

On the Collinearity of three Points on a non-singular Cubic.*

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1. Introductory Remarks.

Every non-singular cubic can be reduced to the canonical form

$$x^3 + y^3 + z^3 + 6mxyz = 0;$$

and if three points (x_i, y_i, z_i) ($i = 1, 2, 3$) on this cubic be collinear, we have the well-known condition due to Cayley†, viz.

$$x_1x_2x_3 + y_1y_2y_3 + z_1z_2z_3 = 0.$$

We have shown here that this condition is necessary but not sufficient.‡ We have further deduced other necessary conditions and we have been able to enunciate the following theorem:—

THEOREM:—When three points on a non-singular cubic in the canonical form are represented in homogeneous co-ordinates (x, y, z) , there cannot be only one necessary and sufficient condition of collinearity which is a rational integral function of the co-ordinates and symmetrical in x, y, z .

We have also exhibited sets of conditions necessary as well as sufficient and shown that they must be at least two in number.

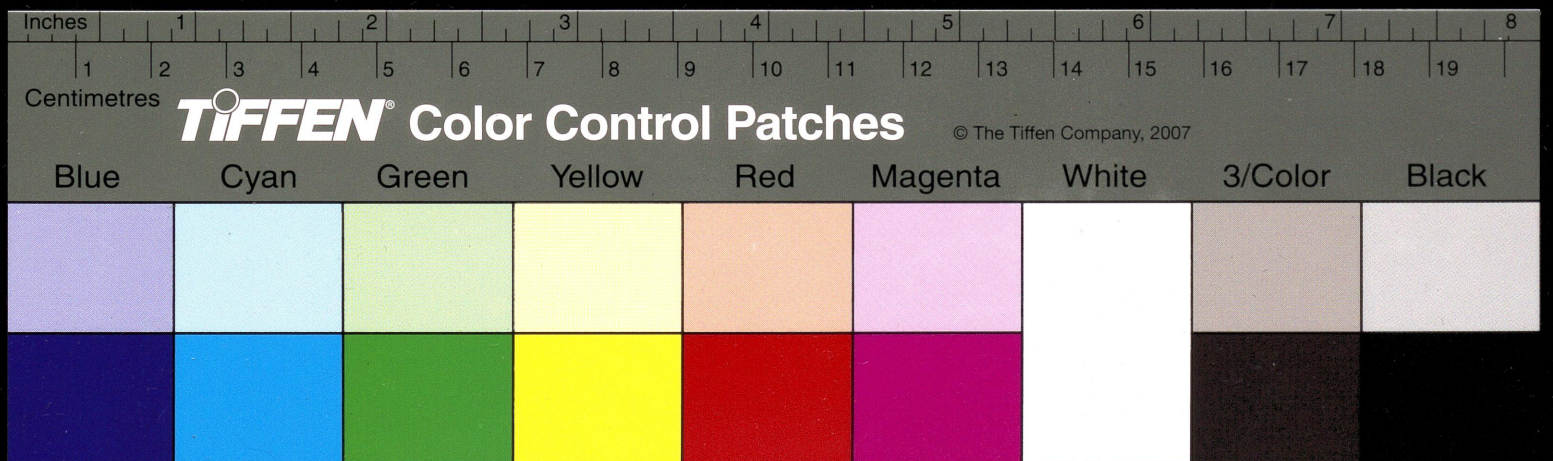
2. The General Problem.

We first proceed to show by *a priori* reasoning that there must be a single such condition for collinearity. In fact, we can choose any three points on a cubic in a 3-fold infinity of ways whereas there are only a 2-fold infinity of straight lines in the plane. Thus the imposing of a single condition is necessary and moreover ought to be sufficient for the collinearity of the three points on the cubic. This fact about the existence of a

* A paper read before the South Indian Science Association, Bangalore, on 30th Sep. 1923.

† See Cayley: *Collected Mathematical Papers*, Vol. 2 : p. 404.

‡ See Ganguli: *Higher Plane Curves*; Vol 2, where it is stated that this condition is necessary as well as sufficient.



single condition has also been proved from other points of view. Representing the co-ordinates of any point on

$$x^3 + y^3 + z^3 + 6mxyz = 0$$

rationally in terms of the elliptic functions of a parameter u , Clebsch* has shown that, if u_1, u_2, u_3 be the parameters corresponding to the three points the *necessary* and *sufficient* condition for collinearity is

$$u_1 + u_2 + u_3 \equiv 0 \pmod{2\omega, 2\omega'}$$

or

$$u_1 + u_2 + u_3 \equiv 0 \pmod{2k, 2k'i}$$

according as the Elliptic Functions are either Weierstrassian or Jacobian.

Let us consider the same question when the co-ordinates of the three points are simply taken as (x_1, y_1, z_1) , (x_2, y_2, z_2) and (x_3, y_3, z_3) . The general condition of collinearity for any three points in a plane is given by

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$$

and the condition we are searching for is the form to which this reduces, when we make use of the facts that

$$x_1^3 + y_1^3 + z_1^3 + 6mx_1y_1z_1 = 0,$$

$$x_2^3 + y_2^3 + z_2^3 + 6mx_2y_2z_2 = 0,$$

$$x_3^3 + y_3^3 + z_3^3 + 6mx_3y_3z_3 = 0.$$

We can therefore solve for z_1, z_2, z_3 from each of the three equations above; substituting their values in the determinant, we obtain a relation among $(x_1, y_1, x_2, y_2, x_3, y_3)$ which is an algebraic function in them since the expression of z in terms of x and y from

$$x^3 + y^3 + z^3 + 6mxyz = 0$$

would involve only square root and cube root. We can thus deduce † a single necessary and sufficient condition even when the points are taken as (x_i, y_i, z_i) ; but it will not be a rational integral function of the co-ordinates and symmetrical in (x, y, z) , and the actual labour involved in obtaining it is very great and, perhaps, not worth the trouble.

What we are concerned with in this paper is a condition which must involve x, y, z symmetrically and be also algebraic in them and the conclusion arrived at is embodied in the theorem of § 1.

* See Clebsch: *Lecons sur la Géométrie*: t.2 (Fr. Edition).

† This point of view is pointed out to us by Prof. Hilton.

If the three points are collinear, the necessary condition is

In virtue of the above, the condition of verification,* can be written as

Any straight line

will meet it in points

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Now $x_1 x_2 x_3$

which is Cayley's result.

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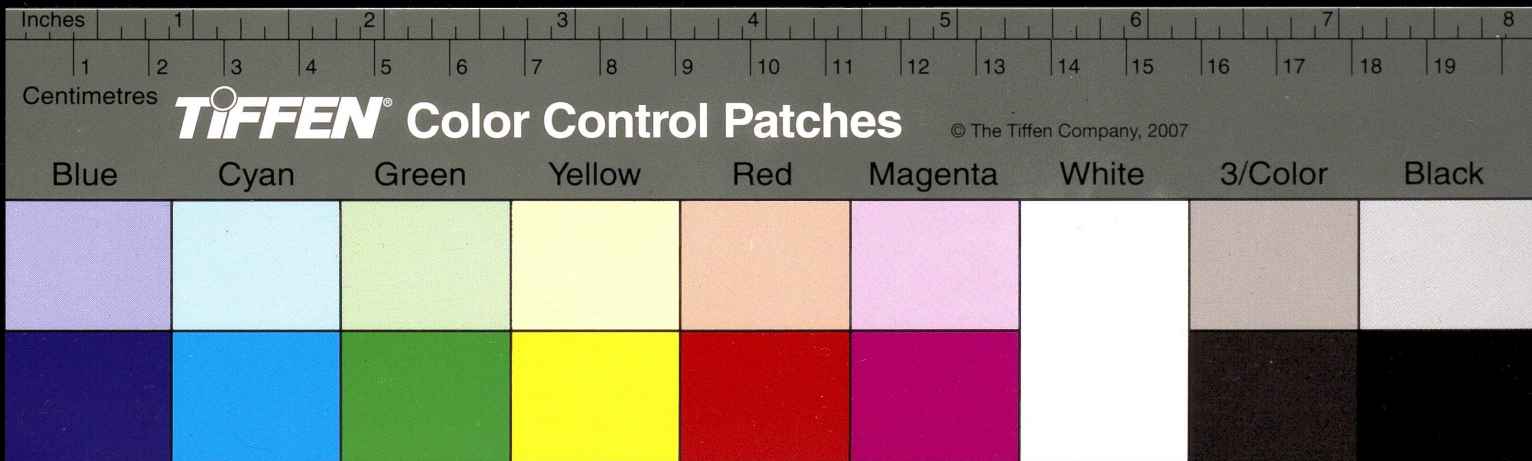
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Collinearity of three Points on a non-singular Cubic. 3

3. Cayley's Condition.

If the three points (x_i, y_i, z_i) ($i = 1, 2, 3$) be collinear, we have the necessary condition due to Cayley, viz.

$$x_1x_2x_3 + y_1y_2y_3 + z_1z_2z_3 = 0.$$

In virtue of the importance of this result we just add an easy method of verification,* consider the cubic

$$x^3 + y^3 + z^3 + 3kxyz = 0.$$

$$\text{Any straight line } \frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = r,$$

will meet it in points given by

$$\Phi(r) = \Sigma (a + lr)^3 + 3k \Pi (a + lr)$$

i.e.,

$$Ar^3 + 3Br^2 + 3Cr + D = 0,$$

where

$$A = l^3 + m^3 + n^3 + 3klmn$$

$$B = \Sigma (al^2) + k \Sigma (amn)$$

$$C = \Sigma (a^2l) + k \Sigma (abn)$$

$$D = a^3 + b^3 + c^3 + 3kabc.$$

$$\text{Now } x_1 x_2 x_3 = (a + lr_1)(a + lr_2)(a + lr_3)$$

$$= \frac{(an - cl)^3 - (bl - am)^3}{A}$$

∴

$$\Sigma x_1 x_2 x_3 = 0,$$

which is Cayley's result.

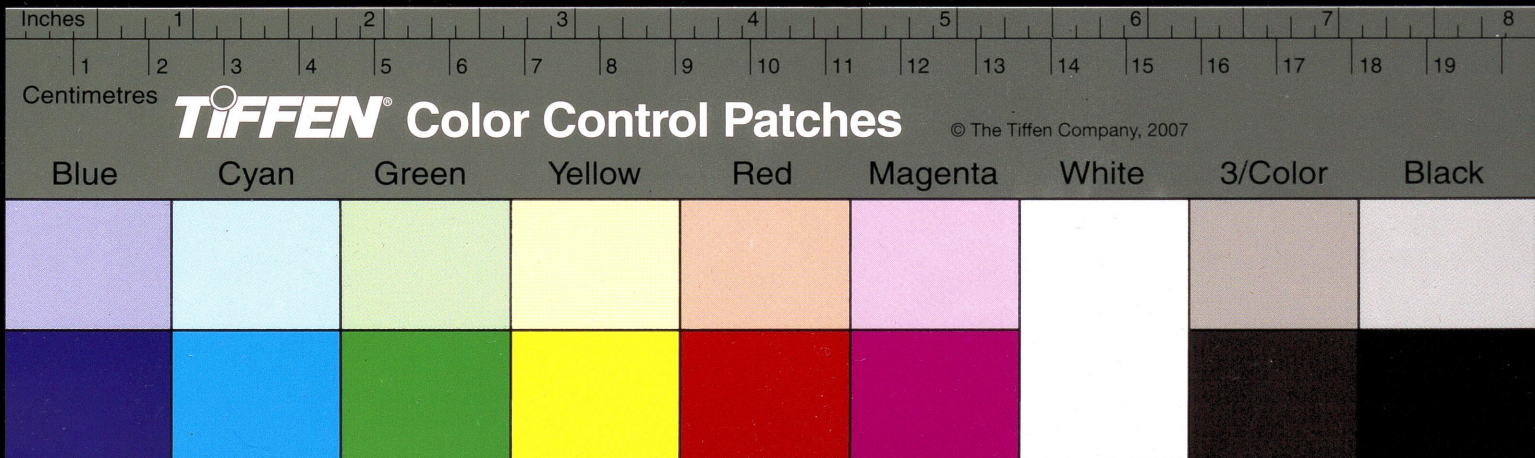
It is quite easy to show that Cayley's condition is not sufficient. That is,

if

$$\Sigma x_1 x_2 x_3 = 0,$$

it does not necessarily follow that the three points are collinear. In fact, consider the line $xx_1x_2 + yy_1y_2 + zz_1z_2 = 0$ which passes through (x_3, y_3, z_3) if we suppose Cayley's condition to be satisfied. This intersects the cubic in two points other than (x_3, y_3, z_3) which are not collinear with (x_1, y_1, z_1) and (x_2, y_2, z_2) but yet satisfy the condition in question. Thus the condition although necessary is not sufficient and we can even specify that the order of insufficiency is two, inasmuch as there are two points which satisfy the condition but do not ensure collinearity.

* Kindly suggested by Prof. M. T. Naraniengar.



It follows therefore that the full geometrical import of Cayley's condition must be wider in scope than mere collinearity. We derive the geometrical significance of the condition from the following considerations. We know that if $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in the plane of the cubic

$$x^3 + y^3 + z^3 + 6mxyz = 0 \quad \dots \dots \dots (i)$$

the equation of the apolar line of P and Q with respect to the cubic is

$$x x_1 x_2 + y y_1 y_2 + z z_1 z_2 + 2m \{ x (y_1 z_2 + y_2 z_1) + \dots \} = 0. (ii)$$

Now if $R(x_3, y_3, z_3)$ is any point on (ii) the relation between the co-ordinates is symmetrical and the triangle PQR is apolar to the cubic.

When P and Q are fixed points, we have an apolar line associated with each cubic of the syzygetic family (i), the apolar lines forming a pencil. When P and Q are points on a syzygetic cubic, the vertex of the pencil is the third point where PQ meets that cubic again.

Let us consider P and Q , two points on the cubic (i). The apolar line of P and Q with respect to the equi-anharmonic cubic

$$x^3 + y^3 + z^3 = 0$$

is

$$x_1 x_2 x + y_1 y_2 y + z_1 z_2 z = 0; \quad \dots (iii)$$

the triangle PQR where R is any point on (iii) is apolar to $x^3 + y^3 + z^3 = 0$. The line (iii) meets the cubic (i) in three points, one of which is the vertex of the apolar lines and is therefore the point where PQ meets the cubic (i) again. Hence the triangle formed by P , Q and any one of these points is apolar with respect to $x^3 + y^3 + z^3 = 0$. The geometrical significance of three points on a cubic satisfying Cayley's condition is therefore that "the triangle formed by these points is apolar with respect to the equi-anharmonic cubic of the first species with its Hessian consisting of three real straight lines."

The property is an invariant one and as such, if it holds for the cubic (i) it also holds for the cubic in the general form.

4. Other linear conditions.

In addition to Cayley's condition, we have also a condition due to Hilton* viz.,

$$x_3 (y_1 z_2 + y_2 z_1) + y_3 (z_1 x_2 + z_2 x_1) + z_3 (x_1 y_2 + x_2 y_1) = 0,$$

* See Hilton : *Plane Algebraic Curves*, p. 238; Ex. 13. We give a proof of this in § 5.

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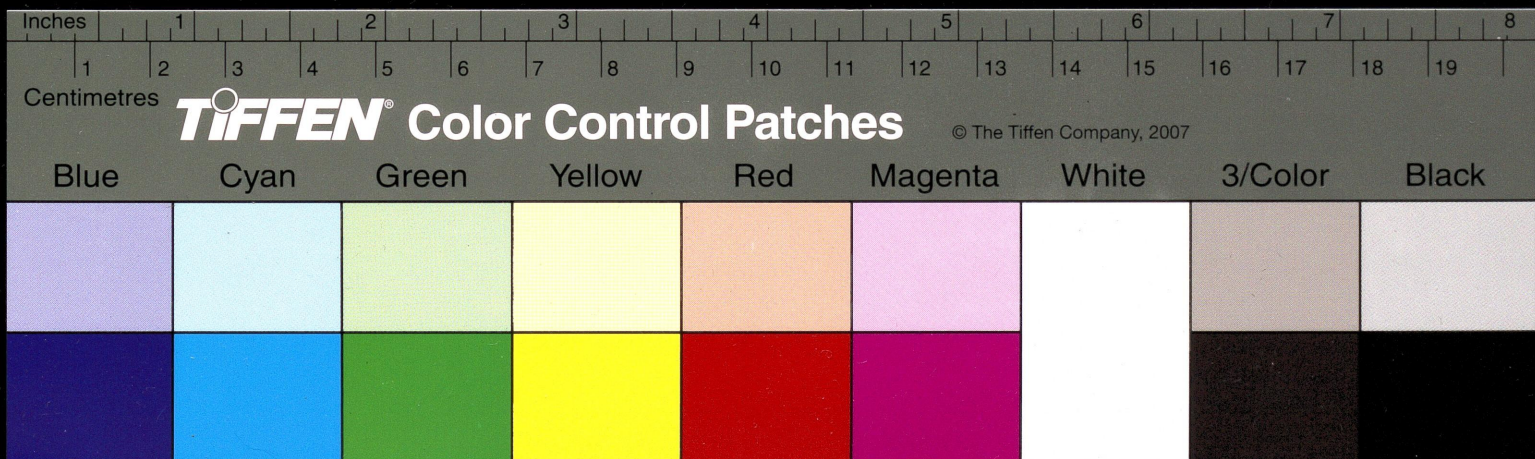
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$$x(y_1z_2 + y_2z_1) + \dots + \dots = 0;$$

we find two points in addition to (x_3, y_3, z_3) which satisfy the condition but are not collinear with (x_1, y_1, z_1) and (x_2, y_2, z_2) .

We now proceed to establish a theorem, viz. :

THEOREM: *Cayley's and Hilton's conditions taken together are necessary and sufficient for the collinearity of the three points.*

For, if the join of A (x_1, y_1, z_1) and B (x_2, y_2, z_2) be not collinear with C (x_3, y_3, z_3) let it meet the cubic again at a point C' (x_3', y_3', z_3') . Since A, B, C' are collinear we have the two conditions

$$\begin{aligned} x_1x_2x_3' + y_1y_2y_3' + z_1z_2z_3' &= 0 \\ (y_1z_2 + y_2z_1)x_3' + \dots + \dots &= 0 \end{aligned}$$

but since C also satisfies both the conditions we deduce that C and C' both lie on the lines

$$\begin{aligned} xx_1x_2 + \dots + \dots &= 0; \\ (y_1z_2 + y_2z_1)x + \dots &= 0; \end{aligned}$$

i.e. C and C' are one and the same point. Thus the two conditions are together necessary and sufficient.

It is to be observed that the conditions due to Hilton and Cayley are linear in any one set of letters (x_1, y_1, z_1) , (x_2, y_2, z_2) and (x_3, y_3, z_3) . We now proceed to determine other such conditions which can be called *linear conditions*. We have the general condition of collinearity of any three points in the plane

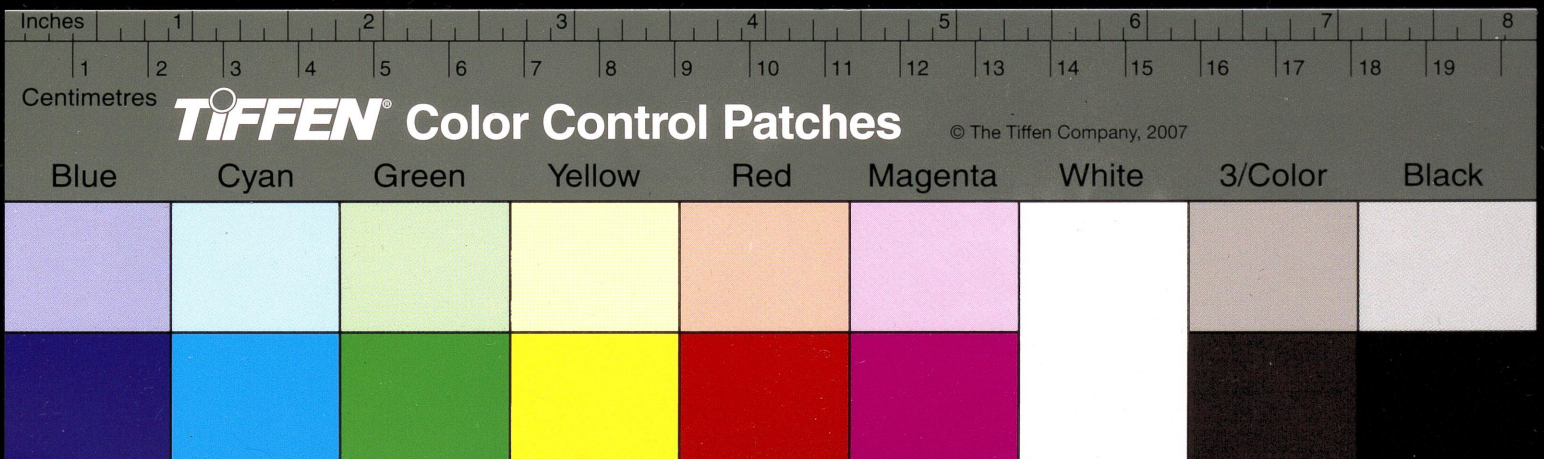
$$w_3(y_1z_2 - y_2z_1) + \dots + \dots = 0$$

which is certainly a necessary condition. Combining this with Hilton's condition, viz.,

$$w_3(y_1z_2 + y_2z_1) + \dots + \dots = 0$$

which is also a necessary condition, we obtain two other necessary conditions by addition and subtraction in the form

$$\begin{aligned} w_3y_1z_2 + y_3z_1w_2 + w_3w_1y_2 &= 0, \\ w_3y_2z_1 + y_3z_2w_1 + w_3w_2y_1 &= 0; \end{aligned}$$



while each of these conditions taken separately is merely necessary, we can establish, by a proof analogous to that used above that these two taken together are necessary as well as sufficient. These two conditions are further worthy of notice inasmuch as they give us remarkably simple expressions for the co-ordinates of the collinear point on the cubic. In fact, we have from the two equations above*

$$\frac{x_3}{(y_1 z_1 x_2^2 - y_2 z_2 x_1^2)} = \frac{y_3}{(z_1 x_1 y_2^2 - z_2 x_2 y_1^2)} = \frac{z_3}{(x_1 y_1 z_2^2 - x_2 y_2 z_1^2)}.$$

We can go still further and state the elegant result that of the three conditions

$$\left. \begin{aligned} x_1 x_2 x_3 + y_1 y_2 y_3 + z_1 z_2 z_3 &= 0 \\ y_1 z_2 x_3 + y_3 z_1 x_2 + y_2 z_3 x_1 &= 0 \\ x_3 y_2 z_1 + y_3 z_2 x_1 + z_3 x_2 y_1 &= 0 \end{aligned} \right\}$$

any two taken together constitute the necessary and sufficient conditions for the collinearity of the three points in question.

5. Necessary conditions of higher degree.

We have hitherto considered only conditions which are linear in terms of the co-ordinates of any one of the three points. We next proceed to determine other necessary conditions which are of higher degree, incidentally indicating a general method of deducing necessary conditions.

Let the cubic

$$x^3 + y^3 + z^3 + 6mxyz = 0$$

be cut by the line

$$\alpha x + \beta y + \gamma z = 0$$

in the points (x_1, y_1, z_1) ; (x_2, y_2, z_2) and (x_3, y_3, z_3) .

Let us join with these two another arbitrary relation

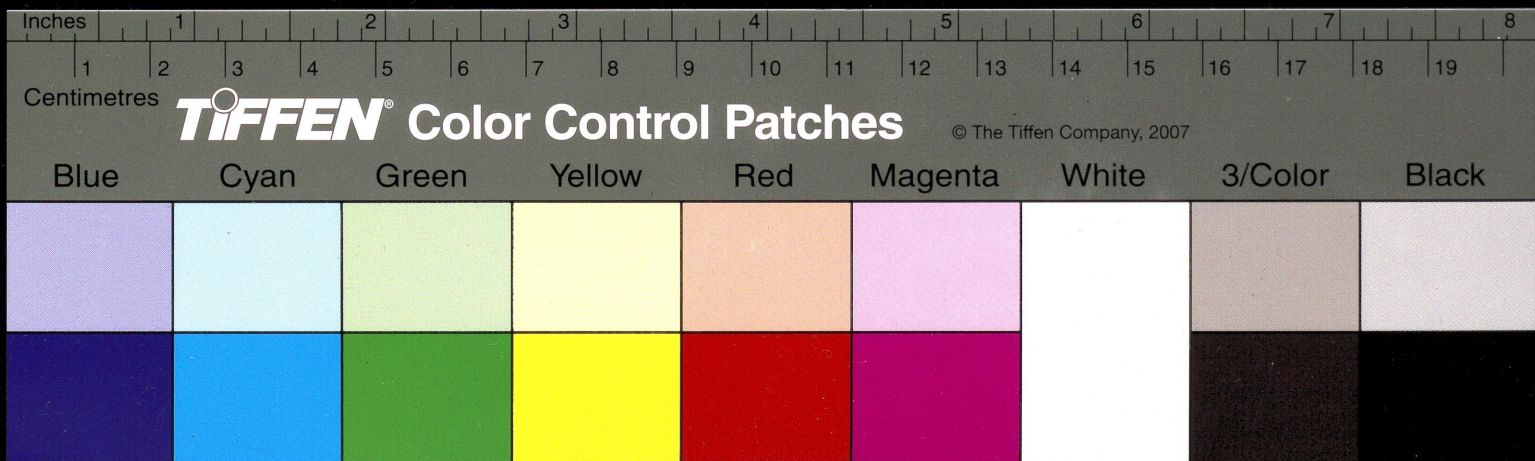
$$px + qy + rz = 0.$$

The result of the elimination† of $x : y : z$ from these three expressions is

$$\begin{aligned} (\beta r - \gamma q)^3 + (\gamma p - \alpha r)^3 + (\alpha q - \beta p)^3 \\ + 6m(\beta r - \gamma q)(\gamma p - \alpha r)(\alpha q - \beta p) = 0, \end{aligned}$$

* These are the expressions for the collinear, otherwise deduced by Sylvester in his *Collected Mathematical Papers*: Vol. 3, pp. 354-5.

† This method of elimination is due to **Schläfi**: See his memoir "*Ueber die Resultante eines Systemes mehrerer algebraischer Gleichungen*" which has been expounded by **Cayley**. See his *Coll. Papers*, Vol. 2, pp. 454-64.



We have further, since

$$F = \gamma(\beta\gamma + 2k\alpha^2) \text{ and } -I = \beta(\delta\gamma + 2k\alpha^2),$$

$$\frac{F}{I} = -\frac{\gamma}{\beta}.$$

Similarly

$$\frac{G}{J} = -\frac{\alpha}{\gamma'}$$

and

$$\frac{H}{K} = -\frac{\beta}{\alpha};$$

so that

$$\frac{F}{I} \cdot \frac{G}{J} \cdot \frac{H}{K} = -1,$$

i.e.,

$$FGH + IJK = 0.$$

This is therefore a necessary condition which is of degree three in the co-ordinates of any one set, since F, G, H, I, J, K are all linear by themselves. We can similarly deduce conditions of degrees six, nine, etc., but we shall take

$$FGH + IJK = 0,$$

as typical of the higher degree conditions. That this is a necessary condition is obvious from its very method of derivation. We now proceed to show that it is really a condition independent of the linear conditions deduced in § 5 and that it is not also a sufficient condition. The condition can be written in the form

$$27(FGH + IJK) = 0:$$

and putting

$$l = x_1 x_2; m = y_1 y_2; n = z_1 z_2$$

$$l' = y_1 z_2 + y_2 z_1; m' = z_1 x_2 + z_2 x_1; n' = x_1 y_2 + x_2 y_1$$

this can be written in the form

$$\begin{aligned} & 2(lmn + l'm'n')x_3y_3z_3 + m(m'n' + l'l')z_3^2x_3 \\ & + m(l'm' + nn')z_3x_3^2 + n(n'l' + mm')x_3^2y_3 \\ & + n(m'n' + ll')x_3y_3^2 + l(l'm' + nn')y_3^2z_3 \\ & + l(n'l' + mm')y_3z_3^2 = 0. \end{aligned}$$

Using the conditions

$$A + B + C = 0, \quad L = 0$$

i.e.,

$$lx_3 + my_3 + nz_3 = 0$$

and

$$l'x_3 + m'y_3 + n'z_3 = 0$$

this reduces on rearrangement to the symmetrical form

$$\begin{aligned} & lmn'z_3^3 + mn'l'x_3^3 + nlm'y_3^3 \\ & + (mn'l'^2 + nn'l'^2 + ll'^2 - 2lmn - 2l'm'n')x_3y_3z_3 = 0. \end{aligned}$$

Collinear

Making x_3 , cubic and hence expression as o

Thus the

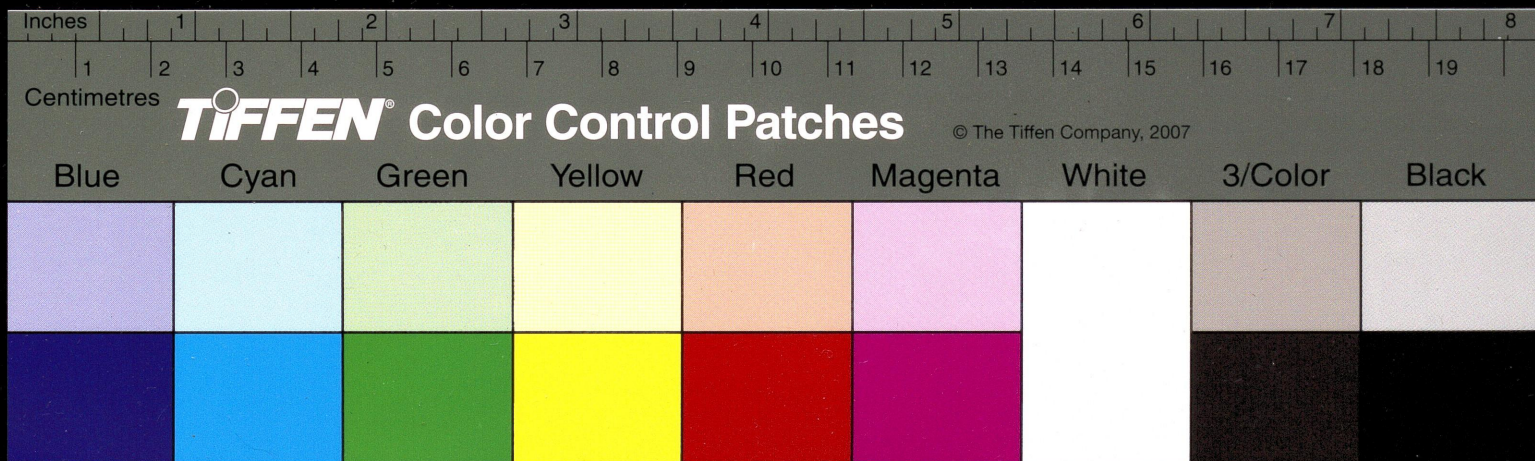
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That this considered above (x_3, y_3, z_3) along other eight point

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Making x_3, y_3, z_3 the current co-ordinates, this represents a non-singular cubic and hence the condition cannot reduce to the cube of a linear expression as otherwise the corresponding cubic would have a triple point.

Thus the condition

$$FGH + IJK = 0$$

is really a condition independent of the linear conditions. This same conclusion can also be reached by examining the cubic

$$\sum mn'l'x^3 + \{ \sum W'^2 - 2lmn - 2l'm'n' \} xyz = 0$$

for a triple point at (x_3, y_3, z_3) , i.e., taking Sylvester's form for the collinear at

$$(y_1z_1x_2^2 - y_2z_2x_1^2, \dots, \dots).$$

That this condition is insufficient follows very easily. The cubic considered above intersects the original cubic in nine points of which (x_3, y_3, z_3) alone is collinear with (x_1, y_1, z_1) and (x_2, y_2, z_2) , whereas the other eight points while they satisfy the condition

$$FGH + IJK = 0$$

are non-collinear with them. We can thus state that the order of insufficiency is 8 in this case. Similarly any condition of a higher degree can be proved to be insufficient. In fact, we can state that for a condition

$$f_n(x, y, z)_{i=1,2,3} = 0$$

the order of insufficiency is $3n - 1$; for making (x_3, y_3, z_3) the current co-ordinates the equation would represent an n -ic which intersects the original cubic in $3n$ -points of which (x_3, y_3, z_3) is one. Thus any one of the remaining $(3n - 1)$ points taken with the points (x_1, y_1, z_1) and (x_2, y_2, z_2) would satisfy the above condition but would not be collinear with those points.

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 m'y₃z₃² = 0.
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