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COMMUTATION RULES FOR MATRICES RELATED TO PARTICLES OF HIGHER SPIN—PART III

BY

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COMMUTATION RULES FOR MATRICES RELATED TO PARTICLES OF HIGHER SPIN—PART III

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1. *Introduction*

IN two earlier papers (1942) (referred to here as I and II), I have considered the question of deriving commutation rules for the matrices β appearing in the wave equation

$$\partial_\mu \beta_\mu \psi + \chi \psi = 0 \tag{1}$$

for a relativistic particle of arbitrary spin value, under suitable assumptions, *viz.*,

(1) the wave equation is relativistic invariant, *i.e.*,

$$(\beta_\mu,\ t_{\nu\rho}) \equiv (\beta_\mu t_{\nu\rho} - t_{\nu\rho} \beta_\mu) = \delta_{\mu\nu} \beta_\rho - \delta_{\mu\rho} \beta_\nu \tag{2}$$

where $s_{\mu\nu} = -it_{\mu\nu}$ is the usual spin operator;

(2) that the spin operator has in general the form

$$t_{\mu\nu} = k\,(\beta_\mu,\ \beta_\nu) \tag{3}$$

where k is a numerical constant;

(3) that for a particle of spin s, the eigenvalues of the $s_{\mu\nu}$ are $s, s-1, \ldots\ldots -s+1, -s$, *i.e.*, $s_{\mu\nu}$ satisfies the algebraic equation

$$(s_{\mu\nu}{}^2 - s^2)\,(s_{\mu\nu}{}^2 - \overline{s-1}^2) \ldots\ldots = 0 \tag{4}$$

or

$$(t_{\mu\nu}{}^2 + s^2)\,(t_{\mu\nu}{}^2 + \overline{s-1}^2) \ldots\ldots = 0. \tag{4a}$$

From (2) and (3) it follows immediately that the general commutation rules, valid for all spins, are given by

$$k\,[\beta_\mu,\ (\beta_\nu,\ \beta_\rho)] = \delta_{\mu\nu} \beta_\rho - \delta_{\mu\rho} \beta_\nu. \tag{5}$$

Using this method it has been shown in I that it is possible to get the commutation rules for the cases of spins 3/2 and 2. In II it was shown that the further condition that (1) should lead to the second order wave equation

$$\partial_\mu \partial_\mu \psi = \chi^2 \psi \tag{6}$$

in virtue of the commutation rules gives in the cases of spins $\frac{1}{2}$ and 1 definite values for k, *viz.*,

$$s_{\mu\nu} = \frac{1}{4i}(\beta_\mu\beta_\nu - \beta_\nu\beta_\mu)$$

$$s_{\mu\nu} = \frac{1}{i}(\beta_\mu\beta_\nu - \beta_\nu\beta_\mu)$$

these being the usual expressions for these cases respectively.

The application of the same method to $s = 3/2$ and 2 leads to fourth order wave equations, and putting $\delta_\mu\delta_\mu\psi = \chi^2\psi$ in these fourth order equations gives *two* alternative values of k as mentioned in II (p. 63). These values of k do not however alter the essential fact that what one gets are fourth order wave equations and not second order ones. This situation is therefore to be interpreted, as done by Bhabha (1945), as giving two values to χ^2 the square of the rest mass of the particle or four values to χ. The same conclusion holds whatever the value of k which can therefore remain quite arbitrary.

We derive the higher order wave equations for the case of general spin following the method of Kemmer (1939) for derivation of the second order wave equation in the meson case, and thus obtain the several values of the rest mass.

We also show that the commutation rules in the case of spin s are identically satisfied by the rules for the lower spins $s - 1$, $s - 2$, etc.

2. *Higher Order Wave Equations*

Let us first consider the case of spin 3/2. The commutation rules are given, retaining k, as [I, equation (26)]

$$\begin{aligned} 2k^2(\beta_\lambda\beta_\mu\beta_\nu\beta_\rho + \beta_\lambda\beta_\rho\beta_\nu\beta_\mu + \beta_\mu\beta_\nu\beta_\rho\beta_\lambda + \beta_\rho\beta_\nu\beta_\mu\beta_\lambda) &= 3k(\beta_\lambda\beta_\mu + \beta_\mu\beta_\lambda)\delta_{\nu\rho} + 3k(\beta_\lambda\beta_\rho + \beta_\rho\beta_\lambda)\delta_{\mu\nu} \\ &+ k(\beta_\nu\beta_\rho + \beta_\rho\beta_\nu)\delta_{\lambda\mu} + k(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu)\delta_{\lambda\rho} \\ &+ k(\beta_\mu\beta_\rho + \beta_\rho\beta_\mu)\delta_{\lambda\nu} + k(\beta_\lambda\beta_\nu + \beta_\nu\beta_\lambda)\delta_{\mu\rho} \\ &- \tfrac{3}{2}(\delta_{\lambda\mu}\delta_{\nu\rho} + \delta_{\lambda\nu}\delta_{\mu\rho} + \delta_{\mu\nu}\delta_{\lambda\rho}). \end{aligned} \tag{7}$$

We operate on ψ by the two sides of (7), and then by $\partial_\lambda\partial_\mu\partial_\nu\partial_\rho$, perform the summation, and use (1) four times successively. This gives

$$8k^2\chi^4\psi = 20k\chi^2\delta_\lambda{}^2\psi - \tfrac{9}{2}\partial_\lambda{}^4\psi$$

i.e.,

$$\left(\chi^2 - \frac{1}{4k}\partial_\lambda{}^2\right)\left(\chi^2 - \frac{9}{4k}\partial_\lambda{}^2\right)\psi = 0 \tag{8}$$

whatever the value of k. We can therefore interpret this by saying that the particle appears with four values of the rest mass, *viz.*,

$$\pm \chi/\sqrt{\left(\frac{1}{4k}\right)},\ \pm \chi/\sqrt{\left(\frac{9}{4k}\right)}. \tag{9}$$

The effect of changing k is to multiply χ by a numerical constant, but χ would still have four values.

For $k = 1$, the four values are $\pm 2\chi,\ \pm 2\chi/3$

$k = \frac{1}{4}$,, $\pm \chi,\ \pm \chi/3$

In the case of spin 2, the commutation rules are given by I, Equation (34) and proceeding as above. Operating by $\partial_\lambda \partial_\mu \partial_\nu \partial_\rho \partial_\epsilon$, taking into account the number of terms and numerical coefficients on the right-hand side of the commutation rule, and using (1) five times, we get

$$-24k^2\chi^5\psi = -120k\chi^3\partial_\lambda{}^2\psi + 96\chi\partial_\lambda{}^4\psi$$

i.e.,
$$\left(\chi^2 - \frac{1}{k}\partial_\lambda{}^2\right)\left(\chi^2 - \frac{4}{k}\partial_\lambda{}^2\right)\chi\psi = 0 \tag{10}$$

the values of the rest mass being

$$\pm \chi/\sqrt{\left(\frac{1}{k}\right)},\ \pm \chi/\sqrt{\left(\frac{4}{k}\right)}, \tag{11}$$

and for $k = 1$, $\pm\chi$, $\pm\chi/2$. (11*a*)

In the case of arbitrary spin, the actual commutation rules are very complicated, but the form of the rules can be generalised from the forms for $s = 3/2$ and 2 given in I, and these general forms are sufficient to extend the previous considerations to the general case. The essential steps in the deduction of the commutation rules for $s, = 3/2$ and 2 are represented by the equations (18) and (27) of I, *viz.*,

$$\left.\begin{aligned} &\beta_\lambda{}^4 - 5/2\,\beta_\lambda{}^2 + 9/16 = 0 \\ &\beta_\lambda{}^5 - 5\beta_\lambda{}^3 + 4\beta_\lambda = 0 \end{aligned}\right\}$$

starting respectively from the algebraic equations (12, *a*, *b* of I)

$$\left.\begin{aligned} &t_{\lambda\mu}{}^4 + 5/2\,t_{\lambda\mu}{}^2 + 9/16 = 0 \\ &t_{\lambda\mu}{}^5 + 5t_{\lambda\mu}{}^3 + 4t_{\lambda\mu} = 0 \end{aligned}\right\}$$

and in the general case one can deduce (putting the arbitrary $k = 1$)

$$(\beta_\mu{}^2 - s^2)(\beta_\mu{}^2 - \overline{s-1}^2)\ldots. = 0 \tag{12}$$

from
$$(t_{\mu\nu}{}^2 + s^2)(t_{\mu\nu}{}^2 + \overline{s-1}^2)\ldots. = 0 \tag{4a}$$

In fact, from (3) and (5) (equations 9*a* and 11*a* of I)

$$\left.\begin{aligned} t_{\mu\nu} &= (\beta_\mu,\ \beta_\nu) \\ \beta_\nu &= (\beta_\mu,\ t_{\mu\nu}) \\ \beta_\mu &= (t_{\mu\nu},\ \beta_\nu) \end{aligned}\right\}$$

and denoting β_μ, β_ν, $-it_{\mu\nu}$ by α, β, γ respectively, these are equivalent to

$$\left.\begin{aligned} \alpha\beta - \beta\alpha &= i\gamma \\ \beta\gamma - \gamma\beta &= i\alpha \\ \gamma\alpha - \alpha\gamma &= i\beta \end{aligned}\right\}$$

The symmetry of the above relations shows, as mentioned by Dirac (1930) who has considered this very example, that α, β, γ have the same eigen values. Thus (12) is an immediate consequence of (4*a*).

The other types of the commutation rule can then be deduced by taking Poisson brackets of (12) successively with $t_{\mu\nu}$ as done in I. The actual form is complicated but the general structure of the rules is evident from those for 3/2 and 2. The number of indices for the β's appearing in the rules would be $2n$ or $2n+1$ for the cases of half-integral spin $s = \frac{1}{2}(2n-1)$ or integral spin $s = n$ respectively, and the rules consist in expressing products of $2n$ or $(2n+1)$ β's in terms of products of $(2n-2, 2n-4, \ldots.)$ or $(2n-1, 2n-3, \ldots.)$ β's. The form of the rules would therefore be, retaining k,

Case (*a*) $s = \frac{1}{2}(2n-1)$

$$\begin{aligned} k^n \Sigma\, a_{2n} (\beta_{\lambda_1}\beta_{\lambda_2}\ldots.\beta_{\lambda_{2n}}) &= k^{n-1} \Sigma\, a_{2n-2} (\beta_{\lambda_1}\beta_{\lambda_2}\ldots.\beta_{\lambda_{2n-2}}) (\delta_{\lambda_{2n},\lambda_{2n-1}}) \\ &\quad + k^{n-2} \Sigma\, a_{2n-4} (\beta_{\lambda_1}\ldots.\beta_{\lambda_{2n-4}}) (\delta_{\lambda_{2n},\lambda_{2n-1}} \cdot \delta_{\lambda_{2n-2},\lambda_{2n-3}}) \\ &\quad + \ldots. + \Sigma\, a_0 (\delta_{\lambda_{2n},\lambda_{2n-1}} \ldots. \delta_{\lambda_2,\lambda_1}) \end{aligned} \tag{13a}$$

Case (*b*) : $s = n$

$$\begin{aligned} k^n \Sigma\, a_{2n+1} (\beta_{\lambda_1}\beta_{\lambda_2}\ldots.\beta_{\lambda_{2n+1}}) &= k^{n-1} \Sigma a_{2n-1} (\beta_{\lambda_1}\beta_{\lambda_2}\ldots.\beta_{\lambda_{2n-1}}) (\delta_{\lambda_{2n+1},\lambda_{2n}}) \\ &\quad + k^{n-2} \Sigma\, a_{2n-3} (\beta_{\lambda_1}\ldots.\beta_{\lambda_{2n-3}}) (\delta_{\lambda_{2n+1},\lambda_{2n}} \cdot \delta_{\lambda_{2n-1},\lambda_{2n-3}}) \\ &\quad + \ldots. + \Sigma\, a_1 \beta_{\lambda_1} (\delta_{\lambda_{2n+1},\lambda_{2n}} \ldots. \delta_{\lambda_3,\lambda_2}). \end{aligned} \tag{13b}$$

The constants in (13*a*) and (13*b*) are such that when all the indices are equal to μ, the commutation rules reduce to

$$(k\beta_\mu{}^2 - s^1)(k\beta_\mu{}^2 - \overline{s-1}^2)\ldots. = 0 \tag{12'}$$

To derive the higher order wave equations, we follow the same procedure as in the cases of spins 3/2 and 2 by using the operations $\partial_{\lambda_1} \partial_{\lambda_2} \ldots. \partial_{\lambda_{2n}}$ and $\partial_{\lambda_1} \partial_{\lambda_2} \ldots. \partial_{\lambda_{2n+1}}$ in (13*a*) and (13*b*) and equation (1) $2n$ or $2n+1$ times.

Since this procedure is equivalent to the procedure of putting all indices equal in (13) thereby deriving (12′), the higher order wave equation reduces to

$$(k\chi^2 - s^2\partial^2_\mu)(k\chi^2 - \overline{s-1}^2\partial_\mu^2)\ldots.(\qquad)\psi = 0 \tag{14}$$

Case (a):

$$\left(k\chi^2 - \frac{\overline{2n-1}^2}{4}\partial_\mu^2\right)\left(k\chi^2 - \frac{\overline{2n-3}^2}{4}\partial_\mu^2\right)\ldots.\left(k\chi^2 - \frac{1}{4}\partial_\mu^2\right)\psi = 0 \tag{14a}$$

Case (b):

$$(k\chi^2 - n^2\partial_\mu^2)(k\chi^2 - \overline{n-1}^2\partial_\mu^2)\ldots.(k\chi^2 - \partial_\mu^2)\chi\psi = 0. \tag{14b}$$

Thus the different values of the rest mass would be:

$$(a)\quad \pm\chi\Big/\frac{(2n-1)}{2\sqrt{k}},\ \pm\chi\Big/\frac{(2n-3)}{2\sqrt{k}},\ldots\ldots\pm\chi\Big/\frac{1}{2\sqrt{k}} \tag{15a}$$

With $k = 1,\ \pm 2\chi/(2n-1),\ \pm 2\chi/(2n-3),\ldots\ldots\pm 2\chi$

$k = \frac{1}{4},\ \pm\ \chi/(2n-1),\ \pm\ \chi/(2n-3),\ldots\ldots\pm\chi$

$$(b)\quad \pm\chi\Big/\frac{n}{\sqrt{k}},\qquad \pm\chi\Big/\frac{n-1}{\sqrt{k}},\qquad \ldots\ldots\pm\chi\Big/\frac{1}{\sqrt{k}} \tag{15b}$$

With $k = 1\ \pm\chi/n,\qquad \pm\chi/(n-1),\quad \ldots\ldots\pm\chi.$

The simplest type of the commutation rule (12′) from which the others are deduced shows at once that (12′) is identically satisfied by the corresponding rule for all lower spins. As an illustration we will show that the general rule (7) for spin 3/2 is identically satisfied by the Dirac commutation rule. This last, retaining k, is

$$\beta_\mu\beta_\nu + \beta_\nu\beta_\mu = \frac{1}{2k}\delta_{\mu\nu}. \tag{II, (10)}$$

With this substitution the right-hand side of (7) reduces to

$$2\delta_{\lambda\mu}\delta_{\nu\rho} + 2\delta_{\lambda\rho}\delta_{\mu\nu} + \delta_{\lambda\nu}\delta_{\mu\rho} - \tfrac{3}{2}(\delta_{\lambda\mu}\delta_{\nu\rho} + \delta_{\lambda\rho}\delta_{\mu\nu} + \delta_{\lambda\rho}\delta_{\mu\rho}). \tag{16}$$

Also,

$$\begin{aligned}
&\beta_\lambda\beta_\mu\beta_\nu\beta_\rho + \beta_\lambda\beta_\mu\left(\frac{1}{2k}\delta_{\nu\rho} - \beta_\rho\beta_\nu\right)\\
&= \frac{1}{2k}\beta_\lambda\beta_\mu\delta_{\nu\rho} - \beta_\lambda\left(\frac{1}{2k}\delta_{\mu\rho} - \beta_\rho\beta_\mu\right)\beta_\nu\\
&= \frac{1}{2k}\beta_\lambda\beta_\mu\delta_{\nu\rho} - \frac{1}{2k}\beta_\lambda\beta_\nu\delta_{\mu\rho} + \beta_\lambda\beta_\rho\left(\frac{1}{2k}\delta_{\mu\nu} - \beta_\nu\beta_\mu\right)
\end{aligned}$$

i.e.,

$$\beta_\lambda\beta_\mu\beta_\nu\beta_\rho + \beta_\lambda\beta_\rho\beta_\nu\beta_\mu = \frac{1}{2k}(\beta_\lambda\beta_\mu\delta_{\nu\rho} - \beta_\lambda\beta_\nu\delta_{\mu\rho} + \beta_\lambda\beta_\rho\delta_{\mu\nu})$$

Similarly

$$\beta_\mu\beta_\nu\beta_\rho\beta_\lambda + \beta_\rho\beta_\nu\beta_\mu\beta_\lambda = \frac{1}{2k}\,(\beta_\mu\beta_\lambda\delta_{\nu\rho} - \beta_\nu\beta_\lambda\delta_{\mu\rho} + \beta_\rho\beta_\lambda\delta_{\mu\nu})$$

Hence the left-hand side of (7) is

$$\tfrac{1}{2}\,(\delta_{\lambda\mu}\delta_{\nu\rho} - \delta_{\lambda\nu}\delta_{\mu\rho} + \delta_{\lambda\rho}\delta_{\mu\nu})$$

which is the same as (16).

In exactly the same way it can be shown that the commutation rule for spin 2 (I, (34)), with k, is identically satisfied by the Duffin commutation rule for spin 1, *viz.*,

$$k\,(\beta_\lambda\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\beta_\lambda) = \delta_{\mu\nu}\beta_\lambda + \delta_{\lambda\mu}\beta_\nu.$$

3. *Conclusion*

It appears therefore that the constant k may be left entirely arbitrary in the derivation of the commutation rules, and a change in k results merely in a change in the value of χ in (1). For purposes of simplification, k can be put equal to 1. In the case of half-integral spin, however, it might perhaps be an advantage to put $k = \tfrac{1}{4}$, since this would result in the case $s = \tfrac{1}{2}$ in the Dirac commutation rule with the standard form.

REFERENCES

Bhabha .. *Current Science*, 1945, **14,** 89–90.
Dirac .. *Principles of Quantum Mechanics*, 1st Edition, 1930, 88–91.
Kemmer .. *Proc. Roy. Soc.*, A, 1939, **173,** 91–116.
Madhava Rao .. *Proc. Ind. Acad. Sci.*, A, 1942, **15,** 139–47 (I).
———— .. *J. Mys. Univ.*, 1942, **3,** 59–63 (II).

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