# Non-linearity in quantum mechanics.

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by

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## (1) Introduction.

It is perhaps the most fundamental principle of quantum mechanics that the system of states forms a linear manifold in which a unitary scalar product is defined. The states are generally represented by wave functions (in general complex quantities) in such a way that $\varphi$ and constant multiples of $\varphi$ represent the same ~~state~~ physical state. The normalisation of the wave function leaves only a constant factor of modulus 1, the so called phase, undetermined in $\varphi$. The linear character of the wave function is ~~the superposition~~ called the superposition principle which forms the fundamental idea of quantum mechanics.

Developments during the last 5 or 6 years have however given an indication that one may ~~to~~ have to admit the possibility of a future non-linear character of quantum mechanics. The first germs of non-linearity were sown by the positron theory of Dirac [1], and later on, the avowedly non-linear field theory of Born [2] was devised to meet the difficulties of ~~self~~- infinite self-energy in quantum electrodynamics. Still more recently the assumption that nuclear forces can be described by some sort of a linear field has, during the course of development of ~~nuclear forces on the basis of~~ the meson theory by Yukawa, Heitler, Bhabha & others [3], led to difficulties which tend to suggest that the right interaction between a heavy particle and a meson ~~will be~~ of may be of a non-linear type.

It is instructive to notice that the three fundamental difficulties facing modern quantum mechanics can be traced in essence to this question of linearity vs non-linearity. These difficulties are, (a) the fusion of relativity and quantum mechanics, (b) the divergences in quantum electro-dynamics, and (c) the problem of the heavy particles. In (a) the question is the fusion of two theories one of which, the general relativity theory is essentially non-linear in character ~~with~~ and ~~the linear quantum mechanics~~ the other linear. In (b) the difficulties are apparently of two kinds, those connected with the "hole" theory of the electron, and those connected with the infinite self-energy of the electron. In the case where the external field is not negligible the former theory leads to field equations which are no longer linear ie where the superposition principle is no longer valid. In view of Born's field theory it appears ~~pos~~ probable that the latter difficulty is also intimately connected with the question of non-linearity, although it must be admitted that non-linearity alone does not seem to be sufficient. ~~It is~~ It is worthy of note that even in the recent classical Dirac theory of the electron proposed to get over these difficulties, the feature of non-linearity still appears to survive. In (c) the main difficulty is to conceive a merging of the different wave equations of the elementary particles, light and heavy, into one comprehensive theory if the theory is to be linear.

The question of non-linearity also appears to be connected intimately ~~[illegible]~~ with the motion of singularities in a field theory, the non-linear character being responsible for the existence of the equations of motion of the singularities ~~without~~ which can be deduced from the field equations alone.

I propose to give a brief idea of several ~~theories~~ of these theories wherein non-linearity plays an important role.

## (2) Preliminary ~~calculation~~ questions

Before entering into a description of such theories, it is perhaps useful to give a short account of some important problems [4] which have engaged attention during the last 4 or 5 years, although it might be mentioned at the outset that much progress has not been achieved during this period. The "hole theory" of the positron has been placed on a more satisfactory basis by the subtraction formalism of Dirac & Heisenberg [5]. We can give coherent explanations of the phenomena of pair production and annihilation, but difficulties arise when the external field ~~is~~ cannot be treated as weak. The self-energy problem of quantum electrodynamics is still as intractable as ever, but the recent ~~the~~ classical theory of the electron by Dirac [6] which attempts at a formal modification of the classical Lorentz equations of motion appears to offer some hopes of solving this problem.

All the hopes raised by the unitary field theory of Born have proved false. While this theory succeeded beautifully in solving the classical self-energy problem it has ~~floundered on the rock~~ failed in two important respects. One is the impossibility of assigning a unique action function out of the doubly infinite number all of which satisfy the requirements of the theory. The other is its inability to assign half-integral spin to electrons when considered as constructed out of light quanta with integral spins.

The converse problem based on de Broglie's idea of constructing light quanta from neutrinos of spin 1/2 were originally developed by Jordan and Krönig but were later on the theory was not found to satisfy the postulate of Lorentz-invariance[7]. Also the theory of de Broglie[8] and his school regarding neutrinos has recently been examined by Pauli[9] and it is found that this theory does not really describe photons but a particle of spin 0, and a particle of spin 1 with equal non-vanishing rest masses connected with each other in a specially interesting manner, and it does not mathematically formulate the assumption that the particle is made up of two particles of spin 1/2. Other neutrino theories free from contradictions have been devised but they can be hardly said to ~~tt~~ apply to the Pauli-Fermi neutrinos of the $\beta$-decay theory.

The phenomenological theory of nuclear forces has achieved great successes thanks to the fundamental ideas of Bohr[10], but the same cannot be said of the field theory of nuclear forces. It was originally suggested that the nuclear field is the same as the $\beta$-field of Fermi, but this was soon given up because of its fundamental contradictions. The theory of the meson ~~originally~~ however appears to offer a very satisfactory solution of the problem of nuclear forces and to be in consonance with the phenomonological theory by providing explanation for the Heisenberg and Majorana forces & their short-range character. Also the anomaly of magnetic moments of the heavy particles receives a simple explanation. A possibility of the application of the meson theory to that of $\beta$-decay has been opened up by the suggestion that the $\beta$-decay is really a double process in which the transition of a neutron to the proton state gives rise to a meson as a virtual intermediate state which in turn decomposes into an ordinary electron and neutrino. The postulated mean life-time of $10^{-6}$ sec of the meson in this process also agrees with cosmic ray data. The difficulties in this theory are ~~its~~ The difficulty of ~~presentation~~ presenting it in a form symmetrical in electrons & neutrinos, and also its failure to explain the asymmetry of the $\beta$-spectrum. Again the question of interaction of mesons with heavy nuclear particles gives rise to the same, if not more serious, types of ~~difficulties~~ divergence difficulties so characteristic of quantum electrodynamics

(3) The positron theory.

The simplest application of this theory is to the adiabatic insertion of an external field into a "vacuum" which induces a sort of vacuum polarisation in other words a charge distribution equivalent to the creation of pairs. A close connection of this theory with Born's field theory is brought out by the fact that such a vacuum polarisation is inherent in the field equations of the latter theory. The next simple application is the case where the external field is so weak that only first order terms in the field strengths and its derivatives can be retained in the "Störungsrechnung". In this approximation additional terms appear in Maxwell's equations equivalent to saying that Coulomb's law breaks down for distances less than the Compton wave length $h/mc$.

For higher approximations the corresponding field equations are no longer linear, and the superposition principle is no longer valid. This deviation from linearity also gives rise to the phenomenon of scattering of light by light which is explained on the basis of the creation of pairs in intermediate states. All calculations made on the assumption of this non-linearity show that the field equations can be derived from a Lagrangian of the form

$$L = L_1 + L_2 + \cdots + L_n + \cdots$$

where $L_n$ is of the $2n^{th}$ order in the field equations. In particular $L_2$ is of the form

$$L_2 = \alpha (E^2 - H^2) + \beta (EH)^2.$$

It is again remarkable that this form of the Lagrangian is the same as that postulated on the Unitary field theory of Born which is purely classical. By a comparison with the Dirac-Heisenberg theory one could calculate $\alpha$ and $\beta$, and we have

$$\beta = 7\alpha, \quad \alpha = \frac{1}{720\pi^3} \cdot \frac{e^4 h}{m^4 c^7}.$$

The deviation from the superposition principle is unimportant as long as the field strength remains less than $(8\pi\alpha)^{-1/2}$. This critical field strength is remarkably numerically equal to the field strength on the classical Lorentz theory at the boundary of the electron ie $e^2/r_0^2$ where $r_0 = e^2/mc^2$. The numerical correspondence depends on the fact that the numerical factor $90\pi^2$ appearing $1/(8\pi\alpha)$ is of the same order of magnitude as the dimensionless constant $hc/e^2$ ($= 2\pi \cdot 137$) & this connection has obviously a deep significance which appears to ~~define~~ characterise the stage at which non-linearity sets in.

Mention might also be made here of several attempts made to get over the mathematical ~~conceptions of non-linearity~~ complications of the positron theory by modifying the theory so as to get the energy as always positive definite. An attempt of Majorana [11] in this direction was ~~partial~~ only partially successful in that the conception of electric charge does not at all enter into the theory. The next notable attempt is the Pauli-Weisskopf [12] theory which ~~also~~ showed that for particles of spin zero one could dispense with the hole theory and obtain particles and antiparticles with the energy density as positive but not the charge density. The work of Pauli-Weisskopf for particles of spin zero has ~~excited~~ encouraged the theory of particles with ~~arbitrarily~~ arbitrary spins. After Dirac's general theory [13] of such general spins comes Proca's theory [14] for particles of spin 1 just like the photon ~~but with~~ with equations just like Maxwell's equation but with this difference that the Proca particles might have a non-vanishing rest mass and charge. The recent ~~meson~~ theory of mesons which are also such Einstein-Bose particles has been based on Proca's equations. The most complete investigations in this direction are due to a penetrating analysis recently given by Pauli [9] and his main conclusions are

(i) Considerations of guage and relativistic invariance show that, in the absence of external fields, the wave equation is essentially determined uniquely by specifying the mass and spin of the particle.

(ii) The definition of a definite particle density (4-current) which transforms like 4-component of a vector is not possible for integral spin, and a positive definite every density and also a positive definite total energy is not capable defined for half-integral spin. These are not positive statements since they do not say that for integral spins there is a definite energy-density, and for half-integral spins a definite charge density. In fact this is no longer the case for spins $> \cancel{\tfrac{1}{2}}\, 1$. It is only in the case of the small spins 0, ½ and 1 that such positive statements could be made. Thus the spin ½ is characterised by the fact of the possibility of a definite charge density, and spins 0, 1 of definite energy density and thus these spins are distinguished from all other higher spins.

(iii) The explanation of the connection between spin and statistics belongs to the most important ~~possible~~ possibilities of application of the special theory of ~~relation~~ relativity, and go to show that for ~~half-integer~~ integral spin quantisation quantisation according to exclusion principle is not possible, while on the other hand for half-integral spin, quantisation by Bose-statistics ~~formally~~ is possible formally, but then the energy of the system is not ~~neg~~ positive. Since this is necessary from the physical point of view, the exclusion principle is necessary for half-integral spin.

(iv) For spin values > 1, and ~~in~~ by the result of external forces there may arise transitions with changes of mass and spin, giving curious transitions to elementary particles ~~not foun~~ apparently not found in nature.

(v) For gravitational quanta with spin = 2, it can be shown that the total energy is positive definite, and the ~~Lagrangian function of~~ general relativity equations ~~taken as~~ ~~linear~~ taken ~~as~~ in a limiting ~~law~~ form linear can be derived as a limit from the quantum side. This limitation is connected with the wellknown divergence difficulties of field theory

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## (4) Born's field theory & Dirac's classical theory.

In addition to the difficulties connected with the positron theory, quantum electrodynamics meets with other difficulties connected with the infinite self-energy of the electron, and wave lengths $\lambda < r_0 \ (= e^2/mc^2)$ are excluded from the theory. This difficulty is perhaps not exactly quantum-theoretic in its origin; in fact, the classical (Lorentz) theory of the electron yields, when the electron radius goes to zero, an infinite quantity, the "electro-magnetic mass". In other words, a point electron cannot, on account of the infinite inertia of its associated electro-magnetic field, be accelerated by a finite external force. In the Heisenberg-Pauli theory also which starts with point electrons one obtains for the electrons an infinite inertial mass. Born's non-linear field theory was devised to get over these difficulties, and although it has not been able to achieve this purpose satisfactorily, it has nevertheless added new ideas to quantum electrodynamics, and given a beautiful solution to the old classical problem of the structure and inertial mass of the electron. The theory is unitary in its nature in that it considers an electron as a point singularity in an electrodynamic field, and Maxwell's equations of the field are modified so as to make the energy around the singularity representing the electron's self-energy a finite quantity. It has been known for quite a long time that, in the self-energy problem, the infinity could be got over by the assumption of an extended electron, but then the theory would become non-invariant relativistically. In addition to this, the assumption of an extended electron would introduce a good deal of arbitrariness (in the choice of charge distribution) in the theory and would also go against the postulate of atomic constitution of electricity. Born's field theory overcomes all these objections since the equations of the theory are relativistically invariant. It might also be noticed that Born's theory is guage-invariant which condition has been shown to be essential by Weyl for any satisfactory field theory. This is achieved in Born's theory since the Lagrangian used in the variational principle on which the whole theory is based contains only the field strengths and not the potentials.

Considering first the classical part of the theory, one starts with the Lagrangian $\mathcal{L}(g_{kl}, f_{kl}) = L(F, G)\sqrt{-g}$, where $g_{kl}$ and $f_{kl}$ are the metric and electro-magnetic field tensors, and $F$, $G$ the usual invariants $\frac{1}{2} f_{kl} f^{kl}$ and $\frac{1}{4} f_{kl} f^{*kl}$. The existence of the potential vector $\phi_k$ is assumed such that

$$f_{kl} = \frac{\partial \phi_l}{\partial x^k} - \frac{\partial \phi_k}{\partial x^l}$$

giving the identity

$$\frac{\partial(\sqrt{-g}\, f^{*kl})}{\partial x^l} = 0. \qquad (1)$$

We introduce a second kind of antisymmetric field tensor $p_{kl}$ by the relation $\sqrt{-g}\, p^{kl} = \partial \mathcal{L}/\partial f_{kl}$, and the variation principle for the Lagrangian gives the Eulerian equations

$$\frac{\partial(\sqrt{-g}\,p^{kl})}{\partial x^l} = 0. \qquad \text{(ii)}$$

(i) and (ii) are the complete set of field equations. The energy-impulse tensor is given by

$$\sqrt{-g}\,T_{kl} = -2\partial\mathcal{L}/\partial g^{kl} = \sqrt{-g}\left(L g_{kl} - f_{ks}\, p_{lr}\, g^{sr}\right).$$

It is easy to generalise the action principle in such a way that it contains Einstein's gravitational laws: one has only to add to the action integral the term $\int R\sqrt{-g}\,d\tau$, where $R$ is the scalar of curvature.

Leaving out of account the gravitational laws, the essence of the theory consists in not taking the charge distribution $\rho$ as fundamental but two sets of field strengths $\vec{E}, \vec{H}$ and $\vec{D}, \vec{B}$, the equations connecting these being non-linear and involving a certain universal constant $b$ (limiting field strength) connected with the electron radius $r_0$ by $b = e/r_0^2$. For weak fields $\vec{E} \equiv \vec{D}$; $\vec{H} \equiv \vec{B}$. By imposing the condition of self-conjugacy on the action function used in the variation principle, Born has shown that there are four equal representations of the theory according as $(\vec{B}, \vec{E})$, $(\vec{D}, \vec{H})$, $(\vec{B}, \vec{D})$, or $(\vec{E}, \vec{H})$ are taken as fundamental. With the third or $U$-representation in the special form of Born's action function

$$L = \sqrt{1 + F/b^2 - G^2/b^4} - 1$$

the energy density $u$ is given by

$$4\pi u = b^2\left\{1 + (\vec{D}^2 + \vec{B}^2)/b^2 - [\vec{D}\vec{B}]^2/b^4\right\}^{1/2} - b^2$$

and the ~~tot~~ total energy and momentum $U$ and $\vec{G}$ here form a four-vector thus successfully exploiting the notion of electro-magnetic origin of mass.

The field equations applied to the special case of an electro-static field with spherical symmetry, and a singularity at the origin give for the self-energy of the electron the finite value $mc^2$. Another important case where the field equations have been solved is that of a ring-singularity[15] where also one obtains finite values for the total energy and angular momentum

In the further development of Born's ~~action~~ field theory, the question of the form of the action function has been discussed in great detail. A great number of action functions have been proposed, and they all appear to serve the purpose of giving finite ~~e~~ energy for the electron equally well. Even the symmetry conditions of self-duality and the conditions of existence of simple algebraic relations between the $f_{kl}$ and $p_{kl}$-fields do not uniquely specify the action functions. It has been shown in fact[16] that there exist a two-fold infinity of such action functions.

Coming now to the quantisation of the field equations, Born has attempted the same on the lines of the Heisenberg-Pauli theory, taking $\vec{D}$ and $\vec{B}$ as the fundamental variables in place of the potentials $\phi_k$. A coherent theory has been built up but its successes must be considered very meagre indeed. The most formidable objection to the quantised theory is the result due to Pryce[17] that the electro-magnetic angular momentum has its eigen-values as integral numbers thus showing that the theory cannot explain spin. Another purely classical attempt[18] to explain spin of the electron on a ring model has met with the same fate.

The most serious objection that can be brought against ~~the~~ a theory which postulates an electro-magnetic origin of mass is the existence of the neutron, and also the theory of the positron in which negative and positive mass play symmetrical roles is not consistent with a theory which does not envisage negative mass even abstractly. Thus although there appears a close correlation between Born's field theory and the positron theory in the domain of the limiting case of the ~~former~~ latter, there exists this fundamental difference ~~of~~ in the question of ~~the~~ definiteness of energy density. The similarities appear to have been achieved by the assumptions of non-linearity in both cases. In view of Pauli's work ~~of~~ on the relation of definiteness of energy density to spin, and the failure of Born's theory to explain spin, it looks ~~that~~ as if non-linearity as a fundamental ~~principle~~ physical principle does not really go very deep being more in the nature of a mathematical device.

Another purely classical attempt to get rid of the infinities has recently been made by Dirac[6] by retaining the Maxwell's equations as they are but trying to remove the infinities by a subtraction formalism of the type used in positron theory giving the finite mass as the difference between the infinite negative mass at the centre and the infinite positive mass of the surrounding Coulomb field. The field equations being Maxwellian are of course linear but the equations of motion of the electron given in the form

$$\cancel{m\ddot{v}_\mu - \frac{2}{3} e^2 \ddot{v}_\mu}$$

$$m\dot{v}_\mu - \frac{2}{3} e^2 \ddot{v}_\mu - \frac{2}{3} e^2 \dot{\vec{V}}^2 v_\mu = e v_\mu F^\nu_{\mu,in}$$

are non-linear. In Born's field theory where the field equations are ~~linear~~ non-linear, the equations of motion cannot be derived from them alone but require additional hypotheses. Here we have the field equations are linear, the equations of motion do not depend for their derivation on other extraneous considerations, but they are non-linear. Thus the non-linearity which comes up in Dirac's theory is only a derived or secondary type of non-linearity and not of the fundamental type as in Born's theory. Nevertheless this non-linearity gives rise on Dirac's theory to striking departures from the usual ideas of mechanics classical and relativistic. Thus one has to obtain solutions of the equations of motion for which the initial position and velocity of the electron are prescribed together with its final acceleration, instead of solutions in which all the initial conditions are prescribed. Again it appears possible for a ~~light~~ signal to be transmitted faster than ~~a~~ light ~~signal~~ through the interior of an electron, and the finite size of the electron reappears in a new sense in that the interior of an electron is a region of failure not of the field equations of electromagnetic theory, but some of the elementary properties of space-time.

(5) The meson theory.

This theory was originally proposed by Yukawa [3] ~~(1935)~~ to explain ~~nuclear forces as exchange forces~~ the exchange character of nuclear forces on the basis of the transitions

$$P \rightleftarrows N + Y^{+}$$
$$N \rightleftarrows P + Y^{-}$$

which show that they obey Bose-statistics and have integral spin. The basic heuristic assumption behind the theory [19] is that nuclear forces can be described by some sort of <u>linear field</u>, and the main ideas can be very simply put. If a nuclear particle surrounded by its static field moves, the field is retarded and assumes the nature of propogating waves. These waves when quantised give rise to quanta which, for example, can be emitted in a collision between two nuclear particles. The dispersion law of these waves ie the relation between energy and momenta of these quanta can be derived from the range of the forces. If the static field has extension $\Delta x = \frac{1}{\lambda}$, then all wave lengths $> \Delta x$ will occur in the Fourier expansion. For the field being static all these waves must be independent of wave lengths ie

$$\nu = \text{const for wave lengths} > \Delta x$$

$$\text{or } h\nu = \text{const for } p < \frac{h}{\Delta x} \left( = h\lambda \right)$$

Using $E^2 = c^2 p^2 + \mu^2 c^4$, we find

$$\mu \simeq h\lambda,$$

and these quanta have therefore rest energy $\mu c^2 = hc\lambda$ or a rest mass $\mu = \frac{h\lambda}{c}$. Inserting for $1/\lambda$ the electronic radius we have $\mu \simeq 137$. electron mass. $\frac{1}{\lambda}$ is a universal constant of the dimension of length entering the formalism.

Yukawa's wave equation for the nuclear field for a scalar field function was

$$\nabla^2 \varphi - \ddot{\varphi} - \lambda^2 \varphi = 0, \qquad \left( \dot{\varphi} = \frac{1}{c} \partial\varphi/\partial t \right)$$

and a static solution of this with a point singularity is

$$\varphi = g \cdot e^{-\lambda r}/r$$

showing that the range of the forces is $1/\lambda$, and $g$ is a characteristic strength of field, and corresponds to charge in the electro-magnetic case, what might be called the meson charge. To explain exchange forces $\varphi$ itself must carry an electric charge, and quanta arising from quantising the field $\varphi$ will then be $-^{ve}$ and $+^{ve}$ charged particles.

This scalar formalism of Yukawa does not however explain all details like dependence ~~of~~ on spin direction of proton or neutron; but such dependence is however necessary to explain magnetic moments. The field is therefore taken a vector, and the field theory is built from Proca's equations analogous to Maxwell's equations in the form

$$\sum_\alpha \frac{\partial \chi_{\alpha\beta}}{\partial x_\alpha} = \lambda^2 \varphi_\beta \qquad (\lambda = 0 \text{ in Maxwell's theory})$$

where $\chi_{\alpha\beta} = \frac{\partial \varphi_\beta}{\partial x_\alpha} - \frac{\partial \varphi_\alpha}{\partial x_\beta}$, and here $\varphi$ and $\chi_{\alpha\beta}$ are on the same footing.

The Lorentz-condition $\Sigma\, \partial\varphi_\beta/\partial x_\beta = 0$ follows here as a consequence, and need not be assumed as in Maxwell's theory, i.e. the ~~guag~~ theory is guage invariant. All field quantities can be expressed by transverse and longitudinal vectors. The quantisation is simply carried out by associating the suitable L, exchange relations as for Bose particles, also a H and hence the field equations. In contrast to Maxwell's theory the longitudinal part does not reduce to static field only. The Pauli-Weisskopf method of double quantisation leads to positive and negative mesons as can be seen easiest in momentum space. The charged character of the field is essentially connected with the complex nature of the field quantities. Neutrettos can however be described by real $\varphi_\alpha$, $\chi_{\alpha\beta}$ and quantisation leads to one sort of quanta only.

The essential differences with the Maxwell field are (i) the meson field is charged, and (ii) $\varphi$ and $\chi_{\alpha\beta}$ are on the same footing and interaction can therefore depend on either set of quantities. This makes the field due to a magnetic point dipole also very simple, and one can assume that (p) and (n) are characterised by a meson charge $g$ and a meson dipole moment $f/\lambda$, the latter having the direction of spin. The theory can now be applied satisfactorily to slowly moving protons and neutrons.

The new universal constants $f$ and $g$ can be interpreted in a simple way. By applying the theory successfully to the explanation of magnetic moments, we could derive the relation

$$f^2/\hbar c \simeq \mu/M \qquad (i)$$

which appears more than accidental. For consider the size of the deuteron. This depends on strength and mass of proton more than the range of nuclear forces. For a rough discussion, assume a Coulomb law $g^2/r$; the radius of the first Bohr orbit could then be $\hbar^2/Mg^2$ analogous to $\hbar^2/me^2$. Since experimentally the size of the deuteron is of order of magnitude $e^2/mc^2 \simeq \hbar/\mu c$, we have the equation

$$\frac{\hbar^2}{Mg^2} \simeq \frac{\hbar}{\mu c} \quad \text{or} \quad \frac{g^2}{\hbar c} \simeq \frac{\mu}{M}$$, which like (i) above is really more than accidental. Again identifying the range $1/\lambda$ of nuclear forces with $e^2/mc^2$ leads to

$$\frac{e^2}{\hbar c} \sim \frac{M}{\mu}$$

Numerically, $g^2/\hbar c \simeq \frac{1}{6}$ in contrast to $e^2/\hbar c = \frac{1}{137}$.

* The recent ~~purely~~ classical theory of Bhabha has ~~sho~~ thrown much light on the nature of this interaction between the meson & the nuclear particle & that in certain cases non-linearity is really not ~~ne~~ necessary.

We might now mention the applications of the meson theory where the feature of non-linearity enters. ~~Those~~ This arises when we consider collision processes with large collision cross-sections and infinite interaction potential, and the theory appears to break down for distances $< \frac{1}{\lambda}$. Let us first consider processes of large $Y$ - (N or P) interaction, for example the passage of fast mesons through matter. A striking feature of this theory is the occurrence of multiple processes like

$$Y^{+} + N \rightarrow P + Y^{+} + Y^{-}$$

the cross-section for which increases more rapidly than for simple processes, showing that the formulae are not valid in the region $\varepsilon \sim (2.5)\mu c^2$. Unfortunately there is no experimental evidence for the behaviour of mesons in the region $\mu c^2$ (80 M.V). There are experimental data for the region $2.5\mu c^2$ (200 M.V) and the theoretical cross-sections in this region are much larger than the experimental values. The very existence of multiple processes shows that the ordinary methods of quantum theory can no longer be applied in this region. Moreover the difference between Maxwell's theory and the meson theory is not in $1/\lambda$ but in exchange character (also spin dependent $f$) and the results differ from anything based on the correspondence principle or the analogy of meson theory to quantum electrodynamics.* ~~Thus~~ But in the region of these high ~~energy~~ energies the difficulties are more serious than in quantum electrodynamics. In the $(p\text{-}n)$ potential there occur terms $\propto \frac{1}{r^3}$. Inserting these into the Schrödinger equation some eigen-values become $\infty$. Some are finite (all singlet terms) but it is not possible to exclude the infinite ones because the finite ones do not form a completely orthogonal system. Thus the breakdown of the theory for distances $< \frac{1}{\lambda}$ appears to have a deep fundamental significance, and in all probability the right interaction between a heavy particle and meson will be of a <u>non-linear type</u> and this fact marks the limit of the applicability of the present quantum mechanics.

The position regarding non-linearity is however different from that in quantum electrodynamics where the deviations from linear laws are purely of a theoretical nature and cannot be verified experimentally. Thus all radiation effects of an electron could very well be treated by a first-order approximation theory and there would be no experiments contradicting such a theory. Here the behaviour of mesons in traversing matter provide direct experimental tests for the interaction in question, ie the non-linearity is of a determinable type.

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(6) <u>Born's reciprocity.</u>

We might also mention here another attempt at a non-linear theory in trying to build up a comprehensive one for all the elementary particles. Such a theory, as has been shown by Heisenberg, should contain a new universal constant like an absolute length, or momentum or field-strength. Basing his considerations on the fact that one has to rely only on theoretic grounds for introducing ~~linear~~ non-linearity, Born[20] has recently introduced his principle of reciprocity. The only natural and unique way of introducing non-linearity into a field theory is that used by Einstein in his theory of gravitation namely the postulate of general invariance which leads us to consider space-time as "curved". But Einstein's theory has to do with very small curvatures imperceptible in the region of laboratory dimensions. It is also clear from the smallness of the gravitational constant that cosmological curvature has nothing to do with atomic effects. These latter are bound to be ~~small~~ extremely small of order $\underline{a}$. But small $\underline{a}$ means large momentum $\underline{b} = \hbar/a$. Born's principle is that the domain of the elementary particle has to be considered from the standpoint of momentum-space in which a non-linear geometry with small curvature ~~so~~ reigns. The curvature equation should contain solely $\underline{a}$ and $\underline{b}$ (or $\underline{a}$ & $\hbar$) but not the gravitational constant.

It is difficult to assess the value of this theory until the electrodynamic laws have been completely reformulated on the basis of reciprocity. But recently on the assumption that the expression for the ~~electrodynamic~~ electrostatic energy of a resting charge is likely to hold in the new electrodynamics also, Born [21] has amplified ~~a suggestion of~~ an idea of Landé by giving a generalised wave mechanics suited to ~~the~~ describe the existence of particles with finite dimensions. The quantum constant divided by the product of an absolute energy and an absolute time appears as the parameter in a linear homogeneous integral equation of the Fredholm type, which actually amounts to a condition for the Planck's constant $\hbar$ on the basis of the reciprocity principle. The numerical determination of the fine-structure constant is reduced to the calculation of an eigen-value of this integral equation

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## (7) Non-linearity and equations of motion.

The question whether the field equations of a field theory are capable by themselves of determining the equations of motion of material particles considered as singularities of the field seems to be connected with the question of non-linearity but does not appear to be capable of an obvious answer. For instance ~~ws~~ in the wellknown theory of Helmholtz on the motion of vortices in a non-viscous fluid the motion of line-singularities is actually determined by the partial differential equations alone which are there non-linear. In the gravitational theory where also the field equations are non-linear the recent "approximation method" of Einstein - Infeld - Hoffmann[22] shows that such a thing is also possible. In ordinary Maxwells equations for empty space in which electrical particles are regarded as point singularities of the field, the motion of these singularites is not determined by the linear field equations. It appears as if linearity of the field equations is a criterion for the determination of the equations of motion without extraneous assumptions. But there are two exceptions. Thus in the recent Dirac theory of the electron the equations of motion follow immediately ~~although~~ and the field equations are linear. In Born's field theory on the other hand the equations of motion are not determined by the field equations alone which are non-linear.

Mention might be made here of another non-linear theory ~~due to~~ of the elementary particles due to Rosen[23] built on the model of Born's theory but with this difference that the Lagrangian contains besides the field strengths $f_{\mu\nu}$ also the potentials $\varphi_\mu$ so as to avoid singularities. At the same time the theory is made invariant for guage-transformations

$$\varphi'_\mu = \varphi_\mu + \partial\lambda/\partial x_\mu$$

The theory though classical offers in principle a possibility ~~for~~ of accounting for the Sommerfeld fine-structure constant, and shows some slight resemblances to the recent work in Born's reciprocity

## (8) Conclusion.

We have described a number of non-linear theories in ~~which~~ some of which non-linearity enters as a fundamental factor, and in some others as a secondary feature. Again in some ~~they are of~~ it is purely of a theoretic origin and in others of a type which can be decided by observation. As regards non-linearity as a principle in theoretical physics it is difficult to say anything in general. Judged by the principle of simplicity [24] linear laws appear to be the most suitable ones. On the score of the principle of beauty [24] it is difficult to decide between linear and non-linear theories, for mathematically the latter might be as rich in concepts as the former.

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