

SYMPOSIUM ON UNIFIED FIELD THEORIES

21st Conference of the Indian Mathematical Society
held at Banaras, 1955.

SEARCH FOR "EVERY WHERE REGULAR" SOLUTIONS OF THE FIELD EQUATIONS

P. C. Vaidya,
Gujarat College, Ahmedabad.

It is well known that in the scheme of general relativity, the mass particles are represented by the singularities in the field. The equations of motion are obtained as prescribing the motions of these singularities. In a purely gravitation theory, singularities will always remain at points occupied by particles. This is because the closer we approach the particles the more pronounced will be the forces other than gravitation become and so the gravitational field must break down near mass-particles. Only when we have a field theory which describes not only the gravitational field, but the total physical field (gravitational, electro-magnetic, nuclear and what not !) shall we have a field free from singularities. This was one of the points of view which led Einstein to generalise the gravitation theory to get the total field theory.

The history of various attempts at such a generalisation from 1923 onwards is quite interesting. Ultimately in 1946 Einstein and in 1947 Schrödinger independently arrived at the present form of the field equations.

$$R_{ik} = 0; \quad R_{ik,l} + R_{kl,i} + R_{li,k} = 0; \quad g_{ik},_{k} = 0.$$

One criterion for deciding whether these equations represent the total field or not would be to see if they allow significant solutions which would be regular every where (i. e. which would have no singularities). After describing these equations in his autobiographical notes, Einstein ends with this query : "What are the every where regular solutions of these equations ?" To get such solutions without any singularity anywhere, mere approximations will not do. One must get a rigorous (non-approximate) solution in order to be satisfied that it is regular every-where.

The well known rigorous solutions of the field equations of general relativity exhibit spherical symmetry round a point. Hence attempts have been made to get rigorous solutions of the field equations of the unified theory which are spherically symmetric round a point. The field equations contain the Einstein tensor They are ultimately differential equations in the affinities These affinities in their turn, are connected in an implicit manner with the fundamental non-symmetric tensor g_{ik} by the 64 algebraic equations. To find a rigorous solution means to find these 16 quantities g_{ik} satisfying the differential equations $R_{ik} = 0$, $R_{[ik,l]} = 0$ via the 64 algebraic equations for Γ_{ik}^l . It is clear that this problem is a very complicated one.

Papapetrou (1948) made a beginning in this direction by first deriving a form of g_{ik} which would exhibit spherical symmetry round a point. This he did by considering the rotation of a sphere through a right angle and finding out the components of g_{ik} at a

point on the axis of rotation which remained unaltered due to this rotation. Thus he reached the conclusion that if of the anti-symmetric part g_{ik} is to exhibit spherical symmetry round a point only g_{14} and g_{23}/μ_0 can survive ($x^1 = r, x^2 = \theta, x^3 = \varphi, x^4 = t$) as functions of r and t . To this antisymmetric tensor was added the spherically symmetric tensor of general relativity and thus the complete tensor g_{ik} was formed. Papapetrou worked out a rigorous solution with $g_{14} \neq 0, g_{23} = 0$ but this solution was neither free from singularity nor "significant". Later Wyman (1950) found out a method of getting a rigorous solution in the other alternative case $g_{14} = 0, g_{23} \neq 0$ which method was fully exploited by Bonnor (1951, 1952) who completely solved Papapetrou's problem by getting the rigorous solution with $g_{14} \neq 0, g_{23} \neq 0$. But all these solutions were either not "regular every-where" or were not "significant".

Now in general relativity there is a rigorous spherically symmetric nonstatic solution which is a generalisation of Schwarzschild's solution and which represents the gravitational field of a radiating star (Vaidya 1951). The electromagnetic field outside such a star is described by the field tensor $F^{\mu\nu}$ with non-vanishing "transverse" components $F_{12}, F_{24}, F_{13}, F_{34}$ (Narlikar and Vaidya 1947). It was clear that such a skew tensor with spherical symmetry round a point was not included in Papapetrou's scheme. This led to a reconsideration of the criterion of spherical symmetry of a second rank tensor g_{ik} . (Vaidya, 1953). The method of infinitesimal rotation of a sphere about a diameter was used to derive the criterion of spherical symmetry of a tensor in the form

$$g_{ik, \alpha} \xi^\alpha + g_{i\alpha} \xi_{,k}^\alpha + g_{\alpha k} \xi_{,i}^\alpha = 0$$

Using this criterion the following general form of spherically symmetric skew tensor J_{ik} was derived

$$\begin{pmatrix} 0 & 0 & \beta \sin \theta & H \\ 0 & 0 & E \sin \theta & 0 \\ -\beta \sin \theta & -E \sin \theta & 0 & q \sin \theta \\ -H & 0 & -q \sin \theta & 0 \end{pmatrix},$$

where p, q, E, H are functions of r and t and the orientations of the axes have been so chosen as to make J_{12} and J_{23} vanish. It was clear that Papapetron's tensor was a particular case of this more general form, obtained from it by putting $p = q = 0$.

To this skew tensor, the spherically symmetric form of the symmetric tensor was added and the complete tensor J_{ik} was formed as

$$J_{ik} = \begin{pmatrix} 0 & 0 & \beta \sin \theta & a+H \\ 0 & -\beta & E \sin \theta & 0 \\ -\beta \sin \theta & -E \sin \theta & -\beta \sin \theta & q \sin \theta \\ a-H & 0 & -q \sin \theta & \gamma \end{pmatrix},$$

Using this J_{ik} it was found that the equations $J^{ik}_{;k} = 0$ lead to $\beta E H = 0$. Since $\beta = 0$ would lead one back to Papapetron's case, one started with $E H = 0$. But

one could find that there also the other field equations ultimately led to $p = 0$ (Vaidya 1954). And the case $p = 0$ is the case fully treated by Bonnor. Hence one concludes that the only solutions of the field equations exhibiting spherical symmetry round a point are the solutions of Papapetron, Wyman or Bonnor. As these solutions are known not to be "everywhere regular" one may further conclude that it is no use searching for everywhere regular solutions of the field equations in terms of a spherically symmetric fundamental tensor J_{ik} .

A point regarding the significance of the solutions obtained, may be noted here. If a field which is spherically symmetric round the origin tends to become flat at large distances from the origin, this very boundary condition (as $\lambda \rightarrow 0$) is usually ~~responsible~~ responsible for introducing a singularity at $r = 0$ in the solution. One may, as well try to search for rigorous solutions which are not adjusted to reduce to flat space-time at large distances but are adjusted to, say, the expanding universe model. An extreme case, in which the condition of flat space-time at large distances is abandoned, has been worked out (Vaidya, 1955) and a solution which is regular everywhere is obtained. But when the condition at large distances is abandoned, there does not remain any criterion to ascertain the significance of such solutions.

REFERENCES

- | | | |
|---------------------|------|--|
| Bonnor | 1951 | Proc. Roy. Soc., <u>A209</u> , 353. |
| | 1952 | Proc. Roy. Soc., <u>A210</u> , 427. |
| Narlikar and Vaidya | 1947 | Nature <u>159</u> , 642. |
| Papapetrou | 1947 | Proc. Roy. Irish Acad. <u>A51</u> , 191. |
| Vaidya | 1951 | Proc. Ind. Acad. Sci., <u>A33</u> , 264. |
| | 1953 | Phys. Rev., <u>90</u> , 695. |
| | 1954 | Phys. Rev., <u>96</u> , 5. |
| | 1955 | Proc. Phys. Soc. (London), <u>A68</u> , 260. |
| Wyman | 1950 | Can. J. Math. <u>2</u> , 427. |
-

SEARCH FOR "EVERYWHERE REGULAR" SOLUTIONS OF THE FIELD EQUATIONS

P. C. Vaidya

It is well-known that in the scheme of general relativity, the mass particles are represented by ^{sin}irregularities in the field. In a purely gravitation theory, singularities will always remain at points occupied by particles. This is because, the closer we approach, the particles the more pronounced will the forces other than gravitation become and so the gravitational field must break down near mass particles. Only when we have a field theory which describes the total physical field, shall we have a field free from singularities. One criterion, therefore, for deciding whether the field equations of the unified theory represent the total field or not would be to see if they allow significant solutions which would be regular everywhere. After describing these field equations in his autobiographical notes, Einstein ends with this query: "~~What~~ ^{What} are the everywhere regular solutions of these equations?" To get such solutions without any singularity anywhere, mere approximations will not do. One must get a rigorous (non-approximate) solution in order to be satisfied that it is regular everywhere.

The well-known rigorous solutions of the field equations of general relativity exhibit spherical symmetry round a point. Hence attempts were made to get spherically symmetric rigorous solutions of the field equations of the unified field theory Papapetrou initiated the investigations by finding the spherically symmetric form of the asymmetric tensor g_{ik} . Wyman ~~had~~ got rigorous solutions for a static field and the problem of finding rigorous static spherically

symmetric solutions was completely solved by Bonnor. But all these solutions were either not "regular everywhere" or were not significant.

Now in general relativity there is a rigorous spherically symmetric nonstatic solution which is a generalisation of Schwarzschild's solution and which represents the field of a radiating star. The electromagnetic field tensor in this spherically symmetric field has transverse components $F_{12}, F_{24}, F_{13}, F_{34}$. But Papapetrou's form of J_{ik} did not have such transverse components. This led to a reconsideration of the criterion of spherical symmetry of a second rank tensor J_{ik} . Using this the method of infinitesimal rotation of a sphere about a diameter a more general form of the spherically symmetric tensor J_{ik} was evaluated. But when this tensor was used for solving the field equations, it was found that only solutions possible were those known already. Hence one can conclude that the only spherically symmetric rigorous solutions of the field equations are the ~~solutions of the field equations are the solutions of~~ Papapetrou, Wyman and Bonnor. And these solutions are known to not to be "regular everywhere" or significant.

If a field which is spherically symmetric round the origin tends to become flat at large distances, this very boundary condition (as $r \rightarrow \infty$) is usually responsible for introducing a singularity at $r = 0$. One may, as well try to search for rigorous solutions which are not adjusted to flat space-time as $r \rightarrow \infty$, but are adjusted to, say, the expanding universe model. But before one can do that one must accommodate cosmological models in the scheme of the unified field theory.