Lecture notes for 10/10/67 on Riemannian Geometry.

(1) Preliminary on covariant & contravariant vectors – covariant, contravariant & mixed tensors of the second order – Kronecker δ's – Symmetric & antisymmetric tensors of 2nd order & any tensor being sum of such two. [no proofs of these since already done]

$$\left[\lambda'^i = \lambda^j \frac{\partial x'^i}{\partial x^j};\ \lambda'_i = \lambda_j \frac{\partial x^j}{\partial x'^i};\ a'^{ij} = a^{kl} \frac{\partial x'^i}{\partial x^k}\cdot\frac{\partial x'^j}{\partial x^l};\ a'_{ij} = a_{kl} \frac{\partial x^k}{\partial x'^i}\cdot\frac{\partial x^l}{\partial x'^j}\right.$$

$$\left. a'^i_j = a^k_l \cdot \frac{\partial x'^i}{\partial x^k}\cdot\frac{\partial x^l}{\partial x'^j};\ a_{ij} = \tfrac{1}{2}(a_{ij}+a_{ji}) + \tfrac{1}{2}(a_{ij}-a_{ji}) \right]$$

(2) Conjugate symmetric tensors of the second order – ~~Christoffel 3-index symbols~~ – Dual tensors – Christoffel 3-index symbols & their relations – Riemann-Christoffel tensor and Ricci tensor – $\underset{\text{}}{\overset{\text{scalar }R}{}}$ Einstein tensors – Generalised Div, Curl, Grad & generalised Δ i.e. $\Box$ – Covariant differentiation w.r.t a tensor g_{ij}.

(3) Riemannian Geometry; introduction of a metric generalizing $ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ or $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$ to n-dimensions by Riemann in a V_n by $ds^2 = g_{ij}\dot x^i \dot x^j$ – length of curve & Angle between two directions [E, p. 36] [p. 38] – Geodesics and parallelism [Both E & P] – Space curvature – Bianchi identity & Schur's theorem [as in E] – Riemannian Coordinates – Euclidean space & space of constant curvature.

(2)(a) g_{ij} (symmetric) with its determinant g. If g^{ij} = cofactor of g_{ij}/g; $g^{ij}g_{kj} = \delta^i_k$

If λ^i be arbitrary vector, let $g_{ij}\lambda^i = \mu_j$. Then $g^{kj}\mu_j = g^{kj}g_{ij}\lambda^i = \delta^k_i\lambda^i = \lambda^k$ & since μ_j is arbitrary it follows that g^{ij} are components of a symmetric contravariant tensor. Similarly if g^{ij} be ~~tr~~ a symmetric ~~covariant~~ contravariant tensor and $\bar g$ its determinant, the cofactors of g^{ij} in $\bar g$ are symmetric covariant tensors g_{ij}. This leads to notion of conjugate tensors g_{ij} & g^{ij}. Also $g^{ij}g_{ij} = \delta^i_i = n$, and $g\bar g = 1$, using rule of multiplication of determinants & $g^{ij}g_{kj} = \delta^i_k$. Raising & lowering of indices of tensors by means of g^{ij} and g_{ij}

(b) Dual tensors.

$$\xi^{*14} = \frac{1}{\sqrt g}\xi_{23},\ \ldots \qquad \xi^*_{14} = \sqrt g\, \xi^{23},\ \ldots$$
$$\xi^{*23} = \frac{1}{\sqrt g}\xi_{14},\ \ldots \qquad \xi^*_{23} = \sqrt g\, \xi^{14},\ \ldots$$

(c) Christoffel 3-index symbols

(i) First kind $= \frac{1}{2}\left(\frac{\partial g_{ir}}{\partial x^s} + \frac{\partial g_{is}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^i}\right) = \Gamma_{i,rs}$

(ii) Second kind $= g^{ik}\Gamma_{k,rs} = \Gamma^i_{rs}$

$$\Gamma_{i,rs} = g_{ik}\,\Gamma^{k}_{rs}$$

$$\left[R.H.S = g_{ik}\, g^{kl}\,\Gamma_{l,rs} = \delta^{l}_{i}\,\Gamma_{l,rs} = \Gamma_{i,rs} \right]$$

Also, $\Gamma^{i}_{ir} = g^{is}\,\Gamma_{s,ir} = \frac{1}{2} g^{is}\left\{ \frac{\partial g_{si}}{\partial x^r} + \frac{\partial g_{sr}}{\partial x^i} - \frac{\partial g_{ir}}{\partial x^s} \right\}$

$= \frac{1}{2} g^{is}\,\frac{\partial g_{is}}{\partial x^r}$ since $g^{is}\,\frac{\partial g_{sr}}{\partial x^i} = g^{is}\,\frac{\partial g_{ir}}{\partial x^s}$

interchanging the dummy $\underline{i}$ & $\underline{s}$

and $\frac{\partial g}{\partial x^r} = g\, g^{is}\,\frac{\partial g_{is}}{\partial x^r}$. [using law of diff^n of dets & defn of g^{is}]

$\therefore\ \Gamma^{l}_{ir} = \frac{1}{2g}\,\frac{\partial g}{\partial x^r} = \frac{1}{\sqrt{g}}\cdot\frac{\partial \sqrt{g}}{\partial x^r} = \frac{\partial \log\sqrt{g}}{\partial x^r}$. [Christoffel symbols are not ~~invariant~~] tensors since g is not an invariant]

~~(Curvature tensor)~~

(ii) ~~Riemann-Christoffel tensor is a curvature tensor~~.

~~defined by R^{h}_{ijk} =~~

Law of transformation of the Christoffel symbols.

$$\Gamma'_{\sigma,\mu\nu} = \Gamma_{k,ij}\,\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\cdot\frac{\partial x^k}{\partial x'^\sigma} + g_{ij}\,\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial^2 x^j}{\partial x'^\mu\,\partial x'^\nu} \quad \text{——— (A)}$$

using $g'_{\mu\nu} = g_{ij}\,\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}$.

(ii) ~~Curvature tensor~~: Multiplying (A) by $g'^{\sigma\lambda}\,\frac{\partial x^l}{\partial x'^\lambda}$ & summing for σ & using $g'^{\sigma\lambda}\,\frac{\partial x^k}{\partial x'^\sigma}\cdot\frac{\partial x^l}{\partial x'^\lambda} = g^{kl}$

$$\Gamma'^{\lambda}_{\mu\nu}\cdot\frac{\partial x^l}{\partial x'^\lambda} = \Gamma^{l}_{ij}\,\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} + \frac{\partial^2 x^l}{\partial x'^\mu\,\partial x'^\nu} \quad \text{——— (B)}$$

Since $\Gamma'^{\lambda}_{\mu\sigma}\cdot\frac{\partial x^l}{\partial x'^\lambda} = \Gamma^{l}_{ij}\,\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma} + \frac{\partial^2 x^l}{\partial x'^\mu\,\partial x'^\sigma}$ ——— (B$'$)

Differentiate (B$'$) w.r.t x'^ν and (B) w.r.t. x'^σ & eliminate $\frac{\partial^3 x^l}{\partial x'^\mu\,\partial x'^\nu\,\partial x'^\sigma}$

$$\frac{\partial}{\partial x'^\nu}\left[\Gamma'^{\lambda}_{\mu\sigma}\,\frac{\partial x^l}{\partial x'^\lambda} - \Gamma^{l}_{ij}\,\frac{\partial x^i}{\partial x'^\mu}\,\frac{\partial x^j}{\partial x'^\sigma}\right] - \frac{\partial}{\partial x'^\sigma}\left[\Gamma'^{\lambda}_{\mu\nu}\,\frac{\partial x^l}{\partial x'^\lambda} - \Gamma^{l}_{ij}\,\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\right] = 0$$

or $R'^{\lambda}_{\mu\sigma\nu}\,\frac{\partial x^l}{\partial x'^\lambda} = R^{l}_{ijk}\,\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\,\frac{\partial x^k}{\partial x'^\nu}$ ——————— (C)

where $R^{l}_{ijk} = \frac{\partial}{\partial x^j}\,\Gamma^{l}_{ik} - \frac{\partial}{\partial x^k}\,\Gamma^{l}_{ij} + \Gamma^{m}_{ik}\,\Gamma^{l}_{mj} - \Gamma^{m}_{ij}\,\Gamma^{l}_{mk}$ — (C$'$)

If (C) be multiplied by $\frac{\partial x'^\alpha}{\partial x^l}$ & summed for l, we have

$$R'^{\alpha}_{\mu\sigma\nu} = R^{l}_{ijk}\,\frac{\partial x'^\alpha}{\partial x^l}\cdot\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu} \quad \text{——————— (D)}$$

(2) Eqn (D) shows R^{l}_{ijk} is a ~~tens~~ mixed tensor of the 4th order, contravariant of the first order & covariant of the 3rd order. It is called the mixed Riemann or Riemann–Chr or curvature tensor of 4th order (2nd kind). ~~By~~ (C')' ~~it follows~~ or −(C') is defined this curvature tensor, and from (C') it follows that R^{l}_{ijk} is skew symmetric in j and k.

Using $R_{hijk} = g_{lh} R^{l}_{ijk}$ we get the curvature tensor of the first kind (rather Riemann symbols of first & second kinds). Also $R^{l}_{ijk} = g^{lh} R_{hijk}$

If (C) be multiplied by $g_{lh} \dfrac{\partial x^h}{\partial x'^\tau}$ and use is made of defn of R_{hijk} and of transformation law for $g'^{\mu\nu}$, we get

$$R'_{\tau\mu\sigma\nu} = R_{hijk} \frac{\partial x^h}{\partial x'^\tau} \cdot \frac{\partial x^i}{\partial x'^\mu} \cdot \frac{\partial x^j}{\partial x'^\sigma} \cdot \frac{\partial x^k}{\partial x'^\nu} \qquad \text{—— (D')}$$

$$\left.\begin{aligned} &\text{Now, } g_{lh} \Gamma^{l}_{ij} = g_{lh} g^{lk} \Gamma_{k,ij} = \delta^{k}_{h} \Gamma_{k,ij} = \Gamma_{h,ij} \\ &\text{and } \quad \frac{\partial g_{ik}}{\partial x^j} = \Gamma_{k,ij} + \Gamma_{kj,i} \quad \text{from defn of Christoffel symbols of 1st kind} \end{aligned}\right\} \text{(E)}$$

$$\text{Using (E)} \quad g_{lh} \frac{\partial}{\partial x^j} \Gamma^{l}_{ik} = \frac{\partial}{\partial x^j}\left(g_{lh} \Gamma^{l}_{ik}\right) - \Gamma^{l}_{ik} \frac{\partial g_{lh}}{\partial x^j}$$

$$= \frac{\partial}{\partial x^j} \Gamma_{h,ik} - \Gamma^{l}_{ik}\left(\Gamma_{h,lj} + \Gamma_{l,hj}\right) \qquad \text{—— (E')}$$

Using (E'), (C') and defn of R_{hijk}, we have

$$R_{hijk} = \frac{\partial}{\partial x^j} \Gamma_{h,ik} - \frac{\partial}{\partial x^k} \Gamma_{h,ij} + \Gamma^{l}_{ij} \Gamma_{l,hk} - \Gamma^{l}_{ik} \Gamma_{l,hj} \qquad \text{—— (F)}$$

& using defns of Christoffel Symbols, this reduces to

$$R_{hijk} = \frac{1}{2}\left(\frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} - \frac{\partial^2 g_{hj}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} \right)$$

$$+ g^{lm}\left\{ \Gamma_{m,ij} \Gamma_{l,hk} - \Gamma_{m,ik} \Gamma_{l,hj} \right\} \qquad \text{—— (F')}$$

From (F') follows

$$\left.\begin{aligned} R_{hijk} &= -R_{ihjk} \\ R_{hijk} &= -R_{hikj} \\ R_{hijk} &= R_{jkhi} \end{aligned}\right\} \text{—— (F'')}$$

$$\text{and } R_{hijk} + R_{hjki} + R_{hkij} = 0 \qquad \text{—— (F''')}$$

$$\cancel{\Gamma'_{\lambda,\mu\nu} = \Gamma_{i,jk}\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\mu}\cdot\frac{\partial x^k}{\partial x'^\nu} + g_{ij}\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial^2 x^j}{\partial x'^\mu\partial x'^\nu}}$$

$$\cancel{\Gamma'_{\lambda,\mu\sigma} = \Gamma_{i,jk}\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\mu}\cdot\frac{\partial x^k}{\partial x'^\sigma} + g_{ij}\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial^2 x^j}{\partial x'^\mu\partial x'^\sigma}}$$

$$\cancel{\frac{\partial}{\partial x'^\sigma}\left(\Gamma'_{\lambda,\mu\nu}\right) - \frac{\partial}{\partial x'^\nu}\left(\Gamma'_{\lambda,\mu\sigma}\right) = \frac{\partial}{\partial x'^\sigma}\Big(}$$

$$+\ \Gamma^l_{mj}\,\Gamma^m_{ik}\,\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu}$$

ie $i \rightleftarrows m$, $n \to k$.

$$R^l_{ijk} = \frac{\partial}{\partial x^j}\left(\Gamma^l_{ik}\right) - \frac{\partial}{\partial x^k}\left(\Gamma^l_{ij}\right) + \Gamma^m_{ik}\Gamma^l_{mj} - \Gamma^m_{ij}\Gamma^l_{mk}$$

$$R'^\lambda_{\mu\sigma\nu}\frac{\partial x^l}{\partial x'^\lambda} = R^l_{ijk}\cdot\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu}$$

$$R'^\lambda_{\mu\sigma\nu}\cdot\frac{\partial x^l}{\partial x'^\lambda}\cdot\frac{\partial x'^\alpha}{\partial x^l} = R^l_{ijk}\cdot\frac{\partial x'^\alpha}{\partial x^l}\cdot\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\frac{\partial x^k}{\partial x'^\nu}$$

$$R'^\alpha_{\mu\sigma\nu} = R^l_{ijk}\cdot$$

or $R'^\lambda_{\mu\sigma\nu} = R^l_{ijk}\dfrac{\partial x'^\lambda}{\partial x^l}\cdot\dfrac{\partial x^i}{\partial x'^\mu}\cdot\dfrac{\partial x^j}{\partial x'^\sigma}\cdot\dfrac{\partial x^k}{\partial x'^\nu}$ showing tensor character R^l_{ijk}

Again from beginning.

$$\cancel{\Gamma'_{\lambda,\mu\nu} = \frac{1}{2}\left(\frac{\partial g'_{\lambda\mu}}{\partial x'^\nu} + \frac{\partial g'_{\lambda\nu}}{\partial x'^\mu} - \frac{\partial g'_{\mu\nu}}{\partial x'^\lambda}\right)}$$

$$g'_{\lambda\mu} = g_{ij}\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\mu}$$

$$\cancel{= \frac{1}{2}\left\{\frac{\partial}{\partial x'^\nu}\left(g_{ij}\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\mu}\right) + \frac{\partial}{\partial x'^\mu}\left(g_{ij}\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\nu}\right) - \frac{\partial}{\partial x'^\lambda}\left(g_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\right)\right.}$$

$$\cancel{= \frac{1}{2}\left\{\frac{\partial}{\partial x'^\nu}(g_{ij}) + \frac{\partial}{\partial x'^\mu}(g_{ij}) - \frac{\partial}{\partial x'^\lambda}(g_{ij}) + \frac{\partial}{\partial x'^\nu}\left(\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^j}{\partial x'^\mu}\right) + \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^j}{\partial x'^\nu}\right) - \frac{\partial}{\partial x'^\lambda}\left(\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}\right)\right.}$$

$$\cancel{= \frac{1}{2}\left[\frac{\partial}{\partial x'^\nu}(g_{ij})\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^j}{\partial x'^\mu} + \frac{\partial}{\partial x'^\mu}(g_{ij})\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^j}{\partial x'^\nu} - \frac{\partial}{\partial x'^\lambda}(g_{ij})\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}\right.}$$

$$\cancel{+ g_{ij}\frac{\partial}{\partial x'^\nu}\left(\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\mu}\right) + g_{ij}\frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\nu}\right) - g_{ij}\frac{\partial}{\partial x'^\lambda}\left(g_{ij}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}\right)}$$

$\Gamma_{i,jk}$

$$\cancel{= \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\left(\frac{\partial x^k}{\partial x'^\nu}\right) + \frac{1}{2}\,\partial g_i}$$

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i, j, k (1)

(A) $\Gamma'_{\lambda,\mu\nu} = \frac{1}{2}\left(\frac{\partial g'_{\lambda\nu}}{\partial x'^\mu} + \frac{\partial g'_{\lambda\mu}}{\partial x'^\nu} - \frac{\partial g'_{\mu\nu}}{\partial x'^\lambda}\right)$

$$g'_{\lambda\nu} = g_{ik}\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^k}{\partial x'^\nu}$$
$$g'_{\lambda\mu} = g_{ij}\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^j}{\partial x'^\mu}$$
$$g_{\mu\nu} = g_{jk}\frac{\partial x^j}{\partial x'^\mu}\frac{\partial x^k}{\partial x'^\nu}$$

$$= \frac{1}{2}\Bigg[\frac{\partial}{\partial x'^\mu}(g_{ik})\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^k}{\partial x'^\nu} + \frac{\partial}{\partial x'^\nu}(g_{ij})\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^j}{\partial x'^\mu} - \frac{\partial}{\partial x'^\lambda}(g_{jk})\frac{\partial x^j}{\partial x'^\mu}\frac{\partial x^k}{\partial x'^\nu}$$
$$+ g_{ij}\frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^j}{\partial x'^\nu}\right) + g_{ij}\frac{\partial}{\partial x'^\nu}\left(\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\mu}\right) - g_{ij}\frac{\partial}{\partial x'^\lambda}\left(\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}\right)$$

$$= \Gamma_{i,jk}\cdot\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^j}{\partial x'^\mu}\cdot\frac{\partial x^k}{\partial x'^\nu} + g_{ij}\Bigg\{\cancel{\frac{\partial^2 x^i}{\partial x'^\lambda\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}} + \frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial^2 x^j}{\partial x'^\mu\partial x'^\nu}$$
$$+ \cancel{\frac{\partial^2 x^i}{\partial x'^\nu\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\mu}} + \frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial^2 x^j}{\partial x'^\nu\partial x'^\mu}$$
$$- \cancel{\frac{\partial^2 x^i}{\partial x'^\lambda\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}} - \cancel{\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial^2 x^j}{\partial x'^\lambda\partial x'^\nu}}$$

(\ by interchanging i & j)

$$\Gamma'_{\lambda,\mu\nu} = \Gamma_{i,jk}\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^j}{\partial x'^\mu}\cdot\frac{\partial x^k}{\partial x'^\nu} + g_{ij}\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial^2 x^j}{\partial x'^\mu\partial x'^\nu} \quad \text{———— (1)}$$

Multiplying by $g'^{\tau\lambda}\dfrac{\partial x^l}{\partial x'^\tau}$

$$\Gamma'^\tau_{\mu\nu}\frac{\partial x^l}{\partial x'^\tau} = \Gamma_{i,jk}\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\mu}\cdot\frac{\partial x^k}{\partial x'^\nu}\cdot g'^{\tau\lambda}\frac{\partial x^l}{\partial x'^\tau} + g'^{\tau\lambda}g_{ij}\frac{\partial x^l}{\partial x'^\tau}\cdot\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial^2 x^j}{\partial x'^\mu\partial x'^\nu}$$

$$= \Gamma_{ijk}g^{il}\frac{\partial x^j}{\partial x'^\mu}\cdot\frac{\partial x^k}{\partial x'^\nu} + g^{il}g_{ij}\frac{\partial^2 x^j}{\partial x'^\mu\partial x'^\nu}$$

ie $$\Gamma'^\lambda_{\mu\nu}\frac{\partial x^l}{\partial x'^\lambda} = \Gamma^l_{jk}\frac{\partial x^j}{\partial x'^\mu}\frac{\partial x^k}{\partial x'^\nu} + \frac{\partial^2 x^l}{\partial x'^\mu\partial x'^\nu}$$

$$= \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} + \frac{\partial^2 x^l}{\partial x'^\mu\partial x'^\nu} \quad \text{———— (2)}$$

$$\frac{\partial^2 x^l}{\partial x'^\mu\partial x'^\nu} = \Gamma'^\lambda_{\mu\nu}\frac{\partial x^l}{\partial x'^\lambda} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu} \quad \text{———— (2')}$$

$$\frac{\partial^2 x^l}{\partial x'^\mu\partial x'^\sigma} = \Gamma'^\lambda_{\mu\sigma}\frac{\partial x^l}{\partial x'^\lambda} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\sigma} \quad \text{———— (2'')}$$

Lecture on 5/12/67

(1) $\Gamma^{\alpha}_{44} \simeq \Gamma_{\alpha,44}$ $\qquad \Gamma^{\alpha}_{44} = g^{\alpha\beta}\,\Gamma_{\beta,44}$ $\qquad$ only cases of interest are: ($\alpha = 1, 2, 3$) $\quad g^{\alpha\beta} \cong \delta^{\alpha\beta}$

(2) $g_{44} = -1 - \dfrac{2\Phi}{c^2}$ chosen because when $\Phi = 0$, g_{44} should have value in sp. relativity

(3) Expt. tests of general theory.

(i) for a clock at rest $ds^2 = g_{44}(dx^4)^2$, $s = ic\tau$, $t = \dfrac{x^4}{c}$.

$$ds = \sqrt{-g_{44}}\,dx^4 = \sqrt{\left(1 + \frac{2\phi}{c^2}\right)}\,dx^4 = \left(1 + \frac{\phi}{c^2}\right)dx^4.$$

$$d\tau = \left(1 + \frac{\phi}{c^2}\right)dt \qquad \frac{t-\tau}{\tau} \text{ or } \frac{t}{\tau} - 1 = -\frac{\phi}{c^2}$$

$$\frac{\Delta t}{\tau} = -\frac{\Phi}{c^2} \quad ; \quad \frac{\Delta\nu}{\nu} = \frac{\phi_E - \phi_S}{c^2} \text{ (Earth \& Sun)} \qquad \phi_\sigma = -\frac{\gamma M}{R}$$

$$\frac{\Delta\nu}{\nu} = \frac{\gamma M}{Rc^2}.$$

(ii) Precession of perihelion of Mercury

$$\cancel{\Delta\Pi = \frac{6\pi m}{a(1-e^2)} \text{ per revolution}}$$

$$\cancel{\frac{4\pi^2 a^3}{T^2}} \qquad \Delta\Pi = \frac{2\pi\,\pi^3 a^3}{c^2 T^2 (1-e^2)} \cong 42.89'' \text{ per century.}$$

(iii) Bending of light ray in gravitational field

$$\epsilon = \frac{4m}{\Delta} \quad (\Delta = \text{distance of origin from asymptotic dirn of the path}$$

$$\cong 1.75''.$$

(4) Relativity & Cosmology: (no need for cosmological term λg_{ik}

(i) New solns of Einstein's eqns of Friedmann used by Lemaitre for expanding Universe

(World homogeneous in space with time dependent metric!

(ii) Hubble's discovery of a red-shift of spectral lines emitted by nebulae proportional to their distance. — Hubble constant H — Age of Earth 3×10^9 years — Age of Universe found recently by fn of H giving $t_H \cong 6 \times 10^9$ years

(1) Covariant differentiation:

(a) of $\lambda^i \rightarrow$ a mixed tensor $\lambda^i_{;j} = \dfrac{\partial \lambda^i}{\partial x^j} + \lambda^h \Gamma^i_{hj}$

(b) of $\lambda_i \rightarrow$ a covariant tensor $\lambda_{i;j} = \dfrac{\partial \lambda_i}{\partial x^j} - \lambda_h \Gamma^h_{ij}$. } to be proved

(c) General rules.

$$a_{ij;k} = \frac{\partial a_{ij}}{\partial x^k} - a_{ih}\Gamma^h_{jk} - a_{jh}\Gamma^h_{ik}$$

$$a^{ij}_{\ ;k} = \frac{\partial a^{ij}}{\partial x^k} + a^{ih}\Gamma^j_{hk} + a^{jh}\Gamma^i_{hk}$$

$$a^i_{j;k} = \frac{\partial a^i_j}{\partial x^k} + a^h_j \Gamma^i_{hk} - a^i_h \Gamma^h_{jk}.$$

Mention that R^h_{ijk} & R_{hijk} etc could be defined in terms of covariant differentiation & properties already obtained could be proved more easily for eg: that R^h_{ijk} is a tensor. — g_{ij} & g^{ij} are constants in covariant differentiation.

(2) Riemannian Geometry (a) $ds^2 = g_{ij}\, dx^i dx^j$ (Riemannian metric) — cf. Euclidean metric, cf $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta . d\phi^2$ — A curve given by $x^i = x^i(t)$ and $ds = \sqrt{g_{ij}\dfrac{dx^i}{dt}\cdot\dfrac{dx^j}{dt}}\, . dt$ (b) Geodesics: cond^n that $s = \int_0^t \sqrt{\ }\, . dt$ may be a minimum is given by $\dfrac{d^2x^l}{dt^2} + \Gamma^l_{jk}\dfrac{dx^j}{dt}\cdot\dfrac{dx^k}{dt} - \dfrac{dx^l}{dt}\cdot\dfrac{d^2s/dt^2}{ds/dt} = 0$ or using s in place of t

$\dfrac{d^2x^l}{ds^2} + \Gamma^l_{jk}\dfrac{dx^j}{ds}\cdot\dfrac{dx^k}{ds} = 0$. Any integral curve satisfying these diff eq^ns is called a geodesic, and has the property that along any such curve $g_{ij}\dfrac{dx^i}{ds}\cdot\dfrac{dx^j}{ds} = 1$. This can be interpreted, by using the unit tangent vector $u^i = \dfrac{dx^i}{ds}$ at any point of a curve, as $u_i u^i = g_{ij}\dfrac{dx^i}{ds}\cdot\dfrac{dx^j}{ds} = 1$. This gives another definition of a geodesic line as a curve always maintaining its direction.

(c) Riemannian coordinates — Eq^n to a geodesic thro' P_0 & with direc. $\xi^i = \left(\dfrac{dx^i}{ds}\right)_0$ can be written as $x^i = x^i_0 + \xi^i s + \dfrac{1}{2!}\left(\dfrac{d^2x^i}{ds^2}\right)_0 s^2 + \ldots$ which can be simplified to

$x^i = x^i_0 + \xi^i s - \dfrac{1}{2!}\left(\Gamma^i_{jk}\right)_0 \xi^j \xi^k s^2 - \ldots$ If $y^i = \xi^i s$, the geodesic can be written as

$x^i = x^i_0 + y^i - \frac{1}{2}\left(\Gamma^i_{\alpha\beta}\right)_0 y^\alpha y^\beta -$ & inverting this series $y^i = (x^i - x_0) + F^i(x^1 - x^1_0, \ldots x^n - x^n_0)$

The y's are called Riemannian Coordinates & in their geodesics have the form of a

258 st. line in Euclidean Geometry. If in Riemannian coordinates the fundamental form is $\overline{g}_{ij}\, dy^i dy^j$ it can be shown that geodesics are given by $\overline{\Gamma}^i_{jk}\, y^j y^k = 0$.

(d). Parallelism & parallel displacement of a vector: Generalisation of notion of parallelism is that all vectors ∥l to one another make the same angle with a geodesic. If λ^i be components of a unit vectors at points of a geodesic C. The condition that the cosine of the angle between λ^i and tangent to C be constant along C is given by

$$\lambda^i_{,k}\frac{dx^k}{ds} = \left(\frac{\partial \lambda^i}{\partial x^k} + \lambda^l \Gamma^i_{kl}\right)\frac{dx^k}{ds} = 0.$$ If in this we put $\lambda^i = \frac{dx^i}{ds}$, we get

→ taken also as ∥lm for any C.

$$\therefore\ \cancel{\frac{d^2x^i}{}}\ \frac{\partial}{\partial x^k}\left(\frac{dx^i}{ds}\right)\frac{dx^k}{ds} = \frac{d^2x^i}{ds^2},$$ the eqn $\frac{d^2x^i}{ds^2} + \Gamma^i_{kl}\frac{dx^l}{ds}\frac{dx^k}{ds} = 0$

i.e. zero for geodesics tangents are ∥l w.r.t the curve. (st. line having same dirn)

For a general parameter t, the ∥lm condn becomes

$$\frac{d\lambda^i}{dt} + \lambda^l \Gamma^i_{kl}\frac{dx^k}{dt} = 0 \quad —(1)$$

This along with $\frac{d}{dt}\left(g_{ik}\lambda^i\lambda^k\right) = \frac{d}{dt}\left(\lambda_i\lambda^i\right) = 0 \quad —(2)$

define the ∥l displacement of a vector λ^i along a curve $x^i = x^i(t)$. It can be shown that a vector can be parallely displaced round a closed circuit if $R^h_{ijk} = 0$.

(e) Curvature at a point in V_n — Let $\lambda^i_{1|}$ & $\lambda^i_{2|}$ be components of two contravariant vector fields. The vectors at a point P determine a pencil of dirns defined by

$$\xi^i = \alpha\lambda^i_{1|} + \beta\lambda^i_{2|} \quad (\alpha, \beta \text{ parameters})$$

The geodesics thro' P in this pencil constitute a geodesic surface S. The Gaussian curvature of S at P was taken by Riemann as the curvature of V_n at P for the given orientation, and the formula for this is

$$K = \frac{R_{hijk}\lambda^h_{1|}\lambda^i_{2|}\lambda^j_{1|}\lambda^k_{2|}}{\left(g_{hj}g_{ik} - g_{hk}g_{ij}\right)\lambda^h_{1|}\lambda^i_{2|}\lambda^j_{1|}\lambda^k_{2|}}$$

[If K at each point is the same for every orientation, it does not vary from point to point & space is said to be of constant Riemannian curvature]

(f) Bianchi identity & Schur's theorem.

$$\text{From } R^h_{ijk} = \frac{\partial}{\partial x^j}\left(\Gamma^h_{ik}\right) - \frac{\partial}{\partial x^k}\left(\Gamma^h_{ij}\right) + \Gamma^h_{mj}\Gamma^m_{ik} - \Gamma^h_{mk}\Gamma^m_{ij}$$

Choosing geodesic coordinates (those for which at a point first covariant derivatives are ordinary derivatives) at a point P, then at P

$$\left[\Gamma_{i,jk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x_k} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_i}\right)\right]$$

$$R^h_{ijk,l} = \frac{\partial^2}{\partial x^j \partial x^l}\left(\Gamma^h_{ik}\right) - \frac{\partial^2}{\partial x^k \partial x^l}\left(\Gamma^h_{ij}\right)$$ (3)

From this follows

$$R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} = 0$$ (Bianchi's identity)

Since g_{ij} & g^{ij} behave like constants in covariant diff^n

$$R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0$$

Using identities $R_{hijk} = -R_{ihjk}$, $R_{hijk} = -R_{hikj}$, $R_{hijk} = R_{jkhi}$

& ~~R_{hijk}~~

Bianchi's identities leads to

$$R^h_{ijk,l} - R^h_{ilk,j} + g^{hm} R_{milj,k} = 0$$

Contracting for h & k, $R_{ij,l} - R_{il,j} + g^{hm} R_{milj,h} = 0$ (R_{ij} = Ricci tensor)

Multiplying this eq^n by g^{il} & summing for i & l, we get

$$R^l_{j,l} = \frac{1}{2}\frac{\partial R}{\partial x^j} \quad \text{where } R = g^{il} R_{il}$$ (R = curvature invariant or scalar curvature).

$$\cancel{R^l_{j,l} = \frac{1}{2}\frac{\partial R}{\partial x^j}}$$

$$\cancel{R} \quad \frac{1}{2}$$

$$\frac{\Delta t - \Delta \tau_E}{\tau_0} = -\frac{\phi_E}{c^2}$$

$$T^l_{j,l} = 0$$

$$\frac{\Delta t - \nabla \tau_S}{\tau_0} = -\frac{\phi_S}{c^2}$$

$$\cancel{\frac{t-\tau}{\tau} = \frac{\phi}{c^2}}$$

$$R_{ij} - \frac{1}{2} g_{ij} R$$

$$dt = d\tau\left(1 - \frac{\phi}{c^2}\right)$$

$$\cancel{\left(R^l_j - \frac{1}{2}\frac{\partial R}{\partial x^j}\right)_{,l}}$$

$$R^l_{j,l} - \frac{\partial}{\partial x_j}\left(\frac{1}{2} g^{il} R_{il}\right)$$

$$\left(R^l_j - \frac{1}{2} g^l_j R\right)_{,l}$$

$$= R^l_{j,l} - \frac{1}{2} g^l_j \frac{\partial R}{\partial x^l} = R^l_{j,l} - \frac{1}{2}\frac{\partial R}{\partial x^j} = 0$$

$$\lambda^i = \lambda'^\mu \frac{\partial x^i}{\partial x'^\mu} = \lambda'^\sigma \frac{\partial x^i}{\partial x'^\sigma}$$

$$\frac{\partial \lambda^i}{\partial x^k} = \frac{\partial \lambda'^\mu}{\partial x^k} \cdot \frac{\partial x^i}{\partial x'^\mu} + \lambda'^\sigma \frac{\partial^2 x^i}{\partial x'^\sigma}$$

$$\frac{\partial \lambda^i}{\partial x^j} = \frac{\partial \lambda'^\mu}{\partial x'^\nu} \cdot \frac{\partial x'^\nu}{\partial x^j} \cdot \frac{\partial x^i}{\partial x'^\mu} + \lambda'^\sigma \frac{\partial^2 x^i}{\partial x'^\sigma \partial x'^\nu} \cdot \frac{\partial x'^\nu}{\partial x^j}$$

$$= \quad " \quad + \lambda'^\sigma \frac{\partial x'^\nu}{\partial x^j} \left\{ \Gamma^\mu_{\sigma\nu} \frac{\partial x^i}{\partial x'^\mu} - \Gamma^i_{hk} \frac{\partial x^h}{\partial x'^\sigma} \cdot \frac{\partial x^k}{\partial x'^\nu} \right\}$$

$$= \left(\frac{\partial \lambda'^\mu}{\partial x'^\nu} + \lambda'^\sigma \Gamma^\mu_{\sigma\nu} \right) \frac{\partial x'^\nu}{\partial x^j} \cdot \frac{\partial x^i}{\partial x'^\mu} - \lambda'^\sigma \Gamma^i_{hk} \underbrace{\frac{\partial x'^\nu}{\partial x^j} \cdot \frac{\partial x^k}{\partial x'^\nu}}_{\delta_{jk}} \cdot \frac{\partial x^h}{\partial x'^\sigma}$$

$$= \quad " \quad - \lambda'^\sigma \frac{\partial x^h}{\partial x'^\sigma} \cdot \Gamma^i_{hj}$$

ie $\left(\frac{\partial \lambda^i}{\partial x^j} + \lambda^h \Gamma^i_{hj} \right) = \left(\frac{\partial \lambda'^\mu}{\partial x'^\nu} + \lambda'^\sigma \Gamma^\mu_{\sigma\nu} \right) \frac{\partial x'^\nu}{\partial x^j} \cdot \frac{\partial x^i}{\partial x'^\mu}$

$\lambda^i_{,j} = \lambda'^\mu_{\nu} \cdot \frac{\partial x'^\nu}{\partial x^j} \cdot \frac{\partial x^i}{\partial x'^\mu}$ ie $\lambda^i_{,j}$ is a mixed tensor, Covariant derivative of λ^i w.r.t x^j

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$$\lambda_i = \lambda'_\mu \frac{\partial x'^\mu}{\partial x^i} = \lambda'_\sigma \frac{\partial x'^\sigma}{\partial x^i}$$

$$\frac{\partial \lambda_i}{\partial x^j} = \frac{\partial \lambda'_\mu}{\partial x'^\nu} \cdot \frac{\partial x'^\nu}{\partial x^j} \cdot \frac{\partial x'^\mu}{\partial x^i} + \lambda'_\sigma \frac{\partial^2 x'^\sigma}{\partial x^i \partial x'^\nu} \frac{\partial x'^\nu}{\partial x^j} \cdot$$

$$= \quad " \quad + \lambda'_\sigma \frac{\partial x'^\nu}{\partial x^j} \{$$

Since $\lambda_{i,j} = \frac{\partial \lambda_i}{\partial x^j} - \lambda_h \Gamma^h_{ij}$ is a covariant tensor

$$\lambda^i_{,j} = \frac{\partial \lambda^i}{\partial x^j} + \lambda^h \Gamma^i_{hj}$$

$$a^{ij}_{,k} = \frac{\partial a^{ij}}{\partial x^k} + a^{ih} \Gamma^j_{hk} + a^{jh} \Gamma^i_{hk}$$

$$a_{ij,k} = \frac{\partial a_{ij}}{\partial x^k} - a_{ih} \Gamma^h_{jk} - a_{jh} \Gamma^h_{ik}$$

$$a^i_{j,k} = \frac{\partial a^i_j}{\partial x^k} + a^h_j \Gamma^i_{hk} - a^i_h \Gamma^h_{jk}$$

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\cdot\frac{dx^k}{ds} = 0 \qquad \left[u^i = \frac{dx^i}{ds}\right] \qquad g_{ij}\frac{dx^i}{ds}\cdot\frac{dx^j}{ds} = 1 \qquad u^i u_i = 1$$

unit tangent vector

$$\left(\frac{ds}{dt}\right)^2 = g_{ij}\frac{dx^i}{dt}\cdot\frac{dx^j}{dt}$$

$$ds = \sqrt{}\cdot dt \qquad s = \int_0^t \sqrt{}\cdot dt$$

minimum length geodesic

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - g_{ih}\Gamma^h_{jk} - g_{jh}\Gamma^h_{ik} = 0 \qquad \Gamma^h_{jk} = g^{ih}\Gamma_{i,jk}$$

Further $R^l_{ijk} = - R^l_{ikj}$ &c (F''') and $R^l_{ijk} = g^{lh} R_{hijk}$

& if R^l_{ijk} be contracted for $\underline{l}$ & $\underline{k}$ we have using $\Gamma^i_{il} = \dfrac{\partial \log \sqrt{g}}{\partial x^l}$, & (C')

tensor $R_{ij} = R^k_{ijk} = \dfrac{\partial}{\partial x^j} \Gamma^k_{ik} - \dfrac{\partial}{\partial x^k} \Gamma^k_{ij} + \Gamma^m_{ik} \Gamma^k_{mj} - \Gamma^m_{ij} \Gamma^k_{mk}$

$$= \frac{\partial^2 \log \sqrt{g}}{\partial x^i \partial x^j} - \frac{\partial}{\partial x^k} \Gamma^k_{ij} + \Gamma^m_{ik} \Gamma^k_{mj} - \Gamma^m_{ij} \frac{\partial \log \sqrt{g}}{\partial x^m} \quad \text{—(G)}$$

R_{ij} is the <u>Ricci-tensor</u> which is evidently symmetric

A further contraction leads to the <u>curvature invariant</u>

$$R = g^{ij} R_{ij} \quad \text{———(G')}$$

(e) The tensor $G_{ik} = R_{ik} - \frac{1}{2} g_{ik} R$ is known as the <u>Einstein tensor</u> & is of great importance in general relativity ———(H)

(f) <u>Covariant differentiation</u> w.r.t g_{ij}

from $\lambda^i = \lambda'^\mu \dfrac{\partial x^i}{\partial x'^\mu} = \lambda'^\sigma \dfrac{\partial x^i}{\partial x'^\sigma}$

differentiating w.r.t x_j

$$\frac{\partial \lambda^i}{\partial x^j} = \frac{\partial \lambda'^\mu}{\partial x'^\nu} \cdot \frac{\partial x'^\nu}{\partial x^j} \cdot \frac{\partial x^i}{\partial x'^\mu} + \lambda'^\sigma \frac{\partial^2 x^i}{\partial x'^\sigma \partial x'^\nu} \cdot \frac{\partial x'^\nu}{\partial x^j}$$

$$= \quad '' \quad + \lambda'^\sigma \frac{\partial x'^\nu}{\partial x^j} \left(\Gamma'^\mu_{\sigma\nu} \frac{\partial x^i}{\partial x'^\mu} - \Gamma^i_{hk} \frac{\partial x^h}{\partial x'^\sigma} \cdot \frac{\partial x^k}{\partial x'^\nu} \right)$$

[using (B)]

$$= \left(\frac{\partial \lambda'^\mu}{\partial x'^\nu} + \lambda'^\sigma \Gamma'^\mu_{\sigma\nu} \right) \frac{\partial x'^\nu}{\partial x^j} \cdot \frac{\partial x^i}{\partial x'^\mu} - \lambda^h \Gamma^i_{hj}$$

Putting $\lambda^i_{,j} = \dfrac{\partial \lambda^i}{\partial x^j} + \lambda^h \Gamma^i_{hj}$, the above eqn becomes ———(I)

$$\lambda^i_{,j} = \lambda'^\mu_{,\nu} \cdot \frac{\partial x'^\nu}{\partial x^j} \frac{\partial x^i}{\partial x'^\mu}$$

ie $\lambda^i_{,j}$ are components of a mixed tensor of 2^{nd} order called covariant derivative of λ^i w.r.t to g_{ij}. Similarly covariant derivative $\lambda_{i,j}$ is given by

$$\lambda_{i,j} = \frac{\partial \lambda_i}{\partial x^j} - \lambda_h \Gamma^h_{ij} \quad \text{———(I')}$$

$\lambda_{i,j} - \lambda_{j,i} = \dfrac{\partial \lambda_i}{\partial x^j} - \dfrac{\partial \lambda_j}{\partial x^i} = \text{curl}\, \lambda_i$ & $\lambda_{i,j}$ <u>is symmetric if λ_i be a gradient</u>.

Further differentiations lead to

$$a_{ij,k} = \frac{\partial a_{ij}}{\partial x^k} - a_{ih}\Gamma^h_{jk} - a_{hj}\Gamma^h_{ik}$$

$$a^{ij}_{,k} = \frac{\partial a^{ij}}{\partial x^k} + a^{ih}\Gamma^j_{hk} + a^{hj}\Gamma^i_{hk}$$

$$a^i_{j,k} = \frac{\partial a^i_j}{\partial x^k} + a^h_j\Gamma^i_{hk} - a^i_h\Gamma^h_{jk}$$

indicating general rule.

(i) covariant differentiation obeys same rules as ordinary diff^n

(ii) g_{ij}, g^{ij} & δ^i_j behave like constants in covariant diff^n

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$$\left[\Gamma'^{\lambda}_{\mu\nu}\cdot\frac{\partial x^l}{\partial x'^\lambda} = \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} + \frac{\partial^2 x^l}{\partial x'^\mu\partial x'^\nu}\right. -$$

$$\lambda^i = \lambda'^\mu\frac{\partial x^i}{\partial x'^\mu} = \lambda'^\sigma\frac{\partial x^i}{\partial x'^\sigma}$$

$$\frac{\partial\lambda^i}{\partial x^j} = \frac{\partial\lambda'^\mu}{\partial x'^\nu}\cdot\frac{\partial x'^\nu}{\partial x^j}\cdot\frac{\partial x^i}{\partial x'^\mu} + \lambda'^\sigma\frac{\partial^2 x^i}{\partial x'^\sigma\partial x'^\nu}\cdot\frac{\partial x'^\nu}{\partial x^j}$$

$$= \quad " \quad + \cancel{\lambda'^\mu}$$

$$= \quad " \quad + \lambda'^\sigma\left(\Gamma'^\mu_{\sigma\nu}\frac{\partial x^i}{\partial x'^\mu} - \Gamma^i_{hk}\frac{\partial x^h}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu}\right)\frac{\partial x'^\nu}{\partial x^j}$$

$$= \left(\frac{\partial\lambda'^\mu}{\partial x'^\nu} + \lambda'^\sigma\Gamma'^\mu_{\sigma\nu}\right)\frac{\partial x'^\nu}{\partial x^j}\cdot\frac{\partial x^i}{\partial x'^\mu} - \lambda'^\sigma\Gamma^i_{hk}\frac{\partial x^h}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu}\cdot\frac{\partial x'^\nu}{\partial x^j}$$

$$= \quad " \quad - \lambda'^\sigma\frac{\partial x^h}{\partial x'^\sigma}\cdot\Gamma^i_{hk}\frac{\partial x^k}{\partial x'^\nu}\cdot\frac{\partial x'^\nu}{\partial x^j}$$

$$\frac{\partial\lambda^i}{\partial x^j} = \quad " \quad - \lambda^h\Gamma^i_{hj}$$

Putting $\frac{\partial\lambda^i}{\partial x^j} + \lambda^h\Gamma^i_{hj} = \lambda^i_{;j}$

$\cancel{\lambda^i_{;j}}$ $\lambda^i_{;j} = \lambda'^\mu_{,\nu}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x'^\nu}{\partial x^j}$ ie $\lambda^i_{,j}$ is a mixed tensor

Similarly $\frac{\partial\lambda_i}{\partial x^j} - \lambda_h\Gamma^h_{ij} = \lambda_{i;j} \longrightarrow$ a covariant tensor of order 2

Remainder of last lecture on matrices.

(a) Definition of unitary matrix: $UU^+ = U^+U = 1$ $\{U = (U^+)^{-1}\}$

(b) Theorems without proof:—

(i) If H be hermitian, then for any S, S^+HS is also hermitian

(ii) If H be hermitian, U is unitary, then $U^{-1}HU$ is also hermitian.

(c) Proof of canonical transitions preserving quantum cond^ns & hermiticity are unitary:

$$\text{If } Q = S^{-1}qS,\ P = S^{-1}pS, \text{ then } f(Q,P) = S^{-1}f(q,p)S \quad \dots (\alpha)$$

(α) ensures preservation of quantum cond^ns

Further if q_r, p_r be hermitian so are S^+q_rS & S^+p_rS -- (β)

(α) & (β) together give $S^+ = S^{-1}$ ie S is unitary.

(d) A hermitian matrix can be reduced to diagonal form ~~[illegible]~~ by a unitary matrix & roots of $|a_{ij} - \lambda| = 0$ are the eigen values (real).

(2) Wave mechanics (Schrödinger)

(a) General remarks: $E = h\nu$, $\lambda = \frac{h}{p}$ ie corpuscular aspect of light, there should exist a wave aspect of matter (de Broglie's hypothesis) [photo-electric effect & Compton effect]

Experimental basis of Davisson & Germer's expts on diffraction of electrons by crystals and G. P. Thomson's expt. of diffraction of electrons passing through a celluloid film.

(b) General Principles.

a(1) Uncertainty Principle of Heisenberg: $\Delta x . \Delta p_x \sim h$, $\Delta\phi . \Delta J_\phi \sim h$, $\Delta t . \Delta E \sim h$.

(Verification by expt. of scattering of photons by an electron: $\Delta x \sim \frac{\lambda}{\sin\epsilon}$, $\Delta p_x \sim \frac{h}{\lambda}\sin\epsilon$.

(2) Principle of complementarity (Bohr) - Behaviour of atomic systems cannot be described independently of the means by which they are observed. [(1) & (2) $\to$ abandoning notion of causality]

(3) Correspondence Principle ie as $h \to 0$, Q.m. $\to$ cl. m.

(3) Nature of the wave formalism for a particle.

Eq. of $\psi = Ae^{2\pi i(\nu t - \vec{r}/\lambda)}$* [* or $Ae^{2\pi i(K\vec{r} - \nu t)}$] corresponding to a moving particle of energy E & momentum p in consonance with $E = h\nu$, $p = h/\lambda$ has no meaning since corresponding to infinite ~~[illegible]~~ extension in space, location of particle is completely uncertain (ie p is completely determined $\Delta p = 0$ & $\Delta x = \infty$). Hence the formalism should consist of superposition of waves of different wave lengths (Ex. of 2 waves superposed). Such superposition of waves to give wave packets is in consonance with the uncertainty principles & the equation $p = h/\lambda$. Also (i) ψ can interfere with itself (ii) ψ is large in magnitude where the particle is likely to be, ~~[illegible]~~ and small elsewhere, (iii) ψ describes behaviour of a single particle.

(4) Derivation of the wave equation.

Need for a wave equation to describe the motion of a particle, specially when there are external forces present – two basic properties necessary: (i) Eq^n must be linear so that its solns can be superposed to produce interference effects, (ii) the coeffts of the eq^n must only involve constants like h, e & m (say) and not E or p or ν since superposition of solns corresponds to different values of these & should not therefore appear in the wave eq^n itself.

(1) One dimensional case – use of linear eq^n like that corresp. to motion of transverse vibrations along a string viz $\frac{\partial^2\psi}{\partial t^2} = v^2\frac{\partial^2\psi}{\partial x^2}$ [v = wave velocity] & putting $\psi = Ae^{2\pi i(Kx - \nu t)}$ gives [P.T.O.]

$v^2 = E^2/p^2$ ie involving E & p. – Use of linear eq^n with first derivative in t viz $\frac{\partial\psi}{\partial t} = \gamma\frac{\partial^2\psi}{\partial x^2}$ gives [P.T.O.]

$$\psi = A e^{2\pi i (kx - \nu t)}$$

$$\frac{\partial \psi}{\partial x} = A e^{2\pi i(kx-\nu t)} \cdot 2\pi i k \qquad \frac{\partial^2 \psi}{\partial x^2} = -4\pi^2 k^2 \psi \qquad \lambda = \frac{1}{p}$$

$$\frac{\partial^2 \psi}{\partial t^2} = -4\pi^2 \nu^2 \psi.$$

$$-4\pi^2\nu^2 = v^2 \cdot 4\pi^2 k^2$$

$$v^2 = \frac{\nu^2}{k^2} = \frac{E^2/h^2}{p^2/h^2} = \frac{E^2}{p^2} \qquad \cancel{\frac{h^2\nu^2}{h^2k^2}}$$

$$k = \frac{1}{\lambda} = \frac{p}{h}$$

$$E = \frac{p^2}{2m}$$

$$\cancel{2}\pi i \nu \cdot \cancel{\psi} = \cancel{4}\pi^2 k^2 \cancel{\psi} \cdot \gamma \qquad E = h\nu \qquad p = hk$$

$$\gamma = \frac{i\nu}{2\pi k^2} = \frac{iE/h}{2\pi \cdot p^2/h^2} = \frac{iE\hbar}{p^2} = \frac{i\hbar}{2m} \quad (\text{using } E = p^2/2m)$$

$\int \bar{u}_{E'} u_E \, d\tau = \delta_{EE'}$ orthonormality.

$\psi = \sum A_E u_E$ completeness

$$\int \bar{u}_{E'} \psi \, d\tau = \int \sum A_E \bar{u}_{E'} u_E \, d\tau = \sum A_E \delta_{EE'} = A_{E'}$$

$$\int \bar{u}_E \psi \, d\tau = A_E$$

Closure: $\sum_E \bar{u}_E(r') u_E(r) = 0$ if $r \neq r'$

$$\int \sum \bar{u}_E(r) u_E(r) \cdot d\tau = 1 \text{ if } r = r'.$$

$$\sum \bar{u}_E(r) u_E(r') = \delta(r - r')$$

$$\left\{ \begin{array}{l} \delta(x) = 0 \text{ if } x \neq 0 \\ \int_{-\infty}^{\infty} \delta x = 1 \qquad \delta x = \lim_{g \to \infty} \frac{\sin g x}{\pi x} \end{array} \right.$$

$$\psi(r) = \sum A_E u_E$$

$$= \sum \int \bar{u}_E(r') \psi(r') \, d\tau' \cdot u_E = \int \psi(r') \left[\sum \bar{u}_E(r') u_E(r) \right] d\tau'$$

$$P(E) = |A_E|^2 \qquad \sum_E P(E) = 1 \qquad \sum A_E \bar{A}_E = \sum \iint \bar{u}_E(r) \psi(r) u_E(r') \bar{\psi}(r') \, d\tau \, d\tau'$$

$$= \iint \psi(r) \bar{\psi}(r') \sum \bar{u}_E(r) u_E(r') \, d\tau \, d\tau'$$

$$= \int \psi \bar{\psi} \cdot d\tau = 1$$

that γ can be chosen suitably, using the classical relation $E = p^2/2m$ in a heuristic sense,

ie $\gamma = \frac{i\hbar}{2m}$ ie not involving E or p.

ie Schrödinger eqn in one dimension is $\frac{\partial\psi}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2\psi}{\partial x^2}$ or $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$ [As an operator eqn with $E \to i\hbar\frac{\partial}{\partial t}$, $p = -i\hbar\frac{\partial}{\partial x}$ in $E = \frac{p^2}{2m}$

(2) Extension to three dimensions – $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$ [As an operator eqn with $E \to i\hbar\frac{\partial}{\partial t}$, $p \to -i\hbar\,\overrightarrow{grad}$ in $E = p^2/2m$]

(3) Extension including external forces is $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{r},t)\psi$.

~~Note that all these~~

$$\text{or } i\hbar\frac{\partial\psi}{\partial t} = \left(\frac{p^2}{2m} + V\right)\psi = H\psi$$

The derivation of $i\hbar\frac{\partial\psi}{\partial t} = H\psi$ is based on several heuristic considerations, like analogy with classical mechanics, wave optics and so on & might well have been taken ~~automatically~~ axiomatically as the wave eqn [heuristic = serving to find out], but physical considerations would not have served any purpose.

(5) ~~Interpretation of the wave eq~~

Separation of the wave equation – If V is independent of t, putting $\psi(\vec{r},t) = u(\vec{r})f(t)$ & substituting in the wave eqn and taking separation constant $= E$, a particular soln is given by $\psi = ue^{-iEt/\hbar}$ [P.T.O]

and $\frac{\partial\psi}{\partial t} = -\frac{iE}{\hbar}\psi$ or $i\hbar\frac{\partial\psi}{\partial t} = E\psi$ ie since $i\hbar\frac{\partial}{\partial t}$ is the energy operator, this ~~gives~~ shows that E is the energy. & $i\hbar\frac{\partial\psi}{\partial t} = E\psi$ is called an eigen-value eqn, ψ an eigenfn of the operator $i\hbar\frac{\partial}{\partial t}$ & E the corresponding eigenvalue or energy eigen-values. Also $Hu = Eu$ is also an eigen-value equation.

(6) Interpretation of the wave equation

(1) Constn of ψ suggests a probability interpretation due to Max Born – $P(\vec{r},t) = |\psi(\vec{r},t)|^2 = \psi\bar{\psi}$ is the probability of finding a particle in dV about $\vec{r}$ at time t. Since this probability $= 1$, $\int|\psi(\vec{r},t)|^2 d\tau = 1$ ie wave fn is normalised.

(2) Interpretative postulates – (α) each dynamical variable relating to motion of a particle can be represented by a linear operator, and with each operator Ω can be associated a linear eigen-value eqn $\Omega u_\omega = \omega u_\omega$ where u_ω is the eigenfn corresponding to eigen-value ω. (β) one or other of the eigenvalues ω is the only possible result of a precise measurement of the dynamical variable represented by Ω, (γ) Eigen-fns constitute a complete set, and if any ψ be expanded in terms of u_ω, the number of measurements that result in the eigen-value ω is $\propto$ to the square of the magnitude of the coeff of u_ω in the expn of ψ & (γ) is equivalent to Born's probability postulate.

(3) Energy eigen-fns – $Hu_E = Eu_E$ – orthogonal eigenfunctions $\int \bar{u}_{E'} u_E\, d\tau = 0$; ~~eigenvalues are real~~ – orthonormal set of eigenfunctions (each normalised & mutually orthogonal) – eigenvalues of energy real – for any $\psi(r) = \sum_E A_E u_E(r)$, (i) $\int \bar{u}_E\psi(r)\,d\tau = A_E$, (ii) $\sum_E \bar{u}_E(r')u_E(r) = 0\ \forall\ r \neq r'$ and

(ii) → $\sum_E = \delta(\vec{r}-\vec{r}')$ [P.T.O]

$\int \sum_E \bar{u}_E(r)u_E(r)\,d\tau = 1$. (Closure property), (iii) $P(E) = |A_E|^2$ since using (ii) we can show that $\sum_E P(E) = 1$. (iv) Problem of solving the Schrödinger eqn is completely equivalent to the problem of diagonalising the Hamiltonian matrix. ~~also~~ The ~~energy~~ eigenvalues of the Hamiltonian matrix are the energy eigen-values of the Schrödinger eqn and the unitary matrix that serves to diagonalise H serves to give the energy eigen-fns $u_E(r)$. This proves equality of the matrix & wave formulations

(7) Appln to H-atom: In the eqn $H\psi = E\psi$, we have in H, $V = -e^2/r$ and using spherical coords and separation of variables we get results in consonance with the Bohr's old quantum theory. For energy eigen-values if we take E, $-$ve we get the discrete spectra & E, $+$ve the continuous spectra, a ~~specially~~

$i\hbar \frac{\partial \psi}{\partial t} = H\psi$

$i\hbar \frac{df}{dt} u = Hu \cdot f$

$\frac{i\hbar}{f} \cdot \frac{df}{dt} = H = E$ (separation const)

$\frac{df}{f} = -\frac{iE}{\hbar} dt$, $\log f = -\frac{iE}{\hbar} t + \log c$

$\frac{f}{c} = e^{-iEt/\hbar}$

$f = ce^{-iEt/\hbar}$

$f(t) = ce^{-iEt/\hbar}$

$Hu = Eu$ $\qquad \psi(\vec{r}, t) = ce^{-iEt/\hbar} \cdot u$

$\bar{\psi}(\vec{r}, t) = ce^{iEt/\hbar} \cdot u$ $\qquad \cancel{\psi\bar{\psi} = 1 = c^2 u \bar{u}}$

$\psi = ue^{-iEt/\hbar}$

$\frac{\partial \psi}{\partial t} = -\frac{iE}{\hbar} ue^{-iEt/\hbar} = -\frac{i}{\hbar} E\psi$ $\qquad \Sigma a_r^{(1)} H'_{rs} = \Sigma a_r^{(1)} H'_{rs} - a_s^{(1)} H'_{nn}$

$i\hbar \frac{\partial \psi}{\partial t} = E\psi.$

with $s = n$, $E_2 = \Sigma a_r^{(1)} H'_{rn} - H'_{nn} a_n^{(1)}$

$= \Sigma'' a_r^{(1)} H'_{rn}$

Diracs δ-fn $\delta(x) = 0$ if $x \neq 0$ & $\int \delta(x) \cdot dx = 1$ $\qquad a_s^{(2)} (E_n - E_s) = \Sigma a_r^{(1)} H'_{rs} - E_1 a_s^{(1)} - E_2 \delta_{ns}.$

$\delta(x) \to \lim_{g \to \infty} \frac{\sin gx}{\pi x}$

$= \Sigma' \frac{H'_{nr} H'_{rn}}{E_r - E_n}$

$= \Sigma' \frac{|H'_{rn}|^2}{E_r - E_n}$

$H = H_0 + \lambda H'$ $\qquad E = E_0$ $\quad H_0 u_n = E_n u_n$

$E_0 = E_n.$

$E = E_0 + E_1 + E_2$

$H\psi = E\psi$

$(H_0 + \lambda H')(\psi_0 + \lambda \psi_1 + \lambda^2 \psi_2 + \cdots) = (E_0 + \lambda E_1 + \lambda^2 E_2 + \cdots)(\psi_0 + \lambda \psi_1 + \lambda^2 \psi_2 + \cdots)$

$H_0 \psi_0 = E_0 \psi_0 \longrightarrow H_0 u_n = E_n u_n$ — (1)

$\psi_0 = u_n]$

$H_0 \psi_1 + H' \psi_0 = E_0 \psi_1 + E_1 \psi_0$ $\qquad \psi_1 = \Sigma' a_r^{(1)} u_r$, $\psi_2 = \Sigma a_r^{(2)} u_r$

$\Sigma H_0 a_r^{(1)} u_r + H' \psi_0 = E_0 \Sigma a_r^{(1)} u_r + E_1 \psi_0.$

$\int \Sigma H_0 a_r^{(1)} u_r \bar{u}_s d\tau + \int H' \psi_0 \bar{u}_s d\tau = E_0 \int \Sigma a_r^{(1)} u_r \bar{u}_s d\tau + \int E_1 \psi_0 \bar{u}_s \cdot d\tau.$

$\int \Sigma a_r^{(1)} E_r u_r \bar{u}_s d\tau + \int H' u_n \bar{u}_s d\tau =$ $\qquad$ " $\qquad + \int E_1 u_n \bar{u}_s d\tau.$

$a_s^{(1)} E_s + H'_{ns} = E_0 a_s^{(1)} + E_1 \delta_{ns}$

$a_s^{(1)} (E_n - E_s) + E_1 \delta_{ns} = H'_{ns}$ $\qquad n = s \to E_1 = H'_{nn}$ — (2) (A)

$n \neq s \to a_s^{(1)} = \frac{H'_{ns}}{E_n - E_s}$ — (B)

$H_0 \psi_2 + H' \psi_1 = E_0 \psi_2 + E_1 \psi_1 + E_2 \psi_0$

$\Sigma H_0 a_r^{(2)} u_r + \Sigma H' a_r^{(1)} u_r = E_0 \Sigma a_r^{(2)} u_r + E_1 \Sigma a_r^{(1)} u_r + E_2 u_n$

$\int \Sigma a_r^{(2)} E_r u_r \bar{u}_s d\tau + \int \Sigma H' a_r^{(1)} u_r \bar{u}_s \cdot d\tau = \int E_0 \Sigma a_r^{(2)} u_r \bar{u}_s d\tau + \int E_2 u_n \bar{u}_s \cdot d\tau$

$+ E_1 a_s^{(1)}$

$a_s^{(2)} E_s + \Sigma' H'_{rs} a_r^{(1)} = a_s^{(2)} E_n + E_2 \delta_{ns}$

a theory which was not possible in the Bohr theory – ~~Bohr~~ Sommerfeld's remark (by uniform analytic process d connected to C spectra) (3)

(7) For the L.H.O. Schrodinger's theory gives same result as the Heisenberg theory.

(8) Perturbation theory

For many electron systems $H\psi = E\psi$, Be $H = -\underbrace{\frac{\hbar^2}{2m}\sum_{j=1}^{N}\nabla_j^2}_{K.E} - \underbrace{Ze^2\sum_j \frac{1}{r_j}}_{C.I. with n} + \underbrace{\sum_{i\neq j}\frac{e^2}{r_{ij}}}_{C.I. between e's}$ $(r_{ij} = |\vec{r}_i - \vec{r}_j|)$

Solved only $N = 1$ i.e. hydrogen atom.

For case $N = 2$, very accurate approx. methods are known

$N > 2$, less exact methods

(9) (i) Bound state perturbation theory – $H = H_0 + H'$ (H' small) & $H_0 u_n = E_n u_n$

Effect of pert^n H' is such that E is much closer to E_n than to $E_{n\pm1}$.

usual method is to write $E = E_0 + E_1 + E_2$

where $E_0 = E_n$

$E_1 = H'_{nn}$

$E_2 = \sum_m' \frac{|H'_{nm}|^2}{E_n - E_m}$ $(m \neq n)$.

where $H'_{nm} = \int u_n^* H' u_m \cdot d\tau$.

$H u_n = E u_n$

$(H_0 + H')u_n = (E_0 + E_1 + E_2)u_n$

$E_n u_n + H' u_n = (E_0 + E_1 + E_2) u_n$

$H' u_n = (E_1 + E_2) u_n$

$H'(u_n + \lambda u_n' + \lambda^2 u_n'' + \dots)$

$= \lambda E$

(ii) ~~Time-dependent perturbation theory:~~

~~$i\hbar \frac{\partial\Psi}{\partial t} = H\Psi$~~

$H\psi = E\psi$, $H = H_0 + H'$, $H_0 u_n = E_n u_n$.

$= H_0 + \lambda H''$

~~$(H_0 + H')$~~

$(H_0 + \lambda H'')(\psi_0 + \lambda\psi_1 + \lambda^2\psi_2 + \dots) = (E_0 + \lambda E_1 + \lambda^2 E_2 + \dots)(\psi_0 + \lambda\psi_1 + \lambda^2\psi_2 + \dots)$

$\left.\begin{array}{l}\psi_0 = u_n \\ E_0 = E_n\end{array}\right\}$

$H_0\psi_0 = E_0\psi_0$

$H_0\psi_1 + H''\psi_0 = E_0\psi_1 + E_1\psi_0$

$H_0\psi_2 + H''\psi_1 = E_0\psi_2 + E_1\psi_1 + E_2\psi_0$

$\psi_2 = S a_r^{(2)} u_r$

~~H_0~~ $S a_r^{(2)} H_0 u_r + H'' S a_r^{(1)} u_r$

$= E_n S a_r^{(2)} u_r + E_1 S a_r^{(1)} u_r + E_2 u_n$

$a_s^{(2)} E_s + S a_r^{(1)} H''_{rs}$

$= E_n a_s^{(2)} + E_1 a_s^{(1)} + E_2 \delta_{ns}$

$a_s^{(2)}(E_n - E_s) = S a_r^{(1)} H''_{rs} - E_1 a_s^{(1)} - E_2\delta_{ns}$

with $n = s \rightarrow$ $S' a_r^{(1)} H''_{rn} = E_2$ using (A) for

$E_2 = S' \frac{H''_{rn} H''_{nr}}{E_n - E_r} = \sum'' \frac{|H''_{mn}|^2}{E_n - E_m}$

using (B)

Let $\psi_1 = S a_r^{(1)} u_r$

$S a_r^{(1)} H_0 u_r + H'' u_n = E_n S a_r^{(1)} u_r + E_1 u_n$

$\int S a_r^{(1)} E_r u_r \bar{u}_s \, d\tau + H''_{ns} = E_n \int S a_r^{(1)} u_r \bar{u}_s \cdot d\tau \, E_1 \delta_{ns}$

$a_s^{(1)} E_s + H''_{ns} = E_n a_s^{(1)} + E_1 \delta_{ns}$

$a_s^{(1)}(E_n - E_s) + E_1\delta_{ns} = H''_{ns}$

with $n = s \rightarrow$ $H''_{nn} = E_1$. —— (A)

for $n \neq s$, $a_s^{(1)} = \frac{H''_{ns}}{E_n - E_s}$ —— (B)

$S a_r^{(1)} H'_{rn} - a_n^{(1)} H_{nn} = \Sigma' a_r^{(1)} H'_{rn}$

$r \neq n$ $E_2 =$

Lecture on 3/1/68.

(1) Remaining topics (6), (7), & (8) of the previous lecture

(2) Angular momentum & spin.

(3) Identical particles & spin

(γ) $\psi = \Sigma A_E u_E$

$$\int \psi \bar{u}_{E'}\, d\tau = \int \Sigma A_E u_E \bar{u}_{E'} d\tau = \Sigma \int A_E \bar{u}_{E'} u_E\, d\tau = \Sigma A_E \delta_{EE'} = A_{E'}.$$

$$\text{or } \int \psi \bar{u}_E\, d\tau = A_E.$$

(δ) $\psi(\psi) = \sum_E \left[\int \bar{u}_E(r')\, \psi(r')\, d\tau' \right] u_E(r)$

$$= \int \psi(r') \left[\Sigma \bar{u}_E(r')\, u_E(r) \right] d\tau'$$

If $r \neq r'$, $\Sigma \bar{u}_E(r')\, u_E(r) = 0$ since ψ is arbitrary

If $r = r'$, $\int \Sigma \bar{u}_E(r')\, u_E(r) \cdot d\tau' = 1$

(ε) $\Sigma P(E) = \Sigma |A_E|^2 = \Sigma \int \bar{u}_E(r)\, \psi(r)\, d\tau \int \bar{u}_E(r')\, \bar{\psi}(r')\, d\tau'$

$$= \iint \bar{\psi}(r')\, \psi(r) \left[\Sigma \bar{u}_E(r)\, \bar{u}_E(r') \right] d\tau\, d\tau' =$$

$$= \int |\psi|^2 d\tau = 1.$$

$(yp_z - zp_y)(zp_x - xp_z) - (zp_x - xp_z)(yp_z - zp_y)$

$(M_x, M_y) = i\hbar M_z$

$= yp_z z p_x - yp_z xp_z - zp_y zp_x + zp_y xp_z$

$- zp_x yp_z + zp_x zp_y + xp_z yp_z - xp_z zp_y$

$= yp_x(p_z z) - xp_y$

only $[x, p_x] = \dots = i\hbar$

$\cancel{(3) = z^2(p_x y + p_y x)p_z}$

$= 0$

$= yp_z zp_x - \cancel{yp_z xp_z} - z\cancel{p_y zp_x} + zp_y xp_z$

$\cancel{[x, p_y] = [x, p_z] \neq 0 \text{ etc}}$

$[x, p_y] = [x, p_z] = 0$ etc

$- zp_x yp_z + zp_x \cancel{zp_y} + \cancel{xp_z yp_z} - xp_z zp_y$. but $xp_y - yp_x = 0$. (p_x not $\neq i\hbar\frac{\partial}{\partial x}$)

$xp_y - p_y x$

$= yp_x p_z z + zp_z p_y x$

$- yp_x zp_z - xp_y p_z z$

$= zp_z(xp_y - yp_x) - p_z z(xp_y - yp_x)$

$= (zp_z - p_z z)(xp_y - yp_x) = i\hbar(xp_y - yp_x) = i\hbar M_z$

$\underline{r}$ & $\underline{p}$ Hermitian and $r \times p = M$ is hermitean

$R_z\phi = 1 + \frac{\phi}{i\hbar}M_z$ · more formal defn

$[M_z, M^2] = M_z(M_x^2 + M_y^2 + M_z^2) - (M_x^2 + M_y^2 + M_z^2)M_z$

$= M_z M_x^2 + M_z M_y^2 - M_x^2 M_z - M_y^2 M_z$

$M_z M_x^2 = M_z M_x \cdot M_x$

$M_x^2 M_z = M_x \cdot M_x M_z$

$= \underline{M_z M_x^2 - M_x^2 M_z} + \underline{M_z M_y^2 - M_y^2 M_z}$

$i\hbar(M_x M_y + M_y M_x) = \cancel{(M_z M_x - M_x M_z)} M_x$

$= [M_y, M_z]M_y + M_y[M_y, M_z]$

$\cancel{(M_y M_z - M_z M_y)M_y}$

$= (M_y M_z - M_z M_y)M_y + M_y(M_y M_z - M_z M_y)$

$= M_y^2 M_z - M_z M_y^2$.

$= M_x[M_z, M_x] + [M_z, M_x]M_x = M_x(M_z M_x - M_x M_z) + (M_z M_x - M_x M_z)M_x$

$= M_z M_x^2 - M_x^2 M_z$

Similarly $i\hbar(M_y M_x + M_x M_y) = M_y(M_y M_z - M_z M_y) + (M_y M_z - M_z M_y)M_y$

$= M_y^2 M_z - M_z M_y^2$.

Hence $\underline{[M_z, M^2] = i\hbar(M_x M_y + M_y M_x) - i\hbar(M_y M_x + M_x M_y) = 0}$.

$\cancel{A = M_x + iM_y = \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\hbar i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$

a_{21}

$a_{11}\ a_{22}$

$\cancel{= \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$

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$$\frac{h}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} + \frac{hi}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}$$

$$\text{"} \qquad + \frac{h}{\sqrt{2}}\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix}$$

$$= \frac{h}{\sqrt{2}}\begin{pmatrix}0&2&0\\0&0&2\\0&0&0\end{pmatrix} \qquad a_{12} \quad a_{23}.$$

$$\vec{M} = \vec{r} \times \vec{p} \qquad M_x = y p_z - z p_y \,; \; M_y = z p_x - x p_z \,, \; M_z = x p_y - y p_x$$

$$\boxed{[x, p_x] = [y, p_y] = [z, p_z] = i\hbar}$$

with others zero

$$[M_x, M_y] = M_x M_y - M_y M_x$$

$$= (y p_z - z p_y)(z p_x - x p_z) - (z p_x - x p_z)(y p_z - z p_y)$$

$$= y p_z z p_x - \cancel{y p_z x p_z} - \cancel{z p_y z p_x} + z p_y x p_z$$

$$- z p_x y p_z + \cancel{z p_x z p_y} + \cancel{x p_z y p_z} - x p_z z p_y$$

$$= - p_z z (x p_y - y p_x) + z p_z (x p_y - y p_x)$$

$$= (z p_z - p_z z)(x p_y - y p_x) = i\hbar M_z .$$

$$[M_x, M_z^2 + M_y^2 + M_z^2] \qquad [M_x, M_y^2] + [M_x, M_z^2)$$

$$[M_x, M^2] = [M_x^2 + M_y^2 + M_z^2 , M_x] = \cancel{[M_y^2, M_x] + [M_z^2, M_x]}$$

$$= M_y^2 M_z$$

$$\cancel{M_y M_z \quad M_y^2 M_x - M_x M_y^2 = M_y (M_y M_x) - (M_x M_y) M_y}$$

$$\cancel{i\hbar (M_x M_y + M_y M_x) = M_x [M_z M_x] + [M_z M_x] M_x}$$

$$\cancel{= M_x (M_z M_x - M_x M_z) + (M_z M_x - M_x M_z) M_x}$$

$$\cancel{= M_x M_z M_x - M_x^2 M_z + M_z M_x^2 - M_x M_z M_x}$$

$$i\hbar (M_y M_z + M_z M_y) = M_y [M_x M_y - M_y M_x) + (M_x M_y - M_y M_x) M_y$$

$$= M_y \cancel{M_x} M_y - M_y^2 M_x + M_x M_y^2 - M_y \cancel{M_x} M_y = [M_x, M_y^2]$$

$$i\hbar (M_z M_y + M_y M_z) = M_z [M_z, M_x] + [M_z, M_x] M_z$$

$$= M_z (M_z M_x - M_x M_z) + [M_z M_x - M_x M_z) M_z$$

$$= M_z^2 M_x - M_z \cancel{M_x} M_z + M_z \cancel{M_x} M_z - M_x M_z^2 = - [M_x, M_z^2]$$

$$\therefore \; [M_x, M^2] = i\hbar (M_y M_z + M_z M_y) - i\hbar (M_z M_y + M_y M_z) = 0 .$$

Lecture for 19/1/68.

(1) Born's probability postulate $- P(\vec{r}, t) = \psi \bar{\psi} = |\psi(\vec{r}, t)|^2 =$ probability of finding a particle in dV about $\vec{r}$ at time t – Hence $\int |\psi|^2 \, d\tau = 1$ ie wave fn is normalised.

(2) Interpretative postulates – (α) Dynamical variable $\rightarrow$ linear operator $\Omega \rightarrow \Omega u_\omega = \omega u_\omega$

(β) one or other of eigen ω is the only possible result of exptl. detn of Ω, (γ) u_ω's form a complete set ie any ψ can be expanded in terms of u_ω's, (δ) no. of measurements that result in ω is $\propto$ to square of the coefft in the expansion of ψ in terms of u_ω's – (δ) equivalent to Born's postulate.

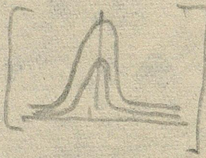
(3) Energy eigen-functions – $H u_E = E u_E$ – (α) orthonormality of the u_E's ie $\int \bar{u}_{E'} u_E \, d\tau = \delta_{EE'}$

(β) Energy eigenvalues real, (γ) Completeness ie $\psi = \Sigma A_E u_E$ ~~(δ) Closure property ie~~

and $\int \bar{u}_E \psi \, d\tau = A_E$ following from orthonormality (δ) Closure property – (i) $\Sigma \bar{u}_E(r') u_E(r) = 0 \;\forall\; r \neq r'$

(ii) $\int \Sigma \bar{u}_E(r') u_E(r) \, d\tau = 1 \;\forall\; r = r'$ & (i) & (ii) can be put in the form

$$\Sigma \bar{u}_E(r') u_E(r) = \delta(\vec{r} - \vec{r}\,')$$ where $\delta(x)$ is Dirac's δ-fn ie $\delta(x) = 0 \;\forall\; x \neq 0$

and $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$ & $\delta(x) = \lim_{g \to \infty} \dfrac{\sin gx}{\pi x}$ [] (ε) $P(E) = |A_E|^2$

and $\Sigma P(E) = 1$.

(4) Equality of wave & matrix formulations – Solving Schroedinger eqn is equivalent to problem of diagonalising the H-matrix. The energy eigenvalues of the H-matrix are the energy eigenvalues of the Schroedinger eqn. The unitary matrix diagonalising H gives $u_E(r)$.

(5) Linear harmonic oscillator – one dimn with $V = \frac{1}{2} kx^2$. with $\omega_e = (k/m)^{1/2}$, energy eigenvalues given by $E_n = (n + 1/2)\hbar\omega_e$ $(n = 0, 1, 2, \ldots)$ and eigen-functions are given by

$$u_n(x) = N_n H_n(\alpha x) e^{-\frac{1}{2}\alpha^2 x^2} \left[N_n = \left(\frac{\alpha}{\sqrt{\pi}\, 2^n \cdot n!} \right)^{1/2}, \right.$$ H_n is Hermitian polynomial

and $\left. \alpha^4 = \dfrac{mk}{\hbar^2} \right]$.

(6) Hydrogen atom $\overset{H\psi = E\psi}{V = -e^2/r}$ – using sph. polar coords & separation of variables*, we get results in accordance with Bohr's theory of discrete spectra ie E –ve in $H\psi = E\psi$. For E +ve we get continuous spectra (not possible in old Bohr's theory) – This is a special feature of the Schroedinger technique – Hadamard's remark about same analytical technique giving both cases.

(7) Perturbation theory – Bound state (or Stationary) – $H = H_0 + H'$ (H' small), $H_0 u_n = E_n u_n$ & effect of H' is such that E is closer to E_n than to $E_{n \pm 1}$. If $E = E_0 + E_1 + E_2$, where $E_0 = E_n$, $E_1 = H'_{nn}$

$$E_2 = \sum_m{}' \frac{|H'_{nm}|^2}{E_n - E_m} \quad (m \neq n)$$ [Fermi's Golden Rule]

[* Principal quantum number n integer
orbital A.M " l "
magnetic " m "]

$$\cancel{\frac{\partial^2 x^l}{\partial x'^\mu \partial x'^\sigma} = \Gamma'^\lambda_{\mu\sigma} - \Gamma^l_{ij} \; \Gamma'^\lambda_{\mu\sigma} \frac{\partial x^l}{\partial x'^\lambda} - \Gamma^l_{ij} \frac{\partial x^i}{\partial x'^\mu} \frac{\partial x^j}{\partial x'^\nu}}$$

$$\cancel{\frac{\partial^2 x'_\sigma}{\partial x_\mu \partial x_\nu} - \Gamma^\rho_{\mu\nu} \frac{\partial x'_\sigma}{\partial x_\rho} = -\Gamma'^\sigma_{\alpha\beta} \frac{\partial x_\alpha}{\partial x_\mu} \frac{\partial x_\beta}{\partial x_\nu}}$$

$$g'_{\lambda\mu} = g_{ij} \frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial x^j}{\partial x'^\mu}$$

$$\Gamma'_{\lambda,\mu\nu} = \frac{1}{2}\left(\frac{\partial g'_{\lambda\mu}}{\partial x'^\nu} + \frac{\partial g'_{\lambda\nu}}{\partial x'^\mu} - \frac{\partial g'_{\mu\nu}}{\partial x'^\lambda} \right)$$

$$= \frac{1}{2}\left\{ \frac{\partial}{\partial x'^\nu}\left(g_{ij} \frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial x^j}{\partial x'^\mu} \right) + \frac{\partial}{\partial x'^\mu}\left(g_{ik} \frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial x^k}{\partial x'^\nu} \right) - \frac{\partial}{\partial x'^\lambda}\left(g_{jk} \frac{\partial x^j}{\partial x'^\mu} \cdot \frac{\partial x^k}{\partial x'^\nu} \right) \right\}$$

$$= \frac{1}{2}\Bigg[\frac{\partial g_{ij}}{\partial x'^\nu} \cdot \frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial x^j}{\partial x'^\mu} + g_{ij} \frac{\partial}{\partial x'^\nu}\left(\frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial x^j}{\partial x'^\mu} \right)$$
$$+ \frac{\partial g_{ik}}{\partial x'^\mu} \cdot \frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial x^k}{\partial x'^\nu} + g_{ik} \frac{\partial}{\partial x'^\mu}\left(\frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial x^k}{\partial x'^\nu} \right)$$
$$- \frac{\partial g_{jk}}{\partial x'^\lambda} \frac{\partial x^j}{\partial x'^\mu} \cdot \frac{\partial x^k}{\partial x'^\nu} - g_{jk} \frac{\partial}{\partial x'^\lambda}\left(\frac{\partial x^j}{\partial x'^\mu} \cdot \frac{\partial x^k}{\partial x'^\nu} \right) \Bigg]$$

$$= \Gamma_{i,jk} \cdot \frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial x^j}{\partial x'^\mu} \cdot \frac{\partial x^k}{\partial x'^\nu} + \frac{1}{2}\Bigg[g_{ij}\left(\cancel{\frac{\partial^2 x^i}{\partial x'^\lambda \partial x'^\nu} \cdot \frac{\partial x^j}{\partial x'^\mu}} + \frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial^2 x^j}{\partial x'^\mu \partial x'^\nu} \right)$$
$$+ g_{ik}\left(\cancel{\frac{\partial^2 x^i}{\partial x'^\lambda \partial x'^\mu} \cdot \frac{\partial x^k}{\partial x'^\nu}} + \frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial^2 x^k}{\partial x'^\mu \partial x'^\nu} \right)$$
$$- g_{jk}\left(\cancel{\frac{\partial^2 x^j}{\partial x'^\lambda \partial x'^\mu} \cdot \frac{\partial x^k}{\partial x'^\nu}} + \cancel{\frac{\partial x^j}{\partial x'^\mu} \cdot \frac{\partial^2 x_k}{\partial x'^\lambda \partial x'^\nu}} \right)$$

$$\Gamma'_{\lambda,\mu\nu} = \Gamma_{i,jk} \frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial x^j}{\partial x'^\mu} \cdot \frac{\partial x^k}{\partial x'^\nu} + g_{ij} \frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial^2 x^j}{\partial x'^\mu \partial x'^\nu} \qquad \text{—— (1)}$$

$$g'^{\lambda\sigma} \Gamma'_{\lambda,\mu\nu} \frac{\partial x^l}{\partial x'^\sigma} = \Gamma_{i,jk} \cdot \frac{\partial x^l}{\partial x'^\sigma} \cdot \frac{\partial x^i}{\partial x'^\lambda} \frac{\partial x^j}{\partial x'^\mu} \frac{\partial x^k}{\partial x'^\nu} \cdot g'^{\sigma\lambda} + g_{ij}\, g'^{\lambda\sigma} \cdot \frac{\partial x^i}{\partial x'^\lambda} \cdot \frac{\partial x^l}{\partial x'^\sigma} \frac{\partial^2 x^j}{\partial x'^\mu \partial x'^\nu}$$

$$\Gamma'^\sigma_{\mu\nu} \frac{\partial x^l}{\partial x'^\sigma} = g^{il} \Gamma_{i,jk} \cdot \frac{\partial x^j}{\partial x'^\mu} \cdot \frac{\partial x^k}{\partial x'^\nu} + g_{ij}\, g^{il} \frac{\partial^2 x^j}{\partial x'^\mu \partial x'^\nu}$$

$$= \Gamma^l_{jk} \cdot \frac{\partial x^j}{\partial x'^\mu} \frac{\partial x^k}{\partial x'^\nu} + \frac{\partial^2 x^l}{\partial x'^\mu \partial x'^\nu} \qquad \text{—— (2)}$$

$$\Gamma'^\sigma_{\mu\nu} = \Gamma^l_{jk} \frac{\partial x'^\sigma}{\partial x^l} \cdot \frac{\partial x^j}{\partial x'^\mu} \cdot \frac{\partial x^k}{\partial x'^\mu} + \frac{\partial^2 x_l}{\partial x'^\mu \partial x'^\nu} \cdot \frac{\partial x'^\sigma}{\partial x^l} \qquad \text{—— (1')}$$

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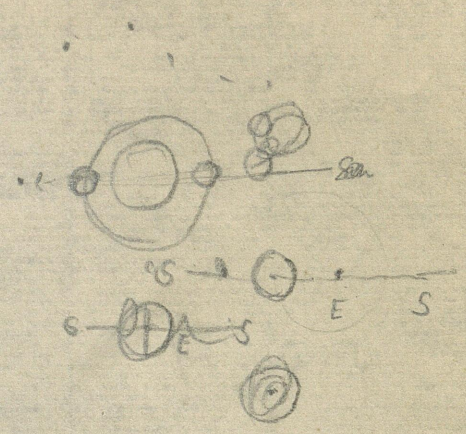
$$\Gamma'^{\sigma}_{\mu\nu}\frac{\partial x^l}{\partial x'^\sigma} = \Gamma^l_{jk}\frac{\partial x^j}{\partial x'^\mu}\cdot\frac{\partial x^k}{\partial x'^\nu} + \frac{\partial^2 x^l}{\partial x'^\mu \partial x'^\nu}$$

$$\frac{\partial^2 x^l}{\partial x'^\mu\partial x'^\nu} = \Gamma'^\lambda_{\mu\nu}\frac{\partial x^l}{\partial x'^\lambda} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}$$

$$\frac{\partial^2 x^l}{\partial x'^\mu\partial x'^\sigma} = \Gamma'^\lambda_{\mu\sigma}\frac{\partial x^l}{\partial x'^\lambda} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}$$



$$\frac{\partial}{\partial x'^\sigma}\left(\Gamma'^\lambda_{\mu\nu}\frac{\partial x^l}{\partial x'^\lambda}\right) - \frac{\partial}{\partial x'^\sigma}\left(\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\right)$$

$$= \frac{\partial}{\partial x'^\nu}\left(\Gamma'^\lambda_{\mu\sigma}\frac{\partial x^l}{\partial x'^\lambda}\right) - \frac{\partial}{\partial x'^\nu}\left(\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\right)$$

$$-\Gamma^l_{ij}\frac{\partial^2 x^i}{\partial x'^\mu \partial x'^\sigma}\cdot\frac{\partial x^j}{\partial x'^\nu}$$

$$\frac{\partial}{\partial x'^\sigma}\left(\Gamma'^\lambda_{\mu\nu}\right)\frac{\partial x^l}{\partial x'^\lambda} + \Gamma'^\lambda_{\mu\nu}\frac{\partial^2 x^l}{\partial x'^\lambda\partial x'^\sigma} - \frac{\partial}{\partial x'^\sigma}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} \cancel{-\Gamma^l_{ij}\frac{\partial}{\partial x'^\sigma}\left(\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\right)}$$

$$= \frac{\partial}{\partial x'^\nu}\left(\Gamma'^\lambda_{\mu\sigma}\right)\frac{\partial x^l}{\partial x'^\lambda} + \Gamma'^\lambda_{\mu\sigma}\frac{\partial^2 x^l}{\partial x'^\lambda\partial x'^\nu} - \frac{\partial}{\partial x'^\nu}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma} \cancel{-\Gamma^l_{ij}\frac{\partial}{\partial x'^\nu}\left(\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\right)}$$

$$-\Gamma^l_{ij}\frac{\partial^2 x^i}{\partial x'^\mu\partial x'^\nu}\cdot\frac{\partial x^j}{\partial x'^\sigma}$$

$$\frac{\partial}{\partial x'^\sigma}\left(\Gamma'^\lambda_{\mu\nu}\right)\frac{\partial x^l}{\partial x'^\lambda} + \Gamma'^\lambda_{\mu\nu}\left\{\Gamma'^\rho_{\lambda\sigma}\frac{\partial x^l}{\partial x'^\rho} - \Gamma^l_{ij}\cancel{\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\sigma}}\right\}$$

$$-\frac{\partial}{\partial x'^\nu}\left(\Gamma'^\lambda_{\mu\sigma}\right)\frac{\partial x^l}{\partial x'^\lambda} - \Gamma'^\lambda_{\mu\sigma}\left\{\Gamma'^\rho_{\lambda\nu}\frac{\partial x^l}{\partial x'^\rho} - \Gamma^l_{ij}\cancel{\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\nu}}\right\}$$

$$= \frac{\partial}{\partial x'^\sigma}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu} + \Gamma^l_{ij}\frac{\partial x^j}{\partial x'^\nu}\left\{\cancel{\Gamma'^\rho_{\mu\sigma}\frac{\partial x^i}{\partial x'^\rho}} - \Gamma^i_{mn}\frac{\partial x^m}{\partial x'^\mu}\frac{\partial x^n}{\partial x'^\sigma}\right\}$$

$$-\frac{\partial}{\partial x'^\nu}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma} - \Gamma^l_{ij}\frac{\partial x^j}{\partial x'^\sigma}\left\{\cancel{\Gamma'^\rho_{\mu\nu}\frac{\partial x^i}{\partial x'^\rho}} - \Gamma^i_{mn}\frac{\partial x^m}{\partial x'^\mu}\frac{\partial x^n}{\partial x'^\nu}\right\}$$

$$\left\{\frac{\partial}{\partial x'^\sigma}\left(\Gamma'^\lambda_{\mu\nu}\right) - \frac{\partial}{\partial x'^\nu}\left(\Gamma'^\lambda_{\mu\sigma}\right) + \Gamma'^\rho_{\mu\nu}\Gamma'^\lambda_{\rho\sigma} - \Gamma'^\rho_{\mu\sigma}\Gamma'^\lambda_{\rho\nu}\right\}\frac{\partial x^l}{\partial x'^\lambda}$$ interchanging ρ & λ in the last two terms

$$= \frac{\partial}{\partial x^k}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\cdot\frac{\partial x^k}{\partial x'^\sigma} - \frac{\partial}{\partial x^k}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu}$$

$$-\Gamma^l_{ij}\Gamma^i_{mn}\frac{\partial x^j}{\partial x'^\nu}\frac{\partial x^m}{\partial x'^\mu}\cdot\frac{\partial x^n}{\partial x'^\sigma} + \Gamma^l_{ij}\Gamma^i_{mn}\frac{\partial x^j}{\partial x'^\sigma}\frac{\partial x^m}{\partial x'^\mu}\cdot\frac{\partial x^n}{\partial x'^\nu}$$

$$= \frac{\partial}{\partial x^j}\left(\Gamma^l_{ik}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu} - \frac{\partial}{\partial x^k}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\frac{\partial x^k}{\partial x'^\nu} - \Gamma^l_{mk}\Gamma^m_{ij}\frac{\partial x^k}{\partial x'^\nu}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}$$

$(j \rightleftarrows k)$ $\qquad$ $j \to k,\ i \rightleftarrows m,\ n \to j$

(A)

$$\Gamma'_{\lambda,\mu\nu} = \frac{1}{2}\left(\frac{\partial g'_{\lambda\mu}}{\partial x'^{\nu}} + \frac{\partial g'_{\lambda\nu}}{\partial x'^{\mu}} - \frac{\partial g'_{\mu\nu}}{\partial x'^{\lambda}}\right)$$

$$= \frac{1}{2}\left\{\frac{\partial}{\partial x'^{\nu}}\left(g_{ij}\frac{\partial x^i}{\partial x'^{\lambda}}\frac{\partial x^j}{\partial x'^{\mu}}\right) + \frac{\partial}{\partial x'^{\mu}}\left(g_{ik}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^k}{\partial x'^{\nu}}\right) - \frac{\partial}{\partial x'^{\lambda}}\left(g_{jk}\frac{\partial x^j}{\partial x'^{\mu}}\cdot\frac{\partial x^k}{\partial x'^{\nu}}\right)\right.$$

$$= \frac{1}{2}\left\{\frac{\partial}{\partial x'^{\nu}}(g_{ij})\frac{\partial x^i}{\partial x'^{\lambda}}\frac{\partial x^j}{\partial x'^{\mu}} + \frac{\partial}{\partial x'^{\mu}}(g_{ik})\frac{\partial x^i}{\partial x'^{\lambda}}\frac{\partial x^k}{\partial x'^{\nu}} - \frac{\partial}{\partial x'^{\lambda}}(g_{jk})\frac{\partial x^j}{\partial x'^{\mu}}\cdot\frac{\partial x^k}{\partial x'^{\nu}}\right.$$

$$\left. + g_{ij}\frac{\partial}{\partial x'^{\nu}}\left(\frac{\partial x^i}{\partial x'^{\lambda}}\frac{\partial x^j}{\partial x'^{\mu}}\right) + g_{ik}\frac{\partial}{\partial x'^{\mu}}\left(\frac{\partial x^i}{\partial x'^{\lambda}}\frac{\partial x^k}{\partial x'^{\nu}}\right) - g_{jk}\frac{\partial}{\partial x'^{\lambda}}\left(\frac{\partial x^j}{\partial x'^{\mu}}\cdot\frac{\partial x^k}{\partial x'^{\nu}}\right)\right\}$$

$$= \left\{\Gamma_{i,jk}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\mu}}\cdot\frac{\partial x^k}{\partial x'^{\nu}}\right\} + \frac{1}{2}g_{ij}\left\{\frac{\partial}{\partial x'^{\nu}}\left(\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\mu}}\right) + \frac{\partial}{\partial x'^{\mu}}\left(\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\nu}}\right)\right.$$

$$\left. - \frac{\partial}{\partial x'^{\lambda}}\left(\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\nu}}\right)\right\}$$

$$= \quad '' \quad + \frac{1}{2}g_{ij}\left\{\cancel{\frac{\partial^2 x^i}{\partial x'^{\nu}\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\mu}}} + \frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial^2 x^j}{\partial x'^{\nu}\partial x'^{\mu}}\right.$$

$$+ \cancel{\frac{\partial^2 x^i}{\partial x'^{\mu}\partial x'^{\lambda}}}\cdot\frac{\partial x^j}{\partial x'^{\nu}} + \cancel{\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial^2 x^j}{\partial x'^{\mu}\partial x'^{\nu}}}$$

/ $i \rightleftarrows j$

$$\left. - \cancel{\frac{\partial^2 x^i}{\partial x'^{\lambda}\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\nu}}} - \cancel{\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial^2 x^j}{\partial x'^{\lambda}\partial x'^{\nu}}}\right\}$$

$$\Gamma'_{\lambda,\mu\nu} = \Gamma_{i,jk}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\mu}}\cdot\frac{\partial x^k}{\partial x'^{\nu}} + g_{ij}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial^2 x^j}{\partial x'^{\mu}\partial x'^{\nu}} \quad\text{———— (1)}$$

$$g'^{\tau\lambda}\Gamma'_{\lambda,\mu\nu}\frac{\partial x^l}{\partial x'^{\tau}} = \Gamma_{i,jk}\cdot g'^{\tau\lambda}\cdot\frac{\partial x^l}{\partial x'^{\tau}}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\mu}}\cdot\frac{\partial x^k}{\partial x'^{\nu}} + g_{ij}\cdot g'^{\tau\lambda}\frac{\partial x^l}{\partial x'^{\tau}}\cdot\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial^2 x^j}{\partial x'^{\mu}\partial x'^{\nu}}$$

$$\Gamma'^{\tau}_{\mu\nu}\frac{\partial x^l}{\partial x'^{\tau}} = \Gamma_{i,jk}\, g^{il}\cdot\frac{\partial x^j}{\partial x'^{\mu}}\cdot\frac{\partial x^k}{\partial x'^{\nu}} + g_{ij}\, g^{il}\frac{\partial^2 x^j}{\partial x'^{\mu}\partial x'^{\nu}}$$

$$\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^l}{\partial x'^{\lambda}} = \Gamma^{l}_{jk}\cdot\frac{\partial x^j}{\partial x'^{\mu}}\cdot\frac{\partial x^k}{\partial x'^{\nu}} + \frac{\partial^2 x^l}{\partial x'^{\mu}\partial x'^{\nu}}$$

$$= \Gamma^{l}_{ij}\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\nu}} + \frac{\partial^2 x^l}{\partial x'^{\mu}\partial x'^{\nu}}$$

$$\frac{\partial^2 x^l}{\partial x'^{\mu}\partial x'^{\nu}} = \Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^l}{\partial x'^{\lambda}} - \Gamma^{l}_{ij}\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\nu}} \quad\text{———— (2)}$$

(2)

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B

$$\frac{\partial^2 x^l}{\partial x'^\mu \partial x'^\nu} = \Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^l}{\partial x'^\lambda} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}$$

$$\frac{\partial^2 x^l}{\partial x'^\mu \partial x'^\sigma} = \Gamma'^{\lambda}_{\mu\sigma}\frac{\partial x^l}{\partial x'^\lambda} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}$$

$$\frac{\partial}{\partial x'^\sigma}\left\{\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^l}{\partial x'^\lambda} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\right\} = \frac{\partial}{\partial x'^\nu}\left\{\Gamma'^{\lambda}_{\mu\sigma}\frac{\partial x^l}{\partial x'^\lambda} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\right\}$$

$$\frac{\partial}{\partial x'^\sigma}\left(\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^l}{\partial x'^\lambda}\right) - \frac{\partial}{\partial x'^\nu}\left(\Gamma'^{\lambda}_{\mu\sigma}\frac{\partial x^l}{\partial x'^\lambda}\right)$$

$$= \frac{\partial}{\partial x'^\sigma}\left(\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}\right) - \frac{\partial}{\partial x'^\nu}\left(\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\right)$$

$$\frac{\partial}{\partial x'^\sigma}\left(\Gamma'^{\lambda}_{\mu\nu}\right)\frac{\partial x^l}{\partial x'^\lambda} + \Gamma'^{\lambda}_{\mu\nu}\frac{\partial^2 x^l}{\partial x'^\sigma \partial x'^\lambda} - \frac{\partial}{\partial x'^\nu}\left(\Gamma'^{\lambda}_{\mu\sigma}\right)\frac{\partial x^l}{\partial x'^\lambda} + \Gamma'^{\lambda}_{\mu\sigma}\frac{\partial^2 x^l}{\partial x'^\nu \partial x'^\lambda}$$

$$= \frac{\partial}{\partial x'^\sigma}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} + \Gamma^l_{ij}\frac{\partial}{\partial x'^\sigma}\left(\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\right) - \frac{\partial}{\partial x'^\nu}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma} - \Gamma^l_{ij}\frac{\partial}{\partial x'^\nu}\left(\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\right)$$

$$= \frac{\partial}{\partial x'^\sigma}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} + \Gamma^l_{ij}\left(\frac{\partial^2 x^i}{\partial x'^\sigma \partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} + \cancel{\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial^2 x^j}{\partial x'^\sigma \partial x'^\nu}}\right)$$

$$- \frac{\partial}{\partial x'^\nu}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma} - \Gamma^l_{ij}\left(\frac{\partial^2 x^i}{\partial x'^\nu \partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma} + \cancel{\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial^2 x^j}{\partial x'^\nu \partial x'^\sigma}}\right)$$

ie $$\frac{\partial}{\partial x'^\sigma}\left(\Gamma'^{\lambda}_{\mu\nu}\right)\frac{\partial x^l}{\partial x'^\lambda} - \frac{\partial}{\partial x'^\nu}\left(\Gamma'^{\lambda}_{\mu\sigma}\right)\frac{\partial x^l}{\partial x'^\lambda} + \Gamma'^{\lambda}_{\mu\nu}\left(\Gamma'^{\rho}_{\sigma\lambda}\frac{\partial x^l}{\partial x'^\rho} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial x^j}{\partial x'^\lambda}\right)$$

$$- \Gamma'^{\lambda}_{\mu\sigma}\left(\Gamma'^{\rho}_{\lambda\nu}\frac{\partial x^l}{\partial x'^\rho} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\nu}\cdot\frac{\partial x^j}{\partial x'^\lambda}\right)$$

$a_\sigma b_{\nu\mu}$ $= 0$
$- a_\nu b_\sigma$ $- a_\sigma b_\nu$
$= a_\nu b_\sigma$
$\cancel{2} a_\nu b_\sigma$
$= \cancel{2} a_\sigma b_\nu$

$$= \frac{\partial}{\partial x'^\sigma}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} - \frac{\partial}{\partial x'^\nu}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma} + \frac{\partial x^j}{\partial x'^\nu}\Gamma^l_{ij}\left(\Gamma'^{\rho}_{\sigma\mu}\frac{\partial x^i}{\partial x'^\rho} - \Gamma^i_{mn}\frac{\partial x^m}{\partial x'^\sigma}\cdot\frac{\partial x^n}{\partial x'^\mu}\right)$$

$$- \frac{\partial x^j}{\partial x'^\sigma}\Gamma^l_{ij}\left(\Gamma'^{\rho}_{\nu\mu}\frac{\partial x^i}{\partial x'^\rho} - \Gamma^i_{mn}\frac{\partial x^m}{\partial x'^\nu}\cdot\frac{\partial x^n}{\partial x'^\mu}\right)$$

$\cancel{\Gamma'^{\lambda}_{\sigma\rho}\frac{\partial x^i}{\partial x'^\lambda}}$

$$\cancel{-\Gamma'^{\lambda}_{\mu\nu}\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial x^j}{\partial x'^\lambda} + \Gamma'^{\rho}_{\sigma\mu}\Gamma^l_{ij}\frac{\partial x^j}{\partial x'^\nu}\frac{\partial x^i}{\partial x'^\rho}}$$

$$\cancel{-\Gamma'^{\lambda}_{\mu\nu}\Gamma^l_{ij}\frac{\partial x^i}{\partial x}}$$

$$\cancel{-\Gamma^l_{ij}\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial x^j}{\partial x'^\lambda}} + \Gamma^l_{ij}\Gamma'^{\rho}_{\sigma\mu}\frac{\partial x^i}{\partial x'^\rho}\cdot\frac{\partial x^j}{\partial x'^\nu}$$

$$\left[\begin{matrix} 2\Gamma^l_{ij}\Gamma'^{\lambda}_{\sigma\mu}\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\nu} \\ -2\Gamma^l_{ij}\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial x^j}{\partial x'^\lambda}\end{matrix}\right.$$

= 0, because of symmetry between ν & σ

$$2\Gamma^l_{ij}\Gamma'^{\lambda}_{\sigma\mu}\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^j}{\partial x'^\nu}$$

$$-2\Gamma^l_{ij}\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial x^j}{\partial x'^\lambda}$$

$$\left.\begin{matrix} -\Gamma^l_{ij}\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial x^j}{\partial x'^\lambda} + \Gamma^l_{ij}\Gamma'^{\lambda}_{\sigma\mu}\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^j}{\partial x'^\nu} \\ + \Gamma^l_{ij}\Gamma'^{\lambda}_{\mu\sigma}\frac{\partial x^i}{\partial x'^\nu}\frac{\partial x^j}{\partial x'^\lambda} - \Gamma^l_{ij}\Gamma'^{\lambda}_{\nu\mu}\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\sigma}\end{matrix}\right\}$$

$= \leftarrow$

$$L.H.S = \frac{\partial}{\partial x'^\sigma}\left(\Gamma'^\lambda_{\mu\nu}\right)\frac{\partial x^l}{\partial x'^\lambda} - \frac{\partial}{\partial x'^\nu}\left(\Gamma'^\lambda_{\mu\sigma}\right)\frac{\partial x^l}{\partial x'^\lambda} + \Gamma'^\lambda_{\mu\nu}\Gamma'^\rho_{\sigma\lambda}\frac{\partial x^l}{\partial x'^\rho} - \Gamma'^\lambda_{\mu\sigma}\Gamma'^\rho_{\nu\lambda}\frac{\partial x^l}{\partial x'^\rho}$$

$$= \quad " \quad\quad " \quad + \Gamma'^\rho_{\mu\nu}\Gamma'^\lambda_{\sigma\rho}\frac{\partial x^l}{\partial x'^\lambda} - \Gamma'^\rho_{\mu\sigma}\Gamma'^\lambda_{\nu\rho}\frac{\partial x^l}{\partial x'^\lambda} \quad \text{by } (\lambda \rightleftarrows \rho).$$

$$= \Gamma'^\lambda_{\mu\sigma\nu}\frac{\partial x^l}{\partial x'^\lambda}.$$

$$R.H.S = \frac{\partial}{\partial x'^\sigma}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} - \frac{\partial}{\partial x'^\nu}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma} + \Gamma^l_{ij}\Gamma^i_{mn}\frac{\partial x^j}{\partial x'^\sigma}\frac{\partial x^m}{\partial x'^\nu}\cdot\frac{\partial x^n}{\partial x'^\mu} \begin{pmatrix} i \rightleftarrows m \\ m = k \end{pmatrix}$$

$$- \Gamma^l_{ij}\Gamma^i_{mn}\frac{\partial x^j}{\partial x'^\nu}\frac{\partial x^m}{\partial x'^\sigma}\frac{\partial x^n}{\partial x'^\mu}$$

$$= \frac{\partial}{\partial x'^\sigma}\left(\Gamma^l_{ik}\right)\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^k}{\partial x'^\nu}$$

$$= \frac{\partial}{\partial x^j}\left(\Gamma^l_{ik}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\frac{\partial x^k}{\partial x'^\nu} - \frac{\partial}{\partial x^k}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu}$$

$j \to k$

$$+ \Gamma^n_{ik}\Gamma^l_{nj}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu} - \Gamma^l_{ik}\Gamma^i_{mn}\frac{\partial x^k}{\partial x'^\nu}\cdot\frac{\partial x^m}{\partial x'^\sigma}\cdot\frac{\partial x^n}{\partial x'^\mu}$$

$$\frac{\partial}{\partial x^j}\left(\Gamma^l_{ij}\right) = \frac{\partial}{\partial x^k}\left(\Gamma^l_{ik}\right)\frac{\partial x^k}{\partial x'^\nu}$$

$$" \quad - \Gamma^l_{nk}\Gamma^n_{ji}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu} \quad \begin{pmatrix} i \rightleftarrows n \\ m = j \end{pmatrix}$$

$$\Gamma'^\lambda_{\mu\sigma\nu}\frac{\partial x^l}{\partial x'^\lambda} = \Gamma^l_{ijk}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\frac{\partial x^k}{\partial x'^\nu}$$

Riemann tensor of R-C. tensor or mixed curvature tensor R^l_{ijk}

Multiply $\frac{\partial x'^\alpha}{\partial x^l}$; $\Gamma'^\alpha_{\mu\sigma\nu} = \Gamma^l_{ijk}\cdot\frac{\partial x'^\alpha}{\partial x^l}\cdot\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu}$ (tensor)

$$\Gamma'^\lambda_{\mu\sigma\nu} = \Gamma^l_{ijk}\cdot\frac{\partial x'^\lambda}{\partial x^l}\,..$$

$$\frac{\partial x^l}{\partial x'^\lambda}\cdot\frac{\partial x'^\alpha}{\partial x^l} = \delta^{\alpha\lambda}.$$

$$\partial x'^\alpha = \frac{\partial x'^\alpha}{\partial x^l}\cdot\frac{\partial x^l}{\partial x'^\lambda}$$

$$\frac{\partial x'^\alpha}{\partial x'^\lambda} = \frac{\partial x'^\alpha}{\partial x^l}\cdot\frac{\partial x^l}{\partial x'^\lambda} = \delta^{\alpha\lambda}$$

$$\frac{\partial x^l}{\partial x'^\lambda}\cdot\frac{\partial x'^\alpha}{\partial x^l}\,\Gamma'^\lambda_{\mu\sigma\nu}$$

$$= \frac{\partial x'^\alpha}{\partial x'^\lambda}\cdot\Gamma'^\lambda_{\mu\sigma\nu} = \delta^{\alpha\lambda}\,\Gamma'^\lambda_{\mu\sigma\nu} = \Gamma'^\alpha_{\mu\sigma\nu}$$

$$\Gamma'^\rho_{\sigma\mu}\frac{\partial x^j}{\partial x'^\nu} = \frac{1}{2}\left(\frac{\partial g_{\lambda\sigma}}{\partial x^\mu} + \frac{\partial g_{\lambda\mu}}{\partial x^\sigma} - \frac{\partial g_{\sigma\mu}}{\partial x^\lambda}\right) g$$

$$\frac{\partial x^i}{\partial x'^\lambda}\,\Gamma'^\lambda_{\sigma\mu} = \frac{\partial x^i}{\partial x'^\sigma}\, g^{\lambda l}\,\Gamma_{l,\mu\nu}$$

$$R^l_{ijk} = \frac{\partial}{\partial x_j}\left(\Gamma^l_{ik}\right) - \frac{\partial}{\partial x_k}\left(\Gamma^l_{ij}\right) + \Gamma^l_{mj}\Gamma^m_{ik} - \Gamma^l_{mk}\Gamma^m_{ij}$$

$$(1-x/a)^{-n} = \left(\frac{a-x}{a}\right)^{-n} = \int (a-x)^{-n}\cdot a^{n}\cdot dx$$

$$= \left[-\frac{(a-x)^{n+1}}{n+1}\,a^{n}\right]_0^L = -\frac{(a-L)^{n+1}-a^{n+1}}{(n+1)}\cdot a^{n}$$

$$= \cancel{a^{n+1}-(a-L)^{n+1}}$$

$$= \frac{a^{2n+1}-a^{n}(a-L)^{n+1}}{n+1}$$

$$\Gamma^l_{ij}\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\sigma}}$$
$$=\Gamma^l_{ij}\Gamma'^{\lambda}_{\mu\sigma}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\nu}}$$

$$\frac{\cos \pi x}{(1-x/a)^n} =$$

$$\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\nu}} = \cancel{\frac{\partial x^l}{\partial x'^{\lambda}\partial x'^{\nu}}}$$
$$= \Gamma'^{\rho}_{\lambda\nu}\frac{\partial x^l}{\partial x'^{\rho}} - \frac{\partial^2 x^l}{\partial x'^{\lambda}\partial x'^{\nu}}$$

$$\Gamma^l_{ij}\frac{\partial x^i\,\partial x^j}{\partial x'^{\sigma}\partial x'^{\lambda}} = \Gamma'^{\rho}_{\sigma\lambda}\cdot\frac{\partial x^l}{\partial x'^{\rho}} - \frac{\partial^2 x^l}{\partial x'^{\sigma}\partial x'^{\lambda}}$$

$$-\Gamma'^{\lambda}_{\mu\nu}\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^{\sigma}}\frac{\partial x^j}{\partial x'^{\lambda}}$$
$$+\Gamma^l_{ij}\Gamma'^{\rho}_{\sigma\mu}\frac{\partial x^i}{\partial x'^{\rho}}\frac{\partial x^j}{\partial x'^{\nu}}$$
$$+\Gamma'^{\lambda}_{\mu\sigma}\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^{\nu}}\frac{\partial x^j}{\partial x'^{\lambda}}$$
$$-\Gamma'^{\rho}_{\mu\nu}\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^{\rho}}\frac{\partial x^j}{\partial x'^{\sigma}}$$

$$2\left(\Gamma'^{\rho}_{\lambda\nu}\frac{\partial x^l}{\partial x'^{\rho}} - \frac{\partial^2 x^l}{\partial x'^{\lambda}\partial x'^{\nu}}\right)\Gamma'^{\lambda}_{\sigma\mu}$$
$$-2\left(\Gamma'^{\rho}_{\sigma\lambda}\frac{\partial x^l}{\partial x'^{\rho}} - \frac{\partial^2 x^l}{\partial x'^{\sigma}\partial x'^{\lambda}}\right)\Gamma'^{\lambda}_{\mu\nu}$$

$$\cancel{2\Gamma^l_{ij}\Gamma'^{\lambda}_{\sigma\mu}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\nu}} - 2\Gamma^l_{ij}\Gamma'^{\lambda}_{\nu\mu}\frac{\partial x^l}{\partial x'^{\lambda}}}$$
$$\cancel{2\Gamma^l_{\lambda\nu}\Gamma'^{\lambda}_{\sigma\mu} - 2\Gamma^l_{\lambda\sigma}\Gamma'^{\lambda}_{\nu\mu}}$$

$$2\Gamma^l_{ij}\Gamma'^{\lambda}_{\sigma\mu}\frac{\partial x^i}{\partial x'^{\lambda}}\frac{\partial x^j}{\partial x'^{\nu}} - 2\Gamma^l_{ij}\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\sigma}}$$

$$\Gamma^l_{ij}\Gamma'^{\lambda}_{\sigma\mu}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\nu}} + \Gamma^l_{ij}\Gamma'^{\lambda}_{\sigma\mu}\frac{\partial x^i}{\partial x'^{\nu}}\frac{\partial x^j}{\partial x'^{\lambda}}$$
$$-\Gamma^l_{ij}\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^i}{\partial x'^{\sigma}}\frac{\partial x^j}{\partial x'^{\lambda}} - \Gamma^l_{ij}\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^i}{\partial x'^{\lambda}}\frac{\partial x^j}{\partial x'^{\sigma}}$$

$$\cancel{2\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^{\lambda}}\left(\Gamma'^{\lambda}_{\sigma\mu}\frac{\partial x^j}{\partial x'^{\nu}} - \Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^j}{\partial x'^{\sigma}}\right)}$$

$$2\Gamma^l_{ij}\Gamma'^{\lambda}_{\sigma\mu}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\nu}} - 2\Gamma^l_{ij}\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^i}{\partial x'^{\lambda}}\frac{\partial x^j}{\partial x'^{\sigma}}$$

$$\cancel{\Gamma'^{\lambda}_{\sigma\mu} = g^{m\lambda}\Gamma_{m,\sigma\mu}}$$
$$\cancel{\Gamma'^{\lambda}_{\mu\nu} = g^{n\lambda}\Gamma_{n,\mu\nu}}$$

$$\cancel{\Gamma^l_{ij}g'^{m\lambda}\left(\frac{\partial g'_{\sigma m}}{\partial x'^{\mu}} + \frac{\partial g'_{\mu m}}{\partial x'^{\sigma}} - \frac{\partial g'_{\sigma\mu}}{\partial x'^{m}}\right)\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\nu}}}$$
$$\cancel{-g'^{n\lambda}\left(\frac{\partial g'_{\mu n}}{\partial x'^{\nu}} + \frac{\partial g'_{n\nu}}{\partial x'^{\mu}} - \frac{\partial g'_{\mu\nu}}{\partial x'^{n}}\right)\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\nu}}}$$

$$g'^{\lambda\rho}\Gamma_{\rho,\sigma\mu}\,g^{lk}\Gamma_{kij}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\nu}} - g^{lk}\Gamma_{kij}\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\sigma}}\cdot g'^{\lambda\rho}\Gamma_{\rho,\mu\nu}$$

$$\cdot\ \cancel{g'^{\lambda\rho}\Gamma_{\rho,\sigma\mu}} \qquad -$$

$$\Gamma^l_{ij}\Gamma'^\lambda_{\sigma\mu}\frac{\partial x^i}{\partial x'^\lambda}\frac{\partial x^j}{\partial x'^\nu} - \Gamma^l_{ij}\Gamma'^\lambda_{\mu\nu}\frac{\partial x^i}{\partial x'^\lambda}\cdot\frac{\partial x^j}{\partial x'^\sigma}$$

$$\cancel{\Gamma'^\rho_{\lambda\mu}\frac{\partial x^l}{\partial x'^\rho}\Gamma'^\lambda_{\sigma\mu} - \Gamma'^\rho_{\lambda\sigma}\frac{\partial x^l}{\partial x'^\rho}\Gamma'^\lambda_{\mu\nu} - \Gamma'^\lambda_{\sigma\nu}\frac{\partial^2 x^l}{\partial x'^\lambda\partial x'^\mu}} + \Gamma'^\lambda_{\mu\nu}\frac{\partial^2 x^l}{\partial x'^\lambda\partial x'^\sigma}$$

$$\left.\begin{aligned}&-\Gamma'^\lambda_{\mu\nu}\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial x^j}{\partial x'^\lambda} - \Gamma'^\rho_{\nu\mu}\Gamma^l_{ij}\frac{\partial x^j}{\partial x'^\sigma}\frac{\partial x^i}{\partial x'^\rho}\\ &+\Gamma'^\lambda_{\mu\sigma}\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\nu}\frac{\partial x^j}{\partial x'^\lambda} + \Gamma'^\rho_{\sigma\mu}\Gamma^l_{ij}\frac{\partial x^j}{\partial x'^\nu}\frac{\partial x^i}{\partial x'^\rho}\end{aligned}\right\} \begin{aligned}&\cancel{\Gamma}^l_{ij}\Gamma'^\lambda_{\mu\sigma}\frac{\partial x^i}{\partial x'^\nu}\cdot\frac{\partial x^j}{\partial x'^\lambda}\\ &-\cancel{\Gamma}^l_{ij}\Gamma'^\lambda_{\mu\nu}\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial x^j}{\partial x'^\lambda}\end{aligned}$$

$$\frac{\partial^2 x^l}{\partial x'^\mu\partial x'^\nu} - \Gamma'^\lambda_{\mu\nu}\frac{\partial x^l}{\partial x'^\lambda} = -\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}$$

$$\begin{aligned}&= \cancel{\Gamma}^l_{ij}\Gamma'^\lambda_{\mu\nu}\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial x^j}{\partial x'^\lambda}\\ &-\cancel{\Gamma}^l_{ij}\Gamma'^\lambda_{\mu\sigma}\frac{\partial x^i}{\partial x'^\nu}\frac{\partial x^j}{\partial x'^\lambda} \quad \underline{\underline{=0}}\end{aligned}$$

Mere symmetry between σ & ν proves that crossterms vanish

$g_{lh}R^l_{ijk} = R_{hijk}$ (Def^n). (Covariant curvature tensor)

$$\cancel{R'^\lambda_{\mu\sigma\nu} = R^l_{ijk}} \qquad R'^\lambda_{\mu\sigma\nu}\frac{\partial x^l}{\partial x'^\lambda} = R^l_{ijk}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\cdot\frac{\partial x^k}{\partial x'^\nu}$$

$$g_{lh}R'^\lambda_{\mu\sigma\nu}\frac{\partial x^l}{\partial x'^\lambda}\cdot\frac{\partial x^h}{\partial x'^\tau} = g_{lh}R^l_{ijk}\frac{\partial x^h}{\partial x'^\tau}\cdots$$

$$g_{\lambda\tau}R'^\lambda_{\mu\sigma\nu} = R_{hijk}\cdot\frac{\partial x^h}{\partial x'^\tau}\cdots$$

ie $R'_{\tau\mu\sigma\nu} =$ ie R_{hijk} a tensor

(1) $$R_{hijk} = \frac{\partial}{\partial x_j}\left(\Gamma_{h,ik}\right) - \frac{\partial}{\partial x_k}\left(\Gamma_{h,ij}\right) + \Gamma^l_{ij}\Gamma_{l,hk} - \Gamma^l_{ik}\Gamma_{l,hj}$$

(2) $$R_{hijk} = \frac{1}{2}\left(\frac{\partial^2 g_{hk}}{\partial x^i\partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^h\partial x^k} - \frac{\partial^2 g_{hj}}{\partial x^i\partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^h\partial x^j}\right) + g^{lm}\left\{\Gamma_{m,ij}\Gamma_{l,hk} - \Gamma_{m,ik}\Gamma_{l,hj}\right\}$$

(3) $$\left.\begin{aligned}R_{hijk} &= -R_{ihjk}\\ R_{hijk} &= -R_{hikj}\\ R_{hijk} &= R_{jkhi}\\ R_{hijk} + R_{hjki} &+ R_{hkij} = 0\end{aligned}\right\}$$

(4) Contracting R^l_{ijk} by putting $l = k$ and using $\Gamma^i_{il} = \frac{\partial \log\sqrt{g}}{\partial x^l}$.

we get R_{ik} (Ricci tensor) $= R^k_{ijk} = \frac{\partial^2 \log\sqrt{g}}{\partial x^i \partial x^j} - \frac{\partial}{\partial x_k}\left(\Gamma^k_{ij}\right) + \Gamma^m_{ik}\Gamma^k_{mj}$
$- \Gamma^m_{ij} \frac{\partial \log\sqrt{g}}{\partial x^m}$.

$$\left(R_{ij} = \frac{\partial}{\partial x^j}\left(\Gamma^k_{ik}\right) - \frac{\partial}{\partial x^k}\left(\Gamma^k_{ij}\right) + \Gamma^m_{ik}\Gamma^k_{mj} - \Gamma^k_{ij}\Gamma^m_{km} \right)$$

(5) curvature invariant (~~scalar~~ curvature) is given by

$$R = g^{ij} R_{ij}$$

~~(6) Bianchi's identity~~

(6) Einstein tensor

$$G_{ij} = R_{ij} - \tfrac{1}{2} g_{ij} R.$$

(7) Covariant differentiation w.r.t. g_{ik}

$\lambda_{i;j} = \frac{\partial \lambda_i}{\partial x^j} - \lambda_h \Gamma^h_{ij}$ (of a covariant vector)

Riemannian Geometry. defined by $ds^2 = g_{ik}\, dx^i dx^k$

(1) Geodesics are curves always maintaining their direction i.e. given vector u^i at an arbitrary point P_0, one obtains the direction vectors at other points by a $\|^l$ displacement of u^i along the geodesic line.

Eqns are $\frac{d^2x^i}{ds^2} + \Gamma^i_{rs}\frac{dx^r}{ds}\frac{dx^s}{ds} = 0$ $\left(\frac{d^2x^l}{ds^2} + \Gamma^l_{jk}\frac{dx^j}{ds}\cdot\frac{dx^k}{ds} = 0 \right)$

(2) ~~Covariant diffe~~ Bianchi identities & Schur's theorem.

$$a_{ij;k} = \frac{\partial a_{ij}}{\partial x^k} - a_{ih}\Gamma^h_{jk} - a_{hj}\Gamma^h_{ik}$$

$$a^{ij}_{;k} = \frac{\partial a^{ij}}{\partial x^k} + a^{ih}\Gamma^j_{hk} + a^{hj}\Gamma^i_{hk}$$

$$a^i_{j;k} = \frac{\partial a^i_j}{\partial x^k} + a^h_j\Gamma^i_{hk} - a^i_h\Gamma^h_{jk}$$

& similarly higher order derivatives

Same rules for sum, diffce & outer & inner multiplication of tensors.

g_{ij}, g^{ij}, δ^i_j behave as constants in covariant diffn

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finally leading to

$$R'^{\lambda}_{\mu\sigma\nu} \cdot \frac{\partial x^l}{\partial x'^\lambda} = R^l_{ijk} \frac{\partial x^i}{\partial x'^\mu} \cdot \frac{\partial x^j}{\partial x'^\sigma} \cdot \frac{\partial x^k}{\partial x'^\nu} .$$

Multiplying this by $\frac{\partial x'^\alpha}{\partial x^l}$ & summing over l, and changing j to h & k to j (dummy

$$R'^{\alpha}_{\mu\sigma\nu} = R^l_{ihj} \frac{\partial x'^\alpha}{\partial x^l} \cdot \frac{\partial x^i}{\partial x'^\mu} \cdot \frac{\partial x^h}{\partial x'^\sigma} \cdot \frac{\partial x^j}{\partial x'^\nu}$$ ~~[illegible]~~ proving that $R^{l}_{\cdot ihj}$ is a tensor

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$$-\frac{\partial}{\partial x'^{\sigma}}\left(\Gamma^{l}_{ij}\right) - \Gamma^{l}_{ij}\left[\frac{\partial x^{j}}{\partial x'^{\nu}}\left\{\Gamma'^{\rho}_{\sigma\mu}\frac{\partial x^{i}}{\partial x'^{\rho}} - \Gamma^{i}_{mn}\frac{\partial x^{m}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{n}}{\partial x'^{\mu}}\right\}\right]$$

$$+\frac{\partial}{\partial x'^{\nu}}\left(\Gamma^{l}_{ij}\right) + \Gamma^{l}_{ij}\left[\frac{\partial x^{j}}{\partial x'^{\sigma}}\left\{\Gamma'^{\rho}_{\mu\nu}\frac{\partial x^{i}}{\partial x'^{\rho}} - \Gamma^{i}_{mn}\frac{\partial x^{m}}{\partial x'^{\mu}}\frac{\partial x^{n}}{\partial x'^{\nu}}\right\}\right] = 0.$$

$$\frac{\partial x^{l}}{\partial x'^{\lambda}}\cdot\frac{\partial}{\partial x'^{\sigma}}\left(\Gamma'^{\lambda}_{\mu\nu}\right) - \frac{\partial}{\partial x'^{\nu}}\left(\Gamma'^{\lambda}_{\mu\sigma}\right)\cdot\frac{\partial x^{l}}{\partial x'^{\lambda}} + \left(\Gamma'^{\lambda}_{\mu\nu}\Gamma'^{\rho}_{\lambda\sigma} - \Gamma'^{\lambda}_{\mu\sigma}\Gamma'^{\rho}_{\lambda\nu}\right)\frac{\partial x^{l}}{\partial x'^{\rho}} - \cancel{\Gamma'^{\lambda}_{\mu\nu}\Gamma^{l}_{ij}\frac{\partial x^{i}}{\partial x'^{\sigma}}\frac{\partial x^{j}}{\partial x'^{\lambda}}} + \cancel{\Gamma'^{\lambda}_{\mu\sigma}\Gamma^{l}_{ij}\frac{\partial x^{i}}{\partial x'^{\nu}}\frac{\partial x^{j}}{\partial x'^{\lambda}}}$$

$$-\frac{\partial}{\partial x'^{\sigma}}\left(\Gamma^{l}_{ij}\right)\cdot\frac{\partial x^{i}}{\partial x'^{\mu}}\cdot\frac{\partial x^{j}}{\partial x'^{\nu}} + \frac{\partial}{\partial x'^{\nu}}\left(\Gamma^{l}_{ij}\right)\cdot\frac{\partial x^{i}}{\partial x'^{\mu}}\cdot\frac{\partial x^{j}}{\partial x'^{\sigma}} - \cancel{\Gamma^{l}_{ij}\Gamma'^{\rho}_{\sigma\mu}\frac{\partial x^{j}}{\partial x'^{\nu}}\frac{\partial x^{i}}{\partial x'^{\rho}}} + \cancel{\Gamma^{l}_{ij}\Gamma'^{\rho}_{\mu\nu}\frac{\partial x^{j}}{\partial x'^{\sigma}}\frac{\partial x^{i}}{\partial x'^{\rho}}}$$

$$+\Gamma^{l}_{ij}\Gamma^{i}_{mn}\frac{\partial x^{j}}{\partial x'^{\nu}}\frac{\partial x^{m}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{n}}{\partial x'^{\mu}} - \Gamma^{l}_{ij}\Gamma^{i}_{mn}\frac{\partial x^{j}}{\partial x'^{\sigma}}\frac{\partial x^{m}}{\partial x'^{\mu}}\cdot\frac{\partial x^{n}}{\partial x'^{\nu}} = 0$$

$$\left[\begin{array}{l}\Gamma^{l}_{ij}\Gamma'^{\rho}_{\sigma\mu}\dfrac{\partial x^{i}}{\partial x'^{\rho}}\cdot\dfrac{\partial x^{j}}{\partial x'^{\nu}} = \Gamma^{l}_{ij}\Gamma'^{\lambda}_{\mu\sigma}\dfrac{\partial x^{i}}{\partial x'^{\lambda}}\cdot\dfrac{\partial x^{j}}{\partial x'^{\nu}} = \Gamma^{l}_{ij}\Gamma'^{\lambda}_{\mu\sigma}\dfrac{\partial x^{i}}{\partial x'^{\nu}}\cdot\dfrac{\partial x^{j}}{\partial x'^{\lambda}} \\ \Gamma^{l}_{ij}\Gamma'^{\rho}_{\mu\nu}\dfrac{\partial x^{j}}{\partial x'^{\sigma}}\dfrac{\partial x^{i}}{\partial x'^{\rho}} = \Gamma^{l}_{ij}\Gamma'^{\lambda}_{\mu\nu}\dfrac{\partial x^{i}}{\partial x'^{\lambda}}\cdot\dfrac{\partial x^{j}}{\partial x'^{\sigma}} = \Gamma^{l}_{ij}\Gamma'^{\lambda}_{\mu\nu}\dfrac{\partial x^{i}}{\partial x'^{\sigma}}\cdot\dfrac{\partial x^{j}}{\partial x'^{\lambda}}\end{array}\right]$$

$$\therefore \left\{\frac{\partial}{\partial x'^{\sigma}}\left(\Gamma'^{\lambda}_{\mu\nu}\right) - \frac{\partial}{\partial x'^{\nu}}\left(\Gamma'^{\lambda}_{\mu\sigma}\right) + \Gamma'^{\rho}_{\mu\nu}\Gamma'^{\lambda}_{\rho\sigma} - \Gamma'^{\rho}_{\mu\sigma}\Gamma'^{\lambda}_{\nu\rho}\right\}\frac{\partial x^{l}}{\partial x'^{\lambda}}$$

$$= \frac{\partial}{\partial x^{k}}\left(\Gamma^{l}_{ij}\right)\frac{\partial x^{i}}{\partial x'^{\mu}}\cdot\frac{\partial x^{j}}{\partial x'^{\nu}}\cdot\frac{\partial x^{k}}{\partial x'^{\sigma}} - \frac{\partial}{\partial x^{k}}\left(\Gamma^{l}_{ij}\right)\frac{\partial x^{i}}{\partial x'^{\mu}}\cdot\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{k}}{\partial x'^{\nu}}$$

$$+\Gamma^{l}_{ij}\Gamma^{i}_{mn}\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{m}}{\partial x'^{\mu}}\cdot\frac{\partial x^{n}}{\partial x'^{\nu}} - \Gamma^{l}_{ij}\Gamma^{i}_{mn}\frac{\partial x^{j}}{\partial x'^{\nu}}\cdot\frac{\partial x^{m}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{n}}{\partial x'^{\mu}}$$

$$= \frac{\partial}{\partial x^{j}}\left(\Gamma^{l}_{ik}\right)\frac{\partial x^{i}}{\partial x'^{\mu}}\cdot\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{k}}{\partial x'^{\nu}} - \frac{\partial}{\partial x^{k}}\left(\Gamma^{l}_{ij}\right)\frac{\partial x^{i}}{\partial x'^{\mu}}\cdot\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{k}}{\partial x'^{\nu}}$$

$$\cancel{+\Gamma^{k}_{mj}\Gamma^{l}_{in}\frac{\partial x^{i}}{\partial x'^{\mu}}\frac{\partial x^{m}}{\partial x'^{\sigma}}\frac{\partial x^{n}}{\partial x'^{\nu}} - \Gamma^{k}_{nj}\Gamma^{l}_{mi}\frac{\partial x^{j}}{\partial x'^{\nu}}\cdot\frac{\partial x^{m}}{\partial x'^{\sigma}}\frac{\partial x^{i}}{\partial x'^{\mu}}}$$

$$= \quad '' \quad - \quad '' \quad - \Gamma^{l}_{mk}\Gamma^{m}_{ij}\frac{\partial x^{i}}{\partial x'^{\mu}}\frac{\partial x^{j}}{\partial x'^{\sigma}}\frac{\partial x^{k}}{\partial x'^{\nu}} \longleftarrow \quad -\Gamma^{l}_{nm}\Gamma^{n}_{ij}\frac{\partial x^{i}}{\partial x'^{\mu}}\frac{\partial x^{j}}{\partial x'^{\sigma}}\frac{\partial x^{m}}{\partial x'^{\nu}}$$

$$\cancel{+\Gamma^{l}_{jm}\Gamma^{l}_{ik}\frac{\partial x^{i}}{\partial x'^{\mu}}\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{k}}{\partial x'^{\nu}} - \Gamma^{l}_{jn}\Gamma^{l}_{mi}\frac{\partial x^{n}}{\partial x'^{\nu}}\cdot\frac{\partial x^{m}}{\partial x'^{\sigma}}} \quad \cancel{-\Gamma^{l}_{nk}\Gamma^{k}_{ji}\frac{\partial x^{k}}{\partial x'^{\nu}}\cdot\frac{\partial x^{j}}{\partial x'^{\sigma}}\frac{\partial x^{i}}{\partial x'^{\mu}}}$$

$$\cancel{+\Gamma^{l}_{ij}\Gamma^{l}_{ik}\frac{\partial x^{j}}{\partial x'^{\sigma}}\frac{\partial x^{i}}{\partial x'^{\mu}}\cdot\frac{\partial x^{k}}{\partial x'^{\nu}}} \quad \Gamma^{l}_{mj}\Gamma^{m}_{ik}\frac{\partial x^{j}}{\partial x'^{\sigma}}\frac{\partial x^{i}}{\partial x'^{\mu}}\cdot\frac{\partial x^{k}}{\partial x'^{\nu}} - \Gamma^{l}_{im}\Gamma^{i}_{jk}\frac{\partial x^{m}}{\partial x'^{\nu}}\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{k}}{\partial x'^{\nu}}$$

$a_{1j}x^{1} + a_{2j}x^{2}$

$a_{ij}x^{i}$

$= a_{ji}x^{j}$

$a_{j1}x^{1} + a_{j2}x^{2}$

(1) $\lambda'^i = \lambda^j \frac{\partial x'^i}{\partial x^j}$, $\lambda'_i = \lambda_j \frac{\partial x^j}{\partial x'^i}$.

$$a'^{ij} = a^{kl}\frac{\partial x'^i}{\partial x^k}\cdot\frac{\partial x'^j}{\partial x^l}\ ; \quad a'_{ij} = a_{kl}\frac{\partial x^k}{\partial x'^i}\cdot\frac{\partial x^l}{\partial x'^j}.$$

$$a'^i_j = a^k_l \frac{\partial x'^i}{\partial x^k}\cdot\frac{\partial x^l}{\partial x'^j}$$

$$a_{ij} = a_{ji}\ ,\ a_{ij} = -a_{ji}\ ,\ a_{ij} = \tfrac{1}{2}(a_{ij}+a_{ji}) + \tfrac{1}{2}(a_{ij}-a_{ji})$$

(2) (a) g_{ij} $g = |g_{ij}|$ & cofactor of $g_{ij}/g = g^{ij}$ then g^{ij} is a contravariant vector.

symmetric:

$$g^{ij}g_{kj} = \delta^i_k\ .\ \text{Let } g_{ij}\lambda^i = \mu_j\ ,\ \text{then } g^{kj}\mu_j = g^{kj}g_{ij}\lambda^i = \delta^k_i\lambda^i = \lambda^k$$

& since μ_j being arbitrary g^{ij} is contravariant.

Similarly if $\bar{g} = |g^{ij}|$ cofactor of $g^{ij}/\bar{g} = g_{ij}$ conjugate tensors

$$g^{ij}g_{ij} = \delta^i_i = n,\ g\bar{g} = \begin{vmatrix} 1 & 0 & & \\ 0 & 1 & & \\ 0 & \cdots & & 1 \end{vmatrix} = 1$$

$g^{ij}g_{kj} = \delta^i_k$. Raising & lowering of indices using these determinants

~~(3) Christoffel symbols:~~

(b) Dual tensors. $\xi^{*14} = \frac{1}{\sqrt{g}}\xi_{23}$, $\xi^{*}_{14} = \sqrt{g}\,\xi^{23}$

$\xi^{*23} = \frac{1}{\sqrt{g}}\xi_{14}$, $\xi^{*}_{23} = \sqrt{g}\,\xi^{14}$

(c) Christoffel 3-index symbols.

First kind $\Gamma_{i,jk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)$

2nd kind $\Gamma^i_{jk} = g^{il}\Gamma_{l,jk}$

$$\Gamma_{i,jk} = g_{il}\Gamma^l_{jk} \qquad R.H.S. = g_{il}g^{lm}\Gamma_{m,jk} = \delta^m_i\Gamma_{m,jk} = \Gamma_{i,jk}$$

$$\Gamma^i_{ij} = g^{il}\Gamma_{l,ij} = \frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{li}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right)$$

$$= \frac{1}{2}g^{il}\frac{\partial g_{il}}{\partial x^j}$$

$$\frac{\partial g}{\partial x^j} = g\cdot g^{il}\frac{\partial g_{il}}{\partial x^j} \qquad \Gamma^i_{ij} = \frac{1}{2g}\cdot\frac{\partial g}{\partial x^j} = \frac{1}{\sqrt{g}}\cdot\frac{\partial\sqrt{g}}{\partial x^j} = \frac{\partial\log\sqrt{g}}{\partial x^j}.$$

$$\left(\frac{\partial g'_{\sigma\mu}}{\partial x'^{\nu}} + \quad - \right)$$

$$g'_{\sigma\mu} = g_{ij} \frac{\partial x^{i}}{\partial x'^{\sigma}} \frac{\partial x^{j}}{\partial x'^{\mu}}$$

$$\frac{\partial g_{ij}}{\partial x'^{\nu}} \left(\quad \right) + g_{ik} \, \partial x'$$

$$\Gamma'_{\sigma,\mu\nu} = \Gamma_{k,ij} \frac{\partial x^{i}}{\partial x'^{\mu}} \cdot \frac{\partial x^{j}}{\partial x'^{\mu}} \cdot \frac{\partial x^{k}}{\partial x'^{\sigma}} + g_{ij} \frac{\partial x^{i}}{\partial x'^{\sigma}} \cdot \frac{\partial^2 x^{j}}{\partial x'^{\mu} \partial x'^{\nu}}$$

$$g_{ij} \, g'^{\sigma\lambda} \frac{\partial x^{l}}{\partial x'^{\lambda}} \cdot \frac{\partial x^{i}}{\partial x'^{\sigma}} \cdot \frac{\partial^2 x^{j}}{\partial x'^{\mu} \partial x'^{\nu}} \cdot = g_{ij} \, g^{li} \cdot \frac{\partial^2 x^{j}}{\partial x'^{\mu} \partial x'^{\nu}} = \frac{\partial^2 x^{l}}{\partial x'^{\mu} \partial x'^{\nu}}$$

∂^2

$$\Gamma'_{\sigma,\mu\nu} = \frac{1}{2} \left(\frac{\partial g'_{\sigma\mu}}{\partial x'^{\nu}} + \frac{\partial g'_{\sigma\nu}}{\partial x'^{\mu}} - \frac{\partial g'_{\mu\nu}}{\partial x'^{\sigma}} \right)$$

$$= \frac{1}{2} \left\{ \frac{\partial}{\partial x'^{\nu}} \left(g_{ik} \frac{\partial x^{i}}{\partial x'^{\sigma}} \cdot \frac{\partial x^{k}}{\partial x'^{\mu}} \right) + \frac{\partial}{\partial x'^{\mu}} \left(g_{jk} \frac{\partial x^{j}}{\partial x'^{\sigma}} \frac{\partial x^{k}}{\partial x'^{\nu}} \right) - \frac{\partial}{\partial x'^{\sigma}} \left(g_{ij} \frac{\partial x^{i}}{\partial x'^{\mu}} \cdot \frac{\partial x^{j}}{\partial x'^{\nu}} \right) \right\}$$

$$= \frac{1}{2} \left\{ \frac{\partial g_{ik}}{\partial x^{j}} \left(\frac{\partial x^{j}}{\partial x'^{\nu}} \cdot \frac{\partial x^{i}}{\partial x'^{\sigma}} \cdot \frac{\partial x^{k}}{\partial x'^{\mu}} \right) + \frac{\partial g_{jk}}{\partial x^{i}} \left(\frac{\partial x^{i}}{\partial x'^{\mu}} \frac{\partial x^{j}}{\partial x'^{\sigma}} \frac{\partial x^{k}}{\partial x'^{\nu}} \right) - \frac{\partial g_{ij}}{\partial x^{k}} \left(\frac{\partial x^{k}}{\partial x'^{\sigma}} \cdot \frac{\partial x^{i}}{\partial x'^{\mu}} \frac{\partial x^{j}}{\partial x'^{\nu}} \right) \right\}$$

$$+ \frac{1}{2} g_{ik} \frac{\partial}{\partial x'^{\nu}} \left(\frac{\partial x^{i}}{\partial x'^{\sigma}} \frac{\partial x^{k}}{\partial x'^{\mu}} \right) + \frac{1}{2} g_{jk} \frac{\partial}{\partial x'^{\mu}} \left(\frac{\partial x^{j}}{\partial x'^{\sigma}} \frac{\partial x^{k}}{\partial x'^{\nu}} \right) - \frac{1}{2} g_{ij} \cdot \frac{\partial}{\partial x'^{\sigma}} \left(\frac{\partial x^{i}}{\partial x'^{\mu}} \cdot \frac{\partial x^{j}}{\partial x'^{\nu}} \right)$$

$$= \frac{1}{2} \left\{ \frac{\partial g_{ik}}{\partial x^{j}} \left(\frac{\partial x^{i}}{\partial x'^{\mu}} \cdot \frac{\partial x^{j}}{\partial x'^{\nu}} \cdot \frac{\partial x^{k}}{\partial x'^{\sigma}} \right) + \frac{\partial g_{jk}}{\partial x^{i}} \left(\frac{\partial x^{i}}{\partial x'^{\mu}} \frac{\partial x^{j}}{\partial x'^{\nu}} \cdot \frac{\partial x^{k}}{\partial x'^{\sigma}} \right) - \frac{\partial g_{ij}}{\partial x^{k}} \left(\frac{\partial x^{i}}{\partial x'^{\mu}} \cdot \frac{\partial x^{j}}{\partial x'^{\nu}} \cdot \frac{\partial x^{k}}{\partial x'^{\sigma}} \right) \right\}$$

$$\left.\begin{aligned} &+ \frac{1}{2} g_{ij} \left(\cancel{\frac{\partial^2 x^{i}}{\partial x'^{\nu} \partial x'^{\sigma}} \cdot \frac{\partial x^{j}}{\partial x'^{\mu}}} + \frac{\partial^2 x^{j}}{\partial x'^{\nu} \partial x'^{\mu}} \cdot \frac{\partial x^{i}}{\partial x'^{\sigma}} \right) \checkmark \\ &+ \frac{1}{2} g_{ij} \left(\cancel{\frac{\partial^2 x^{i}}{\partial x'^{\mu} \partial x'^{\sigma}} \cdot \frac{\partial x^{j}}{\partial x'^{\nu}}} + \frac{\partial^2 x^{j}}{\partial x'^{\mu} \partial x'^{\nu}} \cdot \frac{\partial x^{i}}{\partial x'^{\sigma}} \right) \checkmark \\ &- \frac{1}{2} g_{ij} \left(\cancel{\frac{\partial^2 x^{i}}{\partial x'^{\sigma} \partial x'^{\mu}} \cdot \frac{\partial x^{j}}{\partial x'^{\nu}}} + \cancel{\frac{\partial^2 x^{j}}{\partial x'^{\sigma} \partial x'^{\nu}} \cdot \frac{\partial x^{i}}{\partial x'^{\mu}}} \right) \end{aligned}\right\} = g_{ij} \frac{\partial x^{i}}{\partial x'^{\sigma}} \cdot \frac{\partial^2 x^{j}}{\partial x'^{\mu} \partial x'^{\nu}}$$

(2)

Differentiate (2') w.r.t x'^{σ} & (2'') w.r.t x'^{ν} & subtract

$$0 = \frac{\partial}{\partial x'^{\sigma}}\left(\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^l}{\partial x'^{\lambda}} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^{\mu}}\frac{\partial x^j}{\partial x'^{\nu}}\right) - \frac{\partial}{\partial x'^{\nu}}\left(\Gamma'^{\lambda}_{\mu\sigma}\frac{\partial x^l}{\partial x'^{\lambda}} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^{\mu}}\frac{\partial x^j}{\partial x'^{\sigma}}\right)$$

$$\frac{\partial}{\partial x'^{\sigma}}\left(\Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^l}{\partial x'^{\lambda}}\right) - \frac{\partial}{\partial x'^{\nu}}\left(\Gamma'^{\lambda}_{\mu\sigma}\frac{\partial x^l}{\partial x'^{\lambda}}\right) = \frac{\partial}{\partial x'^{\sigma}}\left(\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\nu}}\right) - \frac{\partial}{\partial x'^{\nu}}\left(\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^{\mu}}\frac{\partial x^j}{\partial x'^{\sigma}}\right)$$

$$\frac{\partial}{\partial x'^{\sigma}}\left(\Gamma'^{\lambda}_{\mu\nu}\right)\frac{\partial x^l}{\partial x'^{\lambda}} + \Gamma'^{\lambda}_{\mu\nu}\frac{\partial^2 x^l}{\partial x'^{\sigma}\partial x'^{\lambda}} - \frac{\partial}{\partial x'^{\nu}}\left(\Gamma'^{\lambda}_{\mu\sigma}\right)\frac{\partial x^l}{\partial x'^{\lambda}} - \Gamma'^{\lambda}_{\mu\sigma}\frac{\partial^2 x^l}{\partial x'^{\nu}\partial x'^{\lambda}}$$

$$= \frac{\partial}{\partial x'^{\sigma}}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\nu}} + \Gamma^l_{ij}\frac{\partial}{\partial x'^{\sigma}}\left(\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\nu}}\right) - \frac{\partial}{\partial x'^{\nu}}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\sigma}} - \Gamma^l_{ij}\frac{\partial}{\partial x'^{\nu}}\left(\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\sigma}}\right)$$

$$= \frac{\partial}{\partial x'^{\sigma}}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\nu}} + \Gamma^l_{ij}\left(\frac{\partial^2 x^i}{\partial x'^{\sigma}\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\nu}} + \cancel{\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial^2 x^j}{\partial x'^{\sigma}\partial x'^{\nu}}}\right)$$

$$- \frac{\partial}{\partial x'^{\nu}}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\sigma}} - \Gamma^l_{ij}\left(\frac{\partial^2 x^i}{\partial x'^{\mu}\partial x'^{\nu}}\cdot\frac{\partial x^j}{\partial x'^{\sigma}} + \cancel{\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial^2 x^j}{\partial x'^{\nu}\partial x'^{\sigma}}}\right)$$

$$\text{L.H.S} = \frac{\partial}{\partial x'^{\sigma}}\left(\Gamma'^{\lambda}_{\mu\nu}\right)\frac{\partial x^l}{\partial x'^{\lambda}} - \frac{\partial}{\partial x'^{\nu}}\left(\Gamma'^{\lambda}_{\mu\sigma}\right)\frac{\partial x^l}{\partial x'^{\lambda}} + \Gamma'^{\lambda}_{\mu\nu}\left\{\Gamma'^{\rho}_{\lambda\sigma}\frac{\partial x^l}{\partial x'^{\rho}} - \Gamma^l_{ij}\cancel{\frac{\partial x^i}{\partial x'^{\lambda}}\cdot\frac{\partial x^j}{\partial x'^{\sigma}}}\right\}$$

$$- \Gamma'^{\lambda}_{\mu\sigma}\left\{\Gamma'^{\rho}_{\lambda\nu}\frac{\partial x^l}{\partial x'^{\rho}} - \Gamma^l_{ij}\cancel{\frac{\partial x^i}{\partial x'^{\lambda}}\frac{\partial x^j}{\partial x'^{\nu}}}\right\}$$

$$\text{R.H.S} = \frac{\partial}{\partial x'^{\sigma}}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\nu}} - \frac{\partial}{\partial x'^{\nu}}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\sigma}} + \frac{\partial x^j}{\partial x'^{\nu}}\Gamma^l_{ij}\left\{\Gamma'^{\rho}_{\sigma\mu}\cancel{\frac{\partial x^i}{\partial x'^{\rho}}} - \Gamma^i_{mn}\frac{\partial x^m}{\partial x'^{\sigma}}\cdot\frac{\partial x^n}{\partial x'^{\mu}}\right\}$$

$$- \frac{\partial x^j}{\partial x'^{\sigma}}\Gamma^l_{ij}\left\{\cancel{\Gamma'^{\rho}_{\mu\nu}\frac{\partial x^i}{\partial x'^{\rho}}} - \Gamma^i_{mn}\frac{\partial x^m}{\partial x'^{\mu}}\cdot\frac{\partial x^n}{\partial x'^{\nu}}\right\}$$

$$\left\{\frac{\partial}{\partial x'^{\sigma}}\left(\Gamma'^{\lambda}_{\mu\nu}\right) - \frac{\partial}{\partial x'^{\nu}}\left(\Gamma'^{\lambda}_{\mu\sigma}\right)\right\}\frac{\partial x^l}{\partial x'^{\lambda}} + \left\{\Gamma'^{\lambda}_{\mu\nu}\Gamma'^{\rho}_{\lambda\sigma} - \Gamma'^{\lambda}_{\mu\sigma}\Gamma'^{\rho}_{\lambda\nu}\right\}\frac{\partial x^l}{\partial x'^{\rho}}$$

$$= \frac{\partial}{\partial x'^{\sigma}}\left(\Gamma^l_{ik}\right)\frac{\partial x^i}{\partial x'^{\mu}}\frac{\partial x^k}{\partial x'^{\nu}} - \frac{\partial}{\partial x'^{\nu}}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^{\mu}}\frac{\partial x^j}{\partial x'^{\sigma}} + \Gamma^l_{ij}\Gamma^i_{mn}\frac{\partial x^j}{\partial x'^{\sigma}}\frac{\partial x^m}{\partial x'^{\mu}}\cdot\frac{\partial x^n}{\partial x'^{\nu}}$$

$$- \Gamma^l_{ij}\Gamma^i_{mn}\frac{\partial x^j}{\partial x'^{\nu}}\cdot\frac{\partial x^m}{\partial x'^{\sigma}}\cdot\frac{\partial x^n}{\partial x'^{\mu}}$$

$$\left\{\frac{\partial}{\partial x'^{\sigma}}\left(\Gamma'^{\lambda}_{\mu\nu}\right) - \frac{\partial}{\partial x'^{\nu}}\left(\Gamma'^{\lambda}_{\mu\sigma}\right) + \Gamma'^{\rho}_{\mu\nu}\Gamma'^{\lambda}_{\rho\sigma} - \Gamma'^{\rho}_{\mu\sigma}\cdot\Gamma'^{\lambda}_{\rho\nu}\right\}\frac{\partial x^l}{\partial x'^{\lambda}}$$

$$= \frac{\partial}{\partial x^k}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^{\mu}}\frac{\partial x^j}{\partial x'^{\nu}}\frac{\partial x^k}{\partial x'^{\sigma}} - \frac{\partial}{\partial x^k}\left(\Gamma^l_{ij}\right)\frac{\partial x^i}{\partial x'^{\mu}}\cdot\frac{\partial x^j}{\partial x'^{\sigma}}\cdot\frac{\partial x^k}{\partial x'^{\nu}} + \Gamma^l_{ij}\Gamma^i_{mn}\frac{\partial x^j}{\partial x'^{\sigma}}\cdot\frac{\partial x^m}{\partial x'^{\mu}}\cdot\frac{\partial x^n}{\partial x'^{\nu}}$$

$$- \Gamma^l_{ij}\Gamma^i_{mn}\frac{\partial x^j}{\partial x'^{\nu}}\cdot\frac{\partial x^m}{\partial x'^{\sigma}}\cdot\frac{\partial x^n}{\partial x'^{\mu}}$$

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$$R^{l}_{ijk} = \frac{\partial}{\partial x^{j}}\left(\Gamma^{l}_{ik}\right) - \frac{\partial}{\partial x^{k}}\left(\Gamma^{l}_{ij}\right) + \Gamma^{m}_{ik}\Gamma^{l}_{mj} - \Gamma^{m}_{ij}\Gamma^{l}_{mk} \quad \text{—— B}$$ (3)

$$R'^{\lambda}_{\mu\sigma\nu}\cdot\frac{\partial x^{l}}{\partial x'^{\lambda}} = \frac{\partial}{\partial x^{j}}\left(\Gamma^{l}_{ik}\right)\frac{\partial x^{i}}{\partial x'^{\mu}}\cdot\frac{\partial x^{j}}{\partial x'^{\sigma}}\frac{\partial x^{k}}{\partial x'^{\nu}} - \frac{\partial}{\partial x^{k}}\left(\Gamma^{l}_{ij}\right)\frac{\partial x^{i}}{\partial x'^{\mu}}\cdot\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{k}}{\partial x'^{\nu}}$$

$$+\ \Gamma^{l}_{mj}\Gamma^{m}_{in}\cdot\frac{\partial x^{i}}{\partial x'^{\mu}}\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{k}}{\partial x'^{\nu}}\ \cancel{\Gamma^{l}_{mj}\Gamma^{m}_{in}\frac{\partial x^{i}}{\partial x'^{\mu}}\frac{\partial x^{j}}{\partial x'^{\sigma}}\frac{\partial x^{n}}{\partial x'^{\mu}}}$$

$$-\ \Gamma^{l}_{ij}\Gamma^{i}_{mn}\cdot\frac{\partial x^{j}}{\partial x'^{\nu}}\cdot\frac{\partial x^{m}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{n}}{\partial x'^{\mu}}$$

$$\cancel{l \rightleftarrows m \to \Gamma^{l}_{im}\Gamma^{i}_{jn}\frac{\partial x^{m}}{\partial x'^{\nu}}\frac{\partial x^{j}}{\partial x'^{\sigma}}\frac{\partial x^{n}}{\partial x'^{\mu}}}$$

$$\cancel{m \to k \to \Gamma^{l}_{ik}\Gamma^{i}_{jn}\frac{\partial x^{k}}{\partial x'^{\nu}}\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{n}}{\partial x'^{\mu}}}$$

$$\cancel{n \rightleftarrows i \to \Gamma^{l}_{nk}\Gamma^{n}_{ij}\frac{\partial x^{k}}{\partial x'^{\nu}}\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{i}}{\partial x'^{\mu}}}$$

$$\cancel{\therefore R'^{\lambda}_{\mu\sigma\nu} = \Gamma^{l}_{ijk}} \qquad \cancel{i \rightleftarrows m \ -\Gamma^{l}_{mj}\Gamma^{m}_{in}\frac{\partial x^{j}}{\partial x'^{\nu}}\frac{\partial x^{i}}{\partial x'^{\sigma}}\frac{\partial x^{n}}{\partial x'^{\mu}}}$$

$$i \to k,\ -\Gamma^{l}_{ik}\Gamma^{i}_{mn}\frac{\partial x^{k}}{\partial x'^{\nu}}\frac{\partial x^{m}}{\partial x'^{\sigma}}\frac{\partial x^{n}}{\partial x'^{\mu}}$$

$$m \to j \to -\Gamma^{l}_{ik}\Gamma^{i}_{jn}\frac{\partial x^{k}}{\partial x'^{\nu}}\cdot\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{n}}{\partial x'^{\mu}}$$

$$i \rightleftarrows n, \to -\Gamma^{l}_{nk}\Gamma^{n}_{ij}\frac{\partial x^{k}}{\partial x'^{\nu}}\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{i}}{\partial x'^{\mu}}$$

$$n \to m \to -\Gamma^{l}_{mk}\Gamma^{m}_{ij}\frac{\partial x^{i}}{\partial x'^{\mu}}\cdot\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{k}}{\partial x'^{\mu}}$$

ie $$R'^{\lambda}_{\mu\sigma\nu}\cdot\frac{\partial x^{l}}{\partial x'^{\lambda}} = \Gamma^{l}_{ijk}\frac{\partial x^{i}}{\partial x'^{\mu}}\cdot\frac{\partial x^{j}}{\partial x'^{\sigma}}\frac{\partial x^{k}}{\partial x'^{\nu}}$$

$$R'^{\lambda}_{\mu\sigma\nu}\frac{\partial x^{l}}{\partial x'^{\lambda}}\cdot\frac{\partial x'^{\alpha}}{\partial x^{l}} = \Gamma^{l}_{ijk}\frac{\partial x'^{\alpha}}{\partial x^{l}}\cdot\frac{\partial x^{i}}{\partial x'^{\mu}}\frac{\partial x^{j}}{\partial x'^{\sigma}}\cdot\frac{\partial x^{k}}{\partial x'^{\nu}}$$

$$R'^{\alpha}_{\mu\sigma\nu} = \ldots$$

$$R'^{\lambda}_{\mu\sigma\nu} = \Gamma^{l}_{ijk}\frac{\partial x'^{\lambda}}{\partial x^{l}}\ldots$$

(ie tensor Character)

$$\frac{\partial^{2} x^{l}}{\partial x'^{\mu}\partial x'^{\nu}} = \Gamma'^{\lambda}_{\mu\nu}\frac{\partial x^{l}}{\partial x'^{\lambda}} - \Gamma^{l}_{ij}\frac{\partial x^{i}}{\partial x'^{\mu}}\frac{\partial x^{j}}{\partial x'^{\nu}} \quad \text{—— A}$$

$$\Gamma'_{\sigma,\mu\nu} = \Gamma_{k,ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\cdot\frac{\partial x^k}{\partial x'^\sigma} + g_{ij}\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial^2 x^j}{\partial x'^\nu \partial x'^\mu}$$

$$g'^{\sigma\lambda}\Gamma'_{\sigma,\mu\nu}\frac{\partial x^l}{\partial x'^\lambda} = \Gamma_{k,ij}\, g'^{\sigma\lambda}\frac{\partial x^l}{\partial x'^\lambda}\cdot\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\cdot\frac{\partial x^k}{\partial x'^\sigma} + g_{ij}\, g'^{\sigma\lambda}\frac{\partial x^l}{\partial x'^\lambda}\cdot\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial^2 x^j}{\partial x'^\mu \partial x'^\nu}$$

$$\Gamma'^\lambda_{\mu\nu}\frac{\partial x^l}{\partial x'^\lambda} = g^{kl}\,\Gamma_{k,ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} + g_{ij}\, g^{li}\frac{\partial^2 x^j}{\partial x'^\mu \partial x'^\nu}$$

$$= \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} + \frac{\partial^2 x^l}{\partial x'^\mu \partial x'^\nu}$$

$$\frac{\partial^2 x^l}{\partial x'^\mu \partial x'^\nu} = \Gamma'^\lambda_{\mu\nu}\frac{\partial x^l}{\partial x'^\lambda} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu} \quad \text{—— (1)}$$

Similarly $$\frac{\partial^2 x^l}{\partial x'^\mu \partial x'^\sigma} = \Gamma'^\lambda_{\mu\sigma}\frac{\partial x^l}{\partial x'^\lambda} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\sigma} \quad \text{—— (2)}$$

Differentiating (1) w.r.t x'^σ & (2) w.r.t x'^ν & subtracting

$$0 = \frac{\partial}{\partial x'^\sigma}\left(\Gamma'^\lambda_{\mu\nu}\frac{\partial x^l}{\partial x'^\lambda}\right) - \frac{\partial}{\partial x'^\sigma}\left(\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\right)$$

$$- \frac{\partial}{\partial x'^\nu}\left(\Gamma'^\lambda_{\mu\sigma}\frac{\partial x^l}{\partial x'^\lambda}\right) + \frac{\partial}{\partial x'^\nu}\left(\Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\right)$$

ie $$\frac{\partial}{\partial x'^\sigma}\left(\Gamma'^\lambda_{\mu\nu}\right) + \Gamma'^\lambda_{\mu\nu}\frac{\partial^2 x^l}{\partial x'^\sigma \partial x'^\lambda} - \frac{\partial}{\partial x'^\nu}\left(\Gamma'^\lambda_{\mu\sigma}\right) - \Gamma'^\lambda_{\mu\sigma}\frac{\partial^2 x^l}{\partial x'^\nu \partial x'^\lambda}$$

$$- \frac{\partial}{\partial x'^\sigma}\left(\Gamma^l_{ij}\right) - \Gamma^l_{ij}\frac{\partial}{\partial x'^\sigma}\left(\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\right) + \frac{\partial}{\partial x'^\nu}\left(\Gamma^l_{ij}\right) + \Gamma^l_{ij}\frac{\partial}{\partial x'^\nu}\left(\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\sigma}\right) = 0$$

=

$$- \frac{\partial}{\partial x'^\sigma}\left(\Gamma^l_{ij}\right) - \Gamma^l_{ij}\left(\frac{\partial^2 x^i}{\partial x'^\sigma \partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\nu}\right) + \frac{\partial}{\partial x'^\nu}\left(\Gamma^l_{ij}\right) + \Gamma^l_{ij}\left(\frac{\partial^2 x^i}{\partial x'^\nu \partial x'^\mu}\cdot\frac{\partial x^j}{\partial x'^\sigma}\right)$$

$$- \Gamma^l_{ij}\cancel{\left(\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial^2 x^j}{\partial x'^\sigma \partial x'^\nu}\right)} + \Gamma^l_{ij}\cancel{\left(\frac{\partial x^i}{\partial x'^\mu}\cdot\frac{\partial^2 x^j}{\partial x'^\nu \partial x'^\sigma}\right)} = 0$$

ie $$\frac{\partial}{\partial x'^\sigma}\left(\Gamma'^\lambda_{\mu\nu}\right) + \Gamma'^\lambda_{\mu\nu}\left\{\Gamma'^\rho_{\lambda\sigma}\frac{\partial x^l}{\partial x'^\rho} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\sigma}\cdot\frac{\partial x^j}{\partial x'^\lambda}\right\} - \frac{\partial}{\partial x'^\nu}\left(\Gamma'^\lambda_{\mu\sigma}\right)$$

$$- \Gamma'^\lambda_{\mu\sigma}\left\{\Gamma'^\rho_{\nu\lambda}\frac{\partial x^l}{\partial x'^\rho} - \Gamma^l_{ij}\frac{\partial x^i}{\partial x'^\nu}\cdot\frac{\partial x^j}{\partial x'^\lambda}\right\}$$

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} \qquad \frac{\partial g}{\partial x^r} = \begin{vmatrix} \frac{\partial g_{11}}{\partial x^r} & \frac{\partial g_{12}}{\partial x^r} & \frac{\partial g_{13}}{\partial x^r} \\ - & - & \\ - & & \end{vmatrix} + \begin{vmatrix} - & & \\ \frac{\partial g_{21}}{\partial x^r} & - & \\ & & \end{vmatrix} + \begin{vmatrix} - & & \\ & & \\ \frac{\partial g_{31}}{\partial x^r} & & \end{vmatrix}$$

$$\frac{\partial g}{\partial x^l} =$$

$$g \Big[g^{11} \frac{\partial g_{11}}{\partial x^r} + g^{12} \frac{\partial g_{12}}{\partial x^r} + g^{13} \frac{\partial g_{13}}{\partial x^r} + g^{21} \frac{\partial g_{21}}{\partial x^r} + g^{22} \frac{\partial g_{22}}{\partial x^r} + g^{23} \frac{\partial g_{23}}{\partial x^r} + g^{31} \frac{\partial g_{31}}{\partial x^r} + g^{32} \frac{\partial g_{32}}{\partial x^r} + g^{33} \frac{\partial g_{33}}{\partial x^r} \Big]$$

$$g^{ij} = \frac{\text{cof}}{g}$$

$$\frac{1}{2} g^{-\frac{1}{2}} \frac{\partial g}{\partial x^l} \qquad = g\, g^{ij} \frac{\partial g_{ij}}{\partial x^l}.$$

$$\frac{1}{2} g^{is} \frac{\partial g_{is}}{\partial x^r} = \frac{1}{2g} \cdot \frac{\partial g}{\partial x^r} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^r}$$

$$\frac{1}{2} g^{-2}.$$

$$\frac{\partial \log \sqrt{g'}}{\partial x'^r} = \frac{\partial \log \sqrt{g}}{\partial x^j} \frac{\partial x'^r}{\partial x^j} \quad \frac{\partial \log \sqrt{g}}{\partial x^j} \cdot \frac{\partial x^j}{\partial x'^r}$$

$$\Gamma'^{\lambda}_{\mu\nu} = g'^{\rho\lambda} \Gamma'_{\rho,\mu\nu} = \frac{1}{2} g'^{\rho\lambda} \left\{ \frac{\partial g'_{\rho\mu}}{\partial x'^{\nu}} + \frac{\partial g'_{\rho\nu}}{\partial x'^{\mu}} - \frac{\partial g'_{\mu\nu}}{\partial x'^{\rho}} \right\}$$

$$= \frac{1}{2} g'^{\rho\lambda} \left\{ \frac{\partial}{\partial x'^{\nu}} \left(g \right.\right.$$

$$R_{hijk} = g_{lh} R^{l}_{ijk} \qquad R^{l}_{ijk} = g^{lh} R_{hijk}$$

also a tensor!

B_1

$$R_{hijk} = g_{lh} R^{l}_{ijk} = g_{lh} \left\{ \frac{\partial}{\partial x^j} \left(\Gamma^{l}_{ik} \right) - \frac{\partial}{\partial x^k} \left(\Gamma^{l}_{ij} \right) + \Gamma^{m}_{ik} \Gamma^{l}_{mj} - \Gamma^{m}_{ij} \Gamma^{l}_{mk} \right\}$$

$$g_{lh} \Gamma^{l}_{ij} = g_{lh} g^{lk} \Gamma$$

(2)

Lecture notes for 21/11/67. Principle of equivalence – gr. mass = inertial mass – Energy & mass & wt.

(1) Three principles of general relativity – for light rays in addition $ds^2 = 0$.

(2) Derivation of Einstein's gravitational equations.

[Generalisation of $\nabla^2 U = -4\pi\gamma\rho$

leading to a relation between R_{ij} & T_{ij}

R_{ij} generalising $\nabla^2 U$ ~~and~~ & T_{ij} generalising ρ

$$T^l_{j,l} = 0. \qquad R^l_{j,l} = \frac{1}{2}\frac{\partial R}{\partial x^j}$$

Connect[ed] T^l_j & $R^l_j - \frac{1}{2} g^l_j R$ $\qquad \left(R^l_j - \frac{1}{2} g^l_j R\right)_{,l} = 0.$

Also from no. of components viz. 10 this connection would be correct.

Linearity in second order derivatives (corresponding to $\nabla^2 U$) means R^l_j & $g^l_j R$.

Also $g^l_{j,l} \equiv 0$, but in absence of masses $F = (m)_{in} \alpha = 0$ & hence

$F = (m_{gr})\, \mathrm{grad}\, U = 0$ i.e. grad $U = 0$. i.e. $R_{kj} = 0$ [Cosmological term]

$$R_{ij} - \frac{1}{2} g_{ij} R = -K T_{ij}$$

$$g^{ik} g_{ik} = n = 4.$$

Contraction $\rightarrow$ $R - \frac{4}{2} R = -KT$

$R = +KT.$

$R - 2R$

$\cancel{\frac{1}{2} g^{l}_{j} R^l_j}$

$$g^{il} R_{ij} - \frac{1}{2} g^{il} g_{ij} R = -K g^{il} T_{ij}$$

$\cancel{R^l_j - \frac{1}{2}\delta^l_j R = -K T^l_j}$

$\cancel{g^l_l R^l_j - \frac{1}{2} g^l_l \delta^l_j R = -K g^l_l T^l_j}$

$$R_{ij} = -K\left(T_{ij} - \frac{1}{2} g_{ij} T\right).$$

R_{ij}

$$R_{44} - \frac{1}{2} g_{44} g^{ik} R_{ik}$$

$$R_{44} - \frac{1}{2} g_{44} g^{44} R_{44} = \frac{1}{2} R_{44}$$

$$= -\frac{1}{2} R$$

$V = \frac{\gamma M}{r}$.

V = gravitational potential

$F = m\alpha$. $\alpha \propto \frac{1}{m}$. F

$\nabla^2 V = -4\pi\gamma\rho$

$\alpha = \text{grad}\, V$.

$\underline{(m_{\text{grav}}) = \frac{F}{\text{grad}\, V} \cdot (m_{\text{inert}}) = \frac{F}{\alpha}}$.

Energy-momentum tensor – for a particle in special relativity

$T_{ik} = \mu_0 u_i u_k$

$\frac{\partial(\mu_0 u^k)}{\partial x^k} = 0$

$\underline{\frac{\partial T_{ik}}{\partial x^k} = f_i}$.

$T^i_{\gamma, i} = 0$.

Lecture on 21/11/67.

(1) Principle of equivalence – Equivalence of inertial & gravitational mass.

Gravitational potential $\overline{U}$. ($\overline{U} = \frac{\gamma M}{r}$, $\nabla^2 \overline{U} = -4\pi\gamma\rho$ i.e. Poisson eqn) – $F = (m)_{in} \cdot \vec{\alpha}$.

$F = (m)_{gr}$ grad $\overline{U}$. – gravitational field property of all masses moving in same manner if initial condns etc. be same – $(m)_{in} = (m)_{gr} \rightarrow \vec{\alpha} = \text{grad}\, \overline{U}$. ($\overline{U}$ is fund. ... in gravitational theory & in defining $(m)_{gr}$.

(2) Fundamental postulates of gen. relativity

(1) Covariance of physical laws in all coord. systems, not necessarily in rel. uniform motion leading to Euclidean geometry → Riemannian geometry

(2) Special relativity valid in a local system where gravitational field vanishes

(3) The path of a test-particle in a purely gravitational field is given by

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\cdot\frac{dx^k}{ds} = 0, \quad \text{where } ds^2 = g_{ik}\,dx^i dx^k \quad (S/c = \text{proper time})$$

and in particular for a ray of light $ds^2 = 0$.

(3) Derivation of Einsteins gravitational equations.

(1) Should be a generalisation of $\nabla^2 \overline{U} = -4\pi\gamma\rho$ (Newtons theory)

Since potentials are g_{ik}'s themselves, L.H.S must be a tensor linear in linear the second derivatives of the g_{ik}'s. R.H.S containing ρ or equivalently energy must contain the second order energy-momentum tensor T_{ik} ($T_{ki} = T_{ik}$).

(2) T_{ik} has ten components & so should the tensor on the L.H.S, R_{hijk} will not do because it has 20 components. Hence R_{ik} (the Ricci tensor) and $g R_{ik}$ and g_{ik} are possible tensors or a linear combination of them.

(3) $T^l_{j,l} = 0$ i.e. $\nabla T_{ik} = 0$ (expressing momentum & energy conservation laws) & hence so should that linear combination have its covariant derivative $= 0$

Consider $R^l_{j,l}$. By Bianchi's identity this $= \frac{1}{2}\frac{\partial R}{\partial x^j}$ (as has been proved)

$$\text{Consider } R^l_j - \frac{1}{2}g^l_j R = G^l_j \text{ say } G^l_{j,l} = R^l_{j,l} - \frac{1}{2}g^l_j\frac{\partial R}{\partial x^l}$$

$$= R^l_{j,l} - \frac{1}{2}\frac{\partial R}{\partial x^j} = 0$$

~~Hence~~ since ~~$g_{ij,l}$~~ $g^l_{j,l} = 0$

4) This leads to putting eqns in the form

$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R = -\kappa T_{ij} \quad (1)$$

(4) Also A term like $c g_{ij}$ could be put on the L.H.S (called the cosmological term) since $g_{ij,l} = 0$., but this would mean that in the absence of masses (i.e. $T_{ij} = 0$), $G_{ij} \neq 0$. Hence this term is discarded in non-cosmological general relativity

(5) Contracting (1) we have $R - \frac{1}{2}g_{ij}g^{ij}R = -\kappa g^{ij}T_{ij}$ i.e. $R - 2R = -\kappa T$

ie $R = KT$

and eqns become $R_{ij} = -KT_{ij} + \frac{1}{2} g_{ij} KT$

$$= -K\left(T_{ij} - \tfrac{1}{2} g_{ij} T\right) \quad \text{— (2)}$$

(4) Newton's theory as a first approximation

$g^{44} = g_{44} = -1$

Consider the 44-component of the eqn $G_{ij} = -K T_{ik}$

& take $T_{ik} = \mu_0 u_i u_k$ (for a particle) (If quantities of order u/c are neglected, all

components of T_{ik} vanish except T_{44} [for eg $\mu_0 u_1 u_2 = \mu_0 \left(\frac{u_1}{c}\right)\left(\frac{u_2}{c}\right) c^2 = 0$, $\mu_0 u_1 u_4 = \mu_0 \left(\frac{u_1}{c}\right) c^2$ $[u_4 = c]$

& $T_{44} = \mu_0 c^2$ & $T = g^{ik} T_{ik} = g^{44} T_{44} = -\mu_0 c^2$

~~& hence $T = g^{ik} T_{ik} = g^{44} T_{44} = -\mu_0 c^2$~~ & hence from (2) $R_{44} = -K(\mu_0 c^2 - \frac{1}{2} g_{44} T)$

$$= -K(\mu_0 c^2 - \tfrac{1}{2}\mu_0 c^4)$$

$$= -\tfrac{1}{2} K \mu_0 c^2 \quad \text{— (3)}$$

From the expn for

$$R_{ij} = \frac{\partial \Gamma^\alpha_{i\alpha}}{\partial x^j} - \frac{\partial \Gamma^\alpha_{ij}}{\partial x^\alpha} + \Gamma^\beta_{i\alpha} \Gamma^\alpha_{j\beta} - \Gamma^\alpha_{ij} \Gamma^\beta_{\alpha\beta}$$

$\mu_0 = \epsilon/c^2$

since we neglect time-derivatives & products of Γ^i_{rs} are neglected.

~~R_{44}~~ $-K$

$$R_{44} = -\frac{\partial \Gamma^\alpha_{44}}{\partial x^\alpha} \simeq -\frac{\partial \Gamma_{\alpha,44}}{\partial x^\alpha} \simeq +\frac{1}{2}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial g_{44}}{\partial x^\alpha}\right)$$

$\frac{1}{2}\left(\frac{\partial g_{\alpha 4}}{\partial x^4} + \frac{\partial g_{\alpha 4}}{\partial x^4} - \frac{\partial g_{44}}{\partial x^\alpha}\right)$

$$R_{44} = \frac{1}{2}\sum_\alpha \frac{\partial^2 g_{44}}{\partial x_\alpha^2} = \frac{1}{2}\nabla^2 g_{44} \qquad \text{\& using } g_{44} = -1 - \frac{2\phi}{c^2}$$

$$= -\frac{\nabla^2 \phi}{c^2}$$

[when $\phi = 0$, $g_{44} = -1$ ie that of special relativity]

$$\therefore \nabla^2 \phi = \tfrac{1}{2} K c^4 . \mu_0, \text{ using (3)} \quad \text{— (4)}$$

ie Poisson's eqn is valid. This is a great achievement of relativity [~~& also explains taking -ve sign in eqn (1)~~] deriving Newton's law of gravitation on the only hypothesis of $G_{ij} = -K T_{ij}$ (in field eqns of gravitation no other material quantities except T_{ij} shall occur)

Cf (3) with $\nabla^2 \phi = 4\pi \gamma \mu_0$ $\quad \frac{1}{2} K c^4 = 4\pi\gamma$ or $K = K c^2 = \frac{8\pi\gamma}{c^2}$ ie K is +ve justifying use of -ve sign. K is not given by theory (since γ in classical mechanics) but by expts.

$\Gamma^\alpha_{44} \supset \quad \Gamma^i_{rs} = g^{ik} \Gamma_{k,rs}$

$$\Gamma_{i,rs} = \frac{1}{2}\left(\frac{\partial g_{ir}}{\partial x^s} + \frac{\partial g_{is}}{\partial x^r} - \frac{\partial g_{rs}}{\partial x^i}\right)$$

$$\Gamma^\alpha_{44} = g^{\alpha k} \Gamma_{k,44}$$

$$\Gamma_{k,44} = \frac{1}{2}\left(\cancel{\frac{\partial g_{k4}}{\partial x^4}} + \cancel{\frac{\partial g_{k4}}{\partial x^4}} - \frac{\partial g_{44}}{\partial x^k}\right)$$

$$= -\frac{1}{2}\frac{\partial g_{44}}{\partial x^k}$$

$$\Gamma^\alpha_{44} = -\frac{1}{2} g^{\alpha k}\frac{\partial g_{44}}{\partial x^k} = -\frac{1}{2}\left(g^{\alpha 1}\frac{\partial g_{44}}{\partial x^1} + g^{\alpha 2}\frac{\partial g_{44}}{\partial x^2} + g^{\alpha 3}\frac{\partial g_{44}}{\partial x^3}\right) \qquad \frac{\partial g_{44}}{\partial x^4} \simeq 0$$

$$\Gamma^{\alpha}_{44} \simeq \Gamma_{\alpha,44}$$

$$g^{\alpha\beta}\Gamma_{\beta,44} = \Gamma_{\alpha,44}$$

$$g^{\alpha 1}\Gamma_{1,44} + g^{\alpha 2}\Gamma_{2,44} + g^{\alpha 3}\Gamma_{3,44} + g^{\alpha 4}\Gamma_{4,44}$$

$$= \frac{1}{2}\left(\frac{\partial g_{\alpha 4}}{\partial x^4} + \frac{\partial g_{\alpha 4}}{\partial x^4} - \frac{\partial g_{44}}{\partial x^\alpha}\right)$$

$$= -\frac{1}{2}\left\{g^{\alpha 1}\frac{\partial g_{44}}{\partial x^1} + g^{\alpha 2}\frac{\partial g_{44}}{\partial x^2} + g^{\alpha 3}\frac{\partial g_{44}}{\partial x^3}\right\}$$

$\cancel{g_{ik} = \mp 1}$

$$g_{11} = g_{22} = g_{33} = 1 = -g_{44}$$

$$g_{ik} = 0 \ (i \neq k)$$

$$\Gamma_{1,44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_1}$$

$$\Gamma_{2,44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x^2}$$

$$\Gamma_{3,44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x^3}$$

$$\Gamma_{4,44} = 0$$

$$g^{11}\Gamma_{1,44} + g^{12}\Gamma_{2,44} + g^{13}\Gamma_{3,44} + g^{14}\Gamma_{4,44}$$

$$= g^{11}\Gamma_{1,44} = \Gamma_{1,44}.$$

$$g^{21}\Gamma_{1,44} + g^{22}\Gamma_{2,44} + \cdots = \Gamma_{2,44}$$

$$g^{31}\Gamma_{1,44} + \cdots = \Gamma_{3,44}$$

$$g^{41}\Gamma_{1,44} + \cdots = 0$$

$g^{\alpha\beta} = \delta^{\alpha\beta}$

except when $\alpha = \beta = 4$

$g^{44} = -1$

Moreover; $g^{\alpha\beta} = \delta^{\alpha\beta}$ for $(\alpha, \beta = 1, 2, 3)$: $g^{\alpha 4}\Gamma_{4,44}$ = for $\alpha = 4$ gives 0

$$\delta^{\alpha\beta}\Gamma_{\beta,44} = \Gamma_{\alpha,44}$$

$\Gamma_{4,44} = 0$

$$\frac{dx^i}{ds} = \frac{dx^i}{dt}\cdot\frac{dt}{ds}$$

$$\frac{d^2x^i}{ds^2} = \frac{d}{dt}(\ )\cdot\frac{dt}{ds} = \left\{\frac{d^2x^i}{dt^2}\cdot\frac{dt}{ds} + \frac{dx^i}{dt}\cdot\frac{d}{dt}\left(\frac{dt}{ds}\right)\right\}\frac{dt}{ds}.$$

$$ds^2 = g_{ik}\,dx^i dx^k = dx'^2 + dx^2 + dx^3 - dx^{4\,2}$$

$$= \qquad {}'' \qquad -c^2\cdot dt^2$$

$g^{\alpha\beta}\Gamma_\beta$

(1) Red shift of spectral lines.

$$s = ic\tau, \quad t = x^4/c$$
$$x^4 = ct$$

$$\tau = \frac{s}{c} \qquad ds = \sqrt{-g_{44}}\, dx^4$$

$$c\,d\tau = \sqrt{-g_{44}}\, c\,dt, \qquad g_{44} = -1 - \frac{2\phi}{c^2}.$$

$ds =$

$$t = \frac{\tau}{\sqrt{\left(1+\frac{2\phi}{c^2}\right)}} = \cancel{2}\,\tau\left(1 - \frac{\phi}{c^2}\right) \qquad \left(1+\frac{2\phi}{c^2}\right)^{-\frac{1}{2}}$$

$$\frac{t-\tau}{\tau} = -\frac{\phi}{c^2}$$

$$\frac{\Delta t}{\tau} = -\frac{\phi}{c^2}. \quad \text{(synchronism clocks having lost).}$$

vibration possessing light as a clock → spectral line in Sun observed on the Earth from

$$\frac{\Delta\nu}{\nu} = \frac{\phi_E - \phi_S}{c^2}. \qquad \phi = -\frac{\gamma M}{R}$$

$$\frac{\Delta\nu}{\nu} = -\frac{\gamma M}{Rc^2}$$

(2) Perihelion precession of Mercury.

$$\frac{\dot u_x}{(1-\dot u_x^2/c^2)^{3/2}} = b.$$

$$\frac{d}{dt}\left(1-u_x^2/c^2\right)^{-\frac12} = -\tfrac12\left(1-u_x^2/c^2\right)^{-3/2}\cdot -2u_x\cdot\dot u_x/c^2$$

$$= \frac{u_x\dot u_x}{c^2(1-u_x^2/c^2)^{3/2}}$$

$$\frac{u_x\dot u_x}{c^2(1-u_x^2/c^2)^{3/2}} = \frac{b}{c^2}\cdot\dot u_x$$

$$\left(1-u_x^2/c^2\right)^{-\frac12} = \frac{b}{c^2}(u_x + k).$$

$$\frac{1}{\sqrt{1-u_x^2/c^2}} = \frac{b(x-x_0)}{c^2}.$$

$$\frac{c^2}{b} = u_x(1-u_x^2/c^2)$$

$$\frac{1}{1-u_x^2/c^2} = \frac{b^2u_x^2}{c^4} \qquad \frac{c^4}{b^2} = u_x^2(1-u_x^2/c^2)$$

$u_x = k$

$x = kt + k$

$x - x_0 = k(t-t_0)$

$$1-\left(\frac{dx}{dt}\right)^2/c^2$$

$$\frac{c\,dt}{\sqrt{c^2dt^2-dx^2}} = \frac{b}{c^2}\cdot\frac{dx}{dt}$$

$$\sqrt{dx^2-c^2dt^2} = ds = ic\,d\tau.$$

$$-i\frac{dt}{d\tau} = \frac{b}{c^2}\cdot\frac{dx}{dt}$$

$$\frac{c^2}{b} = (x-x_0)\sqrt{1-u_x^2/c^2}$$

$$\frac{c^4}{b^2} = (x-x_0)^2 - (x-x_0)^2(1-u_x^2/c^2)$$

$$\frac{c}{\sqrt{c^2-\left(\frac{dx}{dt}\right)^2}} = \frac{b}{c^2}\cdot u_x$$

$$\frac{1}{1-u_x^2/c^2} = \frac{b^2}{c^4}\cdot u_x^2 \qquad \frac{c^2}{c^2-u_x^2} = \frac{b}{c^2}\cdot u_x^2 \qquad \frac{c}{b} = u_x^2$$

$$\frac{c^4}{b^2} = u_x^2(1-u_x^2/c^2)$$

$$\frac{1}{\sqrt{1-u_x^2/c^2}} = \frac{b}{c^2}(x-x_0)$$

$$(x-x_0)^2\left(c^2 - \frac{u_x^2}{c^2}\right) = (t-t_0)^2.$$

$$1-\beta^2 = 1-u^2/c^2 = \frac{c^2-u^2}{c^2}$$

$$(x-x_0)^2(1-u_x^2/c^2) = c^4/b^2$$

$(x-x_0)^2 -$

$$\frac{d^2x}{dt^2}(x-x_0) = c^2 - \left(\frac{dx}{dt}\right)^2$$

$$= c^2 - u^2 = c^2(1-\beta^2)$$

$$dx\sqrt{1-\beta^2} = dt$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{1-\beta^2}}$$

$$u_x\sqrt{1-u_x^2/c^2}$$

$$\frac{x}{2}(1-u_x^2/c^2)^{-\frac12}\cdot\frac{2u_x\dot u_x}{c^2}$$

$$\frac{d^2x}{dt^2} = b$$

$$\frac{dx}{dt} = bt + c \qquad x = \tfrac12 bt^2 + ct + d$$

$$\frac{c^2}{c^2-u_x^2} = \frac{b^2}{c^4}(x-x_0)^2 = A \qquad A(c^2-u_x^2) = c^2$$

$$Au_x^2 = c^2(A-1) \qquad u_x^2 = \frac{c^2(A-1)}{A}$$

$$u_x = c\sqrt{\frac{A-1}{A}}$$

$$\frac{dx}{dt} = c\sqrt{\frac{A-1}{A}}$$

$$\int\frac{x}{\sqrt{x^2-a^2}}$$

$$\frac{\frac{b^2}{c^4}(x-x_0)^2 - 1}{\frac{b^2}{c^4}(x-x_0)^2} = \frac{b^2(x-x_0)^2 - c^4}{b^2(x-x_0)^2}$$

$$\frac{d}{dt} = \int \frac{b^2(x-x_0)}{\ldots}$$

$$(x-x_0)^2 - \frac{c^4}{b^2} = \ldots$$

Accn: $B^i = \frac{du^i}{d\tau} = \frac{d^2x^i}{d\tau^2}$ $\qquad s^2 = x^i x_i$

$$u^i u_i = \frac{u^2}{1-u^2/c^2} - \frac{c^2}{1-u^2/c^2} = \frac{u^2-c^2}{1-u^2/c^2} = -c^2 \qquad 2s\frac{ds}{d\tau} = \frac{dx^i}{d\tau}x_i + x^i\frac{dx_i}{d\tau}$$

$d\tau = \frac{dt}{\sqrt{1-\beta^2}}$

$$u^i \frac{du_i}{d\tau} + u_i \frac{du^i}{d\tau} = 0 \quad \text{ie } u_i \frac{du^i}{d\tau} = 0 \text{ or } u_i B^i = 0$$

$$(B^1, B^2, B^3) = \frac{d\vec{u}}{d\tau} = \frac{d\vec{u}}{dt}\cdot\frac{dt}{d\tau} = \left\{\frac{\dot{\vec{u}}}{\sqrt{1-u^2/c^2}} + \vec{u}\,\tfrac{1}{2}(1-u^2/c^2)^{-3/2}\left(-\frac{2\vec{u}\dot{\vec{u}}}{c^2}\right)\right\}\frac{dt}{d\tau}$$

$$= \frac{\dot{\vec{u}}}{1-\beta^2} + \vec{u}\,\frac{(\vec{u}\cdot\dot{\vec{u}})}{c^2}\cdot\frac{1}{(1-\beta^2)^2} \qquad [\beta = u/c]$$

$$B^4 = ic\frac{d}{dt}\cdot\frac{1}{\sqrt{1-u^2/c^2}}\cdot\frac{dt}{d\tau} = \left\{ic\cdot\left[-\tfrac{1}{2}(1-u^2/c^2)^{-3/2}\left(-2\vec{u}\dot{\vec{u}}/c^2\right)\right]\right\}\frac{1}{\sqrt{1-u^2/c^2}}$$

$$= i\,\frac{\vec{u}\cdot\dot{\vec{u}}}{c}\cdot\frac{1}{(1-\beta^2)^2}$$

K'. $(B'^1, B'^2, B'^3 = \dot{\vec{u}}', B'^4 = 0)$ Rel to K matter moves with vel $\vec{u}$.

K' moves instantaneously with the medium. Taking this as the x-direction we have in K

$$B^1 = \frac{\dot{u}_x}{1-\beta^2} + \frac{\beta}{i}B^4, \quad B^2 = \frac{\dot{u}_y}{1-\beta^2}, \quad B^3 = \frac{\dot{u}_z}{1-\beta^2}.$$

Inverse of this given by

$u'^1 \sin\phi + u'^4 \cos\phi = u^4$

$u'^1 \cos\phi - u'^4 \sin\phi = u^1$

~~$u^4 = \frac{u'^1 + i\beta u'^4}{\sqrt{1-\beta^2}}$~~

$u^4 = \frac{i\beta u'^1 + u'^4}{\sqrt{1-\beta^2}}$

$u^1 = \frac{u'^1 - i\beta u'^4}{\sqrt{1-\beta^2}}$

$$B^1 = \frac{B'^1}{\sqrt{1-\beta^2}}, \quad B^2 = B'^2, \quad B^3 = B'^3, \quad B^4 = \frac{i\beta B'^1}{\sqrt{1-\beta^2}}$$

$$B^1 = \frac{\dot{u}_x}{1-\beta^2} + \frac{\beta}{i}\cdot\frac{i\beta B'^1}{\sqrt{1-\beta^2}}$$

$$\dot{u}'_x(1-\beta^2) = \frac{\dot{u}_x}{1-\beta^2} \quad \text{ie } \dot{u}_x = \dot{u}'_x(1-\beta^2)^{3/2}$$

$$\dot{u}_y = \dot{u}'_y(1-\beta^2)$$

$$\dot{u}_z = \dot{u}'_z(1-\beta^2)$$

$$\frac{d^2x_0}{dt^2} - b = 0 \qquad \frac{dx}{dt} - bt = c \qquad \frac{d(x_0-x_0)}{dt} = b(t-t_0)$$

$1 - u^2/c^2$

~~$dx = bt\,dt + c\,dt$.~~

~~$x - x_0 = b\frac{t^2}{2} + ct + x_0$~~

$$\frac{\frac{du_x}{dt}\cdot\dot{u}_x}{(1-\beta^2)^{3/2}} = b\cdot\dot{u}_x$$

$\frac{du}{dt}$

$2\frac{du}{dt} - 2c^2$

$\frac{d^2x}{dt^2}(x-x_0) = c^2 - \left(\frac{dx}{dt}\right)^2$

$2(x-x_0)\frac{dx}{dt} - c^2(t-t_0) = \text{Const}$ $\quad (1-\beta^2)^{-1/2}$

$$(1-\beta^2)^{-1/2} = b(u-u_0) + c$$

$-\frac{1}{2}(1-\beta^2)^{-3/2}\cdot\frac{-2u\dot{u}}{c^2}$

$$b^2(u-u_0)^2 = \frac{1}{1-u^2/c^2} = \frac{c^2}{c^2-u^2}.$$

$\frac{d^2x}{dt^2}(x-x_0)$

$u\dot{u}$ $\quad (1-\beta^2)^{3/2}$

$$\frac{d^2x}{dt^2}(x-x_0) = c^2 - u^2$$

Lecture for 6/2/68

(1) Dirac Equation:

(a) The K.G. eqn can be written $(E^2 - c^2p^2 - m^2c^4)\psi = 0$, —(1) with E & $\vec{p}$ as operators. This is of second degree & does not hold for particles except spin 0. Dirac tried to get an equivalent linear eqn in the E's & p's in the form, for a particle of spin 1/2

$$(E + c\vec{\alpha}\cdot\vec{p} + \beta mc^2)\psi = 0. \quad \text{—(2)}$$

where $\vec{\alpha}(\alpha_x, \alpha_y, \alpha_z)$ & β are matrices with nos. as elements & commuting therefore with $\vec{p}, E, \vec{r}, t$, not necessarily commuting with each other. Multiplying (2) on the left by $E - c\vec{\alpha}\cdot\vec{p} - \beta mc^2$ & comparing with (1)

$$(E - c\alpha_x p_x - c\alpha_y p_y - c\alpha_z p_z - \beta mc^2)(E + c\alpha_x p_x + c\alpha_y p_y + c\alpha_z p_z + \beta mc^2)\psi = 0$$

or $$E^2 - c^2(\alpha_x^2 p_x^2 + \alpha_y^2 p_y^2 + \alpha_z^2 p_z^2) - c^2(\alpha_x\alpha_y + \alpha_y\alpha_x)p_x p_y - \dots \quad - \dots$$

$$- mc^3(\alpha_x\beta + \beta\alpha_x)p_x - \dots \quad - \dots$$

$$- \beta^2 m^2 c^4 = 0.$$

Comparing this with (1) we have the relations

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1; \quad \alpha_x\alpha_y + \alpha_y\alpha_x = 0, \ -\,-, \ -\,-$$
$$\alpha_x\beta + \beta\alpha_x = 0, \ -\,-, \ -\,-$$

Calling $\alpha_x, \alpha_y, \alpha_z, \beta$ as $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, all these relations can be put in a single eqn

$$\tfrac{1}{2}(\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu) = \delta_{\mu\nu} \quad (\mu, \nu = 1, 2, 3, 4) \quad \text{—(3)}$$

In view of (2) being a linear form of (1), we also have in the Dirac equation +ve as well as -ve energy states. Also the Dirac eqn was devised to include the spin 1/2 of the particle there should be two corresp. wave fns. Thus in (2) the wave fn ψ should have 4 components & consequently the α_μ's should be 4×4 matrices. To find a repn of the α_μ's, we set one

(b) of them viz $\alpha_4 = \beta$ diagonal & since $\beta^2 = 1$, the eigen-values of β are +1 & -1, and we set

$$\alpha_4 = \beta = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \text{ or schematically as } = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ where each 'element' is itself a 'matrix' with 2 rows & cols. Now setting $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ constitute the Pauli spin matrices, it can be shown by using the commutation relations satisfied by σ's that (2) is satisfied by the α's.

$$\left[\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \ \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1\right.$$

[P.T.O

$$\left.\sigma_x\sigma_y + \sigma_y\sigma_x = 0, \ \sigma_x\sigma_y - \sigma_y\sigma_x = 2i\sigma_z, \text{etc}\right]$$

Using these it follows $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \alpha_4^2 = 1$, $\alpha_1\alpha_2 + \alpha_2\alpha_1 = 0$, etc, and $\alpha_1\alpha_4 + \alpha_4\alpha_1 = 0$, etc.

(C) Results relating to algebras or hyper complex systems – A system over a commutative ring R

(1)

$$\alpha_4 = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & -1 & 0 \\ & & 0 & -1 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} & & 0 & 1 \\ & & 1 & 0 \\ 0 & 1 & & \\ 1 & 0 & & \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} & & 0 & -i \\ & & i & 0 \\ 0 & -i & & \\ i & 0 & & \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} & & 1 & 0 \\ & & 0 & -1 \\ 1 & 0 & & \\ 0 & -1 & & \end{pmatrix}$$

$$\alpha_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\left\{\begin{matrix} A_T A = 1 \\ A_T = A^{-1} \\ A A_T = 1 \end{matrix}\right\}$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$\Lambda^{-1} \beta^\mu a_{\mu\lambda} \Lambda = \beta^\lambda$

$\Lambda^{-1} \beta^\mu a_{\mu\lambda} a_{\nu\lambda} \Lambda = a_{\nu\lambda} \beta^\lambda$

(a)

$$a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda} \left\{ \begin{matrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \text{i.e. } A A_T = 1 \end{matrix} \right.$$

$\cancel{a_{1\nu} a_{1\lambda} = \delta_{\nu\lambda}}$

$\cancel{a_{2\nu} a_{2\lambda} = \delta_{\nu\lambda}}$

$x'_\mu x'_\mu = a_{\mu\nu} a_{\mu\lambda} x_\nu x_\lambda \rightarrow a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda}$

$a_{1\nu} a_{1\lambda} + a_{2\nu} a_{2\lambda} = \delta_{\nu\lambda}$

$A A_T = 1 \quad A_T = A^{-1} \quad a (\det A)^2 = 1$ [illegible] (a)

$\frac{\partial \xi'}{\partial x'_\mu} = a_{\mu\nu} \frac{\partial \xi}{\partial x_\nu}$

$\frac{\partial}{\partial x'_\mu} = a_{\mu\lambda} \frac{\partial}{\partial x_\lambda}$ $\quad a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda} \quad x'_\mu = a_{\mu\nu} x_\nu$

$\frac{\partial}{a_{\mu\lambda} \partial x_\lambda} = \left(\frac{\partial \xi'}{\partial x_\lambda} \cdot \frac{\partial x_\lambda}{\partial x'_\mu} \right)$

$\frac{\partial}{a_{\mu\lambda} \partial x_\nu} = a_{\mu\lambda} \cdot \frac{\partial}{\partial x_\lambda}$ [illegible]

$A = (a_{\mu\nu}) \quad \frac{\partial \xi'}{\partial x_\lambda} = a_{\mu\lambda} \frac{\partial \xi}{\partial x}$

$\beta^\mu a_{\mu\nu} \partial x_\lambda \Lambda \psi + \kappa \Lambda \psi = 0$

$\beta^\mu \partial x_\mu + \kappa \psi = 0.$

or $\beta^\lambda \partial x_\lambda + \kappa \psi =$

$\Lambda^{-1} \beta^\mu a_{\mu\lambda} \partial x_\lambda \Lambda \psi + \kappa \psi = 0$ i.e. $\Lambda^{-1} \beta^\mu \Lambda = a_{\mu\lambda} \beta^\lambda$

$A A_T = 1 \rightarrow a_{\nu\mu} a_{\lambda\mu} = \delta_{\nu\lambda}$

$A_T = B$

$A_T A = BA = C$

$C_{\mu\nu} = B_{\mu\lambda} A_{\lambda\nu} = \delta_{\mu\nu}$

$A_{\lambda\mu} A_{\lambda\nu} = \delta_{\mu\nu}$

$a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda}$

$\Lambda^{-1} \beta^\mu a_{\mu\lambda} \partial x_\lambda \Lambda \psi =$

$\beta^\mu \partial x'_\mu \psi' + \kappa \psi' = 0$ $\quad \beta^\alpha a_{\alpha\nu} a_{\alpha\lambda}$

$\beta^\mu a_{\mu\lambda} \partial x_\lambda \Lambda \psi + \kappa \Lambda \psi = 0.$ $\quad \beta^\alpha \delta_{\nu\lambda}$

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{21}^2 & a_{11} a_{12} + a_{21} a_{22} \\ a_{12} a_{11} + a_{22} a_{21}, & a_{12}^2 + a_{22}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0, & 1 \end{pmatrix} = 1$$

$\beta^\mu a_{\mu\nu} \partial x_\lambda \Lambda \psi + \kappa \Lambda \psi = 0$ $\quad \gamma^\mu a_{\mu\lambda} \partial x_\lambda \Lambda \psi + \kappa \Lambda \psi = 0$

$\Lambda^{-1} \beta^\mu a_{\mu\nu} \partial x_\lambda \Lambda \psi + \kappa \psi = 0$ $\quad \Lambda^{-1} \gamma^\mu a_{\mu\lambda} \partial x_\lambda \Lambda \psi + \kappa \psi = 0.$

$\beta^\lambda \partial x_\lambda \psi + \kappa \psi = 0$ $\quad \gamma^\lambda \partial x_\lambda \psi + \kappa \psi = 0.$

$\Lambda^{-1} \beta^\mu a_{\mu\lambda} \Lambda = \beta^\lambda$

$\Lambda^{-1} \beta^\mu a_{\mu\lambda} a_{\mu\lambda} \Lambda = a_{\mu\lambda} \beta^\lambda$

$\Lambda^{-1} \beta^\mu \Lambda = a_{\mu\lambda} \beta^\lambda$

$\Lambda^{-1} \beta^\mu a_{\mu\nu} \partial x_\nu \Lambda \psi + \kappa \Lambda \psi =$

$\Lambda^{-1} \beta^\mu a_{\mu\nu} \Lambda = \beta^\nu$

such that each element a of the algebra is of the form $a = b_1 P + \dots + b_n P$,

and the elements a commute with elements of P ie $a = b_1 \lambda_1 + \dots + b_n \lambda_n = \lambda_1 b_1 + \dots + \lambda_n b_n$ $(\lambda_i \in P)$

If the b_i are linearly independent in relation to P, then n is called the rank of the system. By specification of the basis elements and their multiplication-table, the system is completely determined with respect to a given P. The multiplication table satisfies $b_\lambda (b_\mu b_\nu) = (b_\lambda b_\mu) b_\nu$.

Theorem re. algebras: Burnside's Theorem: A semi-simple algebra of rank n^2 has only one representation of order n but for equivalence.

In view of $\frac{1}{2}(\alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu) = \delta_{\mu\nu}$.

the sixteen elements $1, \alpha_\mu, \alpha_\mu \alpha_\nu, \alpha_\lambda \alpha_\mu \alpha_\nu, \alpha_1 \alpha_2 \alpha_3 \alpha_4$ can be shown to constitute a semi-simple algebra of rank 16, and hence the repn by the Dirac matrices is unique except for equivalence. By equivalence is meant the constn of other sets of matrices by multiplications of the α's among themselves. Consider for example, the set given by $\beta_4 = -\alpha_4, \beta_1 = -i\alpha_4 \alpha_1, \dots$ It can be easily shown that for eg. $\beta_1 \beta_2 + \beta_2 \beta_1 = (-i\alpha_4\alpha_1)(-i\alpha_4\alpha_2) + (-i\alpha_4\alpha_2)(-i\alpha_4\alpha_1) = -\alpha_4\alpha_1\alpha_4\alpha_2 - \alpha_4\alpha_2\alpha_4\alpha_1$

$$= \alpha_1\alpha_2 + \alpha_2\alpha_1 = 0$$

$$\beta_1\beta_4 + \beta_4\beta_1 = (-i\alpha_4\alpha_1)(-\alpha_4) + (-\alpha_4)(-i\alpha_4\alpha_1) = i(\alpha_4\alpha_1\alpha_4 + \alpha_4^2\alpha_1) = i(-\alpha_1 + \alpha_1) = 0.$$

$$\therefore \tfrac{1}{2}(\beta_\mu\beta_\nu + \beta_\nu\beta_\mu) = \delta_{\mu\nu}.$$

This set of β-matrices put the Dirac eqn in a symmetrical form. (2) can be written as

$$\left(c\hbar\frac{\partial}{\partial t} - i\hbar c\alpha_1\frac{\partial}{\partial x_1} - \dots - \dots + \alpha_4 mc^2\right)\psi = 0.$$

or using $x_4 = ict$, $\left(-c\hbar\frac{\partial}{\partial x_4} - \dots + \alpha_4 mc^2\right)\psi = 0$

& multiplying on left throughout by $\beta\alpha_4$

$$\left(-c\hbar\alpha_4\frac{\partial}{\partial x_4} - i\hbar c\alpha_4\alpha_1\frac{\partial}{\partial x_1} - \dots - \dots + mc^2\right)\psi = 0$$

ie $$\left(\beta_4\frac{\partial}{\partial x_4} + \beta_1\frac{\partial}{\partial x_1} + \dots + \frac{mc}{\hbar}\right)\psi = 0$$

or $\beta_\mu \partial_\mu \psi + \varkappa\psi = 0$ $\quad [\varkappa = mc/\hbar$ or $\beta^\mu \partial_\mu \psi + \varkappa\psi = 0$.

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(d) Lorentz-covariance of the Dirac eqn – ~~Assume~~ – Mention to be made of a general theory depending on spinors deriving Dirac's eqn on the basis of invariance under L_4 and (L_4, PTL_4). [P.T.O] &

(e) Free particle solutions

In $(E + c\vec{\alpha}\cdot\vec{p} + \alpha_4 mc^2)\psi$, putting in the exprns for $\vec{\alpha}$ & α_4 and $\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$, we get

$$E_+ \begin{cases} (E + mc^2)\psi_1 + c(p_1 - ip_2)\psi_4 + cp_3\psi_3 = 0 \\ (E + mc^2)\psi_2 + c(p_1 + ip_2)\psi_3 - cp_1\psi_4 = 0 \end{cases}$$

$$E_- \begin{cases} (E - mc^2)\psi_3 + cp_3\psi_1 + c(p_1 - ip_2)\psi_2 = 0 \\ (E - mc^2)\psi_4 + c(p_1 + ip_2)\psi_1 - cp_3\psi_2 = 0 \end{cases}$$

[The determinant of these homogeneous eqns in the ψ's must be zero for ψ's to have soln & det $\to (E^2 - c^2\vec{p}^2 - m^2c^4)^2 = 0$ in consonance with relativist. E & $\vec{p}$.

$$\beta_4 = -\alpha_4 \rightarrow \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\beta_1 = -i\alpha_4\alpha_1 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\beta_2 = -i\alpha_4\alpha_2 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\beta_3 = i\alpha_4\alpha_3 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

β's all different from α's. but rep^ns^ equivalent ie replacement of β^μ by $S\beta^\mu S^{-1}$ where S is any non-singular 4×4 matrix & this gives a new soln $\psi' = S\psi$. Physically this means that the same phenomena are analysed in a diff^t^ way.

Lorentz covariance of Dirac's eqn

~~For a new frame of reference be given by $x'_\mu = a_{\mu\nu}x_\nu$~~

Simple Lorentz transformation $(x_1, x_2, x_3, t) \rightarrow (x'_1, x'_2, x'_3, t)$ for relative vel $\underline{v}$ is

$x'_1 = \eta(x_1 - vt)$, $x'_2 = x_2$, $x'_3 = x_3$, $t' = \eta(t - vx_1/c)$. $[\eta = (1 - v^2/c^2)^{-\frac{1}{2}}]$

Spatial rotations about O_1: $x'_2 = x_2\cos\theta + x_3\sin\theta$; $x'_3 = -x_2\sin\theta + x_3\cos\theta$, $x'_1 = x_1$, $t' = t$

Spatial inversion (Parity) $\vec{x}' = -\vec{x}$, $t' = t$. Time reversal $\vec{x}' = \vec{x}$, $t' = -t$.

These various relations can be brought under $x'_\mu = a_{\mu\nu}x_\nu$ (1) ($a_{\mu\nu}$ consts. $x_4 = ict$). All the tr^ns^ above leave invariant the form $x_\mu^2 = \vec{x}^2 - c^2t^2$. This requires that $a_{\mu\nu}$ (a_{ij}, a_{44} real, a_{i4}, a_{4j} pure imaginary) obey the relation $a_{\mu\nu}a_{\mu\lambda} = \delta_{\nu\lambda}$ (2). Field eqns are unaltered under the relns (1) if this condn is satisfied. Denoting the 4×4 matrix $a_{\mu\nu}$ by A & $\delta_{\mu\nu}$ (identity 4×4 matrix) by I, the condn can be written $A_T A = I$ which gives $A_T = A^{-1}$ and $(\det A)^2 = 1$ ie $\det A = \pm 1$. Tr^ns^ which can be brought together by continuous variation of $a_{\mu\nu}$ cannot have diff. values of $\det A$. Hence simple L.T's & rotns have $\det A = 1$ & superposing on these space inversions or time-reversals give $\det A = -1$.

We expect the fermion to be described in the (x'_μ) frame 4-comp. ψ' such that $\psi'(x'_\mu) = \Lambda\psi(x_\mu)$ where Λ is a non-singular 4×4 matrix which is a fn of $a_{\mu\nu}$ alone. The Dirac eqn in the new frame is $\beta^\mu\partial_{x'_\mu}\psi' + \chi\psi' = 0$. From (2) $\partial_{x'_\mu} = a_{\mu\lambda}\partial_{x_\lambda}$ & a little manipulation shows that this eqn is same as the original if $\Lambda^{-1}\beta^\nu\Lambda = a_{\nu\lambda}\beta^\lambda$ $(\nu = 1, 2, 3, 4)$. For the typical rotation about O_1, a suitable operator is $\Lambda(\theta) = \exp\left(\frac{1}{2}\beta^2\beta^3\theta\right)$ & for spatial reflectn $\Lambda = \gamma^4$. For vel. tr^n^. $\Lambda(v) = \exp\left(\frac{1}{2}i\beta^1\beta^4\phi\right)$ where $\sinh\phi = v\eta/c$. Also $\Lambda(-\theta) = \{\Lambda(\theta)\}^{-1}$ & $\Lambda(-v) = \{\Lambda(v)\}^{-1}$ & $\Lambda(\theta)$, $\Lambda(v)$ are so chosen that they $\rightarrow 1$ as $\theta, v \rightarrow 0$.

These can be written explicitly by choosing a sign for E as ± as E_+ or E_- as

(Using plane wave solns for ψ's viz $\psi_\mu = u_\mu \exp i(\vec{k}\cdot\vec{r} - \omega t)$ these can be written)

$$\psi_1 = \frac{cp_3}{E_+ + mc^2}, \quad \psi_2 = -\frac{c(p_1 + ip_2)}{E_+ + mc^2}, \quad \psi_3 = 1, \quad \psi_4 = 0$$

$$\psi_1 = -\frac{c(p_x - ip_y)}{E_+ + mc^2}, \quad \psi_2 = \frac{cp_3}{E_+ + mc^2}, \quad \psi_3 = 0, \quad \psi_4 = 1.$$

as obtained from the 1st two eqns by putting $\psi_3 = 1, \psi_4 = 0$ & $\psi_3 = 0, \psi_4 = 1$ resp. These ψ's satisfy the second & 3rd eqns also. Similarly putting $u_1 = 1, u_2 = 0$ and $u_1 = 0, u_2 = 1$ in the last two eqns & associating it with E_- we have

$$\psi_1 = 1, \quad \psi_2 = 0, \quad \psi_3 = \frac{-cp_3}{E_- - mc^2}, \quad \psi_4 = \frac{-c(p_x + ip_y)}{E_- - mc^2}$$

$$\psi_1 = 0, \quad \psi_2 = 0, \quad \psi_3 = \frac{-c(p_1 - ip_2)}{E_- - mc^2}, \quad \psi_4 = \frac{cp_3}{E_- - mc^2}$$

& these can be shown to satisfy the first two eqns. [1st 2 → +ve energy solns & last 2 → −ve energy solns]

(f) Spin & mag. moment – Electromagnetic scalar & vector potentials can be introduced into the Dirac eqn relativistically by replacements $E \to E - e\phi$; $c\vec{p} \to c\vec{p} - e\vec{A}$, so that the eqn becomes

[P.T.O]

$$\{E - e\phi + \vec{\alpha}\cdot(c\vec{p} - e\vec{A}) + \alpha_4 mc^2\}\psi = 0.$$

[use Hamilton's defn of $\sigma_{\mu\nu} = \frac{1}{2i}[\alpha_\mu, \alpha_\nu]$]

With $A = 0$, $\phi = \phi(\vec{r})$, this can be written as

$$c\hbar\frac{\partial\psi}{\partial t} = H\psi, \text{ where } H = -c\vec{\alpha}\cdot\vec{p} - \alpha_4 mc^2 + e\phi.$$

[orbital A.M. $L = \vec{r}\times\vec{p}$ doesn't commute with H, for eg. we find

$$i\hbar\frac{dL_x}{dt} = L_xH - HL_x = -c\vec{\alpha}\cdot[(yp_z - zp_y)\vec{p} - \vec{p}(yp_z - zp_y)]$$
$$= i\hbar c(\alpha_z p_y - \alpha_y p_z)] \quad \text{— (A)}$$

Consider the operator σ' defined by $\sigma' = \begin{pmatrix}\vec{\sigma} & 0 \\ 0 & \vec{\sigma}\end{pmatrix}$ i.e a 4×4 matrix. We find that σ_x' commutes with α_x and β, but not with other components of $\vec{\alpha}$

$$\sigma_x'\alpha_y - \alpha_y\sigma_x' = \begin{pmatrix}\sigma_x & 0 \\ 0 & \sigma_x\end{pmatrix}\begin{pmatrix}0 & \sigma_y \\ \sigma_y & 0\end{pmatrix} - \begin{pmatrix}0 & \sigma_y \\ \sigma_y & 0\end{pmatrix}\begin{pmatrix}\sigma_x & 0 \\ 0 & \sigma_x\end{pmatrix}$$

$$= \begin{pmatrix}0 & i\sigma_z \\ i\sigma_z & 0\end{pmatrix} - \begin{pmatrix}0 & -i\sigma_z \\ -i\sigma_z & 0\end{pmatrix} = 2i\alpha_z.$$

and $$i\hbar\frac{d\sigma_x'}{dt} = \sigma_x'H - H\sigma_x' = -2ic(\alpha_z p_y - \alpha_y p_z) \quad \text{— (B)}$$

From (A) & (B) it follows that $\vec{L} + \frac{1}{2}\hbar\vec{\sigma}'$ commutes with H and can be taken as the total A.M. showing that $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}'$ is the spin operator or spin A.M of electron as follows from the Dirac theory without any assumption.

(g) Negative energy solns:

$$(yp_2 - zp_y)\vec{p} - \vec{p}(yp_2 - zp_y)$$

$$(yp_3 - zp_2)(\alpha_1 p_1 + \cdot\ ) - (\alpha_1 p_1 + \cdot)(yp_3 - zp_2)$$

$$y\cancel{p_3}\alpha_1 p_1 + yp_3\cancel{\alpha_2} p_2 + y\cancel{p_3}\cancel{\alpha_3} p_3$$
$$- zp_2\alpha_1 p_1 - zp_2\alpha_2 p_2 - zp_2\alpha_3 p_3$$
$$- \alpha_1\cancel{p_1} yp_3 - \alpha_2\cancel{p_2} yp_3 - \alpha_3\cancel{p_3} yp_3$$
$$+ \alpha_1 p_1 zp_2 + \alpha_2 p_2 zp_2 + \alpha_2 p_2 zp_2$$

$$\alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},\ \alpha_2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$$

$$\sigma_{12}^{\mu\nu} = \frac{1}{2i}[\gamma_\mu, \gamma_\nu]$$

$$\frac{1}{2i}(\alpha_1\alpha_2 - \alpha_2\alpha_1) = \frac{1}{2i}\left\{ \begin{pmatrix} \sigma_1\sigma_2 & 0 \\ 0 & \sigma_1\sigma_2 \end{pmatrix} - \begin{pmatrix} \sigma_2\sigma_1 & 0 \\ 0 & \sigma_2\sigma_1 \end{pmatrix} \right\}$$

$$= \frac{1}{2i}\left\{ \begin{matrix} \sigma_1\sigma_2 - \sigma_2\sigma_1 & 0 \\ 0 & \sigma_1\sigma_2 - \sigma_2\sigma_1 \end{matrix} \right\} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. = \underline{\sigma}$$

$$\sigma_{14} = \frac{1}{2i}(\alpha_1\alpha_4 - \alpha_4\alpha_1) = \frac{1}{2i}\left\{ \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \right\}$$

$$= \frac{1}{2i}\left\{ \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} \right\} = \frac{1}{2i}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Spin operator $\qquad -i\hbar\frac{\partial}{\partial\mu}$

(f) To represent bringing in an external e.m.f is equivalent in quant. mech to the replacement $\partial_\mu \to \partial_\mu + \frac{ie}{\hbar c}A_\mu$ [Corresp. to $p_\mu \to p_\mu + \frac{e}{c}A_\mu$ in classical case, where A_μ is $(\vec{A}, i\phi)$] & takes Dirac's eq^n to the form $\beta^\mu\left(\partial_\mu + \frac{ie}{\hbar c}A_\mu(x)\right)\psi + \kappa\psi = 0$. Let a new operator π_μ be defined by $\pi_\mu = p_\mu + \frac{e}{c}A_\mu$; then Dirac's eq^n now becomes $(\beta^\mu\pi_\mu - imc)\psi = 0$. The —— (1)

π_μ's do not in general commute; their commutator is

$$[\pi_\mu, \pi_\nu] = (\hbar e/ic)f_{\mu\nu} \quad \text{—— (2)}$$

where $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field-strength tensor (anti-sym) [Maxwell's eq^n written in the Lorentz-invariant form $\partial_\mu f_{\mu\nu} = -\frac{4\pi}{c}j_\nu$; $\partial_\lambda f_{\mu\nu} + \partial_\mu f_{\nu\lambda} + \partial_\nu f_{\lambda\mu} = 0$

Operating on (1) with $\beta^\nu\pi_\nu + imc$ gives, using $\beta^\mu\beta^\nu + \beta^\nu\beta^\mu = 2\delta_{\mu\nu}$ & (2),

$$\left\{\pi_\mu^2 + m^2c^2 + \frac{\hbar e}{2c}\sigma^{\mu\nu}f_{\mu\nu}\right\}\psi = 0. \quad \text{—— (3)}$$

where the antisymmetric matrix $\sigma^{\mu\nu}$ is defined by $\sigma^{\mu\nu} = \frac{1}{2i}[\beta^\mu, \beta^\nu]$ —— (4)

The vacuum is defined as the state in which all negative-energy electron levels are occupied ~~and all +ve energy states are empty~~. According to the Pauli Principle no ~~+ve energy electrons~~ ~~were present~~ ~~can transfer~~ no electrons can transfer to the -ve energy levels from the +ve energy states. So a vacuum is one where all -ve energy levels are filled & all the remaining electrons can no longer reduce their energy by making a transition to -ve states. When there are no electrons with +ve energy, electrons can no longer reduce their energy in any way if all the -ve energy states are occupied.

Suppose a quantum with energy $\geq mc^2$ is capable of effecting the transition of an electron from a -ve to a +ve energy state. But after an electron has been removed from a -ve energy state there will remain a "hole" ie an unoccupied level & the hole behaves like an electron with +ve charge and +ve energy. This is the positron of Dirac's theory and was discovered experimentally by Anderson. The attitude before this discovery towards the Dirac theory was suspicious, but changed later on. Further after the discovery of the anti-proton (ie proton with -ve charge) by Segrè once again confirmed the generality of Dirac's conceptions of particles of spin $1/2$.

To prove the "hole" theory mathematically, it is necessary to give up the one-particle picture & go over to the theory of many particles ie relativistic quantized field theories. It is not intended to consider such theories ~~theories~~

The discovery of the positron & the anti-proton were the beginnings of elementary particle physics.

(contd from p.3 reverse) – Considering A.M, the reln between ψ' and ψ for a small rotation about axis O_1 is given by $\psi'(x_1, x_2, x_3) = \psi(x_1, x_2 + x_3\delta\theta, -x_2\delta\theta + x_3) = \{1 + J_1\delta\theta\}\psi(x_1, x_2, x_3)$ where $J_1 = x_2\partial_3 - x_3\partial_2$ is the 1st comp. of the 3-vector operator $\vec{J} = \vec{x} \times \text{grad}$. $\vec{J}$ obeys the commutation rules $\vec{J} \times \vec{J} = -\vec{J}$. The A.M. $\vec{L}$ in wave mechanics is given by $\vec{L} = \vec{x} \times \vec{p} = (\hbar/i)\vec{J}$. ~~Hence~~ and effect of rotating axes thro' $\delta\theta$ about O_1, L_1 is $= (\hbar/i)J_1$ ~~where~~ is described by the operator $(1 + J_1\delta\theta)$. Replacing θ by $\delta\theta$ in $\Lambda(\theta) = \exp(\frac{1}{2}\beta^2\beta^3\theta)$ we get $\Lambda(\delta\theta) = 1 + \frac{1}{2}i\sigma^{23}\delta\theta$, using $\sigma^{\mu\nu} = \frac{1}{2i}[\beta^\mu, \beta^\nu]$ and $\beta^\mu\beta^\nu + \beta^\nu\beta^\mu = 2\delta_{\mu\nu}$. Hence $S_1 = \frac{1}{2}\hbar\sigma^{23}$, where $\vec{S} = (S_1, S_2, S_3)$ is the spin or intrinsic A.M. operator. It is easy to show $(\sigma^{23})^2 = 1$ ie $\sigma^{23} = \pm 1$ (See reverse of p.3)

Hence eigenvalues of spin component in any dirn are $\pm\frac{1}{2}\hbar$

Magnetic moment. Dividing eqn (3) [reverse of p.3] by $2m$, we can write

$$\frac{E + e\phi + mc^2}{2mc^2}(E + e\phi - mc^2)\psi = \left\{\frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 + \frac{e}{2mc}\cdot\frac{1}{2}\hbar\sigma^{\mu\nu}f_{\mu\nu}\right\}\psi \quad \text{--- } 5$$

P.T.O

If $E - mc^2 + \phi$ is small – as it is for a slowly moving particle in a weak electric field, the factor $(E + mc^2 + e\phi)/2mc^2$ on the left of (5) may be replaced by unity (for when $\phi = 0$, & $v = 0$, this is exact). Eqn (5), apart from the term

$$\frac{e}{2mc} \cdot \frac{1}{2} \hbar \sigma^{\mu\nu} f_{\mu\nu} \quad \text{——(6)}$$

is now just the Schrödinger eqn for a spinless particle in an e.m.f. The components $\mu = 4$, $\nu = 1, 2, 3$ of (6) give an effect, which on choosing a repn, can be shown to vanish for particles of low velocity; the components $\mu, \nu = 1, 2, 3$ give $(e/mc)(\vec{S} \cdot \vec{H})$ which is the energy of a magnetic dipole of moment $-(e/mc)\vec{S}$ in the external magnetic field H. The intrinsic magnetic moment of the particle is $e\hbar/2mc$.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^2 + a_{12}^2 & a_{11}a_{21} + a_{22}a_{22} \\ a_{21}a_{11} + a_{22}a_{12}, & a_{21}^2 + a_{22}^2 \end{pmatrix}$$

$$= 1.$$

$$a_{\nu\mu} a_{\lambda\mu} = \delta_{\nu\lambda}$$

$$a_{\nu 1} a_{\lambda 1} + a_{\nu 2} a_{\lambda 2} = \delta_{\nu\lambda}$$

Putting $\nu = \lambda = 1 \to a_{11}^2 + a_{12}^2 = 1$

$\nu = \lambda = 2 \to a_{21}^2 + a_{22}^2 = 1$

Putting $\nu = 1, \lambda = 2 \to a_{11}a_{21} + a_{12}a_{22} = 0$

$\nu = 2, \lambda = 1 \to a_{21}a_{11} + a_{22}a_{12} = 0$

$$\left(\Pi_\mu^2 + m^2c^2 + \frac{\hbar e}{2c} \sigma^{\mu\nu} f_{\mu\nu} \right) \psi = 0$$

$$\Pi_\mu = p_\mu + \frac{e}{c} A_\mu$$

$$\Pi_\mu^2 = \left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 + \left(p_4 + \frac{e}{c}A_4\right)^2$$

$$\Pi_\mu^2 = \left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 + \left(p_4 + \frac{e}{c}A_4\right)^2$$

$$= \quad " \quad + \left(\frac{iE}{c} + \frac{e}{c} . i\phi \right)^2$$

$$= \quad " \quad - \left(\frac{E}{c} + \frac{e}{c} . \phi \right)^2$$

$$\vec{A}^2 = iA_1 + jA_2 + kA_3 = A \cdot A$$
$$= a_1^2 + a_2^2 + a_3^2$$

$$\left\{ \frac{1}{2m} \left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 + \frac{e}{2mc} \cdot \frac{1}{2}\hbar \sigma^{\mu\nu} f_{\mu\nu} \right\} \psi$$

$$= \left(\frac{1}{2m} \left(\frac{E}{c} + \frac{e}{c}\phi \right)^2 - \frac{m^2c^2}{2m} \right) \psi = \frac{1}{2mc^2} \left\{ (E + e\phi)^2 - m^2c^4 \right\}$$

$$= \frac{1}{2mc^2} \cdot (E + e\phi + mc^2)(E + e\phi - mc^2) \psi$$

$$= (E' + e\phi) \psi$$

$\beta^1 \beta^4$

$= -i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= -i \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

β

$= -i \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$

i.e. $\beta^4 \beta^1 - \beta^1 \beta^4 = 0$

Similarly $[\beta^4, \beta^2] = 0$

$[\beta^4, \beta^3] = 0$

In the limit of small ϕ, $\frac{1}{2mc^2} \cong 1$

& slow velocities

and $E = E' + mc^2$

assume E' & $e\phi$ are small cf. mc^2.

$$\frac{1}{2m} \left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 \psi = (E' + e\phi)\psi$$

is the Schrodinger eqn for a spinless particle in an e.m.f

$$\frac{p^2}{2m} \psi = E' \psi$$

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = i\hbar \frac{\partial \psi}{\partial t}.$$

$$\frac{e}{2mc} \frac{1}{2} \hbar \sigma^{\mu\nu} f_{\mu\nu}$$

$$\sigma^{4\nu} f_{4\nu} \to \underbrace{\cancel{\sigma^{44} f_{44}}}_{0} + \sigma^{41} f_{41} + \sigma^{42} f_{42} + \sigma^{43} f_{43}$$

[$f_{\mu\nu}$ is antisymmetric]

$$= -\frac{1}{2i} \left(\beta^4 \beta^1 - \beta^1 \beta^4 \right) E_3 - - - - -$$

$$= 0.$$

$\beta^K = -i\beta\alpha^K$

In the low-vel. repn, $\beta^4 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $\alpha^K = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}$, ($\sigma$ Pauli spin m's)

$$\beta^4 \beta^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$$

$$= -i \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$$

$\Lambda(v) = \exp(\tfrac{1}{2} i \gamma' \gamma^4 \phi)$

$$\text{R.H.S} = 1 + \tfrac{1}{2} i\gamma'\gamma^4\phi + \frac{(\frac{1}{2} i\gamma'\gamma^4\phi)^2}{2!} + \cdots$$

$$= 1 + \tfrac{1}{2} i\gamma'\gamma^4\phi + \frac{\frac{1}{4}\gamma'\gamma^4\phi\gamma'\gamma^4\phi}{2!} + \frac{\frac{1}{8}(-i\gamma'\gamma^4\gamma'\gamma^4\gamma'\gamma^4\phi^3)}{3!}$$

$$= 1 + \tfrac{1}{2} i\gamma'\gamma^4\phi + \frac{\frac{1}{4}\phi^2}{2!} + \frac{\frac{1}{8} i\gamma_1\gamma_4\phi^3}{3!} + \frac{\frac{1}{16}\phi^4}{4!} + \frac{i(\frac{1}{2}\phi)^5\gamma'\gamma^4}{5!} + \cdots$$

$$\left[1 + \frac{(\frac{1}{2}\phi)^2}{2!} + \frac{(\frac{1}{2}\phi)^4}{4!} + \cdots\right] + i\gamma'\gamma^4\left[(\tfrac{1}{2}\phi) + \frac{(\frac{1}{2}\phi)^3}{3!} + \frac{(\frac{1}{2}\phi)^5}{5!} + \cdots\right]$$

$\cosh^2\phi - \sinh^2\phi = 1$

$\sinh\phi = 2\sinh\tfrac{1}{2}\phi\cosh\tfrac{1}{2}\phi$

$\cosh\tfrac{1}{2}\phi + i\gamma'\gamma^4\sinh\tfrac{1}{2}\phi$

$\cosh\phi = 1 + 2\sinh^2\tfrac{1}{2}\phi$

$= 2\cosh^2\tfrac{1}{2}\phi - 1$

$\sinh\phi = \frac{v\eta}{c}$ ✓

$\cosh\phi = \sqrt{1 + v^2\eta^2/c^2} = \eta$ ✓

$\cosh^2\phi - \sinh^2\phi$

$= \eta^2 - \frac{v^2\eta^2}{c^2}$

$= \eta^2(1 - v^2/c^2) = \frac{\eta^2(c^2 - v^2)}{c^2}$

$= 1$ ✓

$1 + \frac{v^2}{c^2(1 - v^2/c^2)}$

$1 + \frac{v^2}{c^2 - v^2}$

$\frac{c^2}{c^2 - v^2} = \frac{1}{1 - v^2/c^2}$

$1 + \cosh\phi = 2\cosh^2\tfrac{1}{2}\phi$

$\frac{1}{2}(1 + \eta) = \cosh^2\tfrac{1}{2}\phi$

$\cosh\tfrac{1}{2}\phi = \frac{1}{\sqrt{2}}(1 + \eta)^{\frac{1}{2}}$

$2\sinh\tfrac{1}{2}\phi\cosh\tfrac{1}{2}\phi = \frac{v\eta}{c}$

$\sinh\tfrac{1}{2}\phi = \frac{v\eta}{2c}\frac{\sqrt{2}}{(1+\eta)^{1/2}} =$

$x'_\mu = a_{\mu\nu} x_\nu \qquad x'_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4$ & $x'_1 = x_1$ gives

$a_{11} = 1, a_{12} = a_{13} = a_{14} = 0$

Since $t' = t$ ie $x'_4 = x_4$ gives $a_{44} = 1, a_{41} = a_{42} = a_{43} = 0$.

$x'_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = x_2\cos\theta + x_3\sin\theta$ ie

$a_{21} = a_{24} = 0,\ a_{22} = \cos\theta,\ a_{23} = \sin\theta$.

Since $x'_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = -x_2\sin\theta + x_3\cos\theta$ ie

$a_{31} = a_{34} = 0,\ a_{32} = -\sin\theta,\ a_{33} = \cos\theta$

ie $a_{11} = a_{44} = 1,\ a_{22} = a_{33} = \cos\theta,\ a_{23} = -a_{32} = \sin\theta$ & all others zero.

$\Lambda^{-1}\beta^\mu\Lambda = a_{\mu\lambda}\beta^\lambda \qquad \Lambda^{-1}\beta^1\Lambda = a_{11}\beta^1 + a_{12}\beta^2 + a_{13}\beta^3 + a_{14}\beta^4$

$\Lambda^{-1}\beta^2\Lambda = a_{21}\beta^1 + a_{22}\beta^2 + a_{23}\beta^3 + a_{24}\beta^4$

$\Lambda^{-1}\beta^3\Lambda = a_{31}\beta^1 + a_{32}\beta^2 + a_{33}\beta^3 + a_{34}\beta^4$

$\Lambda^{-1}\beta^4\Lambda = a_{41}\beta^1 + a_{42}\beta^2 + a_{43}\beta^3 + a_{44}\beta^4$

$$\left.\begin{aligned} \Lambda^{-1}\beta^1\Lambda &= \beta^1 \\ \Lambda^{-1}\beta^2\Lambda &= \beta^2\cos\theta + \beta^3\sin\theta \\ \Lambda^{-1}\beta^3\Lambda &= -\beta^2\sin\theta + \beta^3\cos\theta \\ \Lambda^{-1}\beta^4\Lambda &= \beta^4 \end{aligned}\right\}$$

$x = -iy$

~~$e^{ix} = \cos x + i\sin x$~~

~~$e^{y} = \cos(-iy) + i\sin(-iy)$~~

~~$= \cos(iy) - i\sin(iy)$~~

$\Sigma(\beta_\mu$

~~ie that hold identically for any Λ~~ $\beta^1\Lambda = \Lambda\beta^1$

$$\begin{cases} \beta^2\Lambda = \Lambda(\beta^2\cos\theta + \beta^3\sin\theta) \\ \beta^3\Lambda = \Lambda(-\beta^2\sin\theta + \beta^3\cos\theta) \end{cases}$$

~~$\exp(-\tfrac{1}{2}\beta^2\beta^3\theta)\beta^2\exp(\tfrac{1}{2}\beta^2\beta^3\theta)$~~ $\Lambda(\theta) = \cos\tfrac{1}{2}\theta + \beta^2\beta^3\sin\tfrac{1}{2}\theta$.

$(\cos\tfrac{1}{2}\theta - \beta^2\beta^3\sin\tfrac{1}{2}\theta)\,\beta^2(\cos\tfrac{1}{2}\theta + \beta^2\beta^3\sin\tfrac{1}{2}\theta)$

$= (\cos\tfrac{1}{2}\theta - \beta^2\beta^3\sin\tfrac{1}{2}\theta)(\beta^2\cos\tfrac{1}{2}\theta + \beta^3\sin\tfrac{1}{2}\theta)$

$= \beta^2\cos^2\tfrac{1}{2}\theta + \beta^3\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta - \beta^2\beta^3\beta^2\sin\tfrac{1}{2}\theta - \beta^2\sin^2\tfrac{1}{2}\theta$

$= \beta^2\cos\theta + \beta^3\cos\tfrac{1}{2}\theta\sin\theta + \beta^3\sin\tfrac{1}{2}\theta\cos\theta = \beta^2\cos\theta + \beta^3\sin\theta$.

Since $\Lambda^{-1}\beta^3\Lambda = -\beta^2\sin\theta + \beta^3\cos\theta$

$(\cos\tfrac{1}{2}\theta - \beta^2\beta^3\sin\tfrac{1}{2}\theta)\,\beta^1(\cos\tfrac{1}{2}\theta - \beta^2\beta^3 \ldots$

$)(\beta^1\cos\tfrac{1}{2}\theta - \beta_1\beta_2\beta_3 \ldots$

$= ($

$\beta^1\beta^2\beta^3 = \beta^2\beta^3\beta^1$

$\beta^1(\cos\tfrac{1}{2}\theta - \beta^2\beta^3\sin\tfrac{1}{2}\theta)$

$= (\cos\tfrac{1}{2}\theta - \beta^2\beta^3\sin\tfrac{1}{2}\theta)\beta^1$

$x_1' = \eta(x_1 - v x_4/ic)$, $\frac{x_4'}{ic} = \eta\left(\frac{x_4}{ic} - \frac{v x_1}{c}\right)$ or $x_4' = \eta(x_4 - i v x_1)$

$x_2' = x_2$ and $x_3' = x_3$ give $a_{22} = a_{33} = 1$

$x_1' = \eta(x_1 - v x_4/ic)$ gives $a_{11} = \eta$, $a_{14} = -\eta v/ic = i\eta v/c$

$x_4' = \eta(x_4 - i v x_1)$ gives $a_{41} = -i\eta v$, $a_{44} = \eta$ & all others zero

$$\left.\begin{aligned}\Lambda^{-1}\beta^1\Lambda &= \eta\beta^1 + i\eta v/c\,\beta^4\\ \Lambda^{-1}\beta^2\Lambda &= \beta^2\\ \Lambda^{-1}\beta^3\Lambda &= \beta^3\\ \Lambda^{-1}\beta^4\Lambda &= -i\eta v\beta^1 + \eta\beta^4\end{aligned}\right\}$$

with $\Lambda = \frac{1}{\sqrt{2}}\left\{(1+\eta)^{\frac12} + \frac{iv\eta}{c(1+\eta)^{\frac12}}\beta^1\beta^4\right\}$

all these four can be shown to be satisfied.

for eg.

$$\Lambda(-v)\;\Lambda^{-1} = \frac{\sqrt{2}\left\{(1+\eta)^{\frac12} - \frac{iv\eta}{c(1+\eta)^{\frac12}}\beta^1\beta^4\right\}}{\left\{(1+\eta)^{\frac12} + \frac{iv\eta}{c(1+\eta)^{\frac12}}\beta^1\beta^4\right\}\left\{\text{''} - \text{''}\right\}}$$

$$= \frac{\sqrt{2}\{\qquad\}}{(1+\eta) - \cancel{\frac{iv\eta}{c}\beta^1\beta^4} + \cancel{\frac{iv\eta}{c}\beta^1\beta^4} + \frac{v^2\eta^2}{c^2(1+\eta)}\beta^1\beta^4\beta^1\beta^4}$$

$$= \frac{\text{''}}{(1+\eta) - \frac{v^2\eta^2}{c^2(1+\eta)}}$$ & denom. which $= \frac{c^2(1+\eta)^2 - v^2\eta^2}{c^2(1+\eta)}$

$\eta = \frac{1}{(1-v^2/c^2)^{\frac12}}$

$$= \frac{\{1 + (1-v^2/c^2)^{-\frac12}\}^2 - v^2(1-v^2/c^2)^{-1}}{c^2\{1 + (1-v^2/c^2)^{-1/2}\}}$$

$$= \frac{c^2 + c^2(1-v^2/c^2)^{-1} + 2c^2(1-v^2/c^2)^{-\frac12} - v^2(1-v^2/c^2)^{-1}}{c^2\{1+\eta)} = \frac{c^2 + (1-v^2/c^2)^{-1}\{c^2 - v^2\} + 2c^2\eta}{c^2(1+\eta)}$$

$$= \frac{2c^2(1+\eta)}{c^2(1+\eta)} = 2$$

$$\Lambda^{-1} = \frac{1}{\sqrt{2}}\{\text{''} - \text{''}\} = \Lambda(-v).$$

$$\frac{1}{\sqrt{2}}\left\{(1+\eta)^{\frac12} - \frac{iv\eta}{c(1+\eta)^{\frac12}}\beta^1\beta^4\right\}\beta^1\cdot\frac{1}{\sqrt{2}}\left\{(1+\eta)^{\frac12} + \frac{iv\eta}{c(1+\eta)^{\frac12}}\beta^1\beta^4\right\}$$

$$= \frac12\left\{\text{''} - \text{''}\right\}\left\{(1+\eta)^{\frac12}\beta^1 + \frac{iv\eta}{c(1+\eta)^{\frac12}}\beta^4\right\}$$

$$= \frac12\left\{(1+\eta)\beta^1 + \frac{iv\eta}{c}\beta^4 - \frac{iv\eta}{c}\beta^1\beta^4\beta^1 + \frac{v^2\eta^2}{c^2(1+\eta)}\beta^1\beta^4\beta^4\right\}$$

$$= \frac12\left\{(1+\eta)\beta^1 + \frac{2iv\eta}{c}\beta^4 + \frac{v^2\eta^2}{c^2(1+\eta)}\beta^1\right\}$$

$$= \frac12\left\{2\eta\beta^1 + \frac{2iv\eta}{c}\beta^4\right\}$$

$$= \eta\beta^1 + \frac{iv\eta}{c}\beta^4 \checkmark$$

$\frac{1}{1-v^2/c^2}$

$\eta^2 = \frac{c^2}{c^2-v^2}$

$$\frac{c^2(1+\eta)^2 + v^2\eta^2}{c^2(1+\eta)}$$

$$= \frac{c^2 + 2c^2\eta + \eta^2(c^2+v^2)}{c^2(1+\eta)}$$

$$= 2c^2\eta + c^2 + \frac{c^2+v^2}{1-v^2/c^2}$$

$$= 2c^2\eta + c^2 + \frac{c^2(c^2+v^2)}{(c^2-v^2)}$$

$$= 2c^2\eta + 2c^4/(c^2-v^2)$$

$$= \frac{2c^2\eta + 2c^2\eta^2}{c^2(1+\eta)}$$

$$= 2\eta.$$

Similarly $\Lambda^{-1}\beta^4\Lambda = -i\eta v\beta^1 + \eta\beta^4$

Re $\Lambda^{-1}\beta^2\Lambda$, this is equal to

$$\frac12\left\{(1+\eta)^{\frac12} - \frac{iv\eta}{c(1+\eta)^{\frac12}}\beta^1\beta^4\right\}\left\{(1+\eta)^{\frac12}\beta^2 + \frac{iv\eta}{c(1+\eta)^{\frac12}}\beta^2\beta^1\beta^4\right\}$$

$$= \frac12\Big[(1+\eta)\beta^2 + \cancel{\frac{iv\eta}{c}\beta^2\beta^1\beta^4} - \cancel{\frac{iv\eta}{c}\beta^1\beta^4\beta^2}$$
$$+ \frac{v^2\eta^2}{c^2(1+\eta)}\beta^1\beta^4\beta^2\beta^1\beta^4\Big]$$

$\beta^1\beta^4\beta^2\beta^1\beta^4 = \beta^2\beta^1\beta^4\beta^1\beta^4 = -\beta^2$

$$= \frac12\left[(1+\eta)\beta^2 - \frac{v^2\eta^2}{c^2(1+\eta)}\beta^2\right] = \frac{1}{2c^2(1+\eta)}\beta^2\{c^2(1+\eta)^2 - v^2\eta^2\}$$

$$= \frac{1}{2c^2(1+\eta)}\beta^2\{c^2 + \eta^2(c^2-v^2) + 2c^2\eta\}$$

$$= \frac{1}{2c^2(1+\eta)}\beta^2\{c^2 + c^2 + 2c^2\eta\} = \beta^2.$$

$c^2\left\{1 + \frac{c^2+v^2}{c^2-v^2}\right\}$

$2c^4/c^2-$

Similarly $\Lambda^{-1}\beta^3\Lambda = \beta^3$.

$L = \vec{x} \times \vec{p} = \vec{x} \times -i\hbar\,\mathrm{grad}$

$= \left(\frac{\hbar}{i}\right) I$

$(A \times B) \times (A \times B)$

$y\partial_z - z\partial_x \qquad I_x = y\partial_z - z\partial_y$

$I_y I_z - I_z I_y \qquad I_y =$

$(z\partial_x - x\partial_z)(x\partial_y - y\partial_x) - (x\partial_y - y\partial_x)(z\partial_x - x\partial_z)$

$\cancel{zx\partial_x\partial_y - zy\partial_x^2 - x^2\partial_z\partial_y + xy\partial_z\partial_x}$

$p_1 A_1 + p_2 A_2 + p_3 A_3$

$\Pi_\mu = p_\mu + \frac{e}{c} A_\mu$

$\pi_\mu^2 = (p_\mu + \frac{e}{c}A_\mu)^2 = (p_\mu + \frac{e}{c}A_\mu)(p_\mu + \frac{e}{c}A_\mu)$

$\cancel{(\vec{p} + \frac{e}{c}\vec{A} + p_4 + \frac{e}{c}A_4)^2} \quad = p_\mu p_\mu + \frac{e}{c}(p_\mu A_\mu + A_\mu p_\mu) + \frac{e^2}{c^2} A_\mu A_\mu$

$= \cancel{(\vec{p} + \frac{e}{c}\vec{A})^2 + (p_4 + \frac{e}{c}A_4)^2} \quad = \vec{p}^2 + p_4^2 + \frac{2e}{c}(\vec{p}\cdot\vec{A} + p_4 A_4) + \frac{e^2}{c^2}(\vec{A}^2 + A_4^2)$

$= (\vec{p} + \frac{e}{c}\vec{A})^2 + (p_4 + \frac{e}{c}A_4)^2$

$\left\{\cancel{\frac{1}{2m}(\vec{p} + \frac{e}{c}\vec{A})^2} + \frac{\hbar e}{2mc}\frac{1}{2}k\sigma^{\mu\nu} f_{\mu\nu}\right\}\psi$

$\frac{m^2c^2}{2m}$

$= \left\{-\frac{mc^2}{2m} \cancel{- (p_4 + \frac{e}{c}A_4)^2 - 2(\vec{p} + \frac{e}{c}\vec{A})(p_4 + \frac{e}{c}A_4)}\right\}\psi$

$\frac{1}{2m}\left\{-m^2c^2 - (p_4 + \frac{e}{c}A_4)^2\right\}\psi$

$-\frac{1}{2}mc^2 - \frac{1}{2m}\left\{\frac{iE}{c} + \frac{e}{c}i\phi\right\}^2$

$\frac{1}{2mc^2}(E + e\phi)^2 - \frac{mc^2}{2}$

$\frac{1}{2mc^2}\left\{(E + e\phi)^2 - m^2c^4\right\}$

$2(\sigma^{23} f_{23} + \sigma^{31} f_{31} + \sigma^{12} f_{12})$

$= \frac{1}{2i}(\beta^2\beta^3 - \beta^3\beta^2) H_1 +$

$= \frac{1}{2i}(-2i)\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix} H_1 = 2\cdot\frac{e}{2mc}\cdot\frac{1}{2}\hbar\,\sigma^{\mu\nu} f_{\mu\nu}$

$= \ (\quad) H_1 \qquad = \frac{e}{mc}(\vec{S}\cdot\vec{H})$

$\cancel{\sigma^{23} H_1}$

$\frac{e\hbar}{2mc}$

$\beta^2 = -i\beta\alpha^2$

$= -i\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & \sigma_2\\ \sigma_2 & 0\end{pmatrix} = -i\begin{pmatrix}0 & \sigma_2\\ \sigma_2 & 0\end{pmatrix}$

$\beta^3 = -i\begin{pmatrix}0 & \sigma_3\\ \sigma_3 & 0\end{pmatrix}$

$= i\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix}$

$\beta^2\beta^3 = \begin{pmatrix}\sigma_2\sigma_3 & 0\\ 0 & \sigma_2\sigma_3\end{pmatrix} = -i\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix}$

$\beta^3\beta^2 = \begin{pmatrix}\sigma_3\sigma_2 & 0\\ 0 & \sigma_3\sigma_2\end{pmatrix} = +i\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix}$

$\beta^2\beta^3 - \beta^3\beta^2 = 2i\sigma_1$

$\beta^2\beta^3 + \beta^3\beta^2 = 0$

$\beta^2\beta^3 = i\sigma_1$

$\beta^3\beta^2 = -i\sigma_1$

$\exp(\beta^2\beta^3\theta)$

$1 + \beta^2\beta^3\theta$

$1 + i\sigma_1$

$p_\mu \to p_\mu + \frac{e}{c}A_\mu$ $\quad p_\mu = i\hbar\frac{\partial}{\partial\mu}$ $\quad A_\mu(\vec{A}, i\varphi)$

$-i\hbar\frac{\partial}{\partial\mu} \to -i\hbar\frac{\partial}{\partial\mu} + \frac{e}{c}A_\mu$

$\frac{\partial}{\partial\mu} \to \frac{\partial}{\partial\mu} + \frac{e}{c\cdot i\hbar}A_\mu = \frac{\partial}{\partial\mu} + \frac{ie}{\hbar c}A_\mu$ $\quad \varkappa = \underline{mc}$

$\cancel{-i\hbar\beta^\mu}\ \Pi_\mu\ \beta^\mu\Pi_\mu + \frac{\varkappa\psi}{-i\hbar} = 0\ \beta^\mu\Pi_\mu$

$-i\hbar\ \beta^\mu\Pi_\mu\psi - \frac{mc}{\hbar}\cdot i\hbar\psi = 0$

$(\beta^\nu\beta^\mu + \beta^\mu\beta^\nu)(\Pi_\nu\Pi_\mu - \Pi_\mu\Pi_\nu)$

$(\beta^\nu\Pi_\nu + imc)(\beta^\mu\Pi_\mu - imc)\psi = 0$

$\beta^\nu\beta^\mu\Pi_\nu\Pi_\mu - \beta^\nu\beta^\mu\Pi_\mu\Pi_\nu$
$+\beta^\mu\beta^\nu\Pi_\nu\Pi_\mu - \beta^\mu\beta^\nu\Pi_\mu\Pi_\nu$

$(\beta^\nu\beta^\mu\Pi_\nu\Pi_\mu + m^2c^2)\psi = 0.$

or $(\beta^\mu\beta^\nu\Pi_\mu\Pi_\nu + m^2c^2)\psi = 0$

Cases $\mu = \nu = 1, 2, 3, 4$ give $\quad 2\delta_{\mu\nu}\cdot\frac{\hbar e}{ic}f_{\mu\nu}$

first term $= \Pi_\mu^2$

For cases $\mu \neq \nu$, we have for eg $\nu = 1, \mu = 2$
& $\nu = 2, \mu = 1$

$\beta^\mu\beta^\nu + \beta^\nu\beta^\mu = 2\delta^{\mu\nu}$

$\beta^\mu\beta^\nu - \beta^\nu\beta^\mu = 2i\sigma^{\mu\nu}$

$\beta^1\beta^2\Pi_1\Pi_2 + \beta^2\beta^1\Pi_2\Pi_1 = \beta^1\beta^2[\Pi_1, \Pi_2]$

$\beta^\mu\beta^\nu = \delta^{\mu\nu} + i\sigma^{\mu\nu}$

$\delta^{\mu\nu}\Pi_\mu\Pi_\nu + i\sigma^{\mu\nu}\Pi_\mu\Pi_\nu$ $\quad i\sigma^{\mu\nu}\Pi_\nu\Pi_1$

$\Pi_\mu^2 + \frac{1}{2}i\sigma^{\mu\nu}[\Pi_\mu, \Pi_\nu]$

$\Pi_\mu^2 + \frac{i}{2}\sigma^{\mu\nu}\cdot\frac{\hbar e}{ic}f_{\mu\nu}$

$\Lambda^{-1}\beta^1\Lambda = a_{1\lambda}\beta^\lambda = a_{11}\beta^1, \quad \Lambda = \beta^1$
$\Lambda^{-1}\beta^2\Lambda = a_{2\lambda}\beta^\lambda = a_{22}\beta^2, \quad \Lambda = \beta^2$ X

$\beta^1\Lambda = \Lambda\beta^1$
$\beta^2\Lambda = \Lambda\beta^2$
$\beta^3\Lambda = \Lambda\beta^3$
$\beta^4\Lambda = -\Lambda\beta^4$

$\cancel{\Lambda = \beta^1}, \Lambda^{-1} = -\frac{1}{\beta_1}$

$\Lambda^{-1}\beta^1\Lambda = \beta^1$
$\Lambda^{-1}\beta^4\Lambda = -\beta^4$

$\Lambda = \beta^1\beta^2\beta^3$, 1st eq $\to \beta^2\beta^3 = \beta^1\beta^1\beta^2\beta^3\beta^1$
$= \beta^2\beta^3$ ✓

$\Lambda^{-1}\Lambda = \beta^3\beta^2\beta^1\beta^1\beta^2\beta^3$
$= \beta^3\beta^2\beta^2\beta^3$
$= -1$

2nd eq $\beta^2\beta^1\beta^2\beta^3 = \Lambda\beta^2 = \beta^1\beta^2\beta^3\beta^2$

L.H.S $= -\beta^1\beta^3$

R.H.S $= \beta^1\beta^2\beta^3\beta^2 = -\beta^1\beta^3$ ✓

3rd eq

L.H.S $= \beta^3\beta^1\beta^2\beta^3 = \beta^1\beta^2$, R.H.S $= \beta^1\beta^2\beta^3\beta^3 = \beta^1\beta^2$ ✓

$\Lambda = i\beta_4$
$\Lambda^{-1} = -i\gamma_4$

4th eq.

L.H.S $= \beta^4\beta^1\beta^2\beta^3 = -\beta^1\beta^2\beta^3\beta^4$, R.H.S $= -\beta^1\beta^2\beta^3\beta^4$ ✓

$-i\beta^4\beta^1 i\beta^4$

Hence for time reversal $\Lambda = \beta^1\beta^2\beta^3$, $\Lambda^{-1} = \beta^3\beta^2\beta^1$ * Correct

(i) $\Lambda^{-1}\beta^1\Lambda = \beta^3\beta^2\beta^1\beta^1\beta^1\beta^2\beta^3 = \beta^3\beta^2\beta^1\beta^2\beta^3 = \beta^1$ ✓

(ii) $\Lambda^{-1}\beta^2\Lambda = \beta^3\beta^2\beta^1\beta^2\beta^1\beta^2\beta^3 = \beta^2$ ✓

(iii) $\Lambda^{-1}\beta^3\Lambda = \beta^3\beta^2\beta^1\beta^3\beta^1\beta^2\beta^3 = \beta^3$ ✓

(iv) $\Lambda^{-1}\beta^4\Lambda = \beta^3\beta^2\beta^1\beta^4\beta^1\beta^2\beta^3 = -\beta^4$ ✓

$-\beta^3\beta^2\beta^4\beta^2\beta^3$
$= +\beta^3\beta^4\beta^3$
$= -\beta^4$

* $\Lambda\Lambda^{-1} = \beta^1\beta^2\beta^3\beta^3\beta^2\beta^1 = 1$
$\Lambda^{-1}\Lambda = \beta^3\beta^2\beta^1\beta^1\beta^2\beta^3 = 1$

Lecture for 27/2/68.

(1) Beta-radioactivity – A process in which a ~~charge~~ nucleus of ~~charge~~ atomic no. Z and mass number A transforms, through emission of an electron, into one of ~~charge~~ atomic no. Z+1 and mass number A. Simplest example of this decay is

neutron → proton + electron – – (a)

which has been observed using modern techniques. The next in order of simplicity is that of the triton into a He^3 nucleus and an electron

$$_1H^3 \rightarrow {}_2He^3 + e^{-1} \quad \text{– – –} \quad (b)$$

The energy of e^- emitted may have any energy up to 18 KeV (1000 eV). The energy of emission of e^- should be given by $\{M(H^3) - M(He^3) - m\}c^2$ which is 18 KeV, but in many reactions of type (b) far less energy e^- are emitted. Where has the energy gone in these cases? Also the spin relation is violated in (b) for all these are fermions & R.H.S gives integral spin while L.H.S. gives half-integral spin.

In all beta-radioactive phenomena this apparent disappearance of energy and violation of conservation of A.M. were observed. In artificial radioactivity also for eg. the decay of a sodium atom viz

$$_{11}Na^{22} \rightarrow {}_{10}Ne^{22} + e^+ \quad \text{– – –} \quad (c)$$

in which a positron was emitted, the same two non-conservations were noticed.

(2) Muon & pion decay – In $\mu^- \rightarrow e^-$ or $\mu^+ \rightarrow e^+$, the law of conservation of momentum does not hold ~~unless a positive~~ nor the conservation of energy unless two additional particles are emitted. For pion decay into a muon, conservation of momentum requires only one additional particle to be emitted.

(3) Pauli's neutrino hypothesis (1932). – Later amplified by Fermi theoretically (1936) In beta decay an additional particle (in muon decay two additional particles) are emitted. These particles have very weak interaction with matter, must be fermions, and rest mass should be zero, and were called neutrinos (or little neutrons). From the relation for energy $E = \dfrac{m_0c^2}{\sqrt{(1-v^2/c^2)}}$, if $m_0 = 0$, E could only have a finite value if $v = c$ ie neutrinos move with speed of light. They have no charge.

Using symbol ν to describe a neutrino (b) should be written as

$$_1H^3 \rightarrow {}_2He^3 + e^- + \nu \quad \text{– –} \quad (b')$$

Similarly pion decay is represented by

$$\pi^{\pm} \rightarrow \mu^{\pm} + \nu \quad \text{– –} \quad (d)$$ [only one ν since $\pi^{\pm}$ is a boson & μ a fermion]

and muon decay by

$$\mu^{\pm} \rightarrow e^{\pm} + \nu + \nu \quad \text{– – –} \quad (e)$$ [Two ν's since μ & e^- are fermions]

Also muon absorption by a nucleus is given by

$$\mu^- + p \rightarrow n + \nu \quad \text{– –} \quad (f)$$ (ν taking away a large portion of rest energy of μ^-)

(4) Detection of neutrinos: – This cannot be done by working with beta-decay processes since

to stop there is very little chance of stopping such a neutrino through a block of lead extending from the earth to the nearest star! The only possibility is to make observations of the interaction with matter of free neutrinos in flight for eg. a reaction like

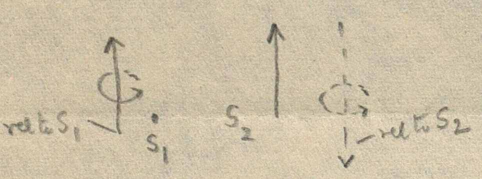
$$\nu + p \rightarrow n + e^{-}$$

This experimental detection has actually been done & the neutrino hypothesis is to be taken as a reality.

(5) ~~Rapid~~ Reflection symmetry and helicity. – Lack of symmetry between existence of particles & anti-particles in Nature necessitates keeping an open mind re. mirror symmetry. In weak interaction phenomena distinction between right handed & lefthandedness, and ideas of mirror images have undergone a drastic revision – $OXYZ$ & $OX'Y'Z'$ as right handed systems while $O\bar{X}\bar{Y}\bar{Z}$ and $OX''Y''Z''$ being lefthanded systems. Test of differentiation is to find whether physical phenomena are changed under $(x, y, z) \rightarrow (-x, -y, -z)$.

Helicity – Right handed & lefthanded screws. Image of a right handed screw in a mirror is a lefthanded screw – Spin of fermions in either of two opposing directions; helicity describes the relation between spin and linear motion – Lee & Yang's observation that no expts. had ever been performed to examine helicity in weak interaction phenomena – Later expts. Check ups confirming – Geometry of expts. relating to Co^{60} nuclei & emissn of electrons

(6) Neutrino as a ghost particle – helicity related to frames of reference relative to observers S_1 & S_2



If S_2 were moving rel. to S_1 in the same sense as the particle or even faster, the particle would appear to move in the opposite sense ie helicity is reversed. Since there is no absolute meaning attached to helicity, it appears meaningless to find dependence of natural phenomena upon it. This is true except for the neutrino. Since it travels with velocity $\underline{c}$, no observer S_2 can move with a velocity > c & hence for a neutron helicity is <u>absolute</u> ie once a lefthanded ~~some~~ neutrino, always a lefthanded neutrino – Suggestion in 1929 of Weyl that a theory was possible in which a neutrino & its anti-particle (which must exist because neutrinos are fermions) ie the anti-neutrino had definite but opposite helicities. This is now accepted & accordingly a neutrino has no mirror image of its own kind, but the mirror image is an anti-neutrino. In a sense it is a ghost which produce a mirror image of its anti-self – Going to the world of particles to anti-particles world, the neutrino → anti-neutrino & from this point of view the reflexion symmetry is also ~~an one~~ part of an overall preference in the Universe for particles as against anti-particles – In the expt. determination of neutrinos, the particles detected were anti-neutrinos. These are emitted with electrons; neutrinos are emitted ~~with~~ only with positrons. Also in beta decay, the spin is assigned in the sense of motion for positrons, and in the opposite sense for electrons.

(7) Lepton conservation – muons, electrons & neutrinos are called <u>leptons</u> & their anti-particles are called <u>anti-leptons</u> (lepton = light particle). In weak particle interactions lepton conservation law holds ie (no. of leptons – no. of anti-leptons) remains constant in any such reaction

(8) Time-reversal symmetry – the assumption of time-reversal symmetry is basic only for Thermodynamics. So far no evidence of failure of this symmetry has been observed & even if it failed it would not affect thermodynamics validity except under condns of extremely high density. It can be shown that if

time-reversal symmetry exists, then complete symmetry must also exist between two systems which have opposite reflection symmetry, provided that the particles in one are replaced by anti-particles in the other. In this way, we could retain mirror symmetry if we suppose that an electron sees itself in a mirror as a positron & vice versa, and that this applies to all particles and anti-particles.

(9) <u>The CPT-theorem of Lee & Yang</u> – This theorem based on what is called the Pauli-Lüders theorem relates to conservation under C (particle-antiparticle transformation), P (parity or space reflection or mirror symmetry) and T (time-reversal), and in its simplest form can be stated as follows:- If one of the operators (P, C, T) is not conserved in a reaction, then at least one other also must not be conserved. Thus there are five possibilities of conservation or non-conservation of P, C, T as in the table below

No	Non-conserved operators	Conserved operators
1	– –	P, C, T
2	C, T	P, CT, TC
3	P, T	C, PT, TP
4	C, P	T, CP, PC
5	P, C, T	PCT & all permutations

Thus the statement under (8) can be brought under No. 4 in the table.

(10) <u>Summary of interactions</u> – Four types viz strong, electromagnetic, weak and gravitational. The strong interactions are those between K-mesons, nucleons & hyperons in which strangeness is conserved. These are on the average 100 times stronger than e.m. interactions between charged particles. Weak interactions are those involving neutrinos, as well as strange particles in which strangeness is not conserved, as in the K-meson or hyperon decays. These weak interactions are about 10^{12} times weaker than the e.m. ones, but even so they are 10^{25} times stronger than the gravitational interaction between fundamental particles. ($1 : 10^{-2} : 10^{-14} : 10^{-39}$)

We are still far from knowing the detailed forms of the strong interactions. Much progress has however been made about knowledge of weak interactions since discovery of reflection ~~asymmetry~~ asymmetry and it may be possible to find breakdown of reflection symmetry even in all cases where ν & $\bar{\nu}$ are not involved. In fact, first clue of this breakdown came from decay of K-mesons of this type, and also this breakdown holds for Λ^0 decay viz

$$\Lambda^0 \rightarrow \pi^- + p.$$

~~The~~ understanding of conservation of baryon number, strangeness and lepton number also demand a deeper understanding of further laws indicating a deeper underlying symmetry in Nature. The following table indicates which symmetries hold or do not hold for strong, e.m & weak interactions of strengths $1 : 10^{-2} : 10^{-14}$

	Strong	e.m	weak
Charge conservation	yes	yes	yes
Baryon "	yes	yes	yes
Lepton "	—	yes	yes
P	yes	yes	no
C	yes	yes	no
T	yes	yes	yes
Strangeness conservation	yes	yes	no

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$(Z, A-Z) \rightarrow Z, A-Z-1$

$\rightarrow Z+1, A-Z-1$