

A Meteoric Mathematician

By F. C. da Gama

WERE Srinivasa Ramanujan alive today he would have been just nine months older than our President, Dr. Radhakrishnan, and his eminence would probably have ranked with that of the noted C. V. Raman, though in a different branch of science. But neither possibility has turned into reality.

Born at Erode on December 22, 1887 the son of a *gumasta* (accountant) to a cloth merchant at Kumbakonam, Srinivassa Iyengar Ramanuja Iyengar (to give his full name) learnt the elements of the three Rs at the Town High School at Kumbakonam and soon became enamoured of the third R. Figures seemed to have cast a spell on him, a spell that brought him both delight and worry.

Too Insipid

Already at school he found the curricular fare in mathematics too insipid and began browsing on such works as Carr's *Synopsis of Pure Mathematics* borrowed from the College Library. An old school fellow of his recalls that due to his extraordinary memory "he could repeat the complete lists of Sanskrit roots *Aatmanepads* and *paramepada*."

Laurelled with the "Junior Subramiam Scholarship" he entered the portals of Government College at Kumbakonam in 1904, but weakness in English forced him to leave it. Another attempt in Madras in 1906 was equally unsuccessful. He therefore pursued his work in mathematics independently, "jotting down his results in two good-sized note books," as his friend Mr. P. V. Seshu Aiyar informs us (*Journal of the Indian Mathematical Society*, 1919).

His marriage in 1909 compelled him to seek employment in the Madras Port Trust where he began working in 1911 on a salary of Rs. 25 a month. "To preserve my brains I want food, and this is now my first consideration," was his candid avowal. As a clerk in the Accounts Department he handled figures galore; but for a genius like him this also meant stagnation.

Through a number of fruitful contacts with men who recognised

his worth as a mathematician, he achieved acquaintance with Prof. G. H. Hardy of Cambridge. A scholarship from the University of Madras and an exhibition from Trinity College, Cambridge, brightened his prospects considerably. After overcoming serious religious difficulties arising from the fact that he was a Brahmin, he was able to sail for England in 1914.

The task that Prof. Hardy faced was by no means an easy one. He had to discipline a young man who was mentally a spirited steed. Ramanujan himself was well aware of this. Once, using the services of a scribe, he had written to Prof. Hardy,

"I have not trodden through the conventional regular course which is followed in a university course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as 'startling'... (They) are not able to understand me in my higher flights."

Disarming Simplicity

And once, after presenting the fruits of his investigations, he added with disarming simplicity, "If I tell you this you will at once point out the lunatic asylum as my goal!"

Prof. Hardy was perplexed and therefore cautious. This is how he describes his predicament: "What was to be done in the way of teaching him modern mathematics? The limitations of his knowledge were as startling as its profundity... All his results, new or old, right or wrong, had been arrived at by a process of mingled argument, intuition and induction, of which he was entirely unable to give any coherent account."

Ramanujan's first papers had to be thoroughly edited, and at times completely rewritten, before they could be submitted for publication. All the while Prof. Hardy was afraid: "If I insisted unduly on matters which Ramanujan found irksome, I might destroy his confidence or break his spell of inspiration." Despite such odds, "in a few years' time he had a very tolerable knowledge of the theory

of functions and the analytic theory of numbers."

While in England Ramanujan manifested interest in subjects other than mathematics too. But they showed the strangest contrasts. "Alike in literature, philosophy, and mathematics," Prof. Hardy tells us, "he had a passion for what was unexpected, strange and odd; he had quite a small library of books by circle-squarers and other cranks." The leopard had not changed his spots! Ramanujan himself had published in the *Journal of the Indian Mathematical Society* (1913) an article captioned "Squaring the Circle."



Srinivasa Ramanujan

His research won him growing recognition leading up to his election in 1918 to the Royal Society and to a Trinity Fellowship. From 1914 till the year of his death and even posthumously several mathematical journals (English, French and German) carried his papers and shorter notes. Such an experience was no doubt stimulating.

In the spring of 1917 Ramanujan took ill; it soon became necessary to hospitalise him; later he spent some time recovering in sanatoria at Wells, Matlok and London. Only in the autumn of 1918 did he show clear signs of improvement and early in 1919

he was thought fit to voyage back to India. Back home, he resumed his mathematical work. A letter of February, 1920, to his patron, Prof. Hardy, bore no discouraging mention of his health; so the news of his death on April 26, 1920 took the professor completely by surprise.

The findings of Ramanujan have been the subject of a number of articles by such mathematicians as L. J. Mordell, H. B. C. Darling, P. A. MacMahon and others. But the one obviously in the best position to evaluate his achievement is his guide and collaborator, Prof. G. H. Hardy.

Hardy's obituary notice in the *Proceedings of the Royal Society* (1921) devotes several pages to some of Ramanujan's remarkable papers. His assessment awards special kudos to his "algebraic approximations of π ", his theory of partitions and the allied parts of the theories of elliptic functions, an "continued fractions," because in these "Ramanujan shows at his very best."

Ramanujan's talent has intrigued many, who have been led to believe that he had some special secret or that there was something abnormal in his mode of thought.

Prodigious Memory

Prof. Hardy admits that he had a prodigious memory and that he could "remember the idiosyncracies of numbers in an almost uncanny way." "Every positive integer," Mr. Littlewood once remarked, "was one of his personal friends." Actually, Prof. Hardy is of the opinion that Ramanujan's work would have been greater had it been less strange; but he cannot deny that it possessed "profound and invincible originality."

That is precisely what made Ramanujan and not just another great mathematician. Had he been caught and tamed in his youth he might have turned out to be a greater mathematician; but circumstances so fashioned his genius that in the brief span of 32 years he flashed through the sky more like a meteor than like an orbiting star.

(A commemorative stamp is being issued this week in Ramanujan's honour.)

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Srinivasa Ramanujan: A biographical sketch and a glimpse of his work

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There have been three truly great algorists in the entire history of mathematics. An algorist is a mathematician who is known for his manipulative ability in a jungle-like algebra of symbols. For him the solution of problems of unusual kinds comes very naturally. He can devise ingenious tricks like the replacement of one or more of the variables in an equation by functions of other variables and thus while apparently complicating the problem actually can throw light on difficult situations which have defied superb intellects of the past. He can manipulate formulae involving infinite processes without being constrained by the puristic need to pay attention to rigour, convergence and mathematical existence and through his extraordinary intuition can arrive most often at the right formulae. Even when he errs the resulting formula is so elegant that his successor mathematicians spend all their time in perfecting or salvaging that beautiful false result of his. He has an unexplainable faith in his own intuition. Ramanujan was one such mathematician. The other two algorists were Leonard Euler (1707-1783) and Carl Gustav Jacobi (1804-1851). But these two had the advantage of a complete university education behind them and unlike Ramanujan they went through the trodden path of academics to acquire their reputation. But Ramanujan did not have this good fortune of a formal university training. He became a mathematician before anybody could think of training him. It is well that he was not trained early because it is debatable whether he could have been so prolific if he had been trained to watch every mathematical step of his. One may say that he was 'uneducated' if I may be permitted to use that word, compared to an Euler or a Jacobi or for that matter any mathematician in the world. Ramanujan was a self-taught genius. He could dispense with all the technical elaborations of the 18th and 19th century mathematics and still have much to say, which continues to occupy the attention of several mathematicians in the world. His dramatic rise to world recognition and his very short career of formalised activity in the then-best of the universities of the world constitute a thrilling success story, so far as the world of mathematics and the pride of India are concerned. However, but for a succession of a few accidents which can be named the world might have missed him.

Born in 1887, Ramanujan was brought up in an orthodox traditional South Indian environment. He was an enigma to his teachers even at school because of his prodigious memory and unusual mathematical talent which began to show even before he was ten. That was the age when he topped the whole district at the Primary examination and this procured him a half-fee concession at school, namely Town High School, Kumbakonam.

At the age of 12, he borrowed Loney's *Trigonometry*, Part II, from a student of the B.A. class who was his neighbour. That student was amazed to find that this young boy, about 7 to 8 years his junior, had not only finished mastering the book at one reading but he had taught himself to do every problem in it. This book, though called *Trigonometry*, has some of the advanced topics of mathematics in it. The treatment of these subjects is weak but the results belong to that part of mathematics called *Analysis*, which deals with continuous processes and expressions which grow in numerical value boundlessly. Topics such as the exponential function, logarithm of a complex variable, hyperbolic functions, infinite products and infinite series expansions of trigonometric functions are dealt with in the book. This book was Ramanujan's first contact with these advanced topics. It is an irony of fate that a better and modern treatment of these areas was not available to Ramanujan. Whittaker's *Modern Analysis* had just arrived but had not reached up to Ramanujan's environment. Bromwich's *Infinite series*, Carslaw's *Fourier series and integrals*, Pierpoint's *Theory of functions of a real variable* and Gibson's *Calculus* were just being written. If Ramanujan had had any one of these books at that age when he was gulping Loney's *Trigonometry* and Carr's *Synopsis*, would it not have made a difference in the mathematical style of Ramanujan? Perhaps. Mathematicians are divided on the answer to this question. There have not been enough geniuses in the world, on whom you can perform controlled experiments in order to answer such questions decisively!

I just mentioned Carr's *Synopsis*. It was this book, called *A synopsis of elementary results in pure mathematics* that created an imperishable record for itself in history, by passing through the hands of Ramanujan in his early teens. Ramanujan was captivated by its contents. It brought forth all his powers, not because it was a great book, but because it was just a compilation of about 6000 theorems with very sketchy proofs, if at all. The challenge to Ramanujan was irresistible and he started working out the proofs of results there in his own way out of his own thinking. Not only could he supply proofs to innumerable results there but he proceeded further to improve them and create his own theorems and results. He began writing theorem after theorem on the pages of quarto notebooks which are today collectively called Ramanujan's *Notebooks*.

He passed the Matriculation examination of the University of Madras in December 1903, secured a first class, and earned for himself the Subramaniam scholarship in the FA (First Examination in Arts) class at Government college, Kumbakonam. His subjects were English, Mathematics, Physiology, Roman and Greek History and Sanskrit. But Mathematics absorbed all his time and energy and he duly failed in the annual examination because of poor marks in the subjects other than mathematics and thus lost his scholarship. He left Kumbakonam, got himself lost somewhere in Andhra region, came back to Government College, Kumbakonam, after a year but could not get the necessary attendance certificate in December 1905 for the examination and thus was lost to Kumbakonam College for ever. Later, he completed the second year FA at Pachaiyappa's college, Madras and sat for the examination in December 1907. Again he failed for the same reason as before.

Prof. S. R. Ranganathan, the first Librarian of the University of Madras and a mathematician himself, calls the period 1907-11 the first period of super-activity in the life of Ramanujan, and writes: 'Inner light began to lead him. The urge for the pursuit of

mathematics became irrepressible. The depression due to failure in the FA examination could not repress it. Failure to get employed could not shake it. Poverty and penury could not obstruct it. His research marched on undeterred by environmental factors—physical, personal, economic or social; magic squares, continued fractions, hypergeometric series, properties of numbers—prime as well as composite, partition of numbers, elliptic integrals and several other such regions of mathematics engaged his thought'. He had to do all this by discovering them *de novo*, because his immediate neighbourhood contained no person or book knowledgeable in these areas. He recorded his results in his notebooks. Proofs were often absent. The profundity of contents of these notebooks as they are being analysed today reveal more and more staggering complexities. Intuition played a large part in these researches. There are three such notebooks in all, containing 212, 352 and 33 pages respectively. Exact facsimiles of these notebooks have now, since 1957, been published in two volumes by the co-operative effort of the University of Madras, the Tata Institute of Fundamental Research and Sir Dorabji Tata Trust.

It was during this period at the age of 22 that Ramanujan was married to Srimathi Janaki, then 9 years old. In 1910 Ramanujan heard of the Indian Mathematical Society which had been founded just three years earlier by Prof. V. Ramaswamy Iyer, a Deputy Collector by profession. Ramanujan ran to him at Tirukkovilur for help. To Ramaswamy Iyer goes the credit of being the first among the chain of discoverers of the genius that was Ramanujan. With his introduction Ramanujan went to Prof. Seshu Iyer and the latter put him on to Dewan Bahadur R. Ramachandra Rao, Collector of Nellore District. This historic meeting that took place in December 1910 between the genius and his patron, should be described in Ramachandra Rao's own words: 'Suspending judgment I asked him to come over again and he did. And then he had gauged my ignorance and showed me some of his simpler results. These transcended existing books and I had no doubt that he was a remarkable man. Then step by step he led me to elliptic integrals and hypergeometric series and at last his theory of divergent series not yet announced to the world converted me'. Ramachandra Rao undertook to pay Ramanujan's expenses for a time. After a few months, being unwilling to be supported by any one for any length of time, Ramanujan accepted a clerk's appointment in the office of the Madras Port Trust. But mathematical work did not slacken. His earliest contribution to the *Journal of the Indian Mathematical Society* appeared in 1911. By this time, the Chairman of the Madras Port Trust, Sir Francis Spring, also took interest in him. The clerk in the Madras Port Trust office had become the subject of talk in the academic circles of Madras. Several attempts were made to get for him a regular scholarship from the University of Madras. Mr. R. Ramachandra Rao, Prof. C. S. T. Griffith of the Madras Engineering College, Prof. M. G. M. Hill, University College, London to whom some of Ramanujan's results were communicated, Dr. Gilbert Walker, a senior Wrangler and then Head of the India Meteorological Department, Prof. B. Hanumantha Rao, Chairman of the Board of Studies of the University of Madras, and Justice P. R. Sundaram Iyer—all had a role to play in the succession of events that finally brought Ramanujan to the University of Madras as a Research Scholar on May 1, 1913 at the age of 26 on a stipend of Rs. 75 per month.

Ramanujan thus became a professional mathematician and remained as such for the rest of his short life. He was now above want and had the academic setting to work on his mathematics. Upon the suggestion of Prof. Seshu Iyer and others Ramanujan began a correspondence with Prof. G. H. Hardy, then Fellow of Trinity College, Cambridge. His first historic letter to Prof. Hardy in January 1913, contained an attachment of 120 theorems, all originally discovered by him. Prof. Hardy's first reaction was to dismiss the letter. But later in the evening he and Prof. Littlewood spent two to three hours on the results in the letter. Several of the results completely floored the two experts. Even assuming that some of them were wrong they could not think of any other explanation but that here was a serious mathematical mind, though uninformed. They decided that the author was not a crack, but a genius. History was made in that decision. They decided to encourage Ramanujan. Their efforts to bring him to England finally materialised in March 1914.

Ramanujan spent four very fruitful years at Cambridge, fruitful certainly to him, but more so to the world of mathematics. Hardy records that the time he spent with Ramanujan from 1914 to 1918 was one of the 'most decisive events' of his life-Hardy's life, that is. Later when Ramanujan died at the unexpected age of 32, Hardy in trying to assess Ramanujan's mathematical work before he arrived in England, found it difficult to conclude whether Ramanujan had been aware of the mathematics contained in such and such well-known books. Hardy regrets that he could have easily asked Ramanujan these biographical questions in a straight-forward manner and 'Ramanujan would have answered them frankly'. But says Hardy, that was not to be. Hardy thought it ridiculous at the time to keep asking whether he had seen this book or that while 'he was showing me half a dozen or more new theorems each day': such was the prolific nature of Ramanujan's creativity. Prof. Hardy did try to 'teach' Ramanujan some of the existing mathematics which 'he ought to know', but Hardy was always in doubt whether by 'teaching' Ramanujan he was doing the right thing or not, to the genius in him. This period of Ramanujan has been well chronicled and suffice it to say that out of this superactivity, Ramanujan published 27 papers, seven of them jointly with Hardy. In 1918, he was elected Fellow of the Royal Society and in the same year was also elected Fellow of Trinity College, both honours coming as the first to any Indian. The University of Madras rose to the occasion and made a permanent provision for Ramanujan by granting him an unconditional allowance of £ 250 a year for five years from April 1, 1919, the date of expiry of the overseas scholarship that he was then drawing. The University was also to be moved, by Prof. Littlehailes, the new Director of Public Instruction, who had just returned from the Bombay Conference of the Indian Mathematical Society for the creation of a University Professorship of Mathematics and Ramanujan to be offered that Professorship, but alas, fate decided otherwise.

Unfortunately, Ramanujan had to spend the fifth year of his stay in England in nursing homes and sanatoria. He returned to India in April 1919 and continued to suffer his incurable illness. All the time his mind was totally absorbed in his mathematics. Thus arose the so-called *Lost Notebook* of Ramanujan, which has been discovered in the last decade. It contains 100 pages of writing and has in it a treasure-house of about 600 fascinating results. Prof. G. E. Andrews of Pennsylvania State University has started

writing a series of papers editing this Lost Notebook of Ramanujan. Ramanujan's discoveries and flights of intuition contained in the four notebooks and in his 32 published papers as well as in the three Quarterly Reports which he submitted to the University of Madras in 1913-14, have thrilled mathematicians the world over. More than two hundred research papers have been published in the world as a result of his discoveries. We shall therefore end this account of Ramanujan by attempting to record some glimpses of his mathematical achievements, even though in a simple and sketchy manner.

A partition of an integer N is a finite sequence a, b, c, d, \dots, r of positive integers, called 'parts' of the partition, such that

$$a + b + c + \dots + r = N.$$

For example, 4, 3, 3, 2, is a partition of 12. We write the partition as 4332 without even the commas separating the integers. 522111 is another partition of 12. Note that we always write a partition in such a way that as we read it, the parts do not increase. How many partitions are there of a given integer n ? The answer is $p(n)$ in standard terminology.

$$p(1) = 1$$

$$p(2) = 2, \text{ for } 2 \text{ and } 11 \text{ are the partitions of } 2.$$

$$p(3) = 3, \text{ for } 3, 21 \text{ and } 111 \text{ are the partitions of } 3.$$

$$p(4) = 5, \text{ for } 4, 31, 22, 211 \text{ and } 1111 \text{ are the partitions of } 4.$$

And so on. $p(200) = 397299029388$. Thus $p(n)$ becomes very large very rapidly.

Very little is known about the arithmetical properties of $p(n)$. Even questions like whether $p(n)$ is odd or even, for a given n , is difficult to answer. Ramanujan was the earliest mathematician to enquire into such properties. Ramanujan observed properties like the following: Whatever integer n might be, $p(5n + 4)$ is divisible by 5; $p(7n + 5)$ is divisible by 7 and similar ones. In connection with these properties, Ramanujan proved a number of identities, one of which is:

$$p(4) + p(9) x + p(14) x^2 + \dots = \frac{5 \{ (1 - x^5) (1 - x^{10}) (1 - x^{15}) \dots \}^5}{\{ (1 - x) (1 - x^2) (1 - x^3) \dots \}^6}.$$

This result has been considered to be representative of the best of Ramanujan's work by Hardy. Hardy says: 'If I had to select one formula for all Ramanujan's work, I would agree with Major Macmahon in selecting the above'.

The practical evaluation of $p(n)$ was, till the time of Ramanujan, done by a formula which goes back to the Euler-Jacobi tradition but was applicable to only small values of n . In 1918 Hardy and Ramanujan published a joint paper in the *Proceedings of the London Mathematical Society* on an exact formula for $p(n)$. It is very complicated to write for a lay audience. It is considered as a crowning achievement in the theory of partitions. The astonishing part of the discovery and establishment of this theorem is that, we can say, on the authority of the collaborator, Prof. Hardy, we would be nowhere

near the final theorem as it is known today but for Ramanujan's unusual intuition and stroke of insight, at two stages of breakthrough in the evolution of the theorem.

In the 120 theorems that he sent to Hardy, in his first letter from India, Ramanujan claimed as one of his theorems that the number of primes less than x is

$$\int_c^x \frac{dt}{\log t} - \frac{1}{2} \int_c^{\sqrt{x}} \frac{dt}{\log t} - \frac{1}{3} \int_c^{\sqrt[3]{x}} \frac{dt}{\log t} - \frac{1}{5} \int_c^{\sqrt[5]{x}} \frac{dt}{\log t} \\ + \frac{1}{6} \int_c^{\sqrt[6]{x}} \frac{dt}{\log t} - \dots, \text{ where } c = 1.45136380 \text{ nearly.}$$

Ramanujan, of course, had not merely guessed his theorems such as this. No flight of imagination could rise to such heights and to such precision. Actually the above result of Ramanujan is false, as shown elaborately by Hardy in his 'Lectures on subjects suggested by the life and work of Ramanujan'. However, the very fact that Ramanujan discovered the above series, better known to mathematicians as Riemann's series, all by himself is a stroke of genius and 'a very astonishing performance'. The error in Ramanujan's statement involves subtleties of complex function theory, a nineteenth century development in the mainstream of mathematics, of which he was not aware until Hardy ventured to 'teach' him. The question about prime numbers that Ramanujan seeks to answer in the above theorem is one of the most fascinating in all of mathematics and the very fact that Ramanujan, as a mere untutored explorer, could rise to the heights of the maturity of Riemann, is Intuition Par Excellence!

Next we shall refer to Entry 29 of Chapter V of Ramanujan's second *Notebook*. It is actually an entry cancelled by Ramanujan himself; and there lies the interest in this story. The entry reads as follows:

$$\frac{1}{(1-x^2)(1-x^3)(1-x^5)(1-x^7)(1-x^{11})(1-x^{13}) \dots} \\ = 1 + \frac{x^2}{1-x} + \frac{x^{2+3}}{(1-x)(1-x^2)} + \frac{x^{2+3+5}}{(1-x)(1-x^2)(1-x^3)} \\ + \frac{x^{2+3+5+7}}{(1-x)(1-x^2)(1-x^3)(1-x^4)} + \dots$$

Let us explain the genius of Ramanujan in cancelling this entry. Suppose both the above expressions are expanded in powers of x , we may then have

$$1 + c_2x^2 + c_3x^3 + c_4x^4 + \dots = 1 + d_2x^2 + d_3x^3 + d_4x^4 + \dots$$

One can calculate c_2 and d_2 and see that they are equal. Similarly $c_3 = d_3$; $c_4 = d_4$ and so on. A natural temptation is to generalise and question whether $c_n = d_n$ for every integer

n . At this point it is necessary to digress and warn the non-mathematical reader on a culture that is unique to a mathematical mind and discipline. Consider the following statement:

(*): n is not divisible by both 2^8 and 5^8 .

This statement (*) is true for all numbers $n = 1, 2, 3, 4, \dots$ up to $n = 99,999,999$. For experimental scientists a rule which is valid for such a large number of cases, is valid for 'all practical purposes' as a general rule. But, for a mathematician, (*) is not true for all n ; for it fails for $n = 10^8$ and all multiples of 10^8 .

Thus, when Ramanujan wrote the above formula he immediately struck it off as not true, because surprisingly, though

$$c_n = d_n \text{ for } n = 1, 2, 3, \dots, 20$$

it happens that

$$c_{21} = 30; d_{21} = 31 \text{ and so } c_{21} \neq d_{21}, \dots$$

So the two sides of the formula are not equal! But then why did he write it all, in the first place? My only guess is that he did not work out his formula on a separate sheet of paper and then transfer it to his notebook for then he would not have started writing the formula at all. As he wrote it thinking it to be true, he must have had his own methods of verifying the truth mentally and by the time he finished writing it, he must have realised the falsity of the formula and he must have struck it off!

Incidentally even this entry has generated much research. Two sequences $\{c_n\}$ and $\{d_n\}$ are said to be a 'Ramanujan pair' nowadays if $\{c_n\}$ takes the place of 2, 3, 5, 7, 11, ... on the left hand side of the above formula and $\{d_n\}$ takes the place of 2, 3, 5, 11, ... on the right hand side and the two sides are equal. As of today it has been proved that there are on the whole only 10 Ramanujan pairs out of a theoretical possibility of an infinite number of pairs. This rarity of Ramanujan pairs shows how Entry 29 almost tantalised Ramanujan to write it and then cancel it in no time.

Our last example is an amusingly delightful entry from the *Lost Notebook* of Ramanujan, not very difficult to visualise for the scientific layman:

$$\begin{aligned} \frac{1}{1-a} + \sum_{n=1}^{\infty} \frac{b^n}{(1-ax^n)(1-ax^{n-1}y)(1-ax^{n-2}y^2) \dots (1-ay^n)} \\ = \frac{1}{1-b} \sum_{n=1}^{\infty} \frac{a^n}{(1-bx^n)(1-bx^{n-1}y)(1-bx^{n-2}y^2) \dots (1-by^n)}. \end{aligned}$$

Prof. G. E. Andrews in his comments on this entry, gives an analytic proof of this using very heavy mathematics initiated by Ramanujan himself. In the same breath Andrews also observes that there is a combinatorial proof of the above using ideas of symmetry. Which method did Ramanujan use to arrive at this formula, or did he use a third method? These are questions for which we may never know the answer.

Such was Ramanujan and such was his genius. One may ask: In what sense is his mathematics relevant? Shall one reply that R. J. Baxter of the Australian National University has found that some of Ramanujan's work was exactly what he needed to solve the hard hexagon model in statistical mechanics? Or shall one quote Carlos Moreno of the City University of New York that Ramanujan's work in the area of modular forms is exactly what physicists need when they work on the 26-dimensional mathematical models of string theory? No. The question about relevance is irrelevant, as far as an assessment of Ramanujan's work is concerned. Ramanujan is great not because his work can also be used in modern technology but because his ideas and innovative genius have not been surpassed ever before or even 100 years after him. William Gosper of Symbolics, Inc., while recently devising a new computer algorithm to calculate π for 17.5 million digits finds that his best ideas had already been discovered by Ramanujan. We shall only quote B. C. Berndt who has just completed editing and analysing the 21 chapters of Ramanujan's second *Notebook* — Part I of Berndt's analysis was published by Springer-Verlag in 1985 — and this will give the layman a quick glimpse of the phenomenon that was Ramanujan:

'Because of the unique circumstances shaping Ramanujan's career, inevitable queries arise about his greatness. Here are three brief assessments of Ramanujan and his work.

'Paul Erdős has passed on to us Hardy's personal ratings of mathematicians. Suppose that we rate mathematicians on the basis of pure talent on a scale from 0 to 100. Hardy gave himself a score of 25, Littlewood 30, Hilbert 80 and Ramanujan 100.

'Neville began a broadcast in Hindustani in 1941 with the declaration, "Srinivasa Ramanujan was a mathematician so great that his name transcends jealousies, the one superlatively great mathematician whom India has produced in the last thousand years.

'In notes left by Wilson, he tells us George Polya was captivated by Ramanujan's formulae. One day in 1925 while Polya was visiting Oxford, he borrowed from Hardy his copy of Ramanujan's *Notebooks*. A couple of days later, Polya returned them in almost a state of panic explaining that however long he kept them, he would have to keep attempting to verify the formulae therein and never again would have time to establish an original result of his own.

'To be sure, India has produced other great mathematicians, and Hardy's views may be moderately biased. But even though the pronouncements of Neville and Hardy and overstated, the excess is insignificant, for Ramanujan reached a pinnacle scaled by few'.

Ramanujan's birth, his super activity in Madras and Cambridge, his glorious rise and his unfortunate death — all seem to have happened in a flash. He came and went like a meteor. When comes such another?

Madras

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5th Aug. 1913.

From S. Ramanujan, Scholarshipholder in Mathematics.

To the Board of Studies in Mathematics.

Through The Registrar, University of Madras.

Gentlemen,

With reference to para. 2 of the University Registrar's letter no. 1631 dated the 9th April, 1913, I beg to submit herewith my quarterly Progress Report for the quarter ended the 31st July, 1913.

The Progress Report is merely the exposition of a new theorem I have discovered in Integral Calculus. At present there are many definite integrals the values of which we know to be finite but still not possible of evaluation by the present known methods. This theorem will be an instrument by which at least some of the definite integrals whose values are at present not known can be evaluated. For instance, the integral treated in Ex. (v) note Art. 5 in the paper, Mr. G. H. Hardy, M.A., F.R.S., of Trinity College, Cambridge, considers to be "new and interesting" Similarly the integral connected with the Besselian

Function of the n^{th} order which at present requires many ^[224] complicated manipulations to evaluate can be readily inferred from the theorem given in the paper. I have also utilised this theorem in definite integrals for the expansion of functions which can now be ordinarily done by Lagrange's, Bürmann's, or Abel's theorems. For instance, the expansions marked as examples nos. (3) and (4) Art. 6, in the second part of the paper.

The investigations I have made on the basis of this theorem are not all contained in the attached paper. There is ample scope for new and interesting results out of this theorem. This paper may be considered the first instalment of the results I have got out of the theorem. Other new results based on the theorem I shall communicate in my later reports.

I beg to submit this, my maiden attempt, and I humbly request that the Members of the Board will make allowance for any defect which they may notice to my want of usual training which is now undergone by College Students and view sympathetically my humble effort in the attached paper.

I beg to remain,
Gentlemen,
Your obedient servant,
S. Ramanujan.

1. Subject of the paper.

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If $F(x)$ be a function capable of expansion in positive integral powers of x , then

(A). The value of $\int_0^{\infty} x^{n-1} F(x) dx$ can be found from the coefficient of x^n in the expansion of $F(x)$, and conversely

(B) The expansion of $F(x)$ in powers of x can be found if the value of the integral $\int_0^{\infty} x^{n-1} F(x) dx$ be known.

2. A:-

Let the expansion of $F(x)$ be

$$\phi(0) - \frac{x}{1!} \phi(1) + \frac{x^2}{2!} \phi(2) - \dots$$

then
$$\int_0^{\infty} x^{n-1} F(x) dx = \Gamma(n) \phi(-n).$$

Dem. We know
$$\int_0^{\infty} e^{-rx} x^{n-1} dx = \frac{\Gamma(n)}{r^n}.$$

By giving the values $1, r, r^2, r^3, \dots$ to n on both the sides,

multiplying the results by

$$f(a), \frac{h f'(a)}{1!}, \frac{h^2 f''(a)}{2!}, \frac{h^3 f'''(a)}{3!}, \dots,$$

and adding up all these results, we have

$$\begin{aligned} f(a) \int_0^{\infty} e^{-x} x^{n-1} dx + \frac{h f'(a)}{1!} \int_0^{\infty} e^{-rx} x^{n-1} dx + \frac{h^2 f''(a)}{2!} \int_0^{\infty} e^{-r^2 x} x^{n-1} dx + \frac{h^3 f'''(a)}{3!} \int_0^{\infty} e^{-r^3 x} x^{n-1} dx + \dots \\ = \Gamma(n) \left\{ f(a) + \frac{h}{r^n} \frac{f'(a)}{1!} + \frac{h^2}{r^{2n}} \frac{f''(a)}{2!} + \frac{h^3}{r^{3n}} \frac{f'''(a)}{3!} + \dots \right\}. \end{aligned}$$

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Expanding e^{-x} , e^{-rx} , e^{-r^2x} , ... on the left side in ascending powers of x and collecting all the terms that contain the same power of x , we have, by applying Taylor's theorem,

$$\int_0^{\infty} x^{n-1} \left\{ f(a+h) - \frac{x}{1!} f(a+rh) + \frac{x^2}{2!} f(a+r^2h) - \dots \right\} dx = \Gamma(n) f\left(a + \frac{h}{r^n}\right).$$

Now let us suppose $f(a+hr^n) = \phi(n)$, treating a , h and r as constants. Then we see that $f\left(a + \frac{h}{r^n}\right) = \phi(-n)$; and also $f(a+h)$, $f(a+rh)$, $f(a+r^2h)$, ... are respectively equal to $\phi(0)$, $\phi(1)$, $\phi(2)$, ... Substituting these results in the above we have

$$\int_0^{\infty} x^{n-1} \left\{ \phi(0) - \frac{x}{1!} \phi(1) + \frac{x^2}{2!} \phi(2) - \dots \right\} dx = \Gamma(n) \phi(-n). \quad \text{Q.E.D.}$$

3. When valid?

The above theorem is legitimate if the following conditions are satisfied.

(a). As already stated, $F'(x)$ should be capable of expansion in positive integral powers of x .

(b). $F'(x)$ should be finite and continuous between the limits 0 and ∞ , but not necessarily at 0 and ∞ .

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(c). n should be positive

(d). $x^n F(x)$ should vanish when x becomes infinite.

The first two conditions are evident from the nature of the integral itself.

The third condition is necessary because we have used the Eulerian integral $\int_0^{\infty} x^{n-1} e^{-x} dx = \Gamma(n)$,

which is true only when n is positive.

The fourth condition is also necessary; for if, when $x = \infty$, $x^n F(x)$ does not vanish but be finite, say equal to a , then the greatest term in the expansion of $x^{n-1} F(x)$ is a/x , and consequently the greatest term in $\int x^{n-1} F(x) dx$ is $a \log x$, which is infinite when $x = \infty$. Hence we see that if, when $x = \infty$, $x^n F(x)$ is finite, then

$$\int_0^{\infty} x^{n-1} F(x) dx$$

is infinite; and much more so it will be if $x^n F(x)$ is itself infinite when x becomes infinite.

Although the first three conditions are necessary in case of oscillating functions, such as the circular, Besselian and other functions, yet the fourth condition differs for different functions we take.

4. Generalization: -

(a). The theorem can be used not only in case of Integrals having the limits 0 and ∞ , but also in case of Integrals having any two limits; for any integral $\int_{\alpha}^{\beta} \psi(x) dx$ may be transformed to an Integral of the form $\int_0^{\infty} F(x) dx$ by suitable substitutions such as $\frac{x-\alpha}{\beta-x} = y$, etc.

(b). According to the condition 3(a), $F(x)$ may include all algebraic functions and all transcendental functions which can be expanded in ascending powers of x , such as $\cos x$, $\sin x$, e^{-x} , $\tan^{-1} x$, $\log(1+x)$, etc. but if $F(x)$ contains transcendental of the form $\log x$, etc., which cannot be expressed in powers of x , we can substitute e^y , etc. for x and then apply our theorem.

(c). Similarly, by suitable substitutions, all fractional powers also may be removed.

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