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BSM

ON THE EQUATIONS OF MOTION OF A NON-HOLONOMIC DYNAMICAL SYSTEM.

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The Half-Yearly Journal of the Mysore University
Vol. VII, No. 1, January 1933

Bangalore Press

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Reprinted from The Half-Yearly Journal of the Mysore University, Vol. VII, No. 1.

ON THE EQUATIONS OF MOTION OF A NON-HOLONOMIC DYNAMICAL SYSTEM.

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The extension of Lagrange's equations to non-holonomic systems in the form of the $n + m$ equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q_r + \lambda_1 A_{r1} + \ldots + \lambda_m A_{rm} \ (r = 1, 2, \ldots n)$$

$$A_{1k} \dot{q}_1 + A_{2k} \dot{q}_2 + \ldots + A_{nk} \dot{q}_n + T_k = 0 \ (k = 1, 2, \ldots m)$$

by the introduction of m undetermined multipliers $\lambda_1 \ldots \lambda_m$ is very well known.* I propose in this note to derive a form of the Lagrangian equations of motion without the introduction of these parameters and such that the number of equations is equal to the number of degrees of freedom of the dynamical system.

For this purpose, I make use of Appell's form of the general differential equations of a dynamical system whether holonomic or not. These equations, as is well known, are expressed† in the form

$$\frac{\partial S}{\partial \ddot{p}_r} = P_r \ (r = 1, 2, \ldots n\text{-}m)$$

where S denotes the function $\frac{1}{2} \Sigma m_k (\ddot{x}_k^2 + \ddot{y}_k^2 + \ddot{z}_k^2)$ and

* Whittaker, *Analytical Dynamics* (3rd Edition), p. 215.

† *Ibid.*, pp. 258-59.

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$(p_1 \ldots . p_{n-m})$ are co-ordinates equal in number to the degrees of freedom of the system. While Appell's equations are very compact and elegant in form, they are not immediately applicable to practical problems (*e.g.*, the rolling of solid bodies without sliding on arbitrary surfaces) for, the calculation of the function S of the co-ordinates would generally entail a large amount of labour. By a slight modification of the method of derivation of these equations, I have given them the Lagrangian form involving the kinetic energy T as well as the function S. The notation used below is the same as in Whittaker's book referred to above, pp. 258-59 :—

Let the equations of constraint be

$$A_{1k}\, dq_1 + \ldots . + A_{nk}\, dq_n + T_k\, dt = 0 \quad (k = 1, 2, \ldots . m).$$

By the aid of these equations we can express m of the velocities $(\dot{q}_1 \ldots . \dot{q}_n)$ in terms of the others. Taking the co-ordinates corresponding to these last as $p_1 \ldots . p_{n-m}$, let the remaining co-ordinates be denoted by $Q_1 \ldots . Q_m$. With this notation the equations of constraint could be replaced by

$$dQ_i = a_{i1}\, dp_1 + \ldots . + t_i\, dt \quad (i = 1, \ldots . m) \quad \ldots \quad \ldots \quad (1)$$

A virtual displacement in a co-ordinate x_k at time t is given by

$$\delta x_k = \Sigma \frac{\partial x_k}{\partial p_r} \delta p_r + \Sigma_i \frac{\partial x_k}{\partial Q_i} \delta Q_i$$

$$= \Sigma \left(\frac{\partial x_k}{\partial p_r} + \Sigma_i a_{ir} \frac{\partial x_k}{\partial Q_i} \right) \delta p_r$$

which gives the value for the coefficient $\pi_r = \dfrac{\partial x_k}{\partial p_r}$. To avoid confusion in symbols it is safer to write

$$\pi_r = \frac{\partial x_{ok}}{\partial p_r} + \Sigma_i a_{ir} \frac{\partial x_{ok}}{\partial Q_i}$$

The starting point for the deduction of Appell's equations is the expression

$$\Sigma_k m_k \left(\ddot{x}_k \frac{\partial x_k}{\partial p_r} + \ddot{y}_k \frac{\partial y_k}{\partial p_r} + \ddot{z}_k \frac{\partial z_k}{\partial p_r} \right) = P_r \quad \ldots \quad \ldots \quad (2)$$

In the expression on the left-hand side of this we substitute for $\dfrac{\partial x_k}{\partial p_r} = \pi_r$ its value obtained above and make similar substitutions for $\dfrac{\partial y_k}{\partial p_r}$ and $\dfrac{\partial z_k}{\partial p_r}$. We can therefore consider merely the first term and for simplicity take only a single individual. Thus the expression

$$m_k\left(\ddot{x}_k \frac{\partial x_k}{\partial p_r}\right) \quad .. \quad .. \quad .. \quad .. \quad .. \quad (3)$$

can be written

$$\frac{d}{dt}\left(m_k\, \dot{x}_k \frac{\partial x_k}{\partial p_r}\right) - m_k\, \dot{x}_k \frac{d}{dt}\left(\frac{\partial x_k}{\partial p_r}\right) \quad .. \quad .. \quad (4)$$

$$= m_k \frac{d}{dt}\,(\dot{x}_k \pi_r) - m_k\, \dot{x}_k \frac{d\pi_r}{dt} \quad .. \quad .. \quad .. \quad (5)$$

but $\dot{x}_k = \sum_{r=1}^{n-m} \pi_r\, \dot{p}_r + a$

so that $\frac{\partial \dot{x}_k}{\partial \dot{p}_r} = \pi_r$ and hence the first term of (5) could be written as $m_k \frac{d}{dt}\left(\dot{x}_k \frac{\partial \dot{x}_k}{\partial \dot{p}_r}\right)$

Further $\dot{x}_k \frac{d\pi_r}{dt} = \dot{x}_k \frac{d}{dt}\left(\frac{\partial x_{ok}}{\partial p_r} + \sum_i a_{ir} \frac{\partial x_{ok}}{\partial Q_i}\right)$

$$= \dot{x}_k \frac{d}{dt}\left(\frac{\partial x_{ok}}{\partial p_r}\right) + \dot{x}_k \frac{d}{dt}\left(\sum_i a_{ir} \frac{\partial x_{ok}}{\partial Q_i}\right) \quad .. \quad (6$$

The first term of (6) is equal to

$\dot{x}_k \frac{\partial \dot{x}_{ok}}{\partial p_r}$ and the second term

$$\dot{x}_k \frac{d}{dt}\left(\Sigma\, a_{ir} \frac{\partial x_{ok}}{\partial Q_i}\right) = \dot{x}_k \frac{d}{dt}\left(\Sigma \frac{\partial \dot{Q}_i}{\partial \dot{p}_r} \cdot \frac{\partial \dot{x}_{ok}}{\partial \dot{Q}_i}\right) \quad .. \quad .. \quad (7)$$

for using (1), $a_{ir} = \frac{\partial \dot{Q}_i}{\partial \dot{p}_r}$ and also

$$\dot{x}_{ok} = \frac{\partial x_{ok}}{\partial t} + \sum_r \frac{\partial x_{ok}}{\partial p_r}\, \dot{p}_r + \sum_i \frac{\partial x_{ok}}{\partial Q_i} \cdot \dot{Q}_i \quad .. \quad .. \quad (8)$$

gives $\frac{\partial \dot{x}_{ok}}{\partial \dot{Q}_i} = \frac{\partial x_{ok}}{\partial Q_i}$.

We can now re-write (7) in the form

$$\frac{d}{dt}\left(\dot{x}_k\, \Sigma \frac{\partial \dot{x}_{ok}}{\partial \dot{Q}_i} \cdot \frac{\partial \dot{Q}_i}{\partial \dot{p}_r}\right) - \ddot{x}_k \left(\Sigma \frac{\partial \ddot{x}_{ok}}{\partial \ddot{Q}_i} \cdot \frac{\partial \ddot{Q}_i}{\partial \ddot{p}_r}\right)$$

since from (8) and (1) it follows easily that $\frac{\partial \dot{x}_{ok}}{\partial \dot{Q}_i} = \frac{\partial \ddot{x}_{ok}}{\partial \ddot{Q}_i}$

and $\frac{\partial \dot{Q}_i}{\partial \dot{p}_r} = \frac{\partial \ddot{Q}_i}{\partial \ddot{p}_r}$.

Expression (6) fully written out is now equal to

$$\dot{x}_k \frac{\partial \dot{x}_{ok}}{\partial p_r} + \frac{d}{dt}\left(\dot{x}_k \sum_i \frac{\partial \dot{x}_{ok}}{\partial \dot{Q}_i} \cdot \frac{\partial \dot{Q}_i}{\partial \dot{p}_r}\right) - \ddot{x}_k \left(\sum_i \frac{\partial \ddot{x}_{ok}}{\partial \ddot{Q}_i} \cdot \frac{\partial \ddot{Q}_i}{\partial \ddot{p}_r}\right)$$

and hence the original expression (3) is equal to

$$m_k \frac{d}{dt}\left(\dot{x}_k \frac{\partial \dot{x}_k}{\partial \dot{p}_r} - \dot{x}_k \sum \frac{\partial \dot{x}_{ok}}{\partial \dot{Q}_i} \cdot \frac{\partial \dot{Q}_i}{\partial \dot{p}_r}\right) - m_k \dot{x}_k \frac{\partial \dot{x}_{ok}}{\partial p_r}$$

$$+ m_k \ddot{x}_k \left(\sum_i \frac{\partial \ddot{x}_{ok}}{\partial \ddot{Q}_i} \cdot \frac{\partial \ddot{Q}_i}{\partial \ddot{p}_r}\right)$$

and the first term of the above could be written as

$$m_k \frac{d}{dt}\left(\dot{x}_k \frac{\partial \dot{x}_{ok}}{\partial \dot{p}_r}\right)$$

Finally, the expression (2) now reduces to the form

$$\frac{d}{dt} \sum m_k \left(\dot{x}_k \frac{\partial \dot{x}_{ok}}{\partial \dot{p}_r} + \ldots + \ldots\right) - \sum m_k \left(\dot{x}_k \frac{\partial \dot{x}_{ok}}{\partial p_r} + \ldots + \ldots\right)$$

$$+ \sum m_k \left\{\ddot{x}_k \left(\sum \frac{\partial \ddot{x}_{ok}}{\partial \ddot{Q}_i} \cdot \frac{\partial \ddot{Q}_i}{\partial \ddot{p}_r}\right) + \ldots + \ldots\right\} = P_r.$$

The first term $= \dfrac{d}{dt}\left\{\dfrac{\partial}{\partial \dot{p}_r}\left[\tfrac{1}{2} m_k (\dot{x}_{ok}{}^2 + \ldots)\right]\right. = \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{p}_r}\right)$

The second term $= \dfrac{\partial T}{\partial p_r}$

The third term $= \dfrac{\partial}{\partial \ddot{p}_r}\left[\tfrac{1}{2} m_k (\ddot{x}_k{}^2 + \ldots)\right] = \dfrac{\partial S}{\partial \ddot{p}_r}$

it being understood that, in this case, S is a function only of the $Q_1 \ldots Q_m$, these being, however, determined from (1). With this understanding we can replace S by S′ and finally the equations of motion reduce to the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}_r}\right) - \frac{\partial T}{\partial p_r} + \frac{\partial S'}{\partial \ddot{p}_r} = P_r \qquad (r = 1, \ldots n - m).$$

These equations have the Lagrangian form and it must be noticed that in the expression for T all the n generalised co-ordinates are to be treated as independent.