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BSM

ON THE EQUATIONS OF MOTION OF A  
NON-HOLONOMIC DYNAMICAL SYSTEM.

B. S. MADHAVA RAO, M.Sc.

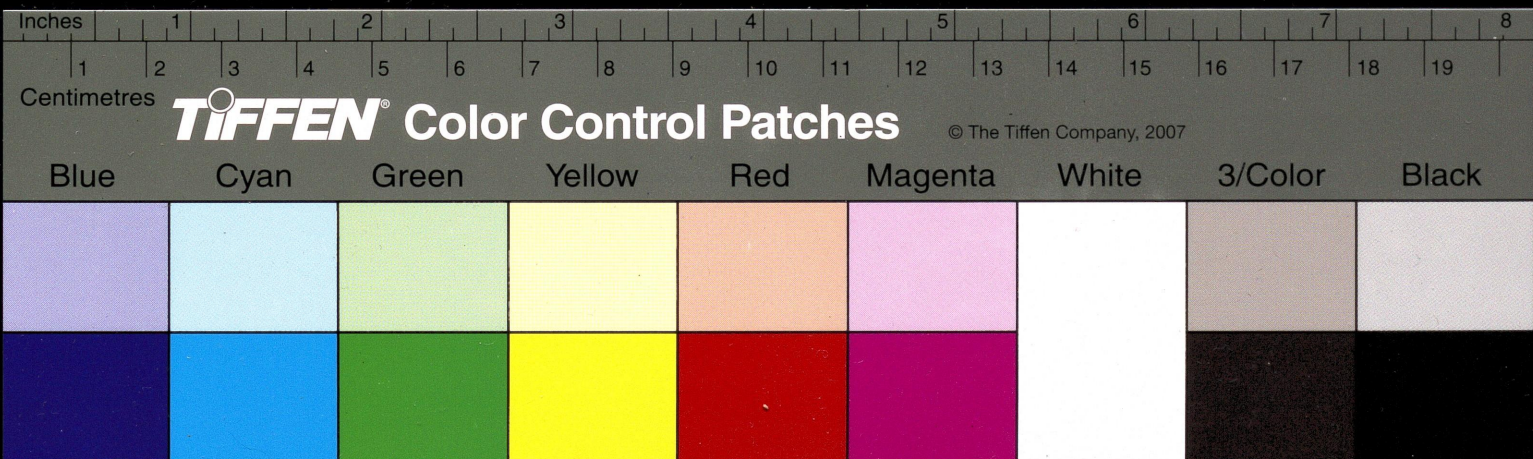
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• ON THE EQUATIONS OF MOTION OF A  
NON-HOLONOMIC DYNAMICAL SYSTEM.

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The extension of Lagrange's equations to non-holonomic systems in the form of the  $n + m$  equations

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_r} \right) - \frac{\partial T}{\partial q_r} = Q_r + \lambda_1 A_{r1} + \dots + \lambda_m A_{rm} \quad (r=1, 2, \dots, n)$$

$$A_{1k} \dot{q}_1 + A_{2k} \dot{q}_2 + \dots + A_{nk} \dot{q}_n + T_k = 0 \quad (k=1, 2, \dots, m)$$

by the introduction of  $m$  undetermined multipliers  $\lambda_1, \dots, \lambda_m$  is very well known.\* I propose in this note to derive a form of the Lagrangian equations of motion without the introduction of these parameters and such that the number of equations is equal to the number of degrees of freedom of the dynamical system.

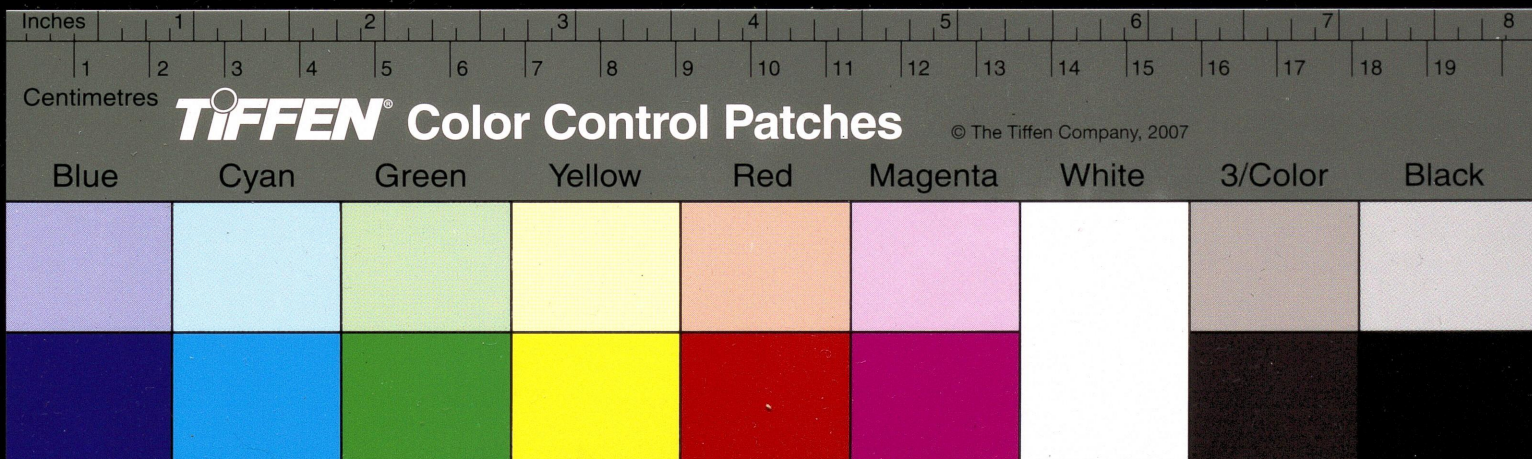
For this purpose, I make use of Appell's form of the general differential equations of a dynamical system whether holonomic or not. These equations, as is well known, are expressed† in the form

$$\frac{\partial S}{\partial \dot{p}_r} = P_r \quad (r=1, 2, \dots, n-m)$$

where  $S$  denotes the function  $\frac{1}{2} \sum m_k (\ddot{x}_k^2 + \ddot{y}_k^2 + \ddot{z}_k^2)$  and

\* Whittaker, *Analytical Dynamics* (3rd Edition), p. 215.

† *Ibid.*, pp. 258-59.







Expression (6) fully written out is now equal to

$$\dot{x}_k \frac{\partial \dot{x}_{ok}}{\partial p_r} + \frac{d}{dt} \left( \dot{x}_k \sum_i \frac{\partial \dot{x}_{ok}}{\partial \dot{Q}_i} \cdot \frac{\partial \dot{Q}_i}{\partial p_r} \right) - \ddot{x}_k \left( \sum_i \frac{\partial \ddot{x}_{ok}}{\partial \ddot{Q}_i} \cdot \frac{\partial \ddot{Q}_i}{\partial \ddot{p}_r} \right)$$

and hence the original expression (3) is equal to

$$m_k \frac{d}{dt} \left( \dot{x}_k \frac{\partial \dot{x}_k}{\partial p_r} - \dot{x}_k \sum_i \frac{\partial \dot{x}_{ok}}{\partial \dot{Q}_i} \cdot \frac{\partial \dot{Q}_i}{\partial p_r} \right) - m_k \dot{x}_k \frac{\partial \dot{x}_{ok}}{\partial p_r} \\ + m_k \ddot{x}_k \left( \sum_i \frac{\partial \ddot{x}_{ok}}{\partial \ddot{Q}_i} \cdot \frac{\partial \ddot{Q}_i}{\partial \ddot{p}_r} \right)$$

and the first term of the above could be written as

$$m_k \frac{d}{dt} \left( \dot{x}_k \frac{\partial \dot{x}_{ok}}{\partial p_r} \right)$$

Finally, the expression (2) now reduces to the form

$$\frac{d}{dt} \sum m_k \left( \dot{x}_k \frac{\partial \dot{x}_{ok}}{\partial p_r} + \dots + \dots \right) - \sum m_k \left( \dot{x}_k \frac{\partial \dot{x}_{ok}}{\partial p_r} + \dots + \dots \right) \\ + \sum m_k \left\{ \ddot{x}_k \left( \sum_i \frac{\partial \ddot{x}_{ok}}{\partial \ddot{Q}_i} \cdot \frac{\partial \ddot{Q}_i}{\partial \ddot{p}_r} \right) + \dots + \dots \right\} = P_r.$$

$$\text{The first term} = \frac{d}{dt} \left\{ \frac{\partial}{\partial p_r} \left[ \frac{1}{2} m_k (\dot{x}_{ok}^2 + \dots) \right] \right\} = \frac{d}{dt} \left( \frac{\partial T}{\partial p_r} \right)$$

$$\text{The second term} = \frac{\partial T}{\partial p_r}$$

$$\text{The third term} = \frac{\partial}{\partial \ddot{p}_r} \left[ \frac{1}{2} m_k (\ddot{x}_k^2 + \dots) \right] = \frac{\partial S}{\partial \ddot{p}_r}$$

it being understood that, in this case, S is a function only of the  $Q_1 \dots Q_m$ , these being, however, determined from (1). With this understanding we can replace S by S' and finally the equations of motion reduce to the form

$$\frac{d}{dt} \left( \frac{\partial T}{\partial p_r} \right) - \frac{\partial T}{\partial p_r} + \frac{\partial S'}{\partial \ddot{p}_r} = P_r \quad (r=1, \dots, n-m).$$

These equations have the Lagrangian form and it must be noticed that in the expression for T all the  $n$  generalised co-ordinates are to be treated as independent.

