^2)\Omega^2 \sin\alpha\cos\alpha$$

$$= + Mga\sin\alpha \;\cancel{-}\; Y \cdot 2a\sin\alpha \;\cancel{+}\; X \cdot 2a\cos\alpha \quad \text{—— (2)}$$

$$+ mg \cdot 2a\sin\alpha$$

Considering the upper rod

$$mg + Mg + Y = 0 \quad \text{—— (3)}$$

Eliminating X and Y from (1) — (3). ie adding (1) & (2)

$$C\Omega n \sin\alpha - (A + Ma^2)\,\Omega^2 \sin\alpha\cos\alpha$$

$$= + Mga\sin\alpha \;\cancel{+2}\; Y \cdot a\sin\alpha + mg \cdot a\sin\alpha$$

$$= 2(M+m)\,ga\sin\alpha.$$

ie $$\cos\alpha = \frac{C\Omega n - 2(M+m)ga}{(A + Ma^2)\Omega^2}$$

and for steady motion being possible ~~this form~~

$0 < \cos\alpha < 1$ which is of the form

req^d

with this choice

$$\omega_1 = \dot\theta \sin\psi - \dot\phi \sin\theta \cos\psi$$

$$\omega_2 = \dot\theta \cos\psi + \dot\phi \sin\theta \sin\psi$$

$$\omega_3 = \dot\psi + \dot\phi \cos\theta$$

$$T_R = \frac{1}{2} mk^2 \left\{ (\dot\theta \cos\psi + \dot\phi \sin\theta \sin\psi)^2 + (\dot\psi + \dot\phi \cos\theta)^2 \right.$$
$$\left. \cancel{+ 2(\dot\psi + \dot\phi\cos\theta)(\dot\theta\cos\psi + \dot\phi\sin\theta\sin\psi)} \right\}$$

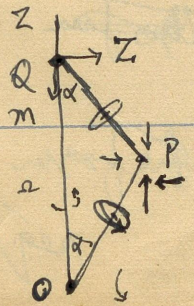
$$\cancel{= \frac{1}{2} mk^2 \left\{ \dot\theta^2 \cos^2\psi + \dot\phi^2 (\sin^2\theta \sin^2\phi + \cos^2\theta - 2\sin\theta\cos\theta\sin\psi) \right.}$$
$$\cancel{\left. + \dot\psi^2 + \dot\theta\dot\phi(2\sin\psi\cos\psi\sin\theta - \cos\theta\cos\psi) \right.}$$

$$-2$$

$$= \frac{1}{2} mk^2 \left\{ \dot\theta^2 \cos^2\psi + \dot\phi^2 (\sin^2\theta \sin^2\psi + \cos^2\theta) + \dot\psi^2 \right.$$
$$\left. + 2\dot\theta\dot\phi \sin\theta \sin\psi \cos\psi + 2\dot\phi\dot\psi \cos\theta \right\}$$

Which is the same form as given.

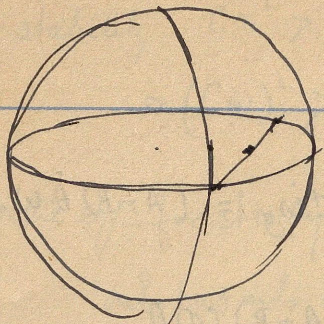
(7)

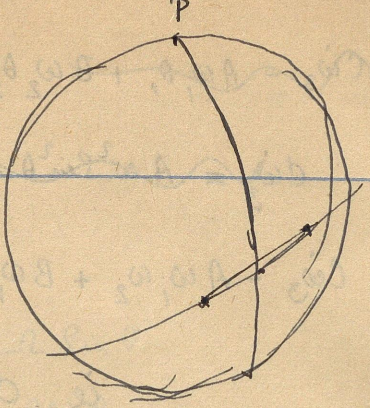


for gyrostat on OP

$$C\Omega n \sin\alpha - (A + Ma^2)\Omega^2 \sin\alpha\cos\alpha$$
$$= + Mga\sin\alpha \mp X \cdot 2a\cos\alpha \mp Y \cdot 2a\sin\alpha \qquad (1)$$

(6)





The coordinates of the middle pt of the rod are

$$\left.\begin{aligned} x &= c\sin\theta\cos\phi \\ y &= c\sin\theta\sin\phi \\ z &= c\cos\theta \end{aligned}\right\}$$

$$\therefore \text{ K.E of translation } = \frac{1}{2}mc^2\left\{\cancel{\sin\dot\theta\cos\phi}\left(\cos\theta\cos\phi\cdot\dot\theta - \sin\theta\sin\phi\cdot\dot\phi\right)^2 + \left(\cos\theta\sin\phi\cdot\dot\theta + \sin\theta\cos\phi\cdot\dot\phi\right)^2 + \left(-\sin\theta\cdot\dot\theta\right)^2\right\}$$

$$= \frac{1}{2}mc^2\left\{\dot\theta^2 + \sin^2\theta\cdot\dot\phi^2\right\}.$$

To find rotational K.E. let $A = 0$, $B = C = mk^2$

~~and $G = H = 0$, $E = B = C$.~~

$$T_R = \frac{1}{2}B\omega_2^2 + \frac{1}{2}C\omega_3^2 \cancel{\;\omega_2\omega_3}.$$

$$\cancel{C\dot\omega_3 - A\omega_1\theta_1 + B\omega_2\theta_2} = 0$$

$$C\dot\omega_3 = \cancel{A\omega^2\sin^2\theta + B\dot\theta^2}$$

$$C\dot\omega_3 - A\omega_1\omega_2 + B\omega_1\omega_2 = 0 \qquad C\dot\omega_3 = (A-B)\,\dot\theta\,\omega\sin\theta.$$

$$\text{ie} \quad C\dot\omega_3 = \cancel{-\omega}\,(A-B)\cos\theta.$$

which shows that (1) can be written in the form

$$B\ddot\theta - (A-C)\,\omega^2\sin\theta\cos\theta = -Mgh\sin\theta$$

A, B, C here are not principal moments

For steady motion $\theta = 0$ which is stable whatever ω.

$$\text{or } (A-C)\,\omega^2\cos\theta = Mgh$$

$$\text{ie } (A-C)\,\omega^2 > Mgh. \quad \omega^2 > \frac{Mgh}{(A-C)}.$$

θ

For stability discussion put $\theta = x$ (small

$$\text{or } \theta = \cos^{-1}\left(\frac{Mgh}{A-C}\right) + x.$$

or

For stability $C^2n^2 > AMgh$

or $(C\omega)^2 > AMgh.$

(4) In the eqn. for steady motion

only has only to put $n = \Omega \sin\theta$

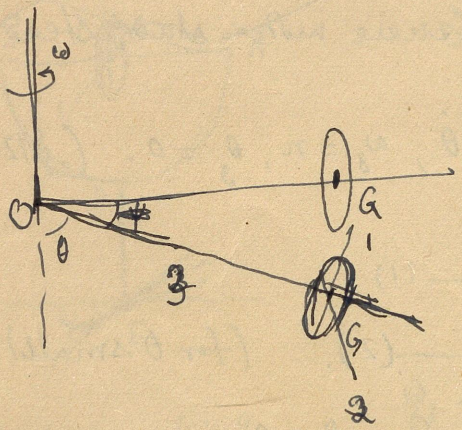
$\alpha = 90 - \theta$

and replace Mgh by Mga

giving $C\Omega^2 \sin\theta - A\Omega^2 \sin\theta = Mga$

or $\sin\theta = \dfrac{Mga}{(C-A)\Omega^2}$.

(5)



$\omega_1 = \dot{\phi}\,\Omega \sin\theta = \theta_1$

$\theta_3 = \Omega \cos\theta$

$\omega_2 = \theta_2 = \dot{\theta}$

Taking eqn of motion about G2

$B\ddot{\theta} + C\omega_3(\omega \sin\theta) \mp A\omega \sin\theta(\omega\cos\theta) = -Mgh\sin\theta.$

$C\dot{\omega}_3 = 0, \ \omega_3 = n$

$\cancel{\Omega Cn \sin\theta - (A+Mh^2)\Omega^2 \sin\theta\cos\theta = Mgh\sin\theta}$.

$\cancel{\Omega Cn - (A+Mh^2)\Omega^2 = Mgh}$

$\cancel{C^2n^2 > 4Mgh}$ $\qquad$ $\cancel{z = h\cos\theta}$

$\cancel{M\ddot{z} = Mg}$

$\cancel{-h\sin\theta\,\ddot{\theta} = Mg}$

$\cancel{-Ah\sin\theta}$

$\cancel{A\ddot{\theta} + Cn\dot{\psi} - Mgh = 0}$, ~~from eqⁿ of steady motion~~

$\cancel{A\ddot{\psi} - Cn\dot{\theta} = 0}$. ~~with~~ $\cancel{\theta = 0}$.

~~but we don't assume motion steady steady~~

$\omega_1 = -\dot{\psi} = \theta_1;\ \omega_2 = \theta_2 = \dot{\theta},\ \omega_3 = n,\ \theta_3 = 0.$ (§12.2).

$\therefore\ -A\ddot{\psi} + Cn\dot{\theta} = 0$ ——— (1)

$A\ddot{\theta} + Cn\dot{\psi} = Mgh\theta$ ——— (2). (for θ small)

from (1) $A\dot{\psi} = Cn\theta$ if $\dot{\psi} = 0$ when $\theta = 0$.

$\therefore\ A\ddot{\theta} + \dfrac{C^2n^2}{A}\theta = Mgh\theta$ $\quad$ ie $\ddot{\theta} + \dfrac{C^2n^2 - AMgh}{A^2}\theta = 0$

i.e. for x small

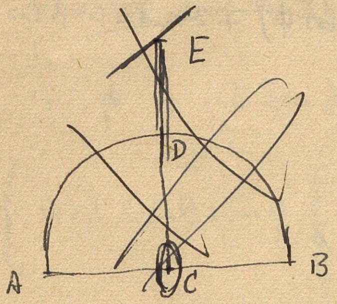
$$\ddot{x} + \Omega(2n\cos\lambda - \Omega^2\cos^2\lambda)x = 0$$

$$\ddot{x} + \Omega\cos\lambda(2n - \Omega\cos\lambda)x = 0.$$

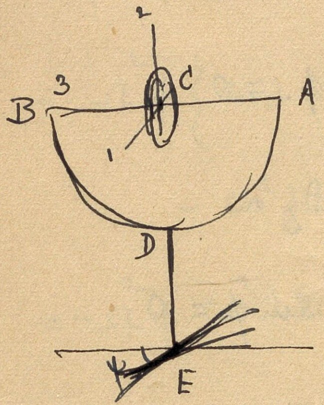
For n large $2n - \Omega\cos\lambda > 0$

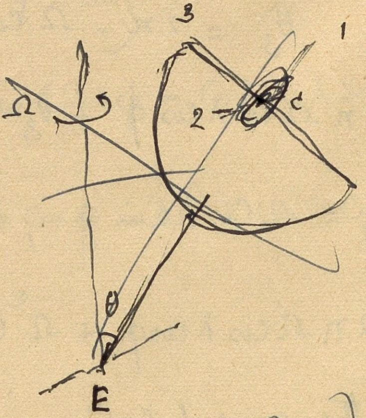
and this gives $\ddot{x} + \mu^2 x = 0.$

(3)



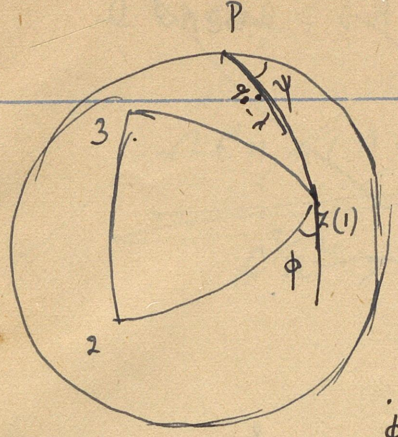
$\cancel{An^2\Omega = -\frac{1}{4}A\Omega^2 - Mgh}$





θ about vertical.

$$\left.\begin{aligned}\omega_1 &= \theta_1 = -\Omega\sin\theta.\\ \omega_2 &= \dot{\theta}_2 = \dot{\theta}\\ \omega_3 &= \Omega\cos\theta.\end{aligned}\right\}$$



$$\omega_3 = \dot\theta + \dot\psi\cos\lambda\sin\phi = \dot\theta + \Omega\cos\lambda\sin\phi = n$$

$$\omega_1 = \dot\phi = \theta_1$$

$$\omega_2 = -\Omega\cos\lambda\cos\phi = \theta_2$$

$$\dot\omega_1 - \omega_2\theta_3 + 2\omega_3\omega_2 = 0 \qquad (1)$$

$$\dot\omega_2 + \omega_1\theta_3 - 2\omega_3\omega_1 = 0.$$

$$\cancel{\ddot\phi + \theta_3(\Omega\cos\lambda\cos\phi) + 2n\Omega\cos\lambda\cos\phi}$$

$$\cancel{\dot\omega_2 = +\Omega\cos\lambda\sin\phi\cdot\dot\phi}$$

$$\cancel{\Omega\cos\lambda\sin\phi\cdot\dot\phi + \dot\phi\cdot\theta_3 - 2n\dot\phi = 0}$$

$$\cancel{\theta_3 = 2n - \Omega\cos\lambda\sin\phi}$$

From (1) $\dot\omega_1 - 2n\Omega\cos\lambda\cos\phi + \theta_3\Omega\cos\lambda\cos\phi. = 0$

$$\theta_3 = \Omega\cos\lambda\sin\phi, \quad \omega_3 = \dot\theta + \theta_3.$$

ie $\ddot\phi - 2n\Omega\cos\lambda\cos\phi + \Omega^2\cos^2\lambda\cos\phi\sin\phi = 0$

Which is the correct eqn.

(3) Put $\phi = \frac{1}{2}\pi + \chi$

$$\ddot\chi + 2n\Omega\cos\lambda\sin\chi \mp \Omega^2\cos^2\lambda\sin\chi\cos\chi = 0$$

$$\cancel{\ddot\chi + 2\Omega\cos\lambda\sin\chi(\dot\theta + \Omega\cos\lambda\cos\chi) - \Omega^2\cos^2\lambda\sin\chi\cos\chi = 0.}$$

$$\omega_1 \cos\theta \cdot \ddot{\phi} + \Omega\omega_1 \cos\lambda \sin\theta \sin\phi \cdot \dot{\phi}$$

$$-\omega_2 \sin\theta \cdot \ddot{\phi} + \Omega\omega_2 \cos\lambda \cos\theta \sin\phi \cdot \dot{\phi} = 0$$

$$\omega_1 \cos\theta - \omega_2 \sin\theta$$

$$= \cos^2\theta \dot{\phi} + \Omega \cos\theta \left(\sin\lambda \cos\theta - \cos\lambda \sin\theta \cos\phi\right)$$

$$+ \sin^2\theta \dot{\phi} + \Omega \sin\theta \left(\sin\lambda \sin\theta + \cos\lambda \cos\theta \cos\phi\right)$$

$$= \dot{\phi} + \Omega \sin\lambda$$

$$\Omega\dot{\phi}\Big\{ \cos\lambda \sin\theta \sin\phi \left\{\cos\theta \cdot \dot{\phi} + \Omega\left(\sin\lambda \cos\theta - \cos\lambda \sin\theta \cos\phi\right)\right\}$$

$$+ \cos\lambda \cos\theta \sin\phi \left\{-\sin\theta \cdot \dot{\phi} - \Omega\left(\sin\lambda \sin\theta + \cos\lambda \cos\theta \cos\phi\right)\right\}$$

$$= \Omega^2 \dot{\phi} \Big\{ \cos\lambda \sin\lambda \sin\theta \cos\theta \sin\phi - \cos\lambda \sin\lambda \sin\theta \cos\theta \sin\phi$$

$$- \cos^2\lambda \sin^2\theta \sin\phi \cos\phi - \cos^2\lambda \cos^2\theta \cos\phi \sin\phi \Big\}$$

$$= -\Omega^2 \dot{\phi} \cos^2\lambda \sin\phi \cos\phi$$

ie $\dot{\phi}\ddot{\phi} + \Omega \sin\lambda \cdot \ddot{\phi} - \Omega^2 \dot{\phi} \cos^2\lambda \sin\phi \cos\phi = 0.$

This is obviously wrong

Take Z as $O1$ the work is simplified

$$\cancel{A}\,\dot\omega_1 - \omega_2\theta_3 + 2\omega_3\omega_2 = 0 \qquad [C' = 2A]$$

$$\dot\omega_1 - \omega_2(\theta_3 - 2\omega_3) = 0. \qquad (1)$$

$$\omega_2 = -\sin\theta\cdot\dot\phi\,\cancel{\phi} - \Omega(\sin\lambda\sin\theta + \cos\lambda\cos\theta\cos\phi).$$

$$\dot\omega_1 = -\sin\theta\cdot\dot\phi\cdot\dot\theta + \cos\theta\cdot\ddot\phi + \Omega(-\sin\lambda\sin\theta\cdot\dot\theta - \cos\lambda\cos\theta\cos\phi\cdot\dot\theta)$$

$$+\,\Omega\cos\lambda\sin\theta\sin\phi\cdot\dot\phi$$

$$\cancel{\pm\Omega} = \cos\theta\cdot\ddot\phi + \omega_2\dot\theta + \Omega\cos\lambda\sin\theta\sin\phi\cdot\dot\phi$$

$$A\,\dot\omega_2 = -\cos\theta\cdot\dot\phi\cdot\dot\theta - \sin\theta\ddot\phi \mp \Omega(\cancel{\phi}\sin\lambda\,\cancel{\sin}\cos\theta\cdot\dot\theta - \cos\lambda\sin\theta\cos\phi\cdot\dot\theta$$

$$+\,\cancel{\cos\lambda}\,\Omega\cos\lambda\cos\theta\sin\phi\cdot\dot\phi$$

$$= -\omega_1\dot\theta - \sin\theta\cdot\ddot\phi + \Omega\cos\lambda\cos\theta\sin\phi\cdot\dot\phi$$

$$\dot\omega_2 - 2\omega_3\omega_1 + \omega_1\theta_3 = 0.$$

$$\dot\omega_2 + \omega_1(\theta_3 - 2\omega_3) = 0 \qquad (2)$$

$$\omega_1\dot\omega_1 + 2\omega_1\omega_2\omega_3 + \omega_2\dot\omega_2 - 2\omega_1\omega_2\omega_3 = 0$$ Eliminating θ_3 bet (1) & (2)

and the eq^n

$$A\dot{\omega}_2 - C\omega_3\theta_1 + A\omega_1\theta_3 = 0 \Big\}\text{ gives}$$

$$A\dot{\theta}_2 - Cn\theta_1 + A\theta_1\theta_3 = 0.$$

$$A\dot{\omega}_1 - A\omega_2\theta_3 + C\omega_3\theta_2 = 0.$$

gives $\dot{\omega}_1 = 0$.

$$\left.\begin{aligned} \theta_1 = \cos P1 &= \sin\lambda\cos\theta - \cos\lambda\sin\theta\cos\phi \\ \cos P2 &= -\sin\lambda\sin\theta - \cos\lambda\cos\theta\cos\phi \end{aligned}\right\}$$

$$\omega_1 = \theta_1 = \Omega\cos P1, \quad \theta_2 = \Omega\cos P2 = \omega_2.$$

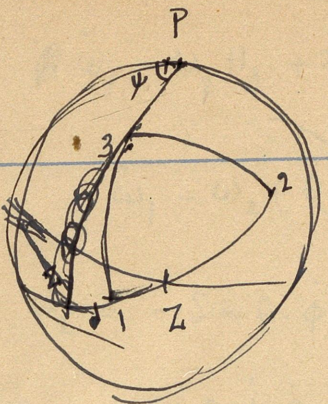
$$\dot{\omega}_1 = \Omega\left(-\sin\lambda\sin\theta\cdot\dot{\theta} - \cos\lambda\cos\theta\cos\phi\cdot\dot{\theta} + \cos\lambda\sin\theta\sin\phi\cdot\dot{\phi}\right)$$

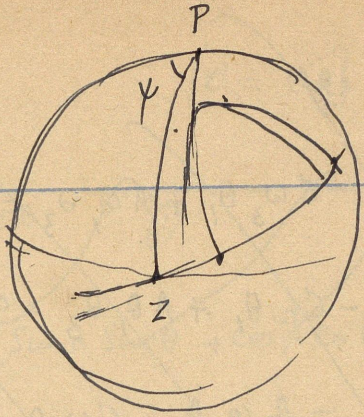
$$= -\Omega\left(n - \Omega\cos\lambda\sin\theta\right)\left(\sin\lambda\sin\theta + \cos\lambda\cos\theta\cos\phi\right).$$

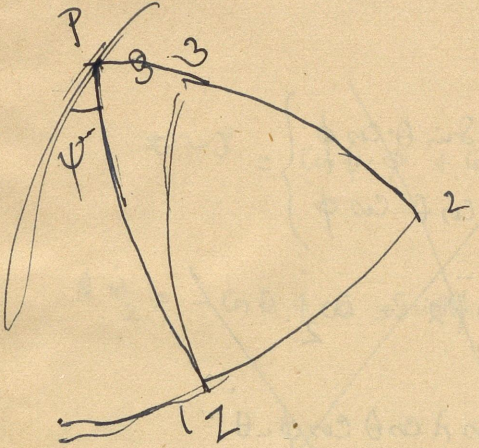
$$+\Omega\cos\lambda\sin\theta\sin\phi\cdot\dot{\phi}$$

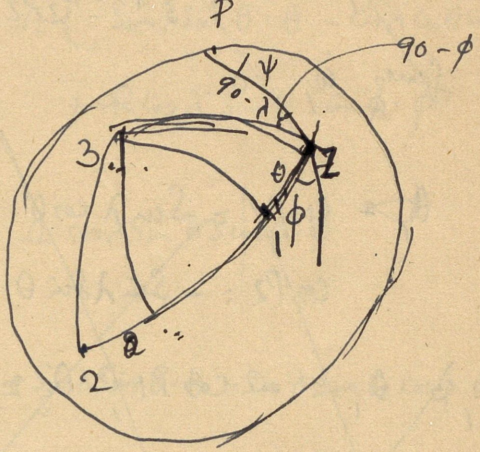
$$\omega_1 = \cos\theta\cdot\dot{\phi} + \Omega\left(\sin\lambda\cos\theta - \cos\lambda\sin\theta\cos\phi\right)$$

$$A\dot{\omega}_1 - A\omega_2\theta_3 + C\omega_3\theta_2 = 0.$$









$\cos P3 = \cos\lambda \sin\phi.$ ie $\omega_3 = \dot\theta + \Omega \cos\lambda \sin\phi.$

Choosing 2 such that $P2 = 90°$ means

$$0 = -\sin\lambda \sin\theta + \cos\lambda \cos\theta \cos\phi$$

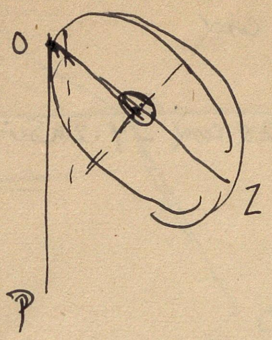
$$\sin\lambda \sin\theta + \cos\lambda \cos\theta \cos\phi = 0.$$

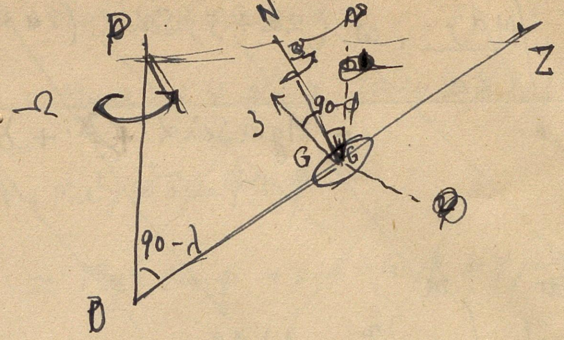
and $\cos P1 = \sin\lambda \cos\theta - \cos\lambda \sin\theta \cos\phi$

ie $\theta_1 = \Omega(\sin\lambda \cos\theta - \cos\lambda \sin\theta \cos\phi)$, $\theta_2 = 0$,

2

(2)





~~Component of OP about OZ = Ω~~

$$\omega_3 = n = \Omega \cos\lambda \sin\phi + \dot{\theta} \qquad (1)$$

Since GN is a $\perp^r$ to OZ lying in the plane OPZ

the component of Ω about GN = $\Omega \cos\lambda$

angle between plane PON & plane of disc = ϕ

$\therefore$ angle between GN and G3 = $90 - \phi$.

This gives (1).

about C:

$$\int_0^{2a} \frac{M\,d\xi}{2a} \cdot \omega^2 (2a+\xi) \sin\alpha \, (2a-\xi)\cos\alpha$$

$$= Mg . a \sin\alpha \cancel{+0} + X . 2a\cos\alpha - Y . 2a \sin\alpha \qquad \text{——— (2)}$$

~~Adda~~

$$\sin\alpha\cos\alpha \frac{M\omega^2}{2a} \cdot \int_0^{2a} (4a^2 - \xi^2)\,d\xi = -$$

$$\text{ie } \frac{M\omega^2}{2a} \sin\alpha\cos\alpha \left(8a^3 - \frac{8a^3}{3}\right) = \hat{}$$

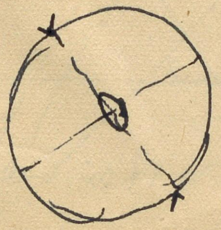
$$4a^2\xi - \xi^3/3 \qquad \frac{8a^2}{3} M\omega^2 \sin\alpha\cos\alpha = - - \qquad (2)$$

Adding (1) & (2), one has

$$4Ma^2\omega^2 \sin\alpha\cos\alpha = 2Mga\sin\alpha$$

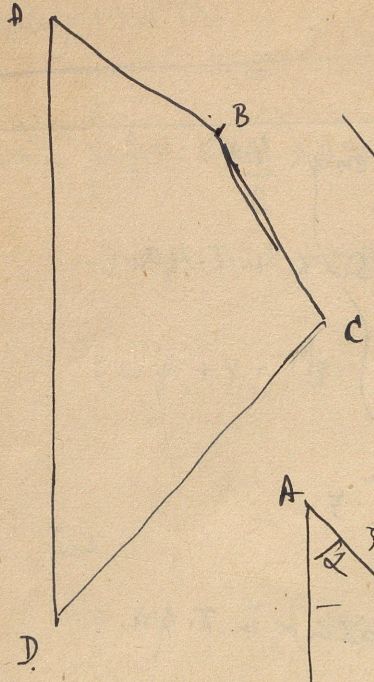
$$\text{ie } \underline{\underline{2a\omega^2\cos\alpha = g}} \text{ : as req}^d$$

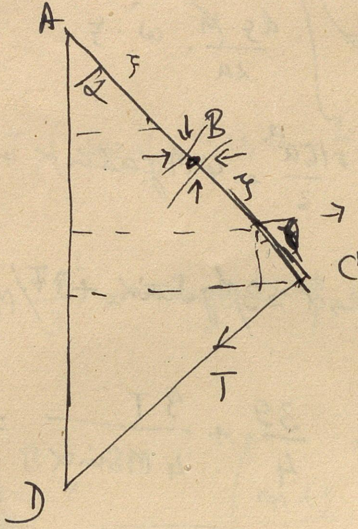
(2)



$$(A + Ma^2)\,\omega^2 \sin\theta \cos\phi$$
$$= Mga\sin\theta - X \cdot 2a\cos\theta + Y \cdot 2a\sin\theta.$$

$$(A + Ma^2)\,\omega^2 \sin\phi \cos\phi$$
$$= Mga\sin\phi + X \cdot 2a\cos\phi + Y \cdot 2a\sin\phi.$$





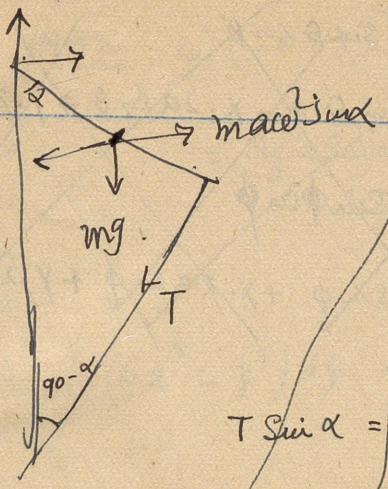
Ex(1):

$$\int_0^{2a} \frac{d\xi \cdot M}{2a} \cdot \xi\omega^2 \xi \sin\alpha \cdot \xi\cos\alpha \cdot d\xi = Mg \cdot a\sin\alpha$$
$$- X \cdot 2a\cos\alpha + Y \cdot 2a\sin\alpha$$

Moments about A: (1)

$$\frac{M\omega^2}{2a} \sin\alpha\cos\alpha \cdot \frac{8a^3}{3} =$$

$$\frac{4a^2}{3} M\omega^2 \sin\alpha\cos\alpha = Mga\sin\alpha - X \cdot 2a\cos\alpha + Y \cdot 2a\sin\alpha.$$



$$m\omega^2 \sin\alpha\cos\alpha \cdot \frac{16a^2}{3}$$

$$= 4\,Mga\sin\alpha + T\cdot 4a$$

$$T\sin\alpha = 2Mg$$

$$T\cos\alpha = \int \frac{d\xi \cdot M}{2a}\cdot \omega^2 \cdot \xi$$

$$2m\omega^2 \sin\alpha\cos\alpha \cdot \frac{16a^2}{3} = 4\,Mga\sin\alpha + T\cdot 4a.$$

$$\frac{8a}{3}\cancel{M}\omega^2 \cancel{\sin\alpha}\cos\alpha = 3\cancel{M}g\cancel{\sin\alpha} + 3T/M\sin\alpha \quad T =$$

$$2a\omega^2\cos\alpha = \frac{3g}{4} + \frac{3T}{4M\sin\alpha} = g.$$

$$\frac{3T}{\cancel{4}M\sin\alpha} = \frac{g}{\cancel{4}}$$

$$T = \frac{1}{3}Mg\sin\alpha :$$

$$\frac{m\omega^2\sin\alpha}{\cancel{2a}} \cdot \frac{\cancel{8}a^2}{\cancel{2}}$$

$$4ma\omega^2\sin\alpha$$

$$2T =$$

$$T\sin\alpha + 2Mg = Y$$

$$T\cos\alpha = X + 4Ma\omega^2\sin\alpha.$$

~~T.4a~~

$$T.\cancel{2a} + X\cdot\cancel{2a}\cos\alpha$$

$$+ Y.\cancel{2a}\sin\alpha = 0.$$

$$T + X\cos\alpha + T\sin^2\alpha + 2Mg\sin\alpha = 0$$

$$T + T\cos^{-1}\alpha + 4Ma\omega^2\sin\alpha\cos\alpha + 2Mg\sin\alpha = 0$$

Adding (1) & (2).

$$ma^2\omega^2\left\{\frac{4}{3}\sin\theta\cos\theta + 4\sin\theta\cos\theta + 2\sin(\theta-\phi) - \frac{4}{3}\sin\phi\cos\phi\right\}$$

$$= -3mga\sin\theta + mga\sin\phi \pm T\cdot 4a\sec\alpha\sin\psi + 2aY\sin\theta$$

$$T\cos\psi + Y = Mg.$$

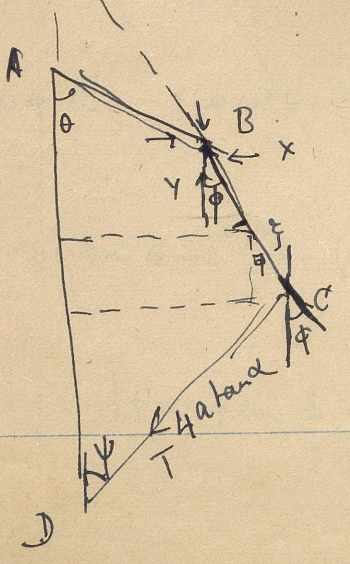
$$\cancel{4a}\tan\alpha\cos\psi$$

$$= \cancel{4a}\sec\alpha - \tfrac{1}{2}\cancel{a}\cos\theta - \tfrac{1}{2}\cancel{a}\cos\phi.$$

T

$$Mg - Y = T\left(\sec\alpha - \frac{L}{2}\cos\theta - \frac{L}{2}\cos\phi\right)\cos\alpha.$$

$$ma\omega^2\sin\phi\cos\phi\cdot\frac{4a^2}{3} = -mga\sin\phi + X\cdot 2a\cos\phi + Y\cdot 2a\sin\phi. \qquad (2)$$



$$\int \frac{m\,d\xi}{2a}\cdot\omega^2\left(4a\tan\alpha\sin\psi - \xi\sin\phi\right)\left(4a\tan\alpha\cos\psi \mp \xi\cos\phi\right)$$

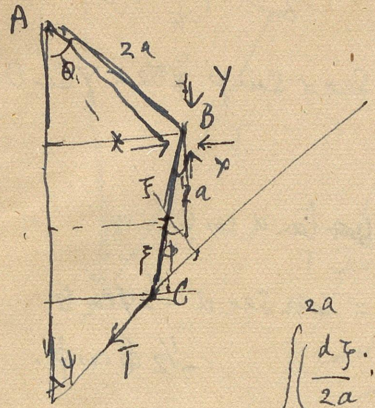
$$= Mg.$$

$\cancel{4a}(1+\tan\alpha) > \cancel{4}a\sec\alpha$

Ramsey . p. 300.

$\cos\alpha + \sin\alpha > 1.$ $\quad \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{1+\sqrt{3}}{2}.$

$\cos 2\alpha > 0, \quad 0 < 2\alpha < \pi/2$



$$m\omega^2 \sin\theta\cos\theta \cdot \frac{4a^2}{3} = mga\sin\theta - X\,2a\cos\theta + Y\cdot 2a\sin\theta. \quad (1)$$

$$\int_0^{2a}\left(\frac{d\xi}{2a}\cdot m\right)\cdot\omega^2(2a\sin\theta - \xi\sin\phi)(2a\cos\theta + \xi\cos\phi)$$

$$= mg(2a\sin\theta - a\sin\phi) + X\cdot 2a\cos\theta + Y\cdot 2a\sin\theta \pm T\cdot 4a\sec\alpha\sin\psi \quad (2)$$

$mg($

$$\frac{m\omega^2}{2a}\int\left(4a^2\sin\theta\cos\theta\, d\xi + (2a\sin\theta\cos\phi - 2a\cos\theta\sin\phi)\xi - \xi^2\sin\phi\cos\phi\right)d\xi$$

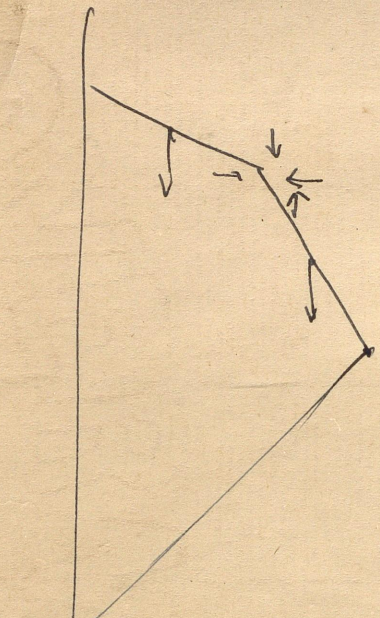
$$= \frac{m\omega^2}{2a}\left\{8a^3\sin\theta\cos\theta + 4a^3\sin(\theta-\phi) - \frac{8a^3}{3}\sin\phi\cos\phi\right\} =$$

$$= 2m\omega^2a^2\left\{2\sin\theta\cos\theta + \sin(\theta-\phi) - \frac{2}{3}\sin\phi\cos\phi\right\}$$

$=$ $\quad$ (2)

$$\int_0^{2a}\left(\frac{d\xi}{2a}\cdot m\right)\cdot\omega^2(2a\sin\theta-\xi\sin\theta)(4a\sec\alpha-2a\cos\theta+\xi\cos\phi)$$

$$=-mg(2a\sin\theta-a\sin\phi)+X\cdot 2$$



$$\frac{2000 \times 2\cancel{0}^{\cancel{6}} \times \cancel{2}}{\cancel{60}\,\cancel{10}\,5}$$

$$\frac{\cancel{4000}^{130} \times 2}{\cancel{3}}$$

$\underline{260}$.

100 per minute.