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1. INTRODUCTION

As is well known, the Lorentz group entered physics through the setting up of Einstein's special theory of relativity, and since this theory is crucial for many a physical theory, the Lorentz group plays a fundamental role in physics. This is especially true in the field of elementary particles where the relativity and quantum theory are brought into close fusion. While the very concept of elementary particles is quantum-mechanical in origin, it is certainly true that relativity, and hence the notion of Lorentz invariance have been crucial for the development of quantum theory.

We can define the general Lorentz transformation as a linear co-ordinate transformation connecting two inertial frames x^i and $x^{i'}$ $(i, i' = 1, 2, 3, 4)$, *viz.*,

$$x^{i'} = a^i + \lambda_j^{\,i} x^j \quad (i, j = 1, 2, 3, 4) \tag{1}$$

which leaves the Minkowskian quadratic differential form

$$ds^2 = (dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad (x^4 = ct) \tag{2}$$

invariant. It need hardly be said that the importance of this transformation is a result of the postulate of special relativity that, in the absence of gravitation, the laws of physics have the same form in all inertial frames of reference, that any two such frames are connected by a transformation specified by (1) and (2), and that c is an invariant in this transformation. It is easily seen that the set of these transformations constitutes a group, and this is called the general Lorentz group, and the essential requirement of special relativity may thus be stated as the invariance of any physical theory under the group of Lorentz transformations. It may happen, however, that for a satisfactory explanation of the theory not all Lorentz transformations of the general type need necessarily be admitted. The minimal requirement so far has been that the theory be invariant with respect to the restricted Lorentz group (denoted here by L_4) consisting of all transformations

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for which $\lambda_4^4 > 0$, and the determinant of the λ_j^i, *viz.*, $|\lambda_j^i| = 1$; in other words, the subgroup not including the transformations of space reflections and time reversal. These latter types can be classified under the three headings:—

$$(a)\ x^{i'} = -x^i\ (i = 1, 2, 3),\ x^{4'} = x^4 \tag{3}$$

being the space reflections denoted by P; this letter is used so as to correspond to the notion of parity which is $\pm$, according as the wave function associated with a particle does not or does change sign under space reflections alone;

$$(b)\ x^{i'} = x^i\ (i = 1, 2, 3),\ x^{4'} = -x^4 \tag{4}$$

being the time reversal denoted by T; and

$$(c)\ x^{i'} = -x^i\ (i = 1, 2, 3, 4) \tag{5}$$

being the total inversion which is the product of P and T.

It is evident that the set of elements of P or T do not constitute subgroups of the general group, and moreover these sets are not of any physical interest by themselves alone since invariance under L_4 is a minimal requirement. We therefore consider the elements PL_4, TL_4, and PTL_4, and build the subgroups (L_4, PL_4), (L_4, TL_4), and (L_4, TPL_4) each containing L_4 as a proper subgroup of itself. In the earlier years of the theory of elementary particles, where free particles were in the centre of interest, invariance under L_4 played a prominent part, but in recent years the centre of interest has shifted to interactions between several types of particles, and it is found that the other three types of subgroups mentioned above have also to be taken into consideration.

2. Relativistic Quantum Theory

Confining oneself for the moment to a theory invariant under L_4 we might mention two striking examples of its early successes. Starting from the notion of a photon, the relation $E = cp$ (including $E = mc^2$) between the energy and momentum of a particle is a relativistic one. The other example is the de Broglie relation $p = h/\lambda$, which follows from Planck's relation $E = h\nu$ on the basis of invariance under L_4. It is however with the introduction of Dirac's equation of the electron inaugurating relativistic quantum mechanics, and the subsequent work of Pauli establishing the connection between spin and statistics on the basis of this theory that the full importance of the Lorentz group in physical applications was clearly brought out, by emphasising the role of the irreducible representations of L_4. In particular, the notion of spin of an elementary particle could be made precise on the basis of such representations.

The relativistic quantum theory consists essentially of two stages in its application, the *c*-number theory with the wave functions and field equations for the particles satisfying the postulates of relativity, and the *q*-number theory making a transition to the particle picture by using the quantum conditions expressing the non-commutation of the field functions at different points of space-time. Such a passage from the *c*-number to the *q*-number theory is called double quantisation.

The essential steps in the *c*-number theory are the setting up of a Lagrangian invariant for transformations of the proper Lorentz group, *i.e.*, the continuous Lorentz group L_4 in which no reflections are included, the derivation of the field equations from a variational principle, the setting up of the energy-momentum and angular momentum tensors, and lastly dividing the field quantities into $U(x)$, their conjugates $U^*(x)$, and the real quantities $V(x)$ and assuming the invariance of the Lagrangian under gauge transformations. Since the field equations are derived from a Lagrangian, the field quantities themselves should transform according to irreducible representations of the group L_4. Such quantities are, as is well known, the spinors, and the field quantity can be written as $U(j, k)$ characterised by two indices j and k corresponding to spinors with $2j$ undotted, and $2k$ dotted indices, and symmetric in them. Using the Clebsch-Gordon rule for the reduction of product representations, and the situation in the case of the subgroup of space rotations, one defines the spin of the particle as $j + k$. For the case $2j + 2k =$ even, the spinors reduce to the ordinary world tensors, but not for the case $2j + 2k =$ odd. Working with these general field quantities, we find the general results that for particles of half-integral spin, the total energy is not necessarily positive, and for integral spin the charge density is not necessarily positive. Going to the *q*-number theory and using the fact that the expressions for the non-commutation of the field quantities at different points should themselves satisfy invariance relations, these relations can be written in terms of the bracket expressions:

$$[U(x),\ U^*(x')]^{\pm} = D(x, x') \tag{6}$$

the + or − being taken according as the particles satisfy the Fermi-Dirac (F.D.) or Einstein-Bose (E.B.) statistics, where further the transformation properties of the U's under L also require that the D's should transform in a certain way. By merely considering the algebraic nature of the transformations of the D's under L_4, with the further requirements that D is a function of the invariant distance between x and x', and that $D = O$ if they be separated by a space-like distance, one is led to the general results that for integral spin quantisation according to F.D. statistics is not possible, and that for half-integral spin there is no algebraic contradiction in either statistics being satisfied, but the removal of the negative energy difficulty is not possible if one uses the E.B. statistics. These results are purely negative in character, but the actual carrying through of the quantisation shows that for half-integral and integral spins a satisfactory theory can be obtained by quantising respectively with the F.D. and E.B. statistics. Thus, the connection between spin and statistics is to be considered as one of the most important physical applications of the Lorentz group.

A deeper insight into the role of L_4, the introduction of the notions of abstract algebra in dealing with elementary particles, and also the need for considering invariance under subgroups like (L_4, PTL_4) more general than L_4 all result when we come to the consideration of a Dirac type of equation of the first order, which might be thought of as a transition from the field to the particle aspect of quantum theory. In order that a theory based on a wave equation set up for establishing the connection between spin and statistics may represent particles of a single spin, it is necessary to assume, besides the symmetry of the spinor in the dotted and undotted indices, that the spinors $p^{\dot{\nu}\rho}a^{\dot{\lambda}\dot{\mu}\ldots}_{\rho\delta\ldots}$ and $p_{\dot{\nu}\delta}a^{\dot{\nu}\dot{\mu}\ldots}_{\lambda\rho\ldots}$ (where $p^{\dot{\nu}\rho}$ is the gradient spinor) should also be symmetrical. This makes it possible to go from the second order wave equation to a system of two first order equations of the type

$$\left.\begin{aligned} p^{\dot{\nu}\rho}a^{\dot{\lambda}\dot{\mu}\ldots}_{\rho\delta\ldots} &= \chi b^{\dot{\nu}\dot{\lambda}\dot{\mu}\ldots}_{\delta\ldots} \\ p_{\dot{\nu}\rho}b^{\dot{\nu}\dot{\lambda}\dot{\mu}\ldots}_{\delta\ldots} &= \chi a^{\dot{\lambda}\dot{\mu}\ldots}_{\rho\delta\ldots} \end{aligned}\right\} \qquad (7)$$

where $h\chi$ is the rest mass of the particle, and the spinor b satisfies the same conditions of symmetry as a. The cases where a has $2k$ undotted and $2k - 1$ dotted indices, or $2k$ dotted and $2k$ undotted indices according as the spin is half-integral or integral respectively are specially simple, and denoting these spinors by $a^{(0)}$ and $b^{(0)}$, they have the property of going over into each other by reflections of space-time. In the integral case $a^{(0)} = b^{(0)}$, and in the half-integral case $a^{(0)} \rightleftarrows b^{(0)}$ under reflections, so that together they form a system invariant under (L_4, PTL_4). The first order wave equations in them are said to be of the Dirac particle type. Using the general result of semi-simple Lie groups (of which L_4 is a particular example) that if the matrix commuting with all the matrices of an infinitesimal product representation is brought to the diagonal form, the product representation is simultaneously split up into its irreducible components, it can be shown that the above type of equations can be reduced to a type involving only one spinor index, *viz.*,

$$\left.\begin{aligned} p^{\dot{\nu}\rho}\psi_{\rho}{}^{A} &= \chi\psi^{\dot{\nu}B} \\ p_{\dot{\nu}\rho}\psi^{\dot{\nu}B} &= \chi\psi_{\rho}{}^{A} \end{aligned}\right\} \tag{8}$$

where A and B indicate that ψ involves magnitudes like $A_{s}{}^{\dot{r}}$ and $B_{j}{}^{i}$, and the equations are to be treated as matrix equations. Finally, the two equations can be combined into one single equation, the famous Dirac equation representing the particle aspect, *viz.*,

$$\partial_{\mu}\beta_{\mu}\psi + \chi\psi = 0; \left(\partial_{\mu} = \frac{\partial}{\partial x_{\mu}}, \mu = 1, 2, 3, 4\right) \tag{9}$$

where for the case of spin $\frac{1}{2}$ the Dirac matrices satisfy the commutation relations

$$\tfrac{1}{2}(\beta_{\mu}\beta_{\nu} + \beta_{\nu}\beta_{\mu}) = \delta_{\mu\nu}. \tag{10}$$

In virtue of these relations, the system of the sixteen numbers 1, β_{μ}, $\beta_{\mu}\beta_{\nu}$, $\beta_{\lambda}\beta_{\mu}\beta_{\nu}$ and $\beta_{1}\beta_{2}\beta_{3}\beta_{4}$ form a hypercomplex system, the Dirac algebra. As is well known the theorems relating to representations of a finite group can be extended to group rings and hence also to an algebra satisfying certain conditions. Applying the theorems which hold in the special case of a semi-simple algebra of which the Dirac algebra is a particular case, it is very easy to prove that that this algebra has only one irreducible representation of order 4, showing that the Dirac equation is unique but for equivalence. I have shown sometime back that the question of setting up a Dirac type of wave equation for particles of spin greater than half can also be dealt with on the basis of the invariance of the wave equation under L_4.

3. Charge Invariance and Gauge Invariance

Another remarkable feature of the Dirac theory is that it has led to the concept of an anti-particle as a hole in a negative energy state, and thus introduced a new type of transformation in physics, the so-called particle-anti-particle conjugation, which we will denote here as C. In the case of a charged particle this transformation merely amounts to a change in the sign of the electric charge, and is referred to as *charge-conjugation*. One is thus led to consider the question of invariance of spinor fields under the transformation C, which, amounting to the replacement of a function by its complex conjugate, is not a linear operator, and is therefore different from the Lorentz type. As mentioned earlier in § 2, the introduction of the notion of charged fields in relativistic quantum theory amounts to the setting up of a current four-vector by dividing the field quantities into U (x), U* (x) and V (x) by assuming the invariance of the Lagrangian against gauge transformations defined by

$$\bar{U}(x) \rightarrow \bar{U}(x)\, e^{i\alpha};\; \bar{U}^*(x) \rightarrow \bar{U}^*(x)\, e^{-i\alpha} \tag{11}$$

(α being a constant when no external fields are present), since such a transformation leads to the possibility of setting up the current vector by expressing the non-measurability of the phase of the complex wave function of a charged particle. The field with a complex U can be written as equivalent to two real fields $V = V^*$ and $W = W^*$ with

$$\bar{U} = \frac{1}{\sqrt{2}}(V + iW);\; \bar{U}^* = \frac{1}{\sqrt{2}}(V - iW) \tag{12}$$

the numerical factor being introduced for the sake of convenience in quantisation. In the particular case of a particle of spin $\frac{1}{2}$ (*e.g.*, an electron), Pauli has shown that this splitting into two real fields is possible, and that one can set up a Lorentz-invariant ordering between the solutions of Dirac's equation with positive and negative frequency, denoted by u_+ and u_-, such that if the former satisfies the wave equation with charge $+ e$, the latter satisfies with charge $- e$. The pair of spinors u_+ and u_- can be easily seen to be of the type of spinors $a^{(0)}$ and $b^{(0)}$ considered under (7) going into each other under space-time reflections in this case of spin $\frac{1}{2}$. Thus the Dirac spinor u_ρ constituting this pair u_+ and u_- can be considered as an irreducible representation of the group (L_4, PTL_4) with the further property of being invariant under C also, and it must be emphasised that this is a consequence of the spin of the electron being half-integral, and hence its satisfying the F.D. statistics. This notion of invariance under C, or that of charge conjugate states can be extended to higher spins also by working with field quantities $\psi_{\rho_1\rho_2\ldots\rho_n}$ which transform like products of Dirac spinors under L_4, and these quantities are called *undors*, u_ρ itself being an undor of rank one. Pauli and Belinfante have shown that in the general case also, one can build up a theory for these undors which is charge-invariant provided one works with a q-number theory.

The special cases of particles of zero charge (neutral particles), and zero mass can be considered under (12) and (11) respectively. In the former case, one can make an abbreviation of the theory due to Majorana by striking out W and its bracket relations, *i.e.*, identifying the charge conjugate states. For the case of general spin, this would amount to considering self-conjugated undors or what are called *neutrettors*, and a consistent theory can be built for these field quantities also. In the latter case of zero mass, the gauge transformation (11) with α a function of space-time, and an additional type of gauge transformation superimposed on it enable the building-up of a theory leading to the physically significant result that a particle of zero rest mass has only two independent and really different plane waves for a specified wave number and frequency.

4. Parity, Charge Conjugation and Time Reversal

In view of the importance of interactions between several types of particles in present-day physics, Pauli has recently returned to the questions of invariance under charge conjugation, space-time reflections, and L_4, and also the spin-statistics connection. Denoting the particle-anti-particle conjugation by AC (which we have denoted by C), the space-time reflection as a weak reflection WR, and the combination of WR with AC as a strong reflection SR, he has derived on the basis of interactions between particles in the simplest cases of spins 0, $\frac{1}{2}$ and 1, the following theorems:—

(*a*) The WR holds whether or not the nouıal connection between spin and statistics is assumed.

(*b*) The transformation law for a quantity with respect to L_4 does not determine uniquely its behaviour for WR; and the invariance with respect to WR imposes further restrictions upon the Lagrangian density of the interactions, besides its invariance for L_4.

(*c*) The SR is uniquely determined as a consequence of L_4 and the spin-statistics connection.

The remarkable result (*c*) that the SR follows from more general results than the WR or AC is referred to as the *Pauli-Ludërs theorem*, and throws light on some problems that have recently arisen in connection with the interaction of elementary particles. As SR is the product of WR and AC, it follows immediately from the above theorem that the results (*a*) and (*b*) are also true for AC with the same additional restrictions imposed on the interaction Lagrangian density, and that the transformation of a certain kind of spinor or tensor for WR uniquely determines its transformation for AC. As an illustration of the Pauli-Ludërs theorem, we might mention the theory of undors dealt with in the previous Section.

Although in the above considerations of Pauli, the conservation properties of P and T have not been separately taken into account, it can be shown, as has been done by Lee, Oehme and Yang (*Phys. Rev.*, 1957, **106**, 340), that we can derive, from (*c*), above some general results regarding conservation under the parity P, the pure time reversal T, and the charge conjugation C. Noting that SR is a combination of C, P and T, these authors have considered the transformation properties (in the Schrodinger representation) of wave functions of the doubly quantised spin fields (both integral and half-integral) under the operations P, C and T with these transformations involving phase factors η_P, η_C, η_T of absolute value unity. Working with a local Hermitian operator invariant under L_4, they have shown that the Pauli-Luders theorem is equivalent to the statement that there always exists a choice of η_P, η_C and η_T such that (A) H commutes with the product of the operators P, C and T taken in any order, and (B) if this choice does not make H commute with P, for example, then no other choice does and the theory is not invariant under P, and the same holds for C and T also. Of course, (B) includes also the possibility that the choice of phases made under (A) may make H commute with P. The statements (A) and (B) constitute the *CPT-theorem* which is thus a simple consequence of the Pauli-Luders theorem. It can be shown from the CPT-theorem that if one of the operators P, C and T is not conserved, at least one other also must not be conserved. Thus, there are five possibilities of conservation or non-conservation of P, C, T as indicated in the table below:—

No.	Non-conserved operators	Conserved operators	
1	..	P, C, T	
2	C, T	P, CT, TC	(13)
3	P, T	C, PT, TP	
4	C, P	T, CP, PC	
5	P, C, T	PCT and permutations	

The CPT-theorem of Lee and Yang has recently obtained a brilliant confirmation in experimental results relating to non-conservation of parity in weak interactions. Considering such an interaction like the emission of neutrinos in β-decay, the experiments of Dr. Wu and collaborators at Columbia University in the β-decay of Cobalt 60 have shown that parity is not conserved in the interaction. Thus, in a correct theory of the neutrino ν^0 (defined to be a particle in the +ve energy state), its spin is always parallel to its momentum, while the spin of an anti-neutrino (defined as a hole in a

—ve energy state) is always anti-parallel to its momentum (that of ν^0). Thus the spin and velocity of ν^0 represent the spiral motion of a right-handed screw, while the spin and velocity of ν^0 represent the spiral motion of a left-handed screw. The theory also shows that there is no invariance under C, and hence as indicated in table (13), there may be invariance under CP, and also under T, or T also may be violated with invariance however under PCT.

Interactions among elementary particles can be classified into three categories, the strong, the electromagnetic, and the weak interactions, the first type having intensities around 1 to 10^{-1}, the electromagentic ones having 10^{-2} and the weak ones having intensities ranging from 10^{-12} to 10^{-14}, and the wide gap separating the third category from the first two is also a remarkable fact still requiring an explanation. Denoting the K-mesons and hyperons as "*strange*" particles, another striking experimental observation is that while the production of strange particles falls into the first category, their decay falls into the third. Regarding the status of the CPT-theorem for strong interactions, experimental evidence indicates that P is conserved in such interactions, and it is usual to assume that C and T also are conserved, so that the first row of table (13) represents strong interactions. Making this assumption, and considering a Hamiltonian which is the sum of a strong part H_s and a weak part H_w, with the former invariant under C, P and T, and both parts invariant under L_4, one can derive interesting results throwing light on the interrelations between non-conservation of C, P and T.

The question of invariance under the transformations C, P, and T can thus be considered as a deeper analysis of the role of the Lorentz group in physical applications, and it becomes necessary and important to examine if many of the recently discovered phenomenological laws of symmetry or invariance, which hold for particle interactions, can be elucidated on the basis of this analysis. One such invariance law is the conservation of the number of nucleons, which necessitates the assignment of a "*heavy particle quantum number*" N to all elementary particles. A consistent assignment is possible if one puts $N = 0$ for the light particles and the L-mesons, $N = \pm 1$ for the nucleons and anti-nucleons respectively, $N = 1$ for the hyperons, and $N = 0$ also for the K-meson class. Another is the conservation of charge in all interactions. A third conservation law is that of "*strangeness*" S which is noticed to hold in the case of the strong and electromagnetic interactions only, if one assigns $S = 0$ to the nucleons and π-mesons, $S = 1$ to K^0 and K^+, $S = -1$ to K^-, K^{-0}, $\overline{K}^0$, Σ^+, Σ^0 and Σ^-, $S = -2$ to the heaviest hyperon Ξ^-. The weak interactions violate this strangeness conservation. Finally taking the strong interactions only into consideration, it is found that the third component τ_3 of the isotopic spin quantum number (given by the Pauli 2×2 matrices) is invariantly connected with N and S by the relation

$$\tau_3 = Q - \frac{N}{2} - \frac{S}{2}, \tag{14}$$

where Q is the charge expressed in multiples of $|e|$. The elucidation of these laws is still an open question. A mention may finally be made of the so-called τ-θ puzzle which appears to have recently received a satisfactory explanation on the basis of the non-conservation of parity in weak interactions. These two strange K-mesons τ^+ and θ^+ have closely identical masses and life-times, but only differ in having different decay processes into π-mesons (the mesons which are responsible for the main features of nuclear forces), *viz.*,

$$\left.\begin{array}{l}\tau^+ \rightarrow \pi^+ + \pi^+ + \pi^- \\ \theta^+ \rightarrow \pi^+ + \pi^0\end{array}\right\} \tag{15}$$

Both these interactions are weak since S is not conserved in them, and non-conservation of parity in such interactions makes it possible to think of τ^+ and θ^+ as one K-particle having a definite parity on production, but decaying into various parities. Also the mass degeneracy of τ^+ and θ^+ can be explained by assuming that they have the same spin but opposite parity, *i.e.*, they are related by an invariance law called the "*parity conjugation*" denoted by C_P, or in other words, that they are parity doublets. C_P would commute with H_S, but not with H_w and one could thus explain the mass degeneracy. Considering the strange particles with odd values of S like the τ and θ, a general law has been established that particles of odd strangeness given by (14) must be parity doublets, while for even S, the operation C leaves the parity invariant, *i.e.*,

$$C_P\, P - (-1)^S\, PC_P = 0. \tag{16}$$

5. CONCLUDING REMARKS

While the questions discussed in the previous Section show that a deeper knowledge of the representations of the full Lorentz group, and its several subgroups, including perhaps the infinite representations also, may yield useful results in interpreting the conservation laws of nature, it is clear that this alone is not sufficient. It has to be correlated with laws relating to charge conjugation, general gauge transformations, laws of isotopic spin, and perhaps others. It also appears that no need has been felt so far radically to modify the quantum principles in the application to elementary particles. Thus, while Bohr's theory of complementarity can be said to have remained inviolate, Einstein's principle of relativity appears to be undergoing modifications in the method of applying it to physical problems. Suggestions of this nature have also been recently made that the inconsistencies discovered in the quantum theory of elementary particles, specially the quantum electro-dynamics, wherein the photon appears, can perhaps be removed when weak gravitational effects of Einstein's general relativity theory are properly taken into consideration. For example, the cosmological asymmetry which results as a violation of parity is perfectly compatible with Riemann space-time of ordinary general relativity. These ideas have given rise to a spate of theories devoted to a re-examination of general relativity. One such theory starts with the assumption that particles with a spin always generate a gravitational field, another attempts to build-up a theory of gravitation by using the neutrino theory, and so on.

An exciting period is now clearly before the experimental as well as the theoretical physicist due to the discovery of the phenomenon of non-conservation of parity, which has resulted from the circumstance that Nature uses simple but strange representations of the Lorentz group to characterise the strange particles.

Summary

The role of special relativity, based on the restricted Lorentz group L_4 consisting of rotations in space-time, in the earlier development of quantum theory, in the connection between spin and statistics established by Pauli, and in Dirac's theory of the electron is briefly indicated. The theory of undors is pointed out as an example where, besides invariance under L_4, invariance under charge-conjugation and space-time reflections also is satisfied.

The Pauli-Ludërs theorem taking into account the interactions between elementary particles, and the subsequent work of Lee and Yang relating to parity, charge-conjugation and time-reversal are explained, and cited as examples of general representations of the Lorentz group.

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