M. A. & M. Sc. EXAMINATIONS

Wednesday, 17*th April*, 1957] [11 *a.m. to* 2 *p.m.*

MATHEMATICS (GROUP B)—PAPER III

Statics, Attractions and Elasticity

N. B.— (*i*) Answer any SIX questions.

(*ii*) Not more than SEVEN questions are to be attempted.

1. State the Laws of Friction.

A uniform rod rests within a fixed vertical circle, subtending an angle 2α at the centre ; its upper end is smooth, and its lower end rough (coefficient of friction $= \tan\lambda$) ; show that the angle which the rod makes with the horizontal cannot be greater than θ, where

$$\tan\theta = \frac{\sin\lambda}{\cos\lambda + \cos(\lambda + 2\alpha)}.$$

2. State and prove the Principle of Virtual work for a system of coplanar forces.

A rhombus A B C D formed of four uniform freely jointed rods each of weight W and length a rests symmetrically in a vertical plane with AB, AD in contact with two smooth pegs in the same horizontal plane at a distance $2c$ apart (the vertex A being downwards) and is kept from collapsing by a light string BD. Prove that the tension of the string is

$$2W\,(a \sin A/2 - C \operatorname{cosec}^2 A/2) \Big/ a \cos A/2.$$

3. Obtain the tension at any point of a heavy string resting on a rough curve.

A heavy uniform chain rests on a rough cycloid, whose axis is vertical and vertex upwards, one end of the chain being at the vertex and the other at a cusp ; if equilibrium be limiting, show that

$$(1+\mu^2)\, e^{\frac{\mu\pi}{2}} = 3.$$

4. Obtain the general conditions of equilibrium of a system of forces, acting on a rigid body.

Derive the equations of the central axis of the system.

A force parallel to the axis of 2 acts at the point $(a, 0, 0)$ and an equal force perpendicular to the axis of Z acts at the point $(-a, 0, 0)$.

Show that the central axis of the system lies on the surface

$$Z^2 (x^2+y^2) = (x^2+y^2-ax)^2.$$

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5. (*a*) Find the co-ordinates of the nul point of the plane $lx+my+nz=1$ with respect to the system $(x, y, z, \mathrm{L}, \mathrm{M}, \mathrm{N})$.

(*b*) A uniform solid is bounded by the surface formed by the revolution of a cycloid about its base and is then cut in halves by a plane through the axis of revolution. Show that the centre of gravity of each half is at a distance $\frac{7a}{3\pi}$ from the plane face, where a is the radius of the generating circle of the cycloid.

6. Find the attraction of a solid uniform sphere at any internal point.

A solid homogeneous sphere is divided by a plane through its centre into two hemispheres. These being placed with their plane faces coincident, show that the force required to pull them apart is $\frac{3}{16}\frac{\mathrm{M}^2}{a^2}$, where M is the mass of the sphere and a its radius.

7. Derive the expression for the potential energy of a system of bodies.

Show that the work done in collecting the particles of a thin circular disc, attracting according to the Newtonian law, from an infinite distance is $\frac{8}{3}\frac{\gamma \mathrm{M}^2}{\pi a}$, where M is the mass of the disc and a its radius.

8. Prove Poisson's equation $\triangle^2 \mathrm{V} = -4\pi\gamma\rho$ at a point inside a gravitating mass.

Find the distribution of matter that gives rise to potentials

$$\mathrm{V}_1 = \frac{4}{3}\mu\pi a^3 + \frac{4}{15}\mu\pi a\,(2x^2-y^2-z^2),\ \mathrm{r}<a\ ;$$

$$\mathrm{V}_2 = \frac{4}{3}\mu\pi\frac{a^4}{r} + \frac{4}{15}\frac{\mu\pi a^6}{r^5}\,(2x^2-y^2-z^2),\ r>a.$$

9. (*a*) Show that a family of right circular cones with a common axis and vertex is a possible family of equipotential surfaces, and find the potential function.

(*b*) A right circular cylinder is of infinite length in one direction and is homogeneous, the finite extremity being perpendicular to the generators.

Prove that the attraction at the centre of this end is $\frac{2\mathrm{M}}{a}$ where M is its mass per unit length and a the radius.

10. Obtain the general equations of equilibrium for a thin rod bent in one plane and an expression for the work done against the stress couples in bending the rod.

Prove that the elastic energy of a uniform rod of weight W, cross section w and length l, hanging from one end is $\frac{1}{6}W^2 l / Ew$.

11. Prove the theorem of three moments.

A uniform slightly flexible rod AC of length $2a$, is supported at its end, and also at its middle point B; the supports being in the same horizontal line, find the thrusts on them and the equation to the curve in which the rod rests.

12. Find the stress strain relations in an isotropic body.

Prove that if a cylindrical tube of radii a and b $(a>b)$ closed at both ends be subject to an internal pressure p_o, the internal radius is increased by

$$\frac{b}{a^2-b^2}\left(\frac{a^2}{2\mu}+\frac{b^2}{3k}\right)p_o,$$ where μ and k have their usual meanings.

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Wednesday, 11*th April*] [11 *a.m. to* 2 *p.m.*

Mathematics (Group B)—Paper III

(STATICS, ATTRACTIONS AND ELASTICITY)

N. B.—(1) Answer any SIX questions.

(2) Not more than SEVEN questions are to be attempted.

1. State the Laws of Friction.

Two rough planes which intersect in a horizontal straight line are inclined at angles α and β to the horizon. A cylindrical ruler whose length is $2a$ and diameter $2b$ rests with one end on each plane the axis of the ruler being at right angles to the line of intersection of the planes. Show that if the ruler be in limiting equilibrium with its axis inclined at an angle θ to the horizon

$\cos(\theta+\phi)\cot(\alpha+\lambda)-\cos(\theta-\phi)\cot(\beta-\lambda)=2\sin\theta\cos\phi$ where $\tan\lambda$ is the coefficient of friction for each end and $\cot\phi=\frac{a}{b}$.

2. State and prove the Principle of Virtual Work for a system of coplanar forces.

A parallelogram $ABCD$ is formed of uniform heavy rods freely jointed at the extremities. AB is held fixed in a horizontal position and the parallelogram is maintained in its form so that ADC is an acute angle α, by means of a string joining A to a point P in DC. Prove that the tension of the string is $W.AP.\cot\alpha/DP$ where W is half the weight of the parallelogram.

3. Obtain the equations of equilibrium of a string lying on a rough curve in a vertical plane under gravity.

A heavy uniform string rests on the upper surface of a rough vertical circle of radius a and partly hangs vertically.

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Prove that if one end be at the highest point of the circle the greatest length that can hang freely is

$$\frac{2\mu a+(\mu^2-1)ae^{\frac{\mu\pi}{2}}}{\mu^2+1},$$ where μ is the coefficient of friction.

4. Show that a system of forces acting on a rigid body can be reduced in general to a force acting at a specified point of the body and a couple. Hence find the equation of the central axis.

A given force acts along the axis of x and another given force along a generator of the cylinder $x^2+y^2=a^2$; prove that the locus of the central axis is an elliptic cylinder.

5. The axes of two wrenches are at right angles and the shortest distance between them is $2a$. Prove that the axis of the resultant wrench divides the shortest distance in the ratio

$$Q\{2aQ+(p-q)Q\} : P\{2aP-(p-q)Q\}$$

where P and Q are the respective intensities of the wrenches and, p and q are their pitches.

A system of forces is such that every tangent to the curve given by $x=a\cos\theta, y=a\sin\theta, z=b\theta$ is a *nul* line. Find the central axis and prove that the pitch of the equivalent wrench is $\frac{a^2}{b}$.

6. Find the condition that a family of surfaces $f(x, y, z, \lambda)=0$ may represent a family of equipotential surfaces.

Show that the family of ellipsoids

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1$$

where λ is a variable parameter and a, b, c are constants is a possible form of equipotential surfaces, and express the potential in terms of a, b, c and λ.

7. Derive the expression for the potential energy of a system of bodies.

The radius of a sphere is a and its mass is M; if the density at any point of it varies inversely as the distance from the centre, show that the work done in bringing the particles of the sphere from infinity under their mutual attraction is $\frac{2\gamma M^2}{3a}$.

8. (*a*) If ρ be the density of a homogeneous solid sphere and a the radius, prove that the force in astronomical units exerted by the rest of the sphere on a conical portion of itself bounded by a right circular cone of angle 2α with its vertex at the centre is $\frac{1}{3}\pi^2\rho^2a^4\sin^2\alpha$.

(*b*) The potential outside a certain cylindrical boundary is zero; inside it is $V=x^3-3xy^2-ax^2+3ay^2$. Find the distribution of matter.

9. Obtain the general equations of equilibrium for a thin rod bent in one plane and an expression for the work done against the stress couples in bending the rod.

A uniform horizontal beam, of length l, and negligible weight, is supported at the ends and carries a weight w at the middle point. Find the deflection at this point and prove that the elastic energy is $\frac{W^2l^3}{96EI}$.

10. Prove the Theorem of Three Moments.

Prove that the deflection of the middle point of a uniform homogeneous elastic beam of length $2a$ supported at two points distant b from the middle point and lying in the same horizontal line is

$$\frac{1}{24}\frac{w}{B}(6a^2b^2-12ab^3+b^4)$$

where w is the weight per unit length of the beam and B the flexural rigidity.

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11. Prove Poisson's equation $\nabla^2 V = -4\pi\gamma\rho$ at a point inside a gravitating mass.

Matter is distributed in a thin layer over the surface of a sphere of radius a and centre at the origin, the density being proportional to $(x+y)^2$ and the whole mass being M. Show that the potential at an external point is given by

$$\frac{\gamma M}{r}\left\{1+\frac{a^2}{10r^4}(x^2+y^2+6xy-2z^2)\right\}$$

and find the value at an internal point.

12. Find the general stress-strain relation in an isotropic body.

The internal and external radii of a thick spherical shell of of an isotropic elastic substance are b and a respectively. If the cavity is filled with gas at pressure intensity p, show that the external radius is increased by $\frac{3ab^3(1-\sigma)p}{2E(a^3-b^3)}$, where σ and E have their usual meanings.