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1. INTRODUCTION

DIRAC matrices are defined by the relations

$$\tfrac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = \delta_{\mu\nu} \tag{1}$$

$$(\delta_{\mu\nu} = 0 \text{ for } \mu \neq \nu,\ \delta_{\mu\mu} = 1;\ \mu,\ \nu = 1, 2, 3, 4)$$

and as is well known the 16 elements 1, γ^μ, $i\gamma^\mu\gamma^\nu$ $(\mu \neq \nu)$, $i\gamma^\lambda\gamma^\mu\gamma^\nu$ $(\lambda, \mu, \nu$ different), and $\gamma^1\gamma^2\gamma^3\gamma^4$ form a system of hypercomplex numbers constituting the Dirac algebra. Following Pauli (1936) we will write the above elements in the form*

$$1,\ \gamma^\mu,\ \gamma^{[\mu\nu]},\ \gamma^{[\lambda\mu\nu]},\ \gamma^5 \tag{2}$$

the square brackets indicating anti-symmetry in the indices, and denote any one of the sixteen elements as γ^A (the latin index A runs through from 1 to 16). The factors i are included in the definition of the $\gamma^{[\mu\nu]}$ and $\gamma^{[\lambda\mu\nu]}$, and are so chosen that for every A, $(\gamma^A)^2 = 1$. The product of any two elements γ^A and γ^B is always equal to a third element γ^C except for a numerical factor which has one of the values ± 1, $\pm i$. The γ^A have thus the group property and the introduction of the co-efficients -1, $\pm i$ in the multiplication table makes the group of the symbols γ^A one of order 64. This group can be shown to have two representations of degree four and thirty-two of degree one. Any finite group can be associated with an algebra (the Frobenius algebra) such that the algebra of the group is the direct sum of the algebras of its representations as groups of linear substitutions. For the group of order 64, it can be shown (D. E. Littlewood, 1934) that the group algebra can be reduced to one single matrix algebra of degree four associated with the γ^A, *viz.*, the Dirac algebra.

The Dirac matrices γ^μ permit the relativistic wave equation of the electron being written in the form

$$\gamma^\mu \partial\psi/\partial x^\mu + \kappa\psi = 0, \tag{3}$$

where γ^μ and ψ are 4×4 and 4×1 matrices respectively, and $\kappa = mc/\hbar$. The Lorentz invariance of (3) can be proved by establishing the existence of a

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* We follow in this paper the notation of that of Pauli (1936) which will be referred to hereafter as P.

matrix Λ specifying the transformation $\psi' = \psi$ of $\Lambda\psi$ under a Lorentz transformation. With the aid of (3) and a function $\psi^{\dagger}$ (a 1×4 matrix) satisfying the wave equation

$$\frac{\partial \psi^{\dagger}}{\partial x^{\mu}} \gamma^{\mu} - \kappa \psi^{\dagger} = 0 \tag{3†}$$

and transforming according to $\psi^{\dagger\prime} = \psi^{\dagger} \Lambda^{-1}$, one can construct the following co-variants:

$$\left.\begin{array}{ll} \text{the invariant} & \psi^{\dagger}\,\psi \equiv i\Omega_1 \\ \text{the vector} & \psi^{\dagger}\gamma^{\mu}\psi \equiv S_{\mu} \\ \text{the antisymmetric tensor} & \psi^{\dagger}\gamma^{[\mu\nu]}\psi \equiv -i\,M_{[\mu\nu]} \\ \text{the pseudo-vector} & \psi^{\dagger}\gamma^{[\lambda\mu\nu]}\psi \equiv S_{\lambda\mu\nu} \\ \text{\& the pseudo-scalar} & \psi^{\dagger}\gamma\,\psi \equiv \Omega_2 \end{array}\right\} \tag{4}$$

Between these co-variants Ω_1, Ω_2, S_{μ}, $M_{[\mu\nu]}$ and $S_{[\lambda\mu\nu]}$ there exist, certain quadratic identities (P. 131), *viz.*

$$\left.\begin{array}{l} -\sum\limits_{\mu} S_{\mu}^2 = \Omega_1^2 + \Omega_2^2 \\ \sum\limits_{[\mu\nu]} M_{[\mu\nu]}^2 = \Omega_1^2 - \Omega_2^2 \\ -\sum\limits_{[\lambda\mu\nu]} S_{[\lambda\mu\nu]}^2 = \Omega_1^2 + \Omega_2^2 \\ -\frac{i}{2}\sum\limits_{[\varkappa\lambda],\,[\mu\nu]} M_{[\varkappa\lambda]}\, M_{[\mu\nu]} = \Omega_1\, \Omega_2 \\ \sum\limits_{\varkappa,\,[\lambda\mu\nu]} S_{\varkappa}\, S_{[\lambda\mu\nu]} = 0 \end{array}\right\} \tag{5, a to e}$$

where the indices $\chi\lambda\mu\nu$ are different from one another.

These identities have been established by Pauli (P. 131–36) in a general manner without making use of any particular representation of the γ^{μ} and also taking ψ & $\psi^{\dagger}$ to be entirely arbitrary one-column and one-row matrices of degree four. The proof consists in generalising the quadratic identities to those bilinear in pairs $(\phi, \phi^{\dagger})$, and $(\psi, \psi^{\dagger})$ of such matrices and putting $\phi = \psi$, $\phi^{\dagger} = \psi^{\dagger}$ in these generalized identities. This method has however been applied to derive (5, b), (5, d), and [(5, a) + (5, c)] while (5, a), and (5, e) are obtained directly without the introduccion of a second pair $(\phi, \phi^{\dagger})$, and by using the properties of the B-matrix (P. 121).

We give here a simple method of deriving identities generalising (5). This method consists in taking products of the γ^{A} with 4×4 matrices of rank unity (*i.e.*, matrices expressed as the product of a one column and a one-row matrix) and expressing these products as linear combinations of the elements of the basis of the algebra by using the commutation rules (1). We show that by suitably choosing these products it is possible to derive not only bilinear identities of the Pauli type, but other 'tensor', multi-linear, and polynomial identities. It is also shown that, in general, products of the five types

$$\psi\psi^\dagger,\ \gamma^\mu\psi\psi^\dagger,\ \gamma^{[\mu\nu]}\psi\psi^\dagger,\ \gamma^{[\lambda\mu\nu]}\psi\psi^\dagger,\ \gamma^5\psi\psi^\dagger \tag{2, a}$$

are sufficient to obtain all possible identities.

Similar results have recently been obtained by Harish-Chandra (1945) by using Eddington's E-numbers. It is clear that the abstract algebra of these E-numbers is the algebra of matrices of degree four, the difference being merely a change of basis of the algebra, and it is doubtful if the change of basis from the γ^A to the E-numbers really leads to any simplification in the algebraic work. Our results show that the identities in question do not depend on any particular choice of the basis of the algebra, but are consequences of expressing particular types of matrices in terms of the basis elements.

2. Method of Derivation of Identities

Since γ^A constitute a basis of the complete matric Dirac algebra any arbitrary matrix of the fourth degree T can be represented in the from

$$T = \sum_A t_A \gamma^A \tag{6}$$

by suitably choosing the numbers t_A. Also the spur of each γ^A is zero except $\gamma^A = 1$. (P. 112, lemma 5), *i.e.*

$$\text{Spur } \gamma^A = 0,\ \gamma^A \neq 1. \tag{7}$$

Multiplying both sides of (6) by a particular γ^A and taking spurs we get

$$\text{Sp } (\gamma^A\, T) = 4\, t_A \tag{8}$$

and we can write (6) as

$$4\,T = \sum_A \text{Sp } (\gamma^A\, T)\, \gamma^A = \Sigma \text{ Sp } (T\, \gamma^A)\, \gamma^A \tag{9}$$

Let $\psi, \phi, \chi \ldots\ldots$ be one-column, and $\psi^\dagger, \phi^\dagger, \chi^\dagger, \ldots\ldots$ one-row matrices of degree four respectively. For purposes of derivation of the identities in question the ψ and $\psi^\dagger$, etc., may be entirely arbitary and independent and not necessarily related as solutions of the wave equation (3) and its adjoint ($3^\dagger$). From these we form 4×4 matrices of the type $\psi\psi^\dagger$, $\psi\phi^\dagger$ etc. The importance of such types arises from the fact that these matrices are of rank unity, and this property is known to be independent of the representation of the algebra but is a property of the abstract algebra itself. We next choose a matrix T as the product of these matrices of rank unity with fixed elements γ^B, γ^C, etc., substitute in (9), and multiply both sides of (9) by a $\psi^\dagger$ on the left and a ψ on the right. By evaluation of the spurs on the right-hand side of (9) using the commutation rules (1), the above process gives rise to an identity of the type under consideration. For example, if we take

$$T = \gamma^\mu \psi\psi^\dagger \qquad (\mu \text{ fixed})$$

the process outlined above gives

$$\left.\begin{aligned} 4\,(\psi^\dagger\gamma^\mu\psi)\,(\psi^\dagger\psi) &= \sum_A \mathrm{Sp}\,(\gamma^\mu\psi\psi^\dagger\gamma^A)\,(\psi^\dagger\gamma^A\psi) \\ \text{or, } 4\,(\phi^\dagger\gamma^\mu\chi)\,(\psi^\dagger\phi) &= \sum_A \mathrm{Sp}\,(\gamma^\mu\psi\psi^\dagger\gamma^A)\,\phi^\dagger\gamma^A\phi) \end{aligned}\right\} \qquad (10, a, b)$$

according as we multiply by $\psi^\dagger$ on the left and ψ on the right, or $\phi^\dagger$ on the left and ϕ on the right. The spur expression on the right of (10, *a*) or (10, *b*) is equal to

$$\mathrm{Sp}\,(\psi^\dagger\gamma^A\gamma^\mu\psi) = (\psi^\dagger\gamma^A\gamma^\mu\psi)$$

since the product inside the bracket is a 1×1 matrix. We can easily evaluate the several $\gamma^A\gamma^\mu$ using the rules (1), and obtain a quadratic identity from (10, *a*) and a bilinear one from (10, *b*). Alternatively we might take $T = \gamma^\mu\,\psi\,\phi^\dagger$ and use the prescription of multiplying on the left by $\psi^\dagger$, and on the right by ϕ. This gives another bilinear identity. The same considerations can be easily extended to more than two matrices.

3. Bilinear and Quadratic Identities

The elements of the basis of the algebra can be grouped into the five types given by (2). Of these the fourth type can be written as $\hat{\gamma}^\times$ dual to $\gamma^\times$ defined in the usual manner (P. 132), and the elements classified as

$$1,\ \gamma^\mu,\ \gamma^{[\mu\nu]},\ \hat{\gamma}^\mu,\ \gamma^5 \qquad (2, b)$$

For purposes of deriving the bilinear identities involving ϕ & ψ, and thereby deducing the quadratic ones by putting $\phi = \psi$ & $\phi^\dagger = \psi^\dagger$, we consider products of the type $\gamma^B\gamma^A\gamma^B$, and keeping A fixed we sum for the index B over the elements in the several types (2, *b*). We then vary the index A itself to refer successively to any one element in each of these types. The several sums that arise thus can be evaluated by using (1). For example,

$$\sum_\mu (\gamma^\mu\gamma^A\gamma^\mu) = 4, -2\gamma^\nu,\ 0,\ 2\hat{\gamma}^\nu, \text{ or } -4\gamma^5$$

according as $\gamma^A = 1,\ \gamma^\nu,\ \gamma^{[\lambda\nu]},\ \hat{\gamma}^\nu$ or γ^5 respectively as can be easily verified. We tabulate below the sums obtained by following the above procedure. This table can be used for deriving the bilinear identities, and can be considered a generalisation of lemma 6 of Pauli's paper (P. 113).

TABLE OF SUMS $\sum_{B} \gamma^{B}\gamma^{A}\gamma^{B}$

Sums $\gamma^A =$	1	γ^μ	$\gamma^{[\mu\nu]}$	$\hat{\gamma}^\mu$	γ^5
$1\cdot\gamma^A . 1$	1	γ^μ	$\gamma^{[\mu\nu]}$	$\hat{\gamma}^\mu$	γ^5
$\Sigma\ \gamma^\mu\gamma^A\gamma^\mu$	4	$-2\gamma^\mu$	0	$2\hat{\gamma}^\mu$	$-4\gamma^5$
$\Sigma\ \gamma^\mu\gamma^A\hat{\gamma}^\mu$	$4i\gamma_5$	$-2\hat{\gamma}^\mu$	0	$-2\gamma^\mu$	$-4i$
$\Sigma\ \hat{\gamma}^\mu\gamma^A\gamma^\mu$	$-4i\gamma^5$	$2\hat{\gamma}^\mu$	0	$-2\gamma^\mu$	$4i$
$\Sigma\ \hat{\gamma}^\mu\gamma^A\hat{\gamma}^\mu$	4	$2\gamma^\mu$	0	$-2\hat{\gamma}^\mu$	$-4\gamma^5$
$\Sigma\ \gamma^{[\mu\nu]}\gamma^A\gamma^{[\mu\nu]}$ $=\Sigma\ \hat{\gamma}^{[\mu\nu]}\gamma^A\hat{\gamma}^{[\mu\nu]}$	6	0	$-2\gamma^{[\mu\nu]}$	0	$6\gamma^5$
$\Sigma\ \gamma^{[\mu\nu]}\gamma^A\hat{\gamma}^{[\mu\nu]}$ $=\Sigma\ \hat{\gamma}^{[\mu\nu]}\gamma^A\gamma^{[\mu\nu]}$	$-6\gamma^5$	0	$-2\hat{\gamma}^{[\mu\nu]}$	0	-6
$\gamma^5\gamma^A\gamma^5$	1	$-\gamma^\mu$	$\gamma^{[\mu\nu]}$	$-\hat{\gamma}^\mu$	γ^5

(11)

We now take $T = \sum_{B} \gamma^B\psi\phi^\dagger\gamma^B$ or $T = \sum_{B} \gamma^B\phi\phi^\dagger\gamma^B$, and apply the procedure outlined in the previous article. The summations over B for the several types of γ^A are taken from the rows of table (11), and the above two expressions for T yield in all 16 identities which we will now write down. In doing so we shall use the tensors dual to $M_{[\mu\nu]}$ and $S_{[\lambda\mu\nu]}$, *viz.* $\hat{M}_{[\varkappa\lambda]} \equiv M_{[\mu\nu]}$ and $\hat{S}_\varkappa \equiv S_{[\lambda\mu\nu]}$, and also indicate by dashes the convariants defined with respect to ϕ & $\phi^\dagger$.

(1) $T = \psi\phi^\dagger$, and $T = \phi\phi^\dagger$ lead respectively [by using the first row of (11)] to:

$$-4\ \Omega_1\Omega_1' = (\phi^\dagger\psi)(\psi^\dagger\phi) + \Sigma\,(\psi^\dagger\gamma^\mu\phi) + \Sigma\,(\phi^\dagger\gamma^{[\mu\nu]}\psi)(\psi^\dagger\gamma^{[\mu\nu]}\phi) + \Sigma\,(\phi^\dagger\hat{\gamma}^\mu\psi) + (\phi^\dagger\gamma^5\psi)(\psi^\dagger\gamma^5\phi) \qquad (12, a)$$

$$4\,(\psi^\dagger\phi)(\phi^\dagger\psi) = -\ \Omega_1\Omega_1' + \Sigma\, S_\mu S'_\mu - \Sigma\, M_{[\mu\nu]}\, M'_{[\mu\nu]} + \Sigma\, S_\mu\, \hat{S}' + \Omega_2\Omega_2' \qquad (12, b)$$

Putting $\phi = \psi$; $\phi^\dagger = \psi^\dagger$ both these lead to

$$3\,\Omega_1^2 + \Omega_2^2 = -\,\Sigma\,S_\mu^2 + \Sigma\,M^2_{[\mu\nu]} - \Sigma\,\hat{S}_\mu^2 \qquad (12,\,c)$$

(2) $T = \Sigma\,\gamma^\mu\psi\phi^\dagger\gamma^\mu$, and $T = \Sigma\,\gamma^\mu\phi\phi^\dagger\gamma^\mu$ [using second row of (11)]:

$$4\,\Sigma\,S_\mu S'_\mu = 4\,(\phi^\dagger\psi)\,(\psi^\dagger\phi) - 2\,\Sigma\,(\phi^\dagger\gamma^\mu\psi)\,(\psi^\dagger\gamma^\mu\phi) + 2\,\Sigma\,(\phi^\dagger\hat{\gamma}^\mu\psi)\,(\psi^\dagger\hat{\gamma}^\mu\phi) - 4\,(\phi^\dagger\gamma^5\phi)\,(\psi^\dagger\gamma^5\phi) \qquad (13,\,a)$$

$$4\,\Sigma\,(\psi^\dagger\gamma^\mu\phi)\,(\phi^\dagger\gamma^\mu\psi) = -\,4\,\Omega_1\Omega_1' - 2\,\Sigma\,S_\mu S_\mu' + 2\Sigma\,\hat{S}_\mu\hat{S}_\mu' - 4\,\Omega_1\Omega_2' \qquad (13,\,c)$$

Both these lead to

$$3\,\Sigma\,S_\mu^2 - \Sigma\,\hat{S}_\mu^2 = -\,2\Omega_1^2 - 2\Omega_2^2 \qquad (13,\,c)$$

(3) $T = \Sigma\,\gamma^\mu\psi\phi^\dagger\hat{\gamma}^\mu$, and $T = \Sigma\,\gamma^\mu\phi\phi^\dagger\hat{\gamma}^\mu$ [using fourth row of (11)]:

$$4\,\Sigma\,S_\mu\hat{S}'_\mu = -\,4i\,(\phi^\dagger\gamma^5\psi)\,(\psi^\dagger\phi) - 2\,\Sigma\,(\phi^\dagger\hat{\gamma}^\mu\psi)\,(\psi^\dagger\gamma^\mu\phi) - 2\,\Sigma\,(\phi^\dagger\gamma^\mu\psi)\,(\psi^\dagger\hat{\gamma}^\mu\phi) + 4i\,(\phi^\dagger\psi)\,(\psi^\dagger\gamma^5\phi) \qquad (14,\,a)$$

$$4\,\Sigma\,(\psi^\dagger\gamma^\mu\phi)\,(\phi^\dagger\hat{\gamma}^\mu\psi) = 4\,\Omega_1\Omega_2' - 2\,\Sigma\,S_\mu\hat{S}_\mu' - 2\,\Sigma\,\hat{S}_\mu S_\mu' - 4\,\Omega_2\Omega_1' \qquad (14,\,b)$$

Both these lead to

$$\Sigma\,S_\mu\hat{S}_\mu = 0. \qquad (5,\,e)\text{ or }(14,\,c)$$

(4) $T = \Sigma\,\hat{\gamma}^\mu\psi\phi^\dagger\gamma^\mu$, and $T = \Sigma\,\hat{\gamma}^\mu\phi\phi^\dagger\gamma^\mu$ [using third row of (11)]:

$$4\,\Sigma\,\hat{S}_\mu S_\mu' = 4i\,(\phi^\dagger\gamma^5\psi)\,(\psi^\dagger\phi) - 2\,\Sigma\,(\phi^\dagger\hat{\gamma}^\mu\psi)\,(\psi^\dagger\gamma^\mu\phi) - 2\,\Sigma\,(\phi^\dagger\lambda^\mu\psi)\,(\psi^\dagger\hat{\gamma}^\mu\phi) - 4i\,(\phi^\dagger\psi)\,(\psi^\dagger\gamma^5\phi) \qquad (15,\,a)$$

$$4\,\Sigma\,(\psi^\dagger\hat{\gamma}^\mu\phi)(\phi^\dagger\gamma^\mu\psi) = -\,4\,\Omega_1\Omega_2' - 2\,\Sigma\,\hat{S}_\mu S_\mu' - 2\,\Sigma\,S_\mu\hat{S}_\mu' + 4\,\Omega_2\,\Omega_1' \qquad (15,\,b)$$

Both these lead to

$$\Sigma\,\hat{S}_\mu S_\mu = 0. \qquad (5,\,e)\text{ or }(15,\,c)$$

(5) $T = \Sigma\,\hat{\gamma}^\mu\psi\phi^\dagger\hat{\gamma}^\mu$, and $T = \Sigma\,\hat{\gamma}^\mu\phi\phi^\dagger\hat{\gamma}^\mu$ [using fifth row of (11)]:

$$4\,\Sigma\,\hat{S}_\mu\hat{S}_\mu' = 4\,(\phi^\dagger\psi)\,(\phi^\dagger\psi) + 2\,\Sigma\,(\phi^\dagger\gamma^\mu\psi)\,(\psi^\dagger\gamma^\mu\phi) - 2\,\Sigma\,(\phi^\dagger\hat{\gamma}^\mu\psi)\,(\psi^\dagger\hat{\gamma}^\mu\phi) - 4\,(\phi^\dagger\gamma^5\psi)\,(\psi^\dagger\gamma^5\phi) \qquad (16,\,a)$$

$$4\,\Sigma\,(\psi^\dagger\hat{\gamma}^\mu\phi)\,(\phi^\dagger\hat{\gamma}^\mu\psi) = -\,4\,\Omega_1\Omega_1' + 2\,\Sigma\,S_\mu S_\mu' - 2\,\Sigma\,\hat{S}_\mu\hat{S}_\mu' - 4\,\Omega_2\Omega_2' \qquad (16,\,b)$$

Both these lead to

$$3\,\Sigma\,\hat{S}_\mu^2 - \Sigma\,S_\mu^2 = -\,2\,\Omega_1^2 - 2\,\Omega_2^2 \qquad (16,\,c)$$

(6) $T = \Sigma\,\gamma^{[\mu\nu]}\psi\phi^\dagger\gamma^{[\mu\nu]}$ or $T = \Sigma\,\hat{\gamma}^{[\mu\nu]}\psi\phi^{\dagger[\mu\nu]}$ and with $\phi\phi^\dagger$ [using the sixth row of (11)]:

$$-\,4\,\Sigma\,M_{[\mu\nu]}\,M'_{[\mu\nu]} = -\,4\,\Sigma\,\hat{M}_{[\mu\nu]}\,\hat{M}'_{[\mu\nu]} = 6\,(\phi^\dagger\psi)\,(\psi^\dagger\phi) - 2\,\Sigma\,(\phi^\dagger\gamma^{[\mu\nu]}\psi)\,(\psi^\dagger\gamma^{[\mu\nu]}\phi) + 6\,(\phi^\dagger\gamma^5\psi)\,(\psi^\dagger\gamma^5\phi) \qquad (17,\,a)$$

$$4\,\Sigma\,(\psi^\dagger\gamma^{[\mu\nu]}\phi)\,(\phi^\dagger\gamma^{[\mu\nu]}\psi)=4\,\Sigma\,(\psi^\dagger\hat{\gamma}^{[\mu\nu]}\phi)\,(\phi^\dagger\hat{\gamma}^{[\mu\nu]}\psi)$$
$$=-6\,\Omega_1\Omega_1'+2\,\Sigma\,\mathrm{M}_{[\mu\nu]}\,\mathrm{M}'_{[\mu\nu]}+6\,\Omega_2\Omega_2' \quad (17,\,b)$$

Both these lead to

$$\Sigma\,\mathrm{M}^2_{[\mu\nu]}=\Omega_1^2-\Omega_2^2 \quad (5,\,b) \text{ or } (17,\,c)$$

(7) $\mathrm{T}=\Sigma\,\gamma^{[\mu\nu]}\phi\phi^\dagger\hat{\gamma}^{[\mu\nu]}$ or $\mathrm{T}=\Sigma\,\hat{\gamma}^{[\mu\nu]}\psi\phi^\dagger\gamma^{[\mu\nu]}$ and with $\phi\phi^\dagger$; [using the seventh row of (11)]:

$$-4\,\Sigma\,\mathrm{M}_{[\mu\nu]}\,\hat{\mathrm{M}}'_{[\mu\nu]}=-4\,\Sigma\,\hat{\mathrm{M}}_{[\mu\nu]}\,\mathrm{M}'_{[\mu\nu]}$$
$$=-6\,(\phi^\dagger\gamma^5\psi)\,(\psi^\dagger\phi)-2\,\Sigma\,(\phi^\dagger\hat{\gamma}^{[\mu\nu]}\psi)\,(\psi^\dagger\gamma^{[\mu\nu]}\phi)$$
$$-6\,(\phi^\dagger\psi)\,(\psi^\dagger\gamma^5\phi) \quad (18,\,a)$$

$$4\,\Sigma\,(\psi^\dagger\gamma^{[\mu\nu]}\phi)\,(\phi^\dagger\hat{\gamma}^{[\mu\nu]}\psi)=4\,\Sigma\,(\psi^\dagger\hat{\gamma}^{[\mu\nu]}\psi)\,(\phi^\dagger\gamma^{[\mu\nu]}\psi)$$
$$=6i\,\Omega_1\Omega_2'+2\,\Sigma\,\mathrm{M}_{[\mu\nu]}\,\hat{\mathrm{M}}'_{[\mu\nu]}+6i\,\Omega_1\Omega_2' \quad (18,\,b)$$

Both these lead to

$$-\frac{i}{2}\,\Sigma\,\mathrm{M}_{[\mu\nu]}\,\hat{\mathrm{M}}_{[\mu\nu]}=\Omega_1\Omega_2 \quad (5,\,d) \text{ or } (18,\,c)$$

(8) $\mathrm{T}=\gamma^5\psi\phi^\dagger\gamma^5$; $\mathrm{T}=\gamma^5\phi\phi^\dagger\gamma^5$, [using the eighth row of (11)]:

$$4\,\Omega_2\Omega_2'=(\phi^\dagger\psi)\,(\psi^\dagger\phi)-\Sigma\,(\phi^\dagger\gamma^\mu\psi)\,(\psi^\dagger\gamma^\mu\phi)+\Sigma\,(\phi^\dagger\gamma^{[\mu\nu]}\psi)\,(\psi^\dagger\gamma^{[\mu\nu]}\phi)$$
$$-\Sigma\,(\phi^\dagger\hat{\gamma}^\mu)\,(\psi^\dagger\hat{\gamma}^\mu\phi)+(\phi^\dagger\gamma^5\psi)\,(\psi^\dagger\gamma^3\phi) \quad (19,\,a)$$

$$4\,(\psi^\dagger\gamma^5\phi)\,(\phi^\dagger\gamma^5\psi)=-\Omega_1\Omega_1'-\Sigma\,\mathrm{S}_\mu\mathrm{S}_\mu'-\Sigma\,\mathrm{M}_{[\mu\nu]}\,\mathrm{M}'_{[\mu\nu]}$$
$$-\Sigma\,\hat{\mathrm{S}}_\mu\hat{\mathrm{S}}_\mu'+\Omega_2\Omega_2' \quad (19,\,b)$$

Both these lead to

$$3\,\Omega_2^2+\Omega_1^2=-\Sigma\,\mathrm{S}_\mu^2-\Sigma\,\hat{\mathrm{M}}^2_{[\mu\nu]}-\Sigma\,\hat{\mathrm{S}}_\mu^2 \quad (19,\,c)$$

Of these identities (14, *c*) or (15, *c*) give the identity (5, *e*); (17, *c*) is (5, *b*); (18, *c*) is (5, *d*). Also by addition and subtraction (13, *c*) & (16, *c*) give

$$\Sigma\,\mathrm{S}_\mu^2=\Sigma\,\hat{\mathrm{S}}_\mu^2=-\Omega_1^2-\Omega_2^2,$$

which proves (5, *a*) and (5, *c*). Thus all the identities (5) are proved without using Pauli's B-matrix. All the generalised identities given by Pauli can be obtained as suitable linear combinations of our identities (12) to (19).

Thus (12, *b*) is the same as [P. (39)]. {(12, *b*) + (19, *b*) or {2 (17, *a*) − (17, *b*)} gives

$$-\Omega_1\Omega'_1+\Omega_2\Omega_2'-\Sigma\,\mathrm{M}_{[\mu\nu]}\,\mathrm{M}'_{[\mu\nu]}=2\,\{(\phi^\dagger\psi)\,(\psi^\dagger\phi)+(\phi^\dagger\gamma^5\psi)\,(\psi^\dagger\gamma^5\phi)\}$$

which is [P. (43)].

{(12, *b*) − (19, *b*)} or {(13, *a*) + (16, *a*)} gives

$$\Sigma\,\mathrm{S}_\mu\mathrm{S}_\mu'+\Sigma\,\hat{\mathrm{S}}_\mu\hat{\mathrm{S}}_\mu'=2\,\{(\phi^\dagger\psi)\,(\psi^\dagger\phi)-(\phi^\dagger\gamma^5\psi)\,(\psi^\dagger\gamma^5\phi)\}$$

which is [P, (44)].

$\{(18, a) - 2\,(18, b)\}$ gives

$$i\,\Omega_1\Omega_2' + i\,\Omega_2\Omega_1' + \Sigma\,\hat{\mathrm{M}}_{[\mu\nu]}\,\mathrm{M}'_{[\mu\nu]} = 2\,\{(\phi^\dagger\psi)\,(\psi^\dagger\gamma^5) + (\psi^\dagger\phi)\,(\phi^\dagger\gamma^5\psi)\}$$

which is [P. (47)].

Finally $\{(14, a) - (15, a)\}$ gives

$$i\,\Sigma\,\hat{\mathrm{S}}_\mu\mathrm{S}_\mu' - i\,\Sigma\,\hat{\mathrm{S}}_\mu\mathrm{S}_\mu' = 2\,\{(\phi^\dagger\psi)\,(\psi^\dagger\gamma^5\phi) - (\psi^\dagger\phi)\,(\phi^\dagger\gamma^5\psi)\}$$

which is [P. (48)].

While these identities of Pauli are generalisations of $(5, h)$; $\{(5, a) + (5, c)\}$ and $(5, d)$ respectively, our identities contain generalisations of all the five in (5), as can be easily shown by taking suitable linear combinations. Similarly all the identities given by Harish-Chandra (pp. 35–36), in particular, his equations (32)–(36) are also linear combinations of our identities.

We note below a few other interesting identities:

$[(13, h) + (16, b)]$ gives

$$\Sigma\,(\psi^\dagger\gamma^\mu\phi)\,(\phi^\dagger\gamma^\mu\psi) + \Sigma\,(\psi^\dagger\hat{\gamma}\phi)\,(\phi^\dagger\hat{\gamma}^\mu\psi) = -2\,(\Omega_1\Omega_1' + \Omega_2\Omega'_2)$$

while $[(13, b) - (16, b)]$ or $[(13, a) - (16, a)]$ gives

$$\Sigma\,(\psi^\dagger\gamma^\mu\phi)\,(\phi^\dagger\gamma^\mu\psi) - \Sigma\,(\psi^\dagger\hat{\gamma}^\mu\psi)\,(\phi^\dagger\hat{\gamma}^\mu\phi) = -\,\Sigma\,\mathrm{S}_\mu\mathrm{S}_\mu' + \Sigma\,\hat{\mathrm{S}}_\mu\hat{\mathrm{S}}_\alpha' \qquad (20, b)$$

Similarly $[(14, a) + (15, a)]$ or $[(14, b) + 15, b)]$ gives

$$\Sigma\,(\psi^\dagger\gamma^\mu\phi)\,(\phi^\dagger\hat{\gamma}^\mu\psi) + \Sigma\,(\psi^\dagger\hat{\gamma}^\mu\phi)\,(\phi^\dagger\gamma^\mu\psi) = -\,\Sigma\,\mathrm{S}_\mu\hat{\mathrm{S}}_\mu' - \Sigma\,\hat{\mathrm{S}}_\mu\mathrm{S}_\mu' \qquad (20, c)$$

and $[(14, b) - (15, b)]$ gives

$$\Sigma\,(\psi^\dagger\gamma^\mu\phi)\,(\phi^\dagger\hat{\gamma}^\mu\psi) - \Sigma\,(\psi^\dagger\hat{\gamma}^\mu\phi)\,(\phi^\dagger\hat{\gamma}^\mu\psi) = 2\,(\Omega_1\Omega_2' - \Omega_2\Omega_1') \qquad (20, d)$$

4. Tensor Identities

The bilinear and quadratic identities derived in the previous article lead to the invariant identities (5) and the corresponding expressions for T which give them might be considered magnitudes of 'invariant' type. It can be shown that by choosing other 'tensor' types for T we can deduce 'tensor' identities which might also involve a single pair $(\psi, \psi^\dagger)$ or two pairs $(\psi, \psi^\dagger)$ and $(\phi, \phi^\dagger)$. We consider here identities of the first kind.

Thus, for example, the 'vector' magnitude

$$\mathrm{T} = \gamma^\mu\psi\psi^\dagger \qquad (21, a)$$

leads to the identity

$$4\,(\psi^\dagger\gamma^\mu\psi)\,(\psi^\dagger\psi) = \sum_{\mathrm{A}}\,(\psi^\dagger\gamma^{\mathrm{A}}\gamma^\mu\psi)\,(\psi^\dagger\gamma^{\mathrm{A}}\psi)$$

and noting that

$$\left.\begin{array}{ll} \gamma^{\mu} = - i\gamma^{[\mu\nu]}\gamma^{\nu} = i\hat{\gamma}^{\nu}\hat{\gamma}^{[\mu\nu]} & (\mu \neq \nu) \\ \text{and} \quad \hat{\gamma}^{\mu} = - i\gamma^{5}\gamma^{\mu} = i\gamma^{\mu}\gamma^{5} & \end{array}\right\} \tag{1'}$$

the right hand side reduces to

$$2\,(\psi^{\dagger}\gamma^{\mu}\psi)\,(\psi^{\dagger}\psi) - 2\sum_{\nu}\,(\psi^{\dagger}\hat{\gamma}^{[\mu\nu]}\psi)\,(\psi^{\dagger}\hat{\gamma}^{\nu}\psi)$$

and we get

$$\Omega_1 S_{\mu} = \sum_{\nu} \hat{M}_{[\mu\nu]}\, \hat{S}_{\nu} \tag{22}$$

which is H-C (37, *a*).*

Similarly

$$T = \sum_{\nu} \hat{\gamma}^{\nu}\psi\psi^{\dagger}\hat{\gamma}^{[\mu\nu]} \tag{21, b}$$

also leads to the same identity.

$$T = \hat{\gamma}^{\mu}\psi\psi^{\dagger} \text{ or } T = \sum_{\nu} \gamma^{\nu}\psi\psi^{\dagger}\hat{\gamma}^{[\mu\nu]} \tag{21, c}$$

leads to

$$\Omega_1 \hat{S}_{\mu} = -\sum_{\nu} \hat{M}_{[\mu\nu]}\, S_{\nu} \tag{23}$$

which is H-C (37, *b*).

Either of the magnitudes

$$T = \hat{\gamma}^{\mu}\psi\psi^{\dagger}\gamma^{5} \text{ or } T = \sum_{\nu} \gamma^{\nu}\psi\psi^{\dagger}\gamma^{[\mu\nu]} \tag{21, d}$$

gives Bhabha's identity [H-C (37, *d*)].

$$i\,\Omega_2 \hat{S}_{\mu} = -\sum_{\nu} M_{[\mu\nu]} S_{\nu}. \tag{24}$$

Similarly,

$$T = \gamma^{\mu}\psi\psi^{\dagger}\gamma^{5} \text{ or } T = \sum_{\nu} \hat{\gamma}^{\nu}\psi\psi^{\dagger}\gamma^{[\mu\nu]} \tag{21, e}$$

gives

$$i\,\Omega_2 S_{\mu} = \sum_{\nu} M_{[\mu\nu]}\, \hat{S}_{\nu} \tag{25}$$

which is H-C (37, *e*).

The above magnitudes are of vector type. We can consider, further, those of a tensor type of the second order. Thus for example each of the following four expressions

$$T = \gamma^{[\mu\nu]}\psi\psi^{\dagger};\ \gamma^{[\mu\nu]}\psi\psi^{\dagger}\gamma^{5};\ \gamma^{\mu}\psi\psi^{\dagger}\hat{\gamma}^{\nu};\ \hat{\gamma}^{\mu}\psi\psi^{\dagger}\gamma^{\nu}\ (\mu \neq \nu) \tag{21, f}$$

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* H-C refers to Harish-Chandra's paper.

† See reference in H-C, p. 36.

gives rise to

$$S_\mu \hat{S} - S_\nu \hat{S}_\mu = i\,\Omega_2\, M_{[\mu\nu]} - \Omega_1\, \hat{M}_{[\mu\nu]} \tag{26}$$

which is H-C (37, *c*).

Similarly

$$T = \gamma^\mu \psi\psi^\dagger \gamma^\nu \text{ or } T = \hat{\gamma}^\mu \psi\psi^\dagger \hat{\gamma}^\nu \tag{21, g}$$

gives

$$S_\mu S_\nu + \hat{S}_\mu \hat{S}_\nu = \sum_\rho M_{[\mu\rho]}\, M_{[\nu\rho]} \quad (\text{for } \mu \neq \nu) \tag{27, a}$$

and

$$S_\mu^2 + \hat{S}_\mu^2 = \sum_\rho M^2_{[\mu\rho]} - \Omega_1^2 \quad (\text{for } \mu = \nu) \tag{27, b}$$

Both (27, *a*) and (27, *b*) are included in H-C (37, *f*).

Equation (26) is identically satisfied for $\mu = \nu$, but it can be shown that by taking

$$T = \gamma^\mu \psi\psi^\dagger \hat{\gamma}^\mu \tag{21, h}$$

we get immediately the quadratic identity (5, *e*). *viz.*, $\Sigma\, S_\mu \hat{S}_\mu = 0$ which was deduced in (14, *c*) by taking for T the expression (21, *h*) summed over μ. In a similar manner, we observe that by taking

$$T = \gamma^{[\mu\nu]} \psi\psi^\dagger \hat{\gamma}^{[\mu\nu]} \tag{21, i}$$

we are led to (5, *d*) without the summation over $[\mu\nu]$ as in (16, *c*).

Finally the 'pseudo-scalar' magnitude

$$T = \gamma^5 \psi\psi^\dagger$$

can be shown to lead to (5, *d*) directly just as (21, *i*). (21, *j*)

We also notice that the quadratic invariant identities can be deduced from the above tensor identities (22)–(27). Multiplying both sides of (22) by $\hat{S}_\mu$ and summing over μ, the right-hand side which becomes a double summation over μ and ν reduces to zero in view of the anti-symmetry of $\hat{M}_{[\mu\nu]}$ and hence $\Sigma\, S_\mu \hat{S}_\mu = 0$, which is (5, *e*). This also results by applying a similar procedure to (23), (24) or (25). Next multiplying both sides of (26) by S_μ, and summing over μ we get, using (23), (24) and (5, *e*) deduced above, that

$$\Sigma\, S_\mu^2 = -\,\Omega_1^2 - \Omega_1^2$$

which is (5, *a*). Multiplying both sides of (22) by S_μ, summing over μ, and using (23) we derive

$$\Sigma\, S_\mu^2 = \Sigma\, \hat{S}_\mu^3$$

and each equal to $-\Omega_1^2 - \Omega_1^2$ which is (5, *c*). Substituting these values in (27, *b*) gives immediately

$$\Sigma\, M^2_{[\mu\nu]} = \Omega_1^2 - \Omega_2^2$$

which is (5, *b*). Finally to obtain (5, *d*) we multiply (25) by $M_{[\mu\nu]}$ and sum over μ and ν. This gives, using (22) and (23),

$$2\,\Omega_2 \sum_{[\mu\nu]} M^2_{[\mu\nu]} + 2\,i\,\Omega_1 \sum_{[\mu\nu]} M_{[\mu\nu]}\,\hat{M}_{[\mu\nu]} = 2\,\Omega_2 \sum_{\mu} (S_\mu^2 + \hat{S}_\mu^2) \qquad (28)$$

Substituting from (5, *a*, *b*, *c*) this leads to

$$-\frac{i}{2}\,\Sigma\, M_{[\mu\nu]}\,\hat{M}_{[\mu\nu]} = \Omega_1\,\Omega_2$$

which is (5, *d*).

The above derivation further enables us to show that the six tensor identities (22) to (27) are not all independent. In fact, (27) can be deduced from the remaining ones. Thus, multiplying (26) by $M_{[\nu\rho]}$ on both sides and summing over ν, we get

$$\sum_{\nu} \Omega_2\, M_{[\mu\nu]}\, M_{[\nu\rho]} + i\,\Omega_1 \sum_{\nu} \hat{M}_{[\mu\nu]}\, M_{[\nu\rho]} + \Omega_2\,(S_\mu S_\rho + \hat{S}_\mu \hat{S}_\rho) = 0,$$

in virtue of (24) and (25). The second summation of the left is easily seen to be identically equal to zero, and we have (27, *a*), *i.e.*, $(\mu \neq \rho)$. To derive (27, *b*), we multiply (24) and (25) by $\hat{S}_\mu$ and S_μ respectively, add and use (25) which yields

$$\Omega_2\,(S_\mu^2 + \hat{S}_\mu^2) = \sum_{\nu} \Omega_2\, M^2_{[\mu\nu]} + i \sum_{\nu} \Omega_1\, M_{[\mu\nu]}\, \hat{M}_{[\mu\nu]} \qquad (29,\ a)$$

Similarly (22) and (23) give

$$\Omega_1\,(S_\mu^2 + \hat{S}_\mu^2) = \sum_{\nu} \Omega_1\, \hat{M}^2_{[\mu\nu]} + i \sum_{\nu} \Omega_2\, M_{[\mu\nu]}\, \hat{M}_{[\mu\nu]} \qquad (29,\ b)$$

Equation (29, *a*) and (29, *b*)

$$i\,(\Omega_1^2 - \Omega_2^2) \sum_{\nu} M_{[\mu\nu]}\, \hat{M}_{[\mu\nu]} = -\,\Omega_1\,\Omega_2 \{\sum_{\nu} (M^2_{[\mu\nu]} + \hat{M}^2_{[\mu\nu]})\}$$
$$= -\,\Omega_1\,\Omega_2 \sum_{[\mu\nu]} M^2_{[\mu\nu]}$$

This along with (28) above determines $\sum_{\nu} M_{[\mu\nu]}\,\hat{M}_{[\mu\nu]}$ which, when substituted in (29, *a*) gives (27, *b*). Thus we have shown that (27, *a*, *b*) can be deduced from (22)-(26), or there are only five independent tensor identities.

5. Multilinear, and Polynomial Identities

The matrices T used in the derivation of the quadratic identities are of the form $\gamma^A \psi \psi^\dagger$, and to derive higher polynomial identities, one has only to take T as the product of two or more such types of expressions. The procedure mentioned in §2 then shows that the identities thus obtained are merely certain multiples of the quadratic identities. For example,

$$\mathrm{T} = \gamma^{\mathrm{A}}\psi\psi^{\dagger}\gamma^{\mathrm{B}}\psi\psi^{\dagger}$$

gives the cubic identity

$$4\,(\psi^{\dagger}\gamma^{\mathrm{A}}\psi)\,(\psi^{\dagger}\gamma^{\mathrm{B}}\psi)\,(\psi^{\dagger}\psi) = \sum_{\mathrm{C}}\,(\psi\gamma^{\mathrm{C}}\gamma^{\mathrm{A}}\psi)\,(\psi\gamma^{\mathrm{B}}\psi)\,(\psi\gamma^{\mathrm{C}}\psi)$$

and this can be obtained, by multiplication with $(\psi^{\dagger}\gamma^{\mathrm{B}}\psi)$, from the quadratic identity

$$4\,(\psi^{\dagger}\gamma^{\mathrm{A}}\psi)\,(\psi^{\dagger}\psi) = \sum_{\mathrm{C}}\,(\psi^{\mathrm{T}}\gamma^{\mathrm{C}}\gamma^{\mathrm{A}}\psi)\,(\psi^{\dagger}\gamma^{\mathrm{C}}\psi)$$

derivable with the aid of $\mathrm{T} = \gamma^{\mathrm{A}}\psi\psi^{\dagger}$. Thus polynomial identities are derivable from quadratic ones. Similarly multilinear identities can be derived from bilinear ones. To illustrate this we consider a slightly complicated example by taking

$$\mathrm{T} = \sum_{\mu}\,(\gamma^{\mu}\phi\psi^{\dagger}\gamma^{\mu}\psi\phi^{\dagger}\gamma^{\mu}) \tag{30}$$

and use $\chi^{\dagger}$ and χ for multiplication on the left and right. (30) yields the trilinear identity

$$4\sum_{\mu}\,(\chi^{\dagger}\gamma^{\mu}\phi)\,(\psi^{\dagger}\gamma^{\mu}\psi)\,(\phi^{\dagger}\gamma^{\mu}\chi) = \sum_{\mathrm{A}}\sum_{\mu}\,(\phi^{\dagger}\gamma^{\mu}\gamma^{\mathrm{A}}\gamma^{\mu}\phi)\,(\psi^{\dagger}\gamma^{\mu}\psi)\,(\chi^{\dagger}\gamma^{\mathrm{A}}\chi) \tag{31}$$

The corresponding bilinear identity would be given by

$$\mathrm{T} = \sum_{\mu}\,\gamma^{\mu}\phi\phi^{\dagger}\gamma^{\mu} \tag{30, a}$$

which on multiplication by $\lambda^{\dagger}$ on the left, and χ on the right yields

$$4\sum_{\mu}\,(\chi^{\dagger}\gamma^{\mu}\phi)\,(\phi^{\dagger}\gamma^{\mu}\chi) = \sum_{\mathrm{A}}\sum_{\mu}\,(\phi^{\dagger}\gamma^{\mu}\gamma^{\mathrm{A}}\gamma^{\mu}\phi)\,(\chi^{\dagger}\chi^{\mathrm{A}}\gamma) \tag{31, a}$$

which is also (13, *b*), On account of the summation over μ, the trilinear identity (31) is not given directly as a mere multiple of (31, *a*) with $\mathrm{S}_{\mu} = (\psi^{\dagger}\gamma^{\mu}\psi)$ as the factor. But it would still be possible to derive (31) by mere multiplication from a different kind of bilinear identity, *viz.*, a tensor one. Thus we could take

$$\mathrm{T} = \gamma^{\mu}\phi\phi^{\dagger}\gamma^{\mu} \qquad \text{(no summation)} \tag{30, b}$$

and derive the 'tensor' bilinear identity

$$4\,(\chi^{\dagger}\gamma^{\mu}\phi)\,(\phi^{\dagger}\gamma^{\mu}\chi) = \sum_{\mathrm{A}}\,(\phi^{\dagger}\gamma^{\mu}\gamma^{\mathrm{A}}\gamma^{\mu}\phi)\,(\chi^{\dagger}\gamma^{\mathrm{A}}\chi) \tag{31, b}$$

which is the generalization of the quadratic tensor identity (27, *b*). Multiplying both sides of (31, *b*) by $\psi^{\dagger}\gamma^{\mu}\psi$ and summing over μ, we go over to (31). The same situation presents itself when considering the cubic identity obtained from (31) by putting $\psi = \phi = \chi$ and $\psi^{\dagger} = \phi^{\dagger} = \chi^{\dagger}$ *viz.*

$$\Sigma\,\mathrm{S}_{\mu}^{3} + \Sigma\,\mathrm{S}_{\mu}\hat{\mathrm{S}}_{\mu}^{2} + \Omega_{1}^{2}\,\Sigma\,\mathrm{S}_{\mu} = \Sigma\,\Sigma\,\mathrm{S}_{\mu}\,\mathrm{M}_{[\mu\nu]}^{2} \tag{32, a}$$

and the analogous one obtained by taking the dual of γ^{μ} in (30),

$$\Sigma\, \hat{S}_{\mu}^{3} + \Sigma\, \hat{S}_{\mu} S_{\mu}^{2} + \Omega\, \Sigma\, \hat{S}_{\mu} = \Sigma\, \Sigma\, \hat{S}_{\mu}\, M_{[\mu\nu]}^{2} \tag{32, b}$$

(32, *a*) and (32, *b*) are not simple multiples of the associated quadratic invariant identity (12, *c*) or (19, *c*), *viz.*

$$\left.\begin{aligned} &\Sigma\, S_{\mu}^{2} + \Sigma\, \hat{S}_{\mu}^{2} + 3\Omega_{1}^{2} + \Omega_{2}^{2} = \sum_{[\mu\nu]} M_{[\mu\nu]}^{2} \\ \text{or} \qquad &\Sigma\, S_{\mu}^{2} + \Sigma\, \hat{S}_{\mu}^{2} + 3\Omega_{2}^{2} + \Omega_{1}^{2} = \sum_{[\nu\mu]} M_{[\mu\nu]}^{2} \end{aligned}\right\}$$

but can be obtained from the tensor identity (27, *b*), *i.e.*

$$S_{\mu}^{2} + \hat{S}_{\mu}^{2} + \Omega_{a}^{2} = \sum_{\nu} M_{[\mu\nu]}^{2}$$

by multiplying with S_{μ} or $\hat{S}_{\mu}$ and summing over μ.

6. Primitive Types of Matrices

We will now show that all the identities considered in this paper can be deduced by considering the five primitive types of matrices

$$T = \psi\psi^{\dagger},\ \gamma^{\mu}\psi\psi^{\dagger},\ \gamma^{[\mu\nu]}\psi\psi^{\dagger},\ \hat{\gamma}^{\mu}\psi\psi^{\dagger},\ \gamma^{5}\psi\psi^{\dagger} \tag{2, a}$$

given as the product of the five types of γ^{A} and matrices of rank unity. Since polynomial and multilinear identities are deducible from quadratic and bilinear ones respectively, we confine ourselves to the latter kinds. Let us consider the quadratic case where only one pair $(\psi, \psi^{\dagger})$ appears;

$T = \psi\psi^{\dagger}$ gives the invariant identity

$$\Sigma\, S_{\mu}^{2} + \Sigma\, \hat{S}_{\mu}^{2} - \sum_{[\mu\nu]} M_{[\mu\nu]}^{2} = -\,3\,\Omega_{1}^{2} - \Omega_{2}^{2} \tag{12, c}$$

$T = \gamma^{\mu}\psi\psi^{\dagger}$ gives the tensor identity

$$\sum_{\nu} \hat{M}_{[\mu\nu]}\, \hat{S}_{\nu} = \Omega_{1}\, S_{\mu} \tag{22}$$

$T = \gamma^{[\mu\nu]}\psi\psi^{\dagger}$ gives the tensor identity

$$\Omega_{2}\, M_{[\mu\nu]} + i\,\Omega_{1}\, \hat{M}_{[\mu\nu]} + i\,(S_{\mu}\hat{S}_{\nu} - S_{\nu}\hat{S}_{\mu}) = 0 \tag{26}$$

$T = \hat{\gamma}^{\mu}\psi\psi^{\dagger}$ gives the tensor identity

$$\sum_{\nu} \hat{M}_{[\mu\nu]} = -\,\Omega_{1}\, \hat{S}_{\mu} \tag{23}$$

and $T = \gamma^{5}\psi\psi^{\dagger}$ gives the invariant identity $-\dfrac{i}{2}\sum_{[\mu\nu]} M_{[\mu\nu]}\, \hat{M}_{[\mu\nu]} = \Omega_{1}\Omega_{2}$ (5, *d*)

As shown in §4, (22) and (23) give $\Sigma\, S_{\mu}^{2} = \Sigma\, \hat{S}_{\mu}^{2}$, and (22) or (23) gives $\Sigma\, S_{\mu}\hat{S}_{\mu} = 0$. Multiplying (26) by $M_{[\mu\nu]}$, using (22) and (23), and summing over μ and ν, and also taking (5, *d*) into account we get a relation between $\Sigma\, S^2_\mu$ and $\Sigma\, M^2_{[\mu\nu]}$ other than (12, *c*). Taking (12, *c*) into consideration therefore gives

$$\Sigma\, S^2_\mu = \Sigma\, \hat{S}^2_\mu = -\,\Omega_1^2 - \Omega_2^2\,;\ \ \Sigma\, M^2_{[\mu\nu]} = \Omega_1^2 - \Omega_2^2$$

and all the invariant quadratic identities are thus derived.

We have next to obtain the tensor identities (24) and (25) since the remaining one, *viz.*, (27) has previously been shown to be a consequence of the others. This is easily done by means of (26). Multiplying it by S_ν and summing over ν we get, using (23) and (5. *e*)

$$\Omega_2 \sum_\nu M_{[\mu\nu]}\, S_\nu + i\,\Omega_1\,(-\,\Omega_1\,\hat{S}_\mu) = i\,\hat{S}_\mu\, \Sigma\, S^2_\nu$$

and since $\Sigma\, S^2_\nu = -\,\Omega_1^2 - \Omega_2^2$, this gives

$$\Sigma\, M_{[\mu\nu]}\, S_\nu = -\, i\,\Omega_2\,\hat{S}_\mu$$

which is (24). Similarly multiplication by $\hat{S}_\nu$ in (26) leads to (25). Hence (33) yield all the quadratic invariant and tensor identities.

The proof that (33) also lead to the bilinear identities of the invariant and tensor forms can be carried out in a similar, though more complicated manner, and need not be explicitly given here. We will notice, however, only one simple case. The last matrix in (33), $T = \gamma^5 \psi \psi^\dagger$ gives the bilinear identity

$$\begin{aligned} 4\,(\phi^\dagger\gamma^5\psi)\,(\psi^\dagger\phi) &= \sum_A\,(\psi^\dagger\gamma^A\gamma^5\psi)\,(\phi^\dagger\gamma^A\phi) \\ &= i\,\Omega_1\Omega_2{}' + i\,\Omega_2\Omega_1{}' - i\,\Sigma\,\hat{S}_\mu S_\mu{}' + i\,\Sigma\, S_\mu\hat{S}_\mu{}' \\ &\qquad - \Sigma\,\hat{M}_{[\mu\nu]}\, M'_{[\mu\nu]} \end{aligned} \qquad (34)$$

Intercharging ϕ and ψ interchanges the dashed and undashed quantities on the right of (34). Adding and subtracting (34) and the expression thus obtained we get immediately (P. 47) and (48).

Summary

It is shown in this paper that by choosing suitable forms for 4×4 matrices as products of Dirac matrices and matrices of rank unity (*i.e.*, products of 4×1 and 1×4 matrices), and expressing them as linear combinations of the sixteen elements γ^A of the basis of the Dirac algebra, one can derive the generalized identities of Pauli which hold in this algebra. Generalizations are given for cases not dealt with by Pauli, and the use of his B-matrix is also avoided. The same method yields further 'tensor', multilinear, and polynomial identities of which it is shown that the last two

kinds of identities are derivable from bilinear and quadratic ones. It is pointed out that all types of identities can be deduced by considering five primitive types of matrices.

References

Harish-Chandra .. *Proc. Ind. Acad. Sci.*, 1945, **22**, 30.

D. E. Littlewood .. *J. L. M. S.*, 1934, **9**, 41.

W. Pauli .. *Annales de l'Institut H. Poincare*, 1936, 109.

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